2.5. Applications to Physics and Engineering

2.5.1. Work. Work is the energy produced by a force F pushing a body along a given distance d. If the force is constant, the work done is the product

$$W = F \cdot d\,.$$

The SI (international) unit of work is the joule (J), which is the work done by a force of one Newton (N) pushing a body along one meter (m). In the American system a unit of work is the foot-pound. Since 1 N = 0.224809 lb and 1 m = 3.28084 ft, we have 1 J = 0.737561 ft lb.

More generally, assume that the force is variable and depends on the position. Let $F(x)$ be the force function. Assume that the force pushes a body from a point $x = a$ to another point $x = b$. In order to find the total work done by the force we divide the interval $[a, b]$ into small subintervals $[x_{i-1}, x_i]$ so that the change of $F(x)$ is small along each subinterval. Then the work done by the force in moving the body from x_{i-1} to x_i is approximately:

$$\Delta W_i \approx F(x_i^*)\,\Delta x\,,$$

where $\Delta x = x_i - x_{i-1} = (b-a)/n$ and x_i^* is any point in $[x_{i-1}, x_i]$. So, the total work is

$$W = \sum_{i=1}^{n} \Delta W_i \approx \sum_{i=1}^{n} F(x_i^*)\,\Delta x\,.$$

As $n \to \infty$ the Riemann sum at the right converges to the following integral:

$$W = \int_a^b F(x)\,dx\,.$$

2.5.2. Elastic Springs. Consider a spring on the x-axis so that its right end is at $x = 0$ when the spring is at its rest position. According to *Hook's Law*, the force needed to stretch the spring from 0 to x is proportional to x, i.e.:

$$F(x) = kx\,,$$

where k is the so called *spring constant*.

The energy needed to stretch the spring from 0 to a is then the integral

$$W = \int_0^a kx\,dx = k\frac{a^2}{2}\,.$$

2.5.3. Work Done Against Gravity. According to Newton's Law, the force of gravity at a distance r from the center of the Earth is

$$F(r) = \frac{k}{r^2}\,,$$

where k is some positive constant.

The energy needed to lift a body from a point at distance R_1 from the center of the Earth to another point at distance R_2 is given by the following integral

$$W = \int_{R_1}^{R_2} \frac{k}{r^2}\,dr = \left[-\frac{k}{r}\right]_{R_1}^{R_2} = k\left(\frac{1}{R_1} - \frac{1}{R_2}\right).$$

Example: Find the energy needed to lift 1000 Km a body whose weight is 1 N at the surface of Earth. The Earth radius is 6378 Km.

Answer: First we must determine the value of the constant k in this case. Since the weight of the body for $r = 6378$ Km is 1 N we have $k/6378^2 = 1$, so $k = 6378^2$. Next we have $R_1 = 6378$, $R_2 = 6378 + 1000 = 7378$, hence

$$W = 6378^2\left(\frac{1}{6378} - \frac{1}{7378}\right) = 864.462 \text{ N Km}\,.$$

Since 1 Km = 1000 m, the final result in joule is

$$864.462 \text{ N Km} = 864.462 \text{ N} \times 1000 \text{ m} = 864462 \text{ J}\,.$$

2.5.4. Work Done Filling a Tank. Consider a tank whose bottom is at some height $y = a$ and its top is at $y = b$. Assume that the area of its cross section is $A(y)$. We fill the tank by lifting from the ground ($y = 0$) tiny layers of thickness dy each. Their mass is $\rho A(y)\,dy$, where ρ is the density (mass per unit of volume) of the liquid that we are putting in the tank. We get their weight dF by multiplying by the acceleration of gravity g ($= 9.8\text{ m/s}^2 = 32\text{ ft/s}^2$), so $dF = \rho g A(y)\,dy$. The work needed to lift each layer is

$$dW = dF \cdot y = \rho g y A(y)\,dy\,.$$

Hence, the work needed to fill the tank is

$$W = \int_a^b \rho g y A(y)\,dy\,.$$

2.5.5. Emptying a Tank. Consider a tank like the one in the previous paragraph. Now we empty it by pumping its liquid to a fix height h. The analysis of the problem is similar to the previous paragraph, but now the work done to pump a tiny layer of thickness dy is

$$dW = dF \cdot (h - y) = \rho g\,(h - y)A(y)\,dy\,.$$

Hence the total work needed to empty the tank is

$$W = \int_a^b \rho g\,(h - y)A(y)\,dy\,.$$

2.5.6. Force Exerted by a Liquid Against a Vertical Wall. The pressure p of an homogeneous liquid of density ρ at depth h is

$$p = \rho g h\,.$$

When the pressure is constant, the force exerted by the liquid against a surface is the product of the pressure and the area of the surface. However the pressure against a vertical wall is not constant because it depends on the depth.

Assume that the surface of the liquid is at $y = c$ and we place a vertical plate of width $w(y)$ between $y = a$ and $y = b$. The force exerted at y (so at depth $h = c - y$) against a small horizontal strip of height dy and width $w(y)$ (area $= w(y)\,dy$) is

$$dF = \rho g\,(c - y)w(y)\,dy\,.$$

hence the total force is

$$F = \int_a^b \rho g\,(c - y)w(y)\,dy\,.$$

Example: A cylindrical tank of radius 1 m and full of water ($\rho = 1000$ Kg/m^3, $g = 9.8$m/s^2) is lying on its side. What is the pressure exerted by the water on its (vertical) bottom?

Answer: We assume the center of the tank is at $y = 0$, so the top of the liquid is at $y = 1$, and its bottom is at $y = -1$. On the other hand we obtain geometrically $w(y) = 2\sqrt{1 - y^2}$. Hence the total force is:

$$F = \int_{-1}^{1} 1000 \cdot 9.8 \cdot (1 - y)2\sqrt{1 - y^2}\,dy = 19800 \cdot \frac{\pi}{2} = 30787.6\ \mathrm{N}\,.$$