2.6. Probability

2.6.1. Continuous Random Variables. A *random variable* is a real-valued function defined on some set of possible outcomes of a random experiment; e.g. the number of points obtained after rolling a dice - which can be 1, 2, 3, 4, 5 or 6. In this example the random variable can take only a discrete set of values. If the variable can take a continuous set of values then it is called a *continuous random variable*, e.g. a person's height.

Given a random variable X, its *probability distribution function* is the function $F(x) = P(X \leq x) =$ probability that the random variable X takes a value less than or equal to x. For instance if $F(x)$ is the probability distribution function of the number of points obtained after rolling a dice, then $F(4.7) = P(X \leq 4.7) =$ probability that the number of points is less than or equal to 4.7, i.e., the number of points is 1, 2, 3 or 4, so the probability is $4/6 = 2/3$, and $F(4.7) = 2/3$.

If the random variable is continuous then we can also define a *probability density function* $f(x)$ equal to the limit as $\Delta x \to 0$ of the probability that the random variable takes a value in a small interval of length Δx around x divided by the length of the interval. This definition means that $f(x) = F'(x)$. The probability that the random variable takes a value in some interval $[a, b]$ is

$$P(a \leq X \leq b) = F(b) - F(a) = \int_a^b f(x)\,dx\,.$$

In general the probability density of a random variable satisfies two conditions:

(1) $f(x) \geq 0$ for every x (probabilities are always non-negative).

(2) $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 1$ (the probability of a sure event is 1).

Example: A probability distribution is called *uniform* on a set S if its probability density is constant on S. Find the probability density of the uniform distribution on the interval $[2, 5]$.

Answer: The probability density function must be constant on $[2, 5]$, so for $2 \leq x \leq 5$ we have $f(x) = c$ for some constant c. On the other hand $f(x) = 0$ for x outside $[2, 5]$, hence:

$$1 = \int_{-\infty}^{\infty} f(x)\,dx = \int_{2}^{5} c\,dx = c(5-2) = 3c\,,$$

so $c = 1/3$. Hence

$$f(x) = \begin{cases} 1/3 & \text{if } 2 \le x \le 5, \\ 0 & \text{otherwise.} \end{cases}$$

2.6.2. Means. The *mean* or *average* of a discrete random variable that takes values $x_1, x_2, \dots, x_n$ with probabilities $p_1, p_2, \dots, p_n$ respectively is

$$\overline{x} = x_1 p_1 + x_2 p_2 + \cdots + x_n p_n = \sum_{i=1}^{n} x_i p_i\,.$$

For instance the mean value of the points obtained by rolling a dice is

$$1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} = \frac{7}{2} = 3.5\,.$$

This means that if we roll the dice many times in average we may expect to get about 3.5 points per roll.

For continuous random variables the probability is replaced with the probability density function, and the sum becomes an integral:

$$\mu = \overline{x} = \int_{-\infty}^{\infty} x f(x)\,dx\,.$$

2.6.3. Waiting Times. The time that we must wait for some event to occur (such as receiving a telephone call) can be modeled with a random variable of density

$$f(t) = \begin{cases} 0 & \text{if } t < 0, \\ ce^{-ct} & \text{if } t \ge 0, \end{cases}$$

were c is a positive constant. Note that, as expected:

$$\int_{-\infty}^{\infty} f(t)\,dt = \int_{0}^{\infty} ce^{-ct}\,dx = \left[-e^{-ct}\right]_0^{\infty} = \lim_{u\to\infty} \{-e^{-cu} - (-e^{0})\} = 1\,.$$

The mean waiting time can be computed like this:

$$\begin{aligned}
\mu &= \int_{-\infty}^{\infty} t f(t)\, dt \\
&= \int_{0}^{\infty} t c e^{-ct}\, dx \\
&= \left[-te^{-ct}\right]_0^{\infty} + \int_0^{\infty} e^{-ct}\, dx \quad \text{(I. by parts)} \\
&= 0 + \left[-\frac{e^{-ct}}{c}\right]_0^{\infty} \\
&= \frac{1}{c}\,,
\end{aligned}$$

hence $\mu = 1/c$. So we can rewrite the density function like this:

$$f(x) = \begin{cases} 0 & \text{if } t < 0, \\ \frac{1}{\mu} e^{-t/\mu} & \text{if } t \geq 0, \end{cases}$$

Example: Assume that the average waiting time for a catastrophic meteorite to strike the Earth is 100 million years. Find the probability that the Earth will suffer a catastrophic meteorite impact in the next 100 years. Find the probability that no such catastrophic event will happen in the next 5 billion years.

Answer: We have $\mu = 10^8$ years, so the probability density function is

$$f(t) = 10^{-8} e^{-10^{-8}\,t} \qquad (t \geq 0)\,.$$

So the answer to the first question is

$$\int_0^{100} 10^{-8} e^{-10^{-8}\,t}\, dt = \left[1 - e^{-10^{-8}\,t}\right]_0^{100} = 1 - e^{-10^{-8}\cdot 100} = 1 - e^{-10^{-6}} \approx 10^{-6}\,,$$

i.e., about 1 in a million. Regarding the second question, the probability is

$$\begin{aligned}
1 - \int_0^{5\cdot 10^9} 10^{-8} e^{-10^{-8}\,t}\, dt &= 1 - \left[1 - e^{-10^{-8}\,t}\right]_0^{5\cdot 10^9} \\
&= 1 - \left\{1 - e^{-10^{-8}\cdot 5\cdot 10^9}\right\} = e^{-50} \approx 2 \cdot 10^{-22}\,,
\end{aligned}$$

which is practically zero.

2.6.4. Normal Distributions. Many important phenomena follow a so called *Normal Distribution*, whose density function is:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} .$$

Its mean is μ. The positive constant σ is called *standard deviation*; it measures how spread out the values of the random variable are.

Example: Intelligent Quotient (IQ) scores are distributed normally with mean $\mu = 100$ and standard deviation $\sigma = 15$. What proportion of the population has an IQ between 70 and 130?

Answer: The integral cannot be evaluated in terms of elementary functions, but it can be approximated with numerical methods:

$$P(70 \leq X \leq 130) = \int_{70}^{130} \frac{1}{15\sqrt{2\pi}} e^{-(x-100)^2/2\cdot 15^2}\, dx \approx \boxed{0.9544997360\cdots} .$$

Another approach for solving these kinds of problems is to use the *error function*, defined in the following way:

$$\phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt .$$

The following are some of its values (rounded to three decimal places):

x	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$\phi(x)$	0.0	0.112	0.223	0.329	0.438	0.520	0.604	0.678	0.742	0.797
x	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
$\phi(x)$	0.843	0.880	0.910	0.934	0.952	0.966	0.976	0.984	0.989	0.993
x	2.0	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9
$\phi(x)$	0.995	0.997	0.998	0.999	0.999	1.000	1.000	1.000	1.000	1.000

Example: Solve the previous problem using the error function.

Answer: We need to transform our integral into another expression containing the error function:

$$
\begin{aligned}
P(70 \leq X \leq 130) &= \int_{70}^{130} \frac{1}{15\sqrt{2\pi}} e^{-(x-100)^2/450}\,dx \\
&= \frac{1}{\sqrt{\pi}} \int_{-\sqrt{2}}^{\sqrt{2}} e^{-u^2}\,du && [u = (x-100)/15\sqrt{2}] \\
&= \frac{2}{\sqrt{\pi}} \int_{0}^{\sqrt{2}} e^{-u^2}\,du && \text{(by symmetry)} \\
&= \phi(\sqrt{2}) \approx \phi(1.4) \approx \boxed{0.952}
\end{aligned}
$$