4.4. Other Convergence Tests

4.4.1. Alternating Series. An *alternating series* is a series whose terms are alternately positive and negative., for instance

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \,.$$

4.4.1.1. *The Alternating Series Test.* If the sequence of positive terms b_n verifies

(1) b_n is decreasing.
(2) $\lim_{n\to\infty} b_n = 0$

then the alternating series

$$\sum_{n=1}^{\infty} (-1)^n b_n = b_1 - b_2 + b_3 - b_4 + \cdots$$

converges.

Example: The alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

converges because $1/n \to 0$. (Its sum is $\ln 2 = 0.6931471806\ldots$.)

4.4.1.2. *Alternating Series Estimation Theorem.* If $s = \sum_{n=1}^{\infty} (-1)^n b_n$ is the sum of and alternating series verifying that b_n is decreasing and $b_n \to 0$, then the remainder of the series verifies:

$$|R_n| = |s - s_n| \leq b_{n+1} \,.$$

4.4.2. Absolute Convergence. A series $\sum_{n=1}^{\infty} a_n$ is called *absolutely convergent* if the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ converges.

Absolute convergence implies convergence, i.e., if a series $\sum a_n$ is absolutely convergent, then it is convergent.

The converse is not true in general. For instance, the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is convergent but it is not absolutely convergent.

Example: Determine whether the series

$$\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$$

is convergent or divergent.

Answer: We see that the series of absolute values $\sum_{n=1}^{\infty} \frac{|\cos n|}{n^2}$ is convergent by comparison with $\sum_{n=1}^{\infty} \frac{1}{n^2}$, hence the given series is absolutely convergent, therefore it is convergent (its sum turns out to be $1/4 - \pi/2 + \pi^2/6 = 0.324137741\ldots$, but the proof of this is beyond the scope of this notes).

4.4.3. The Ratio Test.

(1) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$ then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(2) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ (including $L = \infty$) then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(3) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$ then the test is inconclusive (we do not know whether the series converges or diverges).

Example: Test the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{n!}{n^n}$$

for absolute convergence.

Answer: We have:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!/(n+1)^{n+1}}{n!/n^n} = \frac{n^n}{(n+1)^n} = \frac{1}{\left(1+\frac{1}{n}\right)^n} \xrightarrow[n\to\infty]{} e^{-1} < 1\,,$$

hence by the Ratio Test the series is absolutely convergent.

4.5. Power Series

A *power series* is a series of the form

$$\sum_{n=0}^{\infty} c_0 x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$$

where x is a variable of indeterminate. It can be interpreted as an infinite polynomial. The c_n's are the *coefficients* of the series. The sum of the series is a function

$$f(x) = \sum_{n=0}^{\infty} c_0 x^n$$

For instance the following series converges to the function shown for $-1 < x < 1$:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots = \frac{1}{1-x}.$$

More generally given a fix number a, a *power series in* $(x-a)$, or *centered in* a, or *about* a, is a series of the form

$$\sum_{n=0}^{\infty} c_0 (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots$$

4.5.1. Convergence of Power Series. For a given power series $\sum_{n=1}^{\infty} c_n(x-a)^n$ there are only three possibilities:

(1) The series converges only for $x = a$.
(2) The series converges for all x.
(3) There is a number R, called *radius of convergence*, such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

The *interval of convergence* is the set of values of x for which the series converges.

Example: Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(x-3)^n}{n}.$$

Answer: We use the Ratio Test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(x-3)^{n+1}/(n+1)}{(x-3)^n/n} = (x-3)\frac{n}{n+1} \xrightarrow[n\to\infty]{} x-3\,,$$

So the power series converges if $|x-3| < 1$ and diverges if $|x-3| > 1$. Consequently, the radius of convergence is $R = 1$. On the other hand, we know that the series converges inside the interval $(2,4)$, but it remains to test the endpoints of that interval. For $x = 4$ the series becomes

$$\sum_{n=0}^{\infty} \frac{1}{n}\,,$$

i.e., the harmonic series, which we know diverges. For $x = 2$ the series is

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n}\,,$$

i.e., the alternating harmonic series, which converges. So the interval of convergence is $[2,4)$.