1.2. Predicates, Quantifiers

1.2.1. Predicates. A *predicate* or *propositional function* is a statement containing variables. For instance "$x + 2 = 7$", "X is American", "$x < y$", "p is a prime number" are predicates. The truth value of the predicate depends on the value assigned to its variables. For instance if we replace x with 1 in the predicate "$x+2 = 7$" we obtain "$1+2 = 7$", which is false, but if we replace it with 5 we get "$5 + 2 = 7$", which is true. We represent a predicate by a letter followed by the variables enclosed between parenthesis: $P(x)$, $Q(x, y)$, etc. An *example* for $P(x)$ is a value of x for which $P(x)$ is true. A *counterexample* is a value of x for which $P(x)$ is false. So, 5 is an example for "$x + 2 = 7$", while 1 is a counterexample.

Each variable in a predicate is assumed to belong to a *universe (or domain) of discourse*, for instance in the predicate "n is an odd integer" 'n' represents an integer, so the universe of discourse of n is the set of all integers. In "X is American" we may assume that X is a human being, so in this case the universe of discourse is the set of all human beings.[1]

1.2.2. Quantifiers. Given a predicate $P(x)$, the statement "for some x, $P(x)$" (or "there is some x such that $p(x)$"), represented "$\exists x\, P(x)$", has a definite truth value, so it is a proposition in the usual sense. For instance if $P(x)$ is "$x + 2 = 7$" with the integers as universe of discourse, then $\exists x\, P(x)$ is true, since there is indeed an integer, namely 5, such that $P(5)$ is a true statement. However, if $Q(x)$ is "$2x = 7$" and the universe of discourse is still the integers, then $\exists x\, Q(x)$ is false. On the other hand, $\exists x\, Q(x)$ would be true if we extend the universe of discourse to the rational numbers. The symbol $\exists$ is called the *existential quantifier*.

Analogously, the sentence "for all x, $P(x)$"—also "for any x, $P(x)$", "for every x, $P(x)$", "for each x, $P(x)$"—, represented "$\forall x\, P(x)$", has a definite truth value. For instance, if $P(x)$ is "$x + 2 = 7$" and the

[1]Usually all variables occurring in predicates along a reasoning are supposed to belong to the *same* universe of discourse, but in some situations (as in the so called *many-sorted* logics) it is possible to use different kinds of variables to represent different types of objects belonging to different universes of discourse. For instance in the predicate "σ is a string of length n" the variable σ represents a string, while n represents a natural number, so the universe of discourse of σ is the set of all strings, while the universe of discourse of n is the set of natural numbers.

universe of discourse is the integers, then $\forall x\, P(x)$ is false. However if $Q(x)$ represents "$(x+1)^2 = x^2 + 2x + 1$" then $\forall x\, Q(x)$ is true. The symbol $\forall$ is called the *universal quantifier*.

In predicates with more than one variable it is possible to use several quantifiers at the same time, for instance $\forall x \forall y \exists z\, P(x, y, z)$, meaning "for all x and all y there is some z such that $P(x, y, z)$".

Note that in general the existential and universal quantifiers cannot be swapped, i.e., in general $\forall x \exists y\, P(x, y)$ means something different from $\exists y \forall x\, P(x, y)$. For instance if x and y represent human beings and $P(x, y)$ represents "x is a friend of y", then $\forall x \exists y\, P(x, y)$ means that everybody is a friend of someone, but $\exists y \forall x\, P(x, y)$ means that there is someone such that everybody is his or her friend.

A predicate can be partially quantified, e.g. $\forall x \exists y\, P(x, y, z, t)$. The variables quantified (x and y in the example) are called *bound* variables, and the rest (z and t in the example) are called *free* variables. A partially quantified predicate is still a predicate, but depending on fewer variables.

1.2.3. Generalized De Morgan Laws for Logic. If $\exists x\, P(x)$ is false then there is no value of x for which $P(x)$ is true, or in other words, $P(x)$ is always false. Hence

$$\neg \exists x\, P(x) \equiv \forall x\, \neg P(x)\,.$$

On the other hand, if $\forall x\, P(x)$ is false then it is not true that for every x, $P(x)$ holds, hence for some x, $P(x)$ must be false. Thus:

$$\neg \forall x\, P(x) \equiv \exists x\, \neg P(x)\,.$$

This two rules can be applied in successive steps to find the negation of a more complex quantified statement, for instance:

$$\neg \exists x \forall y\, p(x, y) \equiv \forall x \neg \forall y\, P(x, y) \equiv \forall x \exists y\, \neg P(x, y)\,.$$

Exercise: Write formally the statement "for every real number there is a greater real number". Write the negation of that statement.

Answer: The statement is: $\forall x\, \exists y\, (x < y)$ (the universe of discourse is the real numbers). Its negation is: $\exists x\, \forall y\, \neg(x < y)$, i.e., $\exists x\, \forall y\, (x \not< y)$. (Note that among real numbers $x \not< y$ is equivalent to $x \geq y$, but formally they are different predicates.)