1.9. Numerical Integration

Sometimes the integral of a function cannot be expressed with *elementary functions*, i.e., polynomial, trigonometric, exponential, logarithmic, or a suitable combination of these. However, in those cases we still can find an approximate value for the integral of a function on an interval.

1.9.1. Trapezoidal Approximation. A first attempt to approximate the value of an integral $\int_a^b f(x)\,dx$ is to compute its Riemann sum:

$$R = \sum_{i=1}^{n} f(x_i^*)\,\Delta x\,.$$

Where $\Delta x = x_i - x_{i-1} = (b-a)/n$ and x_i^* is some point in the interval $[x_{i-1}, x_i]$. If we choose the left endpoints of each interval, we get the *left-endpoint approximation*:

$$L_n = \sum_{i=1}^{n} f(x_{i-1}))\Delta x = (\Delta x)\{f(x_0) + f(x_1) + \cdots + f(x_{n-1})\}\,,$$

Similarly, by choosing the right endpoints of each interval we get the *right-endpoint approximation*:

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x = (\Delta x)\{f(x_1) + f(x_2) + \cdots + f(x_n)\}\,.$$

The *trapezoidal approximation* is the average of L_n and R_n:

$$T_n = \frac{1}{2}(L_n+R_n) = \frac{\Delta x}{2}\{f(x_0)+2f(x_1)+2f(x_2)+\cdots+2f(x_{n-1})+f(x_n)\}\,.$$

Example: Approximate $\int_0^1 x^2\,dx$ with trapezoidal approximation using 4 intervals.

Solution: We have $\Delta x = 1/4 = 0.25$. The values for x_i and $f(x_i) = x_i^2$ can be tabulated in the following way:

i	x_i	$f(x_i)$
0	0	0
1	0.25	0.0625
2	0.5	0.25
3	0.75	0.5625
4	1	1

Hence:

$$L_4 = 0.25 \cdot (0 + 0.0625 + 0.25 + 0.5625) = 0.218750\,,$$

$$R_4 = 0.25 \cdot (0.0625 + 0.25 + 0.5625 + 1) = 0.468750\,.$$

So:

$$T_4 = \frac{1}{2}(L_4 + R_4) = \frac{1}{2}(0.218750 + 0.468750) = 0.34375\,.$$

Compare to the exact value of the integral, which is $1/3 = 0.3333\ldots$.

1.9.2. Midpoint Approximation. Alternatively, in the Riemann sum we can use the middle point $\overline{x}_i = (x_{i-1} + x_i)/2$ of each interval $[x_{i-1}, x_i]$. Then the *midpoint approximation* of $\int_a^b f(x)\,dx$ is

$$M_n = (\Delta x)\{f(\overline{x}_1) + f(\overline{x}_2) + \cdots + f(\overline{x}_n)\}\,.$$

Example: Approximate $\int_0^1 x^2\,dx$ with midpoint approximation using 4 intervals.

Solution: We have:

i	$\overline{x}_i$	$f(\overline{x}_i)$
1	0.125	0.015625
2	0.375	0.140625
3	0.625	0.390625
4	0.875	0.765625

Hence:

$$\begin{aligned} M_4 &= 0.25 \cdot (0.015625 + 0.140625 + 0.390625 + 0.765625) \\ &= 0.328125\,. \end{aligned}$$

1.9.3. Simpson's Approximation. *Simpson's approximation* is a weighted average of the trapezoidal and midpoint approximations associated to the intervals $[x_0, x_2]$, $[x_2, x_4]$, $\ldots$, $[x_{n-2}, x_n]$ (of length

$2\Delta x$ each):

$$\begin{aligned}
S_{2n} &= \frac{1}{3}(2M_n + T_n) \\
&= \frac{1}{3}\Bigg[2(2\Delta x)\{f(x_1) + f(x_3) + \cdots + f(x_{2n-1})\} \\
&\qquad + \frac{2\Delta x}{2}\{f(x_0) + 2f(x_2) + 2f(x_4) + \cdots + 2f(x_{n-2}) + f(x_n)\}\Bigg] \\
&= \frac{\Delta x}{3}\{f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots \\
&\qquad\qquad + 2f(x_{2n-2)} + 4f(x_{2n-1}) + f(x_{2n})\}\,.
\end{aligned}$$

Example: Approximate $\int_0^1 x^2\,dx$ with Simpson's approximation using 8 intervals.

Solution: We use the previous results and get:

$$S_8 = \frac{1}{3}(2M_4 + T_4) = \frac{1}{3}(2 \cdot 0.328125 + 0.34375) = 1/3\,.$$

Note: in this particular case Simpson's approximation gives the exact value—in general it just gives a good approximation.

1.9.4. Error Bounds. Here we give a way to estimate the error or difference E between the actual value of an integral and the value obtained using a numerical approximation.

1.9.4.1. *Error Bound for the Trapezoidal Approximation.* Suppose $|f''(x)| \leq K$ for $a \leq x \leq b$. Then the error E_T in the trapezoidal approximation verifies:

$$|E_T| \leq \frac{K(b-a)^3}{12n^2}\,.$$

1.9.4.2. *Error Bound for the Midpoint Approximation.* Suppose $|f''(x)| \leq K$ for $a \leq x \leq b$. Then the error E_M in the trapezoidal approximation verifies:

$$|E_M| \leq \frac{K(b-a)^3}{24n^2}\,.$$

1.9.4.3. *Error Bound for the Simpson's Rule.* Suppose $|f^{(4)}(x)| \leq K$ for $a \leq x \leq b$. Then the error E_S in the Simpson's rule verifies:

$$|E_S| \leq \frac{K(b-a)^5}{180n^4}\,.$$

Example: Approximate the value of π using the trapezoidal, midpoint and Simpson's approximations of

$$\int_0^1 \frac{4}{1+x^2}\,dx$$

for $n = 4$. Estimate the error.

Answer: First note that:

$$4\int_0^1 \frac{1}{1+x^2}\,dx = 4\left[\tan^{-1} x\right]_0^1 = 4\frac{\pi}{4} = \pi\,,$$

so by approximating the given integral we are in fact finding approximated values for π.

Now we find the requested approximations:

(1) Trapezoidal approximation:

$$\begin{aligned} T_4 &= \frac{1/4}{2}\{f(0)+2f(1/4)+2f(1/2)+2f(3/4)+f(1)\} \\ &= \boxed{3.131176470}\,. \end{aligned}$$

For estimating the error we need the second derivative of $f(x) = 4/(1+x^2)$, which is $f''(x) = 8(3x^2-1)/(1+x^2)^3$ so we have

$$\begin{aligned} |f''(x)| = \frac{8|3x^2-1|}{|1+x^2|^3} &\le \frac{8(3x^2+1)}{(1+x^2)^3} \\ &\le \frac{8(3\cdot 1^2+1)}{1} = 32 \end{aligned}$$

for $0 \le x \le 1$, hence

$$|E_T| \le \frac{32\cdot(1-0)^3}{12\cdot 4^2} = 0.1666\ldots$$

(2) Midpoint approximation:

$$\begin{aligned} T_M &= \frac{1}{4}\{f(1/8)+f(3/8)+f(5/8)+f(7/8)\} \\ &= \boxed{3.146800518}\,. \end{aligned}$$

The error estimate is:

$$|E_M| \le \frac{32\cdot(1-0)^3}{24\cdot 4^2} = 0.08333\ldots$$

(3) Simpson's rule:

$$
\begin{aligned}
T_S &= \frac{1/4}{3}\left\{f(0) + 4f(1/4) + 2f(1/2) + 4f(3/4) + f(1)\right\} \\
&= \boxed{3.141568627}
\end{aligned}
$$

For the error estimate we now need the fourth derivative:

$$f^{(4)}(x) = 96(5x^4 - 10x^2 + 1)/(1 + x^2)^5\,,$$

so

$$|f^{(4)}(x)| \le \frac{96(5 + 10 + 1)}{1} = 1536$$

for $0 \le x \le 1$. Hence the error estimate is

$$|E_S| \le \frac{1536 \cdot (1 - 0)^5}{180 \cdot 4^4} = 0.0333\ldots$$