4.5. Power Series

A *power series* is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$$

where x is a variable of indeterminate. It can be interpreted as an infinite polynomial. The c_n's are the *coefficients* of the series. The sum of the series is a function

$$f(x) = \sum_{n=0}^{\infty} c_0 x^n$$

For instance the following series converges to the function shown for $-1 < x < 1$:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots = \frac{1}{1-x}\,.$$

More generally given a fix number a, a *power series in* $(x-a)$, or *centered in* a, or *about* a, is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots$$

4.5.1. Convergence of Power Series.

For a given power series $\sum_{n=1}^{\infty} c_n (x-a)^n$ there are only three possibilities:

(1) The series converges only for $x = a$.
(2) The series converges for all x.
(3) There is a number R, called *radius of convergence*, such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

The *interval of convergence* is the set of values of x for which the series converges.

Example: Find the radius of convergence and interval of convergence of the series

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}\,.$$

Answer: We use the Ratio Test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(x-3)^{n+1}/(n+1)}{(x-3)^n/n} = (x-3)\frac{n}{n+1} \underset{n\to\infty}{\longrightarrow} x-3\,,$$

So the power series converges if $|x-3| < 1$ and diverges if $|x-3| > 1$. Consequently, the radius of convergence is $R = 1$. On the other hand, we know that the series converges inside the interval $(2,4)$, but it remains to test the endpoints of that interval. For $x = 4$ the series becomes

$$\sum_{n=1}^{\infty} \frac{1}{n}\,,$$

i.e., the harmonic series, which we know diverges. For $x = 2$ the series is

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\,,$$

i.e., the alternating harmonic series, which converges. So the interval of convergence is $[2,4)$.