

Probability problem with roulettees.

We have N roulettees, all with the same probability p of stopping at zero.

1. We spin all N roulettees simultaneously. What is the expected number of times that we must spin the roulettees until all of them stop simultaneously at zero?
2. We spin the first roulette until it stops at zero. After that we do the same with the second one, and so on until all roulettees have stopped at zero. What is the expected number of times that we must keep spinning roulettees until all of them have stopped at zero?
3. We spin all N roulettees simultaneously. If any of them stops at zero we keep spinning only the ones that did not stop at zero. We keep doing the same until all of them have stopped at zero. What is the expected number of times that we must spin the roulettees until all of them have stopped at zero?
4. We pick a roulette at random and spin it. After that we pick again a roulette at random and spin it, and so on. What is the expected number of times that we must keep spinning roulettees until n ($1 \leq n \leq N$) of them have stopped at zero at least once?

Solution.

1. If in an experiment an event has probability p , the expected number of times we must repeat it until the event happens is $\frac{1}{p}$. The probability that all the roulettees stop at zero simultaneously is p^N , hence the expected number of times we must spin them until it happens is $\frac{1}{p^N}$.
2. By the linearity of the expected value, the answer is just the sum of the expected number of times we must spin each roulette until it stops at zero. For each roulette the expected time is $\frac{1}{p}$, so for the N roulettees it will be $\frac{N}{p}$.
3. The probability that a roulette does not stop at zero in one spin is $q = 1 - p$. The probability that it does not stop at zero in any of t consecutive spins is q^t . So the probability that it does stop at zero in at most t spins is $1 - q^t$. The probability that all N roulettees stop at zero in at most t spins is $(1 - q^t)^N$. Hence the probability that after exactly t spins all of them end up stopping at zero is $(1 - q^t)^N - (1 - q^{t-1})^N$. Hence, the expected value of the number of spins until they all stop at zero is

$$E[t] = \sum_{t=1}^{\infty} t \{(1 - q^t)^N - (1 - q^{t-1})^N\}$$

The sum can be rewritten as the limit as $T \rightarrow \infty$ of

$$T\{(1 - q^T)^N - 1\} + \sum_{t=0}^{T-1} \{1 - (1 - q^t)^N\}.$$

We have

$$T\{(1 - q^T)^N - 1\} = \sum_{k=1}^N (-1)^k \binom{N}{k} q^{kT} T \xrightarrow{T \rightarrow \infty} 0,$$

hence

$$E[t] = \sum_{t=0}^{\infty} \{1 - (1 - q^t)^N\}.$$

Also:

$$(1 - q^t)^N = \sum_{k=0}^N (-1)^k \binom{N}{k} q^{kt},$$

hence

$$\begin{aligned} E[t] &= \sum_{t=0}^{\infty} \sum_{k=1}^N (-1)^{k+1} \binom{N}{k} q^{kt} = \sum_{k=1}^N (-1)^{k+1} \binom{N}{k} \sum_{t=0}^{\infty} q^{kt} \\ &= \sum_{k=1}^N (-1)^{k+1} \binom{N}{k} \frac{1}{1 - q^k}. \end{aligned}$$

- *Exercise:* Prove $E[t] = \frac{\ln N}{-\ln q} + \frac{\gamma}{p} + O(\frac{1}{N})$, where $\gamma = 0.5772156649 \dots = \text{Euler-Mascheroni constant}$.

4. Let $E_n[t]$ be the expected number of times we must spin roulettes until n of them have stopped at zero at least once. Then $E_1[t] = \frac{1}{p}$, and $E_{k+1}[t] - E_k[t]$ = expected number of spins to get one more roulette stopping at zero for the first time after k of them already did it. The probability of picking a roulette that has not yet stopped at zero is $\frac{N-k}{N}$, and the probability of it stopping at zero is the product $p \frac{N-k}{N}$, so the expected time for that event to occur is $\frac{N}{p(N-k)}$. Hence:

$$E_n[t] = \frac{N}{p} \sum_{k=0}^{n-1} \frac{1}{N - k}.$$

- *Remark:* The following formulas provide suitable approximations for large N :

- (a) If $n = N$ then $E_N[t] = \frac{N}{p} \left\{ \ln(N) + \gamma + \frac{1}{2N} + O(\frac{1}{N^2}) \right\}$.
 (b) The expression can be rewritten:

$$E_n[t] = \frac{1}{p} \sum_{i=0}^{\infty} \left(\frac{1}{N^i} \sum_{k=1}^{n-1} k^i \right) = \frac{n}{p} \left\{ 1 + \frac{n-1}{2N} + \frac{(2n-1)(n-1)}{6N^2} + \dots \right\},$$

which can be used to approximate $E_n[t]$ for N large and n/N small.