Chapter 2

Poisson Processes

2.1 Exponential Distribution

To prepare for our discussion of the Poisson process, we need to recall the definition and some of the basic properties of the exponential distribution. A random variable T is said to have **an exponential distribution with rate** λ, or $T = \text{exponential}(\lambda)$, if

$$P(T \le t) = 1 - e^{-\lambda t} \quad \text{for all } t \ge 0 \tag{2.1}$$

Here we have described the distribution by giving the **distribution function** $F(t) = P(T \le t)$. We can also write the definition in terms of the **density function** $f_T(t)$ which is the derivative of the distribution function.

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t} & \text{for } t \ge 0 \\ 0 & \text{for } t < 0 \end{cases} \tag{2.2}$$

Integrating by parts with $f(t) = t$ and $g'(t) = \lambda e^{-\lambda t}$,

$$\begin{aligned} ET &= \int t\, f_T(t)\, dt = \int_0^\infty t \cdot \lambda e^{-\lambda t}\, dt \\ &= -te^{-\lambda t}\Big|_0^\infty + \int_0^\infty e^{-\lambda t}\, dt = 1/\lambda \end{aligned} \tag{2.3}$$

Integrating by parts with $f(t) = t^2$ and $g'(t) = \lambda e^{-\lambda t}$, we see that

$$\begin{aligned} ET^2 &= \int t^2\, f_T(t)\, dt = \int_0^\infty t^2 \cdot \lambda e^{-\lambda t}\, dt \\ &= -t^2 e^{-\lambda t}\Big|_0^\infty + \int_0^\infty 2te^{-\lambda t}\, dt = 2/\lambda^2 \end{aligned} \tag{2.4}$$

by the formula for ET. So the variance

$$\text{var}\,(T) = ET^2 - (ET)^2 = 1/\lambda^2 \tag{2.5}$$

While calculus is required to know the exact values of the mean and variance, it is easy to see how they depend on λ. Let $T = \text{exponential}(\lambda)$, i.e., have an exponential distribution with rate λ, and let $S = \text{exponential}(1)$. To see that S/λ has the same distribution as T, we use (2.1) to conclude

$$P(S/\lambda \le t) = P(S \le \lambda t) = 1 - e^{-\lambda t} = P(T \le t)$$

Recalling that if c is any number then $E(cX) = cEX$ and $\text{var}\,(cX) = c^2\,\text{var}\,(X)$, we see that

$$ET = ES/\lambda \qquad \text{var}\,(T) = \text{var}\,(S)/\lambda^2$$

Lack of memory property. It is traditional to formulate this property in terms of waiting for an unreliable bus driver. In words, "if we've been waiting for t units of time then the probability we must wait s more units of time is the same as if we haven't waited at all." In symbols

$$P(T > t + s | T > t) = P(T > s) \tag{2.6}$$

To prove this we recall that if $B \subset A$, then $P(B|A) = P(B)/P(A)$, so

$$P(T > t + s | T > t) = \frac{P(T > t + s)}{P(T > t)} = \frac{e^{-\lambda(t+s)}}{e^{-\lambda t}} = e^{-\lambda s} = P(T > s)$$

where in the third step we have used the fact $e^{a+b} = e^a e^b$.

Exponential races. Let $S = \text{exponential}(\lambda)$ and $T = \text{exponential}(\mu)$ be independent. In order for the minimum of S and T to be larger than t, each of S and T must be larger than t. Using this and independence we have

$$\begin{aligned} P(\min(S,T) > t) &= P(S > t, T > t) = P(S > t)P(T > t) \\ &= e^{-\lambda t} e^{-\mu t} = e^{-(\lambda+\mu)t} \end{aligned} \tag{2.7}$$

That is, $\min(S,T)$ has an exponential distribution with rate $\lambda + \mu$. The last calculation extends easily to a sequence of independent random variables $T_1, \ldots, T_n$ where $T_i = \text{exponential}(\lambda_i)$.

$$\begin{aligned} P(\min(T_1, \ldots, T_n) > t) &= P(T_1 > t, \ldots T_n > t) \\ &= \prod_{i=1}^n P(T_i > t) = \prod_{i=1}^n e^{-\lambda_i t} = e^{-(\lambda_1 + \cdots + \lambda_n)t} \end{aligned} \tag{2.8}$$

That is, the minimum, $\min(T_1, \ldots, T_n)$, of several independent exponentials has an exponential distribution with rate equal to the sum of the rates $\lambda_1 + \cdots \lambda_n$.

In the last paragraph we have computed the duration of a race between exponentially distributed random variables. We will now consider: "Who finishes first?" Going back to the case of two random variables, we break things down according to the value of S and then using independence with our formulas (2.1) and (2.2) for the distribution and density functions, to conclude

$$\begin{aligned} P(S < T) &= \int_0^\infty f_S(s) P(T > s)\, ds \\ &= \int_0^\infty \lambda e^{-\lambda s} e^{-\mu s}\, ds \\ &= \frac{\lambda}{\lambda + \mu} \int_0^\infty (\lambda + \mu) e^{-(\lambda+\mu)s}\, ds = \frac{\lambda}{\lambda + \mu} \end{aligned} \tag{2.9}$$

where on the last line we have used the fact that $(\lambda+\mu)e^{-(\lambda+\mu)s}$ is a density function and hence must integrate to 1.

From the calculation for two random variables, you should be able to guess that if $T_1, \ldots, T_n$ are independent exponentials, then

$$P(T_i = \min(T_1, \ldots, T_n)) = \frac{\lambda_i}{\lambda_1 + \cdots + \lambda_n} \tag{2.10}$$

That is, the probability of i finishing first is proportional to its rate.

Proof. Let $S = T_i$ and U be the minimum of T_j, $j \neq i$. (2.8) implies that U is exponential with parameter

$$\mu = (\lambda_1 + \cdots + \lambda_n) - \lambda_i$$

so using the result for two random variables

$$P(T_i = \min(T_1, \ldots, T_n)) = P(S < U) = \frac{\lambda_i}{\lambda_i + \mu} = \frac{\lambda_i}{\lambda_1 + \cdots + \lambda_n}$$

proves the desired result. □

Let I be the (random) index of the T_i that is smallest. In symbols,

$$P(I = i) = \frac{\lambda_i}{\lambda_1 + \cdots + \lambda_n}$$

You might think that the T_i's with larger rates might be more likely to win early. However,

$$I \text{ and } V = \min\{T_1, \ldots T_n\} \text{ are independent.} \tag{2.11}$$

Proof. Let $f_{i,V}(t)$ be the density function for V on the set $I = i$. In order for i to be first at time t, $T_i = t$ and the other $T_j > t$ so

$$\begin{aligned} f_{i,V}(t) &= \lambda_i e^{-\lambda_i t} \cdot \prod_{j \neq i} e^{-\lambda_j t} \\ &= \frac{\lambda_i}{\lambda_1 + \cdots + \lambda_n} \cdot (\lambda_1 + \cdots + \lambda_n) e^{-(\lambda_1 + \cdot + \lambda_n)t} \\ &= P(I = i) \cdot f_V(t) \end{aligned}$$

since V has an exponential$(\lambda_1 + \cdots + \lambda_n)$ distribution. □

Our final fact in this section concerns sums of exponentials.

Theorem 2.1. *Let $\tau_1, \tau_2, \ldots$ be independent exponential(λ). The sum $T_n = \tau_1 + \cdots + \tau_n$ has a gamma(n, λ) distribution. That is, the density function of T_n is given by*

$$f_{T_n}(t) = \lambda e^{-\lambda t} \cdot \frac{(\lambda t)^{n-1}}{(n-1)!} \quad \text{for } t \geq 0 \tag{2.12}$$

and 0 otherwise.

Proof. The proof is by induction on n. When $n = 1$, T_1 has an exponential(λ) distribution. Recalling that the 0th power of any positive number is 1, and by convention we set 0!=1, the formula reduces to

$$f_{T_1}(t) = \lambda e^{-\lambda t}$$

and we have shown that our formula is correct for $n = 1$.

To do the induction step, suppose that the formula is true for n. The sum $T_{n+1} = T_n + \tau_{n+1}$, so breaking things down according to the value of T_n, and using the independence of T_n and t_{n+1}, we have

$$f_{T_{n+1}}(t) = \int_0^t f_{T_n}(s) f_{t_{n+1}}(t-s)\, ds$$

Plugging the formula from (2.12) in for the first term and the exponential density in for the second and using the fact that $e^a e^b = e^{a+b}$ with $a = -\lambda s$ and $b = -\lambda(t-s)$ gives

$$\begin{aligned} \int_0^t \lambda e^{-\lambda s} \frac{(\lambda s)^{n-1}}{(n-1)!} \cdot \lambda e^{-\lambda(t-s)}\, ds &= e^{-\lambda t} \lambda^n \int_0^t \frac{s^{n-1}}{(n-1)!}\, ds \\ &= \lambda e^{-\lambda t} \frac{\lambda^n t^n}{n!} \end{aligned}$$

which completes the proof. □

2.2 Defining the Poisson Process

In this section we will give two definitions of the **Poisson process with rate** λ. The first, which will be our official definition, is nice because it allows us to construct the process easily.

Definition. Let $\tau_1, \tau_2, \ldots$ be independent exponential(λ) random variables. Let $T_n = \tau_1 + \cdots + \tau_n$ for $n \geq 1$, $T_0 = 0$, and define $N(s) = \max\{n : T_n \leq s\}$.

We think of the τ_n as times between arrivals of customers at a bank, so $T_n = \tau_1 + \cdots + \tau_n$ is the arrival time of the nth customer, and $N(s)$ is the number of arrivals by time s. To check the last interpretation, consider the following example:

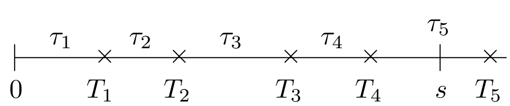

Figure 2.1: Poisson process definitions.

and note that $N(s) = 4$ when $T_4 \leq s < T_5$, that is, the 4th customer has arrived by time s but the 5th has not.

Recall that X has a **Poisson distribution** with mean λ, or $X = \text{Poisson}(\lambda)$, for short, if

$$P(X = n) = e^{-\lambda} \frac{\lambda^n}{n!} \quad \text{for } n = 0, 1, 2, \ldots$$

To explain why $N(s)$ is called the Poisson process rather than the exponential process, we will compute the distribution of $N(s)$.

Lemma 2.2. *$N(s)$ has a Poisson distribution with mean λs.*

Proof. Now $N(s) = n$ if and only if $T_n \le s < T_{n+1}$; i.e., the nth customer arrives before time s but the $(n+1)$th after s. Breaking things down according to the value of $T_n = t$ and noting that for $T_{n+1} > s$, we must have $\tau_{n+1} > s - t$, and τ_{n+1} is independent of T_n, it follows that

$$P(N(s) = n) = \int_0^s f_{T_n}(t) P(t_{n+1} > s - t)\, dt$$

Plugging in (2.12) now, the last expression is

$$\begin{aligned} &= \int_0^s \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!} \cdot e^{-\lambda(s-t)}\, dt \\ &= \frac{\lambda^n}{(n-1)!} e^{-\lambda s} \int_0^s t^{n-1}\, dt = e^{-\lambda s} \frac{(\lambda s)^n}{n!} \end{aligned}$$

which proves the desired result. □

Since this is our first mention of the Poisson distribution, we pause to derive some of its properties.

Theorem 2.3. *For any $k \ge 1$*

$$EX(X-1)\cdots(X-k+1) = \lambda^k \tag{2.13}$$

and hence $var(X) = \lambda$

Proof. $X(X-1)\cdots(X-k+1) = 0$ if $X \le k-1$ so

$$\begin{aligned} EX(X-1)\cdots(X-k+1) &= \sum_{j=k}^{\infty} e^{-\lambda} \frac{\lambda^j}{j!} j(j-1)\cdots(j-k+1) \\ &= \lambda^k \sum_{j=k}^{\infty} e^{-\lambda} \frac{\lambda^{j-k}}{(j-k)!} = \lambda^k \end{aligned}$$

since the sum gives the total mass of the Poisson distribution. Using $\text{var}(X) = E(X(X-1)) + EX - (EX)^2$ we conclude

$$\text{var}(X) = \lambda^2 + \lambda - (\lambda)^2 = \lambda$$ □

Theorem 2.4. *If X_i are independent Poissson(λ_i) then*

$$X_1 + \cdots + X_k = \textit{Poisson}(\lambda_1 + \cdots + \lambda_n).$$

Proof. It suffices to prove the result for $k = 2$, for then the general result follows by induction.

$$\begin{aligned} P(X_1 + X_2 = n) &= \sum_{m=0}^{n} P(X_1 = m) P(X_2 = n - m) \\ &= \sum_{m=0}^{n} e^{-\lambda_1} \frac{(\lambda_1)^m}{m!} \cdot e^{-\lambda_2} \frac{(\lambda_2)^{n-m}}{(n-m)!} \end{aligned}$$

Knowing the answer we want, we can rewrite the last expression as

$$e^{-(\lambda_1+\lambda_2)} \frac{(\lambda_1+\lambda_2)^n}{n!} \cdot \sum_{m=0}^{n} \binom{n}{m} \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^m \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-m}$$

The sum is 1, since it is the sum of all the probabilities for a binomial(n,p) distribution with $p = \lambda_1/(\lambda_1+\lambda_2)$. The term outside the sum is the desired Poisson probability, so have proved the desired result. □

The property of the Poisson process in Lemma 2.2 is the first part of our second definition. To start to develop the second part we prove a Markov property:

Lemma 2.5. *$N(t+s)-N(s)$, $t \geq 0$ is a rate λ Poisson process and independent of $N(r)$, $0 \leq r \leq s$.*

Why is this true? Suppose for concreteness (and so that we can use Figure 2.2 at the beginning of this section again) that by time s there have been four arrivals T_1, T_2, T_3, T_4 that occurred at times t_1, t_2, t_3, t_4. We know that the waiting time for the fifth arrival must have $\tau_5 > s - t_4$, but by the lack of memory property of the exponential distribution (2.6)

$$P(\tau_5 > s - t_4 + t | \tau_5 > s - t_4) = P(\tau_5 > t) = e^{-\lambda t}$$

This shows that the distribution of the first arrival after s is exponential(λ) and independent of T_1, T_2, T_3, T_4. It is clear that $\tau_6, \tau_7, \ldots$ are independent of T_1, T_2, T_3, T_4, and τ_5. This shows that the interarrival times after s are independent exponential(λ), and hence that $N(t+s) - N(s)$, $t \geq 0$ is a Poisson process. □

From Lemma 2.5 we get easily the following:

Lemma 2.6. *$N(t)$ has* **independent increments***: if $t_0 < t_1 < \ldots < t_n$, then*

$$N(t_1) - N(t_0), N(t_2) - N(t_1), \ldots N(t_n) - N(t_{n-1}) \quad \textit{are independent}$$

Why is this true? Lemma 2.5 implies that $N(t_n) - N(t_{n-1})$ is independent of $N(r)$, $r \leq t_{n-1}$ and hence of $N(t_{n-1}) - N(t_{n-2}), \ldots N(t_1) - N(t_0)$. The desired result now follows by induction. □

We are now ready for our second definition. It is in terms of the process $\{N(s) : s \geq 0\}$ that counts the number of arrivals in $[0, s]$.

Theorem 2.7. *If $\{N(s), s \geq 0\}$ is a Poisson process, then*
(i) $N(0) = 0$,
(ii) $N(t+s) - N(s) =$ Poisson(λt), and
(iii) $N(t)$ has independent increments.
Conversely, if (i), (ii), and (iii) hold, then $\{N(s), s \geq 0\}$ is a Poisson process.

Why is this true? Clearly, (i) holds. Lemmas 2.2 and 2.6 prove (ii) and (iii). To start to prove the converse, let T_n be the time of the nth arrival. The first arrival occurs after time t if and only if there were no arrivals in $[0, t]$. So using the formula for the Poisson distribution

$$P(\tau_1 > t) = P(N(t) = 0) = e^{-\lambda t}$$

This shows that $\tau_1 = T_1$ is exponential(λ). For $\tau_2 = T_2 - T_1$ we note that

$$\begin{aligned} P(\tau_2 > t|\tau_1 = s) &= P(\text{ no arrival in } (s, s+t]\,|\tau_1 = s) \\ &= P(N(t+s) - N(s) = 0|N(r) = 0 \text{ for } r < s, N(s) = 1) \\ &= P(N(t+s) - N(s) = 0) = e^{-\lambda t} \end{aligned}$$

by the independent increments property in (iii), so τ_2 is exponential(λ) and independent of τ_1. Repeating this argument we see that $\tau_1, \tau_2, \ldots$ are independent exponential(λ). □

Up to this point we have been concerned with the mechanics of defining the Poisson process, so the reader may be wondering:

Why is the Poisson process important for applications?

Our answer is based on the Poisson approximation to the binomial. Suppose that each of the n students on Duke campus flips coins with probability λ/n of heads to decide if they will go to the Great Hall (food court) between 12:17 and 12:18 . The probability that exactly k students will go during the one-minute time interval is given by the binomial($n, \lambda/n$) distribution

$$\frac{n(n-1)\cdots(n-k+1)}{k!}\left(\frac{\lambda}{n}\right)^k\left(1-\frac{\lambda}{n}\right)^{n-k} \tag{2.14}$$

Theorem 2.8. *If n is large the binomial($n, \lambda/n$) distribution is approximately Poisson(λ).*

Proof. Exchanging the numerators of the first two fractions and breaking the last term into two, (2.14) becomes

$$\frac{\lambda^k}{k!}\cdot\frac{n(n-1)\cdots(n-k+1)}{n^k}\cdot\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-k} \tag{2.15}$$

Considering the four terms separately, we have

(i) $\lambda^k/k!$ does not depend on n.

(ii) There are k terms on the top and k terms on the bottom, so we can write this fraction as

$$\frac{n}{n}\cdot\frac{n-1}{n}\cdots\frac{n-k+1}{n}$$

For any j we have $(n-j)/n \to 1$ as $n \to \infty$, so the second term converges to 1 as $n \to \infty$.

(iii) Skipping to the last term in (2.15), $\lambda/n \to 0$, so $1 - \lambda/n \to 1$. The power $-k$ is fixed so

$$\left(1-\frac{\lambda}{n}\right)^{-k} \to 1^{-k} = 1$$

(iv) We broke off the last piece to make it easier to invoke one of the famous facts of calculus:

$$(1-\lambda/n)^n \to e^{-\lambda} \quad \text{as } n \to \infty.$$

If you haven't seen this before, recall that

$$\log(1-x) = -x + x^2/2 + \ldots$$

so we have $n\log(1-\lambda/n) = -\lambda + \lambda^2/n + \ldots \to \lambda$ as $n\to\infty$.

Combining (i)–(iv), we see that (2.15) converges to

$$\frac{\lambda^k}{k!}\cdot 1\cdot e^{-\lambda}\cdot 1$$

which is the Poisson distribution with mean λ. □

By extending the last argument we can also see why the number of individuals that arrive in two disjoint time intervals should be independent. Using the multinomial instead of the binomial, we see that the probability j people will go between 12:17 and 12:18 and k people will go between 12:31 and 12:33 is

$$\frac{n!}{j!k!(n-j-k)!}\left(\frac{\lambda}{n}\right)^j\left(\frac{2\lambda}{n}\right)^k\left(1-\frac{3\lambda}{n}\right)^{n-(j+k)}$$

Rearranging gives

$$\frac{(\lambda)^j}{j!}\cdot\frac{(2\lambda)^k}{k!}\cdot\frac{n(n-1)\cdots(n-j-k+1)}{n^{j+k}}\cdot\left(1-\frac{3\lambda}{n}\right)^{n-(j+k)}$$

Reasoning as before shows that when n is large, this is approximately

$$\frac{(\lambda)^j}{j!}\cdot\frac{(2\lambda)^k}{k!}\cdot 1\cdot e^{-3\lambda}$$

Writing $e^{-\lambda} = e^{-\lambda/3}e^{-2\lambda/3}$ and rearranging we can write the last expression as

$$e^{-\lambda}\frac{\lambda^j}{j!}\cdot e^{-2\lambda}\frac{(2\lambda)^k}{k!}$$

This shows that the number of arrivals in the two time intervals we chose are independent Poissons with means λ and 2λ.

The last proof can be easily generalized to show that if we divide the hour between 12:00 and 1:00 into any number of intervals, then the arrivals are independent Poissons with the right means. However, the argument gets very messy to write down.

More realistic models.

Two of the weaknesses of the derivation above are:

(i) All students are assumed to have exactly the same probability of going to the Great Hall.

(ii) The probability of going in a given time interval is a constant multiple of the length of the interval, so the arrival rate of customers is constant during the hour. In reality there is a large influx of people between 11:30 and 11:45 soon after the end of 10:10-11:25 classes.

(i) is a very strong assumption but can be weakened by using a more general Poisson approximation result like the following:

Theorem 2.9. *Let $X_{n,m}$, $1\le m\le n$ be independent random variables with $P(X_m=1)=p_m$ and $P(X_m=0)=1-p_m$. Let*

$$S_n = X_1+\cdots+X_n, \quad \lambda_n = ES_n = p_1+\cdots+p_n,$$

and $Z_n = Poisson(\lambda_n)$. Then for any set A

$$|P(S_n \in A) - P(Z_n \in A)| \le \sum_{m=1}^{n} p_m^2$$

Why is this true? If X and Y are integer valued random variables then for any set A

$$|P(X \in A) - P(Y \in A)| \le \frac{1}{2} \sum_n |P(X = n) - P(Y = n)|$$

The right-hand side is called the **total variation distance** between the two distributions and is denoted $\|X - Y\|$. If $P(X = 1) = p$, $P(X = 0) = 1 - p$, and $Y = \text{Poisson}(p)$ then

$$\sum_n |P(X = n) - P(Y = n)| = |(1 - p) - e^{-p}| + |p - pe^{-p}| + 1 - (1 + p)e^{-p}$$

Since $1 \ge e^{-p} \ge 1 - p$ the right-hand side is

$$e^{-p} - 1 + p + p - pe^{-p} + 1 - e^{-p} - pe^{-p} = 2p(1 - e^{-p} \le 2p^2$$

Let $Y_m = \text{Poisson}(p_m)$ be independent. At this point we have shown $\|X_i - Y_i\| \le p_i^2$. With a little work one can show

$$\|(X_1 + \cdots + X_n) - (Y_1 + \cdots + Y_n)\|$$
$$\|(X_1, \cdots, X_n) - (Y_1, \cdots, Y_n)\| \le \sum_{m=1}^{n} \|X_m - Y_m\|$$

and the desired result follows. □

Theorem 2.9 is useful because it gives a bound on the difference between the distribution of S_n and the Poisson distribution with mean $\lambda_n = ES_n$. To bound the bound it is useful to note that

$$\sum_{m=1}^{n} p_m^2 \le \max_k p_k \left(\sum_{m=1}^{n} p_m \right)$$

so the approximation is good if $\max_k p_k$ is small. This is similar to the usual heuristic for the normal distribution: the sum is due to small contributions from a large number of variables. However, here small means that it is nonzero with small probability. When a contribution is made it is equal to 1.

The last results handles problem (i). To address the problem of varying arrival rates mentioned in (ii), we generalize the definition.

Nonhomogeneous Poisson processes. *We say that $\{N(s), s \ge 0\}$ is a Poisson process with rate $\lambda(r)$ if*

(i) $N(0) = 0$,
(ii) $N(t)$ has independent increments, and
(iii) $N(t) - N(s)$ is Poisson with mean $\int_s^t \lambda(r)\, dr$.

The first definition does not work well in this setting since the interarrival times $\tau_1, \tau_2, \ldots$ are no longer exponentially distributed or independent. To demonstrate the first claim, we note that

$$P(\tau_1 > t) = P(N(t) = 0) = e^{-\int_0^t \lambda(s)\, ds}$$

since $N(t)$ is Poisson with mean $\mu(t) = \int_0^t \lambda(s)\,ds$. Differentiating gives the density function

$$P(\tau_1 = t) = -\frac{d}{dt}P(t_1 > t) = \lambda(t)e^{-\int_0^t \lambda(s)\,ds} = \lambda(t)e^{-\mu(t)}$$

Generalizing the last computation shows that the joint distribution

$$f_{T_1,T_2}(u,v) = \lambda(u)e^{-\mu(u)} \cdot \lambda(v)e^{-(\mu(v)-\mu(u))}$$

Changing variables, $s = u$, $t = v - u$, the joint density

$$f_{\tau_1,\tau_2}(s,t) = \lambda(s)e^{-\mu(s)} \cdot \lambda(s+t)e^{-(\mu(s+t)-\mu(s))}$$

so τ_1 and τ_2 are not independent when $\lambda(s)$ is not constant.

2.3 Compound Poisson Processes

In this section we will embellish our Poisson process by associating an independent and identically distributed (i.i.d.) random variable Y_i with each arrival. By independent we mean that the Y_i are independent of each other and of the Poisson process of arrivals. To explain why we have chosen these assumptions, we begin with two examples for motivation.

Example 2.1. Consider the McDonald's restaurant on Route 13 in the southern part of Ithaca. By arguments in the last section, it is not unreasonable to assume that between 12:00 and 1:00 cars arrive according to a Poisson process with rate λ. Let Y_i be the number of people in the ith vehicle. There might be some correlation between the number of people in the car and the arrival time, e.g., more families come to eat there at night, but for a first approximation it seems reasonable to assume that the Y_i are i.i.d. and independent of the Poisson process of arrival times.

Example 2.2. Messages arrive at a central computer to be transmitted across the Internet. If we imagine a large number of users working at terminals connected to a central computer, then the arrival times of messages can be modeled by a Poisson process. If we let Y_i be the size of the ith message, then again it is reasonable to assume $Y_1, Y_2, \ldots$ are i.i.d. and independent of the Poisson process of arrival times.

Having introduced the Y_i's, it is natural to consider the sum of the Y_i's we have seen up to time t:

$$S(t) = Y_1 + \cdots + Y_{N(t)}$$

where we set $S(t) = 0$ if $N(t) = 0$. In Example 2.1, $S(t)$ gives the number of customers that have arrived up to time t. In Example 2.2, $S(t)$ represents the total number of bytes in all of the messages up to time t. In each case it is interesting to know the mean and variance of $S(t)$.

Theorem 2.10. *Let $Y_1, Y_2, \ldots$ be independent and identically distributed, let N be an independent nonnegative integer valued random variable, and let $S = Y_1 + \cdots + Y_N$ with $S = 0$ when $N = 0$.*

(i) If $E|Y_i|$, $EN < \infty$, then $ES = EN \cdot EY_i$.

(ii) If EY_i^2, $EN^2 < \infty$, then $var(S) = EN\,var(Y_i) + var(N)(EY_i)^2$.

(iii) If N is Poisson(λ), then $var(S) = \lambda EY_i^2$.

Why is this reasonable? The first of these is natural since if $N = n$ is nonrandom $ES = nEY_i$. (i) then results by setting $n = EN$. The formula in (ii) is more complicated but it clearly has two of the necessary properties:

If $N = n$ is nonrandom, $\text{var}(S) = n \,\text{var}(Y_i)$.

If $Y_i = c$ is nonrandom $\text{var}(S) = c^2 \,\text{var}(N)$.

Combining these two observations, we see that $EN \,\text{var}(Y_i)$ is the contribution to the variance from the variability of the Y_i, while $\text{var}(N)(EY_i)^2$ is the contribution from the variability of N.

Proof. When $N = n$, $S = X_1 + \cdots + X_n$ has $ES = nEY_i$. Breaking things down according to the value of N,

$$
\begin{aligned}
ES &= \sum_{n=0}^{\infty} E(S|N = n) \cdot P(N = n) \\
&= \sum_{n=0}^{\infty} nEY_i \cdot P(N = n) = EN \cdot EY_i
\end{aligned}
$$

For the second formula we note that when $N = n$, $S = X_1 + \cdots + X_n$ has $\text{var}(S) = n \ \text{var}(Y_i)$ and hence,

$$E(S^2|N = n) = n \ \text{var}(Y_i) + (nEY_i)^2$$

Computing as before we get

$$
\begin{aligned}
ES^2 &= \sum_{n=0}^{\infty} E(S^2|N = n) \cdot P(N = n) \\
&= \sum_{n=0}^{\infty} \{n \cdot \ \text{var}(Y_i) + n^2 (EY_i)^2\} \cdot P(N = n) \\
&= (EN) \cdot \ \text{var}(Y_i) + EN^2 \cdot (EY_i)^2
\end{aligned}
$$

To compute the variance now, we observe that

$$
\begin{aligned}
\text{var}(S) &= ES^2 - (ES)^2 \\
&= (EN) \cdot \ \text{var}(Y_i) + EN^2 \cdot (EY_i)^2 - (EN \cdot EY_i)^2 \\
&= (EN) \cdot \ \text{var}(Y_i) + \ \text{var}(N) \cdot (EY_i)^2
\end{aligned}
$$

where in the last step we have used $\text{var}(N) = EN^2 - (EN)^2$ to combine the second and third terms.

For part (iii), we note that in the special case of the Poisson, we have $EN = \lambda$ and $\text{var}(N) = \lambda$, so the result follows from $\text{var}(Y_i) + (EY_i)^2 = EY_i^2$. □

For a concrete example of the use of Theorem 2.10 consider

Example 2.3. Suppose that the number of customers at a liquor store in a day has a Poisson distribution with mean 81 and that each customer spends an average of \$8 with a standard deviation of \$6. It follows from (i) in Theorem 2.10 that the mean revenue for the day is $81 \cdot \$8 = \$648$. Using (iii), we see that the variance of the total revenue is

$$81 \cdot \left\{(\$6)^2 + (\$8)^2\right\} = 8100$$

Taking square roots we see that the standard deviation of the revenue is \$90 compared with a mean of \$648.

2.4 Transformations

2.4.1 Thinning

In the previous section, we added up the Y_i's associated with the arrivals in our Poisson process to see how many customers, etc., we had accumulated by time t. In this section we will use the Y_i to split one Poisson process into several. Let $N_j(t)$ be the number of $i \le N(t)$ with $Y_i = j$. In Example 2.1, where Y_i is the number of people in the ith car, $N_j(t)$ will be the number of cars that have arrived by time t with exactly j people. The somewhat remarkable fact is:

Theorem 2.11. *$N_j(t)$ are independent Poisson processes with rate $\lambda P(Y_i = j)$.*

Why is this remarkable? There are two "surprises" here: the resulting processes are Poisson and they are independent. To drive the point home consider a Poisson process with rate 10 per hour, and then flip coins to determine whether the arriving customers are male or female. One might think that seeing 40 men arrive in one hour would be indicative of a large volume of business and hence a larger than normal number of women, but Theorem 2.11 tells us that the number of men and the number of women that arrive per hour are independent.

Proof. To begin we suppose that $P(Y_i = 1) = p$ and $P(Y_i = 2) = 1-p$, so there are only two Poisson processes to consider: $N_1(t)$ and $N_2(t)$. We will check the second definition given in Theorem 2.7. It should be clear that the independent increments property of the Poisson process implies that the pairs of increments

$$(N_1(t_i) - N_1(t_{i-1}), N_2(t_i) - N_2(t_{i-1})), \quad 1 \le i \le n$$

are independent of each other. Since $N_1(0) = N_2(0) = 0$ by definition, it only remains to check that the components $X_i = N_i(t+s) - N_i(s)$ are independent and have the right Poisson distributions. To do this, we note that if $X_1 = j$ and $X_2 = k$, then there must have been $j+k$ arrivals between s and $s+t$, j of which were assigned 1's and k of which were assigned 2's, so

$$\begin{aligned} P(X_1 = j, X_2 = k) &= e^{-\lambda t} \frac{(\lambda t)^{j+k}}{(j+k)!} \cdot \frac{(j+k)!}{j!k!} p^j (1-p)^k \\ &= e^{-\lambda p t} \frac{(\lambda p t)^j}{j!} e^{-\lambda(1-p)t} \frac{(\lambda(1-p)t)^k}{k!} \end{aligned} \tag{2.16}$$

so $X_1 = \text{Poisson}(\lambda p t)$ and $X_2 = \text{Poisson}(\lambda(1-p)t)$. For the general case, we use the multinomial to conclude that if $p_j = P(Y_i = j)$ for $1 \le j \le m$ then

$$\begin{aligned} &P(X_1 = k_1, \ldots X_m = k_m) \\ &\quad = e^{-\lambda t} \frac{(\lambda t)^{k_1 + \cdots k_m}}{(k_1 + \cdots k_m)!} \frac{(k_1 + \cdots k_m)!}{k_1! \cdots k_m!} p_1^{k_1} \cdots p_m^{k_m} \quad = \prod_{j=1}^m e^{-\lambda p_j t} \frac{(\lambda p_j)^{k_j}}{k_j!} \end{aligned}$$

which proves the desired result. □

The thinning results generalizes easily to the nonhomogeneous case:

Theorem 2.12. *Suppose that in a Poisson process with rate λ, we keep a point that lands at s with probability $p(s)$. Then the result is a nonhomogeneous Poisson process with rate $\lambda p(s)$.*

For an application of this consider

Example 2.4. M/G/∞ queue. In modeling telephone traffic, we can, as a first approximation, suppose that the number of phone lines is infinite, i.e., everyone who tries to make a call finds a free line. This certainly is not always true but analyzing a model in which we pretend this is true can help us to discover how many phone lines we need to be able to provide service 99.99% of the time.

The argument for arrivals at the Great Hall implies that the beginnings of calls follow a Poisson process. As for the calls themselves, while many people on the telephone show a lack of memory, there is no reason to suppose that the duration of a call has an exponential distribution. So we use a general distribution function G with $G(0) = 0$ and mean μ. Suppose that the system starts empty at time 0. The probability a call started at s has ended by time t is $G(t - s)$, so using Theorem 2.12 the number of calls still in progress at time t is Poisson with mean

$$\int_{s=0}^{t} \lambda(1 - G(t - s))\, ds = \lambda \int_{r=0}^{t} (1 - G(r))\, dr$$

Letting $t \to \infty$ and using (A.22) we see that in the long run the number of calls in the system will be Poisson with mean

$$\lambda \int_{r=0}^{\infty} (1 - G(r))\, dr = \lambda\mu$$

That is, the mean number in the system is the rate at which calls enter times their average duration. In the argument above we supposed that the system starts empty. Since the number of initial calls still in the system at time t decreases to 0 as $t \to \infty$, the limiting result is true for any initial number of calls X_0.

2.4.2 Superposition

Taking one Poisson process and splitting it into two or more by using an i.i.d. sequence Y_i is called **thinning**. Going in the other direction and adding up a lot of independent processes is called **superposition**. Since a Poisson process can be split into independent Poisson processes, it should not be too surprising that when the independent Poisson processes are put together, the sum is Poisson with a rate equal to the sum of the rates.

Theorem 2.13. *Suppose $N_1(t), \dots N_k(t)$ are independent Poisson processes with rates $\lambda_1, \dots, \lambda_k$, then $N_1(t) + \cdots + N_k(t)$ is a Poisson process with rate $\lambda_1 + \cdots + \lambda_k$.*

Proof. Again we consider only the case $k = 2$ and check the second definition given in Theorem 2.7. It is clear that the sum has independent increments and $N_1(0) + N_2(0) = 0$. The fact that the increments have the right Poisson distribution follows from Theorem 2.4. $\square$

We will see in the next chapter that the ideas of compounding and thinning are very useful in computer simulations of continuous time Markov chains. For the moment we will illustrate their use in computing the outcome of races between Poisson processes.

Example 2.5. A Poisson race. Given a Poisson process of red arrivals with rate λ and an independent Poisson process of green arrivals with rate μ, what is the probability that we will get 6 red arrivals before a total of 4 green ones?

Solution. The first step is to note that the event in question is equivalent to having at least 6 red arrivals in the first 9. If this happens, then we have at most 3 green arrivals before the 6th red one. On the other hand if there are 5 or fewer red arrivals in the first 9, then we have had at least 4 red arrivals and at most 5 green.

Viewing the red and green Poisson processes as being constructed by starting with one rate $\lambda + \mu$ Poisson process and flipping coins with probability $p = \lambda/(\lambda + \mu)$ to decide the color, we see that the probability of interest is

$$\sum_{k=6}^{9} \binom{9}{k} p^k (1-p)^{9-k}$$

If we suppose for simplicity that $\lambda = \mu$ so $p = 1/2$, this expression becomes

$$\frac{1}{512} \cdot \sum_{k=6}^{9} \binom{9}{k} = \frac{1 + 9 + (9 \cdot 8)/2 + (9 \cdot 8 \cdot 7)/3!}{512} = \frac{140}{512} = 0.273$$

2.4.3 Conditioning

Let $T_1, T_2, T_3, \dots$ be the arrival times of a Poisson process with rate λ, let $U_1, U_2, \dots U_n$ be independent and uniformly distributed on $[0, t]$, and let $V_1 < \dots V_n$ be the U_i rearranged into increasing order . This section is devoted to the proof of the following remarkable fact.

Theorem 2.14. *If we condition on $N(t) = n$, then the vector $(T_1, T_2, \dots T_n)$ has the same distribution as $(V_1, V_2, \dots V_n)$ and hence the set of arrival times $\{T_1, T_2, \dots, T_n\}$ has the same distribution as $\{U_1, U_2, \dots, U_n\}$.*

Why is this true? We begin by finding the joint density function of (T_1, T_2, T_3) given that there were 3 arrivals before time t. The probability is 0 unless $0 < v_1 < v_2 < v_3 < t$. To compute the answer in this case, we note that $P(N(t) = 4) = e^{-\lambda t}(\lambda t)^3/3!$, and in order to have $T_1 = t_1$, $T_2 = t_2$, $T_3 = t_3$, $N(t) = 4$ we must have $\tau_1 = t_1$, $\tau_2 = t_2 - t_1$, $\tau_3 = t_3 - t_2$, and $\tau > t - t_3$, so the desired conditional distribution is:

$$\begin{aligned} &= \frac{\lambda e^{-\lambda t_1} \cdot \lambda e^{-\lambda(t_2 - t_1)} \cdot \lambda e^{-\lambda(t_3 - t_2)} \cdot e^{-\lambda(t - t_3)}}{e^{-\lambda t}(\lambda t)^3/3!} \\ &= \frac{\lambda^3 e^{-\lambda t}}{e^{-\lambda t}(\lambda t)^3/3!} = \frac{3!}{t^3} \end{aligned}$$

Note that the answer does not depend on the values of v_1, v_2, v_3 (as long as $0 < v_1 < v_2 < v_3 < t$), so the resulting conditional distribution is uniform over

$$\{(v_1, v_2, v_3) : 0 < v_1 < v_2 < v_3 < t\}$$

This set has volume $t^3/3!$ since $\{(v_1, v_2, v_3) : 0 < v_1, v_2, v_3 < t\}$ has volume t^3 and $v_1 < v_2 < v_3$ is one of 3! possible orderings.

Generalizing from the concrete example it is easy to see that the joint density function of $(T_1, T_2, \dots T_n)$ given that there were n arrivals before time t is $n!/t^n$

for all times $0 < t_1 < \ldots < t_n < t$, which is the joint distribution of $(V_1, \ldots, V_n)$. The second fact follows easily from this, since there are $n!$ sets $\{T_1, T_2, \ldots T_n\}$ or $\{U_1, U_2, \ldots U_n\}$ for each ordered vector $(T_1, T_2, \ldots T_n)$ or $(V_1, V_2, \ldots, V_n)$. $\square$

Theorem 2.14 implies that if we condition on having n arrivals at time t, then the locations of the arrivals are the same as the location of n points thrown uniformly on $[0, t]$. From the last observation we immediately get:

Theorem 2.15. *If $s < t$ and $0 \le m \le n$, then*

$$P(N(s) = m | N(t) = n) = \binom{n}{m} \left(\frac{s}{t}\right)^m \left(1 - \frac{s}{t}\right)^{n-m}$$

That is, the conditional distribution of $N(s)$ given $N(t) = n$ is binomial($n, s/t$).

Proof. The number of arrivals by time s is the same as the number of $U_i < s$. The events $\{U_i < s\}$ these events are independent and have probability s/t, so the number of $U_i < s$ will be binomial$(n, s/t)$. $\square$

2.5 Chapter Summary

A random variable T is said to have **an exponential distribution with rate** λ, or T = exponential(λ), if $P(T \le t) = 1 - e^{-\lambda t}$ for all $t \ge 0$. The mean is $1/\lambda$, variance $1/\lambda^2$. The density function is $f_T(t) = \lambda e^{-\lambda t}$. The sum of n independent exponentials has the gamma(n, λ) density

$$\lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}$$

Lack of memory property. "if we've been waiting for t units of time then the probability we must wait s more units of time is the same as if we haven't waited at all."

$$P(T > t + s | T > t) = P(T > s)$$

Exponential races. Let $T_1, \ldots, T_n$ are independent, T_i = exponential(λ_i), and $S = \min(T_1, \ldots, T_n)$. Then S = exponential$(\lambda_1 + \cdots + \lambda_n)$

$$P(T_i = \min(T_1, \ldots, T_n)) = \frac{\lambda_i}{\lambda_1 + \cdots + \lambda_n}$$

$\max\{S, T\} = S + T - \min\{S, T\}$ so taking expected value if S = exponential(μ) and T = exponential(λ) then

$$\begin{aligned} E \max\{S, T\} &= \frac{1}{\mu} + \frac{1}{\lambda} - \frac{1}{\mu + \lambda} \\ &= \frac{1}{\mu + \lambda} + \frac{\lambda}{\lambda + \mu} \cdot \frac{1}{\mu} + \frac{\mu}{\lambda + \mu} \cdot \frac{1}{\lambda} \end{aligned}$$

Poisson(μ) distribution. $P(X = n) = e^{-\mu} \mu^n / n!$. The mean and variance of X are μ.

Poisson process. Let $t_1, t_2, \ldots$ be independent exponential(λ) random variables. Let $T_n = t_1 + \ldots + t_n$ be the time of the nth arrival. Let $N(t) = \max\{n : T_n \le t\}$ be the number of arrivals by time t, which is Poisson(λt). $N(t)$ has

independent increments: if $t_0 < t_1 < \ldots < t_n$, then $N(t_1) - N(t_0), N(t_2) - N(t_1), \ldots N(t_n) - N(t_{n-1})$ are independent.

Thinning. Suppose we embellish our Poisson process by associating to each arrival an independent and identically distributed (i.i.d.) positive integer random variable Y_i. If we let $p_k = P(Y_i = k)$ and let $N_k(t)$ be the number of $i \le N(t)$ with $Y_i = k$ then $N_1(t), N_2(t), \ldots$ are independent Poisson processes and $N_k(t)$ has rate λp_k.

Random sums. Let $Y_1, Y_2, \ldots$ be i.i.d., let N be an independent nonnegative integer valued random variable, and let $S = Y_1 + \cdots + Y_N$ with $S = 0$ when $N = 0$.

(i) If $E|Y_i|$, $EN < \infty$, then $ES = EN \cdot EY_i$.

(ii) If EY_i^2, $EN^2 < \infty$, then $\operatorname{var}(S) = EN \operatorname{var}(Y_i) + \operatorname{var}(N)(EY_i)^2$.

(iii) If N is Poisson(λ) $\operatorname{var}(S) = \lambda E(Y_i^2)$

Superposition. If $N_1(t)$ and $N_2(t)$ are independent Poison processes with rates λ_1 and λ_2 then $N_1(t) + N_2(t)$ is Poisson rate $\lambda_1 + \lambda_2$.

Conditioning. Let $T_1, T_2, T_3, \ldots$ be the arrival times of a Poisson process with rate λ, and let $U_1, U_2, \ldots U_n$ be independent and uniformly distributed on $[0, t]$. If we condition on $N(t) = n$, then the set $\{T_1, T_2, \ldots T_n\}$ has the same distribution as $\{U_1, U_2, \ldots, U_n\}$.

2.6 Exercises

Exponential distribution

2.1. Suppose that the time to repair a machine is exponentially distributed random variable with mean 2. (a) What is the probability the repair takes more than 2 hours. (b) What is the probability that the repair takes more than 5 hours given that it takes more than 3 hours.

2.2. The lifetime of a radio is exponentially distributed with mean 5 years. If Ted buys a 7 year-old radio, what is the probability it will be working 3 years later?

2.3. A doctor has appointments at 9 and 9:30. The amount of time each appointment lasts is exponential with mean 30. What is the expected amount of time after 9:30 until the second patient has completed his appointment?

2.4. Copy machine 1 is in use now. Machine 2 will be turned on at time t. Suppose that the machines fail at rate λ_i. What is the probability that machine 2 is the first to fail?

2.5. Three people are fishing and each catches fish at rate 2 per hour. How long do we have to wait until everyone has caught at least one fish?

2.6. Alice and Betty enter a beauty parlor simultaneously, Alice to get a manicure and Betty to get a haircut. Suppose the time for a manicure (haircut) is exponentially distributed with mean 20 (30) minutes. (a) What is the probability Alice gets done first? (b) What is the expected amount of time until Alice and Betty are both done?

2.7. Let S and T be exponentially distributed with rates λ and μ. Let $U = \min\{S, T\}$ and $V = \max\{S, T\}$. Find (a) EU. (b) $E(V - U)$, (c) EV. (d) Use the identity $V = S + T - U$ to get a different looking formula for EV and verify the two are equal.

2.8. Let S and T be exponentially distributed with rates λ and μ. Let $U = \min\{S, T\}$, $V = \max\{S, T\}$, and $W = V - U$. Find the variances of U, V, and W.

2.9. In a hardware store you must first go to server 1 to get your goods and then go to a server 2 to pay for them. Suppose that the times for the two activities are exponentially distributed with means 6 minutes and 3 minutes. (a) Compute the average amount of time it take Bob to get his goods and pay if when he comes in there is one customer named Al with server 1 and no one at server 2. (b) Find the answer when times for the two activities are exponentially distributed with rates λ and μ.

2.10. Consider a bank with two tellers. Three people, Alice, Betty, and Carol enter the bank at almost the same time and in that order. Alice and Betty go directly into service while Carol waits for the first available teller. Suppose that the service times for each customer are exponentially distributed with mean 4 minutes. (a) What is the expected total amount of time for Carol to complete her businesses? (b) What is the expected total time until the last of the three customers leaves? (c) What is the probability Carol is the last one to leave?

2.11. Consider the set-up of the previous problem but now suppose that the two tellers have exponential service times with rates $\lambda \leq \mu$. Again, answer questions (a), (b), and (c).

2.12. A flashlight needs two batteries to be operational. You start with four batteries numbered 1 to 4. Whenever a battery fails it is replaced by the lowest-numbered working battery. Suppose that battery life is exponential with mean 100 hours. Let T be the time at which there is one working battery left and N be the number of the one battery that is still good. (a) Find ET. (b) Find the distribution of N. (c) Solve (a) and (b) for a general number of batteries.

2.13. A machine has two critically important parts and is subject to three different types of shocks. Shocks of type i occur at times of a Poisson process with rate λ_i. Shocks of types 1 break part 1, those of type 2 break part 2, while those of type 3 break both parts. Let U and V be the failure times of the two parts. (a) Find $P(U > s, V > t)$. (b) Find the distribution of U and the distribution of V. (c) Are U and V independent?

2.14. A submarine has three navigational devices but can remain at sea if at least two are working. Suppose that the failure times are exponential with means 1 year, 1.5 years, and 3 years. What is the average length of time the boat can remain at sea.

2.15. Excited by the recent warm weather Jill and Kelly are doing spring cleaning at their apartment. Jill takes an exponentially distributed amount of time with mean 30 minutes to clean the kitchen. Kelly takes an exponentially distributed amount of time with mean 40 minutes to clean the bath room. The first one to complete their task will go outside and start raking leaves, a task that takes an exponentially distributed amount of time with a mean of one hour.

When the second person is done inside, they will help the other and raking will be done at rate 2. (Of course the other person may already be done raking in which case the chores are done.) What is the expected time until the chores are all done?

2.16. Ron, Sue, and Ted arrive at the beginning of a professor's office hours. The amount of time they will stay is exponentially distributed with means of 1, 1/2, and 1/3 hour. (a) What is the expected time until only one student remains? (b) For each student find the probability they are the last student left. (c) What is the expected time until all three students are gone?

2.17. Let T_i, $i = 1, 2, 3$ be independent exponentials with rate λ_i. (a) Show that for any numbers t_1, t_2, t_3

$$\begin{aligned}\max\{t_1, t_2, t_3\} = t_1 + t_2 + t_3 - \min\{t_1, t_2\} - \min\{t_1, t_3\} \\ - \min\{t_2, t_3\} + \min\{t_1, t_2, t_3\}\end{aligned}$$

(b) Use (a) to find $E \max\{T_1, T_2, T_3\}$. (c) Use the formula to give a simple solution of part (c) of Exercise 2.16.

Poisson approximation to binomial

2.18. Compare the Poisson approximation with the exact binomial probabilities of 1 success when $n = 20$, $p = 0.1$.

2.19. Compare the Poisson approximation with the exact binomial probabilities of no success when (a) $n = 10$, $p = 0.1$, (b) $n = 50$, $p = 0.02$.

2.20. The probability of a three of a kind in poker is approximately 1/50. Use the Poisson approximation to estimate the probability you will get at least one three of a kind if you play 20 hands of poker.

2.21. Suppose 1% of a certain brand of Christmas lights is defective. Use the Poisson approximation to compute the probability that in a box of 25 there will be at most one defective bulb.

Poisson processes: Basic properties

2.22. Suppose $N(t)$ is a Poisson process with rate 3. Let T_n denote the time of the nth arrival. Find (a) $E(T_{12})$, (b) $E(T_{12}|N(2) = 5)$, (c) $E(N(5)|N(2) = 5)$.

2.23. Customers arrive at a shipping office at times of a Poisson process with rate 3 per hour. (a) The office was supposed to open at 8AM but the clerk Oscar overslept and came in at 10AM. What is the probability that no customers came in the two-hour period? (b) What is the distribution of the amount of time Oscar has to wait until his first customer arrives?

2.24. Suppose that the number of calls per hour to an answering service follows a Poisson process with rate 4. (a) What is the probability that fewer (i.e., $<$) than 2 calls came in the first hour? (b) Suppose that 6 calls arrive in the first hour, what is the probability there will be < 2 in the second hour. (c) Suppose that the operator gets to take a break after she has answered 10 calls. How long are her average work periods?

2.25. Traffic on Rosedale Road in Princeton, NJ, follows a Poisson process with rate 6 cars per minute. A deer runs out of the woods and tries to cross the road. If there is a car passing in the next 5 seconds then there will be a collision. (a) Find the probability of a collision. (b) What is the chance of a collision if the deer only needs 2 seconds to cross the road.

2.26. Calls to the Dryden fire department arrive according to a Poisson process with rate 0.5 per hour. Suppose that the time required to respond to a call, return to the station, and get ready to respond to the next call is uniformly distributed between 1/2 and 1 hour. If a new call comes before the Dryden fire department is ready to respond, the Ithaca fire department is asked to respond. Suppose that the Dryden fire department is ready to respond now. Find the probability distribution for the number of calls they will handle before they have to ask for help from the Ithaca fire department.

2.27. A math professor waits at the bus stop at the Mittag-Leffler Institute in the suburbs of Stockholm, Sweden. Since he has forgotten to find out about the bus schedule, his waiting time until the next bus is uniform on (0,1). Cars drive by the bus stop at rate 6 per hour. Each will take him into town with probability 1/3. What is the probability he will end up riding the bus?

2.28. The number of hours between successive trains is T which is uniformly distributed between 1 and 2. Passengers arrive at the station according to a Poisson process with rate 24 per hour. Let X denote the number of people who get on a train. Find (a) EX, (b) $\text{var}\,(X)$.

2.29. Consider a Poisson process with rate λ and let L be the time of the last arrival in the interval $[0,t]$, with $L = 0$ if there was no arrival. (a) Compute $E(t - L)$ (b) What happens when we let $t \to \infty$ in the answer to (a)?

2.30. Customers arrive according to a Poisson process of rate λ per hour. Joe does not want to stay until the store closes at $T =$ 10PM, so he decides to close up when the first customer after time $T - s$ arrives. He wants to leave early but he does not want to lose any business so he is happy if he leaves before T and no one arrives after. (a) What is the probability he achieves his goal? (b) What is the optimal value of s and the corresponding success probability?

2.31. Customers arrive at a sporting goods store at rate 10 per hour. 60% of the customers are men and 40% are women. Women spend an amount of time shopping that is uniformly distributed on $[0, 30]$ minutes, while men spend an exponentially distributed amount of time with mean 30 minutes. Let M and N be the number of men and women in the store. What is the distribution of (M, N) in equilibrium.

2.32. Let T be exponentially distributed with rate λ. (a) Use the definition of conditional expectation to compute $E(T|T < c)$. (b) Determine $E(T|T < c)$ from the identity

$$ET = P(T < c)E(T|T < c) + P(T > c)E(T|T > c)$$

2.33. *When did the chicken cross the road?* Suppose that traffic on a road follows a Poisson process with rate λ cars per minute. A chicken needs a gap of length at least c minutes in the traffic to cross the road. To compute the time the chicken will have to wait to cross the road, let $t_1, t_2, t_3, \ldots$ be the interarrival times for the cars and let $J = \min\{j : t_j > c\}$. If $T_n = t_1 + \cdots + t_n$, then the chicken will start to cross the road at time T_{J-1} and complete his journey at time $T_{J-1} + c$. Use the previous exercise to show $E(T_{J-1} + c) = (e^{\lambda c} - 1)/\lambda$.

Random sums

2.34. Edwin catches trout at times of a Poisson process with rate 3 per hour. Suppose that the trout weigh an average of 4 pounds with a standard deviation of 2 pounds. Find the mean and standard deviation of the total weight of fish he catches in two hours.

2.35. An insurance company pays out claims at times of a Poisson process with rate 4 per week. Writing K as shorthand for "thousands of dollars," suppose that the mean payment is 10K and the standard deviation is 6K. Find the mean and standard deviation of the total payments for 4 weeks.

2.36. Customers arrive at an automated teller machine at the times of a Poisson process with rate of 10 per hour. Suppose that the amount of money withdrawn on each transaction has a mean of \$30 and a standard deviation of \$20. Find the mean and standard deviation of the total withdrawals in 8 hours.

2.37. As a community service members of the Mu Alpha Theta fraternity are going to pick up cans from along a roadway. A Poisson mean 60 members show up for work. 2/3 of the workers are enthusiastic and will pick up a mean of 10 cans with a standard deviation of 5. 1/3 of the workers are lazy and will only pick up an average of 3 cans with a standard deviation of 2. Find the mean and standard deviation of the the number of cans collected.

2.38. Let S_t be the price of stock at time t and suppose that at times of a Poisson process with rate λ the price is multiplied by a random variable $X_i > 0$ with mean μ and variance σ^2. That is,

$$S_t = S_0 \prod_{i=1}^{N(t)} X_i$$

where the product is 1 if $N(t) = 0$. Find $ES(t)$ and $\operatorname{var} S(t)$.

2.39. Messages arrive to be transmitted across the internet at times of a Poisson process with rate λ. Let Y_i be the size of the ith message, measured in bytes, and let $g(z) = Ez^{Y_i}$ be the generating function of Y_i. Let $N(t)$ be the number of arrivals at time t and $S = Y_1 + \cdot + Y_{N(t)}$ be the total size of the messages up to time t. (a) Find the generating function $f(z) = E(z^S)$. (b) Differentiate and set $z = 1$ to find ES. (c) Differentiate again and set $z = 1$ to find $E\{S(S-1)\}$. (d) Compute $\operatorname{var}(S)$.

2.40. Let $\{N(t), t \geq 0\}$ be a Poisson process with rate λ. Let $T \geq 0$ be an independent with mean μ and variance σ^2. Find $\operatorname{cov}(T, N_T)$.

2.41. Let $t_1, t_2, \ldots$ be independent exponential(λ) random variables and let N be an independent random variable with $P(N = n) = (1-p)^{n-1}$. What is the distribution of the random sum $T = t_1 + \cdots + t_N$?

Thinning and conditioning

2.42. Traffic on Snyder Hill Road in Ithaca, NY, follows a Poisson process with rate 2/3's of a vehicle per minute. 10% of the vehicles are trucks, the other 90% are cars. (a) What is the probability at least one truck passes in a hour? (b) Given that 10 trucks have passed by in an hour, what is the expected number of vehicles that have passed by. (c) Given that 50 vehicles have passed by in a hour, what is the probability there were exactly 5 trucks and 45 cars.

2.43. Rock concert tickets are sold at a ticket counter. Females and males arrive at times of independent Poisson processes with rates 30 and 20 customers per hour. (a) What is the probability the first three customers are female? (b) If exactly 2 customers arrived in the first five minutes, what is the probability both arrived in the first three minutes. (c) Suppose that customers regardless of sex buy 1 ticket with probability 1/2, two tickets with probability 2/5, and three tickets with probability 1/10. Let N_i be the number of customers that buy i tickets in the first hour. Find the joint distribution of (N_1, N_2, N_3).

2.44. Ellen catches fish at times of a Poisson process with rate 2 per hour. 40% of the fish are salmon, while 60% of the fish are trout. What is the probability she will catch exactly 1 salmon and 2 trout if she fishes for 2.5 hours?

2.45. Signals are transmitted according to a Poisson process with rate λ. Each signal is successfully transmitted with probability p and lost with probability $1-p$. The fates of different signals are independent. For $t \geq 0$ let $N_1(t)$ be the number of signals successfully transmitted and let $N_2(t)$ be the number that are lost up to time t. (a) Find the distribution of $(N_1(t), N_2(t))$. (b) What is the distribution of $L =$ the number of signals lost before the first one is successfully transmitted?

2.46. A policewoman on the evening shift writes a Poisson mean 6 number of tickets per hour. 2/3's of these are for speeding and cost \$100. 1/3's of these are for DWI and cost \$400. (a) Find the mean and standard deviation for the total revenue from the tickets she writes in an hour. (b) What is the probability that between 2AM and 3AM she writes 5 tickets for speeding and 1 for DWI. (c) Let A be the event that she writes no tickets between 1AM and 1:30, and N be the number of tickets she writes between 1AM and 2AM. Which is larger $P(A)$ or $P(A|N = 5)$? Don't just answer yes or no, compute both probabilities.

2.47. Trucks and cars on highway US 421 are Poisson processes with rate 40 and 100 per hour respectively. 1/8 of the trucks and 1/10 of the cars get off on exit 257 to go to the Bojangle's in Yadkinville. (a) Find the probability that exactly 6 trucks arrive at Bojangle's between noon and 1PM. (b) Given that there were 6 truck arrivals at Bojangle's between noon and 1PM, what is the probability that exactly two arrived between 12:20 and 12:40? (c) Suppose that all trucks have 1 passenger while 30% of the cars have 1 passenger, 50% have 2, and 20% have 4. Find the mean and standard deviation of the number of customers are that arrive at Bojangles' in one hour.

2.48. When a power surge occurs on an electrical line, it can damage a computer without a surge protector. There are three types of surges: "small" surges occur at rate 8 per day and damage a computer with probability 0.001; "medium" surges occur at rate 1 per day and will damage a computer with probability 0.01; "large" surges occur at rate 1 per month and damage a computer with probability 0.1. Assume that months are 30 days. (a) what is the expected number of power surges per month? (b) What is the expected number of computer damaging power surges per month? (c) What is the probability a computer will not be damaged in one month? (d) What is the probability that the first computer damaging surge is a small one?

2.49. Wayne Gretsky scored a Poisson mean 6 number of points per game. 60% of these were goals and 40% were assists (each is worth one point). Suppose he is paid a bonus of 3K for a goal and 1K for an assist. (a) Find the mean and standard deviation for the total revenue he earns per game. (b) What is the probability that he has 4 goals and 2 assists in one game? (c) Conditional on the fact that he had 6 points in a game, what is the probability he had 4 in the first half?

2.50. A copy editor reads a 200-page manuscript, finding 108 typos. Suppose that the author's typos follow a Poisson process with some unknown rate λ per page, while from long experience we know that the copyeditor finds 90% of the mistakes that are there. (a) Compute the expected number of typos found as a function of the arrival rate λ. (b) Use the answer to (a) to find an estimate of λ and of the number of undiscovered typos.

2.51. Two copy editors read a 300-page manuscript. The first found 100 typos, the second found 120, and their lists contain 80 errors in common. Suppose that the author's typos follow a Poisson process with some unknown rate λ per page, while the two copy editors catch errors with unknown probabilities of success p_1 and p_2. Let X_0 be the number of typos that neither found. Let X_1 and X_2 be the number of typos found only by 1 or only by 2, and let X_3 be the number of typos found by both. (a) Find the joint distribution of (X_0, X_1, X_2, X_3). (b) Use the answer to (a) to find an estimates of p_1, p_2 and then of the number of undiscovered typos.

2.52. A light bulb has a lifetime that is exponential with a mean of 200 days. When it burns out a janitor replaces it immediately. In addition there is a handyman who comes at times of a Poisson process at rate .01 and replaces the bulb as "preventive maintenance." (a) How often is the bulb replaced? (b) In the long run what fraction of the replacements are due to failure?

2.53. Starting at some fixed time, which we will call 0 for convenience, satellites are launched at times of a Poisson process with rate λ. After an independent amount of time having distribution function F and mean μ, the satellite stops working. Let $X(t)$ be the number of working satellites at time t. (a) Find the distribution of $X(t)$. (b) Let $t \to \infty$ in (a) to show that the limiting distribution is Poisson($\lambda\mu$).

2.54. Calls originate from Dryden according to a rate 12 Poisson process. 3/4 are local and 1/4 are long distance. Local calls last an average of 10 minutes, while long distance calls last an average of 5 minutes. Let M be the number of local calls and N the number of long distance calls in equilibrium. Find the distribution of (M, N). what is the number of people on the line.

2.55. Ignoring the fact that the bar exam is only given twice a year, let us suppose that new lawyers arrive in Los Angeles according to a Poisson process with mean 300 per year. Suppose that each lawyer independently practices for an amount of time T with a distribution function $F(t) = P(T \le t)$ that has $F(0) = 0$ and mean 25 years. Show that in the long run the number of lawyers in Los Angeles is Poisson with mean 7500.

2.56. Policy holders of an insurance company have accidents at times of a Poisson process with rate λ. The distribution of the time R until a claim is reported is random with $P(R \le r) = G(r)$ and $ER = \nu$. (a) Find the distribution of the number of unreported claims. (b) Suppose each claim has mean μ and variance σ^2. Find the mean and variance of S the total size of the unreported claims.

2.57. Suppose $N(t)$ is a Poisson process with rate 2. Compute the conditional probabilities (a) $P(N(3) = 4|N(1) = 1)$, (b) $P(N(1) = 1|N(3) = 4)$.

2.58. For a Poisson process $N(t)$ with arrival rate 2 compute: (a) $P(N(2) = 5)$, (b) $P(N(5) = 8|N(2) = 3$, (c) $P(N(2) = 3|N(5) = 8)$.

2.59. Customers arrive at a bank according to a Poisson process with rate 10 per hour. Given that two customers arrived in the first 5 minutes, what is the probability that (a) both arrived in the first 2 minutes. (b) at least one arrived in the first 2 minutes.

2.60. Suppose that the number of calls per hour to an answering service follows a Poisson process with rate 4. Suppose that 3/4's of the calls are made by men, 1/4 by women, and the sex of the caller is independent of the time of the call. (a) What is the probability that in one hour exactly 2 men and 3 women will call the answering service? (b) What is the probability 3 men will make phone calls before 3 women do?

2.61. Hockey teams 1 and 2 score goals at times of Poisson processes with rates 1 and 2. Suppose that $N_1(0) = 3$ and $N_2(0) = 1$. (a) What is the probability that $N_1(t)$ will reach 5 before $N_2(t)$ does? (b) Answer part (a) for Poisson processes with rates λ_1 and λ_2.

2.62. Consider two independent Poisson processes $N_1(t)$ and $N_2(t)$ with rates λ_1 and λ_2. What is the probability that the two-dimensional process $(N_1(t), N_2(t))$ ever visits the point (i, j)?