POLYNOMIALS

MIGUEL A. LERMA

(Last updated: February 16, 2005)

Main Results on Polynomials

1. **The Factor Theorem.** If a is a zero of a polynomial $P(x)$, then $x - a$ must be a factor; i.e., $P(x)$ is a product of $x - a$ and another polynomial.

2. **The Fundamental Theorem of Algebra.** Every polynomial with complex coefficients has at least one complex zero.

3. **Rational Roots Theorem.** If a polynomial $P(x)$ with integral coefficients has a rational zero $x = a/b$, where a and b are in lowest terms, then the leading coefficient of $P(x)$ is a multiple of b, and the constant term of $P(x)$ is a multiple of a.

A consequence of this theorem is the following: Any rational zero of a monic polynomial must be an integer. (A monic polynomial is a polynomial with integral coefficients whose leading coefficient is 1)

4. **Relationship between Zeros and Coefficients.** Let $r_1, r_2, \dots, r_n$ the zeros of the polynomial $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$. Then for $k = 1, 2, \dots, n$,

$$\begin{aligned}\frac{a_k}{a_n} &= (-1)^{n-k}(\text{sum of all products of } n-k \text{ different zeros}) \\ &= (-1)^{n-k} \sum_{1 \le i_1 < i_2 < \cdots < i_{n-k} \le n} r_{i_1} r_{i_2} \dots r_{i_{n-k}} .\end{aligned}$$

The function

$$f_k(x_1, \dots, x_n) = \sum_{1 \le i_1 < i_2 < \cdots < i_{n-k} \le n} x_{i_1} x_{i_2} \dots x_{i_{n-k}}$$

is called the kth elementary symmetric function in $x_1, \dots, x_n$. So with this notation the result can be expressed:

$$\frac{a_k}{a_n} = (-1)^{n-k} f_k(r_1, \dots, r_n)\,.$$