2.2. Volumes

2.2.1. Volumes by Slices. First we study how to find the volume of some solids by the method of cross sections (or "slices"). The idea is to divide the solid into slices perpendicular to a given reference line. The volume of the solid is the sum of the volumes of its slices.

2.2.2. Volume of Cylinders. A cylinder is a solid whose cross sections are parallel translations of one another. The volume of a cylinder is the product of its height and the area of its base:

$$V = Ah\,.$$

2.2.3. Volume by Cross Sections. Let R be a solid lying alongside some interval $[a,b]$ of the x-axis. For each x in $[a,b]$ we denote $A(x)$ the area of the cross section of the solid by a plane perpendicular to the x-axis at x. We divide the interval into n subintervals $[x_{i-1},x_i]$, of length $\Delta x = (b-a)/n$ each. The planes that are perpendicular to the x-axis at the points $x_0, x_1, x_2, \dots, x_n$ divide the solid into n slices. If the cross section of R changes little along a subinterval $[x_{i-1},x_i]$, the slab positioned alongside that subinterval can be considered a cylinder of height Δx and whose base equals the cross section $A(x_i^*)$ at some point x_i^* in $[x_{i-1},x_i]$. So the volume of the slice is

$$\Delta V_i \approx A(x_i^*)\,\Delta x\,.$$

The total volume of the solid is

$$V = \sum_{i=1}^{n} \Delta V_i \approx \sum_{i=1}^{n} A(x_i^*)\,\Delta x\,.$$

Once again we recognize a Riemann sum at the right. In the limit as $n \to \infty$ we get the so called *Cavalieri's principle*:

$$V = \int_a^b A(x)\,dx\,.$$

Of course, the formula can be applied to any axis. For instance if a solid lies alongside some interval $[a,b]$ on the y axis, the formula becomes

$$V = \int_a^b A(y)\,dy\,.$$

Example: Find the volume of a cone of radius r and height h.

Answer: Assume that the cone is placed with its vertex in the origin of coordinates and its axis on the x-axis. The x coordinate runs through the interval $[0, h]$. The cross section of the cone at each point x is a circular disk of radius xr/h, hence its area is $A(x) = \pi(xr/h)^2 = \pi r^2 x^2/h^2$. The volume of the cone can now be computed by Cavalieri's formula:

$$V = \int_0^h \frac{\pi r^2}{h^2}\, x^2\, dx = \frac{\pi r^2}{h^2} \left[\frac{x^3}{3}\right]_0^h = \frac{\pi r^2}{h^2}\, \frac{h^3}{3} = \frac{1}{3}\pi r^2 h\,.$$

2.2.4. Solids of Revolution. Consider the plane region between the graph of the function $y = f(x)$ and the x-axis along the interval $[a, b]$. By revolving that region around the x-axis we get a *solid of revolution*. Now each cross section is a circular disk of radius y, so its area is $A(x) = \pi y^2 = \pi[f(x)]^2$. Hence, the volume of the solid is

$$V = \int_a^b \pi y^2\, dx = \int_a^b \pi [f(x)]^2\, dx\,.$$

Example: Find the volume of a cone of radius r and height h.

Answer: Assume that the cone is placed with its vertex in the origin of coordinates and its axis on the x-axis. This cone can be obtained by revolving the area under the line $y = rx/h$ between $x = 0$ and $x = h$ around the x-axis. So its volume is

$$V = \int_0^h \pi \left(\frac{rx}{h}\right)^2 dx = \int_0^h \frac{\pi r^2}{h^2}\, x^2\, dx = \frac{\pi r^2}{h^2} \left[\frac{x^3}{3}\right]_0^h = \frac{\pi r^2}{h^2}\, \frac{h^3}{3} = \frac{1}{3}\pi r^2 h\,.$$

If the revolution is performed around the y-axis, the roles of x and y are interchanged, so in that case the formula is

$$V = \int_a^b \pi x^2\, dy\,,$$

where x must be written as a function of y.

If the region being revolved is the area between two curves $y = f(x)$ and $y = g(x)$, then each cross section is an annular ring (or washer)with outer radius $f(x)$ and inner radius $g(x)$ (assuming $f(x) \geq g(x) \geq 0$.) The area of the annular ring is $A(x) = \pi(f(x)^2 - g(x)^2)$, hence the volume of the solid will be:

$$V = \int_a^b \pi\, \left[(y_T)^2 - (y_B)^2\right]\, dx = \int_a^b \pi\, \left[f(x)^2 - g(x)^2\right]\, dx\,.$$

If the revolution is performed around the y-axis, then:

$$V = \int_a^b \pi \, \left[(x_R)^2 - (x_L)^2\right] \, dy \, .$$

Example: Find the volume of the solid obtained by revolving the area between $y = x^2$ and $y = \sqrt{x}$ around the x-axis.

Solution: First we need to find the intersection points of these curves in order to find the interval of integration:

$$\begin{cases} y = x^2 \\ y = \sqrt{x} \end{cases} \quad \Rightarrow \quad (x, y) = (0, 0) \quad \text{and} \quad (x, y) = (1, 1) \, ,$$

hence we must integrate from $x = 0$ to $x = 1$:

$$\begin{aligned} V &= \pi \int_0^1 \left[(\sqrt{x})^2 - (x^2)^2\right] \, dx = \pi \int_0^1 \left(x - x^4\right) \, dx \\ &= \pi \left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1 = \pi \left(\frac{1}{2} - \frac{1}{5}\right) = \frac{3\pi}{10} \, . \end{aligned}$$

2.2.5. Volumes by Shells. Next we study how to find the volume of some solids by the method of shells. Now the idea is to divide the solid into shells and add up their volumes.

2.2.6. Volume of a Cylindrical Shell. A cylindrical shell is the region between two concentric circular cylinders of the same height h. If their radii are r_1 and r_2 respectively, then the volume is:

$$V = \pi r_2^2 h - \pi r_1^2 h = \pi h (r_2^2 - r_1^2) = \pi h \overbrace{(r_2 + r_1)}^{2\bar{r}} \overbrace{(r_2 - r_1)}^{t} = 2\pi \bar{r} t h \, ,$$

where $\bar{r} = (r_2 + r_1)/2$ is the average radius, and $t = r_2 - r_1$ is the thickness of the shell.

2.2.7. Volumes by Cylindrical Shells. Consider the solid generated by revolving around the y-axis the region under the graph of $y = f(x)$ between $x = a$ and $x = b$. We divide the interval $[a, b]$ into n subintervals $[x_{i-1}, x_i]$ of length $\Delta x = (b - a)/n$ each. The volume V of the solid is the sum of the volumes ΔV_i of the shells determined by the partition. Each shell, obtained by revolving the region under $y = f(x)$ over the subinterval $[x_{i-1}, x_i]$, is approximately cylindrical. Its height is $f(\overline{x}_i)$, where $\overline{x}_i$ is the midpoint of $[x_{i-1}, x_i]$. Its thickness is Δx. Its average radius is $\overline{x}_i$. Hence its volume is

$$\Delta V_i \approx 2\pi \overline{x}_i f(\overline{x}_i) \Delta x \,,$$

and the volume of the solid is

$$V = \sum_{i=1}^{n} \Delta V_i \approx \sum_{i=1}^{n} 2\pi \overline{x}_i f(\overline{x}_i) \Delta x \,.$$

As $n \to \infty$ the right Riemann sum converges to the following integral:

$$V = \int_a^b 2\pi x f(x)\, dx = \int_a^b 2\pi x y \, dx \,.$$

Example: Find the volume of the solid obtained by revolving around the y-axis the plane area between the graph of $y = 1 - x^2$ and the x-axis.

Answer: The graph intersects the positive x-axis at $x = 1$, so the interval is $[0, 1]$. Hence

$$V = \int_0^1 2\pi x y \, dx = \int_0^1 2\pi x \cdot (1 - x^2)\, dx = 2\pi \int_0^1 (x - x^3)\, dx$$
$$= 2\pi \left[\frac{x^2}{2} - \frac{x^4}{4} \right]_0^1 = 2\pi \left(\frac{1}{2} - \frac{1}{4} \right) = \frac{\pi}{2} \,.$$

2.2.8. Revolving the Region Between Two Curves. Here we find the volume of the solid obtained by revolving around the y-axis the area between two curves $y = f(x)$ and $y = g(x)$ over an interval $[a, b]$. The computation is similar, but if $f(x) \geq g(x)$ the shells will have height $f(x_i^*) - g(x_i^*)$, so the volume will be given by the integral:

$$V = \int_a^b 2\pi x (f(x) - g(x))\, dx = \int_a^b 2\pi x (y_T - y_B)\, dx \,.$$

Example: Find the volume of the solid obtained by revolving the plane region limited by the curves $y = x$ and $y = x^2$ over the interval $[0, 1]$.

Answer: In $[0, 1]$ we have $x \geq x^2$, so:

$$V = \int_0^1 2\pi x \, (y_T - y_B)\, dx = 2\pi \int_0^1 x \, (x - x^2)\, dx$$
$$= 2\pi \int_0^1 (x^2 - x^3)\, dx = 2\pi \left[\frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 = 2\pi \left(\frac{1}{3} - \frac{1}{4} \right) = 2\pi \frac{1}{12} = \frac{\pi}{6} \,.$$

If the region is revolved around the x-axis then the variables x and y reverse their roles:

$$V = \int_a^b 2\pi y\,(x_R - x_L)\,dy\,.$$

2.2.9. Revolving Around an Arbitrary Line. If the plane region is revolved around a vertical line $y = c$, the radius of the shell will be $x - c$ (or $c - x$, whichever is positive) instead of x, so the formula becomes:

$$V = \int_a^b 2\pi(x-c)(f(x)-g(x))\,dx = \int_a^b 2\pi(x-c)(y_T - y_B)\,dx\,.$$

Similarly, if the region is revolved around the horizontal line $x = c$, the formula becomes:

$$V = \int_a^b 2\pi(y-c)(f(y)-g(y))\,dy = \int_a^b 2\pi(y-c)(x_R - x_L)\,dy\,,$$

where $y - c$ must be replaced by $c - y$ if $c > y$.