Shilov Linear Algebra 1: Determinants

Michael Lee

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1 Number Fields

Definition A number field is any set of $\mathbb{K}$ of objects, called "numbers," which, when subjected to the four arithmetic operations again give elements of $\mathbb{K}$.

- To every pair of numbers α and β in $\mathbb{K}$ there corresponds a unique number $\alpha + \beta$ in $\mathbb{K}$, called the sum of α and β.
    1. $\alpha + \beta = \beta + \alpha$ for every α and β in $\mathbb{K}$ (addition is commutative).
    2. $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$, for every α, β, γ in $\mathbb{K}$. (addition is associative);
    3. There exists a nbumber 0 (zero) in $\mathbb{K}$ such that $0 + \alpha = \alpha$ for every α in $\mathbb{K}$.
    4. For every α in $\mathbb{K}$ there exists a number (negative element) γ in $\mathbb{K}$ such that $\alpha + \gamma = 0$.
- To every pair of numbers α and β in $\mathbb{K}$ there corresponds a unique number $\alpha \cdot \beta$ in $\mathbb{K}$, called the product of $\alpha and \beta$, where
    1. $\alpha \cdot \beta = \beta \cdot \alpha$ for every α and β in $\mathbb{K}$ (multiplication is commutative).
    2. $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$, for every α, β, γ in $\mathbb{K}$. (multiplication is associative);
    3. There exists a number 1 (identity) in $\mathbb{K}$ such that $1 \cdot \alpha = \alpha$ for every α in $\mathbb{K}$.
    4. For every α in $\mathbb{K}$ there exists a number (inverse element) γ in $\mathbb{K}$ such that $\alpha \cdot \gamma = 1$.
- Two fields $\mathbb{K}$ and $\mathbb{K}$' are called isomorphic if there is a one-to-one correpsondence between the two fields such that the number associated with every sum(difference) or product(quotient) of numbers in $\mathbb{K}$ is the sum(difference) or product(quotient) of the corresponding numbers in $\mathbb{K}$'.

Example Fields

- Field of rational numbers $\mathbb{Q}$
- Field of real numbers $\mathbb{R}$
- Field of complex numbers $\mathbb{C}$

Remark Fundamental Theorem of Algebra We can not onbly carry out the four arithmetic operations in $\mathbb{C}$ but but also solve any algebraic equation $z^n + a_1 z^{n-1} + \ldots + a_n = 0$. The field $\mathbb{R}$ does not have this property. We will use $\mathbb{K}$ to denote an arbitrary number field. If some property is true for the fireld $\mathbb{K}$, then it is automatically true for the field $\mathbb{R}$ and $\mathbb{C}$.

2 Theory of Systems of Linear Equations

Definition A general system of linear equations has the form

$$
\begin{gathered}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\
\ldots\ldots\ldots \\
a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n
\end{gathered}
$$

Here $x_1, x_2, \ldots, x_n$ denote the unknowns (elements of $\mathbb{K}$). The quantities $a_{11}, a_{12}, \ldots, a_{kn}$, taken from the field $\mathbb{K}$ are called the coefficients of the system. The first index of a coefficient indicates the number of the equation in which the coefficient appears, while the second index indicates the number of the unknown with which the coefficient is associated. The numbers $b_1, b_2, \ldots, b_n$ are also from $\mathbb{K}$ and are called the constant terms of the system. A solution of the ststem is a set of numbers $c_1, c_2, \ldots, c_n$ from $\mathbb{K}$ which, when substituted for the unknowns $x_1, x_2, \ldots, x_n$, turns all the equations of the system into identitites.

Definition A system of equations can be

- Compatible: when it has at least one solution
  1. Determinate: when it has a unique solution
  2. Indeterminate: when it has at least two different solutions
- Incompatible: when it has no solutions

Assignment None

3 Determinants of Order n

Definition The number of rows and columns of the matrix is called its order. The numbers a_{ij} are called the elements of the matrix. The first index indicates the row and the second index the column in which a_{ij} appears. The elements $a_{11}, a_{22}, \ldots, a_{nn}$ form the principle diagonal of the matrix.

Definition Consider any product of n elements which appear in differen t rows and different columns of the matrix. Such a product can be written in the form

$$a_{\alpha_1 1} a_{\alpha_2 2} \dots a_{\alpha_n n}$$

An inversion in this sequence α is an arrangement of two indices such that the larger index comes before the smaller index. The total number of inversions will be denoted by $N(\alpha_1, \alpha_2, \dots, \alpha_n)$. If the number of inversions in this sequence is even, we put a plus sign before the product; else if the number is odd, we put a minus sign before the product. Symbolically

$(-1)^{N(\alpha_1,\alpha_2,\dots,\alpha_n)}$ The total number of products of this form whcih can be formed from the elements of a given order n is equal to the total number of permutations of the number $1, 2, \dots, n = n!$.

Definition The determinant D of a matrix is meant the algebraic sum of the $n!$ products of the alternating signed permutations. Henceforth the products of this form will be called the terms of the determinant D. The elements a_{ij} will be called the elements of D

Definition The rule for determining the sign of a given term of a determinant can be formulated somewhat differently (geometrically). Corresponding to the enumeration of elements in the matrix, we can distinguish two natural positive directions, from left to right, from top to bottom. Moreover, the slanting lines joining any two elements of the matrix can be furnished with a direction: we shall say that the line segment joing the element a_{ij} with the elment a_{kl} has positive slope if tis right endpoint lies lower than its left endpoint, and that it has negative slope if its right endpoint lies higher than its left endpoint.

Assignment Shilov 1: 1, 2, 3

Corollary 3.1 *A determinant changes sign when two of its columns are interchanged. First, consider the case where two adjacent columns are interchanged. The determinant which is obtained after these columns are interchanged obviously still consists of the same terms as the original determinant. Consider any terms of original determinant containing an element of the jth column and an element of the $(j+1)$th column.l If the segment joining these two elements originally had negative slope, then after the interchange of columns, its slope becomes positive. Consequently, the number of segments with negative slope joining the elmenets of the given term changes by one when the two columns are interchanged; therefore each term of the determinant, and hence the determinant itself, changes sign when the columns are interchanged. Next consider the general case of two non-adjacent columns. Perform the same procedure, of considering the sign of the elements of the determinant after the interchange. The positive become negative and the negative become positive. At the end of the process, the determinant will have a sign opposite to its original sign.*

Corollary 3.2 *A determinant with two identical columns vanishes.*

Proof Interchanging the columns does not change the determinant D. On the other had, the determinant must change its sign, therefore $D = -D$, which imples that $D = 0$.

Theorem 3.3 *If all the element of the jth column of a determinant D are linear combinations of two columns of numbers, if*

$$a_{ij} = \lambda b_i + \mu c_i$$

where λ and μ are fixed numbers, then D is equal to a linear combination of two determinants:

$$D = \lambda D_1 + \mu D_2$$

Proof Every term of the determinasnt D can be represented in the form $a_{\alpha_1 1} a_{\alpha_2 2} \ldots a_{\alpha_j j} = a_{\alpha_1 1} a_{\alpha_2 2} \ldots (\lambda b_{\alpha_1} + \mu c_{\alpha_j} \ldots a_{\alpha_n n} = \lambda a_{\alpha_1 1} a_{\alpha_2 2} \ldots b_{\alpha_j} \ldots a_{\alpha_n n} + \mu a_{\alpha_1 1} a_{\alpha_2 2} \ldots c_{\alpha_j} \ldots a_{\alpha_n n}$.

Definition It is convenient to write this formula in a somewhat different form. Let D be an arbitrary fixed determinant. Denote by $D_j(p_i)$ the determinant which is obtained by replacing the elmenets of the jth column of D by the numbers $p_i (i = 1, 2, \ldots, n)$.

Corollary 3.4 *Any common factor of a column of a determinant can be factored out of the determinant*

Proof If $a_{ij} = \lambda b_i$, then we have $D_i(a_{ij}) = D_j(\lambda b_i) = \lambda D_j(b_i)$.

Corollary 3.5 *If a column of a determinant consists entirely of zeros, then the determinant vanishes.*

Theorem 3.6 *The value of a determinant is not changed by adding the elements of one column multiplied by an arbitrary number to the corresponding elements of another column.*

4 Cofactors and Minors

Definition Consider the jth column of the determinant D. Let a_{ij} be any element of this column. Add up all the terms containing r5he element appearing the right hand side of the equation for D and then factor out the element a_{ij}. The quantity that remains A_{ij} is called the cofactor of the element a_{ij} of the determinant D. Since every term of the determinant D contains an element from the jth column, the equation for D can be written in the form $D = a_{1j} A_{1j} + a_{2j} A 2j + \ldots + a_{nj} A_{nj}$ called the expansion of the determinant D with respect to the elements of the jth column.

Theorem 4.1 *The sum of all the products of the elements of any column of the determinant D with the corresponding cofactors is equal to the determiant D itself.*

Definition If we delete a row and a column from a matrix of order n, then of course, the remaining elements form a matrix of order $n - 1$. The determinant of this matrix is called a minor of the original nth order matrix. If we delete the ith row and the jth column of D, then the minor so obtained is denoted by $M_{ij}(D)$.

$A_{ij} = (-1)^{i+j} M_{ij}$

Assignment Shilov 1: 4, 5, 6

5 Cramer's Rule

Definition We are now in a position to solvce systems of linear equations. First, we consider a system which has the same number of unknowns and equations. The coefficients $a_{ij}(i, j = 1, 2, \ldots, n)$ form the coefficient matrix of the system. We assume that the determinant of this matrix is different from zero.

Theorem 5.1 *A system whose coefficient matrix has a determinant different from zero is both compatible and determinate.*

Proof Let $c_1, c_2, \ldots, c_n$ be the solution set of the system described by the coefficient matrix. Multiply the first of the equations by the cofactor A_{11} of the element a_{11} in the coefficient matrix, then the second by A_{21}, and so forth. The coefficient of c_1 must be the determinant D itself. The coefficients of all the other c_j vanish. By writting out the expansion of the determinant with respect to its first column, we can now write

$Dc_1 = D_1$. Analogously, we can obtain the expression

$c_j = \frac{D_j}{D}$. where D_j is the determinant obtained from the determinant D by replacing its jth column by the numbers $b_1, b_2, \ldots, b_n$.

Theorem 5.2 *If the determinant of the system is different from zero, then the system has a unique solution, for the value of the unknown $x_j(j = 1, 2, \ldots, n)$, we take the fraction whose denominator is the determinant D of the system and whose numerator is the determinant obtained by replacing the jth column of D by the column consisting of the constant terms of the system.*