$$S_t = -(\frac{\beta_1(I_1+I_2)+\beta_2(I_{21}+I_{12})}{N})S - \mu S + \mu N + \nu(I_{21}+I_{12}) \tag{1a}$$

$$I_{1,t} = (\frac{\beta_1 I_1 + \beta_2 I_{12}}{N})S - \mu I_1 - \gamma I_1 \tag{1b}$$

$$I_{2,t} = (\frac{\beta_1 I_2 + \beta_2 I_{21}}{N})S - \mu I_2 - \gamma I_2 \tag{1c}$$

$$R_{1,t} = \gamma I_1 - (\frac{\beta_1 I_2 + \beta_2 I_{21}}{N})R_1 - \mu R_1 \tag{1d}$$

$$R_{2,t} = \gamma I_2 - (\frac{\beta_1 I_1 + \beta_2 I_{12}}{N})R_2 - \mu R_2 \tag{1e}$$

$$I_{21,t} = (\frac{\beta_1 I_2 + \beta_2 I_{21}}{N})R_1 - (\mu+\nu+\gamma)I_{21} \tag{1f}$$

$$I_{12,t} = (\frac{\beta_1 I_1 + \beta_2 I_{12}}{N})R_2 - (\mu+\nu+\gamma)I_{12} \tag{1g}$$

$$R_t = \gamma(I_{21}+I_{12}) - \mu R \tag{1h}$$

1 Continuous-Time Markov Chain - Shamelessly stolen from Linda Allen

For the one-age dengue model, we have the following possible events:

$$S(t+\Delta t), I_1(t+\Delta t), I_2(t+\Delta t), R_1(t+\Delta t), R_2(t+\Delta t), I_{21}(t+\Delta t), I_{12}(t+\Delta t), R(t+\Delta t) =$$

$$S(t)-1, I_1(t)+1, I_2(t), R_1(t), R_2(t), I_{21}(t), I_{12}(t), R(t) \tag{2a}$$

$$S(t)-1, I_1(t), I_2(t)+1, R_1(t), R_2(t), I_{21}(t), I_{12}(t), R(t) \tag{2b}$$

$$S(t), I_1(t)-1, I_2(t), R_1(t)+1, R_2(t), I_{21}(t), I_{12}(t), R(t) \tag{2c}$$

$$S(t), I_1(t), I_2(t)-1, R_1(t), R_2(t)+1, I_{21}(t), I_{12}(t), R(t) \tag{2d}$$

$$S(t), I_1(t), I_2(t), R_1(t)-1, R_2(t), I_{21}(t)+1, I_{12}(t), R(t) \tag{2e}$$

$$S(t), I_1(t), I_2(t), R_1(t), R_2(t)-1, I_{21}(t), I_{12}(t)+1, R(t) \tag{2f}$$

$$S(t), I_1(t), I_2(t), R_1(t), R_2(t), I_{21}(t)-1, I_{12}(t), R(t)+1 \tag{2g}$$

$$S(t), I_1(t), I_2(t), R_1(t), R_2(t), I_{21}(t), I_{12}(t)-1, R(t)+1 \tag{2h}$$

$$S(t)+1, I_1(t)-1, I_2(t), R_1(t), R_2(t), I_{21}(t), I_{12}(t), R(t) \tag{2i}$$

$$S(t)+1, I_1(t), I_2(t)-1, R_1(t), R_2(t), I_{21}(t), I_{12}(t), R(t) \tag{2j}$$

$$\ldots \tag{2k}$$

$$S(t)+1, I_1(t), I_2(t), R_1(t), R_2(t), I_{21}(t), I_{12}(t)-1, R(t) \tag{2l}$$

$$S(t)+1, I_1(t), I_2(t), R_1(t), R_2(t), I_{21}(t), I_{12}(t), R(t)-1 \tag{2m}$$

(Note: this assumes instant reincarnation to keep population steady.)
We take a (small) time step Δt, such that it is likely that at most one of these events occurs in Δt. Then, the probability of a transition for each of these events satisfies:

event	probability
$S \to I_1$	$(\frac{\beta_1 I_1+\beta_2 I_{12}}{N})S\Delta t + o(\Delta t)$
$S \to I_2$	$(\frac{\beta_1 I_2+\beta_2 I_{21}}{N})S\Delta t + o(\Delta t)$
$I_1 \to R_1$	$\gamma I_1 \Delta t + o(\Delta t)$
$I_2 \to R_2$	$\gamma I_2 \Delta t + o(\Delta t)$
$R_1 \to I_{21}$	$(\frac{\beta_1 I_2+\beta_2 I_{21}}{N})R_1\Delta t + o(\Delta t)$
$R_2 \to I_{12}$	$(\frac{\beta_1 I_1+\beta_2 I_{12}}{N})R_2\Delta t + o(\Delta t)$
$I_{21} \to R$	$\gamma I_{21} \Delta t + o(\Delta t)$
$I_{12} \to R$	$\gamma I_{12} \Delta t + o(\Delta t)$
$I_1 \to S$	$\mu I_1 \Delta t + o(\Delta t)$
$I_2 \to S$	$\mu I_2 \Delta t + o(\Delta t)$
$R_1 \to S$	$\mu R_1 \Delta t + o(\Delta t)$
$R_2 \to S$	$\mu R_2 \Delta t + o(\Delta t)$
$I_{21} \to S$	$(\mu+\nu) I_{21} \Delta t + o(\Delta t)$
$I_{12} \to S$	$(\mu+\nu) I_{12} \Delta t + o(\Delta t)$
$R \to S$	$\mu R \Delta t + o(\Delta t)$

Thus, the probability that NO EVENTS occur during time Δt is:

$$1-\Bigg((\frac{\beta_1 I_1+\beta_2 I_{12}}{N})S + (\frac{\beta_1 I_2+\beta_2 I_{21}}{N})S + \gamma I_1 + \gamma I_2 + (\frac{\beta_1 I_2+\beta_2 I_{21}}{N})R_1 + (\frac{\beta_1 I_1+\beta_2 I_{12}}{N})R_2 +$$
$$\gamma I_{21} + \gamma I_{12} + \mu I_1 + \mu I_2 + \mu R_1 + \mu R_2 + (\mu+\nu)I_{21} + (\mu+\nu)I_{12} + \mu R\Bigg)\Delta t + o(\Delta t) \tag{3}$$

We need to know the time between successive events. Let $T_{(s,i_1,i_2,\ldots)}$ = the time until the next event, given that $S = s$, $I_1 = i_1$, etc. Let the probability that there is no change on $[0, t]$ be

$$G_{s,i_1,\ldots}(t) = \text{Prob}\{T_{s,i_1,\ldots} > t\}$$

Then the probability that no events occur on $[0, t+\Delta t]$ is the product of the probabilities that no events occur on $[0, t]$ and no events occur on $[t, t+\Delta t]$. The first of these is $G_{s,i_1,\ldots}(t)$. The second of these is given by (3). Thus

$$G_{s,i_1,\ldots}(t+\Delta t) = G_{s,i_1,\ldots}(t)\left(1-(\frac{\beta_1 i_1+\beta_2 i_{12}}{N})s\Delta t - \ldots - \mu r\Delta t + o(\Delta t)\right)$$

We may calculate the derivative of $G_{s,i_1,\ldots}(t)$ using

$$\frac{d}{dt}G_{s,i_1,\ldots}(t) = \lim_{\Delta t\to 0}\frac{G_{s,i_1,\ldots}(t+\Delta t) - G_{s,i_1,\ldots}(t)}{\Delta t}$$

Thus

$$\frac{d}{dt}G_{s,i_1,\ldots}(t) = -\left((\frac{\beta_1 i_1+\beta_2 i_{12}}{N})s + \ldots + \mu r\right)$$

Moreover, $G_{s,i_1,...}(0)$ is the probability that no events occur on $[0,0]$. Thus $G_{s,i_1,...}(0) = 1$, and we have a scalar, linear IVP, whose solution is

$$G_{s,i_1,...}(t) = \exp\Bigg(-[(\frac{\beta_1 i_1 + \beta_2 i_{12}}{N})s + ... + \mu r]t\Bigg)$$

Define

$$F_{s,i_1,...}(t) = 1 - G_{s,i_1,...}(t) = \text{Prob}\{T_{s,i_1,...} \leq t\}$$

We note that $F_{s,i_1,...}(t)$ is a strictly increasing function of t (and therefore invertible), so that $U \leq F_{s,i_1,...}(t)$ if and only if $F^{-1}_{s,i_1,...}(U) \leq t$. With that motivation, let U be a uniformly distributed random variable in $[0,1]$, so that $\text{Prob}\{U \leq y\} = y$. Then:

$$\begin{aligned}
\text{Prob}\{F^{-1}_{s,i_1,...}(U) \leq t\} &= \text{Prob}\{U \leq F_{s,i_1,...}(t)\} && (4)\\
&= F_{s,i_1,...}(t) && (5)\\
&= \text{Prob}\{T_{s,i_1,...} \leq t\} && (6)
\end{aligned}$$

and so have that $T_{s,i_1,...} = F^{-1}_{s,i_1,...}(U)$. That is, given a uniformly distributed random variable U, we may model the time between successive events as $T_{s,i_1,...} = F^{-1}_{s,i_1,...}(U)$.

$$U = F_{s,i_1,...}(T_{s,i_1,...}) = 1 - \exp\Bigg(-[(\frac{\beta_1 i_1 + \beta_2 i_{12}}{N})s + ... + \mu r](T_{s,i_1,...})\Bigg)$$

Solving for $T_{s,i_1,...}$, we obtain:

$$T_{s,i_1,...} = -\frac{\ln(1-U)}{(\frac{\beta_1 i_1 + \beta_2 i_{12}}{N})s + ... + \mu r}$$

Observing that if U is a uniformly distributed random variable on $[0,1]$, then so is $1-U$, we may write

$$T_{s,i_1,...} = -\frac{\ln(U)}{(\frac{\beta_1 i_1 + \beta_2 i_{12}}{N})s + ... + \mu r}$$

We not only need to know the time between events, but also, when the next event occurs, which event is it? We again use a uniform random variable in $[0,1]$ to determine this. We divide the interval into subintervals of length $\frac{\text{event}_n}{\Sigma\text{event}_n}$. For a given $(s, i_1, i_2, ...)$ we use:

event 1	$s \to s-1$, $i_1 \to i_1+1$	$(\frac{\beta_1 i_1+\beta_2 i_{12}}{N})s$
event 2	$s \to s-1$, $i_2 \to i_2+1$	$(\frac{\beta_1 i_2+\beta_2 i_{21}}{N})s$
event 3	$I_1 \to R_1$	γi_1
event 4	$I_2 \to R_2$	γi_2
event 5	$R_1 \to I_{21}$	$(\frac{\beta_1 i_2+\beta_2 i_{21}}{N})r_1$
event 6	$R_2 \to I_{12}$	$(\frac{\beta_1 i_1+\beta_2 i_{12}}{N})r_2$
event 7	$I_{21} \to R$	γi_{21}
event 8	$I_{12} \to R$	γi_{12}
event 9	$I_1 \to S$	μi_1
event 10	$I_2 \to S$	μi_2
event 11	$R_1 \to S$	μr_1
event 12	$R_2 \to S$	μr_2
event 13	$I_{21} \to S$	$(\mu+\nu)i_{21}$
event 14	$I_{12} \to S$	$(\mu+\nu)i_{12}$
event 15	$R \to S$	μr

1.1 Octave (MATLAB) Code

2 Stability (no age differentiation)

The Jacobian matrix, evaluated at $S = N$, and $I_1 = I_2 = R_1 = R_2 = I_{21} = I_{12} = R = 0$, is

$$\begin{pmatrix} -\mu & -\beta_1 & -\beta_1 & 0 & 0 & -\beta_2+\nu & -\beta_2+\nu & 0 \\ 0 & \beta_1-(\mu+\gamma) & 0 & 0 & 0 & 0 & \beta_2 & 0 \\ 0 & 0 & \beta_1-(\mu+\gamma) & 0 & 0 & \beta_2 & 0 & 0 \\ 0 & \gamma & 0 & -\mu & 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma & 0 & -\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -(\mu+\nu+\gamma) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -(\mu+\nu+\gamma) & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma & \gamma & -\mu \end{pmatrix}$$

The eigenvalues (with multiplicity) of this matrix are:

$$\{\beta_1-\gamma-\mu, \beta_1-\gamma-\mu, -\mu, -\mu, -\mu, -\mu, -\mu, -(\mu+\nu+\gamma), -(\mu+\nu+\gamma)\}$$

Thus, this is a stable equilibrium provided $\beta_1 < \gamma+\mu$. (All parameters are nonnegative; thus all the other eigenvalues are negative unconditionally.)

Note, this is the only disease-free equilibrium. Any other scenario (say, with some R_1 or R_2 or R people) would eventually have its R_1 people die off and be reborn as S people.

3 Age-Continuous Theoretical Set-Up - Stolen from Herbert Hethcote

Let $U(a,t)$ be the age distribution of the population at time t.

$$\frac{d}{dt}U(a(t),t) = \frac{\partial U}{\partial a}\frac{da}{dt} + \frac{\partial U}{\partial t} = \frac{\partial U}{\partial a} + \frac{\partial U}{\partial t}$$

Then

$$\frac{\partial U}{\partial a} + \frac{\partial U}{\partial t} = -\mu(a)U$$

where $\mu(a)$ is the age-based mortality rate. If $f(a)$ is the fertility of a person of age a, then the number of births at time t is

$$U(0,t) = \int_0^\infty f(a)U(a,t)\,\mathrm{d}a$$

The solution of (3) with above condition is

$$U(a,t) = \begin{cases} U(0,t-a)\exp\left(-\int_0^a \mu(v)\mathrm{d}v\right) & t \geq a \\ U(a-t,0)\exp\left(-\int_{a-t}^a \mu(v)\mathrm{d}v\right) & t < a \end{cases}$$

Assuming U is separable, that is $U(a,t) = T(t)A(a)$, the solution of the PDE is

$$U(a,t) = T(0)e^{qt}A(0)\exp\big(-D(a) - qa\big)$$

where $D(a) = \int_0^a \mu(v)\mathrm{d}v$. Assuming a steady-state age distribution, and normalizing to 1 at $t = 0$, we redefine

$$U(a,t) = \rho e^{qt}e^{-D(a)-qa}, \quad \text{with} \quad \rho = 1/\left[\int_0^\infty e^{-D(a)-qa}\mathrm{d}a\right]$$

$$\frac{\partial S}{\partial a} + \frac{\partial S}{\partial t} = -(\lambda_1 + \lambda_2 + \mu)S \tag{7a}$$

$$\frac{\partial I_1}{\partial a} + \frac{\partial I_1}{\partial t} = \lambda_1 S - (\gamma_1 + \mu)I_1 \tag{7b}$$

$$\frac{\partial I_2}{\partial a} + \frac{\partial I_2}{\partial t} = \lambda_2 S - (\gamma_1 + \mu)I_2 \tag{7c}$$

$$\frac{\partial R_1}{\partial a} + \frac{\partial R_1}{\partial t} = \gamma_1 I_1 - (\tilde{\lambda}_1 + \mu)R_1 \tag{7d}$$

$$\frac{\partial R_2}{\partial a} + \frac{\partial R_2}{\partial t} = \gamma_1 I_2 - (\tilde{\lambda}_2 + \mu)R_2 \tag{7e}$$

$$\frac{\partial I_{21}}{\partial a} + \frac{\partial I_{21}}{\partial t} = \tilde{\lambda}_1 R1 - (\gamma_2 + \nu + \mu)I_{21} \tag{7f}$$

$$\frac{\partial I_{12}}{\partial a} + \frac{\partial I_{12}}{\partial t} = \tilde{\lambda}_2 R2 - (\gamma_2 + \nu + \mu)I_{12} \tag{7g}$$

$$\frac{\partial R}{\partial a} + \frac{\partial R}{\partial t} = \gamma_2(I_{21} + I_{12}) - \mu R \tag{7h}$$

where

$$\lambda_1 = \int_0^\infty \beta_1(a,\tilde{a}) I_1(\tilde{a},t) + \beta_2(a,\tilde{a}) I_{12}(\tilde{a},t)\, d\tilde{a} \tag{8a}$$

$$\lambda_2 = \int_0^\infty \beta_1(a,\tilde{a}) I_2(\tilde{a},t) + \beta_2(a,\tilde{a}) I_{21}(\tilde{a},t)\, d\tilde{a} \tag{8b}$$

$$\tilde{\lambda}_1 = \int_0^\infty \tilde{\beta}_1(a,\tilde{a}) I_1(\tilde{a},t) + \tilde{\beta}_2(a,\tilde{a}) I_{12}(\tilde{a},t)\, d\tilde{a} \tag{8c}$$

$$\tilde{\lambda}_2 = \int_0^\infty \tilde{\beta}_1(a,\tilde{a}) I_2(\tilde{a},t) + \tilde{\beta}_2(a,\tilde{a}) I_{21}(\tilde{a},t)\, d\tilde{a} \tag{8d}$$

$\beta_1(a,\tilde{a})$ encompasses the likelihood of contact between a person of age a and age $\tilde{a}$, as well as transmission rate for a person of age $\tilde{a}$ infected with their first serotype, to a susceptible person of age a. Similarly, β_2 a susceptible person of age a with a person of age $\tilde{a}$ infected with their second serotype. ($\tilde{\beta}_1$ a semi-immune person of age a becoming infected with their first serotype via contact with age $\tilde{a}$; $\tilde{\beta}_2$ a semi-immune person of age a. So, assume separability. For susceptibility/transmissibility, this seems to make sense (one person's transmissibility vs another person's susceptibility... these seem like independent events). On the other hand, this doesn't seem to make sense in terms of mixing... people have a high likelihood of interacting with people their own age. But, let $\beta_1(a,\tilde{a}) = f_1(a)g_1(a)$, $\tilde{\beta}_1(a,\tilde{a}) = \tilde{f}_1(a)\tilde{g}_1(a)$, etc.
The boundary conditions are:

$$S(0,t) = \int_0^\infty f(a) U(a,t)\, da$$

and all other distributions at age 0 are zero. Letting $s = S/U$, $i_1 = I_1/U$, etc, we obtain the following PDEs:

$$\frac{\partial s}{\partial a} + \frac{\partial s}{\partial t} = -(\lambda_1 + \lambda_2)s \tag{9a}$$

$$\frac{\partial i_1}{\partial a} + \frac{\partial i_1}{\partial t} = \lambda_1 s - \gamma_1 i_1 \tag{9b}$$

$$\frac{\partial i_2}{\partial a} + \frac{\partial i_2}{\partial t} = \lambda_2 s - \gamma_1 i_2 \tag{9c}$$

$$\frac{\partial r_1}{\partial a} + \frac{\partial r_1}{\partial t} = \gamma_1 i_1 - \tilde{\lambda}_1 r_1 \tag{9d}$$

$$\frac{\partial r_2}{\partial a} + \frac{\partial r_2}{\partial t} = \gamma_1 i_2 - \tilde{\lambda}_2 r_2 \tag{9e}$$

$$\frac{\partial i_{21}}{\partial a} + \frac{\partial i_{21}}{\partial t} = \tilde{\lambda}_1 r_1 - (\gamma_2 + \nu) i_{21} \tag{9f}$$

$$\frac{\partial i_{12}}{\partial a} + \frac{\partial i_{12}}{\partial t} = \tilde{\lambda}_2 r_2 - (\gamma_2 + \nu) i_{12} \tag{9g}$$

$$\frac{\partial r}{\partial a} + \frac{\partial r}{\partial t} = \gamma_2 (i_{21} + i_{12}) \tag{9h}$$

The boundary conditions are $s(0,t) = 1$, and all others are 0, and we have:

$$\lambda_1(a,t) = \int_0^\infty (f_1(a)g_1(\tilde{a})i_1(\tilde{a},t) + f_2(a)g_2(\tilde{a})i_{12}(\tilde{a},t))\rho e^{-D(\tilde{a})-q\tilde{a}}\,\mathrm{d}\tilde{a} \tag{10a}$$

$$\lambda_2 = ... \tag{10b}$$