Michael Leibert
Math 611
Homework 3

1. A classic Monte Carlo problem is estimating the value of π by randomly generating points in a unit square. Suppose each side of the square has unit length, and thus the area of the square is one. Then the area of the circle is $\frac{\pi}{4}$. If we generate random uniform variables in the unit square, then the fraction of values that land in the circle will be approximately equal to the area of the circle. Use this procedure in R to estimate the value of π. Run your program 1000 times to obtain 1000 estimates of π . Evaluate the mean and variance of your estimates.

We can generate an estimate of π by first considering the geometric probability of the fraction of values that land in the circle, where the circle is inscribed inside the unit square.

$$P(\text{point in circle}) = \frac{\text{Area of circle}}{\text{Area of square}} = \frac{\pi r^2}{(2r)^2} = \frac{\pi \cancel{r^2}}{4\cancel{r^2}}$$

$$4 \cdot P(\text{point in circle}) = \pi$$

In our example the circle is inscribed in the unit square, so $r = 0.5$.

```
r<-.5;m=1000;n=100;PI<-rep(0,m)

for(j in 1:m){circle<-0
for (i in 1:n) {        xy<-c( runif(2,0,1) )
        if ((xy[1]-r)^2+(xy[2]-r)^2 < r^2) {circle<-circle + 1  } }
PI[j]<-4*(circle/n) }

hbarn<-sum(PI)/m;hbarn

## [1] 3.1402

1/m^2 * sum( (PI-hbarn)^2 )

## [1] 2.767356e-05
```

Mean and variance above (cannot type out, estimates vary every time document is recompiled).

2a. Use Monte Carlo integration to evaluate the integral:

$$\int_0^\infty \exp\left(-\frac{4x}{3}\right)\ x^3\ \mathrm{d}x$$

First, in order to sample from the $U(0,1)$ distribution, we change the limits of integration so the domain will be $(0,1)$.

$$
\begin{aligned}
&\int_0^\infty \exp\left(-\frac{4x}{3}\right)\ x^3\ \mathrm{d}x \\
&= \int_1^0 \exp\left(-\frac{4}{3}\frac{1-v}{v}\right)\left(\frac{1-v}{v}\right)^3 \cdot -(1+x)^2\ \mathrm{d}v \\
&= -\int_1^0 \exp\left(\frac{-4+4v}{3v}\right)\left(\frac{1-v}{v}\right)^3\left(1+\frac{1-v}{v}\right)^2\ \mathrm{d}v \\
&= \int_0^1 \exp\left(\frac{-4+4v}{3v}\right)\left(\frac{1-v}{v}\right)^3\left(\frac{1}{v}\right)^2\ \mathrm{d}v
\end{aligned}
$$

$$
\begin{aligned}
\frac{1-v}{v} &= x \\
\frac{1}{v} - 1 &= x \\
\frac{1}{v} &= (1+x) \\
v &= \frac{1}{1+x} \\
\mathrm{d}v &= -\frac{1}{(1+x)^2}\ \mathrm{d}x \\
(1+x)^2\ \mathrm{d}v &= dx
\end{aligned}
$$

Before we use Monte Carlo integration and also analytically evaluate this integral, we evaluate it in R.

```
integrand <- function(x) {exp((-4*x)/3)*x^3 }
integrate(integrand  ,0, Inf )

## 1.898438 with absolute error < 0.0000015

integrand <- function(y) { exp((-4+4*y)/(3*y)) * ((1-y)/y)^3 * (1/y)^2  }
integrate(integrand  ,0, 1 )

## 1.898437 with absolute error < 0.000034
```

It is clear that the integrals are equivalent:

$$
h(x) = \int_0^\infty \exp\left(-\frac{4x}{3}\right)\ x^3\ \mathrm{d}x = \int_0^1 \exp\left(\frac{-4+4v}{3v}\right)\left(\frac{1-v}{v}\right)^3\left(\frac{1}{v}\right)^2\ \mathrm{d}v \approx 1.89843.
$$

We now perform the Monte Carlo integration to get an estimate of the integral. Using the adjusted $h(x)$.

```
n=10000;h<-rep(0,n)
for ( i in 1:n) {  U<-runif(1); h[i]<- exp((-4+4*U)/(3*U)) * ((1-U)/U)^3 * (1/U)^2;   }
barhn<-sum(h)/n;barhn

## [1] 1.911447
```

With 10,000 iterations, the Monte Carlo estimate is very accurate.

2b. Use a known probability distribution to analytically evaluate this integral and compare the exact value to the MC estimate.

This is the kernel of a $Gamma(4, 4/3)$, we multiply it by its normalizing constant and the integral integrates to 1. We do the same to the Monte Carlo estimate and see it is very close to 1.

$$\left(\frac{4}{3}\right)^4 \frac{1}{\Gamma(4)} \int_0^\infty \exp\left(-\frac{4x}{3}\right) \; x^3 \; \mathrm{d}x = 1$$

```
barhn*(4/3)^4 * 1/gamma(4)

## [1] 1.006853
```

3. If $X \sim N(0, \sigma^2)$ show both analytically and by using MC integration that $E\left[\exp(-x^2)\right] = \frac{1}{\sqrt{2\sigma^2+1}}$. For the MC simulation, use the value $\sigma^2 = 4$

First we show $E\left[\exp\left(-x^2\right)\right] = \frac{1}{\sqrt{2\sigma^2+1}}$ analytically.

$$\begin{aligned}
E\left[\exp\left(-x^2\right)\right] &= \int_{-\infty}^{\infty} \frac{\exp\left(-x^2\right)}{\sqrt{2\pi\sigma^2}} \; \exp\left[\frac{-x^2}{2\sigma^2}\right] \; \mathrm{d}x \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \; \exp\left[-x^2 + \frac{-x^2}{2\sigma^2}\right] \; \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \exp\left[-x^2\left(1 + \frac{1}{2\sigma^2}\right)\right] \; \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \exp\left(-ax^2\right) \; \mathrm{d}x
\end{aligned}$$

$$\begin{aligned}
\text{Let a} &= 1 + \frac{1}{2\sigma^2} \\
&= \frac{1+2\sigma^2}{2\sigma^2}
\end{aligned}$$

$$\text{Let I} = \int_{-\infty}^{\infty} \exp\left(-ax^2\right) \; dx$$

$$\begin{aligned}
I^2 &= \int_{-\infty}^{\infty} \exp\left(-ax^2\right) \; dx \cdot \int_{-\infty}^{\infty} \exp\left(-ay^2\right) \; dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left(-ax^2 - ay^2\right) \; dx \; dy \\
&= \int_{-\infty}^{\infty}\int_{0}^{2\pi} r\exp\left(-ar^2\right) \; d\theta \; dr \\
&= \int_{-\infty}^{\infty} r\exp\left(-ar^2\right) \Big[\theta\Big]_0^{2\pi} dr \\
&= 2\pi \int_{-\infty}^{\infty} r\exp\left(-ar^2\right) dr \\
&= \frac{\cancel{2}\pi}{\cancel{2}a} \int_{-\infty}^{\infty} e^{-u} du \\
&= -\frac{\pi}{a}\Big[e^{-u}\Big]_0^{\infty} \\
&= -\frac{\pi}{a}\left[\lim_{b\to\infty} e^{-b} - e^0\right] \\
&= -\frac{\pi}{a}[0-1] \\
I^2 &= \frac{\pi}{a} \\
I &= \frac{\sqrt{\pi}}{\sqrt{a}}
\end{aligned}$$

$$\begin{aligned}
r^2 &= x^2 + y^2 \\
x &= \cos(\theta) \\
y &= \sin(\theta)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial(x,y)}{\partial(r,\theta)} &= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} \\
&= \begin{vmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{vmatrix} \\
&= r
\end{aligned}$$

$$\begin{aligned}
u &= ar^2 \\
du &= 2ar \; dr \\
\frac{du}{2a} &= r \; dr
\end{aligned}$$

$$\begin{aligned}
E\left[\exp\left(-x^2\right)\right] &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \exp\left(-ax^2\right) \; dx \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \frac{\sqrt{\pi}}{\sqrt{a}} \\
&= \frac{1}{\sqrt{2\sigma^2}\cancel{\sqrt{\pi}}} \frac{\cancel{\sqrt{\pi}}}{\sqrt{\frac{1+2\sigma^2}{2\sigma^2}}} \\
&= \frac{1}{\cancel{\sqrt{2\sigma^2}}} \frac{\cancel{\sqrt{2\sigma^2}}}{\sqrt{1+2\sigma^2}} \\
&= \frac{1}{\sqrt{2\sigma^2+1}}
\end{aligned}$$

```r
n=10000;h<-rep(0,n)
for ( i in 1:n){ x<-rnorm(1,mean=0,sd=2); h[i]<-exp(-x^2) }
sum(h)/n
```

```
## [1] 0.3306347
1/(sqrt(2*4+1))
## [1] 0.3333333
```

4. In the social mobility example discussed in class, show that all 3 states are recurrent.

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0.7 & 0.2 & 0.1 \\ 2 & 0.3 & 0.5 & 0.2 \\ 3 & 0.2 & 0.4 & 0.4 \end{array}$$

We note that no matter where X_n is, there is a probability of at least 0.2 of hitting 1 on the next step so

$$P_1(T_1 > n) \leq (0.8)^n \to 0 \text{ as } n \to \infty$$

we will return to 1 with probability 1.

We note that no matter where X_n is, there is a probability of at least 0.2 of hitting 2 on the next step so

$$P_2(T_2 > n) \leq (0.8)^n \to 0 \text{ as } n \to \infty$$

we will return to 2 with probability 1.

We note that no matter where X_n is, there is a probability of at least 0.1 of hitting 3 on the next step so

$$P_3(T_3 > n) \leq (0.9)^n \to 0 \text{ as } n \to \infty$$

we will return to 3 with probability 1.

For each step, the probability that it will take an infinite number of steps to return to any state (starting at that state), goes to zero. For each state here, we will return to that state with probability 1. Thus all three states are recurrent.

5. Use the accept-reject algorithm to generate 10000 observations from the $Beta(2,2)$ distribution.

a. Generate the sample and construct a histogram. Superimpose the density curve of the theoretical model.

b. Comment on the precision of your outcomes, by comparing the empirical with the theoretical quantiles for a sequence of 100 quantiles and a QQ-plot.

Interested in simulating $Beta(2,2)$.

$$f(x) = \frac{x^{2-1}(1-x)^{2-1}}{\mathrm{B}(2,2)} = \frac{1}{6}\,x(1-x), \qquad 0 \leq x \leq 1$$

Using a Uniform $U(0,1)$ distribution as a proposal density, and let $M = \max\big[h(x)\big]$, where,

$$h(x) = \frac{f(x)}{g(x)} = \frac{1}{6}\,x(1-x).$$

$$\frac{dh}{dx} = \frac{d}{dx}\frac{1}{6}x(1-x) \qquad \text{Set to 0} \qquad M = \frac{0.5(1-0.5)}{6} = \frac{1}{24}$$
$$= 1 - 2x \qquad 1 - 2x = 0$$
$$x = \frac{1}{2}$$

$$\text{If } U \leq \frac{f(x)}{M \cdot g(x)} \Longrightarrow U \leq \frac{\frac{1}{6}\,x(1-x)}{\frac{1}{24}\cdot 1} = 4\,x(1-x)$$

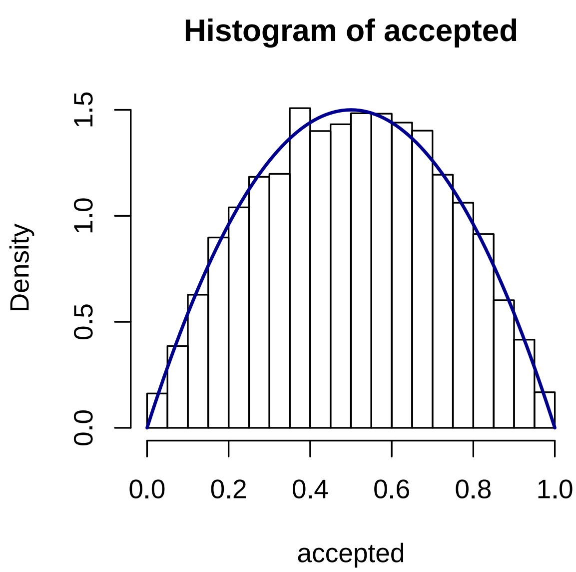
I have combined a & b together here. I have also split the data frame for the sequence of percentiles into two parts to minimize white space (also added another row to get an even amount).

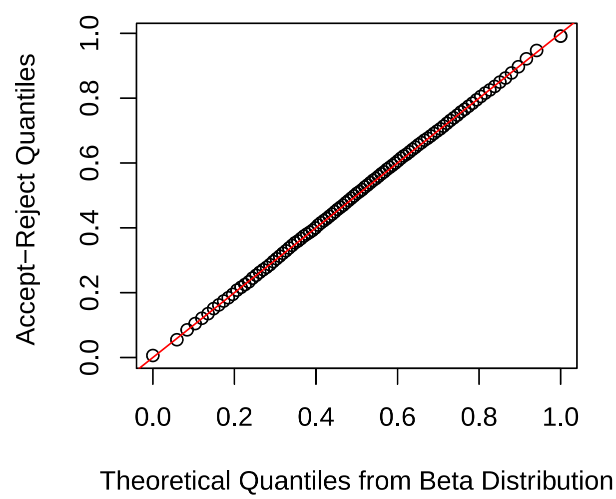
```
accepted<-rep(0,10000);c=1;
while (  0 %in% accepted ) {  y=runif(1) ;  if( runif(1) <= 4* y*(1-y)) {
accepted[c]<-y;c<-c+1} else {next} }

dat<-as.data.frame(as.table(quantile(accepted, probs = seq(0, 1, .01))))
colnames(dat)<-c("percentile","AR")
dat$beta<-(qbeta(seq(0, 1, .01), 2, 2, ))
dat<-rbind(dat, dat[nrow(dat),] )
cbind(dat[1:51,],dat[52:nrow(dat),])
```

```
##    percentile          AR       beta percentile         AR      beta
## 1          0% 0.006315331 0.00000000        51% 0.5116681 0.5066671
## 2          1% 0.054953952 0.05890314        52% 0.5179672 0.5133365
## 3          2% 0.085417109 0.08403770        53% 0.5258475 0.5200107
## 4          3% 0.104670161 0.10364484        54% 0.5321525 0.5266920
## 5          4% 0.121252051 0.12040345        55% 0.5393666 0.5333829
## 6          5% 0.135303352 0.13535036        56% 0.5464426 0.5400859
## 7          6% 0.150913453 0.14901695        57% 0.5521759 0.5468034
## 8          7% 0.162353613 0.16171873        58% 0.5592278 0.5535379
## 9          8% 0.174233306 0.17366118        59% 0.5666322 0.5602922
## 10         9% 0.184337245 0.18498701        60% 0.5727684 0.5670689
## 11        10% 0.194895908 0.19580011        61% 0.5803453 0.5738708
## 12        11% 0.207826420 0.20617891        62% 0.5865285 0.5807008
## 13        12% 0.216624020 0.21618437        63% 0.5930208 0.5875618
## 14        13% 0.225126593 0.22586501        64% 0.5999261 0.5944570
## 15        14% 0.233470107 0.23526018        65% 0.6071924 0.6013897
## 16        15% 0.244828276 0.24440235        66% 0.6149028 0.6083633
## 17        16% 0.253699470 0.25331871        67% 0.6218368 0.6153814
## 18        17% 0.262870187 0.26203228        68% 0.6273959 0.6224479
## 19        18% 0.270946252 0.27056280        69% 0.6344603 0.6295668
## 20        19% 0.278290381 0.27892731        70% 0.6418389 0.6367425
## 21        20% 0.286474084 0.28714073        71% 0.6486764 0.6439796
## 22        21% 0.296080788 0.29521613        72% 0.6563988 0.6512831
## 23        22% 0.304711752 0.30316511        73% 0.6629729 0.6586584
## 24        23% 0.312732858 0.31099803        74% 0.6703022 0.6661113
## 25        24% 0.321534860 0.31872414        75% 0.6763748 0.6736482
## 26        25% 0.329193692 0.32635182        76% 0.6834329 0.6812759
## 27        26% 0.337819614 0.33388866        77% 0.6910602 0.6890020
## 28        27% 0.345807966 0.34134156        78% 0.6985337 0.6968349
## 29        28% 0.353996214 0.34871686        79% 0.7055310 0.7047839
## 30        29% 0.359689295 0.35602038        80% 0.7136250 0.7128593
## 31        30% 0.367027735 0.36325749        81% 0.7225275 0.7210727
```

```
## 32         31% 0.374664990 0.37043319        82% 0.7312505 0.7294372
## 33         32% 0.380651709 0.37755211        83% 0.7393711 0.7379677
## 34         33% 0.386369532 0.38461859        84% 0.7477414 0.7466813
## 35         34% 0.392423268 0.39163671        85% 0.7575787 0.7555976
## 36         35% 0.399841734 0.39861030        86% 0.7656830 0.7647398
## 37         36% 0.408788739 0.40554299        87% 0.7751459 0.7741350
## 38         37% 0.415826006 0.41243821        88% 0.7845122 0.7838156
## 39         38% 0.422219301 0.41929924        89% 0.7949637 0.7938211
## 40         39% 0.428553985 0.42612919        90% 0.8053374 0.8041999
## 41         40% 0.435649080 0.43293108        91% 0.8157847 0.8150130
## 42         41% 0.442507617 0.43970777        92% 0.8254050 0.8263388
## 43         42% 0.449734001 0.44646206        93% 0.8362889 0.8382813
## 44         43% 0.457142939 0.45319663        94% 0.8495245 0.8509830
## 45         44% 0.463520649 0.45991412        95% 0.8622026 0.8646496
## 46         45% 0.469960632 0.46661706        96% 0.8777893 0.8795965
## 47         46% 0.477381885 0.47330798        97% 0.8969731 0.8963552
## 48         47% 0.484444411 0.47998932        98% 0.9214508 0.9159623
## 49         48% 0.491445984 0.48666350        99% 0.9471689 0.9410969
## 50         49% 0.498506047 0.49333294       100% 0.9915108 1.0000000
## 51         50% 0.505914218 0.50000000       100% 0.9915108 1.0000000
```

```r
par(mfrow=c(1,2))
hist(accepted,prob = TRUE)
curve(dbeta(x,2,2), col="darkblue", lwd=2, add=TRUE, yaxt="n")
plot(dat$beta,dat$AR,xlab =     'Theoretical Quantiles from Beta Distribution',
        ylab = "Accept-Reject Quantiles ");abline(0,1,col="red")
```






```r
cor(dat$beta,dat$AR) #£

## [1] 0.9999238
```

The correlation between the theoretical values and the accept-reject values is nearly 1. The precision is remarkable, I am shocked how well the accept-reject algorithm did at generating this beta random variable.

5. Construct a rejection algorithm to generate 1000 observations from a target standard normal distribution, using a candidate double-exponential distribution $g(x) = \dfrac{1}{2}\exp(-|x|)$

Interested in simulating standard normal,

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\,x^2\right) \qquad x \in (-\infty, \infty)\ .$$

Using a double-exponential distribution as a proposal density

$$g(x) = \frac{1}{2} \exp\big(-|x|\big) \qquad x \in (-\infty, \infty)\ .$$

Let $M = \max\big[h(x)\big]$, where

$$h(x) = \frac{\dfrac{1}{\sqrt{2\pi}} \exp\left(-\dfrac{1}{2}\,x^2\right)}{\dfrac{1}{2}\exp\big(-|x|\big)} \qquad x \in (-\infty, \infty)\ .$$

$$\begin{aligned} \frac{dh}{dx} &= \frac{2}{\sqrt{2\pi}}\ \exp\left(-\frac{1}{2}\,x^2 + |x|\right) \\ &= \frac{2}{\sqrt{2\pi}}\ \exp\left(-\frac{1}{2}\,x^2 + |x|\right)\left(-x + \frac{x}{|x|}\right) \end{aligned}$$

Set to 0

$$0 = \underbrace{\frac{2}{\sqrt{2\pi}}\ \exp\left(-\frac{1}{2}\,x^2 + |x|\right)}_{\text{always positive}}\left(-x + \frac{x}{|x|}\right)$$

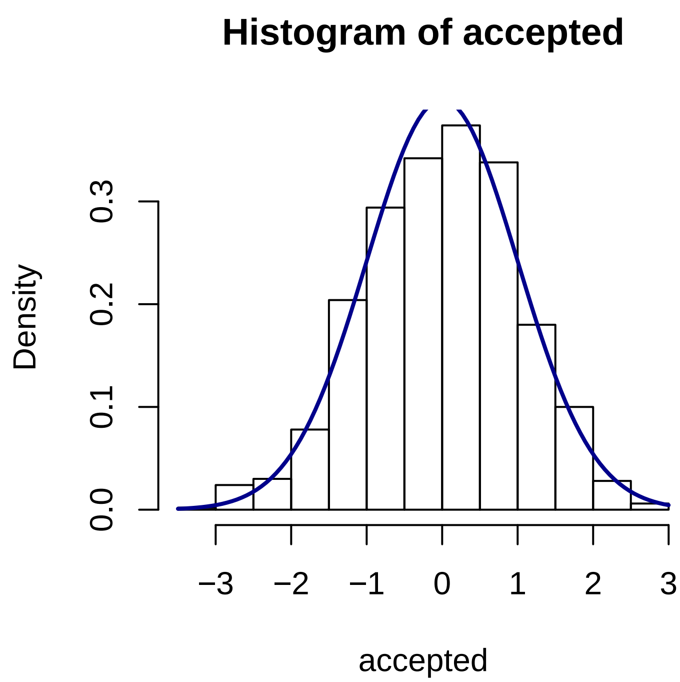
$$x = \pm 1$$

$$M = \frac{\dfrac{1}{\sqrt{2\pi}} \exp\left(-\dfrac{1}{2}\,(-1)^2\right)}{\dfrac{1}{2}\exp\big(-|(-1)|\big)} = \frac{\dfrac{1}{\sqrt{2\pi}} \exp\left(-\dfrac{1}{2}\,(1)^2\right)}{\dfrac{1}{2}\exp\big(-|(1)|\big)} \approx 1.31549\ .$$

$$\text{If } U \le \frac{f(x)}{M \cdot g(x)} \Longrightarrow U \le \frac{\dfrac{1}{\sqrt{2\pi}} \exp\left(-\dfrac{1}{2}\,x^2\right)}{\dfrac{1.31549}{2}\exp\big(-|x|\big)}$$

```
f1<-function(x) {   ( (1/(sqrt(2*pi))) * exp(-.5*x^2) )/(.5*exp(-abs(x))) }
accepted<-rep(0,1000);c=1

while (  0 %in% accepted ) {  y=rlaplace(1, m=0, s=1 );  if( runif(1) <=
(1/(sqrt(2*pi))* exp(-.5*y^2) )/((f1(1)/2) * exp(-abs(y)) ) ) { accepted[c]<-y;c<-c+1}
else {next} }
```

```
hist(accepted,prob=T)
curve(dnorm(x,0,1), col="darkblue", lwd=2, add=TRUE, yaxt="n")
```


Histogram of accepted
Density
0.0
0.1
0.2
0.3
−3
−2
−1
0
1
2
3
accepted