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Math 504

Homework 1

2(a) Let u and v be two column vectors with dimension n. Show $u \cdot v = u^T v$.

$$u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}, \qquad v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \qquad \text{Definition: } \mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{n} a_i b_i = a_1 b_1 + a_2 b_2 + \ldots a_n b_n$$

$$\begin{aligned} u^T v &= \begin{pmatrix} u_1 & u_2 = \cdots & u_n \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \\ &= u_1 v_1 + u_2 v_2 + \cdots + u_n v_n \\ &= \sum_{i=1}^{n} u_i v_i \\ &= \mathbf{u} \cdot \mathbf{v} \end{aligned}$$

2(b) Let u be n dimensional, v by n' dimensional. Let M be a $n \times n'$ dimensional matrix. Show that $u \cdot Mv = \sum_{i=1}^{n} \sum_{j=1}^{n'} M_{ij} u_i v_j$. (Hint: Start by noticing that the kth coordinate of Mv, $(Mv)_k$, is given by $(Mv)_k = \sum_{j=1}^{n} M_{kj} v_j$)

$$u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}, \qquad v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n'} \end{pmatrix}, \qquad M = \begin{pmatrix} m_{1,1} & m_{1,2} & \cdots & m_{1,n'} \\ m_{2,1} & m_{2,2} & \cdots & m_{2,n'} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n,1} & m_{n,2} & \cdots & m_{n,n'} \end{pmatrix}$$

$$u \cdot Mv = u \cdot \begin{pmatrix} m_{1,1} & m_{1,2} & \cdots & m_{1,n'} \\ m_{2,1} & m_{2,2} & \cdots & m_{2,n'} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n,1} & m_{n,2} & \cdots & m_{n,n'} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n'} \end{pmatrix}$$

$$= u \cdot \begin{pmatrix} \sum\limits_{j=1}^{n'} m_{1j} v_j \\ \sum\limits_{j=1}^{n'} m_{2j} v_j \\ \vdots \\ \sum\limits_{j=1}^{n'} m_{nj} v_j \end{pmatrix} \qquad \text{Let } \mu = \begin{pmatrix} \sum\limits_{j=1}^{n'} m_{1j} v_j \\ \sum\limits_{j=1}^{n'} m_{2j} v_j \\ \vdots \\ \sum\limits_{j=1}^{n'} m_{nj} v_j \end{pmatrix}$$

$$\begin{aligned} &= \mathbf{u} \cdot \boldsymbol{\mu} \\ &= \sum_{i=1}^{n} u_i \mu_i = \sum_{i=1}^{n} u_i \left(\sum_{j=1}^{n'} m_{ij} v_j \right) = \sum_{i=1}^{n} \sum_{j=1}^{n'} m_{ij} u_i v_j \end{aligned}$$

2(c) Let b and x be vectors in $\mathbb{R}^n$, Show that $\nabla(b^T x) = b$. (We did this in class.)

$$\begin{aligned}
\nabla\left(b^T x\right) &= \nabla\left(\sum_{i=1}^{n} b_i x_i\right) \\
&= \nabla\left(b_1 x_1 + b_2 x_2 + \cdots + b_n x_n\right) \\
&= \begin{pmatrix} \dfrac{\partial}{\partial x_1}\, b_1 x_1 + b_2 x_2 + \cdots + b_n x_n \\ \dfrac{\partial}{\partial x_2}\, b_1 x_1 + b_2 x_2 + \cdots + b_n x_n \\ \vdots \\ \dfrac{\partial}{\partial x_n}\, b_1 x_1 + b_2 x_2 + \cdots + b_n x_n \end{pmatrix} \\
&= \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \\
&= \mathbf{b}
\end{aligned}$$

2(d) Let A be an $n \times n$ matrix and $x \in \mathbb{R}^n$. Show that $\nabla(x^T A x) = (A + A^T)x$. Hint: Use (b).

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad A^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix}$$

$$x^T A x = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$\begin{aligned}
&= \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} \begin{pmatrix} \sum_{j=1}^{n} a_{1j} x_j \\ \sum_{j=1}^{n} a_{2j} x_j \\ \vdots \\ \sum_{j=1}^{n} a_{1j} x_j \end{pmatrix} \\
&= x_1 \sum_{j=1}^{n} a_{1j} x_j + x_2 \sum_{j=1}^{n} a_{2j} x_j + \ldots + x_n \sum_{j=1}^{n} a_{nj} x_j
\end{aligned}$$

$$\nabla\left(x^TAx\right) = \begin{pmatrix} \frac{\partial}{\partial x_1}\left[x_1\sum_{j=1}^n a_{1j}x_j + x_2\sum_{j=1}^n a_{2j}x_j + ... + x_n\sum_{j=1}^n a_{nj}x_j\right] \\ \frac{\partial}{\partial x_2}\left[x_1\sum_{j=1}^n a_{1j}x_j + x_2\sum_{j=1}^n a_{2j}x_j + ... + x_n\sum_{j=1}^n a_{nj}x_j\right] \\ \vdots \\ \frac{\partial}{\partial x_n}\left[x_1\sum_{j=1}^n a_{1j}x_j + x_2\sum_{j=1}^n a_{2j}x_j + ... + x_n\sum_{j=1}^n a_{nj}x_j\right] \end{pmatrix}$$

$$= \begin{pmatrix} 2a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n + a_{21}x_2 + ... + a_{n1}x_n \\ a_{12}x_1 + a_{21}x_1 + 2a_{22}x_2 + ... + a_{2n}x_n + ... + a_{n2}x_n \\ \vdots \\ a_{1n}x_1 + a_{2n}x_2 + ... + a_{n1}x_1 + a_{n2}x_2 + ... + 2a_{nn}x_n \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n + a_{11}x_1 + a_{21}x_2 + ... + a_{n1}x_n \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n + a_{12}x_1 + a_{22}x_2 + ... + a_{n2}x_n \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n + a_{1n}x_1 + a_{2n}x_2 + ... + a_{nn}x_n \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n \end{pmatrix} + \begin{pmatrix} a_{11}x_1 + a_{21}x_2 + ... + a_{n1}x_n \\ a_{12}x_1 + a_{22}x_2 + ... + a_{n2}x_n \\ \vdots \\ a_{1n}x_1 + a_{2n}x_2 + ... + a_{nn}x_n \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$= \left(A + A^T\right)x$$

2(e) Explain why a general quadratic in $\mathbb{R}^n$ can be written as $f(x) = x^TAx + b^Tx + c$. No proof here, just explain the intuition. https://math.stackexchange.com/questions/2019154/is-the-matrix-a-symmetric-in-the-quadratic-form

??? something about symmetry

2(f) Let

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 1 \end{pmatrix}$$

and let $f(x) = x^T Ax$. Show that $f(x)$ can be rewritten as $f(x) = x^T Bx$ with B symmetric. (Hint: write out $f(x)$ as a sum of terms rather than a matrix expression.) Then explain why for the general quadratic in (e), we can always asssume that A is symmetric.

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 1 \end{pmatrix}, \quad A^T = \begin{pmatrix} 1 & 2 \\ 4 & 1 \end{pmatrix}$$

$$\begin{aligned} \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} x_1 + 4x_2 \\ 2x_1 + x_2 \end{pmatrix} \\ &= x_1(x_1 + 4x_2) + x_2(2x_1 + x_2) \\ &= x_1^2 + 4x_1x_2 + 2x_1x_2 + x_2^2 \\ &= x_1^2 + 6x_1x_2 + x_2^2 \end{aligned}$$

$$\begin{aligned} \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} x_1 + 2x_2 \\ 4x_1 + x_2 \end{pmatrix} \\ &= x_1(x_1 + 2x_2) + x_2(4x_1 + x_2) \\ &= x_1^2 + 2x_1x_2 + 4x_1x_2 + x_2^2 \\ &= x_1^2 + 6x_1x_2 + x_2^2 \end{aligned}$$

2(g) Let $x \in \mathbb{R}^n$ and $f(x) = x^T Ax + b^T x + c$ where A is an $n \times n$ matrix, b is a n dimensional vector and c is a scalar. Show that $\nabla f(x) = (A + A^T)x + b$. Solve for the critical point of $f(x)$. (We essentially did this in class.)

3

```r
library(RCurl)
```

```
## Warning: package 'RCurl' was built under R version 3.3.3
## Loading required package: bitops
```

```r
dat <- getURL("https://raw.githubusercontent.com/mleibert/504/master/economic_data.txt")
dat <- read.table(text = dat ,header=T)
options(scipen = 999)

y<-dat[,ncol(dat)]
B<-as.matrix(dat[,2:(ncol(dat)-1)])
B<-cbind(1,B)

#solve(t(B) %*% B ) %*% t(B) %*% y

cor(B[,-1])
```

```
##           A1        A2         A3         A4        A5        A6
## A1 1.0000000 0.9915892  0.6205874  0.4647442 0.9791634 0.9911492
## A2 0.9915892 1.0000000  0.6042089  0.4464368 0.9910901 0.9952735
## A3 0.6205874 0.6042089  1.0000000 -0.1775504 0.6865068 0.6682035
## A4 0.4647442 0.4464368 -0.1775504  1.0000000 0.3644163 0.4172451
## A5 0.9791634 0.9910901  0.6865068  0.3644163 1.0000000 0.9939528
```

```
## A6 0.9911492 0.9952735  0.6682035  0.4172451 0.9939528 1.0000000
```

```
BB<-B[,-c(7,6)]
cor(BB[,-1])
```

```
##           A1        A2         A3         A4
## A1 1.0000000 0.9915892  0.6205874  0.4647442
## A2 0.9915892 1.0000000  0.6042089  0.4464368
## A3 0.6205874 0.6042089  1.0000000 -0.1775504
## A4 0.4647442 0.4464368 -0.1775504  1.0000000
```

```
solve(t(BB) %*% BB ) %*% t(BB) %*% y
```

```
##              [,1]
##    50135.42355558
## A1    55.34521216
## A2     0.03537306
## A3    -0.85377061
## A4    -0.54975517
```

??? something about multicollinearity

```
dat<-dat[,-1]
( lm(B~.,data=dat) )
```

```
##
## Call:
## lm(formula = B ~ ., data = dat)
##
## Coefficients:
##    (Intercept)             A1              A2              A3
## -3475440.82413        14.78948        -0.03575        -2.02020
##             A4             A5              A6
##       -1.03277        -0.04912      1825.54365
```

4

```
 times<-rep(NA,10)

for ( i in 1:10) {

  times[i]<-system.time( for ( j in 1:(10^i) ) { 1*1 } )[1]
  if ( times[i] > 1 ) { break}}
times
```

```
##  [1] 0.00 0.00 0.00 0.00 0.01 0.27 2.49   NA   NA   NA
```

```
system.time( for( i in 1: (1/times[7] * 10^7) ) { 1*1} )
```

```
##    user  system elapsed
##    1.05    0.00    1.06
```

```
1/times[7] * 10^7
```

```
## [1] 4016064
```