Michael Leibert
Math 611
Homework 7

1. A person can be in one of four states of mental illness (I: chronically insane and hospitalized, II: dead, III: sane and IV: mildly insane and non-hospitalized). States I and II are absorbing states. Of the people in state III, 1.9% will be in state II after one year, 98% will stay in state III and the rest will be in state IV. Of the people in state IV, after one year, 2% will be in state I, 3% in state II and 95% stay in state IV.

$$\begin{array}{c} \\ \text{I} \\ \text{II} \\ \text{III} \\ \text{IV} \end{array} \begin{array}{c} \begin{array}{cccc} \text{I} & \text{II} & \text{III} & \text{IV} \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0.019 & 0.98 & 0.001 \\ 0.02 & 0.03 & 0 & 0.95 \end{pmatrix} \end{array}$$

$$(I-R)^{-1} = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.98 & 0.001 \\ 0 & 0.95 \end{pmatrix} \right]^{-1} = \begin{pmatrix} 0.02 & -0.001 \\ 0 & 0.05 \end{pmatrix}^{-1} = \begin{array}{c} \\ \text{III} \\ \text{IV} \end{array} \begin{array}{c} \begin{array}{cc} \text{III} & \text{IV} \end{array} \\ \begin{pmatrix} 50 & 1 \\ 0 & 20 \end{pmatrix} \end{array}$$

$$(I-R)^{-1} \times S = \begin{pmatrix} 50 & 1 \\ 0 & 20 \end{pmatrix} \begin{pmatrix} 0 & 0.019 \\ 0.02 & 0.03 \end{pmatrix} = \begin{array}{c} \\ \text{III} \\ \text{IV} \end{array} \begin{array}{c} \begin{array}{cc} \text{I} & \text{II} \end{array} \\ \begin{pmatrix} 0.02 & 0.98 \\ 0.4 & 0.6 \end{pmatrix} \end{array}$$

a. Determine the probability that a person who is currently sane, will eventually be chronically insane.

$$P(\text{I}|\text{III}) = 2\%$$

b. For a person in state III, determine the expected number of years, for which the person will be in state III.

50 Years

2. At a 2-year college 60% of the freshmen become sophomores, 20% remain freshmen and 20% drop out of school. Additionally, 70% of the sophomores graduate and transfer to a 4-year college, 15% remain sophomores and 15% drop out.

a. Form the transition probability matrix.

$$\begin{array}{c} \\ \text{Fr} \\ \text{So} \\ \text{Gr} \\ \text{Do} \end{array} \begin{array}{c} \begin{array}{cccc} \text{Fr} & \text{So} & \text{Gr} & \text{Do} \end{array} \\ \begin{pmatrix} 0.2 & 0.6 & 0 & 0.2 \\ 0 & 0.15 & 0.7 & 0.15 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

b. Use the matrix decomposition and the fundamental matrix concepts to derive the probability that a sophomore will eventually graduate.

$$\begin{array}{c} \\ \text{Fr} \\ \text{So} \\ \text{Gr} \\ \text{Do} \end{array} \begin{array}{c} \begin{array}{cccc} \text{Fr} & \text{So} & \text{Gr} & \text{Do} \end{array} \\ \begin{pmatrix} 0.2 & 0.6 & 0 & 0.2 \\ 0 & 0.15 & 0.7 & 0.15 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \sim \begin{array}{c} \\ \text{Gr} \\ \text{Do} \\ \text{Fr} \\ \text{So} \end{array} \begin{array}{c} \begin{array}{cccc} \text{Fr} & \text{So} & \text{Gr} & \text{Do} \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0.2 & 0.6 & 0 & 0.2 \\ 0 & 0.15 & 0.7 & 0.15 \end{pmatrix} \end{array} \sim \begin{array}{c} \\ \text{Gr} \\ \text{Do} \\ \text{Fr} \\ \text{So} \end{array} \begin{array}{c} \begin{array}{cccc} \text{Gr} & \text{Do} & \text{Fr} & \text{So} \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0.2 & 0.2 & 0.6 \\ 0.7 & 0.15 & 0 & 0.15 \end{pmatrix} \end{array}$$

$$(I-R)^{-1} = \left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - \begin{pmatrix}0.2 & 0.6\\ 0 & 0.15\end{pmatrix}\right]^{-1} = \begin{pmatrix}0.8 & -0.6\\ 0 & 0.85\end{pmatrix}^{-1} = \begin{pmatrix}1.25 & 0.8823529\\ 0 & 1.1764706\end{pmatrix}$$

$$(I-R)^{-1} \times S = \begin{pmatrix}1.25 & 0.8823529\\ 0 & 1.1764706\end{pmatrix}\begin{pmatrix}0 & 0.2\\ 0.7 & 0.15\end{pmatrix} = \begin{matrix}\\ \text{Fr}\\ \text{So}\end{matrix}\begin{matrix}\text{Gr} & \text{Do}\\ \end{matrix}\begin{pmatrix}0.6176471 & 0.3823529\\ 0.8235294 & 0.1764706\end{pmatrix}$$

The probability that a sophomore will eventually graduate is 82.35%.