Fourier Transform 1

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Michael Lee

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1 Delta Functions

Definition Kronecker Delta Function.

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if otherwise} \end{cases}$$

Definition Informal Dirac Delta Function.

$$\delta(x) = \begin{cases} \infty & \text{if } x = 0 \\ 0 & \text{if otherwise} \end{cases}$$

Definition Formal definition of Dirac $\delta(x)$. Let $(a, b) \in x$ define an interval of size $\Delta x = b - a$ such that there exists a function $\delta(x)$ such that $\lim_{\Delta x \to 0} \int_a^b \delta(x) \mathrm{d}x = 1$

Definition Riemann Sum. Integration of a function $f(x)$ over an interval $(a, b) \in x$ of n partitions of size $\Delta x = \frac{b-a}{n}$.

$\int_a^b f(x) \mathrm{d}x = \lim_{n \to \infty} \sum_{i=0}^{n} f(i \cdot \Delta x) \Delta x$

2 Finite Dimensional Vector Spaces

Definition A vector space is a set of vectors V that satisfy the field axioms.

Definition A basis is orthogonal if $\langle e_i, e_j \rangle = 0$ and $i \neq j$. A basis is orthonormal if it is orthogonal and contains all normalized vectors (i.e. $\langle e_i, e_j \rangle = \delta_{ij}$).

Theorem 2.1 *Given a real vector space V with an orthogonal basis $\{e_1, e_2, \ldots, e_n\{$, any vector $v \in V$ can be decomposed into a sum of basis vectors. The vector basis $\{e_1, e_2, \ldots, e_n\}$ of V is the representation of a vector as a sum (linear combination) of other vectors. For any vector v, there will be a set of expansion coefficients $\{c_1, c_2, \ldots, c_n\}$, such that $v = c_1 e_1 + c_2 e_2 + \ldots + c_n e_n$.*

$$v = \sum_{i=1}^{n} \langle e_i, v \rangle e_i$$

3 Infinite Dimensional Vector Spaces

Definition We define an infinite dimensional vector space V by extending the definition of its finite dimensional analogue. We simply allow functions to replace vectors. For a finite dimensional vector v, we denote $v(i)$ as the ith index of the vector v. This notation allows us to treat finite dimensional vectors as functions by mapping indices to coordinates.

Remark Observe that no general finite basis exists since any arbitrary function $f\colon \mathbb{R} \to \mathbb{R} \in V$ has infinitely many dimensions and therefore infinitely many varying basis functions.

Definition Let the basis functions of V be the set of functions $e_x\colon \mathbb{R} \to \mathbb{R}$ such that for every $e_x, e_x(x') = \delta_{xx'}$. In other words, for every function e_x in the basis, there is an argument x' such that the basis function's value is the Kronecker delta of the xth basis function in the x' index. Such a finite dimensional vector space is called orthonormal.

Lemma 3.1 $\int_{-\infty}^{\infty} \delta(x'' - x') = 1$

Proof From the Riemann definition of the integral

$\int_{-\infty}^{\infty} \delta(x'' - x')\mathrm{d}x' = \lim_{n\to\infty} \sum_{i=1}^{n} \delta(i \cdot \Delta x')\Delta x'$

Using the piecewise definition of the Dirac delta, if $x' \neq x$, the delta function would be zero and the summand will be zero. However as the interval size gets smaller, the area under the Dirac spike doesn't change and is still 1. Therefore:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \delta(i \cdot \Delta x')\Delta x' = 1 + 0 + 0 \ldots + 0$$
$$= 1$$

WTF?: $0 \cdot \infty = 1$

Theorem 3.2 *Let V be an infinite dimensional vector space of functions $\mathbb{R} \to \mathbb{R}$ with a set of basis functions e_i. If $f \in V$, then the function f can be expressed as the sum of basis functions.*

Proof Let $f\colon \mathbb{R} \to \mathbb{R} \in V$.

$$g(x'') = (\textstyle\int_{-\infty}^{\infty} f(x')e_{x'}\mathrm{d}x')(x'')$$
$$g(x'') = \textstyle\int_{-\infty}^{\infty} f(x')e_{x'}(x'')\mathrm{d}x'$$
$$g(x'') = \textstyle\int_{-\infty}^{\infty} f(x')\delta(x'' - x')\mathrm{d}x'$$

At this point we can substitute $f(x'')$ for $f(x')$ because unless $x'' = x'$, $\delta(x'' - x) = 0$ and the integrand vanishes.

$$g(x'') = f(x'') \textstyle\int_{-\infty}^{\infty} \delta(x'' - x')\mathrm{d}x'$$
$$g(x'') = f(x'') \cdot 1 \text{ (Lemma 3.1)}$$
$$g = f$$

Example Suppose $f(x) = 1$. We want to integrate e_x for every single possible x to set the value of the function to 1 everywhere. Let $g \in V$ such that $g = \int_{-\infty}^{\infty} e_{x'} \mathrm{d}x'$. We want to confirm whether $g = f$ and so we test $g(3)$.

$$g(3) = (\int_{-\infty}^{\infty} e_{x'} \mathrm{d}x')(3)$$
$$g(3) = \int_{-\infty}^{\infty} e_{x'}(3) \mathrm{d}x'$$
$$g(3) = \int_{-\infty}^{\infty} \delta_{x'3} \mathrm{d}x'$$

Contradiction Geometrically, $g(3)$ is the area under the curve of the Kronecker delta function; $g(3) = 0$. Since we can imagine the Kronecker delta as a function of a horizontal line $y = 0$ except at one point where there is a removed discontinuity at $x = 0$, $y = 1$. A single vertical line does not have any area. However, we assumed that $f(x) = 1$, therefore by contradiction, $f \neq g$.
To avoid the problem of nonexistant area, instead of the Kronecker delta, we use the Dirac delta.

$$g(3) = (\int_{-\infty}^{\infty} e_{x'} \mathrm{d}x')(3)$$
$$g(3) = \int_{-\infty}^{\infty} e_{x'}(3) \mathrm{d}x'$$
$$g(3) = \delta(x' - 3) \mathrm{d}x'$$
$$g(3) = 1$$