CS 310-0

Homework Assignment No. 1

Due Fri 4/7/2000

1. Let p and q be primitive statements such that $p \to q$ is false. Find the truth value of the following:
(a) $q \to p$
(b) $\neg p \to \neg q$
(c) $\neg q \to \neg p$

2. We define the connective *nand* by:
$$p \uparrow q \Leftrightarrow \neg(p \wedge q)$$
Make its truth table. Write the following statements using $\uparrow$ only
(a) $\neg p$
(b) $p \wedge q$
(c) $p \vee q$
(d) $p \to q$
(e) $p \leftrightarrow q$
(For instance: $\neg p \Leftrightarrow p \uparrow p$.)

3. Use truth tables to determine if the following logical equivalences are correct:
(a) $p \wedge (q \veebar r) \Leftrightarrow (p \wedge q) \veebar (p \wedge r)$
(b) $(p \veebar q) \veebar r \Leftrightarrow p \veebar (q \veebar r)$
(c) $(p \to q) \to r \Leftrightarrow p \to (q \to r)$
(d) $(p \veebar q) \veebar r \Leftrightarrow (p \leftrightarrow q) \leftrightarrow r$

4. Prove the following logical equivalences by using laws of logic:
(a) $p \to (q \to r) \Leftrightarrow (p \wedge q) \to r$
(b) $(p \to q) \to r \Leftrightarrow (\neg p \to r) \wedge (q \to r)$

5. Alice says that Beth lies. Beth says that Charles lies. Charles says that both Alice and Beth lie. Who lies and who tells the truth?[1] (Formalize the premises and determine which combination of truth values is compatible with them.)

6. Consider the following premises:
   1. If A is large then B is small.
   2. If C is large then D is small.
   3. A is large or C is large.
   4. B is not small.

   Use an argument to prove that D is small.

[1]Each person is talking about what the others are saying; e.g., Alice says that what Beth says, i.e., "Charles lies", is false.

7. Consider the following statements:
   (a) $\forall x \forall y\,(x < y)$.
   (b) $\forall x \exists y\,(x < y)$.
   (c) $\exists x \forall y\,(x < y)$.
   (d) $\exists x \exists y\,(x < y)$.
   Determine their truth value assuming that the universe of discourse is:
   (1) The set of all integers.
   (2) The set of positive integers.
   (3) The set of negative integers.
   (4) The set $A = \{1, 2, 3, 4, 5\}$.

8. Find a model and a countermodel for each of the following statements:
   (a) $\exists x \exists y \exists z\,(x \neq y \wedge y \neq z \wedge x \neq z)$.
   (b) $\forall x \forall y \exists z\,(x < y \to x < z < y)$

9. Assume that the universe of discourse is the set of integers. Prove the following, stating the method or principle being used:
   (a) $\exists x\,(0 < x)$.
   (b) $\forall x\, \forall y\,(x + y < 8 \to x < 3 \vee y < 5)$.

10. Write the negation of the following quantified statement in prenex normal form, leaving the statement inside in conjunctive normal form:
$$\forall n\,\{n > 2 \to \neg[\exists x\,\exists y\,\exists z\,(x^n + y^n = z^n)]\}$$

11. (This problem is not to be graded, just think about it, and write your answer if you can think of any.) Criticize the following "proof" of the existence of Santa Claus: Let p be the statement "Santa Claus exists". Let q be the statement "p and q are false". Let us determine the truth value of q. Assume q is true. Then p and q must be false, in particular q is false, but this is a contradiction, so the assumption that q is true must be wrong. Hence q has to be false. This means that it is not true that p and q are false, so one of them must be true. But q is false, so p is true. Therefore, Santa Claus exists.