

1.10. Improper Integrals

1.10.1. Improper Integrals. Up to now we have studied integrals of the form

$$\int_a^b f(x) dx,$$

where f is a *continuous* function defined on the *closed* and *bounded* interval $[a, b]$. *Improper integrals* are integrals in which one or both of these conditions are not met, i.e.,

- (1) The interval of integration is not bounded:

$$[a, +\infty), \quad (-\infty, a], \quad (-\infty, +\infty),$$

e.g.:

$$\int_1^{\infty} \frac{1}{x^2} dx.$$

- (2) The integrand has an infinite discontinuity at some point c in $[a, b]$:

$$\lim_{x \rightarrow c} f(x) = \pm\infty.$$

e.g.:

$$\int_0^1 \frac{1}{\sqrt{x}} dx.$$

1.10.2. Infinite Limits of Integration. Improper Integrals of Type 1. In case one of the limits of integration is infinite, we define:

$$\int_a^{\infty} f(x) dx = \lim_{t \rightarrow \infty} \int_a^t f(x) dx,$$

or

$$\int_{-\infty}^a f(x) dx = \lim_{t \rightarrow -\infty} \int_t^a f(x) dx.$$

If both limits of integration are infinite, then we choose any c and define:

$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^c f(x) dx + \int_c^{\infty} f(x) dx.$$

If the limits defining the integral exist the integral is called *convergent*, otherwise it is called *divergent*.

Remark: Sometimes we write $[F(x)]_a^{\infty}$ as an abbreviation for

$$[F(x)]_a^{\infty} = \lim_{t \rightarrow \infty} [F(x)]_a^t.$$

Analogously:

$$[F(x)]_{-\infty}^a = \lim_{t \rightarrow -\infty} [F(x)]_t^a ,$$

and

$$[F(x)]_{-\infty}^{\infty} = [F(x)]_{-\infty}^c + [F(x)]_c^{\infty} = \lim_{t \rightarrow -\infty} [F(x)]_t^c + \lim_{t \rightarrow \infty} [F(x)]_c^t .$$

Example:

$$\int_1^{\infty} \frac{1}{x^2} dx = \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x^2} dx = \lim_{t \rightarrow \infty} \left[-\frac{1}{x} \right]_1^t = \lim_{t \rightarrow \infty} \left(-\frac{1}{t} + 1 \right) = 1 ,$$

or in simplified notation:

$$\int_1^{\infty} \frac{1}{x^2} dx = \left[-\frac{1}{x} \right]_1^{\infty} = \lim_{t \rightarrow \infty} \left(-\frac{1}{t} + 1 \right) = 1 .$$

Example: For what values of p is the following integral convergent?:

$$\int_1^{\infty} \frac{1}{x^p} dx .$$

Answer: If $p = 1$ then we have

$$\int_1^t \frac{1}{x} dx = [\ln x]_1^t = \ln t ,$$

so

$$\int_1^{\infty} \frac{1}{x} dx = \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x} dx = \lim_{t \rightarrow \infty} \ln t = \infty ,$$

and the integral is divergent. Now suppose $p \neq 1$:

$$\int_1^t \frac{1}{x^p} dx = \left[\frac{x^{-p+1}}{-p+1} \right]_1^t = \frac{1}{1-p} \left\{ \frac{1}{t^{p-1}} - 1 \right\}$$

If $p > 1$ then $p - 1 > 0$ and

$$\int_1^{\infty} \frac{1}{x^p} dx = \lim_{t \rightarrow \infty} \frac{1}{1-p} \left\{ \frac{1}{t^{p-1}} - 1 \right\} = 0 ,$$

hence the integral is convergent. On the other hand if $p < 1$ then $p - 1 < 0$, $1 - p > 0$ and

$$\int_1^{\infty} \frac{1}{x^p} dx = \lim_{t \rightarrow \infty} \frac{1}{1-p} \{ t^{1-p} - 1 \} = \infty ,$$

hence the integral is divergent. So:

$$\int_1^{\infty} \frac{1}{x^p} dx \text{ is convergent if } p > 1 \text{ and divergent if } p \leq 1 .$$

1.10.3. Infinite Integrands. Improper Integrals of Type 2.

Assume f is defined in $[a, b)$ but

$$\lim_{x \rightarrow b^-} f(x) = \pm\infty.$$

Then we define

$$\int_a^b f(x) dx = \lim_{t \rightarrow b^-} \int_a^t f(x) dx.$$

Analogously, if f is defined in $(a, b]$ but

$$\lim_{x \rightarrow a^+} f(x) = \pm\infty.$$

Then we define

$$\int_a^b f(x) dx = \lim_{t \rightarrow a^+} \int_t^b f(x) dx.$$

Finally, if $f(x)$ has an infinite discontinuity at c inside $[a, b]$, then the definition is

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx.$$

If the limits defining the integral exist the integral is called *convergent*, otherwise it is called *divergent*.

Remark: If the interval of integration is $[a, b)$ sometimes we write $[F(x)]_a^b$ as an abbreviation for $\lim_{t \rightarrow b^-} [F(x)]_a^t$ —and analogously for intervals of the form $(a, b]$.

Example:

$$\int_0^1 \frac{1}{\sqrt{x}} dx = \lim_{t \rightarrow 0^-} \int_t^1 \frac{1}{\sqrt{x}} dx = \lim_{t \rightarrow 0^-} [2\sqrt{x}]_t^1 = \lim_{t \rightarrow 0^-} (2 - 2\sqrt{t}) = 2,$$

or in simplified notation:

$$\int_0^1 \frac{1}{\sqrt{x}} dx = [2\sqrt{x}]_0^1 = \lim_{t \rightarrow 0^-} (2 - 2\sqrt{t}) = 2.$$

Example: Evaluate $\int_0^1 \ln x dx$.

Answer: The function $\ln x$ has a vertical asymptote at $x = 0$ because $\lim_{x \rightarrow 0^+} \ln x = -\infty$. Hence:

$$\begin{aligned} \int_0^1 \ln x \, dx &= \lim_{t \rightarrow 0^+} \int_t^1 \ln x \, dx \\ &= \lim_{t \rightarrow 0^+} [x \ln x - x]_t^1 \\ &= \lim_{t \rightarrow 0^+} \{(1 \ln 1 - 1) - (t \ln t - t)\} \\ &= \lim_{t \rightarrow 0^+} \{t - 1 - t \ln t\} \quad \left(\lim_{t \rightarrow 0^+} t \ln t = 0 \right) \\ &= \boxed{-1}. \end{aligned}$$

1.10.4. Comparison Test for Improper Integrals. Suppose f and g are continuous functions such that $f(x) \geq g(x) \geq 0$ for $x \geq 0$.

- (1) If $\int_a^\infty f(x) \, dx$ is convergent then $\int_a^\infty g(x) \, dx$ is convergent.
- (2) If $\int_a^\infty g(x) \, dx$ is divergent then $\int_a^\infty f(x) \, dx$ is divergent.

A similar statement holds for type 2 integrals.

Example: Prove that $\int_0^\infty e^{-x^2} \, dx$ is convergent.

Answer: We have:

$$\int_0^\infty e^{-x^2} \, dx = \int_0^1 e^{-x^2} \, dx + \int_1^\infty e^{-x^2} \, dx.$$

The first integral on the right hand side is an ordinary definite integral so we only need to show that the second integral is convergent. In fact, for $x \geq 1$ we have $x^2 \geq x$, so $e^{-x^2} \leq e^{-x}$. On the other hand:

$$\int_1^t e^{-x} \, dx = [-e^{-x}]_1^t = -e^{-t} + e^{-1},$$

hence

$$\int_1^\infty e^{-x} \, dx = \lim_{t \rightarrow \infty} (-e^{-t} + e^{-1}) = e^{-1},$$

so $\int_1^\infty e^{-x} \, dx$ is convergent. Hence, by the comparison theorem $\int_1^\infty e^{-x^2} \, dx$ is convergent, QED.