CS 310-0
Homework Assignment No. 3
Due Fri 1/28/2000

1. Let $f, g : \mathbb{R}^+ \to \mathbb{R}^+$ be the functions $f(x) = 1/x$, $g(x) = x + 3$. Find $g \circ f$, $f \circ g$, f^2, g^2, $g \circ f^2$, $f \circ g \circ f$, $f^2 \circ g$, $f \circ g^2$, $g \circ f \circ g$, $g^2 \circ f$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = x^3 - 3x^2 + 2x$. Find $f^{-1}([0, \infty))$.

3. Let $f, g : \mathbb{R} \to \mathbb{R}$ be functions defined by $f(x) = ax + b$, $g(x) = cx + d$, where a, b, c, d are real constants. What relation must be satisfied by a, b, c, d if $f \circ g = g \circ f$?

4. Let $f : \mathbb{R} \to (0, 1)$ be the function $f(x) = e^x/(1 + e^x)$. Prove that f is a one-to-one correspondence. Find its inverse $f^{-1} : (0, 1) \to \mathbb{R}$.

5. Let $f : A \to B$ be any function from a set A to another set B. For every $B' \subseteq B$, we denote $f^{-1}(B') = \{x \in A \mid f(x) \in B'\}$ the preimage set of B' by f. For any subsets $B_1, B_2 \subseteq B$ prove the following:
   (a) $B_1 \subseteq B_2 \Rightarrow f^{-1}(B_1) \subseteq f^{-1}(B_2)$.
   (b) $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$.
   (c) $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$.
   (d) $f^{-1}(B_1 - B_2) = f^{-1}(B_1) - f^{-1}(B_2)$.
   (e) $f^{-1}(B_1 \triangle B_2) = f^{-1}(B_1) \triangle f^{-1}(B_2)$.

6. For each one of the following functions, determine if it is one-to-one (but not onto), onto (but not one-to-one), or a one-to-one correspondence. Justify the answer. If it is a one-to-one correspondence, find its inverse.
   (a) $f : \mathbb{N} \to \mathbb{N}$, $f(x) = x^2$.
   (b) $f : \mathbb{R}^+ \to \mathbb{R}^+$, $f(x) = x^2$.
   (c) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$.
   (d) $f : \mathbb{R} \to [0, 1)$, $f(x) = x - \lfloor x \rfloor$.
   (e) $f : \mathbb{R} \to \mathbb{R}$, defined by cases in the following way:
   $$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 2x + 1 & \text{if } x \in \mathbb{R} - \mathbb{Q} \end{cases}$$

7. Prove that the following is a one-to-one correspondence from $\mathbb{N}$ to $\mathbb{Z}$:
   $$f(n) = \frac{(-1)^n (n + 1/2) - 1/2}{2}.$$
   Find its inverse. Hint: rewrite f in the following way:
   $$f(n) = \begin{cases} \cdots & \text{if } n = 2k \text{ for some } k \in \mathbb{N} \\ \cdots & \text{if } n = 2k + 1 \text{ for some } k \in \mathbb{N} \end{cases}$$

8. Prove that the function $f : \mathbb{N}^2 \to \mathbb{N}$, $f(a, b) = (a + b)^2 + b$, is one-to-one. Is there a one-to-one correspondence from $\mathbb{N}^2$ to $\mathbb{N}$ (or from $\mathbb{N}$ to $\mathbb{N}^2$)?