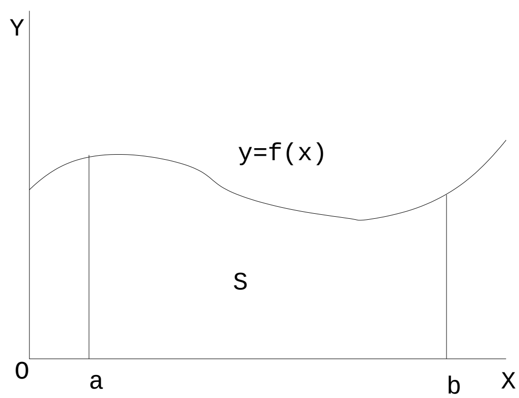
CHAPTER 1

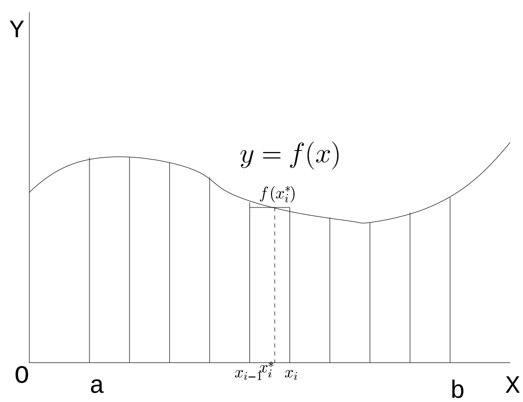
Integrals

1.1. Areas and Distances. The Definite Integral

1.1.1. The Area Problem. The Definite Integral. Here we try to find the area of the region S under the curve $y = f(x)$ from a to b, where f is some continuous function.



In order to estimate that area we begin by dividing the interval $[a, b]$ into n subintervals $[x_0, x_1]$, $[x_1, x_2]$, $[x_2, x_3]$, $\dots$, $[x_{n-1}, x_n]$, each of length $\Delta x = (b - a)/n$ (so $x_i = a + i\Delta x$).



The area S_i of the strip between x_{i-1} and x_i can be approximated as the area of the rectangle of width Δx and height $f(x_i^*)$, where x_i^* is a sample point in the interval $[x_i, x_{i+1}]$. So the total area under the curve is approximately the sum

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x = f(x_1^*)\,\Delta x + f(x_2^*)\,\Delta x + \cdots + f(x_n^*)\,\Delta x\,.$$

This expression is called a *Riemann Sum*.

The estimation is better the thiner the strips are, and we can identify the exact area under the graph of f with the limit:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

As long as f is continuous the value of the limit is independent of the sample points x_i^* used.

That limit is represented $\int_a^b f(x)\,dx$, and is called *definite integral* of f from a to b:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

The symbols at the left historically were intended to mean an infinite sum, represented by a long "S" (the integral symbol $\int$), of infinitely small amounts $f(x)\,dx$. The symbol dx was interpreted as the length of an "infinitesimal" interval, sort of what Δx becomes for infinite n. This interpretation was later abandoned due to the difficulty of reasoning with infinitesimals, but we keep the notation.

Remark: Note that in intervals where $f(x)$ is negative the graph of $y = f(x)$ lies below the x-axis and the definite integral takes a negative value. In general a definite integral gives the *net* area between the graph of $y = f(x)$ and the x-axis, i.e., the sum of the areas of the regions where $y = f(x)$ is above the x-axis minus the sum of the areas of the regions where $y = f(x)$ is below the x-axis.

1.1.2. Evaluating Integrals. We will soon study simple and efficient methods to evaluate integrals, but here we will look at how to evaluate integrals directly from the definition.

Example: Find the value of the definite integral $\int_0^1 x^2\,dx$ from its definition in terms of Riemann sums.

Answer: We divide the interval $[0,1]$ into n equal parts, so $x_i = i/n$ and $\Delta x = 1/n$. Next we must choose some point x_i^* in each subinterval $[x_{i-1}, x_i]$. Here we will use the right endpoint of the interval $x_i^* = i/n$. Hence the Riemann sum associated to this partition is:

$$\sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 1/n = \frac{1}{n^3} \sum_{i=1}^{n} i^2 = \frac{1}{n^3} \frac{2n^3 + 3n^2 + n}{6} = \frac{2 + 3/n + 1/n^2}{6}\,.$$

So:

$$\int_0^1 x^2\,dx = \lim_{n\to\infty} \frac{2 + 3/n + 1/n^2}{6} = \frac{1}{3}\,.$$

In order to check that the result does not depend on the sample points used, let's redo the computation using now the left endpoint of each subinterval:

$$\sum_{i=1}^{n} \left(\frac{i-1}{n}\right)^2 1/n = \frac{1}{n^3} \sum_{i=1}^{n} (i-1)^2 = \frac{1}{n^3} \frac{2n^3 - 3n^2 + n}{6} = \frac{2 - 3/n + 1/n^2}{6}\,.$$

So:

$$\int_0^1 x^2\,dx = \lim_{n\to\infty} \frac{2 - 3/n + 1/n^2}{6} = \frac{1}{3}\,.$$

1.1.3. The Midpoint Rule. The *Midpoint Rule* consists of computing Riemann sums using $\overline{x}_i = (x_{i-1} + x_i)/2 =$ midpoint of each interval as sample point. This yields the following approximation for the value of a definite integral:

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(\overline{x}_i)\,\Delta x = \Delta x\,[f(\overline{x}_1) + f(\overline{x}_2) + \cdots + f(\overline{x}_n)]\,.$$

Example: Use the Midpoint Rule with $n = 5$ to approximate $\int_0^1 x^2\,x$.

Answer: The subintervals are $[0, 0.2]$, $[0.2, 0.4]$, $[0.4, 0.6]$, $[0.6, 0.8]$, $[0.8, 1]$, the midpoints are $0.1, 0.3, 0.5, 0.7, 0.9$, and $\Delta x = 1/5$, so

$$\int_0^1 x^2\,dx \approx \frac{1}{5}\,\left[0.1^2 + 0.3^2 + 0.5^2 + 0.7^2 + 0.9^2\right] = 1.65/5 = 0.33\,,$$

which agrees up to the second decimal place with the actual value $1/3$.

1.1.4. The Distance Problem. Here we show how the concept of definite integral can be applied to more general problems. In particular we study the problem of finding the distance traveled by an object with variable velocity during a certain period of time.

If the velocity v were constant we could just multiply it by the time t: distance $= v \times t$. Otherwise we can approximate the total distance traveled by dividing the total time interval into small intervals so that in each of them the velocity varies very little and can be considered approximately constant. So, assume that the body starts moving at time t_{start} and finishes at time t_{end}, and the velocity is variable, i.e., is a function of time $v = f(t)$. We divide the time interval into n small intervals $[t_{i-1}, t_i]$ of length $\Delta t = (t_{\text{end}} - t_{\text{start}})/n$, choose some instant t_i^* between t_{i-1} and t_i, and take $v = f(t_i^*)$ as the approximate velocity of the body between t_{i-1} and t_i. Then the distance traveled during that time interval is approximately $f(t_i^*)\,\Delta t$, and the total distance can be approximated as the sum

$$\sum_{i=1}^{n} f(t_i^*)\,\Delta t$$

The result will be more accurate the larger the number of subintervals is, and the exact distance traveled will be limit of the above expression as n goes to infinity:

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(t_i^*)\,\Delta t$$

That limit turns out to be the following definite integral:

$$\int_{t_{\text{start}}}^{t_{\text{end}}} f(t)\,dt$$

1.1.5. Properties of the Definite Integral.

(1) Integral of a constant: $\displaystyle\int_a^b c\,dx = c\,(b-a)$.

(2) Linearity:

(a) $\displaystyle\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$.

(b) $\displaystyle\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$.

(3) Interval Additivity

(a) $\displaystyle\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx.$

(b) $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx,$

(c) $\displaystyle\int_a^a f(x)\,dx = 0.$

(4) Comparison:

(a) $\displaystyle f(x) \geq 0 \;\Rightarrow\; \int_a^b f(x)\,dx \geq 0.$

(b) $\displaystyle f(x) \geq g(x) \;\Rightarrow\; \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx.$

(c) $\displaystyle m \leq f(x) \leq M \;\Rightarrow\; m\,(b-a) \leq \int_a^b f(x)\,dx \leq M\,(b-a).$