CHAPTER 2

Applications of Integration

2.1. More about Areas

2.1.1. Area Between Two Curves. The area between the curves $y = f(x)$ and $y = g(x)$ and the lines $x = a$ and $x = b$ (f, g continuous and $f(x) \geq g(x)$ for x in $[a, b]$) is

$$A = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \boxed{\int_a^b [f(x) - g(x)]\,dx}\,.$$

Calling $y_T = f(x)$, $y_B = g(x)$, we have:

$$\boxed{A = \int_a^b (y_T - y_B)\,dx}$$

Example: Find the area between $y = e^x$ and $y = x$ bounded on the sides by $x = 0$ and $x = 1$.

Answer: First note that $e^x \geq x$ for $0 \leq x \leq 1$. So:

$$\begin{aligned} A = \int_0^1 (e^x - x)\,dx &= \left[e^x - \frac{x^2}{2}\right]_0^1 = \left(e^1 - \frac{1^2}{2}\right) - \left(e^0 - \frac{0^2}{2}\right) \\ &= e - \tfrac{1}{2} - 1 = \boxed{e - \tfrac{3}{2}}\,. \end{aligned}$$

The area between two curves $y = f(x)$ and $y = g(x)$ that intersect at two points can be computed in the following way. First find the intersection points a and b by solving the equation $f(x) = g(x)$. Then find the difference:

$$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b [f(x) - g(x)]\,dx\,.$$

If the result is negative that means that we have subtracted wrong. Just take the result in absolute value.

Example: Find the area between $y = x^2$ and $y = 2 - x$. *Solution*: First, find the intersection points by solving $x^2-(2-x) = x^2+x-2 = 0$. We get $x = -2$ and $x = 1$. Next compute:

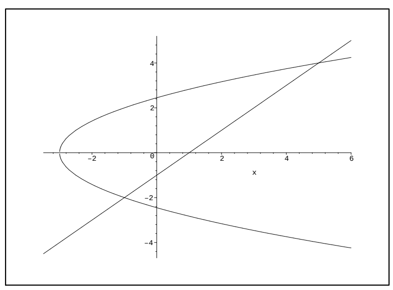
$$\int_{-2}^{1} (x^2 - (2 - x))\, dx = \int_{-2}^{1} (x^2 + x - 2)\, dx = -9/2\,.$$

Hence the area is 9/2.

Sometimes it is easier or more convenient to write x as a function of y and integrate respect to y. If $x_L(y) \le x_R(y)$ for $p \le y \le q$, then the area between the graphs of $x = x_L(y)$ and $x = x_R(y)$ and the horizontal lines $y = p$ and $y = q$ is:

$$\boxed{A = \int_{p}^{q} (x_R - x_L)\, dy}$$

Example: Find the area between the line $y = x-1$ and the parabola $y^2 = 2x + 6$.



Answer: The intersection points between those curves are $(-1, -2)$ and $(5, 4)$, but in the figure we can see that the region extends to the left of $x = -1$. In this case it is easier to write

$$x_L = \tfrac{1}{2}y^2 - 3\,, \qquad x_R = y + 1\,,$$

and integrate from $y = -2$ to $y = 4$:

$$\begin{aligned} A = \int_{-2}^{4} (x_R - x_L)\, dx &= \int_{-2}^{4} \left\{(y + 1) - (\tfrac{1}{2}y^2 - 3)\right\}\, dx \\ &= \int_{-2}^{4} \left(-\tfrac{1}{2}y^2 + y + 4\right)\, dx \\ &= \left[-\frac{y^3}{6} + \frac{y^2}{2} + 4y\right]_{-2}^{4} \\ &= \boxed{18} \end{aligned}$$