Part V

ulti-Factor tudies

Chapter

Two-Factor Studies with Equal Sample Sizes

In Part IV, we considered the design and analysis of experimental and observational studies in which the effects of one factor are investigated. Now we are concerned with investigations of the simultaneous effects of two or more factors. In this chapter, we take up the analysis of variance for two-factor studies where the factors are crossed and all sample sizes are equal. In Chapters 20, 21, 22, and 23, we continue the discussion of two-factor studies by taking up the analysis of factor effects with one case per cell, randomized complete block designs, the analysis of covariance, and two-factor studies with unequal sample sizes. In Chapter 24, we extend the analysis of variance to studies with three or more factors. Finally, in Chapter 25, we take up random and mixed effects models.

19.1 Two-Factor Observational and Experimental Studies

Two-factor studies, like single-factor studies, can be based on experimental or observational data. We begin with three examples of two-factor studies: the first is an experimental study, the second is an observational study, and the third has aspects of both experimental and observational studies.

Examples of Two-Factor Experiments and Observational Studies

Example 1

A company investigated the effects of selling price and type of promotional campaign on sales of one of its products. Three selling prices (55 cents, 60 cents, 65 cents) were studied, as were two types of promotional campaigns (radio advertising, newspaper advertising). Let us consider selling price to be factor A and promotional campaign to be factor B. Factor A here was studied at three price levels; in general, we use the symbol a to denote the number of levels of factor A investigated. Factor B was here studied at two levels; we use the symbol b to denote the number of levels of factor B investigated. Each combination of price and promotional campaign was studied, as shown in the

table below:

Treatment	Description
1	55 price, radio advertising
2	60 price, radio advertising
3	65 price, radio advertising
4	55 price, newspaper advertising
5	60 price, newspaper advertising
6	65 price, newspaper advertising

Each combination of a factor level of A and a factor level of B is a *treatment*. Thus, there are $3 \times 2 = 6$ treatments here altogether. In general, the total number of possible treatments in a two-factor study is ab.

Twelve communities throughout the United States, of approximately equal size and similar socioeconomic characteristics, were selected and the treatments were assigned to them at random, such that each treatment was given to two experimental units. The experiment can be represented by the graph in Figure 19.1. The two experimental units for each treatment combination are represented by the dot with circle circumscribed. Notice that four experimental units are assigned to each price level, as shown by the dot plot along the price (X) axis, and six experimental units are assigned to each mode of advertising, as shown by the dot plot along the advertising (Y) axis.

As before, we use n for the number of units receiving a given treatment when all treatment sample sizes are the same. For the $n = 2$ communities that were assigned treatment 1, for instance, the product price was fixed at 55 cents and radio advertising was employed, and so on for the other communities in the study.

This is an experimental study because control was exercised in assigning the factor A and factor B levels to the experimental units by means of random assignments of the treatments to the communities. The design used was a completely randomized design.

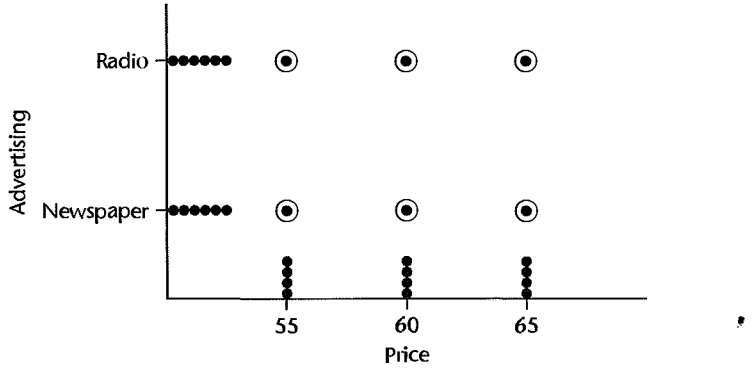

FIGURE 19.1 Experimental Layout—Example 1.

Example 2 An analyst studied the effects of family income (under $15,000, $15,000–$29,999, $30,000–$49,999, $50,000 and more) and stage in the life cycle of the family (stages 1, 2, 3, 4) on appliance purchases. Here, $4 \times 4 = 16$ treatments are defined. These are in part:

Treatment	Description
1	Under $15,000 income, stage 1
2	Under $15,000 income, stage 2
⋮	⋮
16	$50,000 and more income, stage 4

The analyst selected 20 families with the required income and life-cycle characteristics for each of the "treatment" classes for this study, yielding 320 families for the entire study.

This study is an observational one because the data were obtained without assigning income and life-cycle stage to the families. Rather, the families were selected because they had the specified characteristics.

Example 3 A medical investigator studied the relationship between the response to three blood pressure lowering drug types for hypertensive males and females. Here, $3 \times 2 = 6$ treatments are defined. These are:

Treatment	Description
1	Drug type 1, males
2	Drug type 1, females
3	Drug type 2, males
4	Drug type 2, females
5	Drug type 3, males
6	Drug type 3, females

The investigator selected 30 adult males and 30 adult females and randomly assigned 10 males and 10 females to each of the three drug types, yielding 60 total subjects.

This study has one observational factor, gender, and one experimental factor, drug type. This design is referred to as a randomized complete block design where the gender factor is called a block. This design will be discussed in Chapter 21.

Comments

1. When we considered single-factor studies, we did not place any restrictions on the nature of the r factor levels under study. Formally, the ab treatments in a two-factor investigation could be considered as the r factor levels in a single-factor investigation and analyzed according to the methods discussed in Part IV. The reason why new methods of analysis are required is that we wish to analyze the ab treatments in special ways that recognize two factors are involved and enable us to obtain information about the main effects of each of the two factors as well as about any special joint effects.

2. When a completely randomized design is used in a multifactor study, the random assignments of treatments to the experimental units are made in the same manner as for a single-factor study. No new problems are encountered once the treatments are defined in terms of the factor levels of the various factors under study. ■

The One-Factor-at-a-Time (OFAAT) Approach to Experimentation

It is not uncommon for investigators to vary only one factor at a time, holding all others constant, when attempting to understand the effect of a given set of factors on a particular outcome. For example, to maximize sales in Example 1, we might be tempted to first fix price at a particular value such as 60 cents, and then determine which mode of advertising (radio or newspaper) is most effective. If this test reveals that newspaper advertising leads to higher sales, we would then run a second test in which the advertising mode is fixed at "newspaper," and the three price levels are tested. This *one-factor-at-a-time* (OFAAT) experimental approach is depicted in Figure 19.2.

We note a number of deficiencies of the OFAAT approach:

1. The OFAAT approach does not explore the entire space of treatment combinations, and important treatment combinations may therefore be missed. In Figure 19.2, we see that two treatment combinations—(radio, 55 cents) and (radio, 65 cents)—were omitted, or one-third of the total. The fraction of treatment combinations omitted can be much larger for studies involving larger numbers of factors and/or larger numbers of factor levels.

2. Interactions cannot be estimated. As we have seen in regression, an interaction between two predictors is present if the effect (slope) of one predictor changes with the level of the other predictor. With the OFAAT approach, this is impossible to determine, because the slope of one factor is obtained only for a fixed set of levels of the other factors.

3. A full randomization is not possible for the OFAAT approach, because the experiment must be fielded in stages. Thus if certain variables that are not under control of the experimenter change with the stages of the testing, the results may be adversely affected.

4. The OFAAT approach is often more difficult to field logistically, because of the sequence of stages. At each stage, the experimental apparatus is set up, responses are obtained, an analysis is carried out, and the next treatment combinations are determined. Setting up for each experimental phase can be difficult. For example, it may be necessary in an industrial experiment to reserve time on an assembly line or in a pilot plant well in advance. In a field study involving a survey, it may be necessary to preschedule subjects and interviewers. In addition, processing responses can be time-consuming—for example, if complicated laboratory analyses are required—and the subsequent phase of experimentation may be delayed significantly.

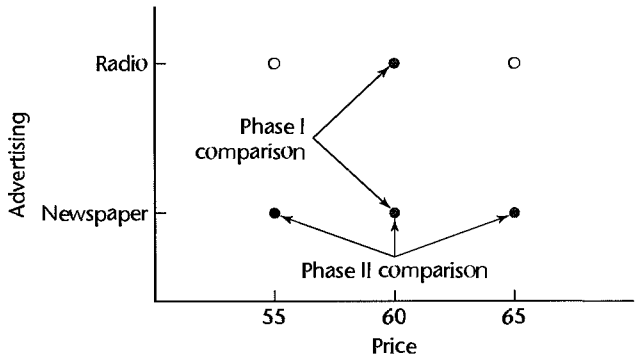

FIGURE 19.2 One-Factor-at-a-Time Approach—Example 1.

Advantages of Crossed, Multi-Factor Designs

Efficiency and Hidden Replication. Multi-factor studies are more efficient than the OFAAT experimental approach. Even though the OFAAT approach devotes all resources to studying the effect of only one factor, it does not yield any more precise information about that factor than a multi-factor experiment of the same size. With reference to Example 1 again, suppose that 12 communities were to be utilized in a traditional study, six assigned to radio advertising and the other six to newspaper advertising, and that the price would be kept constant at 60 cents. For this traditional study, the comparison between the two types of promotional campaigns would be based on two samples of six communities each. The same is true for the two-factor study in Example 1, since each promotional campaign occurs there in three treatments and each treatment has two communities assigned to it. Figure 19.1 reveals what is sometimes called *hidden replication* in a two-factor experiment. While there are only two replicates for each treatment combination, each level of advertising is repeated six times, and each level of price is repeated four times.

The increased efficiency due to hidden replication for main effect tests in multi-factor studies is only present when either unimportant interactions exist or when interaction effects are small relative to main effects. When important interactions are present, multiple comparisons of the individual cell means rather than comparisons of the main effects are usually conducted.

Assessment of Interactions. OFAAT studies provide no information about interactions. Specifically in our previous illustration, it does not provide any information about any special joint effects of price and promotional campaign. For instance, it might be that the price effects are not large when the promotional campaign is in newspapers but are large with radio advertising. Such interaction effects can be readily investigated from cross-classified multifactor studies.

Validity of Findings. In addition to being more efficient and readily providing information about interaction effects, multi-factor studies also can strengthen the validity of the findings. Suppose that in Example 1, management was principally interested in investigating the effects of price on sales. If the promotional campaign used in the price study had been newspaper advertising, doubts might exist as to whether or not the price effects differ for other promotional vehicles. By including type of promotional campaign as another factor in the study, management can get information about the persistence of the price effects with different promotional vehicles, without increasing the number of experimental units in the study. Thus, multifactor studies can include some factors of secondary importance to permit inferences about the primary factors with a greater range of validity.

Comments

1. Multi-factor studies permit a ready evaluation of interaction effects for observational data and economize on the number of cases required for the analysis, just as for experimental studies.

2. The advantages of multi-factor experiments just described should not lead one to think that inclusion of more factors necessarily results in a better study. Experiments involving many factors, each at numerous levels, become complex, costly, and time-consuming. It is often a better research strategy to begin with fewer factors and/or fewer levels for each factor, and then extend the investigation in accordance with the results obtained to date. In this way, resources can be devoted principally to the most promising avenues of investigation, and a better understanding of the effects of the factors can be obtained. ■

19.2 Meaning of ANOVA Model Elements

Before presenting a formal statement of the analysis of variance model for two-factor studies, we shall develop the model elements and discuss their meaning. This will not only be helpful in understanding the ANOVA model but will also provide insights into how the analysis of two-factor studies should proceed. *Throughout this section, we assume that all population means are known and are of equal importance when averages of these means are required.*

Illustration

To illustrate the meaning of the ANOVA model elements, we consider a simple two-factor study in which the effects of gender and age on learning of a task are of interest. For simplicity, the age factor has been defined in terms of only three factor levels (young, middle, old), as shown in Table 19.1a.

Treatment Means

The mean response for a given treatment in a two-factor study is denoted by μ_{ij}, where i refers to the level of factor A ($i = 1, \ldots, a$) and j refers to the level of factor B ($j = 1, \ldots, b$). Table 19.1a contains the true treatment means μ_{ij} for the learning example. Note, for instance, that $\mu_{11} = 9$, which indicates that the mean learning time for young males is 9 minutes. Similarly, we see that $\mu_{22} = 11$, so that the mean learning time for middle-aged females is 11 minutes.

The interpretation of a treatment mean μ_{ij} depends on whether the study is observational, experimental, or a mixture of the two. In an observational study, the treatment mean μ_{ij} corresponds to the population mean for the elements having the characteristics of the ith level of factor A and the jth level of factor B. For instance, in the learning example, the treatment mean μ_{11} is the mean learning time for the population of young males.

In an experimental study, the treatment mean μ_{ij} stands for the mean response that would be obtained if the treatment consisting of the ith level of factor A and the jth level of factor B were applied to all units in the population of experimental units about which

TABLE 19.1 Age Effect but No Gender Effect, with No Interactions—Learning Example.

(a) Mean Learning Times (in minutes)

	Factor *B*—Age			
Factor *A*—Gender	$j = 1$ Young	$j = 2$ Middle	$j = 3$ Old	Row Average
$i = 1$ Male	9 (μ_{11})	11 (μ_{12})	16 (μ_{13})	12 ($\mu_{1\cdot}$)
$i = 2$ Female	9 (μ_{21})	11 (μ_{22})	16 (μ_{23})	12 ($\mu_{2\cdot}$)
Column average	9 ($\mu_{\cdot 1}$)	11 ($\mu_{\cdot 2}$)	16 ($\mu_{\cdot 3}$)	12 ($\mu_{\cdot\cdot}$)

(b) Main Gender Effects (in minutes)

$$\alpha_1 = \mu_{1\cdot} - \mu_{\cdot\cdot} = 12 - 12 = 0$$
$$\alpha_2 = \mu_{2\cdot} - \mu_{\cdot\cdot} = 12 - 12 = 0$$

(c) Main Age Effects (in minutes)

$$\beta_1 = \mu_{\cdot 1} - \mu_{\cdot\cdot} = 9 - 12 = -3$$
$$\beta_2 = \mu_{\cdot 2} - \mu_{\cdot\cdot} = 11 - 12 = -1$$
$$\beta_3 = \mu_{\cdot 3} - \mu_{\cdot\cdot} = 16 - 12 = 4$$

inferences are to be drawn. For instance, in a study where factor A is type of training program (highly structured, partially structured, unstructured) and factor B is time of training (during work, after work), $6n$ employees are selected and n are assigned at random to each of the six treatments. The mean μ_{ij} here represents the mean response, say, mean gain in productivity, if the ith training program administered during the jth time were given to all employees in the population of experimental units.

Factor Level Means

The treatment means in Table 19.1a for the learning example indicate that the mean learning times for men and women are the same for each age group. On the other hand, the mean learning time increases with age for each gender. Thus, gender has no effect on mean learning time, but age does. This can also be seen quickly from the row averages and column averages shown in Table 19.1a, which in this case tell the complete story. The row averages are the gender factor level means, and the column averages are the age factor level means. We denote the column average for the first column by $\mu_{\cdot 1}$, which is the average of μ_{11} and μ_{21}. In general, the column average for the jth column is denoted by $\mu_{\cdot j}$:

$$\mu_{\cdot j} = \frac{\sum_{i=1}^{a} \mu_{ij}}{a} \tag{19.1}$$

and the row average for the ith row is denoted by $\mu_{i\cdot}$:

$$\mu_{i\cdot} = \frac{\sum_{j=1}^{b} \mu_{ij}}{b} \tag{19.2}$$

The overall mean learning time for all ages and both genders is denoted by $\mu_{\cdot\cdot}$, and is defined in the following equivalent fashions:

$$\mu_{\cdot\cdot} = \frac{\sum_i \sum_j \mu_{ij}}{ab} \tag{19.3a}$$

$$\mu_{\cdot\cdot} = \frac{\sum_i \mu_{i\cdot}}{a} \tag{19.3b}$$

$$\mu_{\cdot\cdot} = \frac{\sum_j \mu_{\cdot j}}{b} \tag{19.3c}$$

In Table 19.1a, the gender factor level means are $\mu_{1\cdot} = \mu_{2\cdot} = 12$ for the two genders, the age factor level means are $\mu_{\cdot 1} = 9$, $\mu_{\cdot 2} = 11$, and $\mu_{\cdot 3} = 16$ for the three age groups, and the overall mean learning time is $\mu_{\cdot\cdot} = 12$ minutes.

Main Effects

Main Age Effects. To summarize the main age effects, we shall consider the differences between each factor level mean and the overall mean. These differences are called main age effects. For instance, the main effect for young persons in Table 19.1a is the difference between $\mu_{\cdot 1}$, the mean learning time for young persons, and $\mu_{\cdot\cdot}$, the overall mean. This difference is denoted by β_1:

$$\beta_1 = \mu_{\cdot 1} - \mu_{\cdot\cdot} = 9 - 12 = -3$$

β_1 is called the *main effect* for factor B at the first level. This and the other main effects for factor B are shown in Table 19.1c.

Main Gender Effects. The main gender effects are defined in corresponding fashion, and denoted by α_i. For instance, we have:

$$\alpha_1 = \mu_{1\cdot} - \mu_{\cdot\cdot} = 12 - 12 = 0$$

α_1 is called the main effect for factor A at the first level. The main effects for factor A are shown in Table 19.1b. They are both zero, indicating that gender does not affect mean learning time.

General Definitions. In general, we define the main effect of factor A at the ith level as follows:

$$\alpha_i = \mu_{i\cdot} - \mu_{\cdot\cdot} \tag{19.4}$$

Similarly, the main effect of the jth level of factor B is defined:

$$\beta_j = \mu_{\cdot j} - \mu_{\cdot\cdot} \tag{19.5}$$

It follows from (19.3b) and (19.3c) that:

$$\sum_i \alpha_i = 0 \qquad \sum_j \beta_j = 0 \tag{19.6}$$

Thus, the sum of the main effects for each factor is zero.

Note again that a main effect indicates how much the factor level mean deviates from the overall mean. The greater the main effect, the more the factor level mean differs from the overall mean response averaged over the factor levels for both factors.

Additive Factor Effects

The factor effects in Table 19.1 have an interesting property. Each mean response μ_{ij} can be obtained by adding the respective gender and age main effects to the overall mean $\mu_{\cdot\cdot}$. For instance, we have:

$$\mu_{11} = \mu_{\cdot\cdot} + \alpha_1 + \beta_1 = 12 + 0 + (-3) = 9$$
$$\mu_{23} = \mu_{\cdot\cdot} + \alpha_2 + \beta_3 = 12 + 0 + 4 = 16$$

In general, we have for Table 19.1a:

$$\mu_{ij} = \mu_{\cdot\cdot} + \alpha_i + \beta_j \qquad \text{Additive factor effects} \tag{19.7}$$

which can be expressed equivalently, using the definitions of α_i in (19.4) and of β_j in (19.5), as:

$$\mu_{ij} = \mu_{i\cdot} + \mu_{\cdot j} - \mu_{\cdot\cdot} \qquad \text{Additive factor effects} \tag{19.7a}$$

It can also be shown that each treatment mean μ_{ij} in Table 19.1a can be expressed in terms of three other treatment means:

$$\mu_{ij} = \mu_{ij'} + \mu_{i'j} - \mu_{i'j'} \qquad \text{Additive factor effects} \qquad i \neq i',\ j \neq j' \tag{19.7b}$$

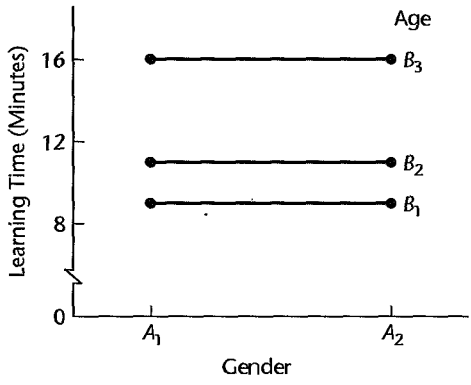

FIGURE 19.3
Age Effect but No Gender Effect, with No Interactions—Learning Example.

For instance, we have:

$$\mu_{11} = \mu_{12} + \mu_{21} - \mu_{22} = 11 + 9 - 11 = 9$$

or:

$$\mu_{11} = \mu_{13} + \mu_{21} - \mu_{23} = 16 + 9 - 16 = 9$$

When all treatment means can be expressed in the form of (19.7), (19.7a), or (19.7b), we say that the *factors do not interact,* or that *no factor interactions are present,* or that the *factor effects are additive.* The significance of no factor interactions is that the effect of either factor does not depend on the level of the other factor. Consequently, the effects of the two factors can be described separately merely by analyzing the factor level means or the factor main effects. For instance, in the learning example in Table 19.1a, the two gender means signify that gender has no influence regardless of age, and the three age means portray the influence of age regardless of gender. The analysis of factor effects is therefore quite simple when there are no factor interactions.

Graphic Presentation. Figure 19.3 presents the mean learning times of Table 19.1a in the form of a *treatment means plot*—also known as an *interaction plot.* The X axis contains the gender factor levels (denoted by A_1 and A_2), and the Y axis contains learning time. Separate curves are drawn for each of the age factor levels (denoted by B_1, B_2, and B_3). The zero slope of each curve indicates that gender has no effect. The differences in the heights of the three curves show the age effects on learning time.

The points on each curve are conventionally connected by straight lines even though the variable on the X axis (gender, in our example) is not a continuous variable. When the variable on the X axis is qualitative, the slopes of the curves have no meaning, except when the slope is zero, which implies there are no factor level effects. If one of the two factors is a quantitative variable, it is ordinarily advisable to place that factor on the X scale.

Note that the treatment means plot in Figure 19.3 corresponds to a conditional effects plot in regression, such as the ones shown in Figure 8.7 on page 307. In each case, the effect of one variable is shown at different levels of the other variable.

A Second Example with Additive Factor Effects. Table 19.2a contains another illustration of factor effects that do not interact, for the same gender-age learning example as before. The situation here differs from that of Table 19.1a in that not only age but also

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(a) Mean Learning Times (in minutes)

	Factor *B*—Age			
Factor *A*—Gender	$j=1$ Young	$j=2$ Middle	$j=3$ Old	Row Average
$i=1$ Male	11 (μ_{11})	13 (μ_{12})	18 (μ_{13})	14 ($\mu_{1\cdot}$)
$i=2$ Female	7 (μ_{21})	9 (μ_{22})	14 (μ_{23})	10 ($\mu_{2\cdot}$)
Column average	9 ($\mu_{\cdot 1}$)	11 ($\mu_{\cdot 2}$)	16 ($\mu_{\cdot 3}$)	12 ($\mu_{\cdot\cdot}$)

(b) Main Gender Effects (in minutes)

$$\alpha_1 = \mu_{1\cdot} - \mu_{\cdot\cdot} = 14 - 12 = 2$$
$$\alpha_2 = \mu_{2\cdot} - \mu_{\cdot\cdot} = 10 - 12 = -2$$

(c) Main Age Effects (in minutes)

$$\beta_1 = \mu_{\cdot 1} - \mu_{\cdot\cdot} = 9 - 12 = -3$$
$$\beta_2 = \mu_{\cdot 2} - \mu_{\cdot\cdot} = 11 - 12 = -1$$
$$\beta_3 = \mu_{\cdot 3} - \mu_{\cdot\cdot} = 16 - 12 = 4$$

FIGURE 19.4
Age and
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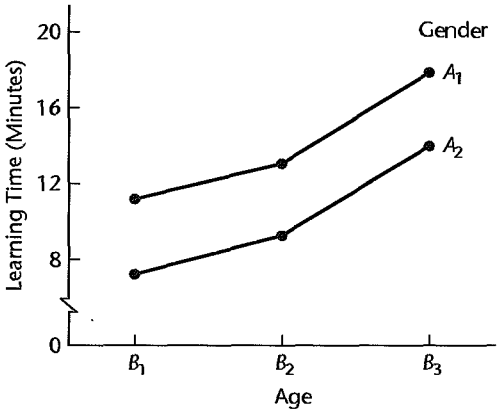

gender affects the learning time. This is evident from the fact that the mean learning times for men and women are not the same for any age group.

In Table 19.2a, as in Table 19.1a, every mean response can be decomposed according to (19.7):

$$\mu_{ij} = \mu_{\cdot\cdot} + \alpha_i + \beta_j$$

For instance:

$$\mu_{11} = \mu_{\cdot\cdot} + \alpha_1 + \beta_1 = 12 + 2 + (-3) = 11$$

Hence, the two factors do not interact, and the factor effects can be analyzed separately by examining the factor level means $\mu_{i\cdot}$ and $\mu_{\cdot j}$, respectively.

Figure 19.4 presents the data from Table 19.2a in the form of a treatment means plot. This time we have placed age on the X axis and used different curves for each gender. Note that the difference in the heights of the two curves reflects the gender difference and the departure from horizontal for each of the curves reflects the age effect. Furthermore, the two curves are parallel, which indicates that no two-factor interactions are present.

Equivalent Statements of Additive Factor Effects. We have said that two factors do not interact if *all* treatment means μ_{ij} can be expressed according to (19.7), (19.7a), or (19.7b). There are a number of other, equivalent, methods of recognizing when two factors do not interact. These are:

1. The difference between the mean responses for any two levels of factor B is the same for all levels of factor A. (For instance, in Table 19.2a, going from young to middle age leads to an increase of two minutes for both males and females, and going from middle age to old leads to an increase of five minutes for both males and females.) Note that it is *not* required that the changes, say, between levels 1 and 2 and between levels 2 and 3 of factor B are the same. These, of course, may differ depending upon the nature of the factor B effect.
2. The difference between the mean responses for any two levels of factor A is the same for all levels of factor B. (For instance, in Table 19.2a, going from male to female leads to a decrease of four minutes for all three age groups.)
3. The curves of the mean responses for the different levels of a factor are all parallel (such as the two gender curves in Figure 19.4).

All of these conditions are equivalent, implying that the two factors do not interact.

Interacting Factor Effects

Table 19.3a contains an illustration for the learning example where the factor effects do interact. The mean learning times for the different gender-age combinations in Table 19.3a indicate that gender has no effect on learning time for young persons but has a substantial effect for old persons. This differential influence of gender, which depends on the age of the person, implies that the age and gender factors interact in their effect on learning time.

TABLE 19.3 **Age and Gender Effects, with Interactions—Learning Example.**

(a) Mean Learning Times (in minutes)

	Factor *B*—Age				
Factor *A*—Gender	$j=1$ Young	$j=2$ Middle	$j=3$ Old	Row Average	Main Gender Effect
$i=1$ Male	9 (μ_{11})	12 (μ_{12})	18 (μ_{13})	13 ($\mu_{1\cdot}$)	1 (α_1)
$i=2$ Female	9 (μ_{21})	10 (μ_{22})	14 (μ_{23})	11 ($\mu_{2\cdot}$)	−1 (α_2)
Column average	9 ($\mu_{\cdot 1}$)	11 ($\mu_{\cdot 2}$)	16 ($\mu_{\cdot 3}$)	12 ($\mu_{\cdot\cdot}$)	
Main age effect	−3 (β_1)	−1 (β_2)	4 (β_3)		

(b) Interactions (in minutes)

	$j=1$	$j=2$	$j=3$	Row Average
$i=1$	−1	0	1	0
$i=2$	1	0	−1	0
Column average	0	0	0	0

Definition of Interaction. We can study the existence of interacting factor effects formally by examining whether or not all treatment means μ_{ij} can be expressed according to (19.7):

$$\mu_{ij} = \mu_{..} + \alpha_i + \beta_j$$

If they can, the factor effects are additive; otherwise, the factor effects are interacting.

For the learning example in Table 19.3a, the main factor effects α_i and β_j are shown in the margins of the table. It is clear that the factors interact. For instance, $\mu_{11} = 9$ while:

$$\mu_{..} + \alpha_1 + \beta_1 = 12 + 1 + (-3) = 10$$

If the two factors were additive, these would be the same.

The difference between the treatment mean μ_{ij} and the value $\mu_{..} + \alpha_i + \beta_j$ that would be expected if the two factors were additive is called the *interaction effect*, or more simply the *interaction*, of the ith level of factor A with the jth level of factor B, and is denoted by $(\alpha\beta)_{ij}$. Thus, we define $(\alpha\beta)_{ij}$ as follows:

$$(\alpha\beta)_{ij} = \mu_{ij} - (\mu_{..} + \alpha_i + \beta_j) \tag{19.8}$$

Replacing α_i and β_j by their definitions in (19.4) and (19.5), respectively, we obtain an alternative definition:

$$(\alpha\beta)_{ij} = \mu_{ij} - \mu_{i\cdot} - \mu_{\cdot j} + \mu_{..} \tag{19.8a}$$

To repeat, the interaction of the ith level of A with the jth level of B, denoted by $(\alpha\beta)_{ij}$, is simply the difference between the treatment mean μ_{ij} and the value that would be expected if the factors were additive. If in fact the two factors are additive, all interactions equal zero; i.e., $(\alpha\beta)_{ij} \equiv 0$.

The interactions for the learning example in Table 19.3a are shown in Table 19.3b. We have, for instance:

$$\begin{aligned}(\alpha\beta)_{13} &= \mu_{13} - (\mu_{..} + \alpha_1 + \beta_3) \\ &= 18 - (12 + 1 + 4) \\ &= 1\end{aligned}$$

Recognition of Interactions. We may recognize whether or not interactions are present in one of the following equivalent fashions:

1. By examining whether all μ_{ij} can be expressed as the sums $\mu_{..} + \alpha_i + \beta_j$.

2. By examining whether the difference between the mean responses for any two levels of factor B is the same for all levels of factor A. (For instance, note in Table 19.3a that the mean learning time increases when going from young to middle-aged persons by three minutes for men but only by one minute for women.)

3. By examining whether the difference between the mean responses for any two levels of factor A is the same for all levels of factor B. (For instance, note in Table 19.3a that there is no difference between genders for young persons, but there is a difference of four minutes for old persons.)

4. By examining whether the treatment means curves for the different factor levels in a treatment means plot are parallel. (Figure 19.5 presents a plot of the treatment means in Table 19.3a, with age on the X axis. Note that the treatment means curves for the two genders are not parallel.)

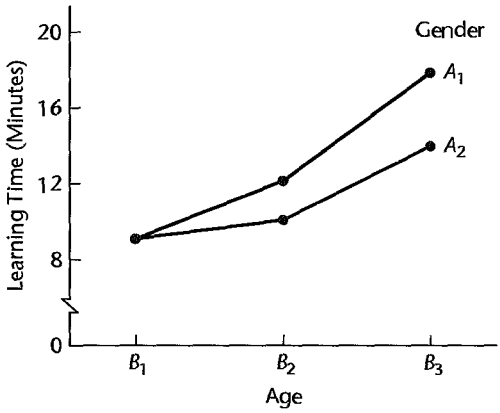

FIGURE 19.5
Age and Gender Effects, with Important Interactions—Learning Example.

Comments

1. Note from Table 19.3b that some interactions are zero even though the two factors are interacting. *All* interactions must equal zero in order for the two factors to be additive.

2. Table 19.3b illustrates that interactions sum to zero when added over either rows or columns:

$$\sum_i (\alpha\beta)_{ij} = 0 \qquad j = 1, \ldots, b \tag{19.9a}$$

$$\sum_j (\alpha\beta)_{ij} = 0 \qquad i = 1, \ldots, a \tag{19.9b}$$

Consequently, the sum of all interactions is also zero:

$$\sum_i \sum_j (\alpha\beta)_{ij} = 0 \tag{19.9c}$$

We show this for (19.9a):

$$\begin{aligned}\sum_i (\alpha\beta)_{ij} &= \sum_{i=1}^{a} (\mu_{ij} - \mu_{\cdot\cdot} - \alpha_i - \beta_j) \\ &= \sum_i \mu_{ij} - a\mu_{\cdot\cdot} - \sum_i \alpha_i - a\beta_j\end{aligned}$$

Now $\sum_i \mu_{ij} = a\mu_{\cdot j}$ by (19.1) and $\sum \alpha_i = 0$ by (19.6). Finally, $\beta_j = \mu_{\cdot j} - \mu_{\cdot\cdot}$ by (19.5). Hence, we obtain:

$$\sum_i (\alpha\beta)_{ij} = a\mu_{\cdot j} - a\mu_{\cdot\cdot} - a(\mu_{\cdot j} - \mu_{\cdot\cdot}) = 0$$

■

Important and Unimportant Interactions

When two factors interact, the question arises whether the factor level means are still meaningful measures. In Table 19.3a, for instance, it may well be argued that the gender factor level means 13 and 11 are misleading measures. They indicate that some difference exists in learning time for men and women, but that this difference is not too great. These factor level means hide the fact that there is no difference in mean learning time between

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Factor *A*—Gender	Factor *B*—Age $j=1$ Young	$j=2$ Middle	$j=3$ Old	Row Average
$i=1$ Male	9.75	12.00	17.25	13.00
$i=2$ Female	8.25	10.00	14.75	11.00
Column average	9.00	11.00	16.00	12.00

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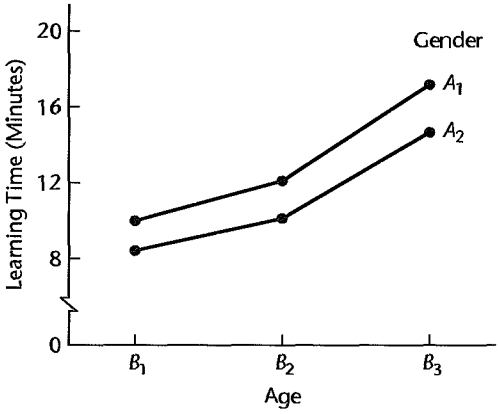

genders for young persons, but there is a relatively large difference for old persons. The interactions in Table 19.3a would therefore be considered *important interactions*, implying that one should not ordinarily examine the effects of each factor separately in terms of the factor level means. A treatment means plot, such as in Figure 19.5, presents effectively a description of the nature of the interacting effects of the two factors.

Sometimes when two factors interact, the interaction effects are so small that they are considered to be *unimportant interactions*. Table 19.4 and Figure 19.6 present such a case. Note from Figure 19.6 that the curves are *almost* parallel. For practical purposes, one may say that the mean learning time for women is two minutes less than that for men, and this statement is approximately true for all age groups. Similarly, statements based on average learning time for different age groups will hold approximately for both genders.

Thus, in the case of unimportant interactions, the analysis of factor effects can proceed as for the case of no interactions. Each factor can be studied separately, based on the factor level means $\mu_{i\cdot}$ and $\mu_{\cdot j}$, respectively. This separate analysis of factor effects is, of course, much simpler than a joint analysis for the two factors based on the treatment means μ_{ij}, which is required when the interactions are important.

Comments

1. The determination of whether interactions are important or unimportant is admittedly sometimes difficult because it depends on the context of the application, just as the determination of whether an effect in a single-factor study is important. The subject area specialist (researcher) needs to play a prominent role in deciding whether an interaction is important or unimportant. The advantage of

unimportant (or no) interactions, namely, that one is then able to analyze the factor effects separately. This is especially great when the study contains more than two factors.

2. Occasionally, it is meaningful to consider the effects of each factor in terms of the factor level means even when important interactions are present. For example, two methods of teaching college mathematics (abstract and standard) were used in teaching students of excellent, good, and moderate quantitative ability. Important interactions between teaching method and student's quantitative ability were found to be present. Students with excellent quantitative ability tended to perform equally well with the two teaching methods, whereas students of moderate or good quantitative ability tended to perform better when taught by the standard method. If equal numbers of students with moderate, good, and excellent quantitative ability are to be taught by one of the two teaching methods, then the method that produces the best average result for all students might be of interest even in the presence of important interactions. A comparison of the teaching method factor level means would then be relevant, even though important interactions are present. ■

Transformable and Nontransformable Interactions

When important interactions exist, they are sometimes the result of the scale on which the response variable is measured. Consider, for instance, factor effects that act multiplicatively, rather than additively as in (19.7):

$$\mu_{ij} = \mu_{..}\alpha_i\beta_j \qquad \text{Multiplicative factor effects} \qquad \textbf{(19.10)}$$

If we were to assume here that the factor effects are additive, we would find that condition (19.7) does not hold and therefore that interactions are present. These interactions can be removed, however, by applying a logarithmic transformation to (19.10):

$$\log \mu_{ij} = \log \mu_{..} + \log \alpha_i + \log \beta_j \qquad \textbf{(19.11)}$$

This result can be restated equivalently as follows:

$$\mu'_{ij} = \mu'_{..} + \alpha'_i + \beta'_j \qquad \textbf{(19.11a)}$$

where:

$$\mu'_{ij} = \log \mu_{ij}$$
$$\mu'_{..} = \log \mu_{..}$$
$$\alpha'_i = \log \alpha_i$$
$$\beta'_j = \log \beta_j$$

The result in (19.11a) suggests that the original measurement scale for the response variable Y may not be the most appropriate one in the sense of leading to easily understood results. Rather, use of $Y' = \log Y$ for the response variable may be better, making the additive model (19.7) then more appropriate.

We say that the interactions present when the factor effects are actually multiplicative are *transformable interactions* because a simple transformation of Y will remove most of these interaction effects and thus make them unimportant.

Another instance of transformable interactions occurs when each interaction effect equals the product of functions of the main effects, for example:

$$\mu_{ij} = \alpha_i + \beta_j + 2\sqrt{\alpha_i}\sqrt{\beta_j} \qquad \text{Multiplicative interactions} \qquad \textbf{(19.12)}$$

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(a) Treatment Means—Original Scale			(b) Treatment Means after Square Root Transformation		
	Factor *B*			Factor *B*	
Factor *A*	$j=1$	$j=2$	Factor *A*	$j=1$	$j=2$
$i=1$	16	64	$i=1$	4	8
$i=2$	49	121	$i=2$	7	11
$i=3$	64	144	$i=3$	8	12

An equivalent form of (19.12) is:

$$\mu_{ij} = \left(\sqrt{\alpha_i} + \sqrt{\beta_j}\right)^2 \tag{19.12a}$$

If we now apply the square root transformation, we obtain an additive effects model:

$$\mu'_{ij} = \alpha'_i + \beta'_j \tag{19.13}$$

where:

$$\mu'_{ij} = \sqrt{\mu_{ij}}$$
$$\alpha'_i = \sqrt{\alpha_i}$$
$$\beta'_j = \sqrt{\beta_j}$$

Some simple transformations that may be helpful in making important interactions unimportant are the square, square root, logarithmic, and reciprocal transformations. When interactions cannot be largely removed by a transformation, they are called *nontransformable interactions*.

Table 19.5a contains an example of important interactions that are transformable. When a square root transformation is applied to these means, the resulting treatment means in Table 19.5b show no interacting effects. Ordinarily, of course, one cannot hope that a simple transformation of scale removes all interactions as in Table 19.5, but only that interactions become unimportant after the transformation.

Interpretation of Interactions

The interpretation of interactions can be quite difficult when the interacting effects are complex. There are many occasions, however, when the interactions have a simple structure, such as in Table 19.3a, so that the joint factor effects can be described in a straightforward manner. Table 19.6 provides several additional illustrations. The corresponding treatment means plots are shown in Figure 19.7.

In Table 19.6a and Figure 19.7a, we have a situation where either raising the pay or increasing the authority of low-paid executives with small authority leads to increased productivity. However, combining both higher pay and greater authority does not lead to any substantial further improvement in productivity than increasing either one alone. Table 19.6b and Figure 19.7b represent a case where both higher pay and greater authority are required before any substantial increase in productivity takes place.

TABLE 19.6
Examples of Different Types of Interactions.

(a) Productivity of Executives

	Factor *B*—Authority	
Factor *A*—Pay	Small	Great
Low	50	72
High	74	75

(b) Productivity of Executives

	Factor *B*—Authority	
Factor *A*—Pay	Small	Great
Low	50	52
High	53	75

(c) Productivity of Executives

	Factor *B*—Authority	
Factor *A*—Pay	Small	Great
Low	50	72
High	72	50

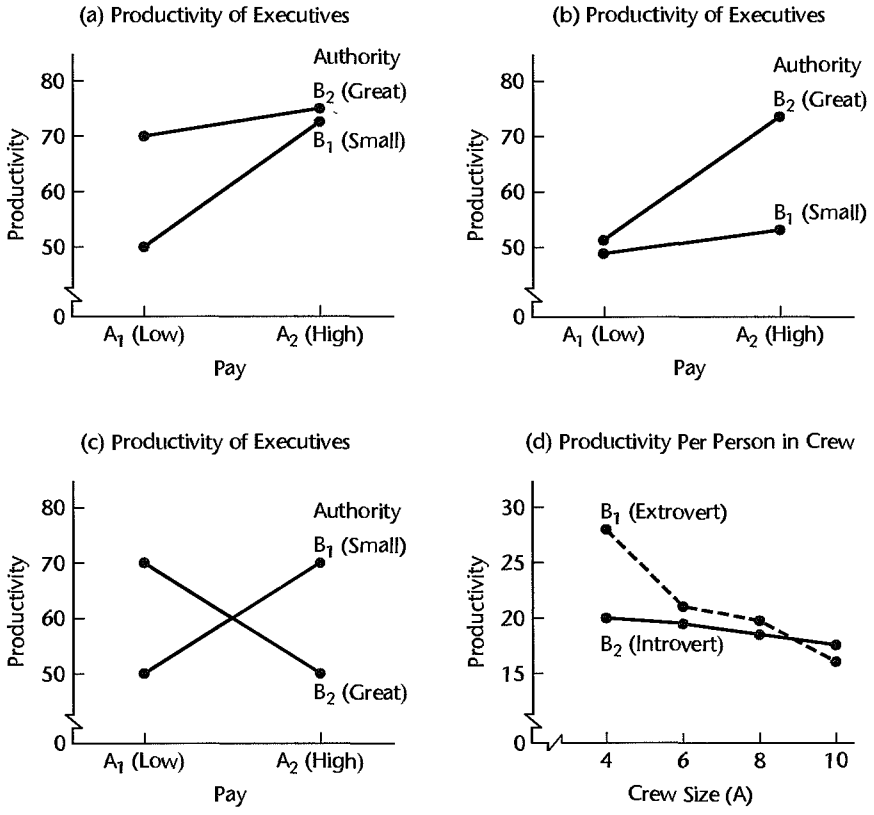
(d) Productivity per Person in Crew

	Factor *B*—Personality of Crew Chief	
Factor *A*—Crew Size	Extrovert	Introvert
4 persons	28	20
6 persons	22	20
8 persons	20	19
10 persons	17	18

It is possible that two factors interact, yet the main effects for one (or both) factors are zero. This would be the result of interactions in opposite directions that balance out over one (or both) factors. Thus, there would be definite factor effects, but these would not be disclosed by the factor level means. Table 19.6c and Figure 19.7c represent this situation where neither factor effect is present and the two factors interact. The case of interacting factors with no main effects for one (or both) factors fortunately is unusual. Typically, interaction effects are smaller than main effects.

Table 19.6d and Figure 19.7d portray a situation where size of crew and personality of crew chief interact in a complex fashion. Productivity with an extrovert crew chief and a crew of four is substantially larger than with an introvert crew chief. The advantage becomes small with crews of six and eight, and with a crew of 10 an introvert crew chief leads to a slightly larger productivity.

IGURE 19.7
Treatment eans Plots—xamples of teractions from able 19.6.



Comment

The terminology of reinforcement and interference interactions described in Chapter 8 for regression models where both predictor variables are quantitative is applicable to analysis of variance models if the two factors are quantitative or can be ordered on a measurement scale. In Figures 19.7a and 19.7b, pay level and authority both can be ordered on a scale. Hence, the interaction in Figure 19.7a can be described as an *interference* or *antagonistic* interaction (the slope decreases for higher levels of factor B), while that in Figure 19.7b can be described as a *reinforcement* or *synergistic* interaction (the slope increases for higher levels of factor B).

Similarly, the terminology of ordinal and disordinal interactions described in Chapter 8 for regression models where one predictor variable is quantitative and the other qualitative is applicable to analysis of variance models if one factor is quantitative or can be ordered on a measurement scale and the other factor is qualitative. In Figure 19.7d, crew size is a quantitative factor and personality is a qualitative factor. Therefore, the interaction in Figure 19.7d can be described as disordinal because the treatment means curves intersect. ■

19.3 Model I (Fixed Factor Levels) for Two-Factor Studies

Having explained the model elements, we are now ready to develop ANOVA model I with fixed factor levels for two-factor studies *when all treatment sample sizes are equal and all treatment means are of equal importance.* This ANOVA model is applicable to observational

studies and to experimental studies based on a completely randomized design. In Part VI we shall consider ANOVA models for some other experimental designs.

The basic situation is as follows: Factor A is studied at a levels, and these are of intrinsic interest in themselves; in other words, the a levels are not considered to be a sample from a larger population of factor A levels. Similarly, factor B is studied at b levels that are of intrinsic interest in themselves. All ab factor level combinations are included in the study. The number of cases for each of the ab treatments is the same, denoted by n, and it is required that $n > 1$. Thus, the total number of cases for the study is:

$$n_T = abn \tag{19.14}$$

The kth observation ($k = 1, \ldots, n$) for the treatment, where A is at the ith level, and B is at the jth level, is denoted by Y_{ijk} ($i = 1, \ldots, a$; $j = 1, \ldots, b$). Table 19.7 on page 833 illustrates this notation for an example where A is at three levels, B is at two levels, and two replications have been made for each treatment.

We shall state the fixed ANOVA model for two-factor studies in two equivalent versions—the cell means version and the factor effects version—and later will use one or the other as convenience dictates.

Cell Means Model

Model Formulation. When we regard the ab treatments without explicitly considering the factorial structure of the study, we express the analysis of variance model in terms of the cell (treatment) means μ_{ij}:

$$Y_{ijk} = \mu_{ij} + \varepsilon_{ijk} \tag{19.15}$$

where:

μ_{ij} are parameters
ε_{ijk} are independent $N(0, \sigma^2)$
$i = 1, \ldots, a$; $j = 1, \ldots, b$; $k = 1, \ldots, n$

Important Features of Model. Some important features of the cell means model are:

1. The parameter μ_{ij} is the mean response for the treatment in which factor A is at the ith level and factor B is at the jth level. This follows because $E\{\varepsilon_{ijk}\} = 0$:

$$E\{Y_{ijk}\} = \mu_{ij} \tag{19.16}$$

2. Since μ_{ij} is a constant, the variance of Y_{ijk} is:

$$\sigma^2\{Y_{ijk}\} = \sigma^2\{\varepsilon_{ijk}\} = \sigma^2 \tag{19.17}$$

3. Since the error terms ε_{ijk} are independent and normally distributed, so are the observations Y_{ijk}. Hence, we can state ANOVA model (19.15) also as follows:

$$Y_{ijk} \text{ are independent } N(\mu_{ij}, \sigma^2) \tag{19.18}$$

4. ANOVA model (19.15) is a linear model because it can be expressed in the form $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$. Consider a two-factor study with each factor having two levels (i.e., $a = b = 2$)

and two trials for each treatment (i.e., $n = 2$). Then $\mathbf{Y}$, $\mathbf{X}$, $\boldsymbol{\beta}$, and $\boldsymbol{\varepsilon}$ are defined as follows:

$$\mathbf{Y} = \begin{bmatrix} Y_{111} \\ Y_{112} \\ Y_{121} \\ Y_{122} \\ Y_{211} \\ Y_{212} \\ Y_{221} \\ Y_{222} \end{bmatrix} \quad \mathbf{X} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \boldsymbol{\beta} = \begin{bmatrix} \mu_{11} \\ \mu_{12} \\ \mu_{21} \\ \mu_{22} \end{bmatrix} \quad \boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{111} \\ \varepsilon_{112} \\ \varepsilon_{121} \\ \varepsilon_{122} \\ \varepsilon_{211} \\ \varepsilon_{212} \\ \varepsilon_{221} \\ \varepsilon_{222} \end{bmatrix} \tag{19.19}$$

Recall that the $\mathbf{E}\{\mathbf{Y}\}$ vector, which consists of the elements $E\{Y_{ijk}\}$, equals $\mathbf{X}\boldsymbol{\beta}$ according to (6.20). This vector here is:

$$\mathbf{E}\{\mathbf{Y}\} = \mathbf{X}\boldsymbol{\beta} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mu_{11} \\ \mu_{12} \\ \mu_{21} \\ \mu_{22} \end{bmatrix} = \begin{bmatrix} \mu_{11} \\ \mu_{11} \\ \mu_{12} \\ \mu_{12} \\ \mu_{21} \\ \mu_{21} \\ \mu_{22} \\ \mu_{22} \end{bmatrix} \tag{19.20}$$

Thus, $E\{Y_{ijk}\} = \mu_{ij}$, as it must according to (19.16), and we have the proper matrix representation for the two-factor ANOVA model (19.15):

$$\mathbf{Y} = \begin{bmatrix} Y_{111} \\ Y_{112} \\ Y_{121} \\ Y_{122} \\ Y_{211} \\ Y_{212} \\ Y_{221} \\ Y_{222} \end{bmatrix} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon} = \begin{bmatrix} \mu_{11} \\ \mu_{11} \\ \mu_{12} \\ \mu_{12} \\ \mu_{21} \\ \mu_{21} \\ \mu_{22} \\ \mu_{22} \end{bmatrix} + \begin{bmatrix} \varepsilon_{111} \\ \varepsilon_{112} \\ \varepsilon_{121} \\ \varepsilon_{122} \\ \varepsilon_{211} \\ \varepsilon_{212} \\ \varepsilon_{221} \\ \varepsilon_{222} \end{bmatrix} \tag{19.21}$$

In view of the error terms being independent with constant variance σ^2, the variance-covariance matrix of the error terms is $\boldsymbol{\sigma}^2\{\boldsymbol{\varepsilon}\} = \sigma^2\mathbf{I}$, as in (16.9) for the single-factor ANOVA model. Also as before, we have $\boldsymbol{\sigma}^2\{\mathbf{Y}\} = \boldsymbol{\sigma}^2\{\boldsymbol{\varepsilon}\}$ for two-factor ANOVA model (19.15).

5. ANOVA model (19.15) is therefore similar to the single-factor ANOVA model (16.2), except for the two subscripts now needed to identify the treatment. Normality, independent error terms, and constant variances for the error terms are properties of the ANOVA models for both single-factor and two-factor studies.

Factor Effects Model

Model Formulation. An equivalent version of cell means model (19.15) can be obtained by replacing each treatment mean μ_{ij} with an identical expression in terms of factor effects based on the definition of an interaction in (19.8):

$$(\alpha\beta)_{ij} = \mu_{ij} - (\mu_{\cdot\cdot} + \alpha_i + \beta_j)$$

Rearranging terms, we obtain the identity:

$$\mu_{ij} \equiv \mu_{..} + \alpha_i + \beta_j + (\alpha\beta)_{ij} \tag{19.22}$$

where:

$$\begin{aligned} \mu_{..} &= \frac{\sum_i \sum_j \mu_{ij}}{ab} \\ \alpha_i &= \mu_{i.} - \mu_{..} \\ \beta_j &= \mu_{.j} - \mu_{..} \\ (\alpha\beta)_{ij} &= \mu_{ij} - \mu_{i.} - \mu_{.j} + \mu_{..} \end{aligned}$$

This formulation indicates that each cell mean μ_{ij} can be viewed as the sum of four component factor effects. Specifically, (19.22) states that the mean response for the treatment where factor A is at the ith level and factor B is at the jth level is the sum of:

1. An overall mean $\mu_{...}$
2. The main effect α_i for factor A at the ith level.
3. The main effect β_j for factor B at the jth level.
4. The interaction effect $(\alpha\beta)_{ij}$ when factor A is at the ith level and factor B is at the jth level.

Replacing μ_{ij} in ANOVA model (19.15) by the expression in (19.22), we obtain an equivalent factor effects ANOVA model for two-factor studies:

$$Y_{ijk} = \mu_{..} + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijk} \tag{19.23}$$

where:

$\mu_{..}$ is a constant
α_i are constants subject to the restriction $\sum \alpha_i = 0$
β_j are constants subject to the restriction $\sum \beta_j = 0$
$(\alpha\beta)_{ij}$ are constants subject to the restrictions:

$$\begin{aligned} \textstyle\sum_i (\alpha\beta)_{ij} &= 0 \qquad j = 1, \ldots, b \\ \textstyle\sum_j (\alpha\beta)_{ij} &= 0 \qquad i = 1, \ldots, a \end{aligned}$$

ε_{ijk} are independent $N(0, \sigma^2)$
$i = 1, \ldots, a;\ j = 1, \ldots, b;\ k = 1, \ldots, n$

Important Features of Model. Some important features of the factor effects model are:

1. ANOVA model (19.23) corresponds to the fixed factor effects ANOVA model (16.62) for a single-factor study except that the single-factor treatment effect is here replaced by the sum of a factor A effect, a factor B effect, and an interaction effect.

2. The properties of the observations Y_{ijk} for factor effects model (19.23) are the same as those for the equivalent cell means model (19.15). Since $E\{\varepsilon_{ijk}\} = 0$, we have:

$$E\{Y_{ijk}\} = \mu_{..} + \alpha_i + \beta_j + (\alpha\beta)_{ij} = \mu_{ij} \tag{19.24}$$

The second equality follows from identity (19.22). Further, we have:

$$\sigma^2\{Y_{ijk}\} = \sigma^2 \tag{19.25}$$

because the error term ε_{ijk} is the only random term on the right-hand side in (19.23) and $\sigma^2\{\varepsilon_{ijk}\} = \sigma^2$. Finally, the Y_{ijk} are independent normal random variables because the error terms are independent normal random variables. Hence, we can also state ANOVA model (19.23) as follows:

$$Y_{ijk} \text{ are independent } N[\mu.. + \alpha_i + \beta_j + (\alpha\beta)_{ij},\ \sigma^2] \qquad \textbf{(19.26)}$$

3. ANOVA model (19.23) is a linear model because it can be stated in the form $\mathbf{Y} = \mathbf{X\beta} + \boldsymbol{\varepsilon}$. We shall show this explicitly in Section 23.2.

19.4 Analysis of Variance

Illustration

Table 19.7 contains an illustration that we shall employ in this chapter and the next. The Castle Bakery Company supplies wrapped Italian bread to a large number of supermarkets in a metropolitan area. An experimental study was made of the effects of height of the shelf display (factor A: bottom, middle, top) and the width of the shelf display (factor B: regular, wide) on sales of this bakery's bread during the experimental period (Y, measured in cases). Twelve supermarkets, similar in terms of sales volume and clientele, were utilized in the study. The six treatments were assigned at random to two stores each according to a completely randomized design, and the display of the bread in each store followed the treatment specifications for that store. Sales of the bread were recorded, and these results are presented in Table 19.7.

TABLE 19.7 Sample Data and Notation for Two-Factor Study—Castle Bakery Example (sales in cases).

Factor A (display height) i	Factor B (display width) j		Row Total	Display Height Average
	B_1 (regular)	B_2 (wide)		
A_1 (bottom)	47 (Y_{111}) 43 (Y_{112})	46 (Y_{121}) 40 (Y_{122})		
Total	90 ($Y_{11\cdot}$)	86 ($Y_{12\cdot}$)	176 ($Y_{1\cdot\cdot}$)	
Average	45 ($\bar{Y}_{11\cdot}$)	43 ($\bar{Y}_{12\cdot}$)		44 ($\bar{Y}_{1\cdot\cdot}$)
A_2 (middle)	62 (Y_{211}) 68 (Y_{212})	67 (Y_{221}) 71 (Y_{222})		
Total	130 ($Y_{21\cdot}$)	138 ($Y_{22\cdot}$)	268 ($Y_{2\cdot\cdot}$)	
Average	65 ($\bar{Y}_{21\cdot}$)	69 ($\bar{Y}_{22\cdot}$)		67 ($\bar{Y}_{2\cdot\cdot}$)
A_3 (top)	41 (Y_{311}) 39 (Y_{312})	42 (Y_{321}) 46 (Y_{322})		
Total	80 ($Y_{31\cdot}$)	88 ($Y_{32\cdot}$)	168 ($Y_{3\cdot\cdot}$)	
Average	40 ($\bar{Y}_{31\cdot}$)	44 ($\bar{Y}_{32\cdot}$)		42 ($\bar{Y}_{3\cdot\cdot}$)
Column total	300 ($Y_{\cdot 1\cdot}$)	312 ($Y_{\cdot 2\cdot}$)	612 ($Y_{\cdot\cdot\cdot}$)	
Display width average	50 ($\bar{Y}_{\cdot 1\cdot}$)	52 ($\bar{Y}_{\cdot 2\cdot}$)		51 ($\bar{Y}_{\cdot\cdot\cdot}$)

Notation

Table 19.7 illustrates the notation we shall use for two-factor studies. It is a straightforward extension of the notation for single-factor studies. An observation is denoted by Y_{ijk}. The subscripts i and j specify the levels of factors A and B, respectively, and the subscript k refers to the given case or trial for a particular treatment (i.e., factor level combination).

A dot in the subscript indicates aggregation or averaging over the variable represented by the index. For instance, the sum of the observations for the treatment corresponding to the ith level of factor A and the jth level of factor B is:

$$Y_{ij\cdot} = \sum_{k=1}^{n} Y_{ijk} \tag{19.27a}$$

The corresponding mean is:

$$\bar{Y}_{ij\cdot} = \frac{Y_{ij\cdot}}{n} \tag{19.27b}$$

The total of all observations for the ith factor level of A is:

$$Y_{i\cdot\cdot} = \sum_{j}^{b} \sum_{k}^{n} Y_{ijk} \tag{19.27c}$$

and the corresponding mean is:

$$\bar{Y}_{i\cdot\cdot} = \frac{Y_{i\cdot\cdot}}{bn} \tag{19.27d}$$

Similarly, for the jth factor level of B the sum of all observations and their mean are denoted by:

$$Y_{\cdot j\cdot} = \sum_{i}^{a} \sum_{k}^{n} Y_{ijk} \tag{19.27e}$$

$$\bar{Y}_{\cdot j\cdot} = \frac{Y_{\cdot j\cdot}}{an} \tag{19.27f}$$

Finally, the sum of all observations in the study is:

$$Y_{\cdot\cdot\cdot} = \sum_{i}^{a} \sum_{j}^{b} \sum_{k}^{n} Y_{ijk} \tag{19.27g}$$

and the overall mean is:

$$\bar{Y}_{\cdot\cdot\cdot} = \frac{Y_{\cdot\cdot\cdot}}{nab} \tag{19.27h}$$

Fitting of ANOVA Model

Cell Means Model (19.15). Fitting the two-factor cell means model (19.15) to the sample data by either the method of least squares or the method of maximum likelihood leads to minimizing the criterion:

$$Q = \sum_{i} \sum_{j} \sum_{k} (Y_{ijk} - \mu_{ij})^2 \tag{19.28}$$

When we perform the minimization of Q, we obtain the least squares and maximum likelihood estimators:

$$\hat{\mu}_{ij} = \bar{Y}_{ij\cdot} \tag{19.29}$$

Thus, the *fitted values* are the estimated treatment means:

$$\hat{Y}_{ijk} = \bar{Y}_{ij\cdot} \tag{19.30}$$

The *residuals,* as usual, are defined as the difference between the observed and fitted values:

$$e_{ijk} = Y_{ijk} - \hat{Y}_{ijk} = Y_{ijk} - \bar{Y}_{ij\cdot} \tag{19.31}$$

Residuals are highly useful for assessing the appropriateness of two-factor ANOVA model (19.15), as they also are for the statistical models considered earlier.

Factor Effects Model (19.23). For the equivalent factor effects model (19.23), the least squares and maximum likelihood methods both lead to minimizing the criterion:

$$Q = \sum_i \sum_j \sum_k [Y_{ijk} - \mu_{\cdot\cdot} - \alpha_i - \beta_j - (\alpha\beta)_{ij}]^2 \tag{19.32}$$

subject to the restrictions:

$$\sum_i \alpha_i = 0 \qquad \sum_j \beta_j = 0 \qquad \sum_i (\alpha\beta)_{ij} = 0 \qquad \sum_j (\alpha\beta)_{ij} = 0$$

When we perform this minimization, we obtain the following least squares and maximum likelihood estimators of the parameters:

Parameter	Estimator	
$\mu_{\cdot\cdot}$	$\hat{\mu}_{\cdot\cdot} = \bar{Y}_{\cdot\cdot\cdot}$	(19.33a)
$\alpha_i = \mu_{i\cdot} - \mu_{\cdot\cdot}$	$\hat{\alpha}_i = \bar{Y}_{i\cdot\cdot} - \bar{Y}_{\cdot\cdot\cdot}$	(19.33b)
$\beta_j = \mu_{\cdot j} - \mu_{\cdot\cdot}$	$\hat{\beta}_j = \bar{Y}_{\cdot j\cdot} - \bar{Y}_{\cdot\cdot\cdot}$	(19.33c)
$(\alpha\beta)_{ij} = \mu_{ij} - \mu_{i\cdot} - \mu_{\cdot j} + \mu_{\cdot\cdot}$	$\widehat{(\alpha\beta)}_{ij} = \bar{Y}_{ij\cdot} - \bar{Y}_{i\cdot\cdot} - \bar{Y}_{\cdot j\cdot} + \bar{Y}_{\cdot\cdot\cdot}$	(19.33d)

The correspondences of these estimators to the definitions of the parameters are readily apparent.

The fitted values and residuals for factor effects model (19.23) are exactly the same as those for cell means model (19.15). Specifically, the fitted values for ANOVA model (19.23) are:

$$\hat{Y}_{ijk} = \bar{Y}_{\cdot\cdot\cdot} + (\bar{Y}_{i\cdot\cdot} - \bar{Y}_{\cdot\cdot\cdot}) + (\bar{Y}_{\cdot j\cdot} - \bar{Y}_{\cdot\cdot\cdot}) + (\bar{Y}_{ij\cdot} - \bar{Y}_{i\cdot\cdot} - \bar{Y}_{\cdot j\cdot} + \bar{Y}_{\cdot\cdot\cdot}) = \bar{Y}_{ij\cdot} \tag{19.34}$$

so that the residuals are again:

$$e_{ijk} = Y_{ijk} - \bar{Y}_{ij\cdot} \tag{19.35}$$

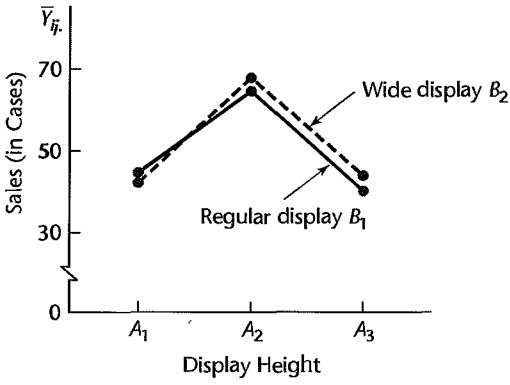


FIGURE 19.8 Estimated Treatment Means Plot—Castle Bakery Example.

Example

For the Castle Bakery example, the fitted values, i.e., the estimated treatment means $\bar{Y}_{ij.}$, are shown in Table 19.7. A plot of these estimated treatment means is presented in Figure 19.8. We see from this estimated treatment means plot that, for both display widths, mean sales for the middle display height are substantially larger than those for the other two display heights. The effect of display width does not appear to be large. Indeed, there may be no effect of display width; the variations between the estimated treatment means for any given display height may be solely of a random nature. In that event, there would be no interactions between display height and display width in their effects on sales.

Figure 19.8 differs from the earlier treatment means plots because the earlier figures presented the true treatment means μ_{ij}, while Figure 19.8 presents sample estimates. We therefore need to test whether or not the effects shown in Figure 19.8 are real effects or represent only random variations. To conduct these tests, we require a partitioning of the total sum of squares, to be discussed next.

Partitioning of Total Sum of Squares

Partitioning of Total Deviation. We shall partition the total deviation of an observation Y_{ijk} from the overall mean $\bar{Y}...$ in two stages. First, we shall obtain a decomposition of the total deviation $Y_{ijk} - \bar{Y}...$ by viewing the study as consisting of ab treatments:

$$\underbrace{Y_{ijk} - \bar{Y}...}_{\text{Total deviation}} = \underbrace{\bar{Y}_{ij.} - \bar{Y}...}_{\text{Deviation of estimated treatment mean around overall mean}} + \underbrace{Y_{ijk} - \bar{Y}_{ij.}}_{\text{Deviation around estimated treatment mean}} \tag{19.36}$$

Note that the deviation around the estimated treatment mean is simply the residual e_{ijk} in (19.35):

$$e_{ijk} = Y_{ijk} - \bar{Y}_{ij.}$$

Treatment and Error Sums of Squares. When we square (19.36) and sum over all cases, the cross-product term drops out and we obtain:

$$SSTO = SSTR + SSE \tag{19.37}$$

where:

$$SSTO = \sum_i \sum_j \sum_k (Y_{ijk} - \bar{Y}...)^2 \tag{19.37a}$$

$$SSTR = n \sum_i \sum_j (\bar{Y}_{ij\cdot} - \bar{Y}...)^2 \tag{19.37b}$$

$$SSE = \sum_i \sum_j \sum_k (Y_{ijk} - \bar{Y}_{ij\cdot})^2 = \sum_i \sum_j \sum_k e_{ijk}^2 \tag{19.37c}$$

SSTR reflects the variability between the *ab* estimated treatment means and is the ordinary *treatment sum of squares,* and *SSE* reflects the variability within treatments and is the usual *error sum of squares.* The only difference between these formulas and those for the single-factor case is the use of the two subscripts i and j to designate a treatment.

Example

For the Castle Bakery example, the decomposition of the total sum of squares in (19.37) is obtained as follows, using the data in Table 19.7:

$$SSTO = (47 - 51)^2 + (43 - 51)^2 + (46 - 51)^2 + \cdots + (46 - 51)^2 = 1{,}642$$
$$SSTR = 2[(45 - 51)^2 + (43 - 51)^2 + (65 - 51)^2 + \cdots + (44 - 51)^2] = 1{,}580$$
$$SSE = (47 - 45)^2 + (43 - 45)^2 + (46 - 43)^2 + \cdots + (46 - 44)^2 = 62$$

Partitioning of Treatment Sum of Squares. Next, we shall decompose the estimated treatment mean deviation $\bar{Y}_{ij\cdot} - \bar{Y}...$ in terms of components reflecting the factor A main effect, the factor B main effect, and the AB interaction effect:

$$\underbrace{\bar{Y}_{ij\cdot} - \bar{Y}...}_{\substack{\text{Deviation of estimated treatment mean around overall mean}}} = \underbrace{\bar{Y}_{i\cdot\cdot} - \bar{Y}...}_{\substack{A \text{ main effect}}} + \underbrace{\bar{Y}_{\cdot j\cdot} - \bar{Y}...}_{\substack{B \text{ main effect}}} + \underbrace{\bar{Y}_{ij\cdot} - \bar{Y}_{i\cdot\cdot} - \bar{Y}_{\cdot j\cdot} + \bar{Y}...}_{\substack{AB \text{ interaction effect}}} \tag{19.38}$$

When we square (19.38) and sum over all treatments and over the n cases associated with each estimated treatment mean $\bar{Y}_{ij\cdot}$, all cross-product terms drop out and we obtain:

$$SSTR = SSA + SSB + SSAB \tag{19.39}$$

where:

$$SSA = nb \sum_i (\bar{Y}_{i\cdot\cdot} - \bar{Y}...)^2 \tag{19.39a}$$

$$SSB = na \sum_j (\bar{Y}_{\cdot j\cdot} - \bar{Y}...)^2 \tag{19.39b}$$

$$SSAB = n \sum_i \sum_j (\bar{Y}_{ij\cdot} - \bar{Y}_{i\cdot\cdot} - \bar{Y}_{\cdot j\cdot} + \bar{Y}...)^2 \tag{19.39c}$$

The interaction sum of squares can also be obtained as a remainder:

$$SSAB = SSTO - SSE - SSA - SSB \tag{19.39d}$$

or from:

$$SSAB = SSTR - SSA - SSB \tag{19.39e}$$

where *SSTO* and *SSTR* are given in (19.37a) and (19.37b), respectively.

SSA, called the *factor A sum of squares*, measures the variability of the estimated factor *A* level means $\bar{Y}_{i\cdot\cdot}$. The more variable they are, the bigger will be *SSA*. Similarly, *SSB*, called the *factor B sum of squares*, measures the variability of the estimated factor *B* level means $\bar{Y}_{\cdot j\cdot}$. Finally, *SSAB*, called the *AB interaction sum of squares*, measures the variability of the estimated interactions $\bar{Y}_{ij\cdot} - \bar{Y}_{i\cdot\cdot} - \bar{Y}_{\cdot j\cdot} + \bar{Y}_{\cdot\cdot\cdot}$ for the *ab* treatments. Since the mean of all estimated interactions is zero, the deviations of the estimated interactions around their mean is not explicitly shown, as it was in *SSA* and *SSB*. The larger absolutely are the estimated interactions, the larger will be *SSAB*.

The partitioning of *SSTR* into the components *SSA*, *SSB*, and *SSAB* is called an *orthogonal decomposition*. An orthogonal decomposition is one where the component sums of squares add to the total sum of squares (*SSTR* here), and likewise for the degrees of freedom. Thus, the decompositions of *SSTO* into *SSTR* and *SSE* for single-factor and two-factor studies are also orthogonal decompositions. While many different orthogonal decompositions of *SSTR* are possible here, the one into the *SSA*, *SSB*, and *SSAB* components is of interest because these three components provide information about the factor *A* main effects, the factor *B* main effects, and the *AB* interactions, respectively, as will be seen shortly.

Example

For the Castle Bakery example, we obtain the following decomposition of *SSTR*, using the data in Table 19.7 and the formulas in (19.39):

$$\begin{aligned} SSA &= 2(2)[(44-51)^2 + (67-51)^2 + (42-51)^2] = 1{,}544 \\ SSB &= 2(3)[(50-51)^2 + (52-51)^2] = 12 \\ SSAB &= 1{,}580 - 1{,}544 - 12 = 24 \end{aligned}$$

Hence, we have:

$$\begin{aligned} 1{,}580 &= 1{,}544 + 12 + 24 \\ SSTR &= SSA + SSB + SSAB \end{aligned}$$

Combined Partitioning. Combining the decompositions in (19.37) and (19.39), we have established that:

$$SSTO = SSA + SSB + SSAB + SSE \tag{19.40}$$

where the component sums of squares are defined in (19.37) and (19.39).

Example

For the Castle Bakery example, we have found:

$$\begin{aligned} 1{,}642 &= 1{,}544 + 12 + 24 + 62 \\ SSTO &= SSA + SSB + SSAB + SSE \end{aligned}$$

Thus, much of the total variability in this instance is associated with the factor *A* (display height) effects.

Partitioning of Degrees of Freedom

We are familiar from single-factor analysis of variance with how the degrees of freedom are divided between the treatment and error components. For two-factor studies with n cases for each treatment, there are a total of $n_T = nab$ cases and $r = ab$ treatments; hence, the degrees of freedom associated with *SSTO*, *SSTR*, and *SSE* are $nab - 1$, $ab - 1$, and $nab - ab = (n - 1)ab$, respectively. These degrees of freedom for the Castle Bakery example are $2(3)(2) - 1 = 11$, $3(2) - 1 = 5$, and $(2 - 1)(3)(2) = 6$, respectively.

Corresponding to the further partitioning of the treatment sum of squares in (19.39), we can also obtain a breakdown of the associated $ab - 1$ degrees of freedom. *SSA* has $a - 1$ degrees of freedom associated with it. There are a factor level deviations $\bar{Y}_{i\cdot\cdot} - \bar{Y}_{\cdot\cdot\cdot}$, but one degree of freedom is lost because the deviations are subject to one restriction, i.e., $\sum(\bar{Y}_{i\cdot\cdot} - \bar{Y}_{\cdot\cdot\cdot}) = 0$. Similarly, *SSB* has $b - 1$ degrees of freedom associated with it. The degrees of freedom associated with *SSAB*, the interaction sum of squares, is the remainder:

$$(ab - 1) - (a - 1) - (b - 1) = (a - 1)(b - 1)$$

The degrees of freedom associated with *SSAB* may be understood as follows: There are ab interaction terms. These are subject to b restrictions since:

$$\sum_i (\bar{Y}_{ij\cdot} - \bar{Y}_{i\cdot\cdot} - \bar{Y}_{\cdot j\cdot} + \bar{Y}_{\cdot\cdot\cdot}) = 0 \qquad j = 1, \ldots, b$$

There are a additional restrictions since:

$$\sum_j (\bar{Y}_{ij\cdot} - \bar{Y}_{i\cdot\cdot} - \bar{Y}_{\cdot j\cdot} + \bar{Y}_{\cdot\cdot\cdot}) = 0 \qquad i = 1, \ldots, a$$

However, only $a - 1$ of these latter restrictions are independent since the last one is implied by the previous b restrictions. Altogether, therefore, there are $b + (a - 1)$ independent restrictions. Hence, the degrees of freedom are:

$$ab - (b + a - 1) = (a - 1)(b - 1)$$

Example For the Castle Bakery example, *SSA* has $3 - 1 = 2$ degrees of freedom associated with it, *SSB* has $2 - 1 = 1$ degree of freedom, and *SSAB* has $(3 - 1)(2 - 1) = 2$ degrees of freedom.

Mean Squares

Mean squares are obtained in the usual way by dividing the sums of squares by their associated degrees of freedom. We thus obtain:

$$MSA = \frac{SSA}{a - 1} \tag{19.41a}$$

$$MSB = \frac{SSB}{b - 1} \tag{19.41b}$$

$$MSAB = \frac{SSAB}{(a - 1)(b - 1)} \tag{19.41c}$$

Example For the Castle Bakery example, these mean squares are:

$$MSA = \frac{1{,}544}{2} = 772$$

$$MSB = \frac{12}{1} = 12$$

$$MSAB = \frac{24}{2} = 12$$

Expected Mean Squares

It can be shown, along the same lines used for single-factor ANOVA, that the mean squares for two-factor ANOVA model (19.23) have the following expectations:

$$E\{MSE\} = \sigma^2 \tag{19.42a}$$

$$E\{MSA\} = \sigma^2 + nb\frac{\sum\alpha_i^2}{a-1} = \sigma^2 + nb\frac{\sum(\mu_{i\cdot} - \mu_{\cdot\cdot})^2}{a-1} \tag{19.42b}$$

$$E\{MSB\} = \sigma^2 + na\frac{\sum\beta_j^2}{b-1} = \sigma^2 + na\frac{\sum(\mu_{\cdot j} - \mu_{\cdot\cdot})^2}{b-1} \tag{19.42c}$$

$$E\{MSAB\} = \sigma^2 + n\frac{\sum\sum(\alpha\beta)_{ij}^2}{(a-1)(b-1)} \tag{19.42d}$$

$$= \sigma^2 + n\frac{\sum\sum(\mu_{ij} - \mu_{i\cdot} - \mu_{\cdot j} + \mu_{\cdot\cdot})^2}{(a-1)(b-1)}$$

These expectations show that if there are no factor A main effects (i.e., if all $\mu_{i\cdot}$ are equal, or all $\alpha_i = 0$), *MSA* and *MSE* have the same expectation; otherwise *MSA* tends to be larger than *MSE*. Similarly, if there are no factor B main effects, *MSB* and *MSE* have the same expectation; otherwise *MSB* tends to be larger than *MSE*. Finally, if there are no interactions [i.e., if all $(\alpha\beta)_{ij} = 0$] so that the factor effects are additive, *MSAB* has the same expectation as *MSE*; otherwise, *MSAB* tends to be larger than *MSE*. This suggests that F^* test statistics based on the ratios *MSA/MSE*, *MSB/MSE*, and *MSAB/MSE* will provide information about the main effects and interactions of the two factors, with large values of the test statistics indicating the presence of factor effects. We shall see shortly that tests based on these statistics are regular F tests.

Analysis of Variance Table

The decomposition of the total sum of squares in (19.40) into the several factor and error components is shown in Table 19.8. Also shown there are the associated degrees of freedom, the mean squares, and the expected mean squares. Table 19.9 contains the two-factor analysis of variance for the Castle Bakery example.

Figure 19.9 presents MINITAB output for the Castle Bakery example. The first output block shows ANOVA results similar to those presented in Table 19.9. The second block presents various estimated means.

TABLE 19.8 ANOVA Table for Two-Factor Study with Fixed Factor Levels.

Source of Variation	SS	df	MS	E{MS}
Factor A	$SSA = nb\sum(\bar{Y}_{i\cdot\cdot} - \bar{Y}_{\cdot\cdot\cdot})^2$	$a-1$	$MSA = \dfrac{SSA}{a-1}$	$\sigma^2 + bn\dfrac{\sum(\mu_{i\cdot} - \mu_{\cdot\cdot})^2}{a-1}$
Factor B	$SSB = na\sum(\bar{Y}_{\cdot j\cdot} - \bar{Y}_{\cdot\cdot\cdot})^2$	$b-1$	$MSB = \dfrac{SSB}{b-1}$	$\sigma^2 + an\dfrac{\sum(\mu_{\cdot j} - \mu_{\cdot\cdot})^2}{b-1}$
AB interactions	$SSAB = n\sum\sum(\bar{Y}_{ij\cdot} - \bar{Y}_{i\cdot\cdot} - \bar{Y}_{\cdot j\cdot} + \bar{Y}_{\cdot\cdot\cdot})^2$	$(a-1)(b-1)$	$MSAB = \dfrac{SSAB}{(a-1)(b-1)}$	$\sigma^2 + n\dfrac{\sum\sum(\mu_{ij} - \mu_{i\cdot} - \mu_{\cdot j} + \mu_{\cdot\cdot})^2}{(a-1)(b-1)}$
Error	$SSE = \sum\sum\sum(Y_{ijk} - \bar{Y}_{ij\cdot})^2$	$ab(n-1)$	$MSE = \dfrac{SSE}{ab(n-1)}$	σ^2
Total	$SSTO = \sum\sum\sum(Y_{ijk} - \bar{Y}_{\cdot\cdot\cdot})^2$	$nab-1$		

TABLE 19.9 ANOVA Table for Two-Factor Study—Castle Bakery Example.

Source of Variation	*SS*	*df*	*MS*
Factor *A* (display height)	1,544	2	772
Factor *B* (display width)	12	1	12
AB interactions	24	2	12
Error	62	6	10.3
Total	1,642	11	

FIGURE 19.9 MINITAB Computer Output for Two-Factor Analysis of Variance—Castle Bakery Example.

Analysis of Variance for Cases Sold

Source	DF	SS	MS	F	P
Height	2	1544.00	772.00	74.71	0.000
Width	1	12.00	12.00	1.16	0.323
Height*Width	2	24.00	12.00	1.16	0.375
Error	6	62.00	10.33		
Total	11	1642.00			

Means

Height	N	Cases So
1	4	44.000
2	4	67.000
3	4	42.000

Width	N	Cases So
1	6	50.000
2	6	52.000

Height	Width	N	Cases So
1	1	2	45.000
1	2	2	43.000
2	1	2	65.000
2	2	2	69.000
3	1	2	40.000
3	2	2	44.000

19.5 Evaluation of Appropriateness of ANOVA Model

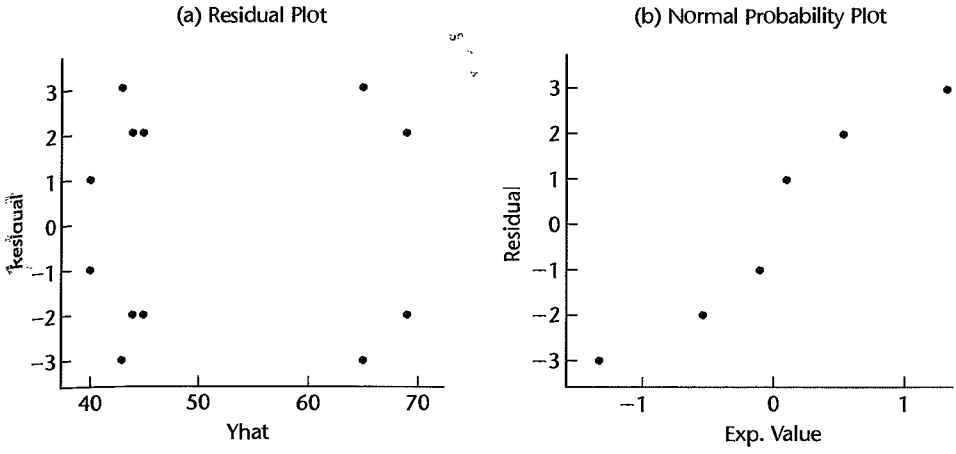
Before undertaking formal inference procedures, we need to evaluate the appropriateness of two-factor ANOVA model (19.23). No new problems arise here. The residuals in (19.35):

$$e_{ijk} = Y_{ijk} - \bar{Y}_{ij.}$$

are examined for normality, constancy of error variance, and independence of error terms in the same fashion as for a single-factor study.

Weighted least squares is a standard remedial measure when the error terms are normally distributed but do not have constant variance. When both the assumptions of normality and constancy of the error variance are violated, a transformation of the response variable may be sought to stabilize the error variance and to bring the distribution of the error terms closer to a normal distribution. Our discussion of these topics in Chapter 18 for single-factor ANOVA applies completely to two-factor ANOVA.

FIGURE 19.10 MINITAB Diagnostic Residual Plots—Castle Bakery Example.



Our earlier discussion on the effects of departures from the single-factor ANOVA model applies fully to two-factor ANOVA. In particular, the employment of equal sample sizes for each treatment minimizes the effect of unequal error variances.

Example

In the Castle Bakery example, there are only two replications for each treatment. Also, the data are rounded to keep the illustrative computations simple. As a result, the analysis of residuals will only be of limited value here. The residuals are obtained according to (19.35). Using the data in Table 19.7, we have, for instance:

$$e_{111} = 47 - 45 = 2$$
$$e_{121} = 46 - 43 = 3$$

A plot of the residuals against the fitted values $\hat{Y}_{ijk} = \bar{Y}_{ij\cdot}$ is presented in Figure 19.10a. There is no strong evidence of unequal error variances for the different treatments here. A normal probability plot of the residuals is presented in Figure 19.10b. The plot is moderately linear; the fact that only six plot points are visible is due to the rounded nature of the data. The coefficient of correlation between the ordered residuals and their expected values under normality is .966, which tends to support the reasonableness of approximate normality.

On the basis of these diagnostics and since the inference procedures for ANOVA model (19.23) are robust, it appears to be reasonable to proceed with tests for factor effects and other inference procedures.

19.6 *F* Tests

In view of the additivity of sums of squares and degrees of freedom, Cochran's theorem (2.61) applies when no factor effects are present. Hence, the F^* test statistics based on the appropriate mean squares then follow the F distribution, leading to the usual type of F tests for factor effects.

Test for Interactions

Ordinarily, the analysis of a two-factor study begins with a test to determine whether or not the two factors interact:

$$\begin{aligned} H_0&: \mu_{ij} - \mu_{i\cdot} - \mu_{\cdot j} + \mu_{\cdot\cdot} = 0 \qquad \text{for all } i, j \\ H_a&: \mu_{ij} - \mu_{i\cdot} - \mu_{\cdot j} + \mu_{\cdot\cdot} \neq 0 \qquad \text{for some } i, j \end{aligned} \tag{19.43}$$

or equivalently:

$$\begin{aligned} H_0&: \text{all } (\alpha\beta)_{ij} = 0 \\ H_a&: \text{not all } (\alpha\beta)_{ij} \text{ equal zero} \end{aligned} \tag{19.43a}$$

As we noted from an examination of the expected mean squares in Table 19.8, the appropriate test statistic is:

$$F^* = \frac{MSAB}{MSE} \tag{19.44}$$

Large values of F^* indicate the existence of interactions. When H_0 holds, F^* is distributed as $F[(a-1)(b-1), (n-1)ab]$. Hence, the appropriate decision rule to control the Type I error at α is:

$$\begin{aligned} &\text{If } F^* \leq F[1-\alpha; (a-1)(b-1), (n-1)ab], \text{ conclude } H_0 \\ &\text{If } F^* > F[1-\alpha; (a-1)(b-1), (n-1)ab], \text{ conclude } H_a \end{aligned} \tag{19.45}$$

where $F[1-\alpha; (a-1)(b-1), (n-1)ab]$ is the $(1-\alpha)100$ percentile of the appropriate F distribution.

Test for Factor *A* Main Effects

Tests for factor A main effects and for factor B main effects ordinarily follow the test for interactions when no important interactions exist. To test whether or not A main effects are present:

$$\begin{aligned} H_0&: \mu_{1\cdot} = \mu_{2\cdot} = \cdots = \mu_{a\cdot} \\ H_a&: \text{not all } \mu_{i\cdot} \text{ are equal} \end{aligned} \tag{19.46}$$

or equivalently:

$$\begin{aligned} H_0&: \alpha_1 = \alpha_2 = \cdots = \alpha_a = 0 \\ H_a&: \text{not all } \alpha_i \text{ equal zero} \end{aligned} \tag{19.46a}$$

we use the test statistic:

$$F^* = \frac{MSA}{MSE} \tag{19.47}$$

Again, large values of F^* indicate the existence of factor A main effects. Since F^* is distributed as $F[a-1, (n-1)ab]$ when H_0 holds, the appropriate decision rule for controlling the risk of making a Type I error at α is:

$$\begin{aligned} &\text{If } F^* \leq F[1-\alpha; a-1, (n-1)ab], \text{ conclude } H_0 \\ &\text{If } F^* > F[1-\alpha; a-1, (n-1)ab], \text{ conclude } H_a \end{aligned} \tag{19.48}$$

Test for Factor *B* Main Effects

This test is similar to the one for factor A main effects. The alternatives are:

$$H_0: \mu_{\cdot 1} = \mu_{\cdot 2} = \cdots = \mu_{\cdot b}$$
$$H_a: \text{not all } \mu_{\cdot j} \text{ are equal} \tag{19.49}$$

or equivalently:

$$H_0: \beta_1 = \beta_2 = \cdots = \beta_b = 0$$
$$H_a: \text{not all } \beta_j \text{ equal zero} \tag{19.49a}$$

The test statistic is:

$$F^* = \frac{MSB}{MSE} \tag{19.50}$$

and the appropriate decision rule for controlling the risk of a Type I error at α is:

$$\text{If } F^* \leq F[1-\alpha; b-1, (n-1)ab], \text{ conclude } H_0$$
$$\text{If } F^* > F[1-\alpha; b-1, (n-1)ab], \text{ conclude } H_a \tag{19.51}$$

Example

We shall investigate in the Castle Bakery example the presence of display height and display width effects, using a level of significance of $\alpha = .05$ for each test. First, we begin by testing whether or not interaction effects are present:

$$H_0: \text{all } (\alpha\beta)_{ij} = 0$$
$$H_a: \text{not all } (\alpha\beta)_{ij} \text{ equal zero}$$

Using the ANOVA results from Table 19.9 in test statistic (19.44), we obtain:

$$F^* = \frac{12}{10.3} = 1.17$$

For $\alpha = .05$, we require $F(.95; 2, 6) = 5.14$, so that the decision rule is:

$$\text{If } F^* \leq 5.14, \text{ conclude } H_0$$
$$\text{If } F^* > 5.14, \text{ conclude } H_a$$

Since $F^* = 1.17 \leq 5.14$, we conclude H_0, that display height and display width do not interact in their effects on sales. The P-value of this test is $P\{F(2, 6) > 1.17\} = .37$.

Since the two factors do not interact, we turn to test for display height (factor A) main effects; the alternative conclusions are given in (19.46). Test statistic (19.47) for our example becomes:

$$F^* = \frac{772}{10.3} = 75.0$$

For $\alpha = .05$, we require $F(.95; 2, 6) = 5.14$. Since $F^* = 75.0 > 5.14$, we conclude H_a, that the factor A level means $\mu_{i\cdot}$ are not equal, or that some definite effects associated with height of display level exist. The P-value of this test is $P\{F(2, 6) > 75.0\} = .0001$.

Next, we test for display width (factor B) main effects; the alternative conclusions are given in (19.49). Test statistic (19.50) becomes for our example:

$$F^* = \frac{12}{10.3} = 1.17$$

For $\alpha = .05$, we require $F(.95; 1, 6) = 5.99$. Since $F^* = 1.17 \leq 5.99$, we conclude H_0, that all $\mu_{\cdot j}$ are equal, or that display width has no effect on sales. The P-value of this test is $P\{F(1, 6) > 1.17\} = .32$.

Thus, the analysis of variance tests confirm the impressions from the estimated treatment means plot in Figure 19.8 that only display height has an effect on sales for the treatments studied. At this point, it is clearly desirable to conduct further analyses of the nature of the display height effects. We shall discuss analyses of the nature of the factor effects in Sections 19.8 and 19.9.

Kimball Inequality

If the test for interactions is conducted with level of significance α_1, that for factor A main effects with level of significance α_2, and that for factor B main effects with level of significance α_3, the level of significance α for the *family* of three tests is greater than the individual levels of significance. From the Bonferroni inequality in (4.4), we can derive the inequality:

$$\alpha \leq \alpha_1 + \alpha_2 + \alpha_3 \tag{19.52}$$

For the case considered here, a somewhat tighter inequality can be used, the *Kimball inequality*, which utilizes the fact that the numerators of the three test statistics are independent and the denominator is the same in each case. This inequality states:

$$\alpha \leq 1 - (1 - \alpha_1)(1 - \alpha_2)(1 - \alpha_3) \tag{19.53}$$

For the Castle Bakery example, where $\alpha_1 = \alpha_2 = \alpha_3 = .05$, the Bonferroni inequality yields as the bound for the family level of significance:

$$\alpha \leq .05 + .05 + .05 = .15$$

while the Kimball inequality yields the bound:

$$\alpha \leq 1 - (.95)(.95)(.95) = .143$$

This illustration makes it clear that the level of significance for the family of three tests may be substantially higher than the levels of significance for the individual tests.

Comment

The F^* test statistics in (19.44), (19.47), and (19.50) can be obtained by the general linear test approach explained in Chapter 2. For example, in testing for the presence of interaction effects, the alternatives are those given in (19.43) and the full model is ANOVA model (19.23):

$$Y_{ijk} = \mu_{\cdot\cdot} + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijk} \qquad \text{Full model} \tag{19.54}$$

Fitting this full model leads to the fitted values $\hat{Y}_{ijk} = \bar{Y}_{ij\cdot}$ and the error sum of squares:

$$SSE(F) = \sum\sum\sum(Y_{ijk} - \hat{Y}_{ijk})^2 = \sum\sum\sum(Y_{ijk} - \bar{Y}_{ij\cdot})^2 = SSE \tag{19.55}$$

which is the usual ANOVA error sum of squares in (19.37c). This error sum of squares has $ab(n-1)$ degrees of freedom associated with it.

The reduced model under H_0: $(\alpha\beta)_{ij} \equiv 0$ is:

$$Y_{ijk} = \mu_{..} + \alpha_i + \beta_j + \varepsilon_{ijk} \qquad \text{Reduced model} \tag{19.56}$$

It can be shown that the fitted values for the reduced model are $\hat{Y}_{ijk} = \bar{Y}_{i..} + \bar{Y}_{.j.} - \bar{Y}_{...}$, so that the error sum of squares for the reduced model is:

$$SSE(R) = \sum\sum\sum(Y_{ijk} - \hat{Y}_{ijk})^2 = \sum\sum\sum(Y_{ijk} - \bar{Y}_{i..} - \bar{Y}_{.j.} + \bar{Y}_{...})^2 \tag{19.57}$$

This error sum of squares can be shown to have $nab - a - b + 1$ degrees of freedom associated with it. Test statistic (2.70) then simplifies to $F^* = MSAB/MSE$ in (19.44). ■

19.7 Strategy for Analysis

Scientific inquiry is often guided by the principle that the simplest explanations of observed phenomena tend to be the most effective. Data analysis is guided by this principle, seeking to obtain a simple, clear explanation of the data. In the context of ANOVA studies, additive effects provide a much simpler explanation of factor effects than do interacting effects. The presence of interacting effects complicates the explanation of the factor effects because they must then be described in terms of the *combined* effects of the two factors. Of course, some phenomena are complex so that the factor effects cannot be described simply by additive effects. The desire for a simple, parsimonious explanation, when possible, suggests the following basic strategy for analyzing factor effects in two-factor studies:

1. Examine whether the two factors interact.
2. If they do not interact, examine whether the main effects for factors A and B are important. For important A or B main effects, describe the nature of these effects in terms of the factor level means $\mu_{i.}$ or $\mu_{.j}$, respectively. In some special cases, there may also be interest in the treatment means μ_{ij}.
3. If the factors do interact, examine if the interactions are important or unimportant.
4. If the interactions are unimportant, proceed as in step 2.
5. If the interactions are important, consider whether they can be made unimportant by a meaningful simple transformation of scale. If so, make the transformation and proceed as in step 2.
6. For important interactions that cannot be made unimportant by a simple transformation, analyze the two factor effects jointly in terms of the treatment means μ_{ij}. In some special cases, there may also be interest in the factor level means $\mu_{i.}$ and $\mu_{.j}$.

A flowchart of this strategy is presented in Figure 19.11.

We have already discussed the testing for interaction effects, the possible diminution of important interactions by a meaningful simple transformation, as well as how to test for the presence of factor main effects. Now we turn to steps 2 and 6 of the strategy for analysis, namely, how to compare factor level means $\mu_{i.}$ or $\mu_{.j}$ when there are no interactions or only unimportant ones, and how to compare treatment means μ_{ij} when there are important interactions. We begin with a discussion of the analysis of factor effects when the factors do not interact or interact only in an unimportant fashion.

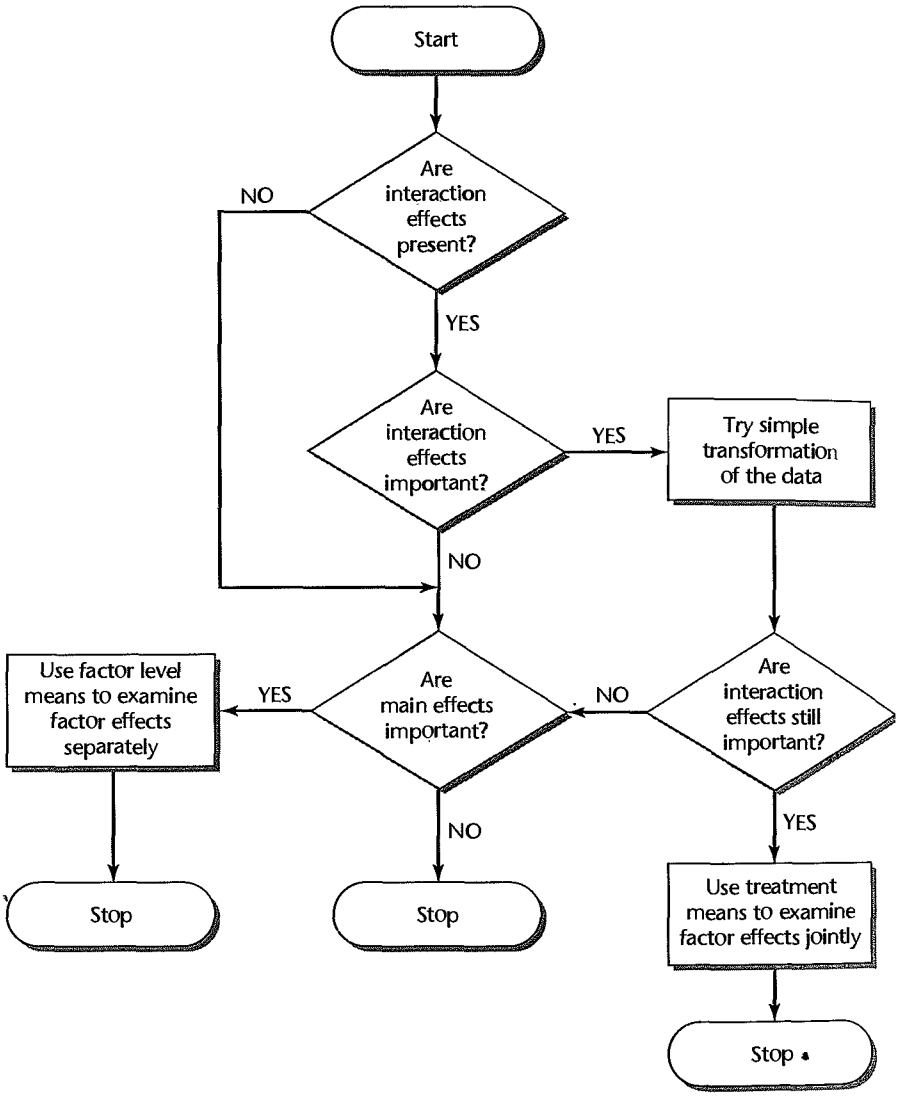


FIGURE 19.11
Strategy for Analysis of Two-Factor Studies.

19.8 Analysis of Factor Effects when Factors Do Not Interact

As just noted, the analysis of factor effects usually only involves the factor level mea and $\mu_{\cdot j}$ when the two factors do not interact, or when they interact only in an unimp fashion.

Estimation of Factor Level Mean

Unbiased point estimators of $\mu_{i\cdot}$ and $\mu_{\cdot j}$ are:

$$\hat{\mu}_{i\cdot} = \bar{Y}_{i\cdot}$$

$$\hat{\mu}_{\cdot j} = \bar{Y}_{\cdot j}$$

where $\bar{Y}_{i\cdot\cdot}$ and $\bar{Y}_{\cdot j\cdot}$ are defined in (19.27d) and (19.27f), respectively. The variance of $\bar{Y}_{i\cdot\cdot}$ is:

$$\sigma^2\{\bar{Y}_{i\cdot\cdot}\} = \frac{\sigma^2}{bn} \tag{19.58a}$$

since $\bar{Y}_{i\cdot\cdot}$ contains bn independent observations, each with variance σ^2. Similarly, we have:

$$\sigma^2\{\bar{Y}_{\cdot j\cdot}\} = \frac{\sigma^2}{an} \tag{19.58b}$$

Unbiased estimators of these variances are obtained by replacing σ^2 with MSE:

$$s^2\{\bar{Y}_{i\cdot\cdot}\} = \frac{MSE}{bn} \tag{19.59a}$$

$$s^2\{\bar{Y}_{\cdot j\cdot}\} = \frac{MSE}{an} \tag{19.59b}$$

Confidence limits for $\mu_{i\cdot}$ and $\mu_{\cdot j}$ utilize, as usual, the t distribution:

$$\bar{Y}_{i\cdot\cdot} \pm t[1-\alpha/2;\,(n-1)ab]s\{\bar{Y}_{i\cdot\cdot}\} \tag{19.60a}$$

$$\bar{Y}_{\cdot j\cdot} \pm t[1-\alpha/2;\,(n-1)ab]s\{\bar{Y}_{\cdot j\cdot}\} \tag{19.60b}$$

The degrees of freedom $(n-1)ab$ are those associated with MSE.

Estimation of Contrast of Factor Level Means

A contrast among the factor level means $\mu_{i\cdot}$:

$$L = \sum c_i \mu_{i\cdot} \qquad \text{where } \sum c_i = 0 \tag{19.61}$$

is estimated unbiasedly by:

$$\hat{L} = \sum c_i \bar{Y}_{i\cdot\cdot} \tag{19.62}$$

Because of the independence of the $\bar{Y}_{i\cdot\cdot}$, the variance of this estimator is:

$$\sigma^2\{\hat{L}\} = \sum c_i^2 \sigma^2\{\bar{Y}_{i\cdot\cdot}\} = \frac{\sigma^2}{bn}\sum c_i^2 \tag{19.63}$$

An unbiased estimator of this variance is:

$$s^2\{\hat{L}\} = \frac{MSE}{bn}\sum c_i^2 \tag{19.64}$$

Finally, the appropriate $1-\alpha$ confidence limits for L are:

$$\hat{L} \pm t[1-\alpha/2;\,(n-1)ab]s\{\hat{L}\} \tag{19.65}$$

To estimate a contrast among the factor level means $\mu_{\cdot j}$:

$$L = \sum c_j \mu_{\cdot j} \qquad \text{where } \sum c_j = 0 \tag{19.66}$$

we use the estimator:

$$\hat{L} = \sum c_j \bar{Y}_{\cdot j\cdot} \tag{19.67}$$

whose estimated variance is:

$$s^2\{\hat{L}\} = \frac{MSE}{an}\sum c_j^2 \tag{19.68}$$

The $1 - \alpha$ confidence limits for L in (19.65) are still appropriate, with $\hat{L}$ and $s\{\hat{L}\}$ now defined in (19.67) and (19.68), respectively.

Estimation of Linear Combination of Factor Level Means

A linear combination of the factor level means $\mu_{i\cdot}$:

$$L = \sum c_i \mu_{i\cdot} \tag{19.69}$$

is estimated unbiasedly by $\hat{L}$ in (19.62). The variance of this estimator is given in (19.63), and an unbiased estimator of this variance is given in (19.64). The appropriate $1 - \alpha$ confidence limits for L are given in (19.65).

Analogous results follow for a linear combination of the factor level means $\mu_{\cdot j}$:

$$L = \sum c_j \mu_{\cdot j} \tag{19.70}$$

Multiple Pairwise Comparisons of Factor Level Means

Usually, more than one pairwise comparison is of interest, and the multiple comparison procedures discussed in Chapter 17 for single-factor ANOVA studies can be employed with only minor modifications for two-factor studies. If all or a large number of pairwise comparisons among the factor level means $\mu_{i\cdot}$ or $\mu_{\cdot j}$ are to be made, the Tukey procedure of Section 17.5 is appropriate. When only a few pairwise comparisons are to be made that are specified in advance of the analysis, the Bonferroni procedure of Section 17.7 may be best. Often, tests for differences between pairs of factor level means precede the construction of interval estimates so that the analysis of the interval estimates can be confined to active comparisons. Finally, when a large number of comparisons among the factor-level means is of interest, the Scheffé method is usually preferred.

Tukey Procedure. The Tukey multiple comparison confidence limits for all pairwise comparisons:

$$D = \mu_{i\cdot} - \mu_{i'\cdot} \tag{19.71}$$

with family confidence coefficient of at least $1 - \alpha$ are:

$$\hat{D} \pm Ts\{\hat{D}\} \tag{19.72}$$

where:

$$\hat{D} = \bar{Y}_{i\cdot\cdot} - \bar{Y}_{i'\cdot\cdot} \tag{19.72a}$$

$$s^2\{\hat{D}\} = \frac{2MSE}{bn} \tag{19.72b}$$

$$T = \frac{1}{\sqrt{2}} q[1 - \alpha; a, (n-1)ab] \tag{19.72c}$$

To use the Tukey procedure to conduct all simultaneous tests of the form:

$$\begin{aligned} H_0&: D = \mu_{i\cdot} - \mu_{i'\cdot} = 0 \\ H_a&: D = \mu_{i\cdot} - \mu_{i'\cdot} \neq 0 \end{aligned} \tag{19.73}$$

the test statistic and decision rule are:

$$q^* = \frac{\sqrt{2}\hat{D}}{s\{\hat{D}\}}; \qquad \text{If } |q^*| > q[1-\alpha; a, (n-1)ab], \text{ conclude } H_a \tag{19.73a}$$

For conciseness in this chapter, we state only the portion of the decision rule leading to conclusion H_a. As for single-factor ANOVA, the family level of significance for all pairwise tests here is $1-\alpha$; in other words, the probability of concluding that there exist any pairwise differences when there are none is α.

For pairwise comparisons of the factor level means $\mu_{\cdot j}$, the only changes are:

$$D = \mu_{\cdot j} - \mu_{\cdot j'} \tag{19.74}$$

$$\hat{D} = \bar{Y}_{\cdot j} - \bar{Y}_{\cdot j'} \tag{19.75}$$

$$s^2\{\hat{D}\} = \frac{2MSE}{an} \tag{19.76}$$

$$T = \frac{1}{\sqrt{2}} q[1-\alpha; b, (n-1)ab] \tag{19.77}$$

$$q^* = \frac{\sqrt{2}\hat{D}}{s\{\hat{D}\}}; \qquad \text{If } |q^*| > q[1-\alpha; b, (n-1)ab], \text{ conclude } H_a \tag{19.78}$$

Bonferroni Procedure. When only a few pairwise comparisons specified in advance are to be made, the Bonferroni method may be best. The simultaneous estimation formulas above still apply, with the Tukey multiple T replaced by the Bonferroni multiple B:

$$B = t[1-\alpha/2g; (n-1)ab] \tag{19.79}$$

where g is the number of statements in the family.

To test simultaneously each of g pairwise differences with the Bonferroni procedure, the test statistic and decision rule are:

$$t^* = \frac{\hat{D}}{s\{\hat{D}\}}; \qquad \text{If } |t^*| > t[1-\alpha/2g; (n-1)ab], \text{ conclude } H_a \tag{19.80}$$

Combined Factor *A* and Factor *B* Family. When important factor A and factor B effects both are present, it is often desired to have a family confidence coefficient $1-\alpha$, or family significance level α, for the joint set of pairwise comparisons involving *both* factor A and factor B means. The Bonferroni method can be used directly for this purpose, with g representing the total number of statements in the joint set.

Alternatively, the Bonferroni method can be used in conjunction with the Tukey method. To illustrate this use, if the pairwise comparisons for factor A are made with the Tukey procedure with a family confidence coefficient of .95, and likewise for the pairwise comparisons for factor B, the Bonferroni inequality then assures us that the family confidence coefficient for the joint set of comparisons for both factors is at least .90.

Multiple Contrasts of Factor Level Means

Scheffé Procedure. When a large number of contrasts among the factor level mean $\mu_{i\cdot}$ or $\mu_{\cdot j}$ are of interest, the Scheffé method should be used. If the contrasts involve the $\mu_{i\cdot}$ as in (19.61), the Scheffé confidence limits are:

$$\hat{L} \pm Ss\{\hat{L}\} \tag{19.81}$$

where:

$$S^2 = (a-1)F[1-\alpha; a-1, (n-1)ab] \tag{19.81a}$$

and $\hat{L}$ is given by (19.62) and $s^2\{\hat{L}\}$ is given by (19.64). The probability is then $1-\alpha$ that every confidence interval (19.81) in the family of all possible contrasts is correct. If the contrasts involve the $\mu_{\cdot j}$ as in (19.66), $\hat{L}$ is given by (19.67), $s^2\{\hat{L}\}$ is given by (19.68), and the Scheffé multiple in (19.81) is defined by:

$$S^2 = (b-1)F[1-\alpha; b-1, (n-1)ab] \tag{19.81b}$$

When the Scheffé procedure is employed to conduct simultaneous tests of the form:

$$\begin{aligned} H_0&: L = 0 \\ H_a&: L \neq 0 \end{aligned} \tag{19.82}$$

for contrasts involving the factor level means $\mu_{i\cdot}$, the test statistic and decision rule are:

$$F^* = \frac{\hat{L}^2}{(a-1)s^2\{\hat{L}\}}; \qquad \text{If } F^* > F[1-\alpha; a-1, (n-1)ab], \text{ conclude } H_a \tag{19.82a}$$

When the contrasts involve the factor level means $\mu_{\cdot j}$, the test statistic and decision rule are:

$$F^* = \frac{\hat{L}^2}{(b-1)s^2\{\hat{L}\}}; \qquad \text{If } F^* > F[1-\alpha; b-1, (n-1)ab], \text{ conclude } H_a \tag{19.82b}$$

Bonferroni Procedure. When the number of contrasts of interest is small and has been specified in advance, the Bonferroni procedure may be best. Confidence limits (19.81) are modified by replacing the Scheffé multiple S with the Bonferroni multiple B:

$$B = t[1-\alpha/2g; (n-1)ab] \tag{19.83}$$

where g is the number of statements in the family.

Simultaneous testing of g tests with the Bonferroni procedure is based on the following test statistic and decision rule:

$$t^* = \frac{\hat{L}}{s\{\hat{L}\}}; \qquad \text{If } |t^*| > t[1-\alpha/2g; (n-1)ab], \text{ conclude } H_a \tag{19.84}$$

Combined Factor *A* and Factor *B* Family. When important factor A and factor B effects are present and contrasts for each of the two factors are of interest, it is often desired that the inference procedure provide assurance for the combined family of factor A and factor B contrasts. Several possibilities exist to accomplish this:

1. The Bonferroni method may be used directly, with g representing the total number of statements in the joint set.

2. The Bonferroni method can be used to join the two sets of Scheffé multiple comparison families in the same way explained earlier for joining two Tukey sets.
3. The Scheffé confidence limits (19.81) can be modified to use the S multiple defined by:

$$S^2 = (a+b-2)F[1-\alpha; a+b-2, (n-1)ab] \qquad \textbf{(19.85)}$$

For simultaneous testing, the test statistics and decision rules in (19.82a) and (19.82b) can be replaced by:

$$F^* = \frac{\hat{L}^2}{(a+b-2)s^2\{\hat{L}\}}; \qquad \text{If } F^* > F[1-\alpha; a+b-2, (n-1)ab], \text{ conclude } H_a \qquad \textbf{(19.86)}$$

Estimates Based on Treatment Means

Occasionally in analyzing the factor effects in a two-factor study when no interactions are present, there is interest in particular treatment means μ_{ij}. For example, in a two-factor study of the effects of price and type of advertisement on sales, interest may exist in estimating the mean sales for two different price levels when a particular advertisement is used. In such cases, the methods of analysis for single-factor studies discussed in Chapter 17 are appropriate. The number of treatments now is simply $r = ab$, the degrees of freedom associated with MSE are $n_T - r = nab - ab = (n-1)ab$, and the estimated treatment means are $\bar{Y}_{ij\cdot}$, based on n observations each.

Example 1—Pairwise Comparisons of Factor Level Means

In the Castle Bakery, the estimated treatment means plot in Figure 19.8 suggested that no interaction effects are present and that display width may not have any effect. The formal analysis of variance based on Table 19.9 supported both of these conclusions. Our interest now is in examining the nature of the display height effects in more detail.

First, we shall obtain a preliminary view of the display height and width effects by plotting bar graphs of the estimated factor level means in Table 19.7. Figure 19.12a contains a bar graph of the estimated factor A level means $\bar{Y}_{i\cdot\cdot}$. For comparison, we show in Figure 19.12b a similar plot for the estimated factor B level means $\bar{Y}_{\cdot j\cdot}$. Figure 19.12a suggests that level 2 of factor A (middle shelf display height) leads to significantly larger sales than the other

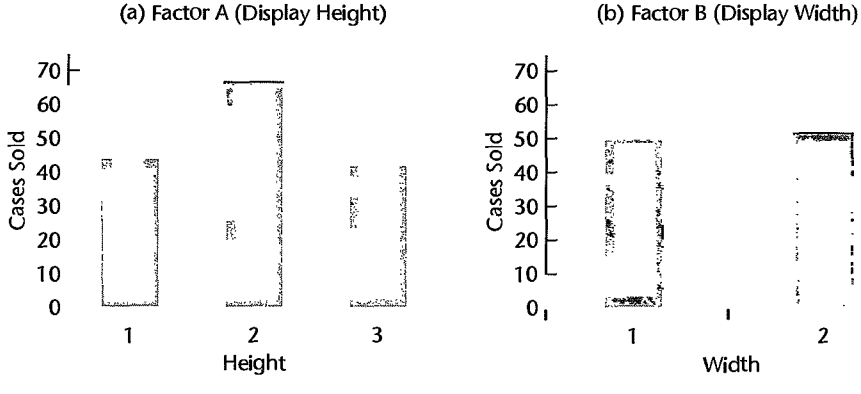


FIGURE 19.12 Bar Graphs of Estimated Factor Level Means—Castle Bakery Example.

TABLE 19.10 **Pairwise Testing of Factor *A* Level Means—Castle Bakery Example.**

(1) Alternatives	(2) Test Statistic (19.73a)	(3) Decision Rule Conclude H_a if $\lvert q^*\rvert >$	(4) Conclusion
H_0: $D_1 = \mu_{2\cdot} - \mu_{1\cdot} = 0$ H_a: $D_1 = \mu_{2\cdot} - \mu_{1\cdot} \neq 0$	$q^* = \frac{\sqrt{2}(23)}{2.27} = 14.33$	$q(.95; 3, 6) = 4.34$	H_a
H_0: $D_2 = \mu_{1\cdot} - \mu_{3\cdot} = 0$ H_a: $D_2 = \mu_{1\cdot} - \mu_{3\cdot} \neq 0$	$q^* = \frac{\sqrt{2}(2)}{2.27} = 1.25$	$q(.95; 3, 6) = 4.34$	H_0
H_0: $D_3 = \mu_{2\cdot} - \mu_{3\cdot} = 0$ H_a: $D_3 = \mu_{2\cdot} - \mu_{3\cdot} \neq 0$	$q^* = \frac{\sqrt{2}(25)}{2.27} = 15.58$	$q(.95; 3, 6) = 4.34$	H_a

two factor levels. In addition, Figure 19.12a also suggests that the mean sales for display height levels 1 and 3 may not be different from each other.

Turning now to formal inference procedures, we shall first test simultaneously all pairwise differences among the shelf height means, using the Tukey multiple comparison procedure with family significance level $\alpha = .05$. The alternatives to be tested for the comparisons of display height means ($i = 1$—bottom, 2—middle, 3—top) are shown in Table 19.10, column 1. From Tables 19.7 and 19.9 we obtain the following information:

$$\hat{D}_1 = \bar{Y}_{2\cdot\cdot} - \bar{Y}_{1\cdot\cdot} = 67 - 44 = 23 \qquad MSE = 10.3$$
$$a = 3$$
$$\hat{D}_2 = \bar{Y}_{1\cdot\cdot} - \bar{Y}_{3\cdot\cdot} = 44 - 42 = 2 \qquad b = 2$$
$$n = 2$$
$$\hat{D}_3 = \bar{Y}_{2\cdot\cdot} - \bar{Y}_{3\cdot\cdot} = 67 - 42 = 25 \qquad (n-1)ab = 6$$

Hence, by (19.72b) we obtain:

$$s^2\{\hat{D}_1\} = s^2\{\hat{D}_2\} = s^2\{\hat{D}_3\} = \frac{2(10.3)}{2(2)} = 5.15$$

so that $s\{\hat{D}_1\} = s\{\hat{D}_2\} = s\{\hat{D}_3\} = 2.27$. The test statistics and decision rules based on (19.73a) are given in Table 19.10, columns 2 and 3, and the conclusions from the tests are shown in column 4.

It can be concluded from the tests in Table 19.10 with family significance level $\alpha = .05$ that for the product studied and the types of stores in the experiment, the middle shelf height is far better than either the bottom or the top heights, and that the latter two do not differ significantly in sales effectiveness. All of these conclusions are covered by the family significance level of .05.

Next, we wish to estimate how much greater are mean sales at the middle shelf height than at either of the other two shelf heights. We shall continue to use the Tukey multiple comparison procedure because the two pairwise comparisons now of interest are the result of the earlier testing of all pairwise comparisons. From our previous work, we have:

$$\hat{D}_1 = \bar{Y}_{2\cdot\cdot} - \bar{Y}_{1\cdot\cdot} = 23 \qquad \hat{D}_3 = \bar{Y}_{2\cdot\cdot} - \bar{Y}_{3\cdot\cdot} = 25 \qquad s\{\hat{D}_1\} = s\{\hat{D}_3\} = 2.27$$

We also require, from (19.72):

$$q(.95; 3, 6) = 4.34$$

$$T = \frac{4.34}{\sqrt{2}} = 3.07$$

$$Ts\{\hat{D}_1\} = Ts\{\hat{D}_3\} = 3.07(2.27) = 7.0$$

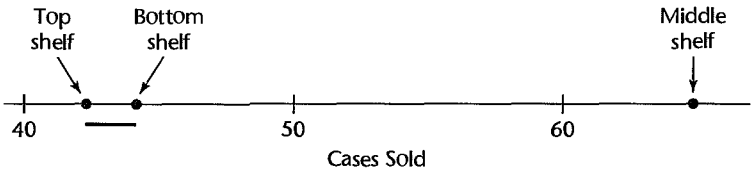
We therefore find the following confidence intervals for the two pairwise comparisons of the shelf height factor level means:

$$16 = 23 - 7.0 \leq \mu_{2\cdot} - \mu_{1\cdot} \leq 23 + 7.0 = 30$$

$$18 = 25 - 7.0 \leq \mu_{2\cdot} - \mu_{3\cdot} \leq 25 + 7.0 = 32$$

With family confidence coefficient of .95, we conclude that mean sales for the middle shelf height exceed those for the bottom shelf height by between 16 and 30 cases and those for the top shelf height by between 18 and 32 cases.

We can summarize the effects of shelf height on mean sales by the following line plot:



Example 2—Estimation of Treatment Means

The manager of a supermarket that has sales volume and clientele similar to the supermarkets included in the Castle Bakery study has room only for the regular shelf display width, and wishes to obtain estimates of mean sales for the middle and top shelf heights. We shall now obtain interval estimates with a 90 percent family confidence coefficient using the Bonferroni procedure.

From Tables 19.7 and 19.9, we have:

$$\bar{Y}_{21\cdot} = 65 \qquad \bar{Y}_{31\cdot} = 40 \qquad MSE = 10.3$$

Hence, we obtain:

$$s^2\{\bar{Y}_{21\cdot}\} = s^2\{\bar{Y}_{31\cdot}\} = \frac{MSE}{n} = \frac{10.3}{2} = 5.15$$

$$s\{\bar{Y}_{21\cdot}\} = s\{\bar{Y}_{31\cdot}\} = 2.27$$

For $g = 2$, we require $B = t[1 - \alpha/2g; (n-1)ab] = t(.975; 6) = 2.447$. Thus, we obtain the confidence limits:

$$65 \pm 2.447(2.27) \qquad 40 \pm 2.447(2.27)$$

and the desired confidence intervals are:

$$59.4 \leq \mu_{21} \leq 70.6 \qquad 34.4 \leq \mu_{31} \leq 45.6$$

19.9 Analysis of Factor Effects when Interactions Are Important

When important interactions exist that cannot be made unimportant by a simple transformation, the analysis of factor effects generally must be based on the treatment means μ_{ij}. Typically, this analysis will involve estimation of multiple comparisons of treatment means or single degree of freedom tests. Furthermore, one often compares the levels of one factor across levels of the other factor, referred to as the comparison of simple effects. For example, in a 2×3 factorial structure study, we compare individual cell means within levels of each factor, e.g., $\mu_{11} = \mu_{12} = \mu_{13}$ and $\mu_{21} = \mu_{22} = \mu_{23}$ and/or $\mu_{11} = \mu_{21}$, $\mu_{12} = \mu_{22}$, and $\mu_{13} = \mu_{23}$.

Multiple Pairwise Comparisons of Treatment Means

If pairs of treatment means μ_{ij} are to be compared, either the Tukey or the Bonferroni multiple comparison procedure may be used, depending on which is more advantageous. In effect, the analysis is equivalent to that for single-factor ANOVA, with the total number of treatments here equal to $r = ab$, the degrees of freedom associated with *MSE* here equal to $n_T - r = (n-1)ab$, and each estimated treatment mean, now denoted by $\bar{Y}_{ij\cdot}$, based on n cases.

Tukey Procedure. The Tukey $1-\alpha$ multiple comparison confidence limits for all pairwise comparisons:

$$D = \mu_{ij} - \mu_{i'j'} \qquad i, j \neq i', j' \tag{19.87}$$

are:

$$\hat{D} \pm Ts\{\hat{D}\} \tag{19.88}$$

where:

$$\hat{D} = \bar{Y}_{ij\cdot} - \bar{Y}_{i'j'\cdot} \tag{19.88a}$$

$$s^2\{\hat{D}\} = \frac{2MSE}{n} \tag{19.88b}$$

$$T = \frac{1}{\sqrt{2}} q[1-\alpha; ab, (n-1)ab] \tag{19.88c}$$

The test statistic and decision rule for all simultaneous Tukey tests of the form:

$$\begin{aligned} H_0&: D = 0 \\ H_a&: D \neq 0 \end{aligned} \tag{19.89}$$

are as follows when the family significance level is controlled at α:

$$q^* = \frac{\sqrt{2}\hat{D}}{s\{\hat{D}\}}; \qquad \text{If } |q^*| > q[1-\alpha; ab, (n-1)ab], \text{ conclude } H_a \tag{19.89a}$$

Bonferroni Procedure. If the Bonferroni method is employed for a family of g comparisons, the multiple T in confidence interval (19.88) is replaced by:

$$B = t[1-\alpha/2g; (n-1)ab] \tag{19.90}$$

and the test statistic and decision rule in (19.89a) become:

$$t^* = \frac{\hat{D}}{s\{\hat{D}\}}; \qquad \text{If } |t^*| > t[1-\alpha/2g;\,(n-1)ab], \text{ conclude } H_a \qquad \textbf{(19.91)}$$

Multiple Contrasts of Treatment Means

Scheffé Procedure. The Scheffé multiple comparison procedure for single-factor studies is directly applicable to the estimation of contrasts involving the treatment means μ_{ij}. The joint confidence limits for contrasts of the form:

$$L = \sum\sum c_{ij}\mu_{ij} \qquad \text{where } \sum\sum c_{ij} = 0 \qquad \textbf{(19.92)}$$

are:

$$\hat{L} \pm Ss\{\hat{L}\} \qquad \textbf{(19.93)}$$

where:

$$\hat{L} = \sum\sum c_{ij}\bar{Y}_{ij\cdot} \qquad \textbf{(19.93a)}$$

$$s^2\{\hat{L}\} = \frac{MSE}{n}\sum\sum c_{ij}^2 \qquad \textbf{(19.93b)}$$

$$S^2 = (ab-1)F[1-\alpha;\,ab-1,\,(n-1)ab] \qquad \textbf{(19.93c)}$$

The test statistic and associated decision rule for all simultaneous Scheffé tests of the form:

$$\begin{aligned} H_0&: L = 0 \\ H_a&: L \neq 0 \end{aligned} \qquad \textbf{(19.94)}$$

are as follows when the family significance level is controlled at α:

$$F^* = \frac{\hat{L}^2}{(ab-1)s^2\{\hat{L}\}}; \qquad \text{If } F^* > F[1-\alpha;\,ab-1,\,(n-1)ab], \text{ conclude } H_a \qquad \textbf{(19.94a)}$$

Bonferroni Procedure. When the number of contrasts is small, the Bonferroni procedure may be preferable. The confidence intervals (19.93) are simply modified by replacing S with B as defined in (19.90). The test statistic and decision rule in (19.94a) are replaced by:

$$t^* = \frac{\hat{L}}{s\{\hat{L}\}}; \qquad \text{If } |t^*| > t[1-\alpha/2g;\,(n-1)ab], \text{ conclude } H_a \qquad \textbf{(19.95)}$$

Example 1—Pairwise Comparisons of Treatment Means

A junior college system studied the effects of teaching method (factor A) and student's quantitative ability (factor B) on learning of college mathematics. Two teaching methods were studied—the standard method of teaching (to be called the standard method) and a method that emphasizes teaching of concepts in the abstract before going into drill routines

TABLE 19.11
Results—Mathematics Learning Example.

(a) Mean Learning Scores ($n = 21$)

Teaching Method i	Quantitative Ability (j)		
	Excellent	Good	Moderate
Abstract	92 ($\bar{Y}_{11\cdot}$)	81 ($\bar{Y}_{12\cdot}$)	73 ($\bar{Y}_{13\cdot}$)
Standard	90 ($\bar{Y}_{21\cdot}$)	86 ($\bar{Y}_{22\cdot}$)	82 ($\bar{Y}_{23\cdot}$)

(b) ANOVA Table

Source of Variation	*SS*	*df*	*MS*
Factor *A* (teaching methods)	504	1	504
Factor *B* (quantitative ability)	3,843	2	1,921.5
AB interactions	651	2	325.5
Error	3,360	120	28
Total	8,358	125	

(to be called the abstract method). The quantitative ability of a student was determined by a standard aptitude test, on the basis of which the student was classified as having excellent, good, or moderate quantitative ability. Thus, factor A (teaching method) has $a = 2$ levels, and factor B (student's quantitative ability) has $b = 3$ levels.

For each quantitative ability group, 42 students were selected and randomly placed into classes according to the designated teaching method, with each class containing equal numbers of students of each quantitative ability level. For simplicity, it is assumed that any effects associated with the classes are negligible.

This study has one experimental factor—teaching method—and one observational factor—quantitative ability. Equal numbers of students with excellent, good, and moderate quantitative ability are randomly selected and then within these categories, students are randomly assigned to a teaching method. Therefore, teaching ability is a blocking factor here with replication within blocks. This experimental study is called a generalized randomized block design and is discussed further in Section 21.6.

The response variable of interest is the amount of learning of college mathematics, as measured by a standard mathematics achievement test. The results of the study are summarized in Table 19.11 (the original data are not shown). The estimated treatment means are shown in Table 19.11a, and the analysis of variance table is presented in Table 19.11b.

Figure 19.13 contains two plots of the estimated treatment means $\bar{Y}_{ij\cdot}$. In Figure 19.13a, the two curves represent the different factor A levels, and in Figure 19.13b, the three curves represent the different factor B levels. The clear lack of parallelism of the curves suggests the presence of interaction effects between teaching method and student's quantitative ability on amount of mathematics learning. A formal test for interactions confirms this. From Table 19.11b, we have $F^* = MSAB/MSE = 325.5/28 = 11.625$. For $\alpha = .01$ we require $F(.99; 2,120) = 4.79$. Since $F^* = 11.625 > 4.79$, we conclude that interaction effects are present. The P-value of this test is 0+.

GURE 19.13
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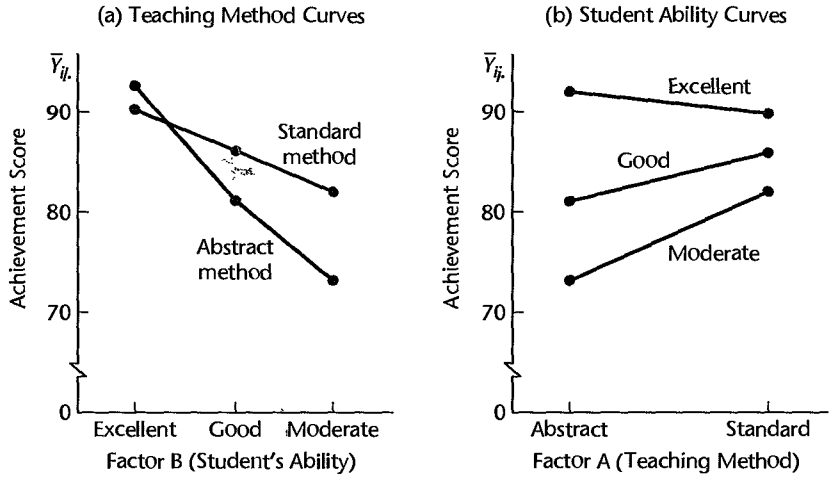

Figure 19.13 suggests that the interactions are important: students with excellent quantitative ability are but little affected by teaching method (perhaps doing slightly better with the abstract method); students with good or moderate abilities learn much better with the standard teaching method. Hence, we shall first investigate whether some simple transformation can make the interactions unimportant. We do this in an approximate fashion by considering the logarithmic and square root transformations of the response. In neither case did the interactions become unimportant, so it appears that the interactions here may be nontransformable.

We now wish to investigate the nature of the interaction effects in Figure 19.13. We shall do this by estimating separately for students with excellent, good, and moderate quantitative abilities how large is the difference in mean learning for the two teaching methods. Thus, we wish to estimate:

$$D_1 = \mu_{11} - \mu_{21}$$

$$D_2 = \mu_{12} - \mu_{22}$$

$$D_3 = \mu_{13} - \mu_{23}$$

We shall employ the Bonferroni multiple comparison procedure with family confidence coefficient .95. (Since only three pairwise comparisons are of interest, the Bonferroni method yields more precise estimates here than the Tukey method.)

For the data in Table 19.11a, the point estimates of the pairwise comparisons are:

$$\hat{D}_1 = 92 - 90 = 2$$

$$\hat{D}_2 = 81 - 86 = -5$$

$$\hat{D}_3 = 73 - 82 = -9$$

We find the estimated variances of these estimates by (19.88b), for $n = 21$:

$$s^2\{\hat{D}_1\} = s^2\{\hat{D}_2\} = s^2\{\hat{D}_3\} = \frac{2(28)}{21} = 2.667$$

so that:

$$s\{\hat{D}_1\} = s\{\hat{D}_2\} = s\{\hat{D}_3\} = 1.633$$

Finally, for family confidence coefficient $1 - \alpha = .95$ and $g = 3$, we require $B = t[1 - .05/2(3); 120] = t(.99167; 120) = 2.428$. Hence, the confidence limits are by (19.88) and (19.90):

$$2 \pm 2.428(1.633) \qquad -5 \pm 2.428(1.633) \qquad -9 \pm 2.428(1.633)$$

and the 95 percent confidence intervals for the family of comparisons are:

$$-1.96 \leq \mu_{11} - \mu_{21} \leq 5.96$$
$$-8.96 \leq \mu_{12} - \mu_{22} \leq -1.04$$
$$-12.96 \leq \mu_{13} - \mu_{23} \leq -5.04$$

For this family of confidence intervals, the following conclusions may be drawn with family confidence coefficient of 95 percent: (1) For students with excellent quantitative ability, the mean learning scores with the two teaching methods do not differ. (2) For students with either good or moderate quantitative abilities, the mean learning score with the abstract teaching method is lower than that with the standard method. The superiority of the standard teaching method may be particularly strong for students with moderate quantitative ability.

Example 2—Contrasts of Treatment Means

In the mathematics learning example, a school administrator also wished to know whether the amount of learning gain with the standard teaching method over the abstract method is greater for students with moderate quantitative ability than for students with good quantitative ability. This question had been raised before the study began. We shall estimate the single contrast:

$$L = (\mu_{23} - \mu_{13}) - (\mu_{22} - \mu_{12})$$

by means of a one-sided lower confidence interval. For the results in Table 19.11a, the point estimate of L is $\hat{L} = (82 - 73) - (86 - 81) = 4$. The estimated variance by (19.93b) is:

$$s^2\{\hat{L}\} = \frac{28}{21}[(1)^2 + (-1)^2 + (-1)^2 + (1)^2] = 5.333$$

so that the estimated standard deviation is $s\{\hat{L}\} = 2.309$. For a 95 percent confidence coefficient, we require $t(.05; 120) = -1.658$. Hence, the lower confidence limit is $4 - 1.658(2.309)$ and the desired confidence interval is:

$$L \geq .17$$

We conclude, therefore, with 95 percent confidence coefficient that the gain in learning with the standard teaching method over the abstract method is greater for students with moderate quantitative ability than for students with good quantitative ability, the difference in the mean gain being at least .17 point.

19.10 Pooling Sums of Squares in Two-Factor Analysis of Variance

The testing approach presented in this chapter assumes that ANOVA model (19.23) is the full model for all tests of factor effects, regardless of the conclusions reached in any of these tests. The rationale for this approach is that ANOVA model (19.23) is based on the identity (19.22) for the treatment means μ_{ij}. Once the analysis of residuals and other diagnostics demonstrate that this model is appropriate, it is used for all tests.

Some statisticians take the view that ANOVA model (19.23) should be revised when the test for interaction effects leads to the conclusion that no interactions are present. With this approach, the full model considered in testing for factor A and factor B main effects when the test for interaction effects leads to the conclusion that no interactions are present is the revised model:

$$Y_{ijk} = \mu.. + \alpha_i + \beta_j + \varepsilon_{ijk} \qquad \text{Revised full model} \qquad \textbf{(19.96)}$$

As we just noted with the regression approach for the Castle Bakery example, the extra sums of squares for factor A and factor B main effects do not depend on the order of the extra sums of squares for factor effects when all treatment sample sizes are equal. Hence, the numerator sums of squares SSA and SSB of the test statistic F^* are not affected by this revision in the full model when the treatment sample sizes are equal. The denominator sum of squares of the F^* test statistic is affected, however, leading to the following error sum of squares for the full model:

$$SSE(F) = SSE + SSAB \qquad \textbf{(19.97)}$$

Thus, the error sum of squares for the full model with this approach involves the *pooling* of the interaction and error sums of squares. Likewise, the degrees of freedom are pooled; the degrees of freedom associated with $SSE(F)$ are:

$$df_F = (a-1)(b-1) + (n-1)ab = nab - a - b + 1$$

For the Castle Bakery example, the pooled error sum of squares for testing factor A and factor B main effects would be (Table 19.9):

$$SSE(F) = 62 + 24 = 86$$

and the pooled degrees of freedom would be:

$$df_F = 6 + 2 = 8$$

Hence, the error mean square for testing factor A or factor B main effects with the model revision approach here would be $86/8 = 10.75$.

This pooling procedure affects both the level of significance and the power of the tests for factor A and factor B main effects, in ways not yet fully understood. It has been suggested

therefore by some statisticians that pooling should not be considered unless: (1) the degrees of freedom associated with *MSE* are small, perhaps 5 or less, and (2) the test statistic *MSAB/MSE* falls substantially below the action limit of the decision rule, perhaps when $MSAB/MSE < 2$ for $\alpha = .05$. Part (1) of this rule is designed to limit pooling to cases where the gains may be substantial, while part (2) is designed to give reasonable assurance that there are indeed no interactions.

19.11 Planning of Sample Sizes for Two-Factor Studies

We introduced the power approach to sample size planning for single-factor studies in Section 16.10, and the estimation approach to sample size planning for single-factor studies was discussed in Section 17.8. We now consider these two approaches in the context of two-factor studies.

Power Approach

Power of *F* Test. Table B.11 can be used for determining the power of tests for multi-factor studies in the same fashion as for single-factor studies. The only differences arise in the definition of the noncentrality parameter and the degrees of freedom. For two-factor fixed effects ANOVA model (19.23) with equal treatment sample sizes, the noncentrality parameter ϕ and the degrees of freedom ν_1 and ν_2 for testing for interaction effects, factor *A* main effects, and factor *B* main effects are as follows:

Test for interactions:

$$\phi = \frac{1}{\sigma}\sqrt{\frac{n\sum\sum(\alpha\beta)_{ij}^2}{(a-1)(b-1)+1}} = \frac{1}{\sigma}\sqrt{\frac{n\sum\sum(\mu_{ij}-\mu_{i\cdot}-\mu_{\cdot j}+\mu_{\cdot\cdot})^2}{(a-1)(b-1)+1}} \qquad (19.98a)$$

$$\nu_1 = (a-1)(b-1) \qquad \nu_2 = ab(n-1)$$

Test for *A* main effects:

$$\phi = \frac{1}{\sigma}\sqrt{\frac{nb\sum\alpha_i^2}{a}} = \frac{1}{\sigma}\sqrt{\frac{nb\sum(\mu_{i\cdot}-\mu_{\cdot\cdot})^2}{a}} \qquad (19.98b)$$

$$\nu_1 = a-1 \qquad \nu_2 = ab(n-1)$$

Test for *B* main effects:

$$\phi = \frac{1}{\sigma}\sqrt{\frac{na\sum\beta_j^2}{b}} = \frac{1}{\sigma}\sqrt{\frac{na\sum(\mu_{\cdot j}-\mu_{\cdot\cdot})^2}{b}} \qquad (19.98c)$$

$$\nu_1 = b-1 \qquad \nu_2 = ab(n-1)$$

Use of Table B.12 for Two-factor Studies. When planning sample sizes for two-factor studies with the power approach, one is concerned typically with both the power of detecting factor *A* main effects and the power of detecting factor *B* main effects. One can first specify the minimum range of factor *A* level means for which it is important to detect factor *A*

main effects, and obtain the needed sample sizes from Table B.12, with $r = a$. The resulting sample size is bn, from which n can be obtained readily. The use of Table B.12 for this purpose is appropriate provided the resulting sample size is not small, specifically provided $a(bn - 1) \geq 20$. If this condition is not met, the ANOVA power tables in Table B.11 should be used. These tables, as noted earlier, require an iterative approach for determining needed sample sizes.

In the same way, the minimum range of factor B level means can then be specified for which it is important to detect factor B main effects, and the needed sample sizes found. If the sample sizes obtained from the factor A and factor B power specifications differ substantially, a judgment will need to be made as to the final sample sizes.

Estimation Approach

The estimation approach to planning sample sizes described in Section 17.8 for single-factor studies is readily adapted for use in two-factor studies. We specify the set of comparisons of interest and determine the expected widths of the confidence intervals for various advance planning values for the standard deviation, σ. Through an iterative, trial-and-error process, we determine a sample size plan that represents an acceptable compromise between the cost of running the study and the precision obtained for comparisons of interest. We illustrate this procedure with a two-factor study example.

Example

In a two-factor study, factor A has $a = 3$ levels and factor B has $b = 2$ levels. No interaction effects are anticipated, and all pairwise comparisons of factor level means are to be made for each of the two factors. A family confidence coefficient of .90 is specified for the $3 + 1 = 4$ pairwise comparisons. Equal treatment sample sizes of n experimental units are to be used. The width of each confidence interval is to be ± 30. A reasonable planning value for the standard deviation of the error terms is $\sigma = 50$.

We know from (19.63) that the variance of a comparison of factor A level means, $\hat{L} = \bar{Y}_{i\cdot\cdot} - \bar{Y}_{i'\cdot\cdot}$, is:

$$\sigma^2\{\hat{L}\} = \frac{\sigma^2}{bn}\sum c_i^2 = \frac{2\sigma^2}{bn} \qquad \text{Factor } A \text{ comparisons}$$

Similarly, the variance of the comparison of the two factor B level means, $\hat{L} = \bar{Y}_{\cdot 1\cdot} - \bar{Y}_{\cdot 2\cdot}$, is:

$$\sigma^2\{\hat{L}\} = \frac{2\sigma^2}{an} \qquad \text{Factor } B \text{ comparison}$$

Since equal precision is specified for all pairwise comparisons and since $a = 3$ and $b = 2$, the variance for the factor A comparisons will be larger for any given treatment sample size n and hence will be the critical consideration.

Suppose that we begin the iterative process with $n = 30$. We then find for the factor A comparisons that $\sigma^2\{\hat{L}\} = 2(50)^2/2(30) = 83.33$ or $\sigma\{\hat{L}\} = 9.13$. For $n_T = 6(30) = 180$, $\alpha = .10$, and $g = 4$ comparisons, the Bonferroni multiple is $B = t(.9875; 174) = 2.26$. Hence, the anticipated width of the confidence intervals is $2.26(9.13) = \pm 20.6$. This

anticipated width is somewhat tighter than the specified width ±30, and a smaller treatment sample size should be tried in the next iteration.

Finding the "Best" Treatment

As we discussed earlier in Section 16.11 in the context of single-factor studies, there are occasions when the chief purpose of the study is to ascertain the treatment with the highest or lowest mean. This is also true for two-factor studies, where the objective is to identify the best of the $r = ab$ factor level combinations. We illustrate the use of this approach with an example.

Two-Factor Study Example. Suppose that in the Castle Bakery example, the chief objective is to identify the combination of shelf height and shelf width that maximizes sales (in cases). There are $3 \times 2 = 6$ treatment combinations. We anticipate that $\sigma = 10$. Further, we want to be able to detect an average difference of $\lambda = 8$ cases between the highest and second highest treatment means with probability $1 - \alpha = .90$ or greater.

The entry in Table B.13 is $\lambda\sqrt{n}/\sigma$. For $r = 6$ and probability $1 - \alpha = .90$, we find from Table B.13 that $\lambda\sqrt{n}/\sigma = 2.7100$. Hence, since $\lambda = 8$, we obtain:

$$\frac{(8)\sqrt{n}}{10} = 2.7100$$

$$\sqrt{n} = 3.3875 \qquad \text{or} \qquad n = 12$$

Thus, when the average number of cases for the best shelf height and shelf width treatment mean exceeds that of the second best by at least 8 cases and $\sigma = 10$, sample sizes of 12 supermarkets for each shelf height and shelf width combination are needed to provide an assurance of at least .90 that the highest estimated mean $\bar{Y}_{ij}$. corresponds to the highest population mean.

Problems

19.1. Refer to the **SENIC** data set in Appendix C.1. An analyst wishes to investigate the effects of medical school affiliation (factor A) and geographic region (factor B) on infection risk. All factor level combinations will be included in the study.

a. How many treatments are being studied?

b. What is the response variable here?

19.2. A student in a class discussion stated: "A treatment is a treatment, whether the study involves a single factor or multiple factors. The number of factors has little effect on the interpretation of the results." Discuss.

19.3. Verify the interactions in Table 19.3b.

*19.4. In a two-factor study, the treatment means μ_{ij} are as follows:

	Factor B		
Factor A	B_1	B_2	B_3
A_1	34	23	36
A_2	40	29	42

a. Obtain the factor A level means.

b. Obtain the main effects of factor A.

c. Does the fact that $\mu_{12} - \mu_{11} = -11$ while $\mu_{13} - \mu_{12} = 13$ imply that factors A and B interact? Explain.

d. Prepare a treatment means plot and determine whether the two factors interact. What do you find?

19.5. In a two-factor study, the treatment means μ_{ij} are as follows:

	Factor B			
Factor A	B_1	B_2	B_3	B_4
A_1	250	265	268	269
A_2	288	273	270	269

a. Obtain the factor B main effects. What do your results imply about factor B?

b. Prepare a treatment means plot and determine whether the two factors interact. How can you tell that interactions are present? Are the interactions important or unimportant?

c. Make a logarithmic transformation of the μ_{ij} and plot the transformed values to explore whether this transformation is helpful in reducing the interactions. What are your findings?

19.6. Three sets of treatment means μ_{ij} for students' grades in a course follow, where factor A is student's major (A_1: computer science; A_2: mathematics) and factor B is student's class affiliation (B_1: junior; B_2: senior; B_3: graduate).

	Set 1				Set 2				Set 3			
	B_1	B_2	B_3		B_1	B_2	B_3		B_1	B_2	B_3
A_1	80	80	80	A_1	75	80	90	A_1	75	80	85
A_2	90	90	90	A_2	80	86	97	A_2	75	85	100

Prepare a treatment means plot for each set of μ_{ij} to study interaction effects. Interpret each plot and state your findings. If interactions are present, describe their nature and indicate whether they are important or unimportant.

*19.7. Refer to Problem 19.4. Assume that $\sigma = 1.4$ and $n = 10$.

a. Obtain $E\{MSE\}$ and $E\{MSA\}$.

b. Is $E\{MSA\}$ substantially larger than $E\{MSE\}$? What is the implication of this?

19.8. Refer to Problem 19.5. Assume that $\sigma = 4$ and $n = 6$.

a. Obtain $E\{MSE\}$ and $E\{MSAB\}$.

b. Is $E\{MSAB\}$ substantially larger than $E\{MSE\}$? What is the implication of this?

19.9. A psychologist stated: "I feel uncomfortable about deciding in a research study whether the interactions are important or unimportant. I would rather have the statistician make that decision." Comment.

*19.10. Refer to **Cash offers** Problem 16.10. Six male and six female volunteers were used in each age group. The observations (in hundred dollars), classified by age (factor A) and gender of owner (factor B), follow.

		Factor B (gender of owner)	
Factor A (age)		$j = 1$ Male	$j = 2$ Female
$i = 1$	Young	21	21
		23	22
		...	...
		23	25
$i = 2$	Middle	30	26
		29	29
		...	...
		27	29
$i = 3$	Elderly	25	23
		22	19
		...	...
		21	20

a. Obtain the fitted values for ANOVA model (19.23).

b. Obtain the residuals. Do they sum to zero for each treatment?

c. Prepare aligned residual dot plots for the treatments. What departures from ANOVA model (19.23) can be studied from these plots? What are your findings?

d. Prepare a normal probability plot of the residuals. Also obtain the coefficient of correlation between the ordered residuals and their expected values under normality. Does the normality assumption appear to be reasonable here?

e. The observations for each treatment were obtained in the order shown. Prepare residual sequence plots and interpret them. What are your findings?

*19.11. Refer to **Cash offers** Problems 16.10 and 19.10. Assume that ANOVA model (19.23) is applicable.

a. Prepare an estimated treatment means plot. Does it appear that any factor effects are present? Explain.

b. Set up the analysis of variance table. Does any one source account for most of the total variability in cash offers in the study? Explain.

c. Test whether or not interaction effects are present; use $\alpha = .05$. State the alternatives, decision rule, and conclusion. What is the P-value of the test?

d. Test whether or not age and gender main effects are present. In each case, use $\alpha = .05$ and state the alternatives, decision rule, and conclusion. What is the P-value of the test? Is it meaningful here to test for main factor effects? Explain.

e. Obtain an upper bound on the family level of significance for the tests in parts (c) and (d); use the Kimball inequality (19.53).

f. Do the results in parts (c) and (d) confirm your graphic analysis in part (a)?

g. What are the relations between the sums of squares in the two-factor analysis of variance in part (b) and the sums of squares in the single-factor analysis of variance in Problem 16.10d? Do the same relations hold for the degrees of freedom?

19.12. **Eye contact effect.** In a study of the effect of applicant's eye contact (factor A) and personnel officer's gender (factor B) on the personnel officer's assessment of likely job success of applicant, 10 male and 10 female personnel officers were shown a front view photograph of an applicant's face and were asked to give the person in the photograph a success rating on a scale of 0 (total failure) to 20 (outstanding success). Half of the officers in each gender group were chosen at random to receive a version of the photograph in which the applicant made eye contact with the camera lens. The other half received a version in which there was no eye contact. The success ratings follow.

		Factor B (gender of officer)	
Factor A (eye contact)		$j = 1$ Male	$j = 2$ Female
$i = 1$	Present	11	15
		7	12
		...	...
		10	16
$i = 2$	Absent	12	14
		16	17
		...	...
		14	18

a. Obtain the fitted values for ANOVA model (19.23).

b. Obtain the residuals. Do they sum to zero for each treatment?

c. Prepare aligned residual dot plots for the treatments. What departures from ANOVA model (19.23) can be studied from these plots? What are your findings?

d. Prepare a normal probability plot of the residuals. Also obtain the coefficient of correlation between the ordered residuals and their expected values under normality. Does the normality assumption appear to be reasonable here?

e. The observations for each treatment were obtained in the order shown. Prepare residual sequence plots and interpret them. What are your findings?

19.13. Refer to **Eye contact effect** Problem 19.12. Assume that ANOVA model (19.23) is applicable.

a. Prepare an estimated treatment means plot. Does it appear that any factor effects are present? Explain.

b. Set up the analysis of variance table. Does any one source account for most of the total variability in the success ratings in the study? Explain.

c. Test whether or not interaction effects are present; use $\alpha = .01$. State the alternatives, decision rule, and conclusion. What is the P-value of the test?

d. Test whether or not eye contact and gender main effects are present. In each case, use $\alpha = .01$ and state the alternatives, decision rule, and conclusion. What is the P-value of each test? Is it meaningful here to test for main factor effects? Explain.

e. Obtain an upper bound on the family level of significance for the tests in parts (c) and (d); use the Kimball inequality (19.53).

f. Do the results in parts (c) and (d) confirm your graphic analysis in part (a)?

*19.14. **Hay fever relief.** A research laboratory was developing a new compound for the relief of severe cases of hay fever. In an experiment with 36 volunteers, the amounts of the two active ingredients (factors A and B) in the compound were varied at three levels each. Randomization was used in assigning four volunteers to each of the nine treatments. The data on hours of relief follow.

		Factor B (ingredient 2)		
Factor A (ingredient 1)		$j = 1$ Low	$j = 2$ Medium	$j = 3$ High
$i = 1$	Low	2.4	4.6	4.8
		...	...	...
		2.5	4.7	4.6
$i = 2$	Medium	5.8	8.9	9.1
		...	...	.
		5.3	9.0	9.4
$i = 3$	High	6.1	9.9	13.5
		...	...	...
		6.2	10.1	13.2

a. Obtain the fitted values for ANOVA model (19.23).

b. Obtain the residuals.

c. Plot the residuals against the fitted values. What departures from ANOVA model (19.23) can be studied from this plot? What are your findings?

d. Prepare a normal probability plot of the residuals. Also obtain the coefficient of correlation between the ordered residuals and their expected values under normality. Does the normality assumption appear to be reasonable here?

*19.15. Refer to **Hay fever relief** Problem 19.14. Assume that ANOVA model (19.23) is applicable.

a. Prepare an estimated treatment means plot. Does your graph suggest that any factor effects are present? Explain.

b. Obtain the analysis of variance table. Does any one source account for most of the total variability in hours of relief in the study? Explain.

c. Test whether or not the two factors interact; use $\alpha = .05$. State the alternatives, decision rule, and conclusion. What is the P-value of the test?

d. Test whether or not main effects for the two ingredients are present. Use $\alpha = .05$ in each case and state the alternatives, decision rule, and conclusion. What is the P-value of each test? Is it meaningful here to test for main factor effects? Explain.

e. Obtain an upper bound on the family level of significance for the tests in parts (c) and (d); use the Kimball inequality (19.53).

f. Do the results in parts (c) and (d) confirm your graphic analysis in part (a)?

19.16. **Disk drive service.** The staff of a service center for electronic equipment includes three technicians who specialize in repairing three widely used makes of disk drives for desktop computers. It was desired to study the effects of technician (factor A) and make of disk drive (factor B) on the service time. The data that follow show the number of minutes required to

complete the repair job in a study where each technician was randomly assigned to five jobs on each make of disk drive.

Factor *A* (technician)	Factor *B* (make of drive) $j = 1$ Make 1	$j = 2$ Make 2	$j = 3$ Make 3
$i = 1$ Technician 1	62	57	59
	48	45	53
	...	...	...
	69	44	47
$i = 2$ Technician 2	51	61	55
	57	58	58
	...	...	...
	39	51	49
$i = 3$ Technician 3	59	58	47
	65	63	56
	...	...	...
	70	60	50

a. Obtain the fitted values for ANOVA model (19.23).

b. Obtain the residuals.

c. Plot the residuals against the fitted values. What departures from ANOVA model (19.23) can be studied from this plot? What are your findings?

d. Prepare a normal probability plot of the residuals. Also obtain the coefficient of correlation between the ordered residuals and their expected values under normality. Does the normality assumption appear to be reasonable here?

e. The observations for each treatment were obtained in the order shown. Prepare residual sequence plots and analyze them. What are your findings?

19.17. Refer to **Disk drive service** Problem 19.16. Assume that ANOVA model (19.23) is applicable.

a. Prepare an estimated treatment means plot. Does your graph suggest that any factor effects are present? Explain.

b. Obtain the analysis of variance table. Does any one source account for most of the total variability? Explain.

c. Test whether or not the two factors interact; use $\alpha = .01$. State the alternatives, decision rule, and conclusion. What is the P-value of the test?

d. Test whether or not main effects for technician and make of drive are present. Use $\alpha = .01$ in each case and state the alternatives, decision rule, and conclusion. What is the P-value of each test? Is it meaningful here to test for main factor effects? Explain.

e. Obtain an upper bound on the family level of significance for the tests in parts (c) and (d); use the Kimball inequality (19.53).

f. Do the results in parts (c) and (d) confirm your graphic analysis in part (a)?

19.18. **Kidney failure hospitalization.** Kidney failure patients are commonly treated on dialysis machines that filter toxic substances from the blood. The appropriate "dose" for effective treatment depends, among other things, on duration of treatment and weight gain between treatments as a result of fluid buildup. To study the effects of these two factors on the number of days hospitalized (attributable to the disease) during a year, a random sample of 10 patients per group who had undergone treatment at a large dialysis facility was obtained. Treatment

duration (factor A) was categorized into two groups: short duration (average dialysis time for the year under four hours) and long duration (average dialysis time for the year equal to or greater than four hours). Average weight gain between treatments (factor B) during the year was categorized into three groups: slight, moderate, and substantial. The data on number of days hospitalized follow.

		Factor B (weight gain)					
Factor A (duration)		$j = 1$ Mild		$j = 2$ Moderate		$j = 3$ Substantial	
$i = 1$	Short	0	2	2	4	15	16
		2	0	4	3	10	7
		...	...		...	...	...
		0	8	15	20	25	27
$i = 2$	Long	0	2	5	1	10	15
		1	7	3	3	8	4
		..			...	.	
		4	3	1	9	7	1

The transformed data $Y' = \log_{10}(Y + 1)$ are to be used for the analysis.

a. Obtain the fitted values and residuals for ANOVA model (19.23) for the transformed data.

b. Prepare aligned residual dot plots for the treatments. What departures from ANOVA model (19.23) can be studied from these plots? What are your findings?

c. Prepare a normal probability plot of the residuals. Also obtain the coefficient of correlation between the ordered residuals and their expected values under normality. Does the normality assumption appear to be reasonable here?

19.19. Refer to **Kidney failure hospitalization** Problem 19.18. Assume that ANOVA model (19.23) is appropriate for the transformed response variable.

a. Prepare an estimated treatment means plot. Does your graph suggest that any factor effects are present? Explain.

b. Obtain the analysis of variance table. Does any one source account for most of the total variability? Explain.

c. Test whether or not the two factors interact; use $\alpha = .05$. State the alternatives, decision rule, and conclusion. What is the P-value of the test?

d. Test whether or not main effects for duration and weight gain are present. Use $\alpha = .05$ in each case and state the alternatives, decision rule, and conclusion. What is the P-value of each test? Is it meaningful here to test for main factor effects? Explain.

e. Obtain an upper bound on the family level of significance for the tests in parts (c) and (d); use the Kimball inequality (19.53).

f. Do the results in parts (c) and (d) confirm your graphic analysis in part (a)?

*19.20. **Programmer requirements.** A computer software firm was encountering difficulties in forecasting the programmer requirements for large-scale programming projects. As part of a study to remedy the difficulties, 24 programmers, classified into equal groups by type of experience (factor A) and amount of experience (factor B), were asked to predict the number of programmer-days required to complete a large project about to be initiated. After this project

was completed, the prediction errors (actual minus predicted programmer-days) were determined. The data on prediction errors follow.

		Factor *B* (years of experience)		
Factor *A* (type of experience)		$j = 1$ Under 5	$j = 2$ 5–under 10	$j = 3$ 10 or more
$i = 1$	Small systems only	240	110	56
		206	118	60
		217	103	68
		225	95	58
$i = 2$	Small and large systems	71	47	37
		53	52	33
		68	31	40
		57	49	45

a. Obtain the fitted values for ANOVA model (19.23).

b. Obtain the residuals.

c. Prepare aligned residual dot plots for the treatments. What departures from ANOVA model (19.23) can be studied from these plots? What are your findings?

d. Prepare a normal probability plot of the residuals. Also obtain the coefficient of correlation between the ordered residuals and their expected values under normality. Does the normality assumption appear to be reasonable here?

*19.21. Refer to **Programmer requirements** Problem 19.20. Assume that ANOVA model (19.23) is applicable.

a. Prepare an estimated treatment means plot. Does your graph suggest that any factor effects are present? Explain.

b. Obtain the analysis of variance table. Does any one source account for most of the total variability? Explain.

c. Test whether or not the two factors interact; use $\alpha = .01$. State the alternatives, decision rule, and conclusion. What is the *P*-value of the test?

d. Test whether or not main effects for type of experience and years of experience are present. Use $\alpha = .01$ in each case and state the alternatives, decision rule, and conclusion. What is the *P*-value of each test? Is it meaningful here to test for main factor effects? Explain.

e. Obtain an upper bound on the family level of significance for the tests in parts (c) and (d); use the Kimball inequality (19.53).

f. Do the results in parts (c) and (d) confirm your graphic analysis in part (a)?

19.22. How does the randomization of treatment assignments in a two-factor study differ when both factors are experimental factors and when only one factor is an experimental factor?

19.23. Refer to **Eye contact effect** Problem 19.12.

a. Explain how you would make the assignments of personnel officers to treatments in this two-factor study. Make all appropriate randomizations.

b. Did you randomize the officers to the factor levels of each factor?

*19.24. Refer to **Hay fever relief** Problem 19.14.

a. Explain how you would make the assignments of volunteers to treatments in this study. Make all appropriate randomizations.

b. Did you randomize the volunteers to the factor levels of each factor?

19.25. Refer to **Disk drive service** Problem 19.16.

a. Is any randomization of treatment assignments called for in this study? Is any randomization utilized? Explain.

b. Would you consider this study to be experimental in nature? Discuss.

19.26. Why is it suggested in the flowchart in Figure 19.11 that a test for interactions should be conducted before tests for main factor effects? Explain.

*19.27. A two-factor study was conducted with $a = 5$, $b = 5$, and $n = 4$. No interactions between factors A and B were noted, and the analyst now wishes to estimate all pairwise comparisons among the factor A level means and all pairwise comparisons among the factor B level means. The family confidence coefficient for the joint set of interval estimates is to be 90 percent.

a. Is it more efficient to use the Bonferroni procedure for the entire family or to use the Tukey procedure for each family of factor level mean comparisons and then to join the two families by means of the Bonferroni procedure?

b. Would your answer differ if each factor had three levels, everything else remaining the same?

19.28. A two-factor study was conducted with $a = 6$, $b = 6$, and $n = 10$. No interactions between factors A and B were found, and it is now desired to estimate five contrasts of factor A level means and four contrasts of factor B level means. The family confidence coefficient for the joint set of estimates is to be 95 percent. Which of the three procedures at the bottom of page 852 and the top of page 853 will be most efficient here?

19.29. Refer to the Castle Bakery example at the top of page 855, where two pairwise comparison estimates were made by means of the Tukey procedure. Why would it not be appropriate to use the Bonferroni procedure here? Discuss.

*19.30. Refer to **Cash offers** Problems 19.10 and 19.11.

a. Estimate μ_{11} with a 95 percent confidence interval. Interpret your interval estimate.

b. Prepare a bar graph of the estimated factor B level means. What does this plot suggest about the equality of the factor B level means?

c. Estimate $D = \mu_{\cdot 1} - \mu_{\cdot 2}$ by means of a 95 percent confidence interval. Is your confidence interval consistent with the test result in Problem 19.11d? Is your confidence interval consistent with your finding in part (b)? Explain.

d. Prepare a bar graph of the estimated factor A level means. What does this plot suggest about the factor A main effects?

e. Obtain all pairwise comparisons among the factor A level means; use the Tukey procedure with a 90 percent family confidence coefficient. Present your findings graphically and summarize your results. Are your conclusions consistent with those in part (d)?

f. Is the Tukey procedure used in part (e) the most efficient one that could be used here? Explain.

g. Estimate the contrast:

$$L = \frac{\mu_{1\cdot} + \mu_{3\cdot}}{2} - \mu_{2\cdot}$$

with a 95 percent confidence interval. Interpret your interval estimate.

h. Suppose that in the population of female owners, 30 percent are young, 60 percent are middle-aged, and 10 percent are elderly. Obtain a 95 percent confidence interval for the mean cash offer in the population of female owners.

19.31. Refer to **Eye contact effect** Problems 19.12 and 19.13.

a. Estimate μ_{21} with a 99 percent confidence interval. Interpret your interval estimate.

b. Estimate $\mu_{1\cdot}$ with a 99 percent confidence interval. Interpret your interval estimate.

c. Prepare a bar graph of the estimated factor B level means. What does this plot suggest about the factor B main effects?

d. Obtain confidence intervals for $\mu_{\cdot 1}$ and $\mu_{\cdot 2}$, each with a 99 percent confidence coefficient. Interpret your interval estimates. What is the family confidence coefficient for the set of two estimates?

e. Prepare a bar graph of the estimated factor A level means. What does this plot suggest about the factor A main effects?

f. Obtain confidence intervals for $D_1 = \mu_{2\cdot} - \mu_{1\cdot}$ and $D_2 = \mu_{\cdot 2} - \mu_{\cdot 1}$; use the Bonferroni procedure and a 95 percent family confidence coefficient. Summarize your findings. Are your findings consistent with those in parts (c) and (e)?

g. Is the Bonferroni procedure used in part (f) the most efficient one that could be used here? Explain.

*19.32. Refer to **Hay fever relief** Problems 19.14 and 19.15.

a. Estimate μ_{23} with a 95 percent confidence interval. Interpret your interval estimate.

b. Estimate $D = \mu_{12} - \mu_{11}$ with a 95 percent confidence interval. Interpret your interval estimate.

c. The analyst decided to study the nature of the interacting factor effects by means of the following contrasts:

$$L_1 = \frac{\mu_{12} + \mu_{13}}{2} - \mu_{11} \qquad L_4 = L_2 - L_1$$

$$L_2 = \frac{\mu_{22} + \mu_{23}}{2} - \mu_{21} \qquad L_5 = L_3 - L_1$$

$$L_3 = \frac{\mu_{32} + \mu_{33}}{2} - \mu_{31} \qquad L_6 = L_3 - L_2$$

Obtain confidence intervals for these contrasts; use the Scheffé multiple comparison procedure with a 90 percent family confidence coefficient. Interpret your findings.

d. The analyst also wished to identify the treatment(s) yielding the longest mean relief. Using the Tukey testing procedure with family significance level $\alpha = .10$, identify the treatment(s) providing the longest mean relief.

e. To examine whether a transformation of the data would make the interactions unimportant, plot separately the transformed estimated treatment means for the reciprocal and square root transformations. Would either of these transformations have made the interaction effects unimportant? Explain.

19.33. Refer to **Disk drive service** Problems 19.16 and 19.17.

a. Estimate μ_{11} with a 99 percent confidence interval. Interpret your interval estimate.

b. Estimate $D = \mu_{22} - \mu_{21}$ with a 99 percent confidence interval. Interpret your interval estimate.

c. The nature of the interaction effects is to be studied by making, for each technician, all three pairwise comparisons among the disk drive makes in order to identify, if possible, the make of disk drive for which the technician's mean service time is lowest. The family confidence coefficient for each set of three pairwise comparisons is to be 95 percent. Use the Bonferroni procedure to make all required pairwise comparisons. Summarize your findings.

d. The service center currently services 30 disk drives of each of the three makes per week, with each technician servicing 10 machines of each make. Estimate the expected total amount of service time required per week to service the 90 disk drives; use a 99 percent confidence interval.

e. How much time could be saved per week, on the average, if technician 1 services only make 2, technician 2 services only make 1, and technician 3 services only make 3? Use a 99 percent confidence interval.

f. To examine whether a transformation of the data would make the interactions unimportant, plot separately the transformed estimated treatment means for the reciprocal and logarithmic transformations. Would either of these transformations have made the interaction effects unimportant? Explain.

19.34. Refer to **Kidney failure hospitalization** Problems 19.18 and 19.19. Continue to work with the transformed observations $Y' = \log_{10}(Y + 1)$.

a. Estimate μ_{22} with a 95 percent confidence interval. Interpret your interval estimate.

b. Estimate $D = \mu_{23} - \mu_{21}$ with a 95 percent confidence interval. Interpret your interval estimate.

c. Prepare separate bar graphs of the estimated factor A and factor B level means. What do these plots suggest about the factor main effects?

d. The researcher wishes to study the main effects of each of the two factors by making all pairwise comparisons of factor level means with a 90 percent family confidence coefficient for the entire set of comparisons. Which multiple comparison procedure is most efficient here?

e. Using the Bonferroni procedure, make all pairwise comparisons called for in part (d). State your findings and prepare a graphic summary. Are your findings consistent with those in part (c)?

f. It is known from past experience that 30 percent of patients have mild weight gains, 40 percent have moderate weight gains, and 30 percent have severe weight gains, and that these proportions are the same for the two duration groups. Estimate the mean number of days hospitalized (in transformed units) in the entire population with a 95 percent confidence interval. Convert your confidence limits to the original units. Does it appear that the mean number of days is less than 7?

*19.35. Refer to **Programmer requirements** Problems 19.20 and 19.21.

a. Estimate μ_{23} with a 99 percent confidence interval. Interpret your interval estimate.

b. Estimate $D = \mu_{12} - \mu_{13}$ with a 99 percent confidence interval. Interpret your interval estimate.

c. The nature of the interaction effects is to be studied by comparing the effect of type of experience for each years-of-experience group. Specifically, the following comparisons are to be estimated:

$$D_1 = \mu_{11} - \mu_{21} \qquad L_1 = D_1 - D_2$$
$$D_2 = \mu_{12} - \mu_{22} \qquad L_2 = D_1 - D_3$$
$$D_3 = \mu_{13} - \mu_{23} \qquad L_3 = D_2 - D_3$$

The family confidence coefficient is to be 95 percent. Which multiple comparison procedure is most efficient here?

d. Use the most efficient procedure to estimate the comparisons specified in part (c). State your findings.

e. Use the Tukey testing procedure with family significance level $\alpha = .05$ to identify the type of experience-years of experience group(s) with the smallest mean prediction errors.

f. For each group identified in part (e), obtain a confidence interval for the mean prediction error. Use the Bonferroni procedure with a 95 percent family confidence coefficient. Does any group have a mean prediction error that could be zero? Explain.

g. To examine whether a transformation of the data would make the interactions unimportant, plot separately the transformed estimated treatment means for the reciprocal and logarithmic transformations. Would either of these transformations have made the interaction effects unimportant? Explain.

19.36. Refer to **Brand preference** Problem 6.5. Suppose the market researcher first wished to employ analysis of variance model (19.23) to determine whether or not moisture content (factor A) and sweetness (factor B) affect the degree of brand liking.

a. State the analysis of variance model for this case.

b. Obtain the analysis of variance table.

c. Test whether or not the two factors interact; use $\alpha = .01$. State the alternatives, decision rule, and conclusion.

d. Study possible curvilinearity of the moisture content effect by estimating the following contrast:

$$L = (\mu_{4\cdot} - \mu_{3\cdot}) - (\mu_{2\cdot} - \mu_{1\cdot})$$

Use a 95 percent confidence interval. What do you conclude?

e. Test whether or not sweetness affects brand liking; use $\alpha = .01$. State the alternatives, decision rule, and conclusion.

19.37. A market research manager is planning to study the effects of duration of advertising (factor A) and price level (factor B) on sales. Each factor has three levels. No important interactions are expected, and the primary analysis is to consist of pairwise comparisons of factor level means for each factor. Equal sample sizes are to be used for each treatment. The precision of each comparison is to be ± 3 thousand dollars. The family confidence coefficient for the joint set of comparisons is to be 90 percent, the Tukey procedure is to be used in making the comparisons for each factor, and the Bonferroni procedure is then to be used to join the two sets of comparisons. Assume that $\sigma = 7$ thousand dollars is a reasonable planning value for the error standard deviation. What sample sizes do you recommend?

*19.38. Refer to **Cash offers** Problem 19.10. Suppose that the sample sizes have not yet been determined but it has been decided to use the same number of "owners" in each age-gender group. What are the required sample sizes if: (1) differences in the age factor level means are to be detected with probability .90 or more when the range of the factor level means is 3 (hundred dollars), and (2) the α risk is to be controlled at .05? Assume that a reasonable planning value for the error standard deviation is $\sigma = 1.5$ (hundred dollars).

19.39. Refer to **Eye contact effect** Problem 19.12. Suppose that the sample sizes have not yet been determined but it has been decided to use equal sample sizes for each treatment. Primary interest is in the two comparisons $L_1 = \mu_{1\cdot} - \mu_{2\cdot}$ and $L_2 = \mu_{\cdot 1} - \mu_{\cdot 2}$. What are the required sample sizes if each of these comparisons is to be estimated with precision not to exceed ± 1.2 with a 95 percent family confidence coefficient, using the most efficient multiple comparison procedure? Assume that a reasonable planning value for the error standard deviation is $\sigma = 2.4$.

*19.40. Refer to **Hay fever relief** Problem 19.14. Suppose that the sample sizes have not yet been determined but it has been decided to use equal sample sizes for each treatment. The chief

objective is to identify the dosage combination that yields the longest mean relief. The probability should be at least .99 that the correct dosage combination is identified when the mean relief duration for the second best combination differs by .5 hour or more. What are the required sample sizes? Assume that a reasonable planning value for the error standard deviation is $\sigma = .29$ hour.

19.41. Refer to **Kidney failure hospitalization** Problem 19.18. Suppose that the sample sizes have not yet been determined but it has been decided to use equal sample sizes for each treatment. The chief objective is to estimate the pairwise comparisons:

$$L_1 = \mu_{1\cdot} - \mu_{2\cdot} \qquad L_3 = \mu_{\cdot 1} - \mu_{\cdot 3}$$
$$L_2 = \mu_{\cdot 1} - \mu_{\cdot 2} \qquad L_4 = \mu_{\cdot 2} - \mu_{\cdot 3}$$

What are the required sample sizes if the precision of each of the estimates should not exceed $\pm .20$ (in transformed units), using the Bonferroni procedure with a family confidence coefficient of 90 percent for the joint set of comparisons? A reasonable planning value for the error standard deviation is $\sigma = .32$ (in transformed units).

*19.42. Refer to **Programmer requirements** Problem 19.20. Suppose that the sample sizes have not yet been determined but it has been decided to use equal sample sizes for each treatment. Primary interest is in identifying the type of experience-years of experience combination for which the mean prediction error is smallest. The probability should be at least .95 that the correct combination is identified when the mean prediction error for the second best combination differs by 8.0 programmer-days or more. Assume that a reasonable planning value for the error standard deviation is $\sigma = 9.1$ days. What are the required sample sizes?

Exercises

19.43. Derive (19.7a) from (19.7).

19.44. Prove the result in (19.9b).

19.45. (Calculus needed.) State the likelihood function for ANOVA model (19.15) when $a = 2$, $b = 2$, and $n = 2$. Find the maximum likelihood estimators.

19.46. (Calculus needed.) Derive (19.29).

19.47. Derive (19.39) from (19.38).

19.48. Show that the point estimator (19.67) is unbiased. Find the variance of this estimator.

19.49. Find the variance of the estimator (19.93a).

19.50. Consider a two-factor study with $a = 2$ and $b = 2$. Show that the interactions $(\alpha\beta)_{12}$ and $(\alpha\beta)_{21}$ are equal.

Projects

19.51. Refer to the SENIC data set in Appendix C.1. The following hospitals are to be considered in a study of the effects of region (factor A: variable 9) and average age of patients (factor B: variable 3) on the mean length of hospital stay of patients (variable 2):

1–44	46	48	51	53	57	58	60	63	66	74
76	79	80	83	84	88	94	101	103	111	

For purposes of this ANOVA study, average age is to be classified into two categories: less than or equal to 53.9 years, 54.0 years or more.

a. Assemble the required data and obtain the fitted values for ANOVA model (19.23).

b. Obtain the residuals.

c. Plot the residuals against the fitted values. What departures from ANOVA model (19.23) can be studied from this plot? What are your findings?

d. Prepare a normal probability plot of the residuals. Also obtain the coefficient of correlation between the ordered residuals and their expected values under normality. Does the normality assumption appear to be reasonable here?

52. Refer to the **SENIC** data set in Appendix C.1 and Project 19.51. Assume that ANOVA model (19.23) is applicable.

a. Prepare an estimated treatment means plot. Does it appear that any factor effects are present? Explain.

b. Obtain the analysis of variance table. Does any one source account for most of the total variability in the study? Explain.

c. Test whether or not interaction effects are present; use $\alpha = .05$. State the alternatives, decision rule, and conclusion. What is the P-value of the test?

d. Test whether or not region and age main effects are present. In each case, use $\alpha = .05$ and state the alternatives, decision rule, and conclusion. What is the P-value of each test? Is it meaningful here to test for main factor effects? Explain.

e. Obtain an upper bound on the family level of significance for the tests in parts (c) and (d); use the Kimball inequality (19.53).

f. Do the results in parts (c) and (d) confirm your graphic analysis in part (a)?

53. Refer to the **CDI** data set in Appendix C.2. The following metropolitan areas are to be considered in a study of the effects of region (factor A: variable 17) and percent below poverty level (factor B: variable 13) on the crime rate (variable 10 ÷ variable 5):

1–5	7	10–17	19–29	32–34	36–42	44	46	49
51–52	54	57	75	84	87	94	136	151
164	178	182	202	218	410	421	434	

For purposes of this ANOVA study, percent of population below poverty level is to be classified into two categories: less than 8 percent, 8 percent or more.

a. Assemble the required data and obtain the fitted values for ANOVA model (19.23).

b. Obtain the residuals.

c. Prepare aligned residual dot plots for the treatments. What departures from ANOVA model (19.23) can be studied from these plots? What are your findings?

d. Prepare a normal probability plot of the residuals. Also obtain the coefficient of correlation between the ordered residuals and their expected values under normality. Does the normality assumption appear to be reasonable here?

54. Refer to the **CDI** data set in Appendix C.2 and Project 19.53. Assume that ANOVA model (19.23) is applicable.

a. Prepare an estimated treatment means plot. Does it appear that any factor effects are present? Explain.

b. Set up the analysis of variance table. Does any one source account for most of the total variability in the study? Explain.

c. Test whether or not interaction effects are present; use $\alpha = .01$. State the alternatives, decision rule, and conclusion. What is the P-value of the test?

d. Test whether or not region and percent of population below poverty level main effects are present. In each case, use $\alpha = .01$ and state the alternatives, decision rule, and conclusion. What is the P-value of each test? Is it meaningful here to test for main factor effects? Explain.

e. Obtain an upper bound on the family level of significance for the tests in parts (c) and (d); use the Kimball inequality (19.53).

f. Do the results in parts (c) and (d) confirm your graphic analysis in part (a)?

19.55. Refer to the **Market share** data set in Appendix C.3. A balanced ANOVA study of the effects of discount price (factor A: variable 5) and package promotion (factor B: variable 6) on the average monthly market share (variable 2) is to be conducted. Order the observations in the four factor-level combination cells from smallest to largest observation number and retain the first 7 observations in each cell for a total of 28 observations. (This process omits cases with identification numbers (variable 1) equal to 24, 25, 27, 28, 30, 33, 34, and 36.)

a. Assemble the required data and obtain the fitted values for ANOVA model (19.23).

b. Obtain the residuals.

c. Plot the residuals against the fitted values. What departures from ANOVA model (19.23) can be studied from this plot? What are your findings?

d. Prepare a normal probability plot of the residuals. Also obtain the coefficient of correlation between the ordered residuals and their expected values under normality. Does the normality assumption appear to be reasonable here?

19.56. Refer to the **Market share** data set in Appendix C.3 and Project 19.55. Assume that ANOVA model (19.23) is applicable.

a. Prepare an estimated treatment means plot. Does it appear that any factor effects are present? Explain.

b. Obtain the analysis of variance table. Does any one source account for most of the total variability in the study? Explain.

c. Test whether or not interaction effects are present; use $\alpha = .05$. State the alternatives, decision rule, and conclusion. What is the P-value of the test?

d. Test whether or not discount price and package promotion main effects are present. In each case, use $\alpha = .05$ and state the alternatives, decision rule, and conclusion. What is the P-value of each test? Is it meaningful here to test for main factor effects? Explain.

e. Obtain an upper bound on the family level of significance for the tests in parts (c) and (d); use the Kimball inequality (19.53).

f. Do the results in parts (c) and (d) confirm your graphic analysis in part (a)?

19.57. Refer to the **SENIC** data set in Appendix C.1 and Projects 19.51 and 19.52.

a. Prepare a bar graph of the estimated factor level means $\bar{Y}_{i\cdot\cdot\cdot}$. What does this plot suggest regarding the region main effects?

b. Analyze the effects of region on mean length of hospital stay by making all pairwise comparisons between regions; use the Tukey procedure and a 90 percent family confidence coefficient. State your findings and present a graphic summary. Are your findings consistent with those in part (a)?

19.58. Refer to the **CDI** data set in Appendix C.2 and Projects 19.53 and 19.54.

a. Prepare a bar graph of the estimated factor level means $\bar{Y}_{i\cdot\cdot\cdot}$. What does this plot suggest regarding the region main effects?

b. Analyze the effects of region on crime rate by making all pairwise comparisons between regions; use the Tukey procedure and a 95 percent family confidence coefficient. State your findings and present a graphic summary. Are your findings consistent with those in part (a)?

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19.59. Refer to the **Real estate sales** data set in Appendix C.7. Carry out a balanced two-way analysis of variance of this data set where the response of interest is sales price (variable 2) and the two crossed factors are quality (variable 10) and style (variable 11). Style is recoded as either 1 or not 1. Order the observations in the six factor-level-combination cells from smallest to largest observation number and retain the first 25 observations in each cell for a total of 150 observations. The analysis should consider transformations of the response variable. Document the steps taken in your analysis and justify your conclusions.

19.60. Refer to the **Ischemic heart disease** data set in Appendix C.9. Carry out a balanced two-way analysis of variance of this data set where the response of interest is total cost (variable 2) and the two crossed factors are number of interventions (variable 5) and number of comorbidities (variable 9). Recode the number of interventions into six categories: 0, 1, 2, 3–4, 5–7, and greater than or equal to 8. Recode the number of comorbidities into two categories: 0–1, and greater than or equal to 2. Order the observations in the twelve factor-level-combination cells from smallest to largest observation number and retain the first 43 observations in each cell for a total of 516 observations. The analysis should consider transformations of the response variable. Document the steps taken in your analysis and justify your conclusions.