

## 1.9. Numerical Integration

Sometimes the integral of a function cannot be expressed with *elementary functions*, i.e., polynomial, trigonometric, exponential, logarithmic, or a suitable combination of these. However, in those cases we still can find an approximate value for the integral of a function on an interval.

**1.9.1. Trapezoidal Approximation.** A first attempt to approximate the value of an integral  $\int_a^b f(x) dx$  is to compute its Riemann sum:

$$R = \sum_{i=1}^n f(x_i^*) \Delta x.$$

Where  $\Delta x = x_i - x_{i-1} = (b-a)/n$  and  $x_i^*$  is some point in the interval  $[x_{i-1}, x_i]$ . If we choose the left endpoints of each interval, we get the *left-endpoint approximation*:

$$L_n = \sum_{i=1}^n f(x_{i-1}) \Delta x = (\Delta x) \{f(x_0) + f(x_1) + \cdots + f(x_{n-1})\},$$

Similarly, by choosing the right endpoints of each interval we get the *right-endpoint approximation*:

$$R_n = \sum_{i=1}^n f(x_i) \Delta x = (\Delta x) \{f(x_1) + f(x_2) + \cdots + f(x_n)\}.$$

The *trapezoidal approximation* is the average of  $L_n$  and  $R_n$ :

$$T_n = \frac{1}{2}(L_n + R_n) = \frac{\Delta x}{2} \{f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\}.$$

*Example:* Approximate  $\int_0^1 x^2 dx$  with trapezoidal approximation using 4 intervals.

*Solution:* We have  $\Delta x = 1/4 = 0.25$ . The values for  $x_i$  and  $f(x_i) = x_i^2$  can be tabulated in the following way:

$i$	$x_i$	$f(x_i)$
0	0	0
1	0.25	0.0625
2	0.5	0.25
3	0.75	0.5625
4	1	1

Hence:

$$L_4 = 0.25 \cdot (0 + 0.0625 + 0.25 + 0.5625) = 0.218750,$$

$$R_4 = 0.25 \cdot (0.0625 + 0.25 + 0.5625 + 1) = 0.468750.$$

So:

$$T_4 = \frac{1}{2}(L_4 + R_4) = \frac{1}{2}(0.218750 + 0.468750) = 0.34375.$$

Compare to the exact value of the integral, which is  $1/3 = 0.3333\dots$

**1.9.2. Midpoint Approximation.** Alternatively, in the Riemann sum we can use the middle point  $\bar{x}_i = (x_{i-1} + x_i)/2$  of each interval  $[x_{i-1}, x_i]$ . Then the *midpoint approximation* of  $\int_a^b f(x) dx$  is

$$M_n = (\Delta x)\{f(\bar{x}_1) + f(\bar{x}_2) + \dots + f(\bar{x}_n)\}.$$

*Example:* Approximate  $\int_0^1 x^2 dx$  with midpoint approximation using 4 intervals.

*Solution:* We have:

$i$	$\bar{x}_i$	$f(\bar{x}_i)$
1	0.125	0.015625
2	0.375	0.140625
3	0.625	0.390625
4	0.875	0.765625

Hence:

$$\begin{aligned} M_4 &= 0.25 \cdot (0.015625 + 0.140625 + 0.390625 + 0.765625) \\ &= 0.328125. \end{aligned}$$

**1.9.3. Simpson's Approximation.** *Simpson's approximation* is a weighted average of the trapezoidal and midpoint approximations associated to the intervals  $[x_0, x_2]$ ,  $[x_2, x_4]$ ,  $\dots$ ,  $[x_{n-2}, x_n]$  (of length

$2\Delta x$  each):

$$\begin{aligned}
 S_{2n} &= \frac{1}{3}(2M_n + T_n) \\
 &= \frac{1}{3} \left[ 2(2\Delta x) \{f(x_1) + f(x_3) + \cdots + f(x_{2n-1})\} \right. \\
 &\quad \left. + \frac{2\Delta x}{2} \{f(x_0) + 2f(x_2) + 2f(x_4) + \cdots + 2f(x_{n-2}) + f(x_n)\} \right] \\
 &= \frac{\Delta x}{3} \{f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots \\
 &\quad + 2f(x_{2n-2}) + 4f(x_{2n-1}) + f(x_{2n})\}.
 \end{aligned}$$

*Example:* Approximate  $\int_0^1 x^2 dx$  with Simpson's approximation using 8 intervals.

*Solution:* We use the previous results and get:

$$S_8 = \frac{1}{3}(2M_4 + T_4) = \frac{1}{3}(2 \cdot 0.328125 + 0.34375) = 1/3.$$

Note: in this particular case Simpson's approximation gives the exact value—in general it just gives a good approximation.

**1.9.4. Error Bounds.** Here we give a way to estimate the error or difference  $E$  between the actual value of an integral and the value obtained using a numerical approximation.

**1.9.4.1. Error Bound for the Trapezoidal Approximation.** Suppose  $|f''(x)| \leq K$  for  $a \leq x \leq b$ . Then the error  $E_T$  in the trapezoidal approximation verifies:

$$|E_T| \leq \frac{K(b-a)^3}{12n^2}.$$

**1.9.4.2. Error Bound for the Midpoint Approximation.** Suppose  $|f''(x)| \leq K$  for  $a \leq x \leq b$ . Then the error  $E_M$  in the trapezoidal approximation verifies:

$$|E_M| \leq \frac{K(b-a)^3}{24n^2}.$$

**1.9.4.3. Error Bound for the Simpson's Rule.** Suppose  $|f^{(4)}(x)| \leq K$  for  $a \leq x \leq b$ . Then the error  $E_S$  in the Simpson's rule verifies:

$$|E_S| \leq \frac{K(b-a)^5}{180n^4}.$$

*Example:* Approximate the value of  $\pi$  using the trapezoidal, midpoint and Simpson's approximations of

$$\int_0^1 \frac{4}{1+x^2} dx$$

for  $n = 4$ . Estimate the error.

*Answer:* First note that:

$$4 \int_0^1 \frac{1}{1+x^2} dx = 4 [\tan^{-1} x]_0^1 = 4 \frac{\pi}{4} = \pi,$$

so by approximating the given integral we are in fact finding approximated values for  $\pi$ .

Now we find the requested approximations:

(1) Trapezoidal approximation:

$$\begin{aligned} T_4 &= \frac{1/4}{2} \{f(0) + 2f(1/4) + 2f(1/2) + 2f(3/4) + f(1)\} \\ &= \boxed{3.131176470}. \end{aligned}$$

For estimating the error we need the second derivative of  $f(x) = 4/(1+x^2)$ , which is  $f''(x) = 8(3x^2 - 1)/(1+x^2)^3$  so we have

$$\begin{aligned} |f''(x)| &= \frac{8|3x^2 - 1|}{|1+x^2|^3} \leq \frac{8(3x^2 + 1)}{(1+x^2)^3} \\ &\leq \frac{8(3 \cdot 1^2 + 1)}{1} = 32 \end{aligned}$$

for  $0 \leq x \leq 1$ , hence

$$|E_T| \leq \frac{32 \cdot (1-0)^3}{12 \cdot 4^2} = 0.1666 \dots$$

(2) Midpoint approximation:

$$\begin{aligned} M_4 &= \frac{1}{4} \{f(1/8) + f(3/8) + f(5/8) + f(7/8)\} \\ &= \boxed{3.146800518}. \end{aligned}$$

The error estimate is:

$$|E_M| \leq \frac{32 \cdot (1-0)^3}{24 \cdot 4^2} = 0.08333 \dots$$

(3) Simpson's rule:

$$\begin{aligned} S_4 &= \frac{1/4}{3} \{f(0) + 4f(1/4) + 2f(1/2) + 4f(3/4) + f(1)\} \\ &= \boxed{3.141568627} \end{aligned}$$

For the error estimate we now need the fourth derivative:

$$f^{(4)}(x) = 96(5x^4 - 10x^2 + 1)/(1 + x^2)^5,$$

so

$$|f^{(4)}(x)| \leq \frac{96(5 + 10 + 1)}{1} = 1536$$

for  $0 \leq x \leq 1$ . Hence the error estimate is

$$|E_S| \leq \frac{1536 \cdot (1 - 0)^5}{180 \cdot 4^4} = 0.0333 \dots$$