**CS 310-0**
**Homework Assignment No. 4**
Due Tue 2/6/2001

1. Let $f, g : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ be the functions defined by $f(x, y) = x + y$ and $g(x, y) = x - y$. Find $f^{-1}(\{2, 3\}) \cap g^{-1}(\{-1, 0, 1\})$ (intersection of *preimages*).

2. Let $A_1, A_2, B_1, B_2$ be non-empty sets such that $A = A_1 \cap A_2 = \emptyset$ and $B_1 \cap B_2 = \emptyset$. Let $f_1 : A_1 \to B_1$, $f_2 : A_2 \to B_2$ two given functions. Define a function $f : A_1 \cup A_2 \to B_1 \cup B_2$ by cases in the following way:
$$f(x) = \begin{cases} f_1(x) & \text{if } x \in A_1 \\ f_2(x) & \text{if } x \in A_2 \end{cases}$$
Prove:
(a) $f$ is one-to-one if and only if $f_1$ and $f_2$ are one-to-one.
(b) $f$ is onto if and only if $f_1$ and $f_2$ are onto.
(c) $f$ is a one-to-one correspondence if and only if $f_1$ and $f_2$ are one-to-one correspondences. In this case, find $f^{-1}$ in terms of $f_1^{-1}$ and $f_2^{-1}$.

3. For each one of the following functions, determine if it is one-to-one (but not onto), onto (but not one-to-one), or a one-to-one correspondence. If it is a one-to-one correspondence, find its inverse.
(a) $f : \mathbb{N} \to \mathbb{N}$, $f(x) = 2x^2 + 1$.
(b) $f : \mathbb{N} \to \mathbb{N}$, $f(x) = \lfloor (x + 2)/3 \rfloor$, where $\lfloor x \rfloor$ = "greatest integer less than or equal to $x$" (*floor* function.)
(c) $f : \mathbb{Z} \to \mathbb{Z}$, defined by cases in the following way:
$$f(x) = \begin{cases} x - 3 & \text{if } x \text{ is even} \\ x + 1 & \text{if } x \text{ is odd} \end{cases}$$

4. Find the properties (commutative, associative, existence of identity element, existence of inverse) verified by the following operations:
(a) $x \circ y = x + y - a$ on $\mathbb{Z}$, where $a$ is a fix integer.
(b) $x * y = x + y - xy$ on $\mathbb{Z}$.
Is there any value of $a$ for which the operations $*$ and $\circ$ defined above verify that $*$ is distributive respect to $\circ$?

5. Let $\mathbb{Q}[\sqrt{2}]$ be the set $\mathbb{Q}[\sqrt{2}] = \{x + y\sqrt{2} \mid x, y \in \mathbb{Q}\}$.
(a) Prove that $(\mathbb{Q}[\sqrt{2}], +)$ is a commutative group.
(b) Prove that $(\mathbb{Q}[\sqrt{2}], +, \cdot)$ is a commutative ring with unity.
(c) Is $(\mathbb{Q}[\sqrt{2}], +, \cdot)$ a field? Prove it or disprove it.
(d) Prove that the function $f : (\mathbb{Q}[\sqrt{2}], +, \cdot) \to (\mathbb{Q}[\sqrt{2}], +, \cdot)$ defined
$$f(x + y\sqrt{2}) = x - y\sqrt{2}$$
is a ring-homomorphism.