4.6. Representation of Functions as Power Series

We have already seen that a power series is a particular kind of function. A slightly different matter is that sometimes a given function can be written as a power series. We already know the example

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n \qquad (|x| < 1)$$

Replacing x with other expressions we may write other functions in the same way, for instance by replacing x with $-2x^2$ we get:

$$\frac{1}{1+2x^2} = 1-2x^2+4x^4-8x^6+\cdots+(-1)^n 2^n x^{2n}+\cdots = \sum_{n=0}^{\infty}(-1)^n 2^n x^{2n} \,,$$

which converges for $|-2x^2| < 1$, i.e., $|x| < 1/\sqrt{2}$.

4.6.1. Differentiation and Integration of Power Series.

Since the sum of a power series is a function we can differentiate it and integrate it. The result is another function that can also be represented with another power series. The main related result is that the derivative or integral of a power series can be computed by *term-by-term differentiation and integration*:

4.6.1.1. *Term-By-Term Differentiation and Integration.* If the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has radius of convergence $R > 0$ then the function

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable on the interval $(a-R, a+R)$ and

(1) $$f'(x) = \sum_{n=0}^{\infty}\{c_n(x-a)^n\}' = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$$

(2) $$\int f(x)\,dx = \sum_{n=0}^{\infty}\int c_n(x-a)^n\,dx = C + \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}$$

The radii of convergence of the series in the above equations is R.

Example: Find a power series representation for the function

$$f(x) = \frac{1}{(1-x)^2} \,.$$

Answer: We have

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\frac{1}{1-x}$$

and

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n ,$$

hence

$$\begin{aligned}\frac{1}{(1-x)^2} &= \frac{d}{dx}\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty}\frac{d}{dx}x^n = \sum_{n=1}^{\infty} nx^{n-1} \\ &= 1 + 2x + 3x^2 + 4x^3 + \cdots = \boxed{\sum_{n=0}^{\infty}(n+1)x^n} \quad \text{(re-indexed)}\end{aligned}$$

The radius of convergence is $R = 1$.

Example: Find a power series representation for $\tan^{-1} x$.

Answer: That function is the antiderivative of $1/(1+x^2)$, hence:

$$\begin{aligned}\tan^{-1} x &= \int \frac{1}{1+x^2}\, dx \\ &= \int \sum_{n=0}^{\infty}(-1)^n x^{2n}\, dx \\ &= \sum_{n=0}^{\infty}\int (-1)^n x^{2n}\, dx \\ &= C + \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1} \\ &= C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots\end{aligned}$$

Since $\tan^{-1} 0 = 0$ then $C = 0$, hence

$$\boxed{\tan^{-1} x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots}$$

The radius of convergence is $R = 1$.

Example: Find a power series representation for $\ln(1+x)$.

Answer: The derivative of that function is $1/(1+x)$, hence

$$\begin{aligned}
\ln(1+x) &= \int \frac{1}{1+x}\,dx \\
&= \int \sum_{n=0}^{\infty} (-1)^n x^n\,dx \\
&= C + \sum_{n=0}^{\infty} \int (-1)^n \frac{x^{n+1}}{n+1}\,dx \\
&= C + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots
\end{aligned}$$

Since $\ln 1 = 0$ then $C = 0$, so

$$\boxed{\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}}$$

The radius of convergence is $R = 1$.