CS 310-0
Homework Assignment No. 4
Due Fri 4/28/2000

1. Find the properties (commutative, associative, existence of identity element, existence of inverse) verified by the following operations:
   (a) Addition on $\mathbb{N}$.
   (b) Multiplication on $\mathbb{Q}^+$.
   (c) $x \circ y = x + y - a$ on $\mathbb{Z}$, where a is a fix integer.
   (d) $x * y = x + y - xy$ on $\mathbb{Z}$.
   Is there any value of a for which the operations $*$ and $\circ$ defined above verify that $*$ is distributive respect to $\circ$?

   In the following two exercises you may use known properties of set operations without proof—for instance, commutativity of set intersection—, as stated in the class notes, and the distributive property of intersection respect to symmetric difference, as proven in homework assignment 2.

2. Let X be a nonempty set. Let $\mathcal{P}(X)$ be the power set of X. Prove that $(\mathcal{P}(X), \triangle)$, where $A \triangle B$ = symmetric difference, is a commutative group. What is the identity element? What is the inverse of a set $A \in \mathcal{P}(X)$?

3. Given a set X with at least two elements, prove that $(\mathcal{P}(X), \triangle, \cap)$ (the power set of X with the operations of symmetric difference and intersection) is a commutative ring with unity, but not a field. Prove that it has proper divisors of zero.

4. Let $(G, *)$ a group with identity element e. Let $H \subseteq G$ be a subset of G. Prove that $(H, *)$ is a subgroup of $(G, *)$ iff $\forall x, y \in H$, $x * y^{-1} \in H$.

5. Let $(G_1, *)$, $(G_2, \circ)$ be two groups, with identity elements e_1 and e_2 respectively. Prove that if $f : G_1 \to G_2$ is a group homomorphism then:
   (a) $f(e_1) = e_2$.
   (b) $\forall x \in G_1,\ f(x^{-1}) = f(x)^{-1}$.
   (c) $f(G_1)$ is a subgroup of $(G_2, \circ)$.
   (d) $\ker f = \{x \in G_1 \mid f(x) = e_2\}$ is a subgroup of $(G_1, *)$.[1]

6. In how many ways can we color the faces of a die with two colors (say red and blue) if two colorings differing only by rotations of the die are considered identical?

7. Ten people go to a movie.
   (a) In how many ways can they be placed in the queue of the movie theater?
   (b) Assume that those people are divided into three groups, one with 2 people, another one with 3 people and another one with 5 people. In how many ways can they be placed in the queue if the members of the same group must remain together?

[1] $\ker f$ is called *kernel* of f.

8. A city has streets in the direction E-W and avenues in the direction N-S, making a perfect grid. Assume that a taxicab has to go from the intersection of 23rd street and 1st avenue to the intersection of 30th street and 7th avenue following a path of minimum length. In how many ways can it be done? Assume that there is an accident at the intersection of 27th street and 5th avenue and the taxicab wants to avoid it. How many (minimal) paths can the taxicab follow now?

9. A cube is made by piling $3 \times 4 \times 5 = 60$ smaller cubes. An insect moves always following the edges of the small cubes (including the edges in the interior of the larger cube) and possibly changing direction only at the vertices of the small cubes. In how many ways can the insect go from one vertex of the large cube to the diametrically opposite vertex following a path of minimum length?

10. How many functions are there from $A = \{a, b, c, d, e\}$ to $B = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$? How many of them are one-to-one? How many assign even numbers to vowels and odd numbers to consonants? How many of the latter are one-to-one?