MATH 214-2 (41) - Fall 2001 - Final Exam (solutions)

SOLUTIONS

1. Find the net distance and the total distance traveled by a particle moving at speed $v(t) = 6(t^2 - 3t + 2)$ between $t = 0$ and $t = 3$.

Solution:

1. Net distance:
   $$\int_0^3 6\,(t^2 - 3t + 2)\,dt = \left[2t^3 - 9t^2 + 12t\right]_0^3 = 54 - 81 + 36 = \boxed{9}$$
2. Total distance:

   First note that $t^2 - 3t + 2 = (t-1)(t-2)$, so $v(t)$ is negative inside the interval $(1, 2)$ and positive outside it. Hence:
   $$\begin{aligned}\int_0^3 |6\,(t^2 - 3t + 2)|\,dt &= \int_0^1 6\,(t^2 - 3t + 2)\,dt - \int_1^2 6\,(t^2 - 3t + 2)\,dt \\ &\quad + \int_2^3 6\,(t^2 - 3t + 2)\,dt \\ &= \left[2t^3 - 9t^2 + 12t\right]_0^1 - \left[2t^3 - 9t^2 + 12t\right]_1^2 \\ &\quad + \left[2t^3 - 9t^2 + 12t\right]_2^3 \\ &= 5 - (-1) + 5 = \boxed{11}\end{aligned}$$

2. (Volumes By Cross sections and Cylindrical Shells) Find the volume of the solid obtained revolving around the x-axis the plane region bounded by the curves $y = 2x^2$ and $y^2 = 4x$. You may use either the method of *cross sections* or that of *cylindrical shells*.

Solution:

First we need to find the intersection points of the given curves:

$$\begin{cases} y &= 2x^2 \\ y^2 &= 4x \end{cases} \quad \Rightarrow \quad (2x^2)^2 = 4x \quad \Rightarrow \quad 4x^4 = 4x \quad \Rightarrow \quad x(x^3 - 1) = 0$$

The solutions are $x = 0, y = 0$ and $x = 1, y = 2$.

1. Solution by *cross sections*:

$$\begin{aligned} V = \int_0^1 \pi \left(y_{\text{top}}^2 - y_{\text{bot}}^2\right) dx &= \int_0^1 \pi \left(4x - 4x^4\right) dx \\ &= \pi \left[2x^2 - \frac{4x^5}{5}\right]_0^1 = \pi \left(2 - \frac{4}{5}\right) = \boxed{\frac{6\pi}{5}} \end{aligned}$$

2. Solution by *cylindrical shells*:

$$\begin{aligned} V = \int_0^2 2\pi y \left(x_{\text{right}} - x_{\text{left}}\right) dy &= 2\pi \int_0^2 \left(\frac{y^{3/2}}{\sqrt{2}} - \frac{y^3}{4}\right) dy \\ &= 2\pi \left[\frac{y^{5/2}\sqrt{2}}{5} - \frac{y^4}{16}\right]_0^2 = 2\pi \left(\frac{2^{5/2}\sqrt{2}}{5} - \frac{2^4}{16}\right) = \boxed{\frac{6\pi}{5}} \end{aligned}$$

3. (Arc Length and Surface Area) Given the arc $y = x^2$ between $x = 0$ and $x = 1$, set up (but do not evaluate) the integrals for finding:

(a) Its length.

(b) The area of the surface generated by revolving it around the x-axis.

Solution:

(a) Length: $\int_0^1 \sqrt{1 + (y')^2}\, dx = \boxed{\int_0^1 \sqrt{1 + 4x^2}\, dx}$

(b) Area: $\int_0^1 2\pi y \sqrt{1 + (y')^2}\, dx = \boxed{2\pi \int_0^1 x^2 \sqrt{1 + 4x^2}\, dx}$

4. (Separable Differential Equations) Solve the following initial value problem:

$$\begin{cases} \dfrac{dy}{dx} = -5(xy)^{3/2} \\ y(1) = 1 \end{cases}$$

Solution:

Separating variables we get:

$$y^{-3/2}\,dy = -5x^{3/2}\,dx\,.$$

Integrating:

$$-2y^{-1/2} = -2x^{5/2} + C\,,$$

The initial condition is $y = 1$ for $x = 1$, hence:

$$-2 = -2 + C \quad \Rightarrow \quad C = 0\,.$$

Consequently:

$$-2y^{-1/2} = -2x^{5/2}\,,$$

which implies:

$$\boxed{y = x^{-5} = \frac{1}{x^5}}$$

5. (Logarithmic Differentiation) Given

$$y = \left(\frac{\sqrt{1+x^2}}{\sqrt[3]{1+x^3}}\right)^3$$

find dy/dx using logarithmic differentiation.

Solution:

We start by taking logarithms and simplifying:

$$\ln y = \ln\left\{\left(\frac{\sqrt{1+x^2}}{\sqrt[3]{1+x^3}}\right)^3\right\} = 3\left(\frac{1}{2}\ln\left(1+x^2\right) - \frac{1}{3}\ln\left(1+x^3\right)\right)$$

Next we differentiate:

$$\frac{y'}{y} = 3\left(\frac{x}{(1+x^2)} - \frac{x^2}{(1+x^3)}\right)$$

Finally we multiply by $y = \left(\frac{\sqrt{1+x^2}}{\sqrt[3]{1+x^3}}\right)^3$:

$$\boxed{y' = 3\left(\frac{\sqrt{1+x^2}}{\sqrt[3]{1+x^3}}\right)^3\left(\frac{x}{(1+x^2)} - \frac{x^2}{(1+x^3)}\right)}$$

6. (L'Hôpital's Rule) Find the following limits using L'Hôpital's Rule:

1. $\lim\limits_{x\to 0} \dfrac{2-e^{-x}-e^{x}}{x^2}$
2. $\lim\limits_{x\to \frac{\pi}{2}} \left(x-\dfrac{\pi}{2}\right)\tan x$
3. $\lim\limits_{x\to 0} (1+\sin x)^{1/x}$

Solution:

1. $\lim\limits_{x\to 0} \dfrac{2-e^{-x}-e^{x}}{x^2} = \lim\limits_{x\to 0} \dfrac{e^{-x}-e^{x}}{2x} = \lim\limits_{x\to 0} \dfrac{-e^{-x}-e^{x}}{2} = \boxed{-1}$
2. $\lim\limits_{x\to \frac{\pi}{2}} \left(x-\dfrac{\pi}{2}\right)\tan x = \lim\limits_{x\to \frac{\pi}{2}} \dfrac{x-\frac{\pi}{2}}{\cot x} = \lim\limits_{x\to \frac{\pi}{2}} \dfrac{1}{-\csc x} = \dfrac{1}{-\csc\frac{\pi}{2}} = \boxed{-1}$
3. If $L = \lim\limits_{x\to 0} (1+\sin x)^{1/x}$, then

$$\ln L = \lim_{x\to 0} \frac{\ln(1+\sin x)}{x} = \lim_{x\to 0} \frac{\frac{\cos x}{1+\sin x}}{1} = 1$$

hence

$$\boxed{L = e^1 = e}$$

7. (Integration by Parts) Find the following integral using integration by parts:

$$\int \tan^{-1} x \, dx =$$

Solution:

We make $u = \tan^{-1} x$, $dv = dx$, so $du = \dfrac{dx}{1+x^2}$, $v = x$:

$$\begin{aligned}
\int \underbrace{\tan^{-1} x}_{u} \underbrace{dx}_{dv} &= \int u \, dv = uv - \int v \, du \\
&= x \tan^{-1} x - \int \frac{x}{1+x^2} \, dx \\
&= x \tan^{-1} x - \frac{1}{2} \int \frac{du}{u} \qquad\qquad (u = 1 + x^2) \\
&= x \tan^{-1} x - \frac{1}{2} \ln |u| + C \\
&= \boxed{x \tan^{-1} x - \frac{1}{2} \ln (1 + x^2) + C}
\end{aligned}$$

8. (Partial Fractions) Find the following integral by decomposing the integrand into partial fractions:

$$\int \frac{1}{x\,(1-x)}\,dx =$$

Solution:

Since the degree of the numerator (0) is less that that of the denominator (2) we do not need to perform long division.

The denominator is already factored, so we write:

$$\frac{1}{x\,(1-x)} = \frac{A}{x} + \frac{B}{1-x}\,.$$

Multiplying by $x\,(1-x)$:

$$1 = A(1-x) + Bx\,.$$

The easiest way to find A and B in this case is by assigning x the values 0 and 1 respectively:

$$\begin{aligned} x = 0 \quad &\Rightarrow \quad 1 = A \\ x = 1 \quad &\Rightarrow \quad 1 = B \end{aligned}$$

hence:

$$\frac{1}{x\,(1-x)} = \frac{1}{x} + \frac{1}{1-x}\,.$$

So:

$$\int \frac{1}{x\,(1-x)}\,dx = \int \left(\frac{1}{x} + \frac{1}{1-x}\right)\,dx = \boxed{\ln|x| - \ln|1-x| + C}$$

9. (Trigonometric Substitution) Evaluate the following integral using an appropriate trigonometric substitution:

$$\int_0^1 \sqrt{1+4x^2}\, dx$$

Solution:

First we find the indefinite integral using the substitution $u = 2x = \tan t$, so $dx = \frac{1}{2}\sec^2 t\, dt$:

$$\int \sqrt{1+4x^2}\, dx = \frac{1}{2}\int \sqrt{1+\tan^2 x}\, \sec^2 t\, dt = \frac{1}{2}\int \sec^3 t\, dx$$

Next we use the reduction formula for the integral of $\sec^n t$

$$\begin{aligned} &= \frac{1}{2}\left\{\frac{\sec t \tan t}{2} + \frac{1}{2}\int \sec t\, dt\right\} \\ &= \frac{1}{4}\sec t \tan t + \frac{1}{4}\ln|\sec u + \tan u| + C \end{aligned}$$

and undo the substitution

$$\begin{aligned} &= \frac{1}{4}\sqrt{1+\tan^2 t}\, \tan t + \frac{1}{4}\ln|\sqrt{1+\tan^2 t} + \tan u| + C \\ &= \frac{x\sqrt{1+4x^2}}{2} + \frac{1}{4}\ln|2x + \sqrt{1+4x^2}| + C\,. \end{aligned}$$

Finally we evaluate the definite integral:

$$\begin{aligned} \int_0^1 \sqrt{1+4x^2}\, dx &= \left[\frac{x\sqrt{1+4x^2}}{2} + \frac{1}{4}\ln\left(2x + \sqrt{1+4x^2}\right)\right]_0^1 \\ &= \boxed{\frac{\sqrt{5}}{2} + \frac{1}{4}\ln\left(2+\sqrt{5}\right)} \end{aligned}$$

10. (Taylor Series) Find the Taylor Series of the function $f(x) = \dfrac{e^x + e^{-x}}{2}$ at $x = 0$. Express the result in summation notation.

Solution:

The simplest solution consists of using the Taylor series for the exponential function:

$$e^x = \frac{1}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

so

$$e^{-x} = \frac{1}{0!} - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}$$

We get the answer by adding those two series and dividing by 2:

$$\boxed{f(x) = \frac{1}{2}(e^x + e^{-x}) = \frac{1}{0!} + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}}$$

Alternatively we can find the successive derivatives:

$f(x) = \frac{e^x + e^{-x}}{2}$, $f'(x) = \frac{e^x - e^{-x}}{2}$, $f''(x) = \frac{e^x + e^{-x}}{2}$, $f'''(x) = \frac{e^x - e^{-x}}{2}$, etc., so:

$f(0) = 1$, $f'(0) = 0$, $f''(0) = 1$, $f'''(0) = 0$, etc.

and apply the formula: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$,

which yields the same final result.

Table of Integrals

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C \ (n \neq -1) \qquad \int \frac{du}{u} = \ln|u| + C$$

$$\int e^u\,du = e^u + C \qquad \int \cos u\,du = \sin u + C$$

$$\int \sin u\,du = -\cos u + C \qquad \int \sec^2 u\,du = \tan u + C$$

$$\int \csc^2 u\,du = -\cot u + C \qquad \int \sec u \tan u\,du = \sec u + C$$

$$\int \csc u \cot u\,du = -\csc u + C \qquad \int \sec u\,du = \ln|\sec u + \tan u| + C$$

$$\int \csc u\,du = \ln|\csc u - \cot u| + C \qquad \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C$$

$$\int \frac{du}{1+u^2} = \tan^{-1} u + C \qquad \int \frac{du}{u\sqrt{u^2-1}}\,du = \sec^{-1}|u| + C$$

Integrals Involving Inverse Hyperbolic Functions

$$\int \frac{du}{\sqrt{u^2+1}} = \sinh^{-1} u + C \qquad \int \frac{du}{\sqrt{u^2-1}} = \cosh^{-1} u + C$$

$$\int \frac{du}{u\sqrt{1-u^2}} = -\operatorname{sech}^{-1}|u| + C \qquad \int \frac{du}{u\sqrt{1+u^2}} = -\operatorname{csch}^{-1}|u| + C$$

Reduction Formulas

$$\int \sin^n u\,du = -\frac{1}{n}\sin^{n-1} u\ \cos u + \frac{n-1}{n}\int \sin^{n-2} u\,du$$

$$\int \cos^n u\,du = \frac{1}{n}\cos^{n-1} u\ \sin u + \frac{n-1}{n}\int \cos^{n-2} u\,du$$

$$\int \tan^n u\,du = \frac{\tan^{n-1} u}{n-1} - \int \tan^{n-2} u\,du\,.$$

$$\int \sec^n u\,du = \frac{\sec^{n-2} u \tan u}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} u\,du\,.$$