MATH 214-2 - Fall 2001 - First Midterm (solutions)

SOLUTIONS

1. (Numerical Methods) Calculate both the *trapezoidal* approximation T_4 and *Simpson's* approximation S_4 for $n = 4$ to the integral

$$\int_0^4 x^3\,dx$$

Solution:

For $n = 4$ we have $\Delta x = (b-a)/n = (4-0)/4 = 1$, and $y_0 = 0^3 = 0$, $y_1 = 1^3 = 1$, $y_2 = 2^3 = 8$, $y_3 = 3^3 = 27$, $y_4 = 4^3 = 64$, hence

Trapezoidal approximation:

$$T_4 = \frac{\Delta x}{2}\,(y_0 + 2y_2 + 2y_2 + 2y_3 + y_4) = \frac{1}{2}\,(0 + 2\cdot 1 + 2\cdot 8 + 2\cdot 27 + 64) = \boxed{68}\,.$$

Simpson's approximation:

$$S_4 = \frac{\Delta x}{3}\,(y_0 + 4y_2 + 2y_4 + 4y_3 + y_4) = \frac{1}{3}\,(0 + 4\cdot 1 + 2\cdot 8 + 4\cdot 27 + 64) = \boxed{64}\,.$$

2. (Setting Up Integrals) Find the net distance and the total distance traveled between time $t = 0$ and $t = 7$ by a particle moving at velocity $v = 60 - 10t$ along a line.

Solution:

Net distance:

$$\int_0^7 v\,dt = \int_0^7 (60 - 10t)\,dt = \left[60t - 5t^2\right]_0^7 = \boxed{175}.$$

Total distance:

$$\int_0^7 |v|\,dt = \int_0^7 |60 - 10t|\,dt = \int_0^6 (60 - 10t)\,dt + \int_6^7 -(60 - 10t)\,dt$$
$$= \left[60t - 5t^2\right]_0^6 + \left[-60t + 5t^2\right]_6^7 = 180 + 5 = \boxed{185}.$$

3. (Volumes by Slices) A solid is generated by revolving the plane region between the curves $y = x^2$ and $y = 2x$ around the x-axis. Find its volume by the method of *slices.*

Solution:

The intersection points of $y = x^2$ and $y = 2x$ are given by the equation $x^2 = 2x$, i.e., $x(x-2) = 0 \Longrightarrow x = 0$ and $x = 2$. Hence

$$V = \int_0^2 \pi \left(y_{\text{top}}^2 - y_{\text{bot}}^2\right) dx = \int_0^2 \pi \left(4x^2 - x^4\right) dx =$$

$$\pi \left[\frac{4x^3}{3} - \frac{x^5}{5}\right]_0^2 = \pi \left(\frac{32}{3} - \frac{32}{5}\right) = \boxed{\frac{64\pi}{15}}.$$

4. (Volumes By Cylindrical Shells) Use the method of *cylindrical shells* to find the volume of the solid defined in the previous problem (generated by revolving the plane region between the curves $y = x^2$ and $y = 2x$ around the x-axis.)

Solution:

By the method of cylindrical shells we must integrate respect to y, so we rewrite the curves in the form $x = y^{1/2}$ and $x = y/2$ respectively. The intersection points are the same as before, but they correspond to $y = 0$ and $y = 4$ respectively. So:

$$V = \int_0^4 2\pi y\,(x_{\text{right}} - x_{\text{left}})\,dy = \int_0^4 2\pi y\,\left(y^{1/2} - \frac{y}{2}\right)\,dy =$$

$$2\pi \int_0^4 \left(y^{3/2} - \frac{y^2}{2}\right)\,dx = 2\pi \left[\frac{2y^{5/2}}{5} - \frac{y^3}{6}\right]_0^4 = 2\pi \left(\frac{64}{5} - \frac{64}{6}\right) = \boxed{\frac{64\pi}{15}}.$$

5. (Arc Length) Set up and simplify the integral that gives the length of the smooth arc $y = \frac{1}{8}x^4 + \frac{1}{4}x^{-2}$ from $x = 1$ to $x = 2$. If possible find the value of the integral.

Solution:

The length is given by the formula:

$$S = \int_1^2 ds = \int_1^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_1^2 \sqrt{1 + (y')^2}\, dx\,.$$

We have:

$$\begin{aligned}
&y' = \left(\tfrac{1}{8}x^4 + \tfrac{1}{4}x^{-2}\right)' = \tfrac{1}{2}x^3 - \tfrac{1}{2}x^{-3}\,,\\
&(y')^2 = \left(\tfrac{1}{2}x^3 - \tfrac{1}{2}x^{-3}\right)^2 = \tfrac{1}{4}x^6 - \tfrac{1}{2} + \tfrac{1}{4}x^{-6}\,,\\
&1 + (y')^2 = \tfrac{1}{4}x^6 + \tfrac{1}{2} + \tfrac{1}{4}x^{-6} = \left(\tfrac{1}{2}x^3 + \tfrac{1}{2}x^{-3}\right)^2\,,\\
&\sqrt{1 + (y')^2} = \tfrac{1}{2}x^3 + \tfrac{1}{2}x^{-3}\,.
\end{aligned}$$

Hence:

$$S = \int_1^2 \left(\frac{1}{2}x^3 + \frac{1}{2}x^{-3}\right) dx = \left[\frac{1}{8}x^4 - \frac{1}{4}x^{-2}\right]_1^2 = \left(2 - \frac{1}{16}\right) - \left(\frac{1}{8} - \frac{1}{4}\right) = \boxed{\frac{33}{16}}.$$

6. (Separable Differential Equations) Solve the following initial value problem:

$$\begin{cases} \dfrac{dy}{dx} = y^2 \sin x \\ y(0) = \frac{1}{3} \end{cases}$$

Solution:

First we separate variables:

$$\frac{1}{y^2}\, dy = \sin x\, dx\, .$$

Next we integrate both sides of the equation:

$$\int \frac{1}{y^2}\, dy = \int \sin x\, dx + C\, ,$$

i.e.:

$$-\frac{1}{y} = -\cos x + C\, .$$

Now we determine the constant C by using the initial condition $x = 0$, $y = 1/3$:

$$-3 = -\cos x + C \quad \Longrightarrow \quad C = -2\, .$$

Hence:

$$-\frac{1}{y} = -\cos x - 2 \quad \Longrightarrow \quad \boxed{y = \frac{1}{2 + \cos x}}\, .$$