CS 310-0
Homework Assignment No. 4
Due Fri 2/4/2000

1. Find the properties (commutative, associative, existence of identity element, existence of inverse) verified by the following operations:
   (a) Addition on $\mathbb{N}$.
   (b) Multiplication on $\mathbb{Q}^+$.
   (c) $x * y = x + y - xy$ on $\{0, 1\}$.
   (d) The *nand* operation $p \uparrow q = \neg(p \wedge q)$ on the set of all statements.
   (e) Function composition on $\mathcal{F} = \{f \mid f : \mathbb{R} \to \mathbb{R}\}$.

   In the following two exercises you may use known properties of set operations without proof—for instance, commutativity of set intersection—, as stated in the class notes.

2. Let X be a nonempty set. Let $\mathcal{P}(X)$ be the power set of X. Prove that $(\mathcal{P}(X), \triangle)$ is a commutative group.[1] What is the identity element? What is the inverse of a set $A \in \mathcal{P}(X)$?

3. Given a set X with at least two elements, prove that $(\mathcal{P}(X), \triangle, \cap)$ (the power set of X with the operations of symmetric difference and intersection) is a commutative ring with unity, but not a field (hint: in a field there are no proper divisors of zero).[2]

4. Let $(G, *)$, $(G', \circ)$ be two groups, with identity elements e and e' respectively. A function $f : G \to G'$, such that $\forall x, y \in G,\ f(x*y) = f(x) \circ f(y)$ is called a *group homomorphism*. Prove that if f is a group homomorphism then:
   (a) $f(e) = e'$.
   (b) $\forall x \in G,\ f(x^{-1}) = f(x)^{-1}$.

5. A group homomorphism that is also a one-to-one correspondence is called a *group isomorphism*. Let a be a positive real number greater than 1. Prove that the function $f(x) = a^x$ is a group isomorphism from $(\mathbb{R}, +)$ to $(\mathbb{R}^+, \cdot)$.

6. Eight people meet for dinner. They will be seated at a round table. Assume that we consider two arrangements equivalent if everybody has exactly the same neighbors; for instance, using letters $A, B, C, ...$ for designing people, and naming then clockwise, the arrangements $ABCDEFGH$, $FGHABCDE$, $HGFEDCBA$ would be equivalent. How many nonequivalent arrangements are possible?

7. In problem 6, assume that the eight people are four couples. How many (non-equivalent) arrangements are possible such that:
   (a) The two members of each couple are together?
   (b) The two members of each couple are seated on diametrically opposite places?

[1] $A \triangle B = (A \cap \overline{B}) \cup (\overline{A} \cap B)$ = symmetric difference of A and B.

[2] In Homework Assignment No. 2 you already proved the distributive property of $\cap$ respect to $\triangle$. You do not need to repeat the proof here.

For instance, if the couples are $\{A_1, A_2\}$, $\{B_1, B_2\}$, $\{C_1, C_2\}$, $\{D_1, D_2\}$, in the arrangement $A_2A_1C_2C_1D_1D_2B_1B_2$ all the couples are together, and in $D_1C_2A_1B_2D_2C_1A_2B_1$ all the couples are on diametrically opposite places.

8. A city has streets in the direction E-W and avenues in the direction N-S, making a perfect grid. Assume that a taxicab has to go from the intersection of 23rd street and 1st avenue to the intersection of 30th street and 7th avenue following a path of minimum length. In how many ways can it be done? Assume that there is an accident at the intersection of 27th street and 5th avenue and the taxicab wants to avoid it. How many (minimal) paths can the taxicab follow now?

9. A cube is made by piling $3 \times 3 \times 3 = 27$ smaller cubes. An insect moves always following the edges of the small cubes (including the edges in the interior of the larger cube) and possibly changing direction only at the vertices of the small cubes. In how many ways can the insect go from one vertex of the large cube to the diametrically opposite vertex following a path of minimum length?

10. In the following questions, two colorings that differ only by rotations of the die are considered identical.
    (a) In how many ways can we color the faces of a die in red, blue, green, hazel, yellow and purple (one color for each face)?
    (b) Same question but now dropping color purple and using any of the other colors on two faces?