

## 1.7. Partial Fractions

**1.7.1. Rational Functions and Partial Fractions.** A rational function is a quotient of two polynomials:

$$R(x) = \frac{P(x)}{Q(x)}.$$

Here we discuss how to integrate rational functions. The idea consists of rewriting the rational function as a sum of simpler fractions called *partial fractions*. This can be done in the following way:

- (1) Use long division of polynomials to get a quotient  $p(x)$  and a remainder  $r(x)$ . Then write:

$$R(x) = \frac{P(x)}{Q(x)} = p(x) + \frac{r(x)}{Q(x)},$$

where the degree of  $r(x)$  is less than that of  $Q(x)$ .

- (2) Factor the denominator  $Q(x) = q_1(x)q_2(x)\dots q_n(x)$ , where each factor  $q_i(x)$  is either linear  $ax+b$ , or irreducible quadratic  $ax^2+bx+c$ , or a power of the form  $(ax+b)^n$  or  $(ax^2+bx+c)^n$ .
- (3) Decompose  $r(x)/Q(x)$  into partial fractions of the form:

$$\frac{r(x)}{Q(x)} = F_1(x) + F_2(x) + F_3(x) + \dots$$

where each fraction is of the form

$$F_i(x) = \frac{A}{(ax+b)^k}$$

or

$$F_i(x) = \frac{Ax+B}{(ax^2+bx+c)^k},$$

where  $1 \leq k \leq n$  ( $n$  is the exponent of  $ax+b$  or  $ax^2+bx+c$  in the factorization of  $Q(x)$ .)

*Example:* Decompose the following rational function into partial fractions:

$$R(x) = \frac{x^3 + x^2 + 2}{x^2 - 1}$$

*Answer:*

$$(1) \frac{x^3 + x^2 + 2}{x^2 - 1} = x + 1 + \frac{x + 3}{x^2 - 1}$$

$$(2) x^2 - 1 = (x + 1)(x - 1).$$

$$(3) \frac{x + 3}{(x + 1)(x - 1)} = \frac{A}{x + 1} + \frac{B}{x - 1}.$$

Multiplying by  $(x + 1)(x - 1)$  we get:

$$x + 3 = A(x - 1) + B(x + 1). \quad (*)$$

Now there are two ways of finding  $A$  and  $B$ :

Method 1. Expand the right hand side of  $(*)$ , collect terms with the same power of  $x$ , and identify coefficients of the polynomials obtained on both sides:

$$x + 3 = (A + B)x + (B - A),$$

Hence:

$$\begin{cases} 1 = A + B & \text{(coefficient of } x) \\ 3 = -A + B & \text{(constant term)} \end{cases}$$

Method 2. In  $(*)$  give  $x$  two different values (as many as the number of coefficients to determine), say  $x = 1$  and  $x = -1$ . We get:

$$\begin{cases} 4 = 2B & (x = 1) \\ 2 = -2A & (x = -1) \end{cases}$$

The solution to the system of equations obtained in either case is  $A = -1$ ,  $B = 2$ , so:

$$\frac{x + 3}{(x + 1)(x - 1)} = -\frac{1}{x + 1} + \frac{2}{x - 1}.$$

Finally:

$$R(x) = \frac{x^3 + x^2 + 2}{x^2 - 1} = x + 1 - \frac{1}{x + 1} + \frac{2}{x - 1}.$$

**1.7.2. Factoring a Polynomial.** In order to factor a polynomial  $Q(x)$  (with real coefficients) into linear or irreducible quadratic factors, first solve the algebraic equation:

$$Q(x) = 0.$$

Then for each real root  $r$  write a factor of the form  $(x - r)^k$  where  $k$  is the multiplicity of the root. For each pair of conjugate complex roots  $r, \bar{r}$  write a factor  $(x^2 - sx + p)^k$ , where  $s = r + \bar{r}$ ,  $p = r \cdot \bar{r}$ , and  $k$  is the

multiplicity of those roots. Finally multiply by the leading coefficient of  $Q(x)$ .

Note that the equation  $Q(x) = 0$  is sometimes hard to solve, or only the real roots can be easily found (when they are integral or rational they can be found by Ruffini's rule, or just by trial and error). In that case we get as many roots as we can, and divide  $Q(x)$  by the factors found. The quotient is another polynomial  $q(x)$  which we must now try to factor. So pose the algebraic equation

$$q(x) = 0$$

and try to solve it for this new (and simpler) polynomial.

*Example:* Factor the polynomial

$$Q(x) = x^6 - x^5 - 15x^4 + 5x^3 + 70x^2 + 12x - 72.$$

*Answer:* The roots of  $Q(x)$  are 1 (simple),  $-2$  (triple) and 3 (double), hence:

$$Q(x) = (x - 1)(x + 2)^3(x - 3)^2.$$

*Example:* Factor  $Q(x) = x^3 + 2x^2 + 2x + 1$ .

*Answer:*  $Q(x)$  has a simple real root  $x = -1$ . After dividing  $Q(x)$  by  $x + 1$  we get the polynomial  $x^2 + x + 1$ , which is irreducible (it has only complex roots), so we factor  $Q(x)$  like this:

$$Q(x) = (x + 1)(x^2 + x + 1).$$

**1.7.3. Decomposing Into Partial Fractions.** Assume that  $Q(x)$  has already been factored and degree of  $r(x)$  is less than degree of  $Q(x)$ . Then  $r(x)/Q(x)$  is decomposed into partial fractions in the following way:

- (1) For each factor of the form  $(x - r)^k$  write

$$\frac{A_1}{x - r} + \frac{A_2}{(x - r)^2} + \frac{A_3}{(x - r)^3} + \cdots + \frac{A_k}{(x - r)^k},$$

where  $A_1 \dots A_k$  are coefficients to be determined.

- (2) For each factor of the form  $(ax^2 + bx + c)^k$  write

$$\frac{B_1x + C_1}{ax^2 + bx + c} + \frac{B_2x + C_2}{(ax^2 + bx + c)^2} + \cdots + \frac{B_kx + C_k}{(ax^2 + bx + c)^k}$$

where  $B_1 \dots B_k$  and  $C_1 \dots C_k$  are coefficients to be determined.

- (3) Multiply by  $Q(x)$  and simplify. This leads to an expression of the form

$$r(x) = \text{some polynomial containing the} \\ \text{indeterminate coefficients } A_i, B_i, C_i.$$

Finally determine the coefficients  $A_i, B_i, C_i$ . One way of doing this is by identifying coefficients of the polynomials on both sides of the last expression. Another way is to write a system of equations with unknowns  $A_i, B_i, C_i$  by giving  $x$  various values.

*Example:* Decompose the following rational function into partial fractions:

$$R(x) = \frac{4x^5 - 2x^4 + 2x^3 - 8x^2 - 2x - 3}{(x-1)^2(x^2+x+1)^2}.$$

*Answer:* The denominator is already factored, so we proceed with the next step:

$$\frac{4x^5 - 2x^4 + 2x^3 - 8x^2 - 2x - 3}{(x-1)^2(x^2+x+1)^2} = \\ \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+x+1} + \frac{Ex+F}{(x^2+x+1)^2}.$$

Next we multiply by the denominator:

$$4x^5 - 2x^4 + 2x^3 - 8x^2 - 2x - 3 = \\ A(x-1)(x^2+x+1)^2 + B(x^2+x+1)^2 \\ + (Cx+D)(x-1)^2(x^2+x+1) + (Ex+F)(x-1)^2 = \\ (A+C)x^5 + (A-C+D+B)x^4 \\ + (A+2B+E-D)x^3 + (-A+3B-C-2E+F)x^2 \\ + (-A+2B+C-D+E-2F)x + (-A+B+D+F).$$

Identifying coefficients on both sides we get:

$$\left\{ \begin{array}{lclclclclcl} A & & & + & C & & & & = & 4 \\ A & + & B & - & C & + & D & & = & -2 \\ A & + & 2B & & & - & D & + & E & = & 2 \\ -A & + & 3B & - & C & & & - & 2E & + & F & = & -8 \\ -A & + & 2B & + & C & - & D & + & E & - & 2F & = & -2 \\ -A & + & B & & & + & D & & & + & F & = & -3 \end{array} \right.$$

The solution to this system of equations is  $A = 2, B = -1, C = 2, D = -1, E = 1, F = 1$ , hence:

$$\frac{4x^5 - 2x^4 + 2x^3 - 8x^2 - 2x - 3}{(x-1)^2(x^2+x+1)^2} = \frac{2}{x-1} - \frac{1}{(x-1)^2} + \frac{2x-1}{x^2+x+1} + \frac{x+1}{(x^2+x+1)^2}$$

**1.7.4. Integration of Rational Functions.** After decomposing the rational function into partial fractions all we need to do is to integrate expressions of the form  $A/(x-r)^k$  and  $(Bx+C)/(ax^2+bx+c)^k$ . For the former we get:

$$\begin{aligned} \int \frac{A}{(x-r)^k} dx &= -\frac{A}{(k-1)(x-r)^{k-1}} + C && \text{if } k \neq 1 \\ \int \frac{A}{x-r} dx &= A \ln|x-r| + C && \text{if } k = 1 \end{aligned}$$

The latter are more involved, but the following are particularly simple special cases:

$$\begin{aligned} \int \frac{1}{x^2+a^2} dx &= \frac{1}{a} \arctan \frac{x}{a} + C \\ \int \frac{x}{x^2+a^2} dx &= \frac{1}{2} \ln(x^2+a^2) + C \\ \int \frac{x}{(x^2+a^2)^k} dx &= -\frac{1}{2(k-1)(x^2+a^2)^{k-1}} + C \quad (k \neq 1) \end{aligned}$$

*Example:* Find the following integral:  $\int \frac{x^3 - x^2 - 7x + 8}{x^2 - 4x + 4} dx$ .

*Answer:* First we decompose the integrand into partial fractions:

$$(1) \quad \frac{x^3 - x^2 - 7x + 8}{x^2 - 4x + 4} = x + 3 + \frac{x-4}{x^2 - 4x + 4}$$

$$(2) \quad x^2 - 4x + 4 = (x-2)^2.$$

$$(3) \quad \frac{x-4}{(x-2)^2} = \frac{A}{x-2} + \frac{B}{(x-2)^2}$$

$$x-4 = A(x-2) + B$$

$$x=2 \quad \Rightarrow \quad -2 = B$$

$$x=3 \quad \Rightarrow \quad -1 = A + B$$

So  $A = 1$ ,  $B = -2$ , and

$$\frac{x-4}{x^2-4x+4} = \frac{1}{x-2} - \frac{2}{(x-2)^2}$$

Hence:

$$\frac{x^3 - x^2 - 7x + 8}{x^2 - 4x + 4} = x + 3 + \frac{1}{x-2} - \frac{2}{(x-2)^2}.$$

Finally we integrate:

$$\begin{aligned} \int \frac{x^3 - x^2 - 7x + 8}{(x-2)^2} dx &= \int (x+3) dx + \int \frac{1}{x-2} dx - \int \frac{2}{(x-2)^2} dx \\ &= \boxed{\frac{x^2}{2} + 3x + \ln|x-2| + \frac{2}{x-2} + C}. \end{aligned}$$

**1.7.5. Completing the Square.** Many integrals containing an irreducible (no real roots) quadratic polynomial  $ax^2 + bx + c$  can be simplified by completing the square, i.e., writing the polynomial as  $u^2 + r$  where  $u = px + q$ , e.g.:

$$x^2 + 2x + 2 = (x+1)^2 + 1.$$

In general:

$$ax^2 + bx + c = \left(x\sqrt{a} + \frac{b}{2\sqrt{a}}\right)^2 - \frac{b^2 - 4ac}{4a}.$$

If  $a = 1$  the formula can be simplified:

$$x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 + \left(c - \frac{b^2}{4}\right).$$

This result is of the form  $u^2 \pm A^2$ , where  $u = x + b/2$ .

*Example:*

$$\begin{aligned} \int \frac{1}{x^2 + 6x + 10} dx &= \int \frac{1}{(x+3)^2 + 1} dx \\ &= \int \frac{1}{u^2 + 1} du \quad (u = x+3) \\ &= \tan^{-1} u + C \\ &= \boxed{\tan^{-1}(x+3) + C}. \end{aligned}$$

*Example:*

$$\begin{aligned}\int \sqrt{x^2 - 4x + 5} \, dx &= \int \sqrt{(x - 2)^2 + 1} \, dx \\&= \int \sqrt{u^2 + 1} \, du && (u = x - 2) \\&= \int \sqrt{\tan^2 t + 1} \cdot \sec^2 t \, dt && (u = \tan t) \\&= \int \sec^3 t \, dt \\&= \frac{\sec t \tan t}{2} + \frac{1}{2} \ln |\sec t + \tan t| + C \\&= \frac{u \sqrt{u^2 + 1}}{2} + \frac{1}{2} \ln |u + \sqrt{u^2 + 1}| + C \\&= \frac{(x - 2) \sqrt{x^2 - 4x + 5}}{2} \\&\quad + \frac{1}{2} \ln \left| (x - 2) + \sqrt{x^2 - 4x + 5} \right| + C.\end{aligned}$$