MATH 214-2 (sec 61) - Fall 2004 - Midterm (solutions)

SOLUTIONS

1. (Integration by Substitution.) Find the following integrals:

Solution:

$$(a)\ \int \frac{3x^2}{\sqrt{1+x^3}}\,dx = \int \frac{1}{\sqrt{u}}\,du \qquad (u = 1 + x^3,\ du = 3x^2\,dx)$$

$$= 2\sqrt{u} + C$$

$$= \boxed{2\sqrt{1+x^3} + C}$$

$$(b)\ \int \frac{\cos(\ln x)}{x}\,dx = \int \cos u\,du \qquad (u = \ln x,\ du = \frac{1}{x}\,dx)$$

$$= \sin u + C$$

$$= \boxed{\sin(\ln x) + C}$$

2. (Trigonometric Integrals.) Find the following integral:

Solution:

$$
\begin{aligned}
\int \cos^5 x\,dx &= \int \cos^4 x \cos x\,dx \\
&= \int (1-u^2)^2\,du && (u = \sin x,\ \ du = \cos x\,dx) \\
&= \int (1 - 2u^2 + u^4)\,du \\
&= u - \frac{2u^3}{3} + \frac{u^5}{5} + C \\
&= \boxed{\sin x - \frac{2\sin^3 x}{3} + \frac{\sin^5 x}{5} + C}
\end{aligned}
$$

3. (Integration by Parts.) Find the following integral:

Solution:

$$\begin{aligned}
\int \ln(1+x^2)\,dx &= \int \underbrace{\ln(1+x^2)}_{u}\;\underbrace{dx}_{dv} && \text{(by parts)}\\
&= \underbrace{x}_{v}\;\underbrace{\ln(1+x^2)}_{u} - \int \underbrace{x}_{v}\;\underbrace{\frac{2x}{1+x^2}\,dx}_{du}\\
&= x\ln(1+x^2) - \int \frac{2x^2}{1+x^2}\,dx\\
&= x\ln(1+x^2) - \int \left\{2 - \frac{2}{1+x^2}\right\}dx\\
&= \boxed{x\ln(1+x^2) - 2x + 2\tan^{-1}x + C}
\end{aligned}$$

4. (Derivative of an Integral.) Compute:

Solution:

$$\frac{d}{dx}\int_1^{\ln x} \sin\left(e^t\right) dt = \sin\left(e^{\ln x}\right) \cdot \frac{d}{dx} \ln x = \boxed{\frac{\sin x}{x}}$$

5. (Numerical Methods.) Calculate both the *trapezoidal* approximation T_4 and *Simpson's* approximation S_4 for $n = 4$ to the integral

$$\int_0^4 e^{-x^2}\, dx$$

Do not make the computations, just write the formula.

Solution:

For $n = 4$ we have $\Delta x = (b - a)/n = (4 - 0)/4 = 1$, and $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, $x_4 = 4$, hence

Trapezoidal approximation:

$$T_4 = \frac{\Delta x}{2}\,(e^{-0^2}+2e^{-1^2}+2e^{-2^2}+2e^{-3^2}+e^{-4^2}) = \frac{1}{2}\,(1+2e^{-1}+2e^{-4}+2e^{-9}+e^{-16})\,.$$

Simpson's approximation:

$$S_4 = \frac{\Delta x}{3}\,(e^{-0^2}+4e^{-1^2}+2e^{-2^2}+4e^{-3^2}+e^{-4^2}) = \frac{1}{3}\,(1+4e^{-1}+2e^{-4}+4e^{-9}+e^{-16})\,.$$

6. (Partial Fractions.) Evaluate the following integral:

$$\int \frac{2}{x^2 - 4x + 3}\, dx =$$

Solution:

1. First, factor the denominator: $x^2 - 4x + 3 = (x-1)(x-3)$.
2. Next, decompose into partial fractions:

$$\frac{2}{x^2 - 4x + 3} = \frac{A}{x-1} + \frac{B}{x-3}\,,$$

$$2 = A(x-3) + B(x-1)\,,$$

$$\begin{aligned} x = 1 \quad &\Rightarrow \quad 2 = -2A \quad \Rightarrow \quad A = -1\,, \\ x = 3 \quad &\Rightarrow \quad 2 = 2B \quad \Rightarrow \quad B = 1\,, \end{aligned}$$

hence

$$\frac{2}{x^2 - 4x + 3} = \frac{-1}{x-1} + \frac{1}{x-3}\,.$$

3. Finally, integrate:

$$\begin{aligned} \int \frac{2}{x^2 - 4x + 3}\, dx &= \int \frac{-1}{x-1}\, dx + \int \frac{1}{x-3}\, dx \\ &= \boxed{-\ln|x-1| + \ln|x-3| + C}\,. \end{aligned}$$

7. (Tables of Integration.) Find the following integral: $\displaystyle\int \frac{\sqrt{25-36x^2}}{x}\,dx$

Solution:

$$\begin{aligned}
\int \frac{\sqrt{25-36x^2}}{x}\,dx &= \int \frac{\sqrt{5^2-(6x)^2}}{6x}\,6\,dx \\
&= \int \frac{\sqrt{a^2-u^2}}{u}\,du \qquad\qquad (u=6x,\ a=5,\ du=6\,dx) \\
&= \sqrt{a^2-u^2} - a\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C \\
&= \boxed{\sqrt{25-36x^2} - 5\ln\left|\frac{5+\sqrt{25-36x^2}}{6x}\right| + C}
\end{aligned}$$

Table of Integrals

23. $\displaystyle\int \frac{\sqrt{a^2+u^2}}{u}\,du = \sqrt{a^2+u^2} - a\ln\left|\frac{a+\sqrt{a^2+u^2}}{u}\right| + C$

32. $\displaystyle\int \frac{\sqrt{a^2-u^2}}{u}\,du = \sqrt{a^2-u^2} - a\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C$

41. $\displaystyle\int \frac{\sqrt{u^2-a^2}}{u}\,du = \sqrt{u^2-a^2} - a\cos^{-1}\frac{a}{|u|} + C$