4.3. The Integral and Comparison Tests

4.3.1. The Integral Test. Suppose f is a continuous, positive, decreasing function on $[1, \infty)$, and let $a_n = f(n)$. Then the convergence or divergence of the series $\sum_{n=1}^{\infty} a_n$ is the same as that of the integral $\int_1^{\infty} f(x)\, dx$, i.e.:

(1) If $\displaystyle\int_1^{\infty} f(x)\, dx$ is convergent then $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent.

(2) If $\displaystyle\int_1^{\infty} f(x)\, dx$ is divergent then $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent.

The best way to see why the integral test works is to compare the area under the graph of $y = f(x)$ between 1 and ∞ to the sum of the areas of rectangles of height $f(n)$ placed along intervals $[n, n+1]$.

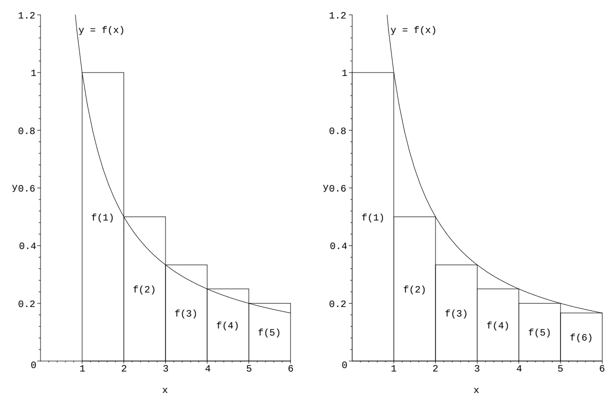

FIGURE 4.3.1

From the graph we see that the following inequality holds:

$$\int_1^{n+1} f(x)\, dx \leq \sum_{i=1}^{n} a_n \leq f(1) + \int_1^{n} f(x)\, dx\,.$$

The first inequality shows that if the integral diverges so does the series. The second inequality shows that if the integral converges then the same happens to the series.

Example: Use the integral test to prove that the harmonic series $\sum_{n=1}^{\infty} 1/n$ diverges.

Answer: The convergence or divergence of the harmonic series is the same as that of the following integral:

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty} \Big[\ln x\Big]_1^t = \lim_{t\to\infty} \ln t = \infty\,,$$

so it diverges.

4.3.2. The p-series. The following series is called p-series:

$$\sum_{n=1}^\infty \frac{1}{n^p}\,.$$

Its behavior is the same as that of the integral $\int_1^\infty \frac{1}{x^p}\,dx$. For $p = 1$ we have seen that it diverges. If $p \neq 1$ we have

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^p}\,dx = \lim_{t\to\infty} \left[\frac{x^{1-p}}{1-p}\right]_1^t = \lim_{t\to\infty} \frac{t^{1-p}}{1-p} - \frac{1}{1-p}\,.$$

For $0 < p < 1$ the limit is infinite, and for $p > 1$ it is zero so:

$$\boxed{\text{The } p\text{-series } \sum_{n=1}^\infty \frac{1}{n^p} \text{ is } \begin{cases} \text{convergent} & \text{if } p > 1 \\ \text{divergent} & \text{if } p \leq 1 \end{cases}}$$

4.3.3. Comparison Test. Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms and suppose that $a_n \leq b_n$ for all n. Then

(1) If $\sum b_n$ is convergent then $\sum a_n$ is convergent.
(2) If $\sum a_n$ is divergent then $\sum b_n$ is divergent.

Example: Determine whether the series $\displaystyle\sum_{n=1}^\infty \frac{\cos^2 n}{n^2}$ converges or diverges.

Answer: We have

$$0 < \frac{\cos^2 n}{n^2} \leq \frac{1}{n^2} \quad \text{for all } n \geq 1$$

and we know that the p-series $\displaystyle\sum_{n=1}^\infty \frac{1}{n^2}$ converges. Hence by the comparison test, the given series also converges (incidentally, its sum is $\frac{1}{2} - \frac{\pi}{2} + \frac{\pi^2}{6} = 0.5736380465\ldots$, although we cannot prove it here).

4.3.4. The Limit Comparison Test. Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c\,,$$

where c is a finite strictly positive number, then either both series converge or both diverge.

Example: Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{1+4n^2}}$ converges or diverges.

Answer: We will use the limit comparison test with the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$. We have

$$\begin{aligned}
\lim_{n\to\infty} \frac{1/n}{1/\sqrt{1+4n^2}} &= \lim_{n\to\infty} \frac{\sqrt{1+4n^2}}{n} \\
&= \lim_{n\to\infty} \sqrt{\frac{1+4n^2}{n^2}} \\
&= \lim_{n\to\infty} \sqrt{\frac{1}{n^2}+4} \quad = \sqrt{4} = 2\,,
\end{aligned}$$

so the given series has the same behavior as the harmonic series. Since the harmonic series diverges, so does the given series.

4.3.5. Remainder Estimate for the Integral Test. The difference between the sum $s = \sum_{n=1}^{\infty} a_n$ of a convergent series and its nth partial sum $s_n = \sum_{i=1} a_i$ is the *remainder*:

$$R_n = s - s_n = \sum_{i=n+1}^{\infty} a_i\,.$$

The same graphic used to see why the integral test works allows us to estimate that remainder. Namely: If $\sum a_n$ converges by the Integral Test and $R_n = s - s_n$, then

$$\boxed{\int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_{n}^{\infty} f(x)\,dx}$$

Equivalently (adding s_n):

$$\boxed{s_n + \int_{n+1}^{\infty} f(x)\,dx \leq s \leq s_n + \int_{n}^{\infty} f(x)\,dx}$$

Example: Estimate $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$ to the third decimal place.

Answer: We need to reduce the remainder below 0.0005, i.e., we need to find some n such that

$$\int_n^{\infty} \frac{1}{x^4}\,dx < 0.0005\,.$$

We have

$$\int_n^{\infty} \frac{1}{x^4}\,dx = \left[-\frac{1}{3x^3}\right]_n^{\infty} = \frac{1}{3n^3}\,,$$

hence

$$\frac{1}{3n^3} < 0.0005 \quad \Rightarrow \quad n > \sqrt[3]{\frac{3}{0.0005}} = 18.17\ldots\,,$$

so we can take $n = 19$. So the sum of the 15 first terms of the given series coincides with the sum of the whole series up to the third decimal place:

$$\sum_{i=1}^{19} \frac{1}{i^4} = 1.082278338\ldots$$

From here we deduce that the actual sum s of the series is between $1.08227\ldots - 0.0005 = 1.08177\ldots$ and $1.08227\ldots + 0.0005 = 1.08277\ldots$, so we can claim $s \approx 1.082$. (The actual sum of the series is $\frac{\pi^4}{90} = 1.0823232337\ldots$.)

4.4. Other Convergence Tests

4.4.1. Alternating Series. An *alternating series* is a series whose terms are alternately positive and negative., for instance

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} .$$

4.4.1.1. *The Alternating Series Test.* If the sequence of positive terms b_n verifies

(1) b_n is decreasing.
(2) $\lim_{n\to\infty} b_n = 0$

then the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n = b_1 - b_2 + b_3 - b_4 + \cdots$$

converges.

Example: The alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

converges because $1/n \to 0$. (Its sum is $\ln 2 = 0.6931471806\dots$.)

4.4.1.2. *Alternating Series Estimation Theorem.* If $s = \sum_{n=1}^{\infty} (-1)^n b_n$ is the sum of and alternating series verifying that b_n is decreasing and $b_n \to 0$, then the remainder of the series verifies:

$$|R_n| = |s - s_n| \le b_{n+1} .$$

4.4.2. Absolute Convergence. A series $\sum_{n=1}^{\infty} a_n$ is called *absolutely convergent* if the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ converges.

Absolute convergence implies convergence, i.e., if a series $\sum a_n$ is absolutely convergent, then it is convergent.

The converse is not true in general. For instance, the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is convergent but it is not absolutely convergent.

Example: Determine whether the series

$$\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$$

is convergent or divergent.

Answer: We see that the series of absolute values $\sum_{n=1}^{\infty} \frac{|\cos n|}{n^2}$ is convergent by comparison with $\sum_{n=1}^{\infty} \frac{1}{n^2}$, hence the given series is absolutely convergent, therefore it is convergent (its sum turns out to be $1/4 - \pi/2 + \pi^2/6 = 0.324137741\ldots$, but the proof of this is beyond the scope of this notes).

4.4.3. The Ratio Test.

(1) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$ then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(2) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ (including $L = \infty$) then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(3) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$ then the test is inconclusive (we do not know whether the series converges or diverges).

Example: Test the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{n!}{n^n}$$

for absolute convergence.

Answer: We have:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!/(n+1)^{n+1}}{n!/n^n} = \frac{n^n}{(n+1)^n} = \frac{1}{\left(1+\frac{1}{n}\right)^n} \xrightarrow[n\to\infty]{} e^{-1} < 1\,,$$

hence by the Ratio Test the series is absolutely convergent.