PUTNAM TRAINING

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Putnam Problems

Putnam 2024-A1. Determine all positive integers n for which there exist positive integers a, b, and c satisfying
$$2a^n + 3b^n = 4c^n.$$

Solution. The answer is $n = 1$.

We look at three cases.

Case 1. When $n = 1$, the equation becomes
$$2a + 3b = 4c.$$
For example, $(a, b, c) = (1, 2, 2)$ satisfies
$$2(1) + 3(2) = 8 = 4(2),$$
so solutions exist when $n = 1$.

Case 2. If $n = 2$, assume for contradiction that there exist positive integers a, b, c satisfying
$$2a^2 + 3b^2 = 4c^2.$$
By dividing by $\gcd(a, b, c)$ if necessary, we may assume $\gcd(a, b, c) = 1$.

Reduce the equation modulo 3. Since $3b^2 \equiv 0 \pmod 3$ and $4c^2 \equiv c^2 \pmod 3$, we obtain
$$2a^2 \equiv c^2 \pmod 3.$$
The quadratic residues modulo 3 are 0 and 1. If $a^2 \equiv 1 \pmod 3$, then $c^2 \equiv 2 \pmod 3$, which is impossible. Hence $a^2 \equiv 0 \pmod 3$, so $3 \mid a$. Then $c^2 \equiv 0 \pmod 3$, so $3 \mid c$.

Substituting into the equation and solving for b^2 gives
$$3b^2 = 4c^2 - 2a^2.$$
Since both a and c are divisible by 3, the right-hand side is divisible by 9, so $3 \mid b$. Therefore $3 \mid a, b, c$, contradicting $\gcd(a, b, c) = 1$.

Thus, there are no solutions for $n = 2$.

Case 3. If $n \geq 3$, again assume for contradiction that $2a^n + 3b^n = 4c^n$ has a solution in positive integers, with $\gcd(a, b, c) = 1$. Reducing the equation modulo 2, we have
$$2a^n + 3b^n \equiv b^n \equiv 0 \pmod 2,$$

Training session conducted by Miguel A. Lerma.

so b is even. Write $b = 2B$. Then

$$2a^n + 3(2B)^n = 4c^n \implies 2a^n + 3 \cdot 2^n B^n = 4c^n.$$

For $n \geq 2$, the term $3 \cdot 2^n B^n$ is divisible by 4. Since $4c^n$ is also divisible by 4, the term $2a^n$ must be divisible by 4, forcing a even. Let $a = 2A$. Substituting again,

$$2(2A)^n + 3(2B)^n = 4c^n \implies 2^{n+1} A^n + 3 \cdot 2^n B^n = 4c^n \implies 2^n (2A^n + 3B^n) = 4c^n.$$

Dividing both sides by 2^n gives

$$2A^n + 3B^n = 4\left(\frac{c}{2}\right)^n.$$

Hence $(A, B, c/2)$ is another solution in positive integers, which implies c is also even. Therefore a, b, c are all even, contradicting $\gcd(a, b, c) = 1$.

Conclusion. The only positive integer n for which $2a^n + 3b^n = 4c^n$ admits a solution in positive integers is $n = 1$.

Putnam 2021-A1. A grasshopper starts at the origin in the coordinate plane and makes a sequence of hops. Each hop has length 5, and after each hop the grasshopper is at a point whose coordinates are both integers; thus, there are 12 possible locations for the grasshopper after the first hop. What is the smallest number of hops needed for the grasshopper to reach the point $(2021, 2021)$?

Solution. The answer is 578.

Each hop corresponds to adding one of the 12 vectors $(0, \pm 5)$, $(\pm 5, 0)$, $(\pm 3, \pm 4)$, $(\pm 4, \pm 3)$ to the position of the grasshopper. Since $(2021, 2021) = 288(3, 4) + 288(4, 3) + (0, 5) + (5, 0)$, the grasshopper can reach $(2021, 2021)$ in $288 + 288 + 1 + 1 = 578$ hops.

On the other hand, let $z = x + y$ denote the sum of the x and y coordinates of the grasshopper, so that it starts at $z = 0$ and ends at $z = 2021 + 2021 = 4042$. Each hop changes the sum of the x and y coordinates of the grasshopper by at most 7, and $4042 > 577 \times 7 = 4039$; it follows immediately that the grasshopper must take more than 577 hops to get from $(0, 0)$ to $(2021, 2021)$.

Remark. This solution implicitly uses the distance function

$$d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$$

on the plane, variously called the *taxicab metric*, the *Manhattan metric*, or the L^1-*norm* (or ℓ_1-norm).

Putnam 2021-B2. Determine the maximum value of the sum

$$S = \sum_{n=1}^{\infty} \frac{n}{2^n} (a_1 a_2 \cdots a_n)^{1/n}$$

over all sequences $a_1, a_2, a_3, \ldots$ of non negative real numbers satisfying

$$\sum_{k=1}^{\infty} a_k = 1.$$

Solution. The answer is $\frac{2}{3}$.

For each n, note that

$$\begin{aligned} 2^{n+1}(a_1 \cdots a_n)^{1/n} &= \left(4^{1+2+\cdots+n}\, a_1 \cdots a_n\right)^{1/n} \\ &= \Big((4a_1)(4^2a_2)\cdots(4^na_n)\Big)^{1/n} \\ &\le \frac{1}{n}\sum_{k=1}^{n} 4^k a_k \qquad \text{(by the AM–GM inequality).,} \end{aligned}$$

where we used $1+2+\cdots+n = \frac{n(n+1)}{2}$. Hence,

$$\begin{aligned} 2S &= \sum_{n=1}^{\infty} \frac{2n}{2^n}(a_1\cdots a_n)^{1/n} = \sum_{n=1}^{\infty} \frac{n}{4^n}\left(2^{n+1}(a_1\cdots a_n)^{1/n}\right) \\ &\le \sum_{n=1}^{\infty} \frac{1}{4^n}\sum_{k=1}^{n} 4^k a_k = \sum_{\substack{n,k\ge 1 \\ n\ge k}} (4^{k-n}a_k) = \sum_{k=1}^{\infty} a_k \sum_{n=k}^{\infty} 4^{k-n} \\ &= \sum_{k=1}^{\infty} a_k \sum_{j=0}^{\infty} 4^{-j} = \Big(\sum_{j=0}^{\infty} 4^{-j}\Big)\Big(\sum_{k=1}^{\infty} a_k\Big) = \frac{4}{3}\cdot 1 = \frac{4}{3}. \end{aligned}$$

Therefore,

$$S \le \frac{2}{3}.$$

Equality in AM–GM holds if and only if all terms inside the inequality are equal, i.e.

$$4^1 a_1 = 4^2 a_2 = \cdots = 4^n a_n = c$$

for some constant $c > 0$. Thus $a_k = c/4^k$. The condition $\sum a_k = 1$ gives

$$1 = c\sum_{k=1}^{\infty} \frac{1}{4^k} = c \cdot \frac{1/4}{1-1/4} = \frac{c}{3} \quad \Rightarrow \quad c = 3.$$

Hence $a_k = \frac{3}{4^k}$ for all k.

For this sequence,

$$(a_1 a_2 \cdots a_n)^{1/n} = \left(\frac{3^n}{4^{1+2+\cdots+n}}\right)^{1/n} = \frac{3}{2^{n+1}},$$

and therefore

$$S = \sum_{n=1}^{\infty} \frac{n}{2^n} \cdot \frac{3}{2^{n+1}} = \frac{3}{2} \sum_{n=1}^{\infty} \frac{n}{4^n}.$$

The sum $\sum_{n=1}^{\infty} \frac{n}{4^n}$ can by found by differentiating the following power series and multiplying by x:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \Rightarrow \quad \sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2} \quad \Rightarrow \quad \sum_{n=1}^{\infty} n x^n = \frac{x}{(1-x)^2},$$

where $|x| < 1$. Setting $x = \frac{1}{4}$ we get:

$$S = \frac{3}{2} \cdot \frac{\frac{1}{4}}{(1-\frac{1}{4})^2} = \frac{3}{2} \cdot \frac{4}{9} = \frac{2}{3}.$$

Hence, the maximum value of S is $\dfrac{2}{3}$, attained when $a_k = \dfrac{3}{4^k}$ for all $k \geq 1$.

Putnam 2018-B3. Find all positive integers $n < 10^{100}$ for which simultaneously n divides 2^n, $n-1$ divides $2^n - 1$, and $n-2$ divides $2^n - 2$.

Solution. The answer is $n = 2^{2^{2^i}}$, $i = 0, 1, 2, 3$.[1]

The first condition $n \mid 2^n$ implies that n must be a power of 2, hence

$$n = 2^k, \qquad k \geq 1.$$

The second condition is $n - 1 \mid 2^n - 1$, and substituting $n = 2^k$, we obtain

$$2^k - 1 \mid 2^{2^k} - 1.$$

To analyze this, we use the following lemma.

Lemma. For positive integers a, b,

$$2^b - 1 \mid 2^a - 1 \iff b \mid a.$$

Proof. Let $m = 2^b - 1$. This implies $2^b = m + 1 \equiv 1 \pmod{m}$.

If $b \mid a$, say $a = bt$,

$$2^a - 1 = (2^b)^t - 1 \equiv 1^t - 1 \equiv 0 \pmod{m},$$

so $m \mid 2^a - 1$.

[1]Based on solution by Fiona Brady, Northwestern University - Edited.

Conversely, suppose $m \mid 2^a - 1$. By Euclidean division we can write $a = qb + r$ with $0 \le r < b$. Since $2^b \equiv 1 \pmod{m}$, we have

$$2^a = 2^{qb+r} \equiv (2^b)^q 2^r \equiv 2^r \pmod{m}.$$

Thus

$$2^a - 1 \equiv 2^r - 1 \pmod{m}.$$

If $m \mid 2^a - 1$, then $m \mid 2^r - 1$. But $0 \le 2^r - 1 < 2^b - 1 = m$, so $2^r - 1 = 0$, i.e. $r = 0$. Hence $b \mid a$. $\square$

Applying the lemma to the second condition we get

$$2^k - 1 \mid 2^{2^k} - 1 \quad \Longleftrightarrow \quad k \mid 2^k.$$

Thus k must itself be a power of 2:

$$k = 2^j, \qquad j \ge 0.$$

The third condition is $n - 2 \mid 2^n - 2$. Substituting $n = 2^k$ we get

$$2^k - 2 \mid 2^{2^k} - 2.$$

Dividing both sides by 2 gives

$$2^{k-1} - 1 \mid 2^{2^k - 1} - 1.$$

By the lemma again,

$$k - 1 \mid 2^k - 1.$$

Substituting $k = 2^j$ the condition becomes

$$2^j - 1 \mid 2^{2^j} - 1 \quad \Longleftrightarrow \quad j \mid 2^j.$$

Hence j must also be a power of 2:

$$j = 2^i, \qquad i \ge 0.$$

So, we have

$$n = 2^k = 2^{2^j} = 2^{2^{2^i}}.$$

Thus all solutions are of the form

$$n = 2^{2^{2^i}}, \qquad i = 0, 1, 2, \ldots.$$

Finally, we use the bounding $n < 10^{100}$. We require

$$2^{2^{2^i}} < 10^{100} \quad \Longleftrightarrow \quad 2^{2^i} < 100 \log_2 10.$$

Since $3 < \log_2 10 < 4$ we have $300 < 100 \log_2 10 < 400$, hence $2^{2^i} \le 256 \Rightarrow 2^i \le 8 \Rightarrow i \le 3$, and $n = 2^{2^{2^i}}$, $i = 0, 1, 2, 3$.

Spanish Mathematical Olympiad 1971, Problem No. 8. We select $n+1$ numbers from the set $\{1,2,3,\dots,2n\}$. Prove that among the chosen numbers there are at least two such that one divides the other.

Solution. Write each chosen number x uniquely in the form

$$x = 2^k m,$$

where m is odd (the *odd part* of x) and $k \geq 0$. We associate each chosen number x with its odd part m.

Among the integers $1,2,\dots,2n$ there are exactly n odd numbers, so there are at most n possible values for m. Since we have chosen $n+1$ numbers, by the pigeonhole principle two of them must share the same odd part. Let these be

$$x = 2^i m \quad \text{and} \quad y = 2^j m,$$

where m is odd and $i < j$.

Then

$$y = 2^j m = 2^{j-i}(2^i m) = 2^{j-i} x,$$

so x divides y.

Therefore, among the $n+1$ chosen numbers, there are at least two such that one divides the other. $\square$

Killer Problem 13. Is it possible to put an equilateral triangle onto a square grid so that all the vertices are in corners?

Solution. The answer is no. This is a direct consequence of the following:

Claim. Any non degenerate equilateral triangle having two vertices with rational coordinates must have irrational area. Consequently, no non degenerate equilateral triangle can have all three vertices with rational coordinates.

Proof. Let the two rational vertices be $A = (x_1, y_1)$ and $B = (x_2, y_2)$ with $x_i, y_i \in \mathbb{Q}$. Set

$$\Delta x := x_2 - x_1 \in \mathbb{Q}, \qquad \Delta y := y_2 - y_1 \in \mathbb{Q}.$$

The side length of the triangle is $s = \sqrt{(\Delta x)^2 + (\Delta y)^2}$, so

$$s^2 = (\Delta x)^2 + (\Delta y)^2 \in \mathbb{Q}^+.$$

For an equilateral triangle, the area depends only on the side length:

$$\text{Area} = \frac{\sqrt{3}}{4} s^2.$$

Since s^2 is a positive rational number and $\sqrt{3} \notin \mathbb{Q}$, the product $\frac{\sqrt{3}}{4}\, s^2$ is irrational. This proves the first statement.

On the other hand, if $C = (x_3, y_3)$ is the third vertex, then by the shoelace formula the area of the triangle is:

$$\text{Area} = \frac{1}{2} \left| \det \begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} \right| = \frac{1}{2} \left| \, x_2 y_3 + x_3 y_1 + x_1 y_2 - x_2 y_1 - x_3 y_2 - x_1 y_3 \, \right|.$$

If all vertices had rational coordinates the area would be rational, a contradiction. Hence no non degenerate equilateral triangle has all three vertices rational. $\square$

The result now follows from the fact that the corners of a square grid can be taken to have integer coordinates.

Useful Results

Arithmetic Mean-Geometric Mean (AM-GM) Inequality

The geometric mean of positive numbers is not greater than their arithmetic mean, i.e., if $x_1, x_2, \ldots, x_n > 0$*, then*

$$\left(\prod_{i=1}^{n} x_i\right)^{1/n} \leq \frac{1}{n}\sum_{i=1}^{n} x_i \,.$$

Equality happens only for $x_1 = \cdots = x_n$.