CS 310-0
Homework Assignment No. 6
Due Fri 2/25/2000

1. Find a close-form formula for the nth term of the Lucas sequence $2, 1, 3, 4, 7, 11, 18\ldots$, recursively defined in the following way:
   $L_0 = 2$, $L_1 = 1$,
   $L_n = L_{n-1} + L_{n-2} \qquad (n \geq 2)$.

2. Solve the following recurrence:
$$x_n = 6\,x_{n-1} - 9\,x_{n-2}$$
   with each of the following initial conditions:
   (a) $x_0 = 1,\ x_1 = 3$.
   (b) $x_0 = 0,\ x_1 = 3$.
   (c) $x_0 = 1,\ x_1 = 6$.

3. Assume that the only operation available on $\mathbb{N}$ is the *successor* function: $S : \mathbb{N} \to \mathbb{N}$, $S(n) = n + 1$.[1] Using this function define recursively the following binary operations on $\mathbb{N}$:
   (a) Addition of natural numbers: $m + n$.
   (b) Multiplication of natural numbers: $m \cdot n$.
   (c) Power: a^n.[2]
   (d) Iterated power: $a \uparrow\uparrow n = a^{a^{\cdot^{\cdot^{\cdot^{a}}}}}$, with n a's piled in the tower of exponents—for instance, $2 \uparrow\uparrow 4 = 2^{2^{2^{2}}} = 2^{2^4} = 2^{16} = 65536$.[3]
   Each already defined operation may be used in the definition of the following one.

4. Ackerman's function $A : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is defined for every $m, n \geq 0$ by the following double recurrence:
   (a) $A(0, n) = n + 1$ for every $n \geq 0$.
   (b) $A(m, 0) = A(m - 1, 1)$ if $m \geq 1$.
   (c) $A(m, n) = A(m - 1, A(m, n - 1))$ if $m, n \geq 1$.
   Find close-form formulae for $A(1, n)$, $A(2, n)$, $A(3, n)$ and $A(4, n)$.[4] Find all other values of $A(m, n)$ $(m \geq 5)$ whose decimal representation can be written in our universe (assume that our universe has 10^{80} atoms.)

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[1] The expression "$n+1$" should not be interpreted as an addition, it just represents the integer that follows n; e.g.: $S(5) = 6$, $S(2178) = 2179$, etc.
[2] Note that this one is already solved in the notes.
[3] This notation is due to Donald Knuth: $a \uparrow n = a^n$ represents the usual power; $a \uparrow\uparrow n$ is the iterated power; $a \uparrow\uparrow\uparrow n =$ the iterated-iterated power, and so on.
[4] The formulae may include any arithmetic operation, powers and iterated powers.

5. Find the general solution to the following Diophantine equation:
$$13x + 15y = 7\,.$$

6. You need to pour exactly 1 tsp of water into a pot, but you only have three containers with capacity for 72 tsp, 98 tsp and 189 tsp respectively. You are allowed to transfer water among the containers as you wish, but you cannot measure directly any amount of water that is a fraction of one of the three containers. Pose the problem as a Diophantine equation, solve it, and use the solution to find a way of measuring exactly 1 tsp with the three containers. (In order to make things easier you may assume that you have a fourth container in which you can store an arbitrarily large amount of water temporarily, although this fourth container is not really indispensable.)

7. We have a number of stamps of various denominations and want to mail a package that requires \$3.25 postage. In each of the following cases determine if we have the appropriate stamps to get exactly the required postage. Justify the answers.
   (a) 1000 8¢-stamps, 500 10¢-stamps and 300 22¢-stamps.
   (b) 20 15¢-stamps and 5 50¢-stamps.
   (c) 100 50¢-stamps and 100 33¢-stamps.
   (d) 4 50¢-stamps and 4 35¢-stamps.

8. Mark writes the letters of his name (M,A,R,K) on the sides of the tires of his car (one letter per tire). He takes care to write the letters in their correct positions, but as soon as he moves his car the letters start rotating with the tires. The tires are not exactly the same size; the length of their circumferences are 560, 400, 675 and 588 tenths of an inch respectively. What is the minimum distance that the car must run in order to get all the letters simultaneously in their original positions again?