6.4. Planar Graphs

6.4.1. Planar Graphs. A graph G is *planar* if it can be drawn in the plane with its edges intersecting at their vertices only. One such drawing is called an *embedding* of the graph in the plane.

A particular planar representation of a planar graph is called a *map*. A map divides the plane into a number of regions or faces (one of them infinite).

6.4.2. Graph Homeomorphism. If a graph G has a vertex v of degree 2 and edges (v, v_1), (v, v_2) with $v_1 \neq v_2$, we say that the edges (v, v_1) and (v, v_2) are in *series*. Deleting such vertex v and replacing (v, v_1) and (v, v_2) with (v_1, v_2) is called a *series reduction*. For instance, in the third graph of figure 6.16, the edges (h, b) and (h, d) are in series. By removing vertex h we get the first graph in the left.

Two graphs are said to be *homeomorphic* if they are isomorphic or can be reduced to isomorphic graphs by a sequence of series reductions (fig. 6.16).

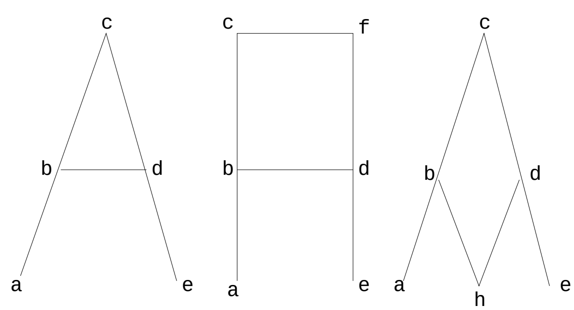

FIGURE 6.16. Three homeomorphic graphs.

Note that if a graph G is planar, then all graphs homeomorphic to G are also planar.

6.4.3. Some Results About Planar Graphs.

1. *Euler's Formula*: Let $G = (V, E)$ be a connected planar graph, and let $v = |V|$, $e = |E|$, and f = number of faces (regions) in which some given embedding of G divides the plane. Then:

$$v - e + f = 2 \,.$$

Note that this implies that all plane embeddings of a given graph define the same number of faces.

2. Let $G = (V, E)$ be a simple connected planar graph with v vertices, $e \geq 3$ edges and f faces. Then $3f \leq 2e$ and $e \leq 3v - 6$.
3. The graph K_5 is non-planar. Proof: in K_5 we have $v = 5$ and $e = 10$, hence $3v - 6 = 9 < e = 10$, which contradicts the previous result.
4. The graph $K_{3,3}$ is non-planar. Proof: in $K_{3,3}$ we have $v = 6$ and $e = 9$. If $K_{3,3}$ were planar, from Euler's formula we would have $f = 5$. On the other hand, each face is bounded by at least four edges, so $4f \leq 2e$, i.e., $20 \leq 18$, which is a contradiction.
5. *Kuratowski's Theorem*: A graph is non-planar if and only if it contains a subgraph that is homeomorphic to either K_5 or $K_{3,3}$.

6.4.4. Dual Graph of a Map. A map is defined by some planar graph $G = (V, E)$ embedded in the plane. Assume that the map divides the plane into a set of regions $R = \{r_1, r_2, \dots, r_k\}$. For each region r_i, select a point p_i in the interior of r_i. The *dual graph* of that map is the graph $G^d = (V^d, E^d)$, where $V^d = \{p_1, p_2, \dots, p_k\}$, and for each edge in E separating the regions r_i and r_j, there is an edge in E^d connecting p_i and p_j. *Warning*: Note that a different embedding of the same graph G may give different (and non-isomorphic) dual graphs. *Exercise*: Find the duals of the maps shown in figure 6.14, and prove that they are not isomorphic.

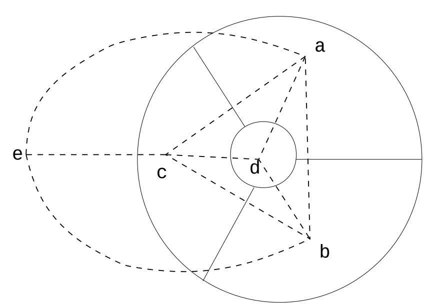

FIGURE 6.17. Dual graph of a map.

6.4.5. Graph Coloring. Consider the problem of coloring a map M in such a way that no adjacent regions (sharing a border) have the same color. This is equivalent to coloring the vertices of the dual map of M in such a way that no adjacent vertices have the same color.

In general, a *coloring* of a graph is an assignment of a color to each vertex of the graph. The coloring is called *proper* if there are no adjacent vertices with the same color. If a graph can be properly colored with n colors we say that it is *n-colorable*. The minimum number of colors needed to properly color a given graph $G = (V, E)$ is called the *chromatic number* of G, and is represented $\chi(G)$. Obviously $\chi(G) \leq |V|$.

6.4.6. Some Results About Graph Coloring.

1. $\chi(K_n) = n$.
2. Let G be a simple graph. The following statement are equivalent:
   (a) $\chi(G) = 2$.
   (b) G is bipartite.
   (c) Every cycle in G has even length
3. *Five Color Theorem (Kempe, Heawood)* (not hard to prove): Every simple, planar graph is 5-colorable.
4. *Four Color Theorem (Appel and Haken, 1976)*, proved with an intricate computer analysis of configurations: Every simple, planar graph is 4-colorable.

Exercise: Find a planar graph G such that $\chi(G) = 4$.