1.6. Trigonometric Integrals and Trigonometric Substitutions

1.6.1. Trigonometric Integrals.

Here we discuss integrals of powers of trigonometric functions. To that end the following *half-angle identities* will be useful:

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)\,,$$
$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)\,.$$

Remember also the identities:

$$\sin^2 x + \cos^2 x = 1\,,$$
$$\sec^2 x = 1 + \tan^2 x\,.$$

1.6.1.1. *Integrals of Products of Sines and Cosines.* We will study now integrals of the form

$$\int \sin^m x \cos^n x \, dx\,,$$

including cases in which $m = 0$ or $n = 0$, i.e.:

$$\int \cos^n x \, dx\,; \qquad \int \sin^m x \, dx\,.$$

The simplest case is when either $n = 1$ or $m = 1$, in which case the substitution $u = \sin x$ or $u = \cos x$ respectively will work.

Example: $\displaystyle\int \sin^4 x \cos x \, dx = \cdots$

($u = \sin x$, $du = \cos x \, dx$)

$$\cdots = \int u^4 \, du = \frac{u^5}{5} + C = \boxed{\frac{\sin^5 x}{5} + C}\,.$$

More generally if at least one exponent is odd then we can use the identity $\sin^2 x + \cos^2 x = 1$ to transform the integrand into an expression containing only one sine or one cosine.

Example:

$$\begin{aligned}\int \sin^2 x \cos^3 x \, dx &= \int \sin^2 x \cos^2 x \cos x \, dx \\ &= \int \sin^2 x \,(1 - \sin^2 x) \cos x \, dx = \cdots\end{aligned}$$

$(u = \sin x,\ du = \cos x \, dx)$

$$\begin{aligned}\cdots = \int u^2 \,(1 - u^2) \, du &= \int (u^2 - u^4) \, du \\ &= \frac{u^3}{3} - \frac{u^5}{5} + C \\ &= \boxed{\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C}.\end{aligned}$$

If all the exponents are even then we use the half-angle identities.

Example:

$$\begin{aligned}\int \sin^2 x \cos^2 x \, dx &= \int \tfrac{1}{2}(1 - \cos 2x)\tfrac{1}{2}(1 + \cos 2x) \, dx \\ &= \frac{1}{4} \int (1 - \cos^2 2x) \, dx \\ &= \frac{1}{4} \int (1 - \tfrac{1}{2}(1 + \cos 4x)) \, dx \\ &= \frac{1}{8} \int (1 - \cos 4x) \, dx \\ &= \frac{x}{8} - \frac{\sin 4x}{32} + C \,.\end{aligned}$$

1.6.1.2. *Integrals of Secants and Tangents.* The integral of $\tan x$ can be computed in the following way:

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = -\int \frac{du}{u} = \boxed{-\ln |u| + C = -\ln |\cos x| + C},$$

where $u = \cos x$. Analogously

$$\int \cot x \, dx = \int \frac{\cos x}{\sin x} \, dx = \int \frac{du}{u} = \ln |u| + C = \boxed{\ln |\sin x| + C},$$

where $u = \sin x$.

The integral of $\sec x$ is a little tricky:

$$\int \sec x\,dx = \int \frac{\sec x\,(\tan x + \sec x)}{\sec x + \tan x}\,dx = \int \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}\,dx =$$
$$\int \frac{du}{u} = \ln|u| + C = \boxed{\ln|\sec x + \tan x| + C},$$

where $u = \sec x + \tan x$, $du = (\sec x \tan x + \sec^2 x)\,dx$.

Analogously:

$$\int \csc x\,dx = \boxed{-\ln|\csc x + \cot x| + C}.$$

More generally an integral of the form

$$\int \tan^m x \sec^n x\,dx$$

can be computed in the following way:

(1) If m is odd, use $u = \sec x$, $du = \sec x \tan x\,dx$.
(2) If n is even, use $u = \tan x$, $du = \sec^2 x\,dx$.

Example: $\int \tan^3 x \sec^2 x\,dx = \cdots$

Since in this case m is odd and n is even it does not matter which method we use, so let's use the first one:

$(u = \sec x,\ du = \sec x \tan x\,dx)$

$$\begin{aligned}
\cdots = \int \underbrace{\tan^2 x}_{u^2-1}\ \underbrace{\sec x}_{u}\ \underbrace{\tan x \sec x\,dx}_{du} &= \int (u^2-1)u\,du \\
&= \int (u^3 - u)\,du \\
&= \frac{u^4}{4} - \frac{u^2}{2} + C \\
&= \boxed{\tfrac{1}{4}\sec^4 x - \tfrac{1}{2}\sec^2 x + C}.
\end{aligned}$$

Next let's solve the same problem using the second method:

$(u = \tan x,\ du = \sec^2 x\, dx)$

$$\int \underbrace{\tan^3 x}_{u^3}\, \underbrace{\sec^2 x\, dx}_{du} = \int u^3\, du = \frac{u^4}{4} + C = \boxed{\tfrac{1}{4}\tan^4 x + C}\,.$$

Although this answer looks different from the one obtained using the first method it is in fact equivalent to it because they differ in a constant:

$$\tfrac{1}{4}\tan^4 x = \tfrac{1}{4}(\sec^2 x - 1)^2 = \underbrace{\tfrac{1}{4}\sec^4 x - \tfrac{1}{2}\sec^2 x}_{\text{previous answer}} + \tfrac{1}{4}\,.$$

1.6.2. Trigonometric Substitutions. Here we study substitutions of the form $x =$ some trigonometric function.

Example: Find $\displaystyle\int \sqrt{1-x^2}\, dx$.

Answer: We make $x = \sin t$, $dx = \cos t\, dt$, hence

$$\sqrt{1-x^2} = \sqrt{1-\sin^2 t} = \sqrt{\cos^2 t} = \cos t\,,$$

and

$$\begin{aligned}
\int \sqrt{1-x^2}\, dx &= \int \cos t\ \cos t\, dt \\
&= \int \cos^2 t\, dt \\
&= \int \tfrac{1}{2}(1+\cos 2t)\, dt && \text{(half-angle identity)} \\
&= \frac{t}{2} + \frac{\sin 2t}{4} + C \\
&= \frac{t}{2} + \frac{2\sin t\cos t}{4} + C && \text{(double-angle identity)} \\
&= \frac{t}{2} + \frac{\sin t\sqrt{1-\sin^2 t}}{2} + C \\
&= \boxed{\frac{\sin^{-1} x}{2} + \frac{x\sqrt{1-x^2}}{2} + C}\,.
\end{aligned}$$

The following substitutions are useful in integrals containing the following expressions:

expression	substitution	identity
$a^2 - u^2$	$u = a\sin t$	$1 - \sin^2 t = \cos^2 t$
$a^2 + u^2$	$u = a\tan t$	$1 + \tan^2 t = \sec^2 t$
$u^2 - a^2$	$u = a\sec t$	$\sec^2 t - 1 = \tan^2 t$

So for instance, if an integral contains the expression $a^2 - u^2$, we may try the substitution $u = a\sin t$ and use the identity $1 - \sin^2 t = \cos^2 t$ in order to transform the original expression in this way:

$$a^2 - u^2 = a^2(1 - \sin^2 t) = a^2\cos^2 t\,.$$

Example:

$$\begin{aligned}
\int \frac{x^3}{\sqrt{9 - x^2}}\,dx &= 27 \int \frac{\sin^3 t\cos t}{\sqrt{1 - \sin^2 t}}\,dt \qquad (x = 3\sin t) \\
&= 27 \int \sin^3 t\,dx \\
&= 27 \int (1 - \cos^2 t)\,\sin t\,dx \\
&= 27\left(-\cos t + \frac{\cos^3 t}{3}\right) + C \\
&= 27\left(-\sqrt{1 - \sin^2 t} + \tfrac{1}{3}(1 - \sin^2 t)^{3/2}\right) + C \\
&= \boxed{-9\sqrt{9 - x^2} + \tfrac{1}{3}(9 - x^2)^{3/2} + C}\,.
\end{aligned}$$

where $x = 3\sin t$, $dx = 3\cos t\,dt$.

Example:

$$\begin{aligned}
\int \sqrt{9+4x^2}\,dx &= 2\int \sqrt{\tfrac{9}{4}+x^2}\,dx \qquad\qquad (x=\tfrac{3}{2}\tan t)\\
&= 2\int \frac{3}{2}\sqrt{1+\tan^2 t}\,\frac{3}{2}\sec^2 t\,dt\\
&= \frac{9}{2}\int \sec^3 t\,dt\\
&= \frac{9}{4}(\sec t\tan t + \ln|\sec t+\tan t|) + C_1\\
&= \frac{9}{4}\left(\tfrac{2}{3}x\sqrt{1+\tfrac{4}{9}x^2} + \ln\left|\tfrac{2}{3}x + \sqrt{1+\tfrac{4}{9}x^2}\right|\right) + C_1\\
&= \boxed{\frac{x\sqrt{9+4x^2}}{2} + \frac{9}{4}\ln|2x+\sqrt{9+4x^2}| + C}.
\end{aligned}$$

where $x=\frac{3}{2}\tan t$, $dx=\frac{3}{2}\sec^2 t\,dt$

Example:

$$\begin{aligned}
\int \frac{\sqrt{x^2-1}}{x}\,dx &= \int \frac{\sqrt{\sec^2 t-1}}{\sec t}\,\sec t\tan t\,dt \qquad\qquad (x=\sec t)\\
&= \int \tan^2 t\,dt\\
&= \tan t - t + C\\
&= \sqrt{\sec^2 t-1} - t + C\\
&= \sqrt{x^2-1} - \sec^{-1} x + C\,.
\end{aligned}$$

where $x=\sec t$, $dx=\sec t\tan t\,dt$.