1.2. Quantifiers

1.2.1. Predicates. A *predicate* or *propositional function*[1] is a statement containing variables. For instance "$x+2=7$", "X is American", "$x<y$", "p is a prime number" are predicates. The truth value of the predicate depends on the value assigned to its variables. For instance if we replace x with 1 in the predicate "$x+2=7$" we obtain "$1+2=7$", which is false, but if we replace it with 5 we get "$5+2=7$", which is true. We represent a predicate by a letter followed by the variables enclosed between parenthesis: $P(x)$, $Q(x,y)$, etc. An *example* for $P(x)$ is a value of x for which $P(x)$ is true. A *counterexample* is a value of x for which $P(x)$ is false. So, 5 is an example for "$x+2=7$", while 1 is a counterexample.

Each variable in a predicate is assumed to belong to a *domain (or universe) of discourse*, for instance in the predicate "n is an odd integer" 'n' represents an integer, so the domain of discourse of n is the set of all integers. In "X is American" we may assume that X is a human being, so in this case the domain of discourse is the set of all human beings.[2]

1.2.2. Quantifiers. Given a predicate $P(x)$, the statement "for some x, $P(x)$" (or "there is some x such that $p(x)$"), represented "$\exists x\, P(x)$", has a definite truth value, so it is a proposition in the usual sense. For instance if $P(x)$ is "$x+2=7$" with the integers as domain of discourse, then $\exists x\, P(x)$ is true, since there is indeed an integer, namely 5, such that $P(5)$ is a true statement. However, if $Q(x)$ is "$2x=7$" and the domain of discourse is still the integers, then $\exists x\, Q(x)$ is false. On the other hand, $\exists x\, Q(x)$ would be true if we extend the domain of discourse to the rational numbers. The symbol $\exists$ is called the *existential quantifier*.

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[1]The term *propositional function* used by Johnsonbaugh is rather obsolete and I have replaced it here with the more currently used *predicate*.

[2]Usually all variables occurring in predicates along a reasoning are supposed to belong to the *same* domain of discourse, but in some situations (as in the so called *many-sorted* logics) it is possible to use different kinds of variables to represent different types of objects belonging to different domains of discourse. For instance in the predicate "σ is a string of length n" the variable σ represents a string, while n represents a natural number, so the domain of discourse of σ is the set of all strings, while the domain of discourse of n is the set of natural numbers.

Analogously, the sentence "for all x, $P(x)$"—also "for any x, $P(x)$", "for every x, $P(x)$", "for each x, $P(x)$"—, represented "$\forall x\, P(x)$", has a definite truth value. For instance, if $P(x)$ is "$x + 2 = 7$" and the domain of discourse is the integers, then $\forall x\, P(x)$ is false. However if $Q(x)$ represents "$(x + 1)^2 = x^2 + 2x + 1$" then $\forall x\, Q(x)$ is true. The symbol $\forall$ is called the *universal quantifier*.

In predicates with more than one variable it is possible to use several quantifiers at the same time, for instance $\forall x \forall y \exists z\, P(x, y, z)$, meaning "for all x and all y there is some z such that $P(x, y, z)$".

Note that in general the existential and universal quantifiers cannot be permuted, i.e., in general $\forall x \exists y\, P(x, y)$ means something different from $\exists y \forall x\, P(x, y)$. For instance if x and y represent human beings and $P(x, y)$ represents "x is married to y", then $\forall x \exists y\, P(x, y)$ means that everybody is married to someone, but $\exists y \forall x\, P(x, y)$ means that there is someone to whom everybody else is married (a extreme form of polygamy!).

A predicate can be partially quantified, e.g. $\forall x \exists y\, P(x, y, z, t)$. The variables quantified (x and y in the example) are called *bound* variables, and the rest (z and t in the example) are called *free* variables. A partially quantified predicate is still a predicate, but depending on fewer variables.

1.2.3. Generalized De Morgan Laws for Logic. If $\exists x\, P(x)$ is false then there is no value of x for which $P(x)$ is true, or in other words, $P(x)$ is always false. Hence

$$\overline{\exists x\, P(x)} \equiv \forall x\, \overline{P(x)}\,.$$

On the other hand, if $\forall x\, P(x)$ is false then it is not true that for every x, $P(x)$ holds, hence for some x, $P(x)$ must be false. Thus:

$$\overline{\forall x\, P(x)} \equiv \exists x\, \overline{P(x)}\,.$$

This two rules can be applied in successive steps to find the negation of a more complex quantified statement, for instance:

$$\overline{\exists x \forall y\, p(x, y)} \equiv \forall x \overline{\forall y\, P(x, y)} \equiv \forall x \exists y\, \overline{P(x, y)}\,.$$

Exercise: Write formally the statement "for every real number there is a greater real number". Write the negation of that statement.

Answer: The statement is: $\forall x\, \exists y\, (x < y)$ (the domain of discourse is the real numbers). Its negation is: $\exists x\, \forall y\, \overline{x < y}$, i.e., $\exists x\, \forall y\, (x \not< y)$. (Note that among real numbers $x \not< y$ is equivalent to $x \geq y$, but formally they are different predicates.)

1.3. Proofs

1.3.1. Mathematical Systems, Proofs. A *Mathematical System* consists of:

1. *Axioms*: propositions that are assumed true.
2. *Definitions*: used to create new concepts from old ones.
3. *Undefined terms*: corresponding to the primitive concepts of the system (for instance in set theory the term "set" is undefined).

A *theorem* is a proposition that has been proved to be true. An argument that establishes the truth of a proposition is called a *proof*.

Example: Prove that if $x > 2$ and $y > 3$ then $x + y > 5$.

Answer: Assuming $x > 2$ and $y > 3$ and adding the inequalities term by term we get: $x + y > 2 + 3 = 5$.

That is an example of *direct proof*. In a direct proof we assume the hypothesis together with axioms and other theorems previously proved and we derive the conclusion from them.

Proof by Contradiction. In a *proof by contradiction* or (*Reductio ad Absurdum*) we assume the hypothesis and the negation of the conclusion, and try to derive a *contradiction*, i.e., a proposition of the form $r \wedge \overline{r}$.

Example: Prove by contradiction that if $x+y > 5$ then either $x > 2$ or $y > 3$.

Answer: We assume the hypothesis $x + y > 5$. From here we must conclude that $x > 2$ or $y > 3$. Assume to the contrary that "$x > 2$ or $y > 3$" is false, so $x \leq 2$ and $y \leq 3$. Adding those inequalities we get $x \leq 2 + 3 = 5$, which contradicts the hypothesis $x + y > 5$. From here we conclude that the assumption "$x \leq 2$ and $y \leq 3$" cannot be right, so "$x > 2$ or $y > 3$" must be true.

A related proof is the *proof by contrapositive*, i.e., instead of proving $p \to q$ we prove the contrapositive $\overline{q} \to \overline{p}$.

1.3.2. Arguments, Rules of Inference. An *argument* is a sequence of propositions $p_1, p_2, \dots, p_n$ called *hypothesis* (or *premises*) followed by a proposition q called *conclusion*. An argument is usually written:

$$\begin{array}{rl} & p_1 \\ & p_2 \\ & \vdots \\ & p_n \\ \hline \therefore & q \end{array}$$

or

$$p_1, p_2, \dots, p_n / \therefore q$$

The argument is called *valid* if q is true whenever $p_1, p_2, \dots, p_n$ are true; otherwise it is called *invalid*.

Rules of inference are certain simple arguments known to be valid and used to make a proof step by step. For instance the following argument is called *modus ponens* or *law of detachment*:

$$\begin{array}{rl} & p \to q \\ & p \\ \hline \therefore & q \end{array}$$

In order to check whether it is valid we must examine the following truth table:

p	q	$p \to q$	p	q
T	T	T	T	T
T	F	F	T	F
F	T	T	F	T
F	F	T	F	F

If we look now at the rows in which both $p \to q$ and p are true (just the first row) we see that also q is true, so the argument is valid.

Other rules of inference are the following:

1. *Modus Ponens* or *Rule of Detachment*:

$$\begin{array}{rl} & p \to q \\ & p \\ \hline \therefore & q \end{array}$$

2. *Modus Tollens*:

$$\begin{array}{rl} & p \to q \\ & \overline{q} \\ \hline \therefore & \overline{p} \end{array}$$

3. *Addition*:

$$\begin{array}{rl} & p \\ \hline \therefore & p \vee q \end{array}$$

4. *Simplification*:

$$\begin{array}{rl} & p \wedge q \\ \hline \therefore & p \end{array}$$

5. *Conjunction*:

$$\begin{array}{rl} & p \\ & q \\ \hline \therefore & p \wedge q \end{array}$$

6. *Hypothetical Syllogism*:

$$\begin{array}{rl} & p \to q \\ & q \to r \\ \hline \therefore & p \to r \end{array}$$

7. *Disjunctive Syllogism*:

$$\begin{array}{rl} & p \vee q \\ & \overline{p} \\ \hline \therefore & q \end{array}$$

Arguments are usually written using three columns. Each row contains a label, a statement and the reason that justifies the introduction of that statement in the argument. That justification can be one of the following:

1. The statement is a *premise*.
2. The statement can be derived from statements occurring earlier in the argument by using a *rule of inference*.

Example: Consider the following statements: "I take the bus or I walk. If I walk I get tired. I do not get tired. Therefore I take the bus." We can formalize this by calling B = "I take the bus", W = "I walk" and T = "I get tired". The premises are $B \vee W$, $W \to T$ and $\overline{T}$, and the conclusion is B. The argument can be described in the following steps:

step	statement	reason
1)	$W \to T$	Premise
2)	$\overline{T}$	Premise
3)	$\overline{W}$	1,2, Modus Tollens
4)	$B \vee W$	Premise
5)	$\therefore B$	4,3, Disjunctive Syllogism

1.3.3. Rules of Inference for Quantified Statements. We state the rules for predicates with one variable, but they can be generalized to predicates with two or more variables.

1. *Universal Instantiation.* If $\forall x\, p(x)$ is true, then $p(a)$ is true for each specific element a in the domain of discourse; i.e.:
$$\begin{array}{rl} & \forall x\, p(x) \\ \hline \therefore & p(a) \end{array}$$
For instance, from $\forall x\,(x+1 = 1+x)$ we can derive $7+1 = 1+7$.

2. *Existential Instantiation.* If $\exists x\, p(x)$ is true, then $p(a)$ is true for some specific element a in the domain of discourse; i.e.:
$$\begin{array}{rl} & \exists x\, p(x) \\ \hline \therefore & p(a) \end{array}$$
The difference respect to the previous rule is the restriction in the meaning of a, which now represents some (not any) element of the domain of discourse. So, for instance, from $\exists x\,(x^2 = 2)$ (the domain of discourse is the real numbers) we derive the existence of some element, which we may represent $\pm\sqrt{2}$, such that $(\pm\sqrt{2})^2 = 2$.

3. *Universal Generalization.* If $p(x)$ is proved to be true for a generic element in the domain of discourse, then $\forall x\, p(x)$ is true; i.e.:
$$\begin{array}{rl} & p(x) \\ \hline \therefore & \forall x\, p(x) \end{array}$$
By "generic" we mean an element for which we do not make any assumption other than its belonging to the domain of discourse. So, for instance, we can prove $\forall x\,[(x+1)^2 = x^2 + 2x + 1]$ (say,

for real numbers) by assuming that x is a generic real number and using algebra to prove $(x+1)^2 = x^2 + 2x + 1$.

4. *Existential Generalization.* If $p(a)$ is true for some specific element a in the domain of discourse, then $\exists x\, p(x)$ is true; i.e.:

$$\begin{array}{c} p(a) \\ \hline \therefore \quad \exists x\, p(x) \end{array}$$

   For instance: from $7 + 1 = 8$ we can derive $\exists x\, (x + 1 = 8)$.

Example: Show that a counterexample can be used to disprove a universal statement, i.e., if a is an element in the domain of discourse, then from $\overline{p(a)}$ we can derive $\overline{\forall x\, p(x)}$. *Answer*: The argument is as follows:

step	statement	reason
1)	$\overline{p(a)}$	Premise
2)	$\exists x\, \overline{p(x)}$	Existential Generalization
3)	$\overline{\forall x\, p(x)}$	Negation of Universal Statement