4.3. Recurrence Relations

Here we look at recursive definitions under a different point of view. Rather than definitions they will be considered as equations that we must solve. The point is that a recursive definition is actually a definition when there is one and only one object satisfying it, i.e., when the equations involved in that definition have a unique solution. Also, the solution to those equations may provide a *closed-form* (explicit) formula for the object defined.

The recursive step in a recursive definition is also called a *recurrence relation*. We will focus on *kth-order linear recurrence relations*, which are of the form

$$C_0\, x_n + C_1\, x_{n-1} + C_2\, x_{n-2} + \cdots + C_k\, x_{n-k} = b_n\,,$$

where $C_0 \neq 0$. If $b_n = 0$ the recurrence relation is called *homogeneous*. Otherwise it is called *non-homogeneous*.

The basis of the recursive definition is also called *initial conditions* of the recurrence. So, for instance, in the recursive definition of the Fibonacci sequence, the recurrence is

$$F_n = F_{n-1} + F_{n-2}$$

or

$$F_n - F_{n-1} - F_{n-2} = 0\,,$$

and the initial conditions are

$$F_0 = 0,\ \ F_1 = 1\,.$$

One way to solve some recurrence relations is by *iteration*, i.e., by using the recurrence repeatedly until obtaining a explicit close-form formula. For instance consider the following recurrence relation:

$$x_n = r\, x_{n-1} \quad (n > 0)\,; \qquad x_0 = A\,.$$

By using the recurrence repeatedly we get:

$$x_n = r\, x_{n-1} = r^2\, x_{n-2} = r^3\, x_{n-3} = \cdots = r^n\, x_0 = A\, r^n\,,$$

hence the solution is $x_n = A\, r^n$.

In the following we assume that the coefficients $C_0, C_1, \ldots, C_k$ are constant.

4.3.1. First Order Recurrence Relations. The homogeneous case can be written in the following way:

$$x_n = r\, x_{n-1} \quad (n > 0)\,; \qquad x_0 = A\,.$$

Its general solution is

$$x_n = A\, r^n\,,$$

which is a *geometric sequence* with *ratio* r.

The non-homogeneous case can be written in the following way:

$$x_n = r\, x_{n-1} + c_n \quad (n > 0)\,; \qquad x_0 = A\,.$$

Using the summation notation, its solution can be expressed like this:

$$x_n = A\, r^n + \sum_{k=1}^{n} c_k\, r^{n-k}\,.$$

We examine two particular cases. The first one is

$$x_n = r\, x_{n-1} + c \quad (n > 0); \qquad x_0 = A\,.$$

where c is a constant. The solution is

$$x_n = A\, r^n + c \sum_{k=1}^{n} r^{n-k} = A\, r^n + c\, \frac{r^n - 1}{r - 1} \qquad \text{if } r \neq 1\,,$$

and

$$x_n = A + c\, n \qquad \text{if } r = 1\,.$$

Example: Assume that a country with currently 100 million people has a population growth rate (birth rate minus death rate) of 1% per year, and it also receives 100 thousand immigrants per year (which are quickly assimilated and reproduce at the same rate as the native population). Find its population in 10 years from now. (Assume that all the immigrants arrive in a single batch at the end of the year.)

Answer: If we call $x_n =$ population in year n from now, we have:

$$x_n = 1.01\, x_{n-1} + 100,000 \quad (n > 0); \qquad x_0 = 100,000,000\,.$$

This is the equation above with $r = 1.01$, $c = 100,000$ and $A = 100,000,00$, hence:

$$\begin{aligned} x_n &= 100,000,000 \cdot 1.01^n + 100,000\, \frac{1.01^n - 1}{1.01 - 1} \\ &= 100,000,000 \cdot 1.01^n + 1000\, (1.01^n - 1)\,. \end{aligned}$$

So:

$$x_{10} = 110,462,317\,.$$

The second particular case is for $r = 1$ and $c_n = c + d\,n$, where c and d are constant (so c_n is an arithmetic sequence):

$$x_n = x_{n-1} + c + d\,n \quad (n > 0)\,; \qquad x_0 = A\,.$$

The solution is now

$$x_n = A + \sum_{k=1}^{n} (c + d\,k) = A + c\,n + \frac{d\,n\,(n+1)}{2}\,.$$

4.3.2. Second Order Recurrence Relations. Now we look at the recurrence relation

$$C_0\,x_n + C_1\,x_{n-1} + C_2\,x_{n-2} = 0\,.$$

First we will look for solutions of the form $x_n = c\,r^n$. By plugging in the equation we get:

$$C_0\,c\,r^n + C_1\,c\,r^{n-1} + C_2\,c\,r^{n-2} = 0\,,$$

hence r must be a solution of the following equation, called the *characteristic equation* of the recurrence:

$$C_0\,r^2 + C_1\,r + C_2 = 0\,.$$

Let r_1, r_2 be the two (in general complex) roots of the above equation. They are called *characteristic roots*. We distinguish three cases:

1. *Distinct Real Roots.* In this case the general solution of the recurrence relation is

$$x_n = c_1\,r_1^n + c_2\,r_2^n\,,$$

   where c_1, c_2 are arbitrary constants.

2. *Double Real Root.* If $r_1 = r_2 = r$, the general solution of the recurrence relation is

$$x_n = c_1\,r^n + c_2\,n\,r^n\,,$$

   where c_1, c_2 are arbitrary constants.

3. *Complex Roots.* In this case the solution could be expressed in the same way as in the case of distinct real roots, but in

order to avoid the use of complex numbers we write $r_1 = r\, e^{\alpha i}$, $r_2 = r\, e^{-\alpha i}$, $k_1 = c_1 + c_2$, $k_2 = (c_1 - c_2)\, i$, which yields:[1]

$$x_n = k_1\, r^n\, \cos n\alpha + k_2\, r^n\, \sin n\alpha\,.$$

Example: Find a closed-form formula for the Fibonacci sequence defined by:

$$F_{n+1} = F_n + F_{n-1} \quad (n > 0)\,; \qquad F_0 = 0,\ F_1 = 1\,.$$

Answer: The recurrence relation can be written

$$F_n - F_{n-1} - F_{n-2} = 0\,.$$

The characteristic equation is

$$r^2 - r - 1 = 0\,.$$

Its roots are:[2]

$$r_1 = \phi = \frac{1+\sqrt{5}}{2}; \qquad r_2 = -\phi^{-1} = \frac{1-\sqrt{5}}{2}\,.$$

They are distinct real roots, so the general solution for the recurrence is:

$$F_n = c_1\, \phi^n + c_2\, (-\phi^{-1})^n\,.$$

Using the initial conditions we get the value of the constants:

$$\left\{ \begin{array}{lll} (n=0) & c_1 + c_2 & = \ 0 \\ (n=1) & c_1\, \phi + c_2\, (-\phi^{-1}) & = \ 1 \end{array} \right. \Rightarrow \left\{ \begin{array}{lcr} c_1 & = & 1/\sqrt{5} \\ c_2 & = & -1/\sqrt{5} \end{array} \right.$$

Hence:

$$F_n = \frac{1}{\sqrt{5}} \left\{ \phi^n - (-\phi)^{-n} \right\}\,.$$

[1]Remainder: $e^{\alpha i} = \cos\alpha + i \sin\alpha$.

[2]$\phi = \frac{1+\sqrt{5}}{2}$ is the *Golden Ratio.*