2.4. Functions

2.4.1. Correspondences. Suppose that to each element of a set A we assign some elements of another set B. For instance, $A = \mathbb{N}$, $B = \mathbb{Z}$, and to each element $x \in \mathbb{N}$ we assign all elements $y \in \mathbb{Z}$ such that $y^2 = x$ (fig. 2.11).

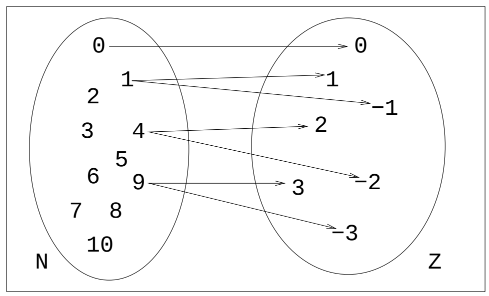

FIGURE 2.11. Correspondence $x \mapsto \pm\sqrt{x}$.

This operation can be interpreted as a relation, but when we want to stress the fact that it is an assignment of some elements to other elements, we call it a *correspondence*.

2.4.2. Functions. A *function* or *mapping* f from a set A to a set B, denoted $f : A \to B$, is a correspondence in which to each element x of A corresponds exactly one element $y = f(x)$ of B (fig. 2.12).

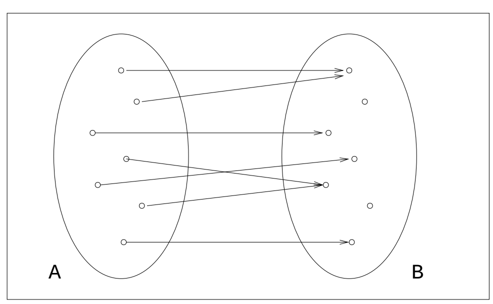

FIGURE 2.12. Function.

Sometimes we represent the function with a diagram like this:

$$\begin{aligned} f : A &\to B \\ x &\mapsto y \end{aligned} \qquad \text{or} \qquad \begin{aligned} A &\xrightarrow{f} B \\ x &\mapsto y \end{aligned}$$

For instance, the following represents the function from $\mathbb{Z}$ to $\mathbb{Z}$ defined by $f(x) = 2x + 1$:

$$\begin{aligned} f : \mathbb{Z} &\to \mathbb{Z} \\ x &\mapsto 2x + 1 \end{aligned}$$

The element $y = f(x)$ is called the *image* of x, and x is a *preimage* of y. For instance, if $f(x) = 2x + 1$ then $f(7) = 2 \cdot 7 + 1 = 15$. The set A is the *domain* of f, and B is its *codomain*. If $A' \subseteq A$, the image of A' by f is $f(A') = \{f(x) \mid x \in A'\}$, i.e., the subset of B consisting of all images of elements of A'. The subset $f(A)$ of B consisting of all images of elements of A is called the *range* of f. For instance, the range of $f(x) = 2x + 1$ is the set of all integers of the form $2x + 1$ for some integer x, i.e., all odd numbers.

Example: Two useful functions from $\mathbb{R}$ to $\mathbb{Z}$ are the following:

1. The *floor* function:

   $$\lfloor x \rfloor = \text{greatest integer less than or equal to } x\,.$$

   For instance: $\lfloor 2 \rfloor = 2$, $\lfloor 2.3 \rfloor = 2$, $\lfloor \pi \rfloor = 3$, $\lfloor -2.5 \rfloor = -3$.

2. The *ceiling* function:

   $$\lceil x \rceil = \text{least integer greater than or equal to } x\,.$$

   For instance: $\lceil 2 \rceil = 2$, $\lceil 2.3 \rceil = 3$, $\lceil \pi \rceil = 4$, $\lceil -2.5 \rceil = -2$.

Example: The *modulus operator* is the function $\mathrm{mod} : \mathbb{Z} \times \mathbb{Z}^+ \to \mathbb{Z}$ defined:

$$x \bmod y = \text{ remainder when } x \text{ is divided by } y.$$

For instance $23 \bmod 7 = 2$ because $23 = 3 \cdot 7 + 2$, $59 \bmod 9 = 5$ because $59 = 6 \cdot 9 + 5$, etc.

Graph: The *graph* of a function $f : A \to B$ is the subset of $A \times B$ defined by $G(f) = \{(x, f(x)) \mid x \in A\}$ (fig. 2.13).

2.4.3. Types of Functions.

1. *One-to-One* or *Injective*: A function $f : A \to B$ is called *one-to-one* or *injective* if each element of B is the image of at most one element of A (fig. 2.14):

   $$\forall x, x' \in A,\ f(x) = f(x') \Rightarrow x = x'\,.$$

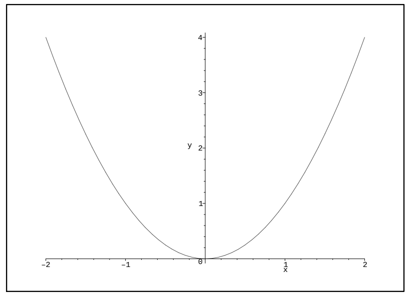

FIGURE 2.13. Graph of $f(x) = x^2$.

For instance, $f(x) = 2x$ from $\mathbb{Z}$ to $\mathbb{Z}$ is injective.

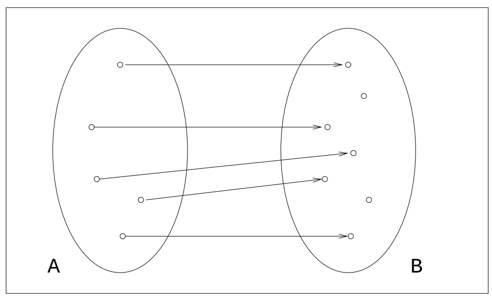

FIGURE 2.14. One-to-one function.

2. *Onto* or *Surjective*: A function $f : A \to B$ is called *onto* or *surjective* if every element of B is the image of some element of A (fig. 2.15):

$$\forall y \in B,\ \exists x \in A \text{ such that } y = f(x)\,.$$

For instance, $f(x) = x^2$ from $\mathbb{R}$ to $\mathbb{R}^+ \cup \{0\}$ is onto.

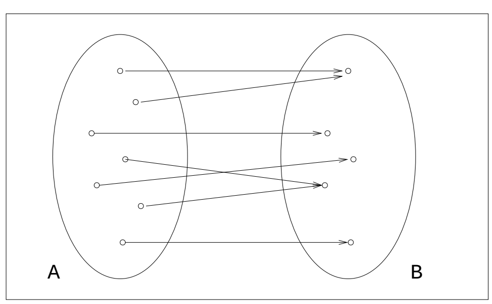

FIGURE 2.15. Onto function.

3. *Bijective Function* or *Bijection*: A function $f : A \to B$ is said to be *bijective* or a *bijection* if it is one-to-one and onto (fig. 2.16). For instance, $f(x) = x + 3$ from $\mathbb{Z}$ to $\mathbb{Z}$ is a bijection.

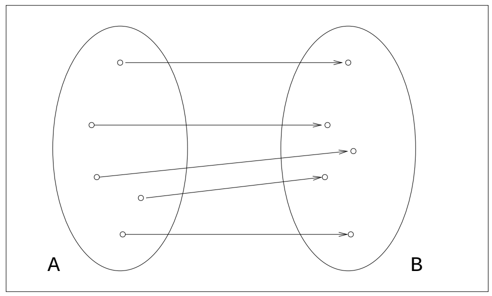

FIGURE 2.16. Bijection.

2.4.4. Identity Function. Given a set A, the function $1_A : A \to A$ defined by $1_A(x) = x$ for every x in A is called the *identity function* for A.

2.4.5. Function Composition. Given two functions $f : A \to B$ and $g : B \to C$, the *composite function* of f and g is the function $g \circ f : A \to C$ defined by $(g \circ f)(x) = g(f(x))$ for every x in A:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend left, "g\circ f"] & B \arrow[r, "g"] & C \\ x \arrow[r, mapsto] & y=f(x) \arrow[r, mapsto] & z=g(y)=g(f(x)) \end{tikzcd}$$

For instance, if $A = B = C = \mathbb{Z}$, $f(x) = x + 1$, $g(x) = x^2$, then $(g \circ f)(x) = f(x)^2 = (x+1)^2$. Also $(f \circ g)(x) = g(x) + 1 = x^2 + 1$ (the composition of functions is not commutative in general).

Some properties of function composition are the following:

1. If $f : A \to B$ is a function from A to B, we have that $f \circ 1_A = 1_B \circ f = f$.

2. Function composition is associative, i.e., given three functions
$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D\,,$$
we have that $h \circ (g \circ f) = (h \circ g) \circ f$.

Function iteration. If $f : A \to A$ is a function from A to A, then it makes sense to compose it with itself: $f^2 = f \circ f$. For instance, if $f : \mathbb{Z} \to \mathbb{Z}$ is $f(x) = 2x + 1$, then $f^2(x) = 2(2x + 1) + 1 = 4x + 3$. Analogously we can define $f^3 = f \circ f \circ f$, and so on, $f^n = f \circ \overset{(n \text{ times})}{\cdots} \circ f$.

2.4.6. Inverse Function. If $f : A \to B$ is a bijective function, its inverse is the function $f^{-1} : B \to A$ such that $f^{-1}(y) = x$ if and only if $f(x) = y$.

For instance, if $f : \mathbb{Z} \to \mathbb{Z}$ is defined by $f(x) = x + 3$, then its inverse is $f^{-1}(x) = x - 3$.

The arrow diagram of f^{-1} is the same as the arrow diagram of f but with all arrows reversed.

A characteristic property of the inverse function is that $f^{-1} \circ f = 1_A$ and $f \circ f^{-1} = 1_B$.

2.4.7. Operators. A function from $A \times A$ to A is called a *binary operator* on A. For instance the addition of integers is a binary operator $+ : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$. In the usual notation for functions the sum of two integers x and y would be represented $+(x, y)$. This is called *prefix* notation. The *infix* notation consists of writing the symbol of the binary operator between its arguments: $x + y$ (this is the most common). There is also a *postfix* notation consisting of writing the symbol after the arguments: $x\, y\, +$.

Another example of binary operator on $\mathbb{Z}$ is $(x, y) \mapsto x \cdot y$.

A *monary* or *unary operator* on A is a function from A to A. For instance the change of sign $x \mapsto -x$ on $\mathbb{Z}$ is a unary operator on $\mathbb{Z}$. An example of unary operator on $\mathbb{R}^*$ (non-zero real numbers) is $x \mapsto 1/x$.