1.2. The Evaluation Theorem

1.2.1. The Evaluation Theorem. If f is a continuous function and F is an antiderivative of f, i.e., $F'(x) = f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)\,.$$

Example: Find $\int_0^1 x^2\,dx$ using the evaluation theorem.

Answer: An antiderivative of x^2 is $x^3/3$, hence:

$$\int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}\,.$$

1.2.2. Indefinite Integrals. If F is an antiderivative of a function f, i.e., $F'(x) = f(x)$, then for any constant C, $F(x) + C$ is another antiderivative of $f(x)$. The family of all antiderivatives of f is called *indefinite integral* of f and represented:

$$\int f(x)\,dx = F(x) + C\,.$$

Example: $\displaystyle\int x^2\,dx = \frac{x^3}{3} + C.$

1.2.3. Table of Indefinite Integrals. We can make an integral table just by reversing a table of derivatives.

(1) $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \ (n \neq -1).$

(2) $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C.$

(3) $\displaystyle\int e^x\,dx = e^x + C.$

(4) $\displaystyle\int a^x\,dx = \frac{a^x}{\ln a} + C.$

(5) $\displaystyle\int \sin x\,dx = -\cos x + C.$

(6) $\displaystyle\int \cos x\,dx = \sin x + C.$

(7) $\displaystyle\int \sec^2 x\,dx = \tan x + C.$

(8) $\displaystyle\int \csc^2 x\, dx = -\cot x + C.$

(9) $\displaystyle\int \sec x \tan x\, dx = \sec x + C.$

(10) $\displaystyle\int \csc x \cot x\, dx = -\csc x + C.$

(11) $\displaystyle\int \frac{dx}{x^2+1} = \tan^{-1} x + C.$

(12) $\displaystyle\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C.$

(13) $\displaystyle\int \frac{dx}{x\sqrt{x^2-1}}\, dx = \sec^{-1}|x| + C.$

1.2.4. Total Change Theorem. The integral of a rate of change is the total change:

$$\int_a^b F'(x)\, dx = F(b) - F(a)\,.$$

This is just a restatement of the evaluation theorem.

As an example of application we find the *net distance* or *displacement*, and the *total distance* traveled by an object that moves along a straight line with position function $s(t)$. The velocity of the object is $v(t) = s'(t)$. The net distance or displacement is the difference between the final and the initial positions of the object, and can be found with the following integral

$$\int_{t_1}^{t_2} v(t)\, dt = s(t_2) - s(t_1)\,.$$

In the computation of the displacement the distance traveled by the object when it moves to the left (while $v(t) \le 0$) is subtracted from the distance traveled to the right (while $v(t) \ge 0$). If we want to find the total distance traveled we need to add all distances with a positive sign, and this is accomplished by integrating the absolute value of the velocity:

$$\int_{t_1}^{t_2} |v(t)|\, dt = \text{total distance traveled}\,.$$

Example: Find the displacement and the total distance traveled by an object that moves with velocity $v(t) = t^2 - t - 6$ from $t = 1$ to $t = 4$.

Answer: The displacement is

$$\begin{aligned}\int_1^4 (t^2 - t - 6)\,dx &= \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_1^4 \\ &= \left(\frac{4^3}{3} - \frac{4^2}{2} - 6\cdot 4\right) - \left(\frac{1^3}{3} - \frac{1^2}{2} - 6\right) \\ &= -\frac{32}{3} - \left(-\frac{37}{6}\right) = \boxed{-\frac{9}{2}}\end{aligned}$$

In order to find the total distance traveled we need to separate the intervals in which the velocity takes values of different signs. Those intervals are separated by points at which $v(t) = 0$, i.e., $t^2 - t - 6 = 0 \Rightarrow t = -2$ and $t = 3$. Since we are interested only in what happens in $[1, 4]$ we only need to look at the intervals $[1, 3]$ and $[3, 4]$. Since $v(1) = -6$, the velocity is negative in $[1, 3]$, and since $v(4) = 6$, the velocity is positive in $[3, 4]$. Hence:

$$\begin{aligned}\int_1^4 |v(t)|\,dt &= \int_1^3 [-v(t)]\,dt + \int_3^4 v(t)\,dt \\ &= \int_1^3 (t^2 - t - 6)\,dt + \int_3^4 (t^2 - t - 6)\,dt \\ &= \left[-\frac{t^3}{3} + \frac{t^2}{2} + 6t\right]_1^3 + \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_3^4 \\ &= \frac{22}{3} + \frac{17}{6} = \boxed{\frac{61}{6}}.\end{aligned}$$