CS 310
Homework Assignment No. 4
Due on Tue 2/10/2004

1. Find $27^{-1} \mod 34$.

2. Let a_n be the sequence defined recursively as follows: $a_0 = 1$, $a_{n+1} = 3^{a_n}$ $(n \geq 0)$. Its first few terms are $a_0 = 1$, $a_1 = 3^1 = 3$, $a_2 = 3^3 = 27$, $a_3 = 3^{27} = 7625597484987$, $a_4 = 3^{7625597484987}$ = a very large number, etc. Find the two rightmost digits of $a_{123456789}$.

3. There are 2 Mathematics books, 3 Physics books and 5 Computer Science books on a shelf (all books are different).

(a) In how many ways can the 10 books be placed on the shelf?

(b) In how many ways can they be placed if the books belonging to the same matter must remain together?

4. A city has streets in the direction E-W and avenues in the direction N-S, making a perfect grid. A taxicab has to go from the intersection of 23rd street and 1st avenue to the intersection of 30th street and 7th avenue following a path of minimum length.

(a) In how many ways can it be done?

(b) Assume that the taxicab needs gas and there is a gas station at the intersection of 27th street and 5th avenue. How many (minimal) paths go through that intersection?

(c) Assume that there is an accident at the intersection of 27th street and 5th avenue and the taxicab wants to avoid it. How many (minimal) paths can the taxicab follow now?

5. Prove the following identities involving binomial coefficients:

(a) $\displaystyle\sum_{k=0}^{n} \binom{n}{k} = 2^n.$

(b) $\displaystyle\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$

(c) $\displaystyle\sum_{k=0}^{n} \binom{n}{k} k = n2^{n-1}.$

(d) $\displaystyle\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$

[Hint: all those identities can be obtained playing around with $(1+x)^n$.]