4.2. Series

A *series* is an infinite sum:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

In order to define the value of this sum we start be defining its sequence of *partial sums*

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n \,.$$

Then, if $\lim_{n\to\infty} s_n = s$ exists the series is called *convergent* and its sum is that limit:

$$\sum_{n=1}^{\infty} a_n = s = \lim_{n\to\infty} s_n \,.$$

Otherwise the series is called *divergent*.

For instance, consider the following series:

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} + \cdots = \sum_{n=1}^{\infty} \frac{1}{2^n} \,.$$

Its partial sums are:

$$s_n = \sum_{i=1}^{n} \frac{1}{2^i} = \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n} \,.$$

Hence its sum is

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2^i} = \lim_{n\to\infty} \left(1 - \frac{1}{2^n}\right) = 1 + 0 = 1 \,.$$

4.2.1. Geometric Series. A series verifying $a_{n+1} = r a_n$, where r is a constant, is called *geometric series*. If the first term is $a \neq 0$ then the series is

$$a + ar + ar^2 + \cdots + ar^n + \cdots = \sum_{n=0}^{\infty} ar^n \,.$$

The partial sums are now:

$$s_n = \sum_{i=0}^{n} ar^i \,.$$

The nth partial sum can be found in the following way:

$$\begin{aligned} s_n &= a + ar + ar^2 + \cdots + ar^n \\ rs_n &= \qquad ar + ar^2 + \cdots + ar^n + ar^{n+1} \end{aligned}$$

hence

$$s_n - rs_n = a + \ 0\ +\ 0\ + \cdots +\ 0\ - ar^{n+1}\,,$$

so:

$$s_n = \frac{a(1-r^{n+1})}{1-r}\,.$$

If $|r| < 1$ we can rewrite the result like this:

$$s_n = \frac{a}{1-r} - \frac{a}{1-r} r^{n+1}\,,$$

and then get the limit as $n \to \infty$:

$$s = \lim_{n\to\infty} s_n = \frac{a}{1-r} - \frac{a}{1-r} \underbrace{\lim_{n\to\infty} r^{n+1}}_{\downarrow \atop 0} = \frac{a}{1-r}$$

So for $|r| < 1$ the series is convergent and

$$\boxed{\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}}\,.$$

For $|r| \geq 1$ the series is divergent.

4.2.2. Telescopic Series. A telescopic series is a series whose terms can be rewritten so that most of them cancel out.

Example: Find $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

Answer: Note that $\dfrac{1}{n(n+1)} = \dfrac{1}{n} - \dfrac{1}{n+1}$. So the nth partial sum is

$$\begin{aligned} s_n &= \sum_{i=1}^{\infty} \left(\frac{1}{i} - \frac{1}{i+1}\right) \\ &= \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{n} - \frac{1}{n+1} \\ &= 1 - \frac{1}{n+1}\,. \end{aligned}$$

Hence, the sum of the series is

$$s = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = \boxed{1}.$$

4.2.3. Theorem. If the series $\sum_{n=0}^{\infty} a_n$ is convergent then $\lim_{n\to\infty} a_n = 0$.

Proof: If the series is convergent then the sequence of partial sums $s_n = \sum_{i=1}^{n} a_i$ have a limit s. On the other hand $a_n = s_n - s_{n-1}$, so taking limits we get $\lim_{n\to\infty} a_n = s - s = 0$.

The converse is not true in general. The harmonic series provides a counterexample.

4.2.4. The Harmonic Series. The following series is called *harmonic series*:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

The main fact about it is that it is *divergent*. In order to prove it we find

$s_1 = 1$

$s_2 = 1 + \frac{1}{2}$

$s_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{1}{2} + \frac{1}{2} = 1 + \frac{2}{2}$

$s_8 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)$

$\quad > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + \frac{3}{2}$

etc., so in general $s_{2^n} > 1 + \frac{n}{2}$, hence the sequence of partial sums grows without limit and the series diverges.

4.2.5. Test for Divergence. If $\lim\limits_{n\to\infty} a_n$ does not exist or if $\lim\limits_{n\to\infty} a_n \neq 0$ then $\sum\limits_{n=1}^{\infty} a_n$ diverges.

Example: Show that $\sum\limits_{n=1}^{\infty} \dfrac{n}{n+1}$ diverges.

Answer: We have $\lim\limits_{n\to\infty} \dfrac{n}{n+1} = 1$. Since the nth term of the series does not tend to 0, the series diverges.

Example: Show that $\displaystyle\sum_{n=1}^{\infty} \sin n$ diverges.

Answer: All we need to show is that $\sin n$ does not tend to 0. If for some value of n, $\sin n \approx 0$, then $n \approx k\pi$ for some integer k, but then

$$\begin{aligned}
\sin(n+1) &= \sin n \cos 1 + \cos n \sin 1 \\
&\approx \sin k\pi \cos 1 + \cos k\pi \sin 1 \\
&= 0 \pm \sin 1 \\
&= \pm 0.84\cdots \neq 0
\end{aligned}$$

So if a term $\sin n$ is close to zero, the next term $\sin(n+1)$ will be far from zero, so it is impossible for $\sin n$ to get permanently closer and closer to 0.

4.2.6. Operations with Series. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series and c is a constant then the following series are also convergent and:

(1) $\displaystyle\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$

(2) $\displaystyle\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

(3) $\displaystyle\sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$