Nonlinear Schrödinger Equation with non-zero boundary conditions

1 IST

The focusing nonlinear Schroödinger equation with symmetric NZBC is

$$iq_t + q_{xx} + 2|q|^2 q = 0\,,$$

where $|q(x,t)| \to q_\pm$ as $x \to \pm\infty$, and $|q_-| = |q_+|$. Related to this PDE are two ODEs, one of which is the "scattering problem,"

$$\phi_x = (ik\sigma_3 + Q)\,\phi\,,$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad Q(x,t) = \begin{pmatrix} 0 & q \\ -q^* & 0 \end{pmatrix}.$$

As $x \to \pm\infty$, the solutions of the scattering problem are approximated by those of the asymptotic scattering problem $\phi_x = X_\pm\,\phi$, where $X_\pm = ik\sigma_3 + Q_\pm$ and $Q_\pm = \lim_{x\to\pm\infty} Q(x,t)$. The eigenvalues of $X_\pm$ are

$$\lambda(k) = (q_o^2 + k^2)^{1/2}\,.$$

The eigenvalues are given by a complex square root, so there is a branch cut along $i[-q_o, q_o]$. We write the eigenvector matrix for $X_\pm$ (dropping the dependence on t) as

$$Y_\pm(k) = I + i\sigma_3 Q_\pm/(k + \lambda(k))\,.$$

The Jost eigenfunctions $\phi_\pm(x,k)$ solve the scattering ODE and satisfy the boundary conditions

$$\phi_\pm(x,k) = Y_\pm(k)\,\mathrm{e}^{i\lambda x\sigma_3} + o(1) \qquad \text{as } x \to \pm\infty\,.$$

(The Jost eigenfunctions are actually functions of t, as well, but in what follows we only need time $t = 0$.) For all $k \in \mathbb{R} \cup i(-q_o, q_o)$ and for all $x \in \mathbb{R}$, $\phi_-(x,k)$ and $\phi_+(x,k)$ are two fundamental matrix solutions of the Lax pair, so there is some matrix $S(k)$ (the "scattering matrix") such that

$$\phi_+(x,k) = \phi_-(x,k)\,S(k)\,.$$

It turns out that the columns of $\phi_\pm$ may be extended to different regions of the complex k-plane. The entries of $S(k)$ may be written as Wronskians of columns of the $\phi_\pm$, so some of them may be extendable as well. In particular, we will care about $s_{11}(k)$, which extends beyond $k \in \mathbb{R} \cup i(-q_o, q_o)$, because the zeros of $s_{11}(k)$ correspond to solitary waves in the solution $q(x,t)$.

2 Piecewise constant potentials

A summary of some results will be given in the case of the IVP with the following piecewise constant, box-like initial conditions:

$$q(x,0) = \begin{cases} 1\,, & |x| > L\,, \\ b\,\mathrm{e}^{i\alpha}\,, & |x| < L\,, \end{cases} \tag{1}$$

where $b > 0$. The case $0 < b < 1$ represents a potential well, the case $b > 1$ a potential barrier. The parameter α represents the phase difference between the value of the potential inside the box and its background value.

This is a convenient potential to consider, because the scattering ODE may be solved exactly, and so the Jost solutions at $t = 0$ have a simple representation in certain subsets of the real axis. Namely,

$$\phi_-(x,0,k) = \left(I + \frac{i}{k+\lambda}\sigma_3 Q_-\right) \mathrm{e}^{i\lambda x\sigma_3} \qquad x < -L\,, \tag{2a}$$

$$\phi_+(x,0,k) = \left(I + \frac{i}{k+\lambda}\sigma_3 Q_+\right) \mathrm{e}^{i\lambda x\sigma_3} \qquad x > L\,. \tag{2b}$$

In addition, a fundamental matrix solution of the scattering problem in the region $|x| < L$ is

$$\phi_0(x,0,k) = \left(I + \frac{i}{k+\mu}\sigma_3 Q_0\right) \mathrm{e}^{i\mu x\sigma_3} \qquad |x| < L\,, \tag{2c}$$

where

$$\mu^2 = k^2 + b^2\,. \tag{3}$$

The expression of all these eigenfunctions beyond these domains is obtained by requiring continuity at the boundary. Thus, there exist matrices $S_+(k)$ and $S_-(k)$, independent of x, such that

$$S_+(k) = \phi_o^{-1}(x,0,k)\phi_+(x,0,k)\,, \tag{4a}$$

$$S_-(k) = \phi_o^{-1}(x,0,k)\phi_-(x,0,k)\,. \tag{4b}$$

The scattering matrix $S(k) = \phi_-^{-1}(x,0,k)\phi_+(x,0,k)$ is given by

$$S(k) = S_-^{-1}(k)S_+(k)\,.$$

Evaluating (**??**) at $x = \pm L$ (respectively),

$$\begin{aligned} s_{1,1}(k) = \mathrm{e}^{-2iL(\mu-\lambda)} \big[&\mathrm{e}^{4iL\mu}\big(b^2(k+\lambda)^2 - 2b\cos\alpha(k+\lambda)(k+\mu) + (k+\mu)^2\big) \\ &+ b^2 + 2b\cos\alpha(k+\lambda)(k+\mu) + (k+\lambda)^2(k+\mu)^2\big] \\ &\qquad\qquad /((k+\lambda)^2+1)((k+\mu)^2+b^2)\,. \end{aligned}$$

Note that $s_{1,1}(k)$ depends on $\mu(k)$, which is defined by (**??**) up to a sign. After some algebra,

$$s_{1,1}(k) = \mathrm{e}^{2iL\lambda}\left(\cos(2L\mu) - \frac{i(b\cos\alpha + k^2)}{\lambda\mu}\sin(2L\mu)\right). \tag{5}$$

2.1 Results

1. If $b < 1$ and $\cos\alpha > b$ there are no discrete eigenvalues (that is, zeros of s_{11}). (So there is a class of perturbations of the constant background that don't have any solitons.)

2. If $b > 1$ and $\cos\alpha > 1/b$, there is always at least one eigenvalue. The eigenvalues are always purely imaginary, in the interval $i(1,b)$. More precisely, there are n discrete eigenvalues in $i(1,b)$ whenever $(n-1)\pi/(2\sqrt{b^2-1}) < L < n\pi/(2\sqrt{b^2-1})$, and these eigenvalues accumulate near ib (on the imaginary axis) as L increases. (This means that there are

3. If $b < 1$ and $\cos\alpha < b$, we don't have analytical results. Numerical evidence suggests there are no zeros for $L < (2n+1)\pi/(4\sqrt{b^2-b_\alpha^2})$. As L increases, an eigenvalues appear on the continuous spectrum and accumulate near ib.

4. If $b > 1$ and $\cos\alpha < 1/b$, $s_{1,1}$ has a zero on the continuous spectrum at ib_α if $L = (2n+1)\pi/(4\sqrt{b^2 - b_\alpha^2})$. Numerical evidence suggests that as L increases, the eigenvalues travel towards the imaginary axis, bifurcating at points in $i(1, b)$. The eigenvalues that travel down disappear at i; those that travel up accumulate on the imaginary axis near ib.

3 Numerical Questions

I have been wondering what any of these solutions look like. In particular,

1. What happens to $q(x,t)$ in the case when there are no solitons? That is, what does $q(x,t)$ look like if $b < 1$ and $\cos\alpha > b$?

2. Is there a difference between the $q(x,t)$ with a purely imaginary eigenvalue and the $q(x,t)$ with a complex eigenvalue, or a purely real "eigenvalue?" (Technically, if a zero of s_{11} is real, then it is on the continuous spectrum, and so is not actually an eigenvalue.)

3. What happens at the L at which eigenvalues bifurcate? That is, in a case where $L - \epsilon$ has one eigenvalue, and $L + \epsilon$ has 2 eigenvalues, what happens to $q(x,t)$ right at L?