MATH 214-2 - Fall 2001 - Second Midterm (solutions)

SOLUTIONS

1. Differentiate the following functions

1. $f(x) = \ln(\sin x^2)$.
2. $f(x) = e^{-\cos^2 x}$.
3. $f(x) = \ln\left(\dfrac{(\sin x)^{1/5}\, e^{-x^2}}{\sqrt[3]{1+x^2}}\right)$ (simplify first).

Solution:

1. $f'(x) = \boxed{2x \cot x^2}$
2. $f'(x) = \boxed{2\sin x \cos x\, e^{-\cos^2 x}}$ or $f'(x) = \boxed{\sin(2x)\, e^{-\cos^2 x}}$
3. $f(x) = \dfrac{1}{5}\ln(\sin x) - x^2 - \dfrac{1}{3}\ln(1+x^2) \quad \Rightarrow$

   $f'(x) = \boxed{\dfrac{1}{5}\cot x - 2x - \dfrac{2x}{3(1+x^2)}}$

2. Use logarithmic differentiation to find the derivatives of the following functions:

1. $y = x^{1/x}$.
2. $y = (\sin x)^x$.

Solution:

1. $\ln y = \dfrac{\ln x}{x}$

   $$\frac{y'}{y} = \frac{\frac{1}{x}x - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$$

   $$y' = y\,\frac{1 - \ln x}{x^2} = \boxed{x^{1/x}\frac{1 - \ln x}{x^2}}$$

2. $\ln y = x \ln(\sin x)$

   $$\frac{y'}{y} = \ln(\sin x) + x \cot x$$

   $$y' = y\left\{\ln(\sin x) + x \cot x\right\} = \boxed{(\sin x)^x \left\{\ln(\sin x) + x \cot x\right\}}$$

3. Find the following indefinite integrals

1. $\displaystyle\int \frac{1+(\ln x)^2}{x}\,dx.$

2. $\displaystyle\int x^6\, e^{1+x^7}\,dx.$

Solution:

1. $u = \ln x,\ du = dx/x;$

$$\int \frac{1+(\ln x)^2}{x}\,dx = \int (1+u^2)\,du = u + \frac{u^3}{3} + C = \boxed{\ln x + \frac{(\ln x)^3}{3} + C}$$

2. $u = 1 + x^7, du = 7x^6\,dx;$

$$\int x^6\, e^{1+x^7}\,dx = \frac{1}{7}\int e^u\,du = \tfrac{1}{7}e^u + C = \boxed{\tfrac{1}{7}e^{1+x^7} + C}$$

4. Solve the following initial value problem:

$$\begin{cases} \dfrac{dx}{dt} = 2x + 1 \\ x(0) = 0 \end{cases}$$

Solution:

First we separate variables:

$$\frac{dx}{2x+1} = dt \,.$$

Next we integrate:

$$\int \frac{dx}{2x+1} = \int dt + C \,,$$

$$\frac{1}{2} \ln{(2x+1)} = t + C \,,$$

hence:

$$x(t) = \frac{1}{2}(-1 + e^{2t+2C}) = \frac{1}{2}(-1 + Ae^{2t}) \,,$$

where $A = e^{2C}$. Now we determine the value of the constant A by using the initial value for $t = 0$:

$$x(0) = \frac{1}{2}(-1 + Ae^{0}) \quad \Rightarrow \quad 0 = \frac{1}{2}(-1 + A) \quad \Rightarrow \quad A = 1 \,.$$

Hence:

$$\boxed{x(t) = \frac{1}{2}(-1 + e^{2t})}$$

5. Find the following integrals using appropriate inverse trigonometric functions:

1. $\displaystyle\int \frac{dx}{\sqrt{36-x^2}}$.
2. $\displaystyle\int \frac{dx}{\sqrt{e^{2x}-1}}$.

Solution:

1. $u = x/6$, $du = dx/6$;

$$\begin{aligned}\int \frac{dx}{\sqrt{36-x^2}} &= \int \frac{6\,du}{6\sqrt{1-u^2}} \\ &= \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C = \boxed{\sin^{-1}\left(\frac{x}{6}\right) + C}\end{aligned}$$

2. $u = e^x$, $du = e^x\,dx$, $dx = du/e^x = du/u$;

$$\int \frac{dx}{\sqrt{e^{2x}-1}} = \int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1}|u| + C = \boxed{\sec^{-1}(e^x) + C}$$

6. Find the following limit using l'Hôpital:

$$L = \lim_{x \to 0} \frac{-2 + \sqrt{1+x} + \sqrt{1-x}}{x^2}$$

Solution:

$$L = \lim_{x \to 0} \frac{-2 + (1+x)^{1/2} + (1-x)^{1/2}}{x^2}$$

$$= \lim_{x \to 0} \frac{\frac{1}{2}(1+x)^{-1/2} - \frac{1}{2}(1-x)^{-1/2}}{2x}$$

$$= \lim_{x \to 0} \frac{-\frac{1}{4}(1+x)^{-3/2} - \frac{1}{4}(1-x)^{-3/2}}{2} = \boxed{-\frac{1}{4}}$$