PUTNAM TRAINING - PROBLEMS

October 31st, 2024

Putnam 2021-A1. A grasshopper starts at the origin in the coordinate plane and makes a sequence of hops. Each hop has length 5, and after each hop the grasshopper is at a point whose coordinates are both integers; thus, there are 12 possible locations for the grasshopper after the first hop. What is the smallest number of hops needed for the grasshopper to reach the point $(2021, 2021)$?

Putnam 2021-A3. Determine all positive integers N for which the sphere
$$x^2 + y^2 + z^2 = N$$
has an inscribed regular tetrahedron whose vertices have integer coordinates.

Putnam 2021-B2. Determine the maximum value of the sum
$$S = \sum_{n=1}^{\infty} \frac{n}{2^n} (a_1 a_2 \cdots a_n)^{1/n}$$
over all sequences $a_1, a_2, a_3, \ldots$ of nonnegative real numbers satisfying
$$\sum_{k=1}^{\infty} a_k = 1.$$

Putnam 2018-B2. Let n be a positive integer, and let $f_n(z) = n + (n-1)z + (n-2)z^2 + \cdots + z^{n-1}$. Prove that f_n has no roots in the closed unit disk $\{z \in \mathbb{C} \colon |z| \leq 1\}$.

Putnam 2018-B3. Find all positive integers $n < 10^{100}$ for which simultaneously n divides 2^n, $n-1$ divides $2^n - 1$, and $n-2$ divides $2^n - 2$.