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Math 611
Homework 4

1. In order to estimate the integral

$$I = \int_0^1 \frac{e^{-x}}{1+x^2}\,\mathrm{d}x$$

Consider the following two choices of an importance function.

a. $f(x) = 1, \quad 0 < x < 1$

b. $g(x) = \dfrac{e^{-x}}{(1-e^{-1})}, \quad 0 < x < 1$

For $n = 10000$ simulations summarize the results in a table showing the values of the estimates and their standard errors.

For $f(x) = 1,\ 0 < x < 1$, we verify that this is a pdf. $\displaystyle\int_0^1 1\,\mathrm{d}x = 1.$

For $g(x) = \dfrac{e^{-x}}{(1-e^{-1})}, \quad 0 < x < 1$, we verify that this is a pdf.

$$\begin{aligned}
\int_0^1 \frac{e^{-x}}{(1-e^{-1})}\,\mathrm{d}x &= -\frac{1}{(1-e^{-1})}\int_0^{-1} \exp(u)\,\mathrm{d}u & u &= -x \\
& & -du &= dx \\
&= \frac{1}{(1-e^{-1})}\int_{-1}^{0} \exp(u)\,\mathrm{d}u \\
&= \frac{1}{(1-e^{-1})}\Big[\exp(u)\Big]_{-1}^{0} \\
&= \frac{1}{(1-e^{-1})}\Big[1-\exp(-1)\Big] \\
&= 1
\end{aligned}$$

It's a pdf, but we need a distribution to sample from. We use the inverse transform and sample from that.

$$\begin{aligned}
G(x) &= \int_0^y \frac{e^{-x}}{(1-e^{-1})}\,\mathrm{d}x & u &= -x \\
& & -du &= dx \\
&= -\frac{1}{(1-e^{-1})}\int_0^{-y} \exp(u)\,\mathrm{d}u \\
&= \frac{1}{(1-e^{-1})}\int_{-y}^{0} \exp(u)\,\mathrm{d}u \\
&= \frac{1}{(1-e^{-1})}\Big[\exp(u)\Big]_{-y}^{0} \\
&= \frac{1-\exp(-y)}{(1-e^{-1})}
\end{aligned}$$

$$
\begin{aligned}
z &= \frac{1-e^{-y}}{(1-e^{-1})} \\
z(1-e^{-1}) &= 1-e^{-y} \\
z(1-e^{-1})-1 &= -e^{-y} \\
-z(1-e^{-1})+1 &= e^{-y} \\
\log\left[-z(1-e^{-1})+1\right] &= -y \\
-\log\left[-z(1-e^{-1})+1\right] &= y \\
&= G^{-1}(x)
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^1 \frac{I}{f(x)}\, f(x)\ \mathrm{d}x \\
&= \int_0^1 \frac{e^{-x}}{1+x^2}\,\frac{1}{1}\ \mathrm{d}x \\
&= \int_0^1 \frac{e^{-x}}{1+x^2}\, f(x)\ \mathrm{d}x
\end{aligned}
$$

Sampling from $U(0,1)$

$$
\begin{aligned}
&\int_0^1 \frac{I}{g(x)}\, g(x)\ \mathrm{d}x \\
&= \int_0^1 \frac{(\cancel{e^{-x}})\,(1-e^{-1})}{(1+x^2)\,(\cancel{e^{-x}})}\,\frac{e^{-x}}{(1-e^{-1})}\ \mathrm{d}x \\
&= \int_0^1 \frac{1-e^{-1}}{1+x^2}\, g(x)\ \mathrm{d}x
\end{aligned}
$$

Using $U(0,1)$ to sample from $-\log\left[-z(1-e^{-1})+1\right]$

```
integrand <- function(x) {(exp(-x)) / (1+x^2 ) }
integrate(integrand ,0,1)

## 0.5247971 with absolute error < 5.8e-15

n=10000;f<-rep(0,n)
for ( i in 1:n) {  U<-runif(1); f[i]<- integrand(U)      }
fbarhn<-sum(f)/n;fbarhn

## [1] 0.5244589

I2<-function(G) { (1-exp(-1))/(1+G^2) }
g<-rep(0,n)
for ( i in 1:n) { U<-runif(1); G=-log(-U*(1-exp(-1))+1 ); g[i]<-I2(G)}
gbarhn<-sum(g)/n;gbarhn

## [1] 0.5237006

dat<-data.frame(c(fbarhn,gbarhn),c(sd(f),sd(g)));row.names(dat) <- c("f(x)","g(x)");
colnames(dat)<-c("mean","se");dat

##           mean         se
## f(x) 0.5244589 0.24438138
## g(x) 0.5237006 0.09743345
```

2. In order to approximate the value of the simple integral

$$I = \int_0^1 e^x\ \mathrm{d}x$$

Consider an antithetic variable approach with $U \sim Uniform(0,1)$. What is the percent reduction in the variance of the estimate that can be achieved using this antithetic approach, compared to simple MC integration?

Our antithetic variable estimate is $\dfrac{\exp(U_i) + \exp(1 - U_i)}{2}$, where $U \sim Uniform(0, 1)$.

```
integrand <- function(x) { exp( x)  }
integrate(integrand ,0,1)

## 1.718282 with absolute error < 1.9e-14

n=10000;AV<-rep(NA,n)
for (i in 1:n){ U<-runif(1);V=1-U; AV[i]<-.5* (exp(U) + exp(V) )        }
mean(AV);var(AV)

## [1] 1.717981
## [1] 0.003861412

MC<-rep(NA,n*2)
#simple mc
for (i in 1:(n*2)){ U<-runif(1) ; MC[i]<- exp(U)        }
mean(MC);var(MC)

## [1] 1.723584
## [1] 0.244473

(var(MC)-var(AV) / var(MC) ) * 100

## [1] 22.86781
```

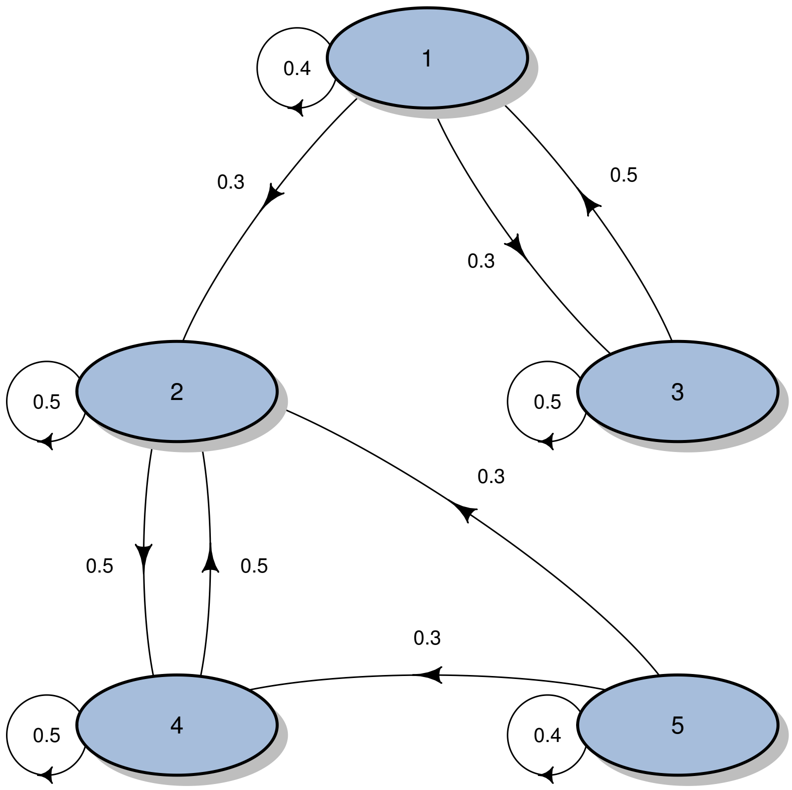
Variance reduction:

$$\frac{var\left(\hat{\theta}_2\right) - var\left(\hat{\theta}_1\right)}{var\left(\hat{\theta}_2\right)} \cdot 100 = \frac{0.244473 - 0.003861412}{0.244473} \cdot 100 = 22.86781\%$$

3. For the transition matrix:

$$\begin{array}{c c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{pmatrix} 0.4 & 0.3 & 0.3 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0.5 & 0 \\ 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0.5 & 0 \\ 0 & 0.3 & 0.3 & 0.3 & 0.4 \end{pmatrix} \end{array}$$

Identify the irreducible closed sets of the Markov Chain. Identify the transient and recurrent states.



$2 \longrightarrow 4$ and $4 \longrightarrow 2$, so $\{2,4\}$ is irreducible.
$p(2,j)=0$ and $p(4,j)=0$ where $j \in \{1,3,5\}$, so the set $\{2,4\}$ is closed.
$\{2,4\}$ is a closed, finite, and irreducible set so then all states in $\{2,4\}$ are recurrent.

$1 \longrightarrow 2,$ but $2 \not\longrightarrow 1$, so 1 is transient.
$3 \longrightarrow 2,$ but $2 \not\longrightarrow 3$, so 3 is transient.
$5 \longrightarrow 4,$ but $4 \not\longrightarrow 5$, so 5 is transient.

4. Find the stationary distribution for the Markov Chain on $\{1,2,3,4\}$ having transition probability matrix:

$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.6 & 0 & 0.4 & 0 \\ 0 & 0.5 & 0 & 0.5 \\ 0 & 0.4 & 0 & 0.6 \end{pmatrix} \end{array}$$

The equation $\pi p = \pi$ says

$$\begin{pmatrix}\pi_1 & \pi_2 & \pi_3 & \pi_4\end{pmatrix}\begin{pmatrix}0.7 & 0 & 0.3 & 0\\ 0.6 & 0 & 0.4 & 0\\ 0 & 0.5 & 0 & 0.5\\ 0 & 0.4 & 0 & 0.6\end{pmatrix} = \begin{pmatrix}\pi_1 & \pi_2 & \pi_3 & \pi_4\end{pmatrix}$$

which translates into four equations

$$\begin{array}{ccccccccc} 0.7\pi_1 & + & 0.6\pi_2 & + & 0\pi_3 & + & 0\pi_4 & = & \pi_1\\ 0\pi_1 & + & 0\pi_2 & + & 0.5\pi_3 & + & 0.4\pi_4 & = & \pi_2\\ 0.3\pi_1 & + & 0.4\pi_2 & + & 0\pi_3 & + & 0\pi_4 & = & \pi_3\\ 0\pi_1 & + & 0.0\pi_2 & + & 0.5\pi_3 & + & 0.6\pi_4 & = & \pi_4 \end{array}.$$

The fourth equation is redundant since if we add up the three equations we get

$$\pi_1 + \pi_2 + \pi_3 + \pi_4 = \pi_1 + \pi_2 + \pi_3 + \pi_4.$$

If we replace the fourth equation by $\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$ and subtract π_1 from each side of the first equation, π_2 from each side of the second equation, and π_3 from each side of the third equation we get

$$\begin{array}{ccccccccc} -0.3\pi_1 & + & 0.6\pi_2 & + & 0\pi_3 & + & 0\pi_4 & = & 0\\ 0\pi_1 & + & -\pi_2 & + & 0.5\pi_3 & + & 0.4\pi_4 & = & 0\\ 0.3\pi_1 & + & 0.4\pi_2 & + & -\pi_3 & + & 0\pi_4 & = & 0\\ \pi_1 & + & \pi_2 & + & \pi_3 & + & \pi_4 & = & 1 \end{array}.$$

At this point we can solve the equations in R.

```
mat<- matrix( c(.7,0,.3,0,.6,0,.4,0,0,.5,0,.5,0,.4,0,.6 ),4,4,byrow = T);mat

##      [,1] [,2] [,3] [,4]
## [1,]  0.7  0.0  0.3  0.0
## [2,]  0.6  0.0  0.4  0.0
## [3,]  0.0  0.5  0.0  0.5
## [4,]  0.0  0.4  0.0  0.6

mat2<-matrix( c(-.3,.6,0,0,0,-1,.5,.4,.3,.4,-1,0,1,1,1,1),4,4,byrow = T);mat2

##      [,1] [,2] [,3] [,4]
## [1,] -0.3  0.6  0.0  0.0
## [2,]  0.0 -1.0  0.5  0.4
## [3,]  0.3  0.4 -1.0  0.0
## [4,]  1.0  1.0  1.0  1.0

PI=solve(mat2,c(0,0,0,1));PI

## [1] 0.3809524 0.1904762 0.1904762 0.2380952

PI%*%mat

##           [,1]      [,2]      [,3]      [,4]
## [1,] 0.3809524 0.1904762 0.1904762 0.2380952
```

$$\begin{pmatrix}\frac{8}{21} & \frac{4}{21} & \frac{4}{21} & \frac{5}{21}\end{pmatrix}\begin{pmatrix}0.7 & 0 & 0.3 & 0\\ 0.6 & 0 & 0.4 & 0\\ 0 & 0.5 & 0 & 0.5\\ 0 & 0.4 & 0 & 0.6\end{pmatrix} = \begin{pmatrix}\frac{8}{21} & \frac{4}{21} & \frac{4}{21} & \frac{5}{21}\end{pmatrix}$$

5. A Markov chain X_n, varies over the state space $S = \{0, 1, 2, 3\}$. At each stage, X_n describes the distribution of X_n: type 1 genes and $N - X_n$: type 2 genes in a certain population. The transition probabilities for having j: type 1 genes in the present state if we had i: type 1 genes in the previous state follow a binomial distribution with probability of success $\frac{i}{N}$. State 0 is an absorbing state. Derive the stationary distribution of this MC for the given state space.

We note that the Markov chain X_n has a transition probability,

$$p(i,j) = \binom{N}{j}\left(\frac{i}{N}\right)^j \left(1 - \frac{i}{N}\right)^{1-j}, \quad j \in \{0, 1, 2, 3\}.$$

And the transition probability matrix is as follows,

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{8}{27} & \frac{12}{27} & \frac{6}{27} & \frac{1}{27} \\ \frac{1}{27} & \frac{6}{27} & \frac{12}{27} & \frac{8}{27} \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

To determine the stationary distribution we set up the following equation:

$$\begin{pmatrix} \pi_1 & \pi_2 & \pi_3 & \pi_4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{8}{27} & \frac{12}{27} & \frac{6}{27} & \frac{1}{27} \\ \frac{1}{27} & \frac{6}{27} & \frac{12}{27} & \frac{8}{27} \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \pi_1 & \pi_2 & \pi_3 & \pi_4 \end{pmatrix}.$$

Performing the same steps in question (4), we end up with this system of equations:

$$\begin{array}{ccccccccc} 0\pi_1 & + & \frac{8}{27}\pi_2 & + & \frac{1}{27}\pi_3 & + & 0\pi_4 & = & 0 \\ 0\pi_1 & - & \frac{5}{9}\pi_2 & + & \frac{6}{27}\pi_3 & + & 0\pi_4 & = & 0 \\ 0\pi_1 & + & \frac{6}{27}\pi_2 & - & \frac{5}{9}\pi_3 & + & 0\pi_4 & = & 0 \\ \pi_1 & + & \pi_2 & + & \pi_3 & + & \pi_4 & = & 1 \end{array}.$$

However, the determinant of the matrix on the LHS is 0, so it is a singular matrix. However, looking back at the equation for the stationary distribution, there are ones in the $[1, 1]$ and $[4, 4]$ entries (along with 0's in their respective rows because the rows sum to 1).

Thus a vector of the form: $\boldsymbol{\pi} = \begin{pmatrix} x & 0 & 0 & y \end{pmatrix}$ will fulfill the stationary distribution equation $\boldsymbol{\pi}\mathbf{P} = \boldsymbol{\pi}$. The components of $\boldsymbol{\pi}$ must also sum to 1, so the stationary distribution equation is $\boldsymbol{\pi} = \begin{pmatrix} 1-q & 0 & 0 & q \end{pmatrix}$, where $0 < q < 1$. Where we start determines q, so $q = \dfrac{X_0}{N}$.

$$\begin{pmatrix} 1-\dfrac{X_0}{N} & 0 & 0 & \dfrac{X_0}{N} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ \dfrac{8}{27} & \dfrac{12}{27} & \dfrac{6}{27} & \dfrac{1}{27} \\ \dfrac{1}{27} & \dfrac{6}{27} & \dfrac{12}{27} & \dfrac{8}{27} \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-\dfrac{X_0}{N} & 0 & 0 & \dfrac{X_0}{N} \end{pmatrix}.$$