Assume $y_i \sim L(\mu, \sigma)$. We can represent this model as a mixture of a normal likelihood and inverse-gamma priors. Thus if we let $y_i \sim N\left(\mu, \frac{4\sigma^2}{\alpha_i}\right)$ where $\sigma^2 \sim IG(a, b)$ and $\alpha_i \overset{iid}{\sim} IG(1, 1/2)$ for fixed hyper-parameters a and b. Using this specification and a flat prior on μ, $\pi(\mu) \propto 1$, state the full posterior and determine the conditional posterior distributions for all model parameters, μ, σ^2, and $\alpha_1, \ldots, \alpha_n$. Write out the steps of a Gibbs Sampler you could use to draw posterior samples. (Hint: the conditional posterior for α_i, with some manipulation, should be recognizable as an inverse-Gaussian.)

$$
\begin{aligned}
\mathcal{L}(y_i|\mu, \sigma^2, \alpha_i) &\propto \prod_{i=1}^{n} \left(\frac{4\sigma^2}{\alpha_i}\right)^{-\frac{1}{2}} \exp\left[-\frac{1}{2\,\frac{4\sigma^2}{\alpha_i}}\,(y_i - \mu)^2\right] \\
&\propto \left(\sigma^2\right)^{-\frac{n}{2}} \left[\prod_{i=1}^{n} \sqrt{\alpha_i}\right] \exp\left[-\frac{1}{8\sigma^2}\sum_{i=1}^{n} \alpha_i (y_i - \mu)^2\right]
\end{aligned}
$$

$$
\begin{aligned}
P(\mu, \sigma^2, \alpha_i|y_i) &\propto \left(\sigma^2\right)^{-\frac{n}{2}} \left[\prod_{i=1}^{n} \sqrt{\alpha_i}\right] \exp\left[-\frac{1}{8\sigma^2}\sum_{i=1}^{n} \alpha_i (y_i - \mu)^2\right] \left(\sigma^2\right)^{-(a+1)} \exp\left(-\frac{b}{\sigma^2}\right) \left[\prod_{i=1}^{n} \alpha_i^{-(1+1)} \exp\left(-\frac{1}{2\alpha_i}\right)\right] \\
&\propto \left(\sigma^2\right)^{-\left(\frac{n}{2}+a+1\right)} \left[\prod_{i=1}^{n} \alpha_i^{-\frac{3}{2}}\right] \exp\left[-\frac{1}{8\sigma^2}\sum_{i=1}^{n} \alpha_i (y_i - \mu)^2 - \frac{b}{\sigma^2} - \sum_{i=1}^{n} \frac{1}{2\alpha_i}\right]
\end{aligned}
$$

$$
\begin{aligned}
p(\alpha_k|\mu, \sigma^2, \alpha_{i\neq k}, y) &\propto \alpha_k^{-\frac{3}{2}} \exp\left[-\frac{1}{8\sigma^2}\,\alpha_k (y_1 - \mu)^2 - \frac{1}{2\alpha_k}\right] \\
&= \alpha_k^{-\frac{3}{2}} \exp\left[-\frac{1}{2}\left(\frac{\alpha_k^2 (y_1 - \mu)^2 + 4\sigma^2}{4\alpha_k \sigma^2}\right)\right] \\
&= \alpha_k^{-\frac{3}{2}} \exp\left[-\frac{1}{2}\left(\frac{\alpha_k^2 (y_1 - \mu)^2 - 4\sigma(y_1 - \mu)\alpha_k + 4\sigma^2 + 4\sigma(y_1 - \mu)\alpha_k}{4\alpha_k \sigma^2}\right)\right] \\
&= \alpha_k^{-\frac{3}{2}} \exp\left[-\frac{1}{2}\left(\frac{(y_1 - \mu)^2 \left(\alpha_k - \frac{4\sigma(y_1-\mu)^2}{2(y_1-\mu)^2}\right)^2}{4\alpha_k \sigma^2} + \frac{4\sigma^2 - \frac{\left(4\sigma(y_1-\mu)\right)^2}{4(y_1-\mu)^2} + 4\sigma(y_1 - \mu)\alpha_k}{4\alpha_k \sigma^2}\right)\right] \\
&= \alpha_k^{-\frac{3}{2}} \exp\left[-\frac{1}{2}\left(\frac{(y_1 - \mu)^2 \left(\alpha_k - 2\sigma\right)^2}{4\alpha_k \sigma^2} + \frac{\cancel{4\sigma^2 - 4\sigma^2} + 4\sigma(y_1 - \mu)\cancel{\alpha_k}}{4\sigma^2 \cancel{\alpha_k}}\right)\right] \\
&\propto \alpha_k^{-\frac{3}{2}} \exp\left[-\frac{(y_1 - \mu)^2 \left(\alpha_k - 2\sigma\right)^2}{2\alpha_k (2\sigma)^2}\right]
\end{aligned}
$$

Suppose we wish to build a more general Bayesian model for a binomial sample. Let $X \sim Binom(N, p)$. Further, let $p \sim Beta(\alpha, \beta)$ where $\alpha \sim Gamma(a_1, b_1)$ and $\beta \sim Gamma(a_2, b_2)$. Find the likelihood, posterior, and the full conditionals. If a full conditional is recognizable, state its name. If they are not recognizable, suggest a potential proposal distribution.

$$\mathcal{L}\left(X|N, p,\right) \propto p^x (1-p)^{N-x}$$

$$p(p, \alpha, \beta, N, X) \propto p^x (1-p)^{N-x} \, \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \, p^{\alpha-1}(1-p)^{\beta-1}\alpha^{a_1-1}\exp(\alpha b_1)\beta^{a2-1}\exp(\beta b_2)$$

$$\begin{aligned} p(p|\alpha, \beta, N, X) &\propto p^x (1-p)^{N-x} p^{\alpha-1}(1-p)^{\beta-1} \\ &= p^{x+\alpha-1}(1-p)^{N+\beta-x-1} \\ p &\sim Beta(x+\alpha, N+\beta-x) \end{aligned}$$

$$p(\alpha|p, \beta, N, X) \propto \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} \, p^{\alpha-1}\alpha^{a_1-1}\exp(\alpha b_1)$$

PROPOSAL DENS ???

$$p(\beta|p, \alpha, N, X) \propto \frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)} \, (1-p)^{\beta-1}\beta^{a_2-1}\exp(\beta b_2)$$

PROPOSAL DENS ???

The Kumaraswamy distribution is a distribution that, like the Beta, can be used to model probabilities. It has as its pdf the following:

$$p(\theta) = ab\theta^{a-1}(1-\theta^a)^{b-1},\ \theta \in (0,1).$$

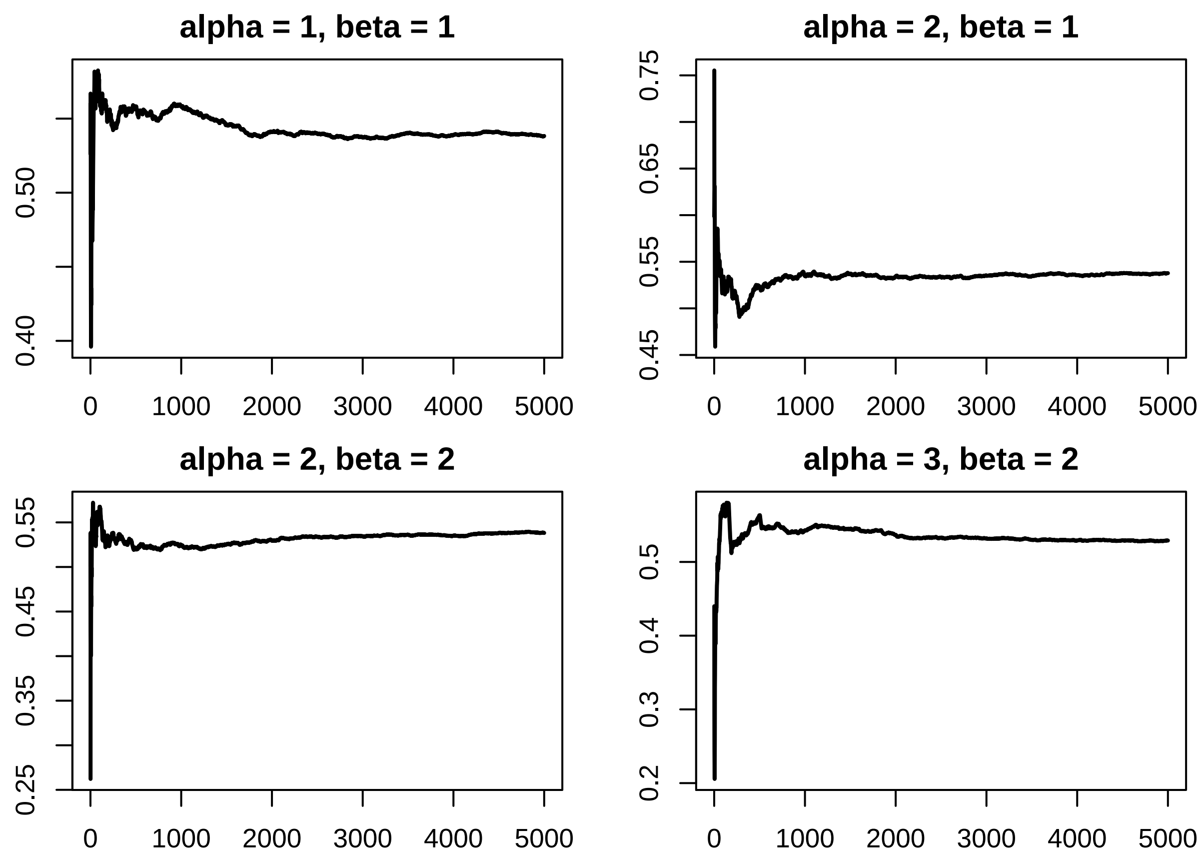
We wish to draw samples from $p(\theta)$ when $a = 2$ and $b = 2$ using a Metropolis-Hastings algorithm. Compare the following proposal densities to each other based on acceptance rate, ACF, and the resulting sampled density: $Beta(1,1)$, $Beta(2,1)$, $Beta(2,2)$, and $Beta(3,2)$. Select the proposal density you think is best out of these four and provide the criteria by which you made your selection. Set the seed to 1218 and take $B = 20000$ samples. Discard the burn-in before examining ACF and the sampled density. Without thinning, do you notice any differences between proposals?

\

```
## Loading required package: MCMCpack

## Loading required package: coda

## Loading required package: MASS

## ##
## ## Markov Chain Monte Carlo Package (MCMCpack)

## ## Copyright (C) 2003-2019 Andrew D. Martin, Kevin M. Quinn, and Jong Hee Park

## ##
## ## Support provided by the U.S. National Science Foundation

## ## (Grants SES-0350646 and SES-0350613)
## ##
```

```
Kumaraswamy <- function(x) { 2*2*x^(2-1)*(1-x^2)^(2-1) }

B <- 10000
xs <- ar <- rep(0, B)
ar <- vector("numeric", B)

Xs <- Ar <- list()

a <- c(1,2,2,3)
b <- c(1,1,2,2)

j = 2

for( j in 1:length(a)) {
    x  <- .5
    xs <- ar <- rep(0, B)
    set.seed(1218)
    for( i in 2:B){
        xstar   <- rbeta(1, a[j], b[j])
        rho <- (  Kumaraswamy(xstar) / Kumaraswamy(x)  ) *
            ( dbeta( x , a[j], b[j] )  /  dbeta( xstar, a[j], b[j] ) )
        rho <- min(1, rho )

        if ( runif(1) < rho ){ x <- xstar; ar[i] <- 1}
        xs[i] <- x }
    Xs[[j]] <- xs[-(1:(B/2))]
```

```r
    Ar[[j]] <- ar
    }

j = 4


par(mfrow=c(2,2) , mar=c(2.1,2.1,2.1,2.1) )
for(k in 1:4){ plot(cumsum(Xs[[k]] )/(1:(B/2)), type = 'l',
    ylab = 'Running Mean', xlab = 'B', lwd = 2,
    main = paste0("alpha = ",a[k], ", beta = ",b[k]) ) }
```




```r
lapply( Xs, geweke.diag )
```

```
## [[1]]
##
## Fraction in 1st window = 0.1
## Fraction in 2nd window = 0.5
##
##  var1
## 1.683
##
##
## [[2]]
##
## Fraction in 1st window = 0.1
## Fraction in 2nd window = 0.5
```

```
##
##    var1
## -1.053
##
##
## [[3]]
##
## Fraction in 1st window = 0.1
## Fraction in 2nd window = 0.5
##
##    var1
## -1.543
##
##
## [[4]]
##
## Fraction in 1st window = 0.1
## Fraction in 2nd window = 0.5
##
##   var1
## 1.502
```

DISCUSSION…