4.8. Applications of Taylor Polynomials

4.8.1. Applications to Physics. Here we illustrate an application of Taylor polynomials to physics.

Consider the following formula from the Theory of Relativity for the total energy of an object moving at speed v:

$$E = \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}}\,,$$

where c is the speed of light and m_0 is the mass of the object at rest. Let's rewrite the formula in the following way:

$$E = m_0 c^2 \left(1 - \frac{v^2}{c^2}\right)^{-1/2}\,.$$

Now we expand the expression using the power series of the binomial function:

$$(1+x)^\alpha = \sum_{n=0} \binom{\alpha}{n} x^n = 1 + \alpha x + \binom{\alpha}{2} x^2 + \binom{\alpha}{3} x^3 + \cdots\,,$$

which for $\alpha = -1/2$ becomes:

$$\begin{aligned}(1+x)^{-1/2} &= 1 - \frac{1}{2}x + \binom{-\frac{1}{2}}{2} x^2 + \binom{-\frac{1}{2}}{3} x^3 + \cdots \\ &= 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \cdots\,,\end{aligned}$$

hence replacing $x = -v^2/c^2$ we get the desired power series:

$$E = m_0 c^2 \left(1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \frac{5}{16}\frac{v^6}{c^6} + \cdots\right)\,.$$

If we subtract the energy at rest $m_0 c^2$ we get the kinetic energy:

$$K = E - m_0 c^2 = m_0 c^2 \left(\frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \frac{5}{16}\frac{v^6}{c^6} + \cdots\right)\,.$$

For low speed all the terms except the first one are very small and can be ignored:

$$K \approx m_0 c^2 \left(\frac{1}{2}\frac{v^2}{c^2}\right) = \frac{1}{2} m_0 v^2\,.$$

That is the expression for the usual (non relativistic or Newtonian) kinetic energy, so this tells us at low speed the relativistic kinetic energy is approximately equal to the non relativistic one.

4.8.2. Using Series to Solve Differential Equations. Some differential equations cannot be solved explicitly. In such cases an alternative is to represent the solution as a power series and try to determine the values of the coefficients that solve the equation. That yields a power series representation of the solution, which often is enough for getting approximations to it.

We start with an equation that we do know how to solve explicitly, so we can compare the power series obtained with the explicit solution:

$$y' = y\,.$$

This equation can be solved by separation of variables:

$$\begin{aligned} \frac{dy}{y} &= dx \\ \int \frac{dy}{y} &= \int dx \\ \ln y &= x + C \\ y &= Ae^x \qquad (A = e^C)\,. \end{aligned}$$

Next we solve it using power series. We start by representing the solution by a power series:

$$y = c_0 + c_1 x + c_2 x^2 + \cdots = \sum_{n=0}^{\infty} c_n x^n\,.$$

Its derivative is

$$y' = c_1 + 2c_2 x + 3c_3 x^2 \cdots = \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n\,.$$

Now we write the differential equation using the series:

$$c_0 + c_1 x + c_2 x^2 + \cdots = c_1 + 2c_2 x + 3c_3 x^2 + \cdots\,,$$

or

$$\sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n\,.$$

In order to be equal the coefficients must be the same on both sides, so:

$$\begin{cases} c_0 &= c_1 \\ c_1 &= 2c_2 \\ c_2 &= 3c_3 \\ & \cdots \\ c_n &= (n+1)c_{n+1} \\ & \cdots \end{cases}$$

This defines a sequence of coefficients in which the first one c_0 is arbitrary, and the following ones verify the recursive relation

$$c_{n+1} = \frac{c_n}{n+1}\,.$$

So the sequence is:

$$\begin{aligned}
c_0 &= \text{(arbitrary)} \\
c_1 &= c_0 \\
c_2 &= \frac{c_1}{2} = \frac{c_0}{2} \\
c_3 &= \frac{c_2}{3} = \frac{c_0}{2\cdot 3} \\
c_4 &= \frac{c_3}{4} = \frac{c_0}{2\cdot 3\cdot 4} \\
&\dots \\
c_n &= \frac{c_0}{n!} \\
&\dots
\end{aligned}$$

and the solution is

$$y = c_0 + c_0 x + \frac{c_0}{2!}x^2 + \frac{c_0}{3!}x^3 + \cdots = c_0 \sum_{n=0}^{\infty} \frac{x^n}{n!}\,,$$

i.e., c_0 (a constant) multiplied by the Maclaurin series of e^x, so the solution is the same one we got explicitly (with $A = c_0$).

Lets look now at a more sophisticated example. Solve the differential equation

$$y'' - 2xy' + y = 0\,.$$

The idea is the same as before, we replace y with a power series, find its derivatives that appear in the equation, pose the equation with the powers series, and find a relation among the coefficients:

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} c_n x^n \\
y' &= \sum_{n=1}^{\infty} n c_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}
\end{aligned}$$

$$
\begin{aligned}
y'' - 2xy' + y &= \sum_{n=0}^{\infty} c_n x^n - x \sum_{n=1}^{\infty} n c_n x^{n-1} + \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} \\
&= \sum_{n=0}^{\infty} c_n x^n - \sum_{n=1}^{\infty} n c_n x^n + \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}
\end{aligned}
$$

After some reindexing and grouping we get that the equation becomes:

$$
\sum_{n=0}^{\infty} \{(n+2)(n+1)c_{n+2} - (2n-1)c_n\} \, x^n = 0 \,,
$$

which implies:

$$
c_{n+2} = \frac{2n-1}{(n+1)(n+2)} \, c_n \,.
$$

The first two coefficients c_0 and c_1 are arbitrary, and the rest can be computed using that relation:

$$
\begin{aligned}
c_2 &= \frac{-1}{2} c_0 \\
c_3 &= \frac{1}{2 \cdot 3} c_0 \\
c_4 &= \frac{3}{3 \cdot 4} c_2 = -\frac{3}{4!} c_0 \\
c_5 &= \frac{5}{4 \cdot 5} c_3 = \frac{5}{5!} c_0 \\
&\cdots
\end{aligned}
$$

In general the even and odd coefficients are:

$$
\begin{aligned}
c_{2n} &= \frac{(-1) \cdot 3 \cdot 7 \cdot 11 \cdot \cdots \cdot (4n-5)}{(2n)!} \, c_0 \\
c_{2n+1} &= \frac{1 \cdot 5 \cdot 9 \cdot \cdots \cdot (4n-3)}{(2n+1)!} \, c_1 \,,
\end{aligned}
$$

and the solution is

$$
\begin{aligned}
y = c_0 & \left\{ 1 + \sum_{n=1}^{\infty} \frac{(-1) \cdot 3 \cdot 7 \cdot 11 \cdot \cdots \cdot (4n-5)}{(2n)!} \, x^{2n} \right\} \\
& + c_1 \left\{ x + \sum_{n=1}^{\infty} \frac{1 \cdot 5 \cdot 9 \cdot \cdots \cdot (4n-3)}{(2n+1)!} \, x^{2n+1} \right\}
\end{aligned}
$$