

## 1.2. The Evaluation Theorem

**1.2.1. The Evaluation Theorem.** If  $f$  is a continuous function and  $F$  is an antiderivative of  $f$ , i.e.,  $F'(x) = f(x)$ , then

$$\int_a^b f(x) dx = F(b) - F(a).$$

*Example:* Find  $\int_0^1 x^2 dx$  using the evaluation theorem.

*Answer:* An antiderivative of  $x^2$  is  $x^3/3$ , hence:

$$\int_0^1 x^2 dx = \left[ \frac{x^3}{3} \right]_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}.$$

**1.2.2. Indefinite Integrals.** If  $F$  is an antiderivative of a function  $f$ , i.e.,  $F'(x) = f(x)$ , then for any constant  $C$ ,  $F(x) + C$  is another antiderivative of  $f(x)$ . The family of all antiderivatives of  $f$  is called *indefinite integral* of  $f$  and represented:

$$\int f(x) dx = F(x) + C.$$

Example:  $\int x^2 dx = \frac{x^3}{3} + C.$

**1.2.3. Table of Indefinite Integrals.** We can make an integral table just by reversing a table of derivatives.

$$(1) \int x^n dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1).$$

$$(2) \int \frac{1}{x} dx = \ln |x| + C.$$

$$(3) \int e^x dx = e^x + C.$$

$$(4) \int a^x dx = \frac{a^x}{\ln a} + C.$$

$$(5) \int \sin x dx = -\cos x + C.$$

$$(6) \int \cos x dx = \sin x + C.$$

$$(7) \int \sec^2 x dx = \tan x + C.$$

$$(8) \quad \int \csc^2 x \, dx = -\cot x + C.$$

$$(9) \quad \int \sec x \tan x \, dx = \sec x + C.$$

$$(10) \quad \int \csc x \cot x \, dx = -\csc x + C.$$

$$(11) \quad \int \frac{dx}{x^2 + 1} = \tan^{-1} x + C.$$

$$(12) \quad \int \frac{dx}{\sqrt{1 - x^2}} = \sin^{-1} x + C.$$

$$(13) \quad \int \frac{dx}{x\sqrt{x^2 - 1}} = \sec^{-1} |x| + C.$$

**1.2.4. Total Change Theorem.** The integral of a rate of change is the total change:

$$\int_a^b F'(x) \, dx = F(b) - F(a).$$

This is just a restatement of the evaluation theorem.

As an example of application we find the *net distance* or *displacement*, and the *total distance* traveled by an object that moves along a straight line with position function  $s(t)$ . The velocity of the object is  $v(t) = s'(t)$ . The net distance or displacement is the difference between the final and the initial positions of the object, and can be found with the following integral

$$\int_{t_1}^{t_2} v(t) \, dt = s(t_2) - s(t_1).$$

In the computation of the displacement the distance traveled by the object when it moves to the left (while  $v(t) \leq 0$ ) is subtracted from the distance traveled to the right (while  $v(t) \geq 0$ ). If we want to find the total distance traveled we need to add all distances with a positive sign, and this is accomplished by integrating the absolute value of the velocity:

$$\int_{t_1}^{t_2} |v(t)| \, dt = \text{total distance traveled}.$$

*Example:* Find the displacement and the total distance traveled by an object that moves with velocity  $v(t) = t^2 - t - 6$  from  $t = 1$  to  $t = 4$ .

*Answer:* The displacement is

$$\begin{aligned}\int_1^4 (t^2 - t - 6) dx &= \left[ \frac{t^3}{3} - \frac{t^2}{2} - 6t \right]_1^4 \\ &= \left( \frac{4^3}{3} - \frac{4^2}{2} - 6 \cdot 4 \right) - \left( \frac{1^3}{3} - \frac{1^2}{2} - 6 \right) \\ &= -\frac{32}{3} - \left( -\frac{37}{6} \right) = \boxed{-\frac{9}{2}}\end{aligned}$$

In order to find the total distance traveled we need to separate the intervals in which the velocity takes values of different signs. Those intervals are separated by points at which  $v(t) = 0$ , i.e.,  $t^2 - t - 6 = 0 \Rightarrow t = -2$  and  $t = 3$ . Since we are interested only in what happens in  $[1, 4]$  we only need to look at the intervals  $[1, 3]$  and  $[3, 4]$ . Since  $v(1) = -6$ , the velocity is negative in  $[1, 3]$ , and since  $v(4) = 6$ , the velocity is positive in  $[3, 4]$ . Hence:

$$\begin{aligned}\int_1^4 |v(t)| dt &= \int_1^3 [-v(t)] dt + \int_3^4 v(t) dt \\ &= \int_1^3 -(t^2 - t - 6) dt + \int_3^4 (t^2 - t - 6) dt \\ &= \left[ -\frac{t^3}{3} + \frac{t^2}{2} + 6t \right]_1^3 + \left[ \frac{t^3}{3} - \frac{t^2}{2} - 6t \right]_3^4 \\ &= \frac{22}{3} + \frac{17}{6} = \boxed{\frac{61}{6}}.\end{aligned}$$