Comment

The result in (5.97a) can be derived directly by using (5.46), since $\hat{Y}_h = \mathbf{X}_h'\mathbf{b}$:

$$\sigma^2\{\hat{Y}_h\} = \mathbf{X}_h'\boldsymbol{\sigma}^2\{\mathbf{b}\}\mathbf{X}_h$$

Hence:

$$\sigma^2\{\hat{Y}_h\} = [1 \quad X_h] \begin{bmatrix} \sigma^2\{b_0\} & \sigma\{b_0, b_1\} \\ \sigma\{b_1, b_0\} & \sigma^2\{b_1\} \end{bmatrix} \begin{bmatrix} 1 \\ X_h \end{bmatrix}$$

or:

$$\sigma^2\{\hat{Y}_h\} = \sigma^2\{b_0\} + 2X_h\sigma\{b_0, b_1\} + X_h^2\sigma^2\{b_1\} \tag{5.99}$$

Using the results from (5.92a), we obtain:

$$\sigma^2\{\hat{Y}_h\} = \frac{\sigma^2}{n} + \frac{\sigma^2\bar{X}^2}{\sum(X_i - \bar{X})^2} + \frac{2X_h(-\bar{X})\sigma^2}{\sum(X_i - \bar{X})^2} + \frac{X_h^2\sigma^2}{\sum(X_i - \bar{X})^2}$$

which reduces to the familiar expression:

$$\sigma^2\{\hat{Y}_h\} = \sigma^2\left[\frac{1}{n} + \frac{(X_h - \bar{X})^2}{\sum(X_i - \bar{X})^2}\right] \tag{5.99a}$$

Thus, we see explicitly that the variance expression in (5.99a) contains contributions from $\sigma^2\{b_0\}$, $\sigma^2\{b_1\}$, and $\sigma\{b_0, b_1\}$, which it must according to (A.30b) since $\hat{Y}_h = b_0 + b_1X_h$ is a linear combination of b_0 and b_1. ■

Prediction of New Observation

The estimated variance $s^2\{\text{pred}\}$, given earlier in (2.38), in matrix notation is:

$$s^2\{\text{pred}\} = MSE(1 + \mathbf{X}_h'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}_h) \tag{5.100}$$

Cited Reference

5.1. Graybill, F. A. *Matrices with Applications in Statistics.* 2nd ed. Belmont, Calif.: Wadsworth, 2002.

Problems

5.1. For the matrices below, obtain (1) $\mathbf{A} + \mathbf{B}$, (2) $\mathbf{A} - \mathbf{B}$, (3) $\mathbf{AC}$, (4) $\mathbf{AB}'$, (5) $\mathbf{B}'\mathbf{A}$.

$$\mathbf{A} = \begin{bmatrix} 1 & 4 \\ 2 & 6 \\ 3 & 8 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 3 \\ 1 & 4 \\ 2 & 5 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 3 & 8 & 1 \\ 5 & 4 & 0 \end{bmatrix}$$

State the dimension of each resulting matrix.

5.2. For the matrices below, obtain (1) $\mathbf{A} + \mathbf{C}$, (2) $\mathbf{A} - \mathbf{C}$, (3) $\mathbf{B}'\mathbf{A}$, (4) $\mathbf{AC}'$, (5) $\mathbf{C}'\mathbf{A}$.

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 3 & 5 \\ 5 & 7 \\ 4 & 8 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 6 \\ 9 \\ 3 \\ 1 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 3 & 8 \\ 8 & 6 \\ 5 & 1 \\ 2 & 4 \end{bmatrix}$$

State the dimension of each resulting matrix.

5.3. Show how the following expressions are written in terms of matrices: (1) $Y_i - \hat{Y}_i = e_i$, (2) $\sum X_ie_i = 0$. Assume $i = 1, \ldots, 4$.

*5.4. **Flavor deterioration.** The results shown below were obtained in a small-scale experiment to study the relation between °F of storage temperature (X) and number of weeks before flavor deterioration of a food product begins to occur (Y).

i:	1	2	3	4	5
X_i:	8	4	0	−4	−8
Y_i:	7.8	9.0	10.2	11.0	11.7

Assume that first-order regression model (2.1) is applicable. Using matrix methods, find (1) $\mathbf{Y}'\mathbf{Y}$, (2) $\mathbf{X}'\mathbf{X}$, (3) $\mathbf{X}'\mathbf{Y}$.

5.5. **Consumer finance.** The data below show, for a consumer finance company operating in six cities, the number of competing loan companies operating in the city (X) and the number per thousand of the company's loans made in that city that are currently delinquent (Y):

i:	1	2	3	4	5	6
X_i:	4	1	2	3	3	4
Y_i:	16	5	10	15	13	22

Assume that first-order regression model (2.1) is applicable. Using matrix methods, find (1) $\mathbf{Y}'\mathbf{Y}$, (2) $\mathbf{X}'\mathbf{X}$, (3) $\mathbf{X}'\mathbf{Y}$.

*5.6. Refer to **Airfreight breakage** Problem 1.21. Using matrix methods, find (1) $\mathbf{Y}'\mathbf{Y}$, (2) $\mathbf{X}'\mathbf{X}$, (3) $\mathbf{X}'\mathbf{Y}$.

5.7. Refer to **Plastic hardness** Problem 1.22. Using matrix methods, find (1) $\mathbf{Y}'\mathbf{Y}$, (2) $\mathbf{X}'\mathbf{X}$, (3) $\mathbf{X}'\mathbf{Y}$.

5.8. Let $\mathbf{B}$ be defined as follows:

$$\mathbf{B} = \begin{bmatrix} 1 & 5 & 0 \\ 1 & 0 & 5 \\ 1 & 0 & 5 \end{bmatrix}$$

a. Are the column vectors of $\mathbf{B}$ linearly dependent?

b. What is the rank of $\mathbf{B}$?

c. What must be the determinant of $\mathbf{B}$?

5.9. Let $\mathbf{A}$ be defined as follows:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 8 \\ 0 & 3 & 1 \\ 0 & 5 & 5 \end{bmatrix}$$

a. Are the column vectors of $\mathbf{A}$ linearly dependent?

b. Restate definition (5.20) in terms of row vectors. Are the row vectors of $\mathbf{A}$ linearly dependent?

c. What is the rank of $\mathbf{A}$?

d. Calculate the determinant of $\mathbf{A}$.

5.10. Find the inverse of each of the following matrices:

$$\mathbf{A} = \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 4 & 3 & 2 \\ 6 & 5 & 10 \\ 10 & 1 & 6 \end{bmatrix}$$

Check in each case that the resulting matrix is indeed the inverse.

5.11. Find the inverse of the following matrix:

$$\mathbf{A} = \begin{bmatrix} 5 & 1 & 3 \\ 4 & 0 & 5 \\ 1 & 9 & 6 \end{bmatrix}$$

Check that the resulting matrix is indeed the inverse.

*5.12. Refer to **Flavor deterioration** Problem 5.4. Find $(\mathbf{X}'\mathbf{X})^{-1}$.

5.13. Refer to **Consumer finance** Problem 5.5. Find $(\mathbf{X}'\mathbf{X})^{-1}$.

*5.14. Consider the simultaneous equations:

$$4y_1 + 7y_2 = 25$$
$$2y_1 + 3y_2 = 12$$

a. Write these equations in matrix notation.
b. Using matrix methods, find the solutions for y_1 and y_2.

5.15. Consider the simultaneous equations:

$$5y_1 + 2y_2 = 8$$
$$23y_1 + 7y_2 = 28$$

a. Write these equations in matrix notation.
b. Using matrix methods, find the solutions for y_1 and y_2.

5.16. Consider the estimated linear regression function in the form of (1.15). Write expressions in this form for the fitted values $\hat{Y}_i$ in matrix terms for $i = 1, \ldots, 5$.

5.17. Consider the following functions of the random variables Y_1, Y_2, and Y_3:

$$W_1 = Y_1 + Y_2 + Y_3$$
$$W_2 = Y_1 - Y_2$$
$$W_3 = Y_1 - Y_2 - Y_3$$

a. State the above in matrix notation.
b. Find the expectation of the random vector $\mathbf{W}$.
c. Find the variance-covariance matrix of $\mathbf{W}$.

*5.18. Consider the following functions of the random variables Y_1, Y_2, Y_3, and Y_4:

$$W_1 = \frac{1}{4}(Y_1 + Y_2 + Y_3 + Y_4)$$
$$W_2 = \frac{1}{2}(Y_1 + Y_2) - \frac{1}{2}(Y_3 + Y_4)$$

a. State the above in matrix notation.
b. Find the expectation of the random vector $\mathbf{W}$.
c. Find the variance-covariance matrix of $\mathbf{W}$.

*5.19. Find the matrix $\mathbf{A}$ of the quadratic form:

$$3Y_1^2 + 10Y_1Y_2 + 17Y_2^2$$

5.20. Find the matrix $\mathbf{A}$ of the quadratic form:

$$7Y_1^2 - 8Y_1Y_2 + 8Y_2^2$$

*5.21. For the matrix:

$$\mathbf{A} = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$$

find the quadratic form of the observations Y_1 and Y_2.

5.22. For the matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 3 & 0 \\ 4 & 0 & 9 \end{bmatrix}$$

find the quadratic form of the observations Y_1, Y_2, and Y_3.

*5.23. Refer to **Flavor deterioration** Problems 5.4 and 5.12.

a. Using matrix methods, obtain the following: (1) vector of estimated regression coefficients, (2) vector of residuals, (3) SSR, (4) SSE, (5) estimated variance-covariance matrix of $\mathbf{b}$, (6) point estimate of $E\{Y_h\}$ when $X_h = -6$, (7) estimated variance of $\hat{Y}_h$ when $X_h = -6$.

b. What simplifications arose from the spacing of the X levels in the experiment?

c. Find the hat matrix $\mathbf{H}$.

d. Find $\mathbf{s}^2\{\mathbf{e}\}$.

5.24. Refer to **Consumer finance** Problems 5.5 and 5.13.

a. Using matrix methods, obtain the following: (1) vector of estimated regression coefficients, (2) vector of residuals, (3) SSR, (4) SSE, (5) estimated variance-covariance matrix of $\mathbf{b}$, (6) point estimate of $E\{Y_h\}$ when $X_h = 4$, (7) $s^2\{\text{pred}\}$ when $X_h = 4$.

b. From your estimated variance-covariance matrix in part (a5), obtain the following: (1) $s\{b_0, b_1\}$; (2) $s^2\{b_0\}$; (3) $s\{b_1\}$.

c. Find the hat matrix $\mathbf{H}$.

d. Find $\mathbf{s}^2\{\mathbf{e}\}$.

*5.25. Refer to **Airfreight breakage** Problems 1.21 and 5.6.

a. Using matrix methods, obtain the following: (1) $(\mathbf{X}'\mathbf{X})^{-1}$, (2) $\mathbf{b}$, (3) $\mathbf{e}$, (4) $\mathbf{H}$, (5) SSE, (6) $\mathbf{s}^2\{\mathbf{b}\}$, (7) $\hat{Y}_h$ when $X_h = 2$, (8) $s^2\{\hat{Y}_h\}$ when $X_h = 2$.

b. From part (a6), obtain the following: (1) $s^2\{b_1\}$; (2) $s\{b_0, b_1\}$; (3) $s\{b_0\}$.

c. Find the matrix of the quadratic form for SSR.

5.26. Refer to **Plastic hardness** Problems 1.22 and 5.7.

a. Using matrix methods, obtain the following: (1) $(\mathbf{X}'\mathbf{X})^{-1}$, (2) $\mathbf{b}$, (3) $\hat{\mathbf{Y}}$, (4) $\mathbf{H}$, (5) SSE, (6) $\mathbf{s}^2\{\mathbf{b}\}$, (7) $s^2\{\text{pred}\}$ when $X_h = 30$.

b. From part (a6), obtain the following: (1) $s^2\{b_0\}$; (2) $s\{b_0, b_1\}$; (3) $s\{b_1\}$.

c. Obtain the matrix of the quadratic form for SSE.

Exercises

5.27. Refer to regression-through-the-origin model (4.10). Set up the expectation vector for $\boldsymbol{\varepsilon}$. Assume that $i = 1, \ldots, 4$.

5.28. Consider model (4.10) for regression through the origin and the estimator b_1 given in (4.14). Obtain (4.14) by utilizing (5.60) with $\mathbf{X}$ suitably defined.

5.29. Consider the least squares estimator $\mathbf{b}$ given in (5.60). Using matrix methods, show that $\mathbf{b}$ is an unbiased estimator.

5.30. Show that $\hat{Y}_h$ in (5.96) can be expressed in matrix terms as $\mathbf{b}'\mathbf{X}_h$.

5.31. Obtain an expression for the variance-covariance matrix of the fitted values $\hat{Y}_i$, $i = 1, \ldots, n$, in terms of the hat matrix.