4.7. Taylor and MacLaurin Series

4.7.1. Polynomial Approximations. Assume that we have a function f for which we can easily compute its value $f(a)$ at some point a, but we do not know how to find $f(x)$ at other points x close to a. For instance, we know that $\sin 0 = 0$, but what is $\sin 0.1$? One way to deal with the problem is to find an *approximate* value of $f(x)$. If we look at the graph of $f(x)$ and its tangent line at $(a, f(a))$, we see that the points of the tangent line are close to the graph, so the y-coordinates of those points are possible approximations for $f(x)$.

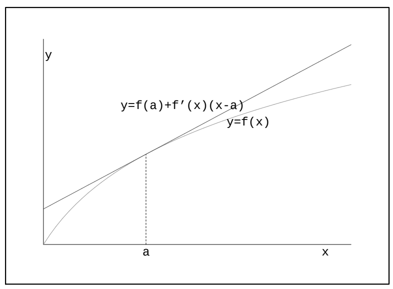

FIGURE 4.7.1. Linear approximation of $f(x)$.

The equation of the tangent line to $y = f(x)$ at $x = a$ is

$$y = f(a) + f'(a)(x - a)\,,$$

hence

$$f(x) \approx f(a) + f'(a)(x - a)\,,$$

for x close to a. For instance:

$$\sin(x) \approx \sin a + \cos a\,(x - a)\,.$$

For $a = 0$ we get:

$$\sin(x) \approx \sin 0 + \cos 0 \cdot (x - 0) = x\,,$$

so $\sin(0.1) \approx 0.1$. In fact $\sin(0.1) = 0.099833416\dots$, which is close to 0.1.

The tangent line is the graph of the first degree polynomial

$$T_1(x) = f(a) + f'(a)(x - a)\,.$$

This polynomial agrees with the value and the first derivative of $f(x)$ at $x = a$:

$$T_1(a) = f(a)$$
$$T_1'(a) = f'(a)$$

We can extend the idea to higher degree polynomials in the hope of obtaining closer approximations to the function. For instance, we may try a second degree polynomial of the from:

$$T_2(x) = c_0 + c_1(x-a) + c_2(x-a)^2\,,$$

with the following conditions:

$$\begin{aligned} T_2(a) &= f(a) \\ T_2'(a) &= f'(a) \\ T_2''(a) &= f''(a) \end{aligned}$$

i.e.:

$$\begin{cases} c_0 = f(a) \\ c_1 = f'(a) \\ 2c_2 = f''(a) \end{cases}$$

After solving the system of equations obtained we get:

$$\begin{aligned} c_0 &= f(a) \\ c_1 &= f'(a) \\ c_2 &= \frac{f''(a)}{2} \end{aligned}$$

hence:

$$T_2(x) = f(a) + f'(a)x + \frac{f''(a)}{2}x^2\,.$$

In general the nth polynomial approximation of $f(x)$ at $x = a$ is an nth degree polynomial

$$T_n(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n$$

verifying

$$\begin{aligned} T_n(a) &= f(a) \\ T_n'(a) &= f'(a) \\ T_n''(a) &= f''(a) \\ &\cdots \\ T_n^{(n)}(a) &= f^{(n)}(a) \end{aligned}$$

From here we get a system of $n+1$ equations with the following solution:

$$\begin{aligned} c_0 &= f(a) \\ c_1 &= f'(a) \\ c_2 &= \frac{f''(a)}{2!} \\ &\dots \\ c_n &= \frac{f^{(n)}(a)}{n!} \end{aligned}$$

hence:

$$\begin{aligned} T_n(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k . \end{aligned}$$

That polynomial is the so called *nth-degree Taylor polynomial of $f(x)$ at $x = a$.*

Example: The third-degree Taylor polynomial of $f(x) = \sin x$ at $x = a$ is

$$T_3(x) = \sin a + \cos a \cdot (x-a)^2 - \frac{\sin a}{2}(x-a)^2 - \frac{\cos a}{3!}(x-a)^3 .$$

For $a = 0$ we have $\sin 0 = 0$ and $\cos 0 = 1$, hence:

$$T_3(x) = x - \frac{x^3}{6} .$$

So in particular

$$\sin 0.1 \approx 0.1 - \frac{0.1^3}{6} = 0.09983333\dots .$$

The actual value of $\sin 0.1$ is

$$\sin 0.1 = 0.099833416 ,$$

which agrees with the value obtained from the Taylor polynomial up to the sixth decimal place.

4.7.2. Taylor's Inequality. The difference between the value of a function and its Taylor approximation is called *remainder*:

$$R_n(x) = f(x) - T_n(x) .$$

The Taylor's inequality states the following: If $|f^{(n+1)}(x)| \leq M$ for $|x-a| \leq d$ then the reminder satisfies the inequality:

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \qquad \text{for } |x-a| \leq d\,.$$

Example: Find the third degree Taylor approximation for $\sin x$ at $x=0$, use it to find an approximate value for $\sin 0.1$ and estimate its difference from the actual value of the function.

Answer: We already found

$$T_3(x) = x - \frac{x^3}{6}\,,$$

and

$$T_3(0.1) = 0.099833333\ldots$$

Now we have $f^{(4)}(x) = \sin x$ and $|\sin x| \leq 1$, hence

$$|R_3(0.1)| =\leq \frac{1}{4!}0.1^4 = 0.0000041666\cdots < 0.0000042 = 4.2 \cdot 10^{-6}\,.$$

In fact the estimation is correct, the approximate value differs from the actual value in

$$|T_3(0.1) - \sin 0.1| = 0.000000083313\cdots < 8.34 \cdot 10^{-8}\,.$$

4.7.3. Taylor Series. If the given function has derivatives of all orders and $R_n(x) \to 0$ as $n \to \infty$, then we can write

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \\ &\quad \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots \end{aligned}$$

The infinite series to the right is called *Taylor series* of $f(x)$ at $x=a$. If $a=0$ then the Taylor series is called *Maclaurin series*.

Example: The Taylor series of $f(x) = e^x$ at $x=0$ is:

$$1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

For $|x| < d$ the remainder can be estimated taking into account that $f^{(n)}(x) = e^x$ and $|e^x| < e^d$, hence

$$|R_n(x)| < \frac{e^d}{(n+1)!}|x|^{n+1}.$$

We know that $\lim_{n\to\infty} x^n/n! = 0$, so

$$\lim_{n\to\infty} \frac{e^d}{(n+1)!}|x|^{n+1} = 0$$

hence $R_n(x) \to 0$ as $n \to \infty$. Consequently we can write:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

For $x = 1$ this formula provides a way of computing number e:

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots = 2.718281828459\ldots$$

The following are Maclaurin series of some common functions:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\ln(1+x) = -\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$(1+x)^\alpha = \sum_{n=0} \binom{\alpha}{n} x^n = 1 + \alpha x + \binom{\alpha}{2} x^2 + \binom{\alpha}{3} x^3 + \cdots$$

where $\displaystyle\binom{\alpha}{n} = \frac{\alpha(\alpha-1)(\alpha-2)\ldots(\alpha-n+1)}{n!}$.

$$\frac{1}{1+x} = (1+x)^{-1} = \sum_{n=0} (-1)^n x^n = 1 - x + x^2 - x^3 + \cdots$$

$$\tan^{-1} x = \sum_{n=0} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

Remark: By letting $x = 1$ in the Taylor series for $\tan^{-1} x$ we get the beautiful expression:

$$\boxed{\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}}.$$

Unfortunately that series converges too slowly for being of practical use in computing π. Since the series for $\tan^{-1} x$ converges more quickly for small values of x, it is more convenient to express π as a combination of inverse tangents with small argument like the following one:

$$\boxed{\frac{\pi}{4} = 4 \tan^{-1} \frac{1}{5} - \tan^{-1} \frac{1}{239}}.$$

That identity can be checked with plain trigonometry. Then the inverse tangents can be computed using the Maclaurin series for $\tan^{-1} x$, and from them an approximate value for π can be found.

4.7.4. Finding Limits with Taylor Series. The following example shows an application of Taylor series to the computation of limits:

Example: Find $\displaystyle\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$.

Answer: Replacing e^x with its Taylor series:

$$\begin{aligned}
\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} &= \lim_{x\to 0} \frac{(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots) - 1 - x}{x^2} \\
&= \lim_{x\to 0} \frac{\frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots}{x^2} \\
&= \lim_{x\to 0} \left\{ \frac{1}{2} + \frac{x}{6} + \frac{x^2}{24} + \dots \right\} = \boxed{\frac{1}{2}}.
\end{aligned}$$

4.8. Applications of Taylor Polynomials

4.8.1. Applications to Physics. Here we illustrate an application of Taylor polynomials to physics.

Consider the following formula from the Theory of Relativity for the total energy of an object moving at speed v:

$$E = \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}},$$

where c is the speed of light and m_0 is the mass of the object at rest. Let's rewrite the formula in the following way:

$$E = m_0 c^2 \left(1 - \frac{v^2}{c^2}\right)^{-1/2}.$$

Now we expand the expression using the power series of the binomial function:

$$(1+x)^\alpha = \sum_{n=0} \binom{\alpha}{n} x^n = 1 + \alpha x + \binom{\alpha}{2} x^2 + \binom{\alpha}{3} x^3 + \cdots,$$

which for $\alpha = -1/2$ becomes:

$$\begin{aligned}(1+x)^{-1/2} &= 1 - \frac{1}{2}x + \binom{-\frac{1}{2}}{2} x^2 + \binom{-\frac{1}{2}}{3} x^3 + \cdots \\ &= 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \cdots,\end{aligned}$$

hence replacing $x = -v^2/c^2$ we get the desired power series:

$$E = m_0 c^2 \left(1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \frac{5}{16}\frac{v^6}{c^6} + \cdots\right).$$

If we subtract the energy at rest $m_0 c^2$ we get the kinetic energy:

$$K = E - m_0 c^2 = m_0 c^2 \left(\frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \frac{5}{16}\frac{v^6}{c^6} + \cdots\right).$$

For low speed all the terms except the first one are very small and can be ignored:

$$K \approx m_0 c^2 \left(\frac{1}{2}\frac{v^2}{c^2}\right) = \frac{1}{2} m_0 v^2 .$$

That is the expression for the usual (non relativistic or Newtonian) kinetic energy, so this tells us at low speed the relativistic kinetic energy is approximately equal to the non relativistic one.