10-701: Introduction to Machine Learning

Lecture 24 –SVMs and Kernels

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* Slides adopted from F24 offering of 10701 by Henry Chai.

Summary Thus Far

- The margin of a linear separator is the distance between it and the nearest training data point
- Questions:
  1. How can we efficiently find a maximal-margin linear separator? By solving a constrained quadratic optimization problem using quadratic programming
  2. Why are linear separators with larger margins better? They're simpler *waves hands*
  3. What can we do if the data is not linearly separable? Next!

Hard-margin SVMs

$$\text{minimize} \quad \frac{1}{2}\boldsymbol{w}^T\boldsymbol{w}$$

$$\text{subject to} \quad y^{(i)}\left(\boldsymbol{w}^T\boldsymbol{x}^{(i)} + b\right) \geq 1 \; \forall \left(\boldsymbol{x}^{(i)}, y^{(i)}\right) \in \mathcal{D}$$

- When $\mathcal{D}$ is not linearly separable, there are no feasible solutions to this optimization problem

Soft-margin SVMs

$$\text{minimize} \quad \frac{1}{2}\boldsymbol{w}^T\boldsymbol{w} + C\sum_{i=1}^{N}\xi^{(i)}$$

$$\text{subject to} \quad y^{(i)}\left(\boldsymbol{w}^T\boldsymbol{x}^{(i)} + b\right) \geq 1 - \xi^{(i)} \;\forall \left(\boldsymbol{x}^{(i)}, y^{(i)}\right) \in \mathcal{D}$$

$$\xi^{(i)} \geq 0 \qquad \forall\, i \in \{1, \ldots, N\}$$

Primal-Dual Optimization

$$\text{minimize} \quad \frac{1}{2}\boldsymbol{w}^T\boldsymbol{w}$$

$$\text{subject to} \quad y^{(i)}\left(\boldsymbol{w}^T\boldsymbol{x}^{(i)} + w_0\right) \geq 1 \; \forall \left(\boldsymbol{x}^{(i)}, y^{(i)}\right) \in \mathcal{D}$$

Primal

$$\Updownarrow$$

$$\text{maximize} \quad -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\alpha^{(i)}\alpha^{(j)}y^{(i)}y^{(j)}\boldsymbol{x}^{(i)^T}\boldsymbol{x}^{(j)} + \sum_{i=1}^{N}\alpha^{(i)}$$

$$\text{subject to} \quad \sum_{i=1}^{N}\alpha^{(i)}y^{(i)} = 0$$

$$\alpha^{(i)} \geq 0 \; \forall \; i \in \{1, \ldots, N\}$$

Dual

Note: Primal-Dual Optimization

- **Lagrange Method** handles constraints by incorporating them into the objective using multipliers.
  - Provides *necessary* optimality conditions for equality-constrained problems. No dual problem.
- **Primal–Dual Optimization** considers the primal problem and its dual—tracking both primal variables and Lagrange multipliers—to find optimality.

SVM

$$\underset{\boldsymbol{w}, w_0}{\text{minimize}} \; \frac{1}{2}\boldsymbol{w}^T\boldsymbol{w} + \underset{\alpha^{(i)} \geq 0}{\text{maximize}} \; \sum_{i=1}^{N} \alpha^{(i)} \left(1 - y^{(i)}\left(\boldsymbol{w}^T\boldsymbol{x}^{(i)} + w_0\right)\right)$$

$$\Updownarrow$$

$$\underset{\boldsymbol{w}, w_0}{\text{minimize}} \; \underset{\alpha^{(i)} \geq 0}{\text{maximize}} \; \frac{1}{2}\boldsymbol{w}^T\boldsymbol{w} + \sum_{i=1}^{N} \alpha^{(i)} \left(1 - y^{(i)}\left(\boldsymbol{w}^T\boldsymbol{x}^{(i)} + w_0\right)\right)$$

$$\Updownarrow$$

$$\underset{\alpha^{(i)} \geq 0}{\text{maximize}} \; \underset{\boldsymbol{w}, w_0}{\text{minimize}} \; \frac{1}{2}\boldsymbol{w}^T\boldsymbol{w} + \sum_{i=1}^{N} \alpha^{(i)} \left(1 - y^{(i)}\left(\boldsymbol{w}^T\boldsymbol{x}^{(i)} + w_0\right)\right)$$

$$\Updownarrow$$

$$\underset{\boldsymbol{\alpha} \geq 0}{\text{maximize}} \; \underset{\boldsymbol{w}, w_0}{\text{minimize}} \; L(\boldsymbol{\alpha}, \boldsymbol{w}, w_0)$$

Karush-Kuhn-Tucker (KKT) Conditions

$$\underset{\boldsymbol{w}, w_0}{\text{minimize}} \quad L(\boldsymbol{\alpha}, \boldsymbol{w}, w_0)$$

$$L(\boldsymbol{\alpha}, \boldsymbol{w}, w_0) = \frac{1}{2}\boldsymbol{w}^T\boldsymbol{w} + \sum_{i=1}^{N} \alpha^{(i)} \left(1 - y^{(i)}\left(\boldsymbol{w}^T\boldsymbol{x}^{(i)} + w_0\right)\right)$$

$$\frac{\partial L(\boldsymbol{\alpha}, \boldsymbol{w}, w_0)}{\partial \boldsymbol{w}} = \boldsymbol{w} - \sum_{i=1}^{N} \alpha^{(i)} y^{(i)} \boldsymbol{x}^{(i)} \rightarrow \widehat{\boldsymbol{w}} = \sum_{i=1}^{N} \alpha^{(i)} y^{(i)} \boldsymbol{x}^{(i)}$$

$$\frac{\partial L(\boldsymbol{\alpha}, \boldsymbol{w}, w_0)}{\partial w_0} = -\sum_{i=1}^{N} \alpha^{(i)} y^{(i)} \qquad \rightarrow \sum_{i=1}^{N} \alpha^{(i)} y^{(i)} = 0$$

Minimizing the Lagrangian

$$\widehat{\boldsymbol{w}} = \sum_{i=1}^{N} \alpha^{(i)} y^{(i)} \boldsymbol{x}^{(i)}$$

$$\sum_{i=1}^{N} \alpha^{(i)} y^{(i)} = 0$$

$$L(\boldsymbol{\alpha}, \widehat{\boldsymbol{w}}, \widehat{w}_0) = \frac{1}{2} \widehat{\boldsymbol{w}}^T \widehat{\boldsymbol{w}} + \sum_{i=1}^{N} \alpha^{(i)} \left(1 - y^{(i)} \left(\widehat{\boldsymbol{w}}^T \boldsymbol{x}^{(i)} + \widehat{w}_0\right)\right)$$

Minimizing the Lagrangian

$$\hat{\boldsymbol{w}} = \sum_{i=1}^{N} \alpha^{(i)} y^{(i)} \boldsymbol{x}^{(i)}$$

$$\sum_{i=1}^{N} \alpha^{(i)} y^{(i)} = 0$$

$$L(\boldsymbol{\alpha}, \hat{\boldsymbol{w}}, \hat{w}_0) = \frac{1}{2} \hat{\boldsymbol{w}}^T \hat{\boldsymbol{w}} + \sum_{i=1}^{N} \alpha^{(i)} \left(1 - y^{(i)} \left(\hat{\boldsymbol{w}}^T \boldsymbol{x}^{(i)} + \hat{w}_0\right)\right)$$

$$= \frac{1}{2} \hat{\boldsymbol{w}}^T \hat{\boldsymbol{w}}$$

$$+ \sum_{i=1}^{N} \alpha^{(i)} - \sum_{i=1}^{N} \alpha^{(i)} y^{(i)} \hat{\boldsymbol{w}}^T \boldsymbol{x}^{(i)} - \hat{w}_0 \sum_{i=1}^{N} \alpha^{(i)} y^{(i)} \quad = 0$$

$$= \frac{1}{2} \left(\sum_{i=1}^{N} \alpha^{(i)} y^{(i)} \boldsymbol{x}^{(i)} \right) \left(\sum_{j=1}^{N} \alpha^{(j)} y^{(j)} \boldsymbol{x}^{(j)} \right)$$

$$+ \sum_{i=1}^{N} \alpha^{(i)} - \sum_{i=1}^{N} \alpha^{(i)} y^{(i)} \left(\sum_{j=1}^{N} \alpha^{(j)} y^{(j)} \boldsymbol{x}^{(j)} \right)^T \boldsymbol{x}^{(i)}$$

Minimizing the Lagrangian

$$\hat{\boldsymbol{w}} = \sum_{i=1}^{N} \alpha^{(i)} y^{(i)} \boldsymbol{x}^{(i)}$$

$$\sum_{i=1}^{N} \alpha^{(i)} y^{(i)} = 0$$

$$L(\boldsymbol{\alpha}, \hat{\boldsymbol{w}}, \hat{w}_0) = \frac{1}{2} \hat{\boldsymbol{w}}^T \hat{\boldsymbol{w}} + \sum_{i=1}^{N} \alpha^{(i)} \left(1 - y^{(i)} \left(\hat{\boldsymbol{w}}^T \boldsymbol{x}^{(i)} + \hat{w}_0\right)\right)$$

$$= \frac{1}{2} \hat{\boldsymbol{w}}^T \hat{\boldsymbol{w}}$$

$$+ \sum_{i=1}^{N} \alpha^{(i)} - \sum_{i=1}^{N} \alpha^{(i)} y^{(i)} \hat{\boldsymbol{w}}^T \boldsymbol{x}^{(i)}$$

$$= -\frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha^{(i)} \alpha^{(j)} y^{(i)} y^{(j)} \boldsymbol{x}^{(i)^T} \boldsymbol{x}^{(j)} + \sum_{i=1}^{N} \alpha^{(i)}$$

Maximizing the Minimum

$$\underset{\boldsymbol{\alpha} \geq 0}{\text{maximize}} \; \underset{\boldsymbol{w}, w_0}{\text{minimize}} \; L(\boldsymbol{\alpha}, \boldsymbol{w}, w_0)$$

$$\Updownarrow$$

$$\text{maximize} \; -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} \alpha^{(i)}\alpha^{(j)}y^{(i)}y^{(j)}\boldsymbol{x}^{(i)^T}\boldsymbol{x}^{(j)} + \sum_{i=1}^{N}\alpha^{(i)}$$

$$\text{subject to} \; \sum_{i=1}^{N}\alpha^{(i)}y^{(i)} = 0$$

$$\alpha^{(i)} \geq 0 \; \forall \; i \in \{1, \ldots, N\}$$

Primal-Dual Optimization

$$\text{minimize} \quad \frac{1}{2}\boldsymbol{w}^T\boldsymbol{w}$$

$$\text{subject to} \quad y^{(i)}\left(\boldsymbol{w}^T\boldsymbol{x}^{(i)} + w_0\right) \geq 1 \; \forall \left(\boldsymbol{x}^{(i)}, y^{(i)}\right) \in \mathcal{D}$$

Primal

$$\text{minimize} \quad \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\alpha^{(i)}\alpha^{(j)}y^{(i)}y^{(j)}\boldsymbol{x}^{(i)^T}\boldsymbol{x}^{(j)} - \sum_{i=1}^{N}\alpha^{(i)}$$

$$\text{subject to} \quad \sum_{i=1}^{N}\alpha^{(i)}y^{(i)} = 0$$

$$\alpha^{(i)} \geq 0 \; \forall \; i \in \{1, \ldots, N\}$$

Dual

Primal-Dual Optimization

- Primal
  - Directly returns the weights, $[\hat{w}_0, \hat{\boldsymbol{w}}]$
  - Support vectors are all $\left(\boldsymbol{x}^{(s)}, y^{(s)}\right) \in \mathcal{D}$ s.t.

$$y^{(s)}\left(\hat{\boldsymbol{w}}^T \boldsymbol{x}^{(s)} + \hat{w}_0\right) = 1$$

- Dual
  - Returns the vector, $\hat{\boldsymbol{\alpha}}$

$$\hat{\boldsymbol{w}} = \sum_{i=1}^{N} \hat{\alpha}^{(i)} y^{(i)} \boldsymbol{x}^{(i)}$$

$$\hat{w}_0 = ???$$

Complementary Slackness

$$\text{minimize} \quad \frac{1}{2}\boldsymbol{w}^T\boldsymbol{w}$$

$$\text{subject to} \quad 1 - y^{(i)}\left(\boldsymbol{w}^T\boldsymbol{x}^{(i)} + w_0\right) \leq 0 \; \forall \left(\boldsymbol{x}^{(i)}, y^{(i)}\right) \in \mathcal{D}$$

$$\Updownarrow$$

$$\underset{\boldsymbol{w}, w_0}{\text{minimize}} \quad \frac{1}{2}\boldsymbol{w}^T\boldsymbol{w} + \underset{\alpha^{(i)} \geq 0}{\text{maximize}} \quad \sum_{i=1}^{N} \alpha^{(i)} \left(1 - y^{(i)}\left(\boldsymbol{w}^T\boldsymbol{x}^{(i)} + w_0\right)\right)$$

- Theorem: $\hat{\alpha}^{(i)} \left(1 - y^{(i)}\left(\hat{\boldsymbol{w}}^T\boldsymbol{x}^{(i)} + \hat{w}_0\right)\right) = 0 \; \forall \left(\boldsymbol{x}^{(i)}, y^{(i)}\right) \in \mathcal{D}$
  - If $\hat{\alpha}^{(s)} > 0$, then $1 - y^{(s)}\left(\hat{\boldsymbol{w}}^T\boldsymbol{x}^{(s)} + \hat{w}_0\right) = 0$

Computing $\hat{w}_0$

$$\hat{\alpha}^{(i)}\left(1 - y^{(i)}\left(\hat{\boldsymbol{w}}^T \boldsymbol{x}^{(i)} + \hat{w}_0\right)\right) = 0 \; \forall \left(\boldsymbol{x}^{(i)}, y^{(i)}\right) \in \mathcal{D}$$

If $\hat{\alpha}^{(s)} > 0 \rightarrow 1 - y^{(s)}\left(\hat{\boldsymbol{w}}^T \boldsymbol{x}^{(s)} + \hat{w}_0\right) = 0$

$$\rightarrow y^{(s)}\left(\hat{\boldsymbol{w}}^T \boldsymbol{x}^{(s)} + \hat{w}_0\right) = 1$$

$$\rightarrow {y^{(s)}}^2\left(\hat{\boldsymbol{w}}^T \boldsymbol{x}^{(s)} + \hat{w}_0\right) = y^{(s)}$$

$$\rightarrow \hat{\boldsymbol{w}}^T \boldsymbol{x}^{(s)} + \hat{w}_0 = y^{(s)}$$

$$\rightarrow \hat{w}_0 = y^{(s)} - \hat{\boldsymbol{w}}^T \boldsymbol{x}^{(s)}$$

Primal-Dual Optimization

- Primal
  - Directly returns the weights, $[\hat{w}_0, \hat{\boldsymbol{w}}]$
  - Support vectors are all $\left(\boldsymbol{x}^{(s)}, y^{(s)}\right) \in \mathcal{D}$ s.t.

$$y^{(s)}\left(\hat{\boldsymbol{w}}^T \boldsymbol{x}^{(s)} + \hat{w}_0\right) = 1$$

- Dual
  - Returns the vector, $\hat{\boldsymbol{\alpha}}$

$$\hat{\boldsymbol{w}} = \sum_{i=1}^{N} \hat{\alpha}^{(i)} y^{(i)} \boldsymbol{x}^{(i)}$$

$$\hat{w}_0 = y^{(s)} - \hat{\boldsymbol{w}}^T \boldsymbol{x}^{(s)} \text{ for any } s \text{ s.t. } \hat{\alpha}^{(s)} > 0$$

  - Support vectors are all $\left(\boldsymbol{x}^{(s)}, y^{(s)}\right) \in \mathcal{D}$ s.t. $\hat{\alpha}^{(s)} > 0$

Primal-Dual Optimization

- Primal
  - $\hat{y} = \text{sign}(\hat{\boldsymbol{w}}^T \vec{x} + \hat{w}_0)$

- Dual
  - $\hat{y} = \text{sign}(\hat{\boldsymbol{w}}^T \vec{x} + \hat{w}_0)$

$$= \text{sign}\left(\left(\sum_{i=1}^{N} \hat{\alpha}^{(i)} y^{(i)} \boldsymbol{x}^{(i)}\right)^T \boldsymbol{x} + \hat{w}_0\right)$$

$$= \text{sign}\left(\sum_{i\,:\,\hat{\alpha}^{(i)} > 0} \hat{\alpha}^{(i)} y^{(i)} \boldsymbol{x}^{(i)^T} \boldsymbol{x} + \hat{w}_0\right)$$

Primal-Dual Soft-Margin SVMs

Primal

$$\text{minimize} \quad \frac{1}{2}\boldsymbol{w}^T\boldsymbol{w} + C\sum_{i=1}^{N}\xi^{(i)}$$

$$\text{subject to} \quad y^{(i)}\left(\boldsymbol{w}^T\boldsymbol{x}^{(i)} + w_0\right) \geq 1 - \xi^{(i)} \quad \forall \left(\boldsymbol{x}^{(i)}, y^{(i)}\right) \in \mathcal{D}$$

$$\xi^{(i)} \geq 0 \quad \forall\, i \in \{1, \ldots, N\}$$

Dual

$$\text{minimize} \quad \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\alpha^{(i)}\alpha^{(j)}y^{(i)}y^{(j)}\boldsymbol{x}^{(i)^T}\boldsymbol{x}^{(j)} - \sum_{i=1}^{N}\alpha^{(i)}$$

$$\text{subject to} \quad \sum_{i=1}^{N}\alpha^{(i)}y^{(i)} = 0$$

$$0 \leq \alpha^{(i)} \leq C \;\forall\, i \in \{1, \ldots, N\}$$

Primal-Dual Soft-Margin SVMs

Primal

$$\text{minimize} \quad \frac{1}{2}\boldsymbol{w}^T\boldsymbol{w} + C\sum_{i=1}^{N}\xi^{(i)}$$

$$\text{subject to} \quad y^{(i)}\left(\boldsymbol{w}^T\boldsymbol{x}^{(i)} + w_0\right) \geq 1 - \xi^{(i)} \quad \forall \left(\boldsymbol{x}^{(i)}, y^{(i)}\right) \in \mathcal{D}$$

$$\xi^{(i)} \geq 0 \quad \forall\, i \in \{1, \ldots, N\}$$

Dual

$$\text{minimize} \quad \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\alpha^{(i)}\alpha^{(j)}y^{(i)}y^{(j)}\boldsymbol{x}^{(i)^T}\boldsymbol{x}^{(j)} - \sum_{i=1}^{N}\alpha^{(i)}$$

$$\text{subject to} \quad \sum_{i=1}^{N}\alpha^{(i)}y^{(i)} = 0$$

$$0 \leq \alpha^{(i)} \leq C \;\forall\, i \in \{1, \ldots, N\}$$

Recall: Nonlinear Transforms

- Decide on some transformation $\Phi: \mathcal{X} \to \mathcal{Z}$
- Given $\mathcal{D} = \left\{\left(\boldsymbol{x}^{(i)}, y^{(i)}\right)\right\}_{i=1}^{N}$, learn a hypothesis, $\tilde{h}(\boldsymbol{z})$, using $\widetilde{\mathcal{D}} = \left\{\left(\boldsymbol{z}^{(i)} = \Phi\left(\boldsymbol{x}^{(i)}\right), y^{(i)}\right)\right\}_{i=1}^{N}$
- Return the corresponding predictor in the original space: $h(\boldsymbol{x}) = \tilde{h}\left(\Phi(\boldsymbol{x})\right)$

Nonlinear SVMs

- Decide on some transformation $\Phi: \mathcal{X} \rightarrow \mathcal{Z}$
- Find a maximal-margin separating hyperplane in the transformed space, $[\widetilde{\widehat{\boldsymbol{w}}}, \widetilde{\widehat{w}_0}]$, by solving the QP:

$$\text{minimize} \quad \frac{1}{2}\widetilde{\boldsymbol{w}}^T\widetilde{\boldsymbol{w}}$$

$$\text{subject to} \quad y^{(i)}\left(\widetilde{\boldsymbol{w}}^T\Phi\left(\boldsymbol{x}^{(i)}\right) + \widetilde{w_0}\right) \geq 1 \; \forall \left(\boldsymbol{x}^{(i)}, y^{(i)}\right) \in \mathcal{D}$$

- Return the corresponding predictor in the original space:

$$h(\boldsymbol{x}) = \text{sign}\left(\widetilde{\widehat{\boldsymbol{w}}}^T\Phi(\boldsymbol{x}) + \widetilde{\widehat{w}_0}\right)$$

Nonlinear Dual SVMs

- Decide on some transformation $\Phi: \mathcal{X} \to \mathcal{Z}$
- Find a maximal-margin separating hyperplane in the transformed space by solving the QP:

$$\text{minimize} \quad \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\alpha^{(i)}\alpha^{(j)}y^{(i)}y^{(j)}\Phi\left(\boldsymbol{x}^{(i)}\right)^{T}\Phi\left(\boldsymbol{x}^{(j)}\right) - \sum_{i=1}^{N}\alpha^{(i)}$$

$$\text{subject to} \quad \sum_{i=1}^{N}\alpha^{(i)}y^{(i)} = 0$$

$$0 \leq \alpha^{(i)} \leq C \;\forall\; i \in \{1, \ldots, N\}$$

- Return the corresponding predictor in the original space:

$$h(\boldsymbol{x}) = \text{sign}\left(\sum_{i:\hat{\alpha}^{(i)}>0}\hat{\alpha}^{(i)}y^{(i)}\Phi\left(\boldsymbol{x}^{(i)}\right)^{T}\Phi(\boldsymbol{x}) + \widetilde{\hat{w}_0}\right)$$

Efficiency

- Depending on the transformation Φ and the dimensionality of the original input space $\mathcal{X}$, computing $\Phi(\boldsymbol{x})$ can be prohibitively computationally expensive
  - Computing $\Phi_2(\boldsymbol{x}) = [x_1, x_2, \dots, x_D, x_1^2, x_1 x_2, \dots, x_D^2]$ requires $D + \binom{D}{2} + D = \frac{D^2+3D}{2} = O(D^2)$ time
  - Computing $\Phi_{10}(\boldsymbol{x})$ requires $O(D^{10})$ time
- Tradeoff:
  - High-dimensional transformations can result in good hypotheses (as long as they don't overfit)
  - High-dimensional transformations are expensive

Nonlinear Dual SVMs

- Insight: Φ only appears in inner products!
- Find a maximal-margin separating hyperplane in the transformed space by solving the QP:

$$\text{minimize} \quad \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\alpha^{(i)}\alpha^{(j)}y^{(i)}y^{(j)}\Phi\left(\boldsymbol{x}^{(i)}\right)^{T}\Phi\left(\boldsymbol{x}^{(j)}\right) - \sum_{i=1}^{N}\alpha^{(i)}$$

$$\text{subject to} \quad \sum_{i=1}^{N}\alpha^{(i)}y^{(i)} = 0$$

$$0 \leq \alpha^{(i)} \leq C \;\forall\; i \in \{1, \ldots, N\}$$

- Return the corresponding predictor in the original space:

$$h(\boldsymbol{x}) = \text{sign}\left(\sum_{i:\hat{\alpha}^{(i)}>0}\hat{\alpha}^{(i)}y^{(i)}\Phi\left(\boldsymbol{x}^{(i)}\right)^{T}\Phi(\boldsymbol{x}) + \widetilde{\hat{w}_0}\right)$$

Nonlinear Dual SVMs

- Insight: Φ only appears in inner products!
- Find a maximal-margin separating hyperplane in the transformed space by solving the QP:

$$\text{minimize} \quad \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\alpha^{(i)}\alpha^{(j)}y^{(i)}y^{(j)}\Phi\left(\boldsymbol{x}^{(i)}\right)^{T}\Phi\left(\boldsymbol{x}^{(j)}\right) - \sum_{i=1}^{N}\alpha^{(i)}$$

$$\text{subject to} \quad \sum_{i=1}^{N}\alpha^{(i)}y^{(i)} = 0$$

$$0 \leq \alpha^{(i)} \leq C \;\forall\; i \in \{1, \dots, N\}$$

- Return the corresponding predictor in the original space:

$$h(\boldsymbol{x}) = \text{sign}\left(\sum_{i:\hat{\alpha}^{(i)}>0}\hat{\alpha}^{(i)}y^{(i)}\Phi\left(\boldsymbol{x}^{(i)}\right)^{T}\Phi(\boldsymbol{x}) + \widetilde{\widehat{w}_0}\right)$$

The Kernel Trick

- Approach: instead of computing $\Phi(\boldsymbol{x})$, find some function K_Φ s.t. $K_\Phi(\boldsymbol{x}, \boldsymbol{x}') = \Phi(\boldsymbol{x})^T \Phi(\boldsymbol{x}') \; \forall \; \boldsymbol{x}, \boldsymbol{x}' \in \mathcal{X}$
  - $K_\Phi(\boldsymbol{x}, \boldsymbol{x}')$ should be cheaper to compute than $\Phi(\boldsymbol{x})$

Example

- Approach: instead of computing $\Phi(\boldsymbol{x})$, find some function K_Φ s.t. $K_\Phi(\boldsymbol{x}, \boldsymbol{x}') = \Phi(\boldsymbol{x})^T \Phi(\boldsymbol{x}') \; \forall \; \boldsymbol{x}, \boldsymbol{x}' \in \mathcal{X}$
  - $K_\Phi(\boldsymbol{x}, \boldsymbol{x}')$ should be cheaper to compute than $\Phi(\boldsymbol{x})$
- Example: $\Phi_2'(\boldsymbol{x}) = \left[x_1, \ldots, x_D, x_1^2, \sqrt{2}x_1x_2, \ldots, \sqrt{2}x_{D-1}x_D, x_D^2\right]$

$$K_{\Phi_2'}(\boldsymbol{x}, \boldsymbol{x}') = \boldsymbol{x}^T\boldsymbol{x}' + (\boldsymbol{x}^T\boldsymbol{x}')^2$$

- Computing $\Phi_2'(\boldsymbol{x})^T \Phi_2'(\boldsymbol{x}')$ requires $O(D^2)$ time whereas computing $K_{\Phi_2'}(\boldsymbol{x}, \boldsymbol{x}')$ only takes $O(D)$!

Example

- Approach: instead of computing $\Phi(\boldsymbol{x})$, find some function K_Φ s.t. $K_\Phi(\boldsymbol{x}, \boldsymbol{x}') = \Phi(\boldsymbol{x})^T \Phi(\boldsymbol{x}') \; \forall \, \boldsymbol{x}, \boldsymbol{x}' \in \mathcal{X}$
  - $K_\Phi(\boldsymbol{x}, \boldsymbol{x}')$ should be cheaper to compute than $\Phi(\boldsymbol{x})$
- Example: $\Phi_2'(\boldsymbol{x}) = \left[x_1, \ldots, x_D, x_1^2, \sqrt{2}x_1x_2, \ldots, \sqrt{2}x_{D-1}x_D, x_D^2\right]$

$$\Phi_2'(\boldsymbol{x})^T \Phi_2'(\boldsymbol{x}') = \sum_{i=1}^{D} x_i x_i' + \sum_{i=1}^{D} x_i^2 {x_i'}^2 + \sum_{i=1}^{D} \sum_{j>i} 2x_i x_i' x_j x_j'$$

$$= \sum_{i=1}^{D} x_i x_i' + \left(\sum_{i=1}^{D} x_i x_i' \right)^2 = \boldsymbol{x}^T \boldsymbol{x}' + (\boldsymbol{x}^T \boldsymbol{x}')^2$$

$$K_{\Phi_2'}(\boldsymbol{x}, \boldsymbol{x}') = \boldsymbol{x}^T \boldsymbol{x}' + (\boldsymbol{x}^T \boldsymbol{x}')^2$$

- Computing $\Phi_2'(\boldsymbol{x})^T \Phi_2'(\boldsymbol{x}')$ requires $O(D^2)$ time whereas computing $K_{\Phi_2'}(\boldsymbol{x}, \boldsymbol{x}')$ only takes $O(D)$!

Common Kernels

- $K_{\Phi_2'}(\boldsymbol{x}, \boldsymbol{x}') = \boldsymbol{x}^T\boldsymbol{x}' + (\boldsymbol{x}^T\boldsymbol{x}')^2$
  - Implied feature transformation:

$$\Phi_2'(\boldsymbol{x}) = \left[x_1, \dots, x_D, x_1^2, \sqrt{2}x_1x_2, \dots, \sqrt{2}x_{D-1}x_D, x_D^2\right]$$

  - Implied dimensionality: $\frac{D^2+3D}{2}$

- $K_{\Phi_2^{(\gamma)}}(\boldsymbol{x}, \boldsymbol{x}') = (1 + \gamma\boldsymbol{x}^T\boldsymbol{x}')^2 - 1$
  - Implied feature transformation:

$$\Phi_2^{(\gamma)}(\boldsymbol{x}) = \left[\sqrt{2\gamma}x_1, \dots, \sqrt{2\gamma}x_D, \gamma x_1^2, \gamma x_1x_2, \dots, \gamma x_D^2\right]$$

    - γ affects the geometry of the transform

  - Implied dimensionality: $\frac{D^2+3D}{2}$

Common Kernels

- Polynomial Kernel: $K_{\Phi_Q^{(\gamma)}}(\boldsymbol{x}, \boldsymbol{x}') = (1 + \gamma \boldsymbol{x}^T \boldsymbol{x}')^Q - 1$
  - Implied dimensionality: $O(D^Q)$
  - γ affects the geometry of the transform
- Gaussian-RBF Kernel: $K_{\Phi_r}(\boldsymbol{x}, \boldsymbol{x}') = e^{-\frac{\|x-x'\|^2}{2r}}$
  - Implied feature transformation: $\Phi_r(\boldsymbol{x}) = \left[e^{-\frac{x_1^2}{2r}}, \dots, e^{-\frac{x_D^2}{2r}}, \right.$

$$\left. e^{-\frac{x_1^2}{2r}} \sqrt{\frac{(x_1)^2}{1! r^1}}, \dots, e^{-\frac{x_D^2}{2r}} \sqrt{\frac{(x_D)^2}{1! r^1}}, e^{-\frac{x_1^2}{2r}} \sqrt{\frac{\left(x_1^2\right)^2}{2! r^2}}, \dots, e^{-\frac{x_D^2}{2r}} \sqrt{\frac{\left(x_D^2\right)^2}{2! r^2}}, \dots \right]$$

Common Kernels

- Polynomial Kernel: $K_{\Phi_Q^{(\gamma)}}(\boldsymbol{x}, \boldsymbol{x}') = (1 + \gamma \boldsymbol{x}^T \boldsymbol{x}')^Q - 1$
  - Implied dimensionality: $O(D^Q)$
  - γ affects the geometry of the transform
- Gaussian-RBF Kernel: $K_{\Phi_r}(\boldsymbol{x}, \boldsymbol{x}') = e^{-\frac{\|x - x'\|^2}{2r}}$
  - Implied feature transformation: $\Phi_r(x) =$

$$\left[\left[e^{-\frac{x_1^2}{2r}}\sqrt{\frac{\left(x_1^d\right)^2}{d! r^d}}, \ldots, e^{-\frac{x_D^2}{2r}}\sqrt{\frac{\left(x_1^d\right)^2}{d! r^d}}\right] : d \in \mathbb{N}\right]$$

  - Implied dimensionality: ∞!

Nonlinear Dual SVMs

- Decide on a (valid) kernel function K_Φ
- Find a maximal-margin separating hyperplane in the transformed space by solving the QP:

$$\text{minimize} \quad \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\alpha^{(i)}\alpha^{(j)}y^{(i)}y^{(j)}K_\Phi\left(\boldsymbol{x}^{(i)},\boldsymbol{x}^{(j)}\right)-\sum_{i=1}^{N}\alpha^{(i)}$$

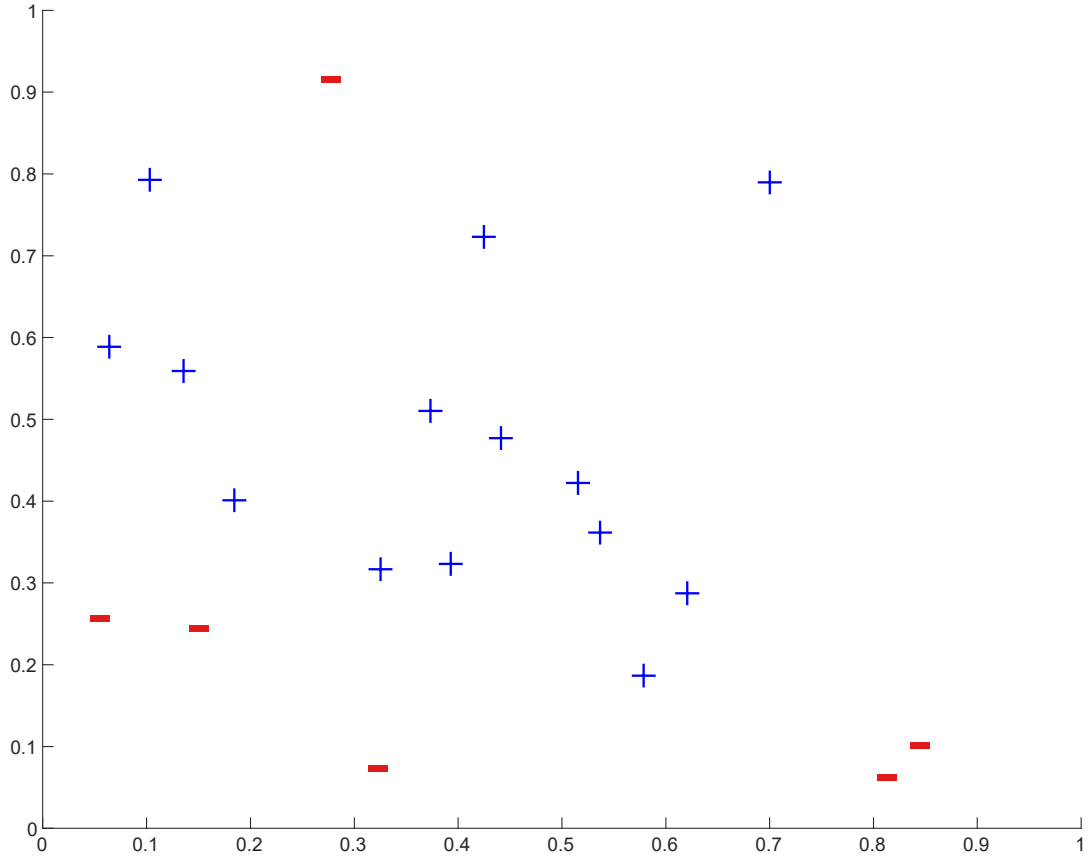
$$\text{subject to} \quad \sum_{i=1}^{N}\alpha^{(i)}y^{(i)}=0$$

$$0\leq\alpha^{(i)}\ \forall\ i\in\{1,\ldots,N\}$$

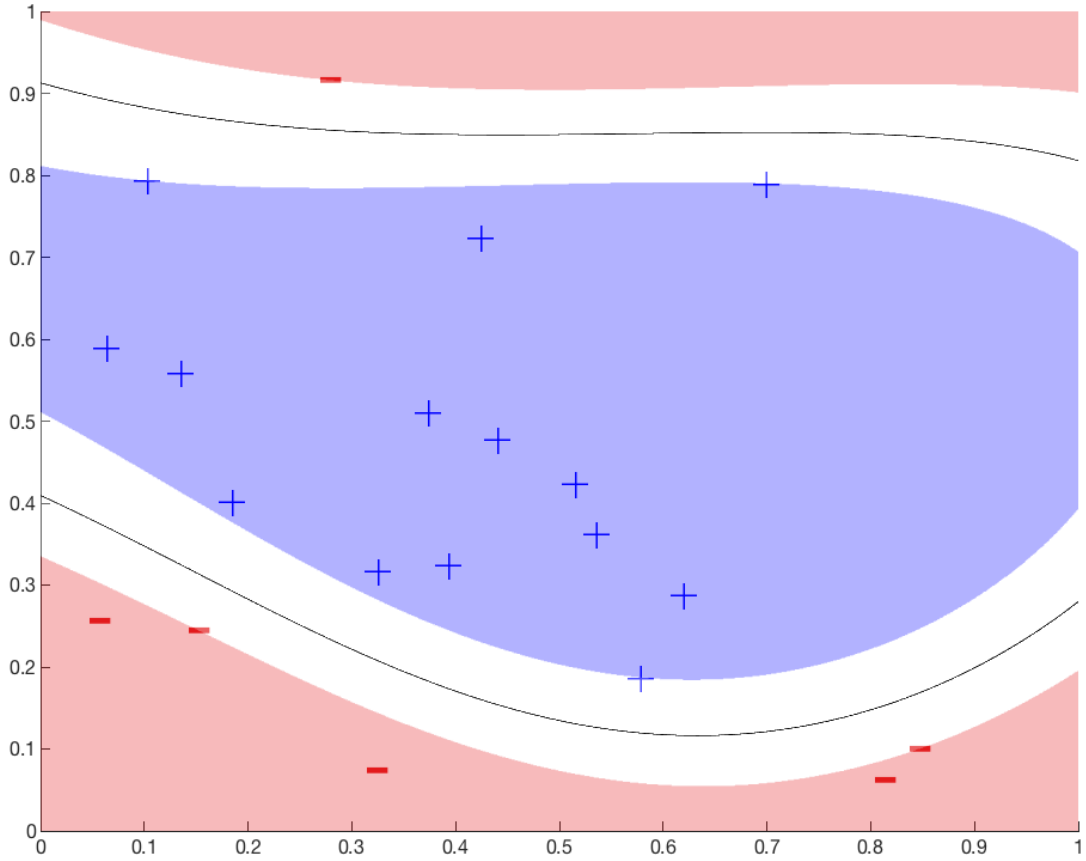
- Return the corresponding predictor in the original space:

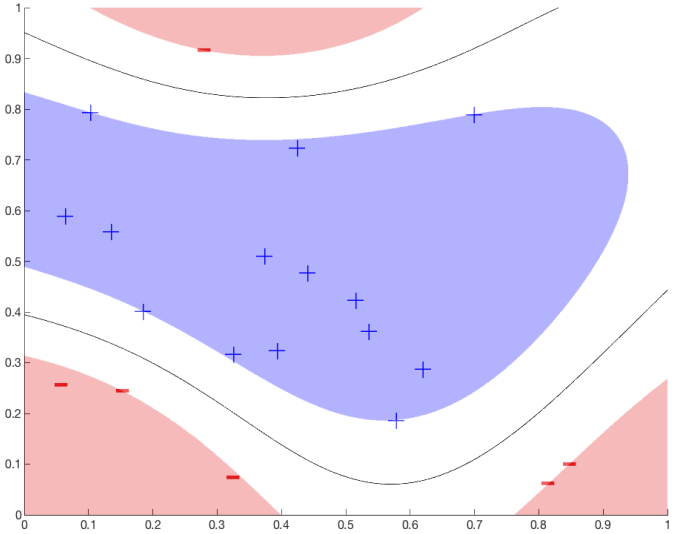
$$h(\boldsymbol{x})=\text{sign}\left(\sum_{i:\hat{\alpha}^{(i)}>0}\hat{\alpha}^{(i)}y^{(i)}K_\Phi\left(\boldsymbol{x}^{(i)},\boldsymbol{x}\right)+\widetilde{\hat{w}_0}\right)$$

Gaussian-RBF Kernel

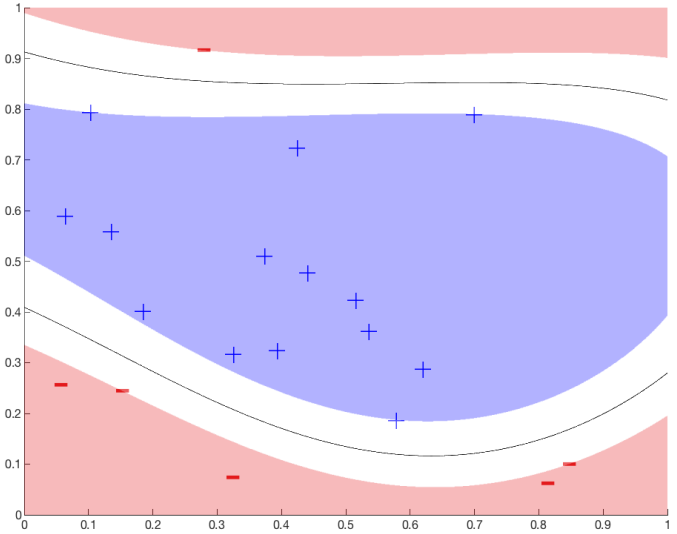


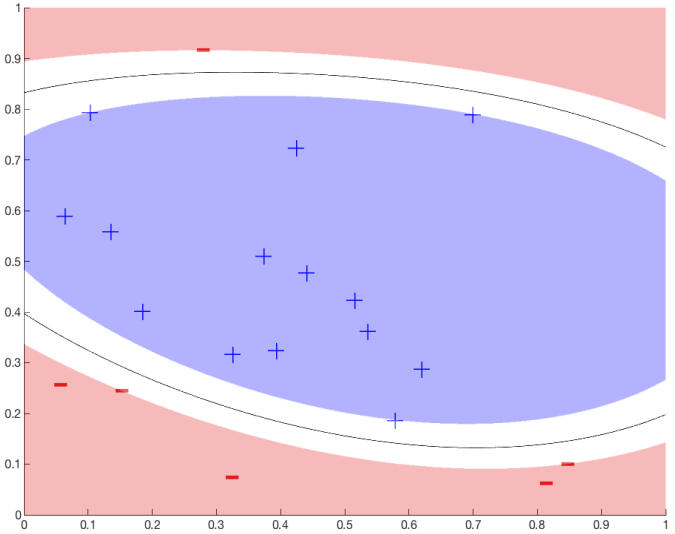
Gaussian-RBF Kernel





Smaller r





Larger r

Gaussian-RBF Kernel

Valid Kernels

- Any function K is a valid kernel if and only if:
  - $\exists$ a transformation Φ s.t.

$$K(\boldsymbol{x}, \boldsymbol{x}') = \Phi(\boldsymbol{x})^T \Phi(\boldsymbol{x}') \; \forall \; \boldsymbol{x}, \boldsymbol{x}'$$

$$\Updownarrow$$

  - the Gram matrix

$$\mathrm{K} = \begin{bmatrix} K(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(1)}) & K(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}) & \cdots & K(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(N)}) \\ K(\boldsymbol{x}^{(2)}, \boldsymbol{x}^{(1)}) & K(\boldsymbol{x}^{(2)}, \boldsymbol{x}^{(2)}) & \cdots & K(\boldsymbol{x}^{(2)}, \boldsymbol{x}^{(N)}) \\ \vdots & \vdots & \ddots & \vdots \\ K(\boldsymbol{x}^{(N)}, \boldsymbol{x}^{(1)}) & K(\boldsymbol{x}^{(N)}, \boldsymbol{x}^{(2)}) & \cdots & K(\boldsymbol{x}^{(N)}, \boldsymbol{x}^{(N)}) \end{bmatrix}$$

    is symmetric and positive semi-definite $\forall$ sets

    $\{\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}, \ldots, \boldsymbol{x}^{(N)}\}$

Building New Kernels

- For any valid kernels K_1, K_2 with implied feature transformations Φ_1, Φ_2 and non-negative coefficients c_1, c_2, the following are all valid kernels:

1. $K(\boldsymbol{x}, \boldsymbol{x}') = c_1 K_1(\boldsymbol{x}, \boldsymbol{x}') + c_2 K_2(\boldsymbol{x}, \boldsymbol{x}')$

   $\Phi(\boldsymbol{x}) = [\sqrt{c_1}\Phi_1(\boldsymbol{x}), \sqrt{c_1}\Phi_2(\boldsymbol{x})]$

2. $K(\boldsymbol{x}, \boldsymbol{x}') = c_1 K_1(\boldsymbol{x}, \boldsymbol{x}') K_2(\boldsymbol{x}, \boldsymbol{x}')$

   $$\Phi(\boldsymbol{x}) = \left[\left\{\sqrt{c_1}\phi_i(\boldsymbol{x})\phi_j(\boldsymbol{x})\right\}_{\phi_i(\boldsymbol{x})\in\Phi_1(\boldsymbol{x}),\phi_j(\boldsymbol{x})\in\Phi_2(\boldsymbol{x})}\right]$$

3. $K(\boldsymbol{x}, \boldsymbol{x}') = e^{K_1(\boldsymbol{x},\boldsymbol{x}')}$

Taylor series: $e^{K_1(\boldsymbol{x},\boldsymbol{x}')} = 1 + K_1(\boldsymbol{x}, \boldsymbol{x}') + \frac{K_1(\boldsymbol{x},\boldsymbol{x}')^2}{2!} + \frac{K_1(\boldsymbol{x},\boldsymbol{x}')^3}{3!} + \cdots$

Kernels Everywhere!

- Any method that only depends on the Euclidean distance between data points is an inner product method:

$$\|\boldsymbol{x} - \boldsymbol{x}'\|_2 = \sqrt{(\boldsymbol{x} - \boldsymbol{x}')^T(\boldsymbol{x} - \boldsymbol{x}')} = \sqrt{\boldsymbol{x}^T\boldsymbol{x} - 2\boldsymbol{x}^T\boldsymbol{x}' + \boldsymbol{x}'^T\boldsymbol{x}'}$$

  - We can kernelize kNN!
- We can also kernelize linear/ridge regression!
  - See Murphy, Chapter 14.4

Key Takeaways

- SVMs provide a principled way of finding linear decision boundaries with maximal margins
  - Larger margins can lead to better generalization
  - Defined as a constrained optimization problem
  - Interpretation of solution and definition of support vectors
  - Soft margins for linearly inseparable data
- Dual formulations
  - Interpretation of solution and definition of support vectors
- Kernels and the “kernel trick” allow for efficient use of feature transformations for inner product methods
  - Definition of valid kernels
  - Common kernels and combining kernels

Primal-Dual Soft-Margin SVMs

Primal

$$\text{minimize} \quad \frac{1}{2}\boldsymbol{w}^T\boldsymbol{w} + C\sum_{i=1}^{N}\xi^{(i)}$$

$$\text{subject to} \quad y^{(i)}\left(\boldsymbol{w}^T\boldsymbol{x}^{(i)} + w_0\right) \geq 1 - \xi^{(i)} \quad \forall \left(\boldsymbol{x}^{(i)}, y^{(i)}\right) \in \mathcal{D}$$

$$\xi^{(i)} \geq 0 \quad \forall\, i \in \{1, \ldots, N\}$$

Dual

$$\text{minimize} \quad \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\alpha^{(i)}\alpha^{(j)}y^{(i)}y^{(j)}\boldsymbol{x}^{(i)^T}\boldsymbol{x}^{(j)} - \sum_{i=1}^{N}\alpha^{(i)}$$

$$\text{subject to} \quad \sum_{i=1}^{N}\alpha^{(i)}y^{(i)} = 0$$

$$0 \leq \alpha^{(i)} \leq C \;\forall\, i \in \{1, \ldots, N\}$$

Soft-Margin SVMs

$$\text{minimize} \quad \frac{1}{2}\boldsymbol{w}^T\boldsymbol{w} + C\sum_{i=1}^{N}\xi^{(i)}$$

$$\text{subject to} \quad y^{(i)}\left(\boldsymbol{w}^T\boldsymbol{x}^{(i)} + w_0\right) \geq 1 - \xi^{(i)} \quad \forall \left(\boldsymbol{x}^{(i)}, y^{(i)}\right) \in \mathcal{D}$$

$$\xi^{(i)} \geq 0 \quad \forall\, i \in \{1, \ldots, N\}$$

$$\Updownarrow$$

$$\underset{\boldsymbol{\alpha}, \boldsymbol{\beta} \geq 0}{\text{maximize}} \quad \underset{\boldsymbol{w}, w_0, \boldsymbol{\xi}}{\text{minimize}} \quad L(\boldsymbol{\alpha}, \boldsymbol{w}, w_0, \boldsymbol{\beta}, \boldsymbol{\xi})$$

$$L(\boldsymbol{\alpha}, \boldsymbol{w}, w_0, \boldsymbol{\beta}, \boldsymbol{\xi}) = \frac{1}{2}\boldsymbol{w}^T\boldsymbol{w} + C\sum_{i=1}^{N}\xi^{(i)} + \sum_{i=1}^{N}\alpha^{(i)}\left(1 - \xi^{(i)} - y^{(i)}\left(\boldsymbol{w}^T\boldsymbol{x}^{(i)} + w_0\right)\right) - \sum_{i=1}^{N}\beta^{(i)}\xi^{(i)}$$

Karush-Kuhn-Tucker (KKT) Conditions

$$\underset{\boldsymbol{w}, w_0, \boldsymbol{\xi}}{\text{minimize}} \quad L(\boldsymbol{\alpha}, \boldsymbol{w}, w_0, \boldsymbol{\beta}, \boldsymbol{\xi})$$

$$L(\boldsymbol{\alpha}, \boldsymbol{w}, w_0, \boldsymbol{\beta}, \boldsymbol{\xi}) = \frac{1}{2}\boldsymbol{w}^T\boldsymbol{w} + C\sum_{i=1}^{N}\xi^{(i)}$$

$$+\sum_{i=1}^{N}\alpha^{(i)}\left(1 - \xi^{(i)} - y^{(i)}\left(\boldsymbol{w}^T\boldsymbol{x}^{(i)} + w_0\right)\right) - \sum_{i=1}^{N}\beta^{(i)}\xi^{(i)}$$

$$\frac{\partial L(\boldsymbol{\alpha}, \boldsymbol{w}, w_0, \boldsymbol{\beta}, \boldsymbol{\xi})}{\partial \boldsymbol{w}} = \boldsymbol{w} - \sum_{i=1}^{N}\alpha^{(i)}y^{(i)}\boldsymbol{x}^{(i)} \rightarrow \widehat{\boldsymbol{w}} = \sum_{i=1}^{N}\alpha^{(i)}y^{(i)}\boldsymbol{x}^{(i)}$$

$$\frac{\partial L(\boldsymbol{\alpha}, \boldsymbol{w}, w_0, \boldsymbol{\beta}, \boldsymbol{\xi})}{\partial w_0} = -\sum_{i=1}^{N}\alpha^{(i)}y^{(i)} \qquad \rightarrow \sum_{i=1}^{N}\alpha^{(i)}y^{(i)} = 0$$

$$\frac{\partial L(\boldsymbol{\alpha}, \boldsymbol{w}, w_0, \boldsymbol{\beta}, \boldsymbol{\xi})}{\partial \xi^{(i)}} = C - \alpha^{(i)} - \beta^{(i)} \qquad \rightarrow \beta^{(i)} = C - \alpha^{(i)}$$

$$\forall\, i \in \{1, \ldots, N\}$$

Minimizing the Lagrangian

$$\hat{\boldsymbol{w}} = \sum_{i=1}^{N} \alpha^{(i)} y^{(i)} \boldsymbol{x}^{(i)}$$

$$\sum_{i=1}^{N} \alpha^{(i)} y^{(i)} = 0$$

$$\beta^{(i)} = C - \alpha^{(i)} \; \forall \, i$$

$$L\left(\boldsymbol{\alpha}, \hat{\boldsymbol{w}}, \hat{w}_0, \boldsymbol{\beta}, \hat{\boldsymbol{\xi}}\right) = \frac{1}{2}\hat{\boldsymbol{w}}^T\hat{\boldsymbol{w}} + C\sum_{i=1}^{N}\hat{\xi}^{(i)}$$

$$+ \sum_{i=1}^{N} \alpha^{(i)}\left(1 - \hat{\xi}^{(i)} - y^{(i)}\left(\hat{\boldsymbol{w}}^T\boldsymbol{x}^{(i)} + \hat{w}_0\right)\right) - \sum_{i=1}^{N}\beta^{(i)}\hat{\xi}^{(i)}$$

$$= \frac{1}{2}\hat{\boldsymbol{w}}^T\hat{\boldsymbol{w}} + C\sum_{i=1}^{N}\hat{\xi}^{(i)} - \sum_{i=1}^{n}(C - \alpha^{(i)})\hat{\xi}^{(i)}$$

$$+ \sum_{i=1}^{N} \alpha^{(i)}\left(1 - \hat{\xi}^{(i)} - y^{(i)}\left(\hat{\boldsymbol{w}}^T\boldsymbol{x}^{(i)} + \hat{w}_0\right)\right)$$

$$= \frac{1}{2}\hat{\boldsymbol{w}}^T\hat{\boldsymbol{w}} + \sum_{i=1}^{N} \alpha^{(i)}\left(1 - y^{(i)}\left(\hat{\boldsymbol{w}}^T\boldsymbol{x}^{(i)} + \hat{w}_0\right)\right)$$

Maximizing the Minimum

$$\text{minimize} \quad \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\alpha^{(i)}\alpha^{(j)}y^{(i)}y^{(j)}\boldsymbol{x}^{(i)^T}\boldsymbol{x}^{(j)} - \sum_{i=1}^{N}\alpha^{(i)}$$

$$\text{subject to} \quad \sum_{i=1}^{N}\alpha^{(i)}y^{(i)} = 0$$

$$\alpha^{(i)} \geq 0 \qquad \forall\, i \in \{1, \dots, N\}$$

$$\beta^{(i)} \geq 0 \qquad \forall\, i \in \{1, \dots, N\}$$

$$\beta^{(i)} = C - \alpha^{(i)} \quad \forall\, i \in \{1, \dots, N\}$$

$$\Updownarrow$$

$$\text{minimize} \quad \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\alpha^{(i)}\alpha^{(j)}y^{(i)}y^{(j)}\boldsymbol{x}^{(i)^T}\boldsymbol{x}^{(j)} - \sum_{i=1}^{N}\alpha^{(i)}$$

$$\text{subject to} \quad \sum_{i=1}^{N}\alpha^{(i)}y^{(i)} = 0$$

$$0 \leq \alpha^{(i)} \leq C \;\forall\, i \in \{1, \dots, N\}$$

Primal-Dual Soft-Margin SVMs

- Primal
  - Directly returns the weights, $[\hat{w}_0, \hat{\boldsymbol{w}}]$
  - Support vectors are all $\left(\boldsymbol{x}^{(s)}, y^{(s)}\right) \in \mathcal{D}$ s.t.

$$y^{(s)}\left(\hat{\boldsymbol{w}}^T \boldsymbol{x}^{(s)} + \hat{w}_0\right) = 1$$

- Dual
  - Returns the vector, $\hat{\boldsymbol{\alpha}}$

$$\hat{\boldsymbol{w}} = \sum_{i=1}^{N} \hat{\alpha}^{(i)} y^{(i)} \boldsymbol{x}^{(i)}$$

$$\hat{w}_0 = ???$$

Complementary Slackness

$$\text{minimize} \quad \frac{1}{2}\boldsymbol{w}^T\boldsymbol{w} + C\sum_{i=1}^{N}\xi^{(i)}$$

$$\text{subject to} \quad y^{(i)}\left(\boldsymbol{w}^T\boldsymbol{x}^{(i)} + w_0\right) \geq 1 - \xi^{(i)} \quad \forall \left(\boldsymbol{x}^{(i)}, y^{(i)}\right) \in \mathcal{D}$$

$$\xi^{(i)} \geq 0 \qquad \forall\, i \in \{1, \ldots, N\}$$

$$\Updownarrow$$

$$\underset{\boldsymbol{\alpha}, \boldsymbol{\beta} \geq 0}{\text{maximize}} \quad \underset{\boldsymbol{w}, w_0, \boldsymbol{\xi}}{\text{minimize}} \quad L(\boldsymbol{\alpha}, \boldsymbol{w}, w_0, \boldsymbol{\beta}, \boldsymbol{\xi})$$

$$L(\boldsymbol{\alpha}, \boldsymbol{w}, w_0, \boldsymbol{\beta}, \boldsymbol{\xi}) = \frac{1}{2}\boldsymbol{w}^T\boldsymbol{w} + C\sum_{i=1}^{N}\xi^{(i)} + \sum_{i=1}^{N}\alpha^{(i)}\left(1 - \xi^{(i)} - y^{(i)}\left(\boldsymbol{w}^T\boldsymbol{x}^{(i)} + w_0\right)\right) - \sum_{i=1}^{N}\beta^{(i)}\xi^{(i)}$$

Complementary Slackness

$$\underset{\boldsymbol{\alpha},\boldsymbol{\beta}\geq 0}{\text{maximize}}\ \underset{\boldsymbol{w},w_0,\boldsymbol{\xi}}{\text{minimize}}\ L(\boldsymbol{\alpha},\boldsymbol{w},w_0,\boldsymbol{\beta},\boldsymbol{\xi})$$

$$L(\boldsymbol{\alpha},\boldsymbol{w},w_0,\boldsymbol{\beta},\boldsymbol{\xi}) = \frac{1}{2}\boldsymbol{w}^T\boldsymbol{w} + C\sum_{i=1}^{N}\xi^{(i)} + \sum_{i=1}^{N}\alpha^{(i)}\left(1-\xi^{(i)}-y^{(i)}\left(\boldsymbol{w}^T\boldsymbol{x}^{(i)}+w_0\right)\right) - \sum_{i=1}^{N}\beta^{(i)}\xi^{(i)}$$

- Theorem: $\hat{\beta}^{(i)}\hat{\xi}^{(i)} = \left(C-\hat{\alpha}^{(i)}\right)\hat{\xi}^{(i)} = 0\ \forall\left(\boldsymbol{x}^{(i)},y^{(i)}\right)\in\mathcal{D}$ and

$$\hat{\alpha}^{(i)}\left(1-\hat{\xi}^{(i)}-y^{(i)}\left(\hat{\boldsymbol{w}}^T\boldsymbol{x}^{(i)}+\hat{w}_0\right)\right) = 0\ \forall\left(\boldsymbol{x}^{(i)},y^{(i)}\right)\in\mathcal{D}$$

  - If $0<\hat{\alpha}^{(s)}$, then $1-\hat{\xi}^{(i)}-y^{(i)}\left(\hat{\boldsymbol{w}}^T\boldsymbol{x}^{(i)}+\hat{w}_0\right)=0$
  - If $\hat{\alpha}^{(s)}<C$, then $\hat{\xi}^{(i)}=0$
  - If $0<\hat{\alpha}^{(s)}<C$, then $1-y^{(i)}\left(\hat{\boldsymbol{w}}^T\boldsymbol{x}^{(i)}+\hat{w}_0\right)=0$

Primal-Dual Soft-Margin SVMs

- Dual
  - Returns the vector, $\hat{\boldsymbol{\alpha}}$

$$\hat{\boldsymbol{w}} = \sum_{i=1}^{N} \hat{\alpha}^{(i)} y^{(i)} \boldsymbol{x}^{(i)}$$

$$\hat{w}_0 = y^{(s)} - \hat{\boldsymbol{w}}^T \boldsymbol{x}^{(s)} \text{ for any } s \text{ s.t. } 0 < \hat{\alpha}^{(s)} < C$$

  - Support vectors are all $\left(\boldsymbol{x}^{(s)}, y^{(s)}\right) \in \mathcal{D}$ s.t. $0 < \hat{\alpha}^{(s)}$
  - If $0 < \hat{\alpha}^{(s)} < C$, then $y^{(i)}\left(\hat{\boldsymbol{w}}^T \boldsymbol{x}^{(i)} + \hat{w}_0\right) = 1 \Rightarrow$ $\left(\boldsymbol{x}^{(s)}, y^{(s)}\right)$ defines the margin
  - If $\hat{\alpha}^{(s)} = C$, then $\hat{\xi}^{(s)} > 0 \Rightarrow \left(\boldsymbol{x}^{(s)}, y^{(s)}\right)$ is inside the margin or misclassified.