1.8. Integration using Tables and CAS

The use of tables of integrals and Computer Algebra Systems allow us to find integrals very quickly without having to perform all the steps for their computation. However we often need to modify slightly the original integral and perhaps complete or simplify the answer.

Example: Find the following integral using the tables at the end of Steward's book:

$$\int \frac{\sqrt{x^2-1}}{x}\,dx = \cdots .$$

Answer: In the tables we find the following formula No. 41:

$$\int \frac{\sqrt{u^2-a^2}}{u}\,du = \sqrt{u^2-a^2} - a\cos^{-1}\frac{a}{|u|} + C\,,$$

hence, letting $a = 1$, $u = x$ we get the answer:

$$\int \frac{\sqrt{x^2-1}}{x}\,dx = \boxed{\sqrt{x^2-1} - \cos^{-1}\frac{1}{|x|} + C}\,.$$

Example: Find the integral:

$$\int \frac{x^2}{\sqrt{9+4x^2}}\,dx = \cdots$$

Answer: In the tables the formula that resembles this integral most is No. 26:

$$\int \frac{u^2\,du}{\sqrt{a^2+u^2}} = \frac{u}{2}\sqrt{a^2+u^2} - \frac{a^2}{2}\ln\left(u+\sqrt{a^2+u^2}\right) + C\,,$$

hence letting $a = 3$, $u = 2x$:

$$\begin{aligned}\int \frac{x^2}{\sqrt{9+4x^2}}\,dx &= \frac{1}{8}\int \frac{u^2\,du}{\sqrt{a^2+u^2}} \\ &= \frac{1}{8}\left\{\frac{u}{2}\sqrt{a^2+u^2} - \frac{a^2}{2}\ln\left(u+\sqrt{a^2+u^2}\right)\right\} + C \\ &= \boxed{\frac{x}{8}\sqrt{9+4x^2} - \frac{9}{16}\ln\left(2x+\sqrt{9+4x^2}\right) + C}\,.\end{aligned}$$

Example: Find the same integral using Maple.

Answer: In Maple we enter at the prompt:

```
> int(x^2/sqrt(9+4*x^2),x);
```

and it returns:

$$\frac{x}{8}\sqrt{9+4x^2} - \frac{9}{16}\operatorname{arcsinh}\left(\frac{2}{3}x\right)$$

First we notice that the answer omits the constant C. On the other hand, it involves an inverse hyperbolic function:

$$\operatorname{arcsinh} x = \ln\left(x + \sqrt{1+x^2}\right),$$

hence the answer provided by Maple is:

$$\frac{x}{8}\sqrt{9+4x^2} - \frac{9}{16}\ln\left(\frac{2x}{3} + \sqrt{1+\frac{4x^2}{9}}\right) =$$
$$\frac{x}{8}\sqrt{9+4x^2} - \frac{9}{16}\ln\left(2x + \sqrt{9+4x^2}\right) + \frac{9}{32}\ln(3),$$

so it differs from the answer found using the tables in a constant $\frac{9}{32}\ln(3)$ which can be absorbed into the constant of integration.

1.9. Numerical Integration

Sometimes the integral of a function cannot be expressed with *elementary functions*, i.e., polynomial, trigonometric, exponential, logarithmic, or a suitable combination of these. However, in those cases we still can find an approximate value for the integral of a function on an interval.

1.9.1. Trapezoidal Approximation. A first attempt to approximate the value of an integral $\int_a^b f(x)\,dx$ is to compute its Riemann sum:

$$R = \sum_{i=1}^{n} f(x_i^*)\,\Delta x\,.$$

Where $\Delta x = x_i - x_{i-1} = (b-a)/n$ and x_i^* is some point in the interval $[x_{i-1}, x_i]$. If we choose the left endpoints of each interval, we get the *left-endpoint approximation*:

$$L_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x = (\Delta x)\{f(x_0) + f(x_1) + \cdots + f(x_{n-1})\}\,,$$

Similarly, by choosing the right endpoints of each interval we get the *right-endpoint approximation*:

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x = (\Delta x)\{f(x_1) + f(x_2) + \cdots + f(x_n)\}\,.$$

The *trapezoidal approximation* is the average of L_n and R_n:

$$T_n = \frac{1}{2}(L_n+R_n) = \frac{\Delta x}{2}\{f(x_0)+2f(x_1)+2f(x_2)+\cdots+2f(x_{n-1})+f(x_n)\}\,.$$

Example: Approximate $\int_0^1 x^2\,dx$ with trapezoidal approximation using 4 intervals.

Solution: We have $\Delta x = 1/4 = 0.25$. The values for x_i and $f(x_i) = x_i^2$ can be tabulated in the following way:

i	x_i	$f(x_i)$
0	0	0
1	0.25	0.0625
2	0.5	0.25
3	0.75	0.5625
4	1	1

Hence:

$$L_4 = 0.25 \cdot (0 + 0.0625 + 0.25 + 0.5625) = 0.218750\,,$$

$$R_4 = 0.25 \cdot (0.0625 + 0.25 + 0.5625 + 1) = 0.468750\,.$$

So:

$$T_4 = \frac{1}{2}(L_4 + R_4) = \frac{1}{2}(0.218750 + 0.468750) = 0.34375\,.$$

Compare to the exact value of the integral, which is $1/3 = 0.3333\ldots$.

1.9.2. Midpoint Approximation. Alternatively, in the Riemann sum we can use the middle point $\overline{x}_i = (x_{i-1} + x_i)/2$ of each interval $[x_{i-1}, x_i]$. Then the *midpoint approximation* of $\int_a^b f(x)\,dx$ is

$$M_n = (\Delta x)\{f(\overline{x}_1) + f(\overline{x}_2) + \cdots + f(\overline{x}_n)\}\,.$$

Example: Approximate $\int_0^1 x^2\,dx$ with midpoint approximation using 4 intervals.

Solution: We have:

i	$\overline{x}_i$	$f(\overline{x}_i)$
1	0.125	0.015625
2	0.375	0.140625
3	0.625	0.390625
4	0.875	0.765625

Hence:

$$\begin{aligned} M_4 &= 0.25 \cdot (0.015625 + 0.140625 + 0.390625 + 0.765625) \\ &= 0.328125\,. \end{aligned}$$

1.9.3. Simpson's Approximation. *Simpson's approximation* is a weighted average of the trapezoidal and midpoint approximations associated to the intervals $[x_0, x_2]$, $[x_2, x_4]$, $\ldots$, $[x_{n-2}, x_n]$ (of length

$2\Delta x$ each):

$$\begin{aligned}
S_{2n} &= \frac{1}{3}(2M_n + T_n) \\
&= \frac{1}{3}\Big[2(2\Delta x)\{f(x_1) + f(x_3) + \cdots + f(x_{2n-1})\} \\
&\qquad + \frac{2\Delta x}{2}\{f(x_0) + 2f(x_2) + 2f(x_4) + \cdots + 2f(x_{n-2}) + f(x_n)\}\Big] \\
&= \frac{\Delta x}{3}\{f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots \\
&\qquad\qquad + 2f(x_{2n-2)} + 4f(x_{2n-1}) + f(x_{2n})\}\,.
\end{aligned}$$

Example: Approximate $\int_0^1 x^2\,dx$ with Simpson's approximation using 8 intervals.

Solution: We use the previous results and get:

$$S_8 = \frac{1}{3}(2M_4 + T_4) = \frac{1}{3}(2\cdot 0.328125 + 0.34375) = 1/3\,.$$

Note: in this particular case Simpson's approximation gives the exact value—in general it just gives a good approximation.

1.9.4. Error Bounds. Here we give a way to estimate the error or difference E between the actual value of an integral and the value obtained using a numerical approximation.

1.9.4.1. *Error Bound for the Trapezoidal Approximation.* Suppose $|f''(x)| \leq K$ for $a \leq x \leq b$. Then the error E_T in the trapezoidal approximation verifies:

$$|E_T| \leq \frac{K(b-a)^3}{12n^2}\,.$$

1.9.4.2. *Error Bound for the Midpoint Approximation.* Suppose $|f''(x)| \leq K$ for $a \leq x \leq b$. Then the error E_M in the trapezoidal approximation verifies:

$$|E_M| \leq \frac{K(b-a)^3}{24n^2}\,.$$

1.9.4.3. *Error Bound for the Simpson's Rule.* Suppose $|f^{(4)}(x)| \leq K$ for $a \leq x \leq b$. Then the error E_S in the Simpson's rule verifies:

$$|E_S| \leq \frac{K(b-a)^5}{180n^4}\,.$$

Example: Approximate the value of π using the trapezoidal, midpoint and Simpson's approximations of

$$\int_0^1 \frac{4}{1+x^2}\,dx$$

for $n = 4$. Estimate the error.

Answer: First note that:

$$4\int_0^1 \frac{1}{1+x^2}\,dx = 4\left[\tan^{-1}x\right]_0^1 = 4\frac{\pi}{4} = \pi\,,$$

so by approximating the given integral we are in fact finding approximated values for π.

Now we find the requested approximations:

(1) Trapezoidal approximation:

$$\begin{aligned} T_4 &= \frac{1/4}{2}\{f(0) + 2f(1/4) + 2f(1/2) + 2f(3/4) + f(1)\} \\ &= \boxed{3.131176470}\,. \end{aligned}$$

For estimating the error we need the second derivative of $f(x) = 4/(1+x^2)$, which is $f''(x) = 8(3x^2-1)/(1+x^2)^3$ so we have

$$\begin{aligned} |f''(x)| = \frac{8|3x^2-1|}{|1+x^2|^3} &\le \frac{8(3x^2+1)}{(1+x^2)^3} \\ &\le \frac{8(3\cdot 1^2+1)}{1} = 32 \end{aligned}$$

for $0 \le x \le 1$, hence

$$|E_T| \le \frac{32\cdot(1-0)^3}{12\cdot 4^2} = 0.1666\dots$$

(2) Midpoint approximation:

$$\begin{aligned} M_4 &= \frac{1}{4}\{f(1/8) + f(3/8) + f(5/8) + f(7/8)\} \\ &= \boxed{3.146800518}\,. \end{aligned}$$

The error estimate is:

$$|E_M| \le \frac{32\cdot(1-0)^3}{24\cdot 4^2} = 0.08333\dots$$

(3) Simpson's rule:

$$\begin{aligned} S_4 &= \frac{1/4}{3}\{f(0) + 4f(1/4) + 2f(1/2) + 4f(3/4) + f(1)\} \\ &= \boxed{3.141568627} \end{aligned}$$

For the error estimate we now need the fourth derivative:

$$f^{(4)}(x) = 96(5x^4 - 10x^2 + 1)/(1 + x^2)^5\,,$$

so

$$|f^{(4)}(x)| \le \frac{96(5 + 10 + 1)}{1} = 1536$$

for $0 \le x \le 1$. Hence the error estimate is

$$|E_S| \le \frac{1536 \cdot (1-0)^5}{180 \cdot 4^4} = 0.0333\ldots$$