Homework solution 3 for APPM4/5560 Markov Process

9.1 A fair coin is tossed repeatedly with results $Y_0, Y_1, Y_2, \cdots$ that are 0 or 1 with probability 1/2 each. For $n \geq 1$ let $X_n = Y_n + Y_{n-1}$ be the number of 1's in the (n-1)th and nth tosses. Is X_n a Markov chain?

X_n is not a Markov chain. Counterexample:

Suppose X_n is a Markov chain. If $X_1 = 0$, $X_2 = 1$, then $Y_0 = 0$, $Y_1 = 0$, $Y_2 = 1$. $P(X_3 = 0|X_2 = 1, X_1 = 0) = 0$. If $X_1 = 2$, $X_2 = 1$, then $Y_0 = 1$, $Y_1 = 1$, $Y_2 = 0$. $P(X_3 = 0|X_2 = 1, X_1 = 2) = 1$. But $P(X_3 = 0|X_2 = 1, X_1 = 0) \neq P(X_3 = 0|X_2 = 1, X_1 = 2)$.

9.2 Five white balls and five black balls are distributed in two urns in such a way that each urn contains five balls. At each step we draw one ball from each urn and exchange them. Let X_n be the number of white balls in the left urn at time n. Compute the transition probability for X_n.

It is obvious that $p(i, j) = 0$ if $|i - j| > 1$.

If $j = i + 1$, the number of white balls increases, which means we picked up a black ball from the left and a white ball from the right. Then

$$p(i, j) = \frac{5-i}{5} \cdot \frac{5-i}{5} = \frac{(5-i)^2}{25}.$$

If $j = i$, the number of white balls is stable, which means we exchanged balls with the same color from both sides. Then

$$p(i, j) = \frac{i}{5} \cdot \frac{5-i}{5} + \frac{5-i}{5} \cdot \frac{i}{5} = \frac{10i - 2i^2}{25}.$$

If $j = i - 1$, the number of white balls decreases, which means we picked up a white ball from the left and a black ball from the right. Then

$$p(i, j) = \frac{i}{5} \cdot \frac{i}{5} = \frac{i^2}{25}.$$

Thus the transition probability is

$$p(i, j) = \begin{cases} \frac{(5-i)^2}{25} & j = i + 1 \\ \frac{10i-2i^2}{25} & j = i \\ \frac{i^2}{25} & j = i - 1 \\ 0 & otherwise. \end{cases}$$

9.3 Suppose that the probability it rains today is .3 if neither of the last two days was rainy, but .6 if at least one of the last two days was rainy. Let the weather on day n, W_n, be R for rain, or S for sun. W_n is not a Markov chain, but the weather for the last two days $X_n = (W_{n-1}, W_n)$ is a Markov chain with four states RR,RS,SR,SS. (a)Compute its transition probability. (b) Compute the two-step transition probability. (c) What is the probability it will rain on Wednesday given that it did not rain on Sunday or Monday.

(a) The transition probability is shown in the table:

	RR	RS	SR	SS
RR	.6	.4	0	0
RS	0	0	.6	.4
SR	.6	.4	0	0
SS	0	0	.3	.7

(b) The two-step transition probability is

$$p^2 = \begin{pmatrix} .6 & .4 & 0 & 0 \\ 0 & 0 & .6 & .4 \\ .6 & .4 & 0 & 0 \\ 0 & 0 & .3 & .7 \end{pmatrix}^2 = \begin{pmatrix} .36 & .24 & .24 & .16 \\ .36 & .24 & .12 & .28 \\ .36 & .24 & .24 & .16 \\ .18 & .12 & .21 & .49 \end{pmatrix}.$$

(c) By using the two-step transition probability in (b), the probability of raining on Wednesday is

$$p^2(SS, SR) + p^2(SS, RR) = .21 + .18 = .39.$$

9.4 Consider a gambler's ruin chain with $N = 4$. That is, if $1 \le i \le 3$, $p(i, i+1) = .4$, and $p(i, i-1) = .6$, but the endpoints are absorbing states: $p(0,0) = 1$ and $p(4,4) = 1$. Compute $p^3(1,4)$ and $p^3(1,0)$.

Method 1: Compute the power of transition matrix:

$$p^3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ .6 & 0 & .4 & 0 & 0 \\ 0 & .6 & 0 & .4 & 0 \\ 0 & 0 & .6 & 0 & .4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}^3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ .744 & 0 & .192 & 0 & .064 \\ .36 & .288 & 0 & .192 & .16 \\ .216 & 0 & .288 & 0 & .496 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$p^3(1,4) = .064, p^3(1,0) = .744.$$

Method 2: We only look at the possible transitions:

$$\begin{aligned} p^3(1,4) &= p(1,2)p(2,3)p(3,4) = (.4)(.4)(.4) \\ &= .064. \end{aligned}$$

$$\begin{aligned} p^3(1,0) &= p(1,0)p(0,0)p(0,0) + p(1,2)p(2,1)p(1,0) \\ &= (.6)(1)(1) + (.4)(.6)(.6) \\ &= .744. \end{aligned}$$

9.5 A taxicab driver moves between the airport A and two hotels B and C according to the following rules. If he is at the airport, he will be at one of the two hotels next with equal probability. If at a hotel then he returns to the airport with probability 3/4 and goes to the other hotel with probability 1/4. (a) Find the transition matrix for the chain. (b) Suppose the driver begins at the airport at time 0. Find the probability for each of his three possible locations at time 2 and the probability he is at hotel B at time 3.

(a) The transition matrix for the chain is

$$\begin{pmatrix} 0 & 1/2 & 1/2 \\ 3/4 & 0 & 1/4 \\ 3/4 & 1/4 & 0 \end{pmatrix}.$$

(b)

$$\begin{pmatrix} 0 & 1/2 & 1/2 \\ 3/4 & 0 & 1/4 \\ 3/4 & 1/4 & 0 \end{pmatrix}^2 = \begin{pmatrix} 3/4 & 1/8 & 1/8 \\ 3/16 & 7/16 & 3/8 \\ 3/16 & 3/8 & 7/16 \end{pmatrix}.$$

At time 2, the probability the driver is at A is 3/4, at B is 1/8, at C is 1/8.

At time 3, the probability the driver is at B is

$$\begin{aligned} p^2(A,A)p(A,B) + p^2(A,C) + p(C,B) &= (3/4)(1/2) + (1/8)(1/4) \\ &= 13/32. \end{aligned}$$

9.6 The Markov chain associated with a manufacturing process may be described as follows: A part to be manufactured will begin the process by entering step 1. After step 1, 20% of the parts must be reworked, i.e., returned to step 1, 10% of the parts are thrown away, and 70% proceed to step 2. After step 2, 5% of the parts must be returned to step 1, 10% to step 2, 5% are scrapped, and 80% emerge to be sold for a profit. (a) Formulate a four-state Markov chain with states 1, 2, 3, and 4 where 3 = a part that was scrapped and 4 = a part that was sold for a profit.

The transition matrix is

$$\begin{pmatrix} .2 & .7 & .1 & 0 \\ .05 & .1 & .05 & .8 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

9.7 Consider the following transition matrices. Which states are recurrent and which are transient? Give reasons for four answers.

(a)	**1**	**2**	**3**	**4**	**5**
1	.4	.3	.3	0	0
2	0	.5	0	.5	0
3	.5	0	.5	0	0
4	0	.5	0	.5	0
5	0	.3	0	.3	.4

Since {2,4} $\to$ {2,4} with equal probability at any time, {2,4} are recurrent. {1,3,5} have probability to go to 2 or 4, they are all transient.

(b)	**1**	**2**	**3**	**4**	**5**	**6**
1	.1	0	0	.4	.5	0
2	.1	.2	.2	0	.5	0
3	0	.1	.3	0	0	.6
4	.1	0	0	.9	0	0
5	0	0	0	.4	0	.6
6	0	0	0	0	.5	.5

By looking at the transitions: $1 \to \{1,4,5\}$, $2 \to \{1,2,3,5\}$, $3 \to \{2,3,6\}$, $4 \to \{1,4\}$, $5 \to \{4,6\}$, $6 \to \{5,6\}$, 1,4,5 is an closed irreducible set, so state 1, 4, 5 are recurrent. State 2 communicates with 1, which does not communicates with it, so state 2 is transient. State 3 communicates with 4, which does not communicates with it, so state 3 is transient.

9.8 Consider the following transition matrices. Identify the transient and recurrent states, and the closed irreducible sets in the Markov chains. Give reasons for your answers.

(a)	**1**	**2**	**3**	**4**	**5**
1	0	0	0	0	1
2	0	.2	0	.8	0
3	.1	.2	.3	.4	0
4	0	.6	0	.4	0
5	.3	0	0	0	.7

The transitions are: $1 \to 5$, $2 \to \{2,4\}$, $3 \to \{1,2,3,4\}$, $4 \to \{2,4\}$, $5 \to \{1,5\}$. {1,5}, {2,4} are closed irreducible sets. State 3 communicates with state 1, which does not communicate with it. So state 1,2,4,5 are recurrent and 3 is transient.

(b)	**1**	**2**	**3**	**4**	**5**	**6**
1	2/3	0	0	1/3	0	0
2	0	1/2	0	0	1/2	0
3	0	0	1/3	1/3	1/3	0
4	1/2	0	0	1/2	0	0
5	0	1/2	0	0	1/2	0
6	1/2	0	0	1/2	0	0

The transitions are: $1 \to \{1,4\}$, $2 \to \{2,5\}$, $3 \to \{3,4,5\}$, $4 \to \{1,4\}$, $5 \to \{2,5\}$, $6 \to \{1,4\}$. {1,4} and {2,5} are closed irreducible sets. State 3 communicates with state 4, which does not communicate with it, and state 6 communicates with state 1, which does not communicate with it. So state 1,2,4,5 are recurrent and 3,6 are transient.

9.9 Six children(Dick, Helen, Joni, Mark, Sam, and Tony) play catch. If Dick has the ball he is equally likely to throw it to Helen, Mark, Sam, and Tony. If Helen has the ball she is equally likely to throw it to Dick, Joni, Sam, and Tony. If Sam has the ball he is equally likely to throw it to Dick, Helen, Mark, and Tony. If either Joni or Tony gets the ball, they keep throwing it to each other. If Mark gets the ball he runs away with it. (a) Find the transition probability and classify the states of the chain.

(a) The transition probability is

	D	H	J	M	S	T
D	0	1/4	0	1/4	1/4	1/4
H	1/4	0	1/4	0	1/4	1/4
J	0	0	0	0	0	1
M	0	0	0	1	0	0
S	1/4	1/4	0	1/4	0	1/4
T	0	0	1	0	0	0

It is obvious that {J,T} and {M} are closed irreducible sets, and D, H, S all have a state to communicate with, while that state does not communicate with them. So J,M,T are recurrent and D, H, S are transient.