CS 310-0
Homework Assignment No. 2
Due Fri 1/21/2000

1. Using the Principle of Extension, prove the following absorption law:
$$A \cap (A \cup B) = A\,.$$

2. Using algebra of sets, prove the following distributive property of the intersection respect to the symmetric difference of sets:
$$A \cap (B \bigtriangleup C) = (A \cap B) \bigtriangleup (A \cap C)\,.$$
Use the following "definition" of *symmetric difference*: $A \bigtriangleup B = (A \cap \overline{B}) \cup (\overline{A} \cap B)$.

3. Find the following: (a) $\displaystyle\bigcup_{n=1}^{\infty} [\tfrac{1}{n}, 1]$, (b) $\displaystyle\bigcap_{n=1}^{\infty} (0, 1 + \tfrac{1}{n})$, (c) $\displaystyle\bigcap_{n=1}^{\infty} [n, \infty)$. Justify the answer.

4. Find the properties (reflexive, transitive, symmetric, antisymmetric) verified by the following relations:
   (a) Strict inequality of integers: $x \mathcal{R} y \Leftrightarrow x < y$.
   (b) Non-equality of integers: $x \mathcal{R} y \Leftrightarrow x \neq y$.
   (c) The following relation on $\mathbb{N}$: $x \mathcal{R} y \Leftrightarrow \exists z \in \mathbb{N},\ x + 3z = y$.
   (d) The following relation on $\mathbb{R}$: $x \mathcal{R} y \Leftrightarrow x - y \in \mathbb{Q}$.

5. We define the following relation on $\mathbb{Z}^*$: $x \mathcal{R} y$ if and only if
   (a) x is negative and y is positive, or
   (b) x and y have the same sign and $|x| \leq |y|$.[1]
   For instance: $2 \mathcal{R} 6$, $(-3) \mathcal{R} (-7)$, $(-5) \mathcal{R} 2$.
   1. Prove that $\mathcal{R}$ is a total order.
   2. For each of the following numbers find a *successor* an *immediate successor*, a *predecessor* and an *immediate predecessor*, or show that there is none: -2, -1, 1, 2.

6. Prove that the following is an *equivalence relation* on $\mathbb{Z} \times \mathbb{Z}^*$:[2]
$$(a, b) \mathcal{R} (a', b') \Leftrightarrow ab' = a'b\,.$$
Is it also an *equivalence relation* on $\mathbb{Z} \times \mathbb{Z}$?

7. On $\mathbb{Z}^+$ we define the following relation: $a \mathcal{R} b \Leftrightarrow \exists n \in \mathbb{N},\ b = 2^n a$. Prove that $\mathcal{R}$ is a *partial order*. Find the minimal elements of $\mathbb{Z}^+$.

8. Let $\mathcal{R}$ be the relation defined in problem 7. Let $\mathcal{S}$ be the following relation on $\mathbb{Z}^+$: $a \mathcal{S} b$ if and only if $a \mathcal{R} b$ or $b \mathcal{R} a$. Show that $a \mathcal{S} b \Leftrightarrow \exists n \in \mathbb{Z},\ b = 2^n a$. Show that $\mathcal{S}$ is an equivalence relation. Describe the equivalence classes.

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[1] $|x|$ = absolute value of x.
[2] $\mathbb{Z}^* = \mathbb{Z} - \{0\}$.