CS 310
Homework Assignment No. 4
Due on Tue 4/27/2005

1. Prove the following by induction:

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{2n^3 + 3n^2 + n}{6}\,.$$

2. We define recursively a sequence x_n in the following way:

$$x_0 = 2\,; \qquad x_{n+1} = \frac{x_n^2 + 1}{2\,x_n} \quad (n \geq 0)\,.$$

Prove by induction: $0 < x_n - 1 \leq 1/2^{2^n - 1}$ for every $n \geq 0$. (Hint: begin by proving that $x_{n+1} - 1 = (x_n - 1)^2/2x_n$.)

3. Prove the following statements using mathematical induction:

(a) $1 \cdot 2^0 + 2 \cdot 2^1 + 3 \cdot 2^2 + 4 \cdot 2^3 + \cdots + n \cdot 2^{n-1} = (n-1)2^n + 1$ for $n \geq 1$.

(b) $10^n < 2^{2^n}$ for $n \geq 4$.

4. A computer virus works in such a way that it creates two copies of itself every second for two seconds and then dies. At time 0 sec one copy of the virus enters the computer. How many copies of the virus will be created at time n sec? How many copies of the virus will there be in the computer at time n sec? (Note: the sequence of copies created at time n sec starts like this: $x_0 = 1, x_1 = 3, x_2 = 8, x_3 = 22, \ldots$.)

5. Find a close-form formula for the nth term of the Lucas sequence $2, 1, 3, 4, 7, 11, 18 \ldots,$ recursively defined in the following way:

$L_0 = 2,\ L_1 = 1,$
$L_n = L_{n-1} + L_{n-2} \qquad (n \geq 2).$