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Math 611
Homework 9

1. Let $N(s)$ be a Poisson process with parameter (λ). Use the properties of a Poisson process to derive:

a. $E\big[N^2(s)\big]$

We note $N(s) \sim Poisson(\lambda s)$ and let $X = N(s)$. If $X \sim Poisson(\lambda s)$, then $E[X] = \lambda s$ and $Var(X) = \lambda s$

$$\begin{aligned} Var(X) &= E\left[X^2\right] - \big(E[X]\big)^2 \\ \lambda s &= E\left[X^2\right] - \big(\lambda s\big)^2 \\ E\left[X^2\right] &= \big(\lambda s\big)^2 + \lambda s \\ E\big[N^2(s)\big] &= \big(\lambda s\big)^2 + \lambda s \end{aligned}$$

b. $E\Big[\big(N(t) - N(s)\big)^2\Big], \quad t > s$

We note $N(t) - N(s) = N(t-s)$ and $N(t-s) \sim Poisson\big(\lambda(t-s)\big)$ and let $Y = N(t-s)$.

If $Y \sim Poisson\big(\lambda(t-s)\big)$, then $E[Y] = \lambda(t-s)$ and $Var(Y) = \lambda(t-s)$

$$\begin{aligned} Var(Y) &= E\left[Y^2\right] - \big(E[Y]\big)^2 \\ \lambda(t-s) &= E\left[Y^2\right] - \Big[\lambda(t-s)\Big]^2 \\ E\left[Y^2\right] &= \Big[\lambda(t-s)\Big]^2 + \lambda(t-s) \\ E\Big[\big(N(t) - N(s)\big)^2\Big] &= \Big[\lambda(t-s)\Big]^2 + \lambda(t-s) \end{aligned}$$

2. Patients arrive at a doctor's office according to a Poisson process with rate $\lambda = \dfrac{1}{5}$ per minute. The doctor will not see a patient until at least 3 patients are in the waiting room.

a. Find the expected waiting time until the first patient is admitted to see the doctor.

Let τ_1, τ_2, τ_3 be the times between arrivals of the first, second, and third patient respectively.

$\tau_1, \tau_2, \tau_3 \overset{iid}{\sim} Exp(\lambda)$. Let T_3 be the arrival time of the 3^{rd} patient. $T_3 = \displaystyle\sum_{i=1}^{3} \tau_i$.

The expected arrival time of the third patient, or equivalently the expected waiting time until the fist patient is admitted, is:

$$\begin{aligned} E[T_3] &= E\left[\sum_{i=1}^{3} \tau_i\right] \\ &= \sum_{i=1}^{3} E[\tau_i] \\ &= \frac{3}{\lambda} \end{aligned}$$

Because $\lambda = \dfrac{1}{5}$, the expected waiting time until the first patient is admitted to see the doctor is

$$\frac{3}{0.2} = 15 \text{ minutes.}$$

b. What is the probability that nobody is admitted to see the doctor during the first hour?

This would be the event that zero or one or two people show within the first 60 minutes. Let $N(t)$ be the number of arrivals by time t. We know $N(t) \sim Poisson\,(\lambda t)$, so $N(60) \sim Poisson\,(12)$.

$$\sum_{k=0}^{2} \frac{12^k \; e^{-12}}{k!} = 0.0005222581$$

3. In an attempt to draw samples from $Beta(2.7, 6.3)$ target density, use an independent Metropolis-Hastings algorithm and the proposal density $Beta(1, 1)$. Derive the acceptance rate of the algorithm.

```
N=10000
X<-rep(NA,N)
acc<-rep(0,N)

X[1]<-rbeta(1,1,1)

for (i in 2:N) {
        Y=rbeta(1,1,1)
        rho=(dbeta(Y,2.7,6.3)/dbeta(X[i-1],2.7,6.3) ) *
                (dbeta(X[i-1],1,1) /dbeta(Y,1,1))

        if (runif(1)<rho) {X[i]<-Y;acc[i]<-1} else {X[i]<-X[i-1]}
}

mean(acc)

## [1] 0.4536
```

4. For the same target density in question 3, use an independent Metropolis-Hastings algorithm and the proposal density $Beta(1, 2)$. Derive the acceptance rate and compare with that of the algorithm employed in question 3.

```
N=10000
X<-rep(NA,N)
acc<-rep(0,N)

X[1]<-rbeta(1,1,2)

for (i in 2:N) {
        Y=rbeta(1,1,2)
        rho=(dbeta(Y,2.7,6.3)/dbeta(X[i-1],2.7,6.3) ) *
                (dbeta(X[i-1],1,2) /dbeta(Y,1,2))

        if (runif(1)<rho) {X[i]<-Y;acc[i]<-1} else {X[i]<-X[i-1]}
```

```
}

mean(acc)

## [1] 0.6361
```

The acceptance rate for this algorithm with the $Beta(1, 2)$ proposal is about 20 percentage points higher than the algorithm with the $Beta(1, 1)$ proposal. 65% vs 45%.