1. Suppose you wish to model a random sample of standard deviations, which you believe to be small, using the half-Normal distributions. The density is then

$$f(s_i) = \left(\frac{2}{\pi\sigma^2}\right)^{1/2} \exp\left(-\frac{1}{2\sigma^2}s_i^2\right)1(s_i > 0)$$

for $\sigma^2 > 0$. Use this to answer the following.

   (a) (5 points) Let s_i be an iid sample, $i = 1, \ldots, n$, from a half-Normal and define the transformation $\tau^2 = 1/\sigma^2$, the precision. State the likelihood in terms of the precision and determine Jeffrey's prior for τ^2.

(b) (5 points) Find the posterior distribution that results from using the prior you found in (a).

(c) (5 points) Explain why placing placing the non-informative prior on the precision, τ^2, is equivalent to placing a non-informative prior on the variance, σ^2.

2. The log-normal distribution is useful for modeling skewed data. Its density has the following form

$$f(x_i) = \frac{1}{x_i\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2\sigma^2}\left(\log\{x_i\} - \mu\right)^2\right]$$

for $\sigma^2 > 0, \mu \in \mathbb{R}$ and $x_i > 0$. The joint non-informative prior for μ and σ^2 is $\pi(\mu, \sigma^2) \propto (\sigma^2)^{-1}$. Use this to answer the following questions.

(a) (5 points) Assuming you have an iid sample of x_i's, find the conditional posterior distribution of $\mu|\sigma^2$. (Hint: it may be useful to define $\bar{x}_\ell$ as $\frac{1}{n}\sum_{i=1}^n \log\{x_i\}$, i.e. the log of the geometric mean.)

(b) (10 points) Now derive the marginal posterior distribution for σ^2.

3. The Gompertz distribution can be used to analyze survival times. It is parameterized by two parameters: a scale parameter b and a shape parameter η. The density for the Gompertz is given by

$$f(t_i) = b\eta e^{bt_i+\eta} \exp\left(-\eta e^{bt_i}\right)$$

for $b, \eta > 0$ and $t_i \in [0, \infty)$. Assume that b is fixed at b_0 but η is unknown. Further assume the prior on η is $\pi(\eta) \propto \eta^{-1}$. Use this to answer the following questions.

(a) (5 points) Suppose we have an iid sample of n survival times that we assume to be Gompertz. Determine the likelihood and state the posterior.

(b) (5 points) Now find the posterior mode.

(c) (5 points) Find the observed information.

(d) (5 points) Using your answers in (b) and (c), develop a strategy for sampling from the posterior distribution of $\eta|t, b_0$.

4. (10 points) Suppose we are running a Phase I Clinical trial to determine toxicity of a new drug. There are four possible responses to the drug: no toxic event, mildly toxic event, moderately toxic event, and severely toxic event. As a stopping rule, we've determined that after $x_0 = 3$ severely toxic events, we will stop the trial. This process can then be modeled using a negative multinomial which has as its mass function

$$P(X_0 = x_0, X_1 = x_1, X_2 = x_2, X_3 = x_3) = \Gamma\left(\sum_{i=0}^{3} x_i\right) \frac{\theta_0^{x_0}}{\Gamma(x_0)} \prod_{i=1}^{3} \frac{\theta_i^{x_i}}{x_i!}$$

where the "failure" count is denoted by X_0. In this case a "failure" is having a severely toxic event. We conduct the trail and reach our stopping rule, giving us a single vector of counts with the first element defined by the stopping rule and the remaining elements determined by the observed counts in the other categories. Determine the likelihood and find a conjugate prior for the vector of probabilities, $\boldsymbol{\theta} = [\ \theta_0 \quad \theta_1 \quad \theta_2 \quad \theta_3\ ]$. Then, using your conjugate prior, suggest a non-informative prior. (Hint: to establish conjugacy, you must state the resulting posterior.)