Michael Leibert
Math 611
Homework 2

1. There are 4 black balls in urn A and 4 white balls in urn B. A ball each selected independently at random from each urn and placed in the other urn (balls are being swapped between urns). Consider the Markov Chain representing the number of black balls in urn A at time n (immediately after the nth trial). Compute the transition probability for X_n.

There are always $N = 4$ balls in each urn. Let x denote the number of black balls B in Urn A at any given time point. The number of white balls W in Urn B can also be denoted by x. Further, $4 - x$ will denote the number of white balls in Urn A and the number of black balls in Urn B.

On each step there four actions that can take place:

1. We can remove a black ball and receive a white ball.

$$\begin{aligned} P(i, i-1) &= P(X_{n+1} = i-1 | X_n = i) \\ &= P(\ B \mid \text{Urn A}\ ) \cap P(\ W \mid \text{Urn B}\ ) \\ &= \frac{x}{4} \cdot \frac{x}{4} \\ &= \frac{x^2}{16} \end{aligned}$$

2. We can remove a black ball and receive a black ball.

$$\begin{aligned} P(i, i) &= P(X_{n+1} = i | X_n = i) \\ &= P(\ B \mid \text{Urn A}\ ) \cap P(\ B \mid \text{Urn B}\ ) \\ &= \frac{x}{4} \cdot \frac{4-x}{4} \\ &= \frac{x(4-x)}{16} \end{aligned}$$

3. We can remove a white ball and receive a white ball.

$$\begin{aligned} P(i, i) &= P(X_{n+1} = i | X_n = i) \\ &= P(\ W \mid \text{Urn A}\ ) \cap P(\ W \mid \text{Urn B}\ ) \\ &= \frac{4-x}{4} \cdot \frac{x}{4} \\ &= \frac{x(4-x)}{16} \end{aligned}$$

Note that steps *2.* and *3.* are mathematically the same, thus $P(i, i) = \dfrac{2x(4-x)}{16}$

4. We can remove a white ball and receive a black ball.

$$\begin{aligned} P(i, i+1) &= P(X_{n+1} = i+1 | X_n = i) \\ &= P(\ W \mid \text{Urn A}\ ) \cap P(\ B \mid \text{Urn B}\ ) \\ &= \frac{4-x}{4} \cdot \frac{4-x}{4} \\ &= \frac{(4-x)^2}{16} \end{aligned}$$

2. Consider a gambler's ruin chain with $N = 4$. that is $0 \leq i \leq 3$, $p(i, i+1) = 0.4$, $p(i, i-1) = 0.6$ and the two endpoints (0 and 3) are absorbing states. Compute $p^2(1,3)$, $p^2(0,2)$, and $p^3(1,0)$. Show all your steps.

$$\begin{aligned} p^2(1,3) &= p(1,2)p(2,3) \\ &= (0.4)(0.4) \\ &= 0.16 \end{aligned}$$

$$p^2(0,2) = 0$$

$$\begin{aligned} p^3(1,0) &= p(1,2)p(2,1)p(1,0) + p(1,0)p(0,0)p(0,0) \\ &= (0.4)(0.6)(0.6) + (0.6)(1)(1) \\ &= 0.744 \end{aligned}$$

3. A trait of animals is governed by a pair of genes, each of which may be of type G or g. An individual may then have a GG combination, or Gg (which is the same as gG), or gg. In the mating of a GG and a Gg animal, The offspring gets a G from the GG and it is equally likely that the offspring gets a G or a g from the Gg . In the mating of a gg and a Gg animal, the offspring gets a g from the gg it is equally likely that the offspring gets a G or a g from the Gg. In the mating of two Gg's the offspring has an equal chance of getting G or g from each parent. Form the transition matrix of the Markov Chain representing the genetic type of the offspring for the next negation.

$$\begin{array}{c} \\ GG \\ Gg \\ gg \end{array} \begin{array}{c} \begin{array}{ccc} GG & Gg & gg \end{array} \\ \begin{pmatrix} .5 & .5 & 0 \\ .25 & .5 & .25 \\ 0 & .5 & .5 \end{pmatrix} \end{array}$$

$$\begin{aligned} P(G|GG) \cap P(G|Gg) &= P(G|GG) \cdot P(G|Gg) \\ &= (1)(0.5) \\ &= 0.5 \end{aligned}$$

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$$\begin{aligned} P(G|Gg) \cap P(G|Gg) &= P(G|Gg) \cdot P(G|Gg) \\ &= (0.5)(0.5) \\ &= 0.25 \end{aligned}$$

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$$\begin{aligned} P(G|Gg) \cap P(g|Gg) \cup P(g|Gg) \cap P(G|Gg) &= P(G|Gg) \cdot P(g|Gg) + P(g|Gg) \cdot P(G|Gg) \\ &= (0.25)(0.25) + (0.25)(0.25) \\ &= 0.5 \end{aligned}$$