1. You are hired as a programmer for a mortgage company. You need to write a program to compute the principal balance P_t of a mortgage after t months at a yearly fixed rate r (so the monthly interest per dollar is $r/1200$) and with a monthly payment m. Of course, before writing the program you need to find a formula that computes P_t from A = initial amount of the loan, r, t and m. In order to do that do the following:
   (a) Find a recurrence satisfied by the sequence $P_0, P_1, P_2, \ldots$, where P_t = principal balance after t monthly payments ($P_0 = A$).
   (b) Solve the recurrence and give an explicit formula for P_t.

2. Find a close-form formula for the nth term of the Lucas sequence $2, 1, 3, 4, 7, 11, 18 \ldots$, recursively defined in the following way:
   $$L_0 = 2, \; L_1 = 1,$$
   $$L_n = L_{n-1} + L_{n-2} \quad (n \geq 2).$$

3. Use the method of generating functions to prove the following relation among binomial coefficients:
   $$\sum_{j=0}^{k} \binom{n}{j} \binom{n}{k-j} = \binom{2n}{k}.$$
   [Hints: $(1+x)^n (1+x)^n = (1+x)^{2n}$;
   $$\left(\sum_{k=0}^{\infty} a_k x^k \right) \left(\sum_{k=0}^{\infty} b_k x^k \right) = \sum_{k=0}^{\infty} c_k x^k, \text{ where } c_k = \sum_{j=0}^{k} a_j b_{k-j}.]$$

4. You need to pour exactly 1 tsp of water into a pot, but you only have two containers with capacity for 36 tsp and 49 tsp respectively. You are allowed to transfer water among the containers as you wish, but you cannot measure directly any amount of water that is a fraction of one of the containers. Pose the problem as a Diophantine equation, solve it, and use the solution to find a way of measuring exactly 1 tsp with the two containers.

5. We have a number of stamps of various denominations and want to mail a package that requires $3.25 postage. In each of the following cases determine if we have the appropriate stamps to get exactly the required postage. Justify the answers.
   (a) 1000 8¢-stamps, 500 10¢-stamps and 300 22¢-stamps.
   (b) 20 15¢-stamps and 5 50¢-stamps.
   (c) 100 50¢-stamps and 100 33¢-stamps.
   (d) 4 50¢-stamps and 4 35¢-stamps.

6. Four siblings visit their mother every 57, 60, 76 and 95 days respectively. They coincide by chance on December 31, 2000. When will they coincide again the next time?