The Nonlinear Schrödinger equation with non-zero boundary conditions and piecewise-constant potential

May 2014

Outline

(abbreviated) Inverse Scattering Transform

NLS equation:

$$iq_t + q_{xx} + 2|q|^2 q = 0$$

NZBCs:

$$\lim_{x\to\pm\infty} q(x,0) = q_\pm, \qquad |q_\pm| = q_o \neq 0$$

(abbreviated) Inverse Scattering Transform

NLS equation:

$$iq_t + q_{xx} + 2|q|^2 q = 0$$

NZBCs:

$$\lim_{x \to \pm\infty} q(x,0) = q_\pm, \qquad |q_\pm| = q_o \neq 0$$

Related ODE (called the "Scattering problem"):

$$\mathbf{v}_x = \left[ik \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & q(x,t) \\ -q^*(x,t) & 0 \end{pmatrix} \right] \mathbf{v}$$

Asymptotic scatttering problem

$$\mathbf{v}_x = \begin{pmatrix} ik & q_\pm \\ -q_\pm^* & -ik \end{pmatrix} \mathbf{v}$$

This is a first order, constant coefficient ODE.

Asymptotic scatttering problem

$$\mathbf{v}_x = \begin{pmatrix} ik & q_\pm \\ -q_\pm^* & -ik \end{pmatrix} \mathbf{v}$$

This is a first order, constant coefficient ODE. Its eigenvalues are

$$\pm i\lambda\,, \qquad \lambda = \sqrt{k^2 + q_o^2}$$

One way to write the eigenvector matrices is

$$\mathbf{E}_\pm(k) = \mathbf{I}_2 + \frac{i}{k + \lambda(k)} \sigma_3 \mathbf{Q}_\pm$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,, \qquad \mathbf{Q}_\pm = \begin{pmatrix} 0 & q_\pm \\ -q_\pm^* & 0 \end{pmatrix}$$

Jost solutions and scattering matrix

"Jost solutions" solve the scattering ODE and asymptotically look like:

$$\Phi_{\pm}(x,k) = \mathbf{E}_{\pm}(k)\,\mathrm{e}^{i\lambda x\sigma_3} + o(1) \qquad \text{as } x \to \pm\infty\,.$$

Jost solutions and scattering matrix

"Jost solutions" solve the scattering ODE and asymptotically look like:

$$\Phi_{\pm}(x,k) = \mathbf{E}_{\pm}(k)\, \mathrm{e}^{i\lambda x\sigma_3} + o(1) \qquad \text{as } x \to \pm\infty\,.$$

Both $\Phi_{\pm}$ solve the same ODE (the scattering problem), so there is a matrix independent of x that relates them:

$$\Phi_{+}(x,k) = \Phi_{-}(x,k)\mathbf{S}(k)$$

Jost solutions and scattering matrix

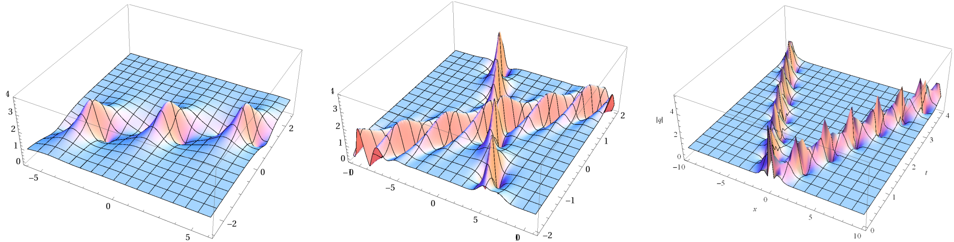
"Jost solutions" solve the scattering ODE and asymptotically look like:

$$\Phi_{\pm}(x,k) = \mathbf{E}_{\pm}(k)\,\mathrm{e}^{i\lambda x \sigma_3} + o(1) \qquad \text{as } x \to \pm\infty\,.$$

Both $\Phi_{\pm}$ solve the same ODE (the scattering problem), so there is a matrix independent of x that relates them:

$$\Phi_{+}(x,k) = \Phi_{-}(x,k)\mathbf{S}(k)$$

Fastforward: The zeros of $s_{11}(k)$ correspond to soliton solutions:



Outline

"Box-like" Potential

Initial condition:

$$q(x,0) = \begin{cases} 1 & : |x| > L \\ b\,\mathrm{e}^{i\alpha} & : |x| < L \end{cases}$$

"Box-like" Potential

Initial condition:

$$q(x,0) = \begin{cases} 1 & : |x| > L \\ b\,\mathrm{e}^{i\alpha} & : |x| < L \end{cases}$$

$b < 1$: Potential well

$b > 1$: Potential barrier

"Box-like" Potential

Initial condition:

$$q(x,0) = \begin{cases} 1 & : |x| > L \\ b\,\mathrm{e}^{i\alpha} & : |x| < L \end{cases}$$

$b < 1$: Potential well

$b > 1$: Potential barrier

Define $b_\alpha = \sqrt{b \cos \alpha}$.

$b_\alpha > \max(1, b)$: "small phase difference,"

$b_\alpha < \max(1, b)$: "large phase difference"

Summary of Results: "small phase"

Potential well, $b_\alpha > b$

No discrete eigenvalues
(i.e., zeros of s_{11})

Potential barrier, $b_\alpha > 1$

Always an eigenvalue in $i(1, b)$
No other eigenvalues
Number of eigenvalues n given by:

$$\frac{(n-1)\pi}{2\sqrt{b^2-1}} < L < \frac{n\pi}{2\sqrt{b^2-1}}$$

Eigenvalues accumulate near ib

Summary of Results: "large phase"

If $L = \dfrac{(2n+1)\pi}{4\sqrt{b^2 - b_\alpha^2}}$, then $s_{11}(k)$ has a zero at ib_α

Potential well, $b_\alpha < b$
No purely real or purely imaginary discrete eigenvalues
Eigenvalues accumulate near ib

Potential barrier, $b_\alpha < 1$
Eigenvalues bifurcate on the imaginary axis; one eigenvalue travels up to i and vanishes.
Eigenvalues accumulate near ib

Note: unlike the "small phase" case, these are not analytical results.

Outline

Questions to ask

- What do any of these solutions even look like?
- In particular, given a small phase, the barrier always has an eigenvalue and the well never does. Contrast two such solutions.
- What is the difference between a soliton from a purely imaginary eigenvalue and one from a complex eigenvalue?
  Note: the symmetries of the problem mean that a complex eigenvalue is paired with its reflection over the imaginary axis. So a complex eigenvalue should correspond to two solitons.
- What happens at L at which eigenvalues bifurcate?

Split Step

Focusing NLS equation:

$$iq_t + q_{xx} + 2|q|^2 q = 0$$

Rewrite the NLS equation in terms of a linear and nonlinear operator:

$$q_t = (\mathcal{L} + \mathcal{N})q$$

$$\mathcal{L} = i(\partial_x)^2\,, \qquad \mathcal{N} = 2i|q|^2$$

The "split step"

$$q(x, t+\Delta t) \sim \mathrm{e}^{\Delta t/2\mathcal{L}}\,\mathrm{e}^{\Delta t\mathcal{N}}\,\mathrm{e}^{\Delta t/2\mathcal{L}} q(x,t)$$

Split Step, continued

Linear Step:

$$q_t = iq_{xx}$$

Treat as a periodic problem; use discrete Fourier transform
Discontinuous potential; use exponential filter to avoid Gibbs

Split Step, continued

Linear Step:

$$q_t = iq_{xx}$$

Treat as a periodic problem; use discrete Fourier transform
Discontinuous potential; use exponential filter to avoid Gibbs

Nonlinear Step:

$$q_t = 2i|q|^2 q$$

RK3

Some pictures

Still looks like Gibbs!

Idea: linearize the PDE near the constant solution

$$q(x,t) = 1 + v(x,t)\,, \qquad v = O(\epsilon)$$