3.3. Exponential Growth and Decay

3.3.1. Natural Growth. Consider population with $P(t)$ individuals at time t and with constant *birth rate* β (births per unit of time) and *death* rate δ (deaths per unit of time). This basically means that if P does not change, then during a unit of time (say, a year), βP births and δP deaths will occur. Since P in fact varies, we need to use smaller intervals of time $[t, t+\Delta t]$ in which P can be considered almost constant. During such interval of time the number of births will be $\beta P \Delta t$, and the number of deaths $\delta P \Delta t$. So the change in the population will be

$$\Delta P = P(t+\Delta t) - P(t) \approx \beta P \Delta t - \delta P \Delta t\,.$$

Dividing by Δt and finding the limit as $\Delta t \to 0$ we get

$$P'(t) = (\beta - \delta) P(t)\,,$$

i.e.,

$$\frac{dP}{dt} = kP\,,$$

where $k = \beta - \delta$. With $x(t)$ in place of $P(t)$ we get the *Natural Growth Equation*:

$$\frac{dx}{dt} = kx\,.$$

This equation can be solved by separation of variables:

$$\begin{aligned} \frac{dx}{x} &= k\,dt \\ \int \frac{dx}{x} &= \int k\,dt + C \\ \ln x &= kt + C \\ x &= e^{kt+C} = Ae^{kt}\,, \end{aligned}$$

where $A = e^C$. Putting $t = 0$ we see that $A = x_0 = x(0)$, hence:

$$x(t) = x_0 e^{kt}\,.$$

Example: The current (year 2000) population of the Earth is 6 billion people, and the yearly birth and death rates are $\beta = 0.021$ and $\delta = 0.009$ respectively. Assuming the birth and death rates remain constant, find the population of the Earth in the year 2100.

Answer: For the purpose of the problem we can take $t = 0$ in the year 2000, so the year 2100 will correspond to $t = 100$. So we have

$x_0 = 6$ billion, and $k = \beta - \delta = 0.021 - 0.009 = 0.012$, hence:

$$P(t) = P_0 \, e^{kt} = 6 \, e^{0.012t}$$

in billions. So the solution is

$$P(100) = 6 \, e^{0.012 \times 100} = 19.92 \text{ billion} \, .$$

3.3.2. Radioactive Decay. Consider a given sample of radioactive material with $N(t)$ atoms at time t. During a given unit of time a fix fraction of these atoms will spontaneously decay, so the sample behaves like a population with a constant death rate and no births:

$$\frac{dN}{dt} = -kN \, ,$$

where $k > 0$ is the *decay constant*. The solution to this equation is

$$N(t) = N_0 \, e^{-kt} \, ,$$

where N_o is the number of atoms at time $t = 0$.

The *half-life* τ of the material is the time required for half of the sample to decay, i.e.:

$$\frac{1}{2} N_0 = N_0 \, e^{-k\tau} \, ,$$

so

$$\tau = \frac{\ln 2}{k} \, .$$

3.3.3. Radiocarbon Dating. The air in the atmosphere contains two carbon isotopes: ^{12}C, which is stable, and ^{14}C, which is radioactive with a half-life of about 5700 years—so $k = \ln 2/\tau = \ln 2/5700 = 0.0001216$.

While an organism is alive, it absorbs both carbon isotopes by breathing air, so the proportion of those isotopes in living matter is the same as in air. But when an organism dies, the ^{14}C in it keeps decaying without being replaced. So by measuring the proportion of ^{14}C in an organism we can estimate for how long it has been dead.

Example: A cadaver found in an old burial site has 80% as much ^{14}C as a current day human body. When did that individual die?

Answer: We have:

$$0.80 = e^{-kt} = e^{-kt}$$

Hence

$$t = -\frac{\ln 0.80}{k} = -\frac{\ln 0.80}{0.0001216} = 1835 \text{ years ago}\,.$$

3.3.4. Continuously Compounded Interest. Consider an account opened with an initial amount of A_0 dollars and an annual interest rate r. Let $A(t)$ be the number of dollars in the account at time t. Assume the interest is compounded after an interval of time Δt. The interest produced is $rA(t)\Delta t$, so

$$A(t+\Delta t) = A(t) + rA(t)\Delta t\,,$$

i.e.

$$\frac{\Delta A}{\Delta t} = rA(t)\,.$$

The limit for $\Delta t \to 0$ is called *continuously compounded interest*. In that case we get:

$$\frac{dA}{dt} = rA(t)\,.$$

The solution to this equation is

$$A(t) = A_0 e^{rt}\,.$$