MATH 214-2 (sec 41) - Fall 2002 - Midterm (solutions)

SOLUTIONS

1. (Integration by Substitution) Find the following integral:

Solution:

$$\begin{aligned}
\int \frac{2x}{\sqrt{1+x^2}}\,dx &= \int \frac{1}{\sqrt{u}}\,du \qquad\qquad (u = 1+x^2,\ du = 2x\,dx) \\
&= 2\sqrt{u} + C \\
&= \boxed{2\sqrt{1+x^2} + C}
\end{aligned}$$

2. (Trigonometric Integrals) Find the following integral:

Solution:

$$\begin{aligned}
\int \sin^3 x\,dx &= \int \sin^2 x \sin x\,dx \\
&= \int (u^2 - 1)\,du && (u = \cos x,\ du = -\sin x\,dx) \\
&= \frac{u^3}{3} - u + C \\
&= \boxed{\frac{\cos^3 x}{3} - \cos x + C}
\end{aligned}$$

3. (Integration by Parts) Find the following integral:

Solution:

$$\begin{aligned}
\int \ln(1+x^2)\,dx &= \int \underbrace{\ln(1+x^2)}_{u}\ \underbrace{dx}_{dv} && \text{(by parts)} \\
&= \underbrace{x}_{v}\ \underbrace{\ln(1+x^2)}_{u} - \int \underbrace{x}_{v}\ \underbrace{\frac{2x}{1+x^2}\,dx}_{du} \\
&= x\ln(1+x^2) - \int \frac{2x^2}{1+x^2}\,dx \\
&= x\ln(1+x^2) - \int \left\{2 - \frac{2}{1+x^2}\right\} dx \\
&= \boxed{x\ln(1+x^2) - 2x + 2\tan^{-1} x + C}
\end{aligned}$$

4. (Derivative of an Integral) Compute:

Solution:

$$\frac{d}{dx}\int_1^{x^2} e^{-1/t}\,dt = e^{-1/x^2}\cdot\frac{d}{dx}x^2 = \boxed{2x\,e^{-1/x^2}}$$

5. (Numerical Integration) Estimate $\displaystyle\int_1^5 \frac{1}{x}\,dx$ using Simpson's Rule with $n = 4$.

Solution:

$$S_4 = \frac{1}{3}\left\{\frac{1}{1} + 4\cdot\frac{1}{2} + 2\cdot\frac{1}{3} + 4\cdot\frac{1}{4} + \frac{1}{5}\right\} = \boxed{\frac{73}{45}}$$

6. (Partial Fractions) Evaluate the following integral:

$$\int \frac{2}{x^2 - 4x + 3}\,dx =$$

Solution:

1. First, factor the denominator: $x^2 - 4x + 3 = (x - 1)(x - 3)$.
2. Next, decompose into partial fractions:
$$\frac{2}{x^2 - 4x + 3} = \frac{A}{x - 1} + \frac{B}{x - 3}\,,$$
$$2 = A(x - 3) + B(x - 1)\,,$$
$$\begin{aligned} x = 1 \quad &\Rightarrow \quad 2 = -2A \quad \Rightarrow \quad A = -1\,, \\ x = 3 \quad &\Rightarrow \quad 2 = 2B \quad \Rightarrow \quad B = 1\,, \end{aligned}$$
hence
$$\frac{2}{x^2 - 4x + 3} = \frac{-1}{x - 1} + \frac{1}{x - 3}\,.$$
3. Finally, integrate:
$$\begin{aligned} \int \frac{2}{x^2 - 4x + 3}\,dx &= \int \frac{-1}{x - 1}\,dx + \int \frac{1}{x - 3}\,dx \\ &= \boxed{-\ln|x - 1| + \ln|x - 3| + C}\,. \end{aligned}$$

7. (Tables of Integration) Using the following formula

$$\int \frac{\sqrt{u^2 - a^2}}{u}\, du = \sqrt{u^2 - a^2} - a\cos^{-1}\frac{a}{|u|} + C$$

find the following integral: $\displaystyle\int \sqrt{9 - \frac{25}{4x^2}}\, dx$

Solution:

First we need to transform the integrand into something resembling the left hand side of the given formula:

$$\begin{aligned}\sqrt{9 - \frac{25}{4x^2}} &= \sqrt{\frac{36x^2}{4x^2} - \frac{25}{4x^2}} = \sqrt{\frac{36x^2 - 25}{4x^2}} \\ &= \frac{\sqrt{36x^2 - 25}}{2|x|} = \operatorname{sgn}(x)\, 3\frac{\sqrt{(6x)^2 - 5^2}}{6x}\,,\end{aligned}$$

were $\operatorname{sgn}(x)$ is the sign of x. Hence (ignoring the subtlety concerning the sign of x) the given integral becomes:

$$\begin{aligned}\frac{1}{2}\int \frac{\sqrt{(6x)^2 - 5^2}}{6x}\, 6\, dx &= \frac{1}{2}\int \frac{\sqrt{u^2 - a^2}}{u}\, du \qquad (u = 6x,\ a = 5,\ du = 6\, dx) \\ &= \frac{1}{2}\left\{\sqrt{u^2 - a^2} - a\cos^{-1}\frac{a}{|u|}\right\} + C \\ &= \boxed{\frac{1}{2}\left\{\sqrt{36x^2 - 25} - 5\cos^{-1}\left(\frac{5}{6|x|}\right)\right\} + C}\end{aligned}$$

(Technically in order to get the correct answer we still need to multiply by the sign of x, but we won't worry about it here.)

8. (Volumes) Find the volume V of the solid obtained by rotating about the y-axis the region bounded by $y = x$ and $y = x^2$.

Solution:

First we find the intersection points of $y = x$ and $y = x^2$ by solving $x = x^2$ $\Rightarrow x = 0$ and $x = 1$. From here we get that the intersection points are $(0, 0)$ and $(1, 1)$. So the limits of integration for both x and y will be 0 and 1.

1. Method of slices (washers):

$$V = \int_0^1 \pi(x_R^2 - x_L^2)\, dy = \int_0^1 \pi\{(\sqrt{y})^2 - y^2\}\, dy = \int_0^1 \pi(y - y^2)\, dy$$
$$= \pi\left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1 = \pi\left(\frac{1}{2} - \frac{1}{3}\right) = \boxed{\frac{\pi}{6}}.$$

2. Method of cylindrical shells:

$$V = \int_0^1 2\pi x(y_T - y_B)\, dx = \int_0^1 2\pi x(x - x^2)\, dx = \int_0^1 2\pi(x^2 - x^3)\, dx$$
$$= 2\pi\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 2\pi\left(\frac{1}{3} - \frac{1}{4}\right) = \boxed{\frac{\pi}{6}}.$$

9. (Arc Length) Find the length of the parametric arc $x = \cos t$, $y = \sin t$ from $t = 0$ to $t = \pi/3$.

Solution:

$$\begin{aligned} L &= \int_0^{\pi/3} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= \int_0^{\pi/3} \left\{\sqrt{\sin^2 t + \cos^2 t}\right\} dt \\ &= \int_0^{\pi/3} 1\, dt = [t]_0^{\pi/3} = \boxed{\frac{\pi}{3}} \end{aligned}$$

10. (Physics) An object is pushed along the x axis from $x = 0$ to $x = 20$ with a force equal to $1/(1+x)$. Find the work done by the force.

Solution:

$$
\begin{aligned}
W &= \int_0^{20} F(x)\,dx = \int_0^{20} \frac{1}{1+x}\,dx = \left[\ln(1+x)\right]_0^{20} \\
&= \ln 21 - \ln 1 = \boxed{\ln 21}
\end{aligned}
$$