## 4.3. The Integral and Comparison Tests

**4.3.1. The Integral Test.** Suppose $f$ is a continuous, positive, decreasing function on $[1, \infty)$, and let $a_n = f(n)$. Then the convergence or divergence of the series $\sum_{n=1}^{\infty} a_n$ is the same as that of the integral $\int_1^{\infty} f(x)\, dx$, i.e.:

(1) If $\displaystyle\int_1^{\infty} f(x)\, dx$ is convergent then $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent.

(2) If $\displaystyle\int_1^{\infty} f(x)\, dx$ is divergent then $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent.

The best way to see why the integral test works is to compare the area under the graph of $y = f(x)$ between 1 and $\infty$ to the sum of the areas of rectangles of height $f(n)$ placed along intervals $[n, n+1]$.
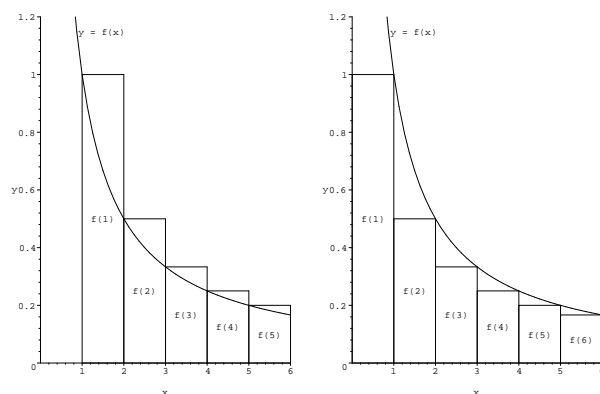


FIGURE 4.3.1

From the graph we see that the following inequality holds:

$$\int_1^{n+1} f(x)\, dx \le \sum_{i=1}^{n} a_n \le f(1) + \int_1^{n} f(x)\, dx\,.$$

The first inequality shows that if the integral diverges so does the series. The second inequality shows that if the integral converges then the same happens to the series.

*Example*: Use the integral test to prove that the harmonic series $\sum_{n=1}^{\infty} 1/n$ diverges.

*Answer*: The convergence or divergence of the harmonic series is the same as that of the following integral:

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty}\left[\ln x\right]_1^t = \lim_{t\to\infty} \ln t = \infty,$$

so it diverges.

**4.3.2. The $p$-series.** The following series is called $p$-series:

$$\sum_{n=1}^\infty \frac{1}{n^p}.$$

Its behavior is the same as that of the integral $\int_1^\infty \frac{1}{x^p}\,dx$. For $p = 1$ we have seen that it diverges. If $p \neq 1$ we have

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^p}\,dx = \lim_{t\to\infty}\left[\frac{x^{1-p}}{1-p}\right]_1^t = \lim_{t\to\infty}\frac{t^{1-p}}{1-p} - \frac{1}{1-p}.$$

For $0 < p < 1$ the limit is infinite, and for $p > 1$ it is zero so:

$$\boxed{\text{The } p\text{-series } \sum_{n=1}^\infty \frac{1}{n^p} \text{ is } \begin{cases} \text{convergent} & \text{if } p > 1 \\ \text{divergent} & \text{if } p \leq 1 \end{cases}}$$

**4.3.3. Comparison Test.** Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms and suppose that $a_n \leq b_n$ for all $n$. Then

(1) If $\sum b_n$ is convergent then $\sum a_n$ is convergent.

(2) If $\sum a_n$ is divergent then $\sum b_n$ is divergent.

*Example*: Determine whether the series $\sum_{n=1}^\infty \frac{\cos^2 n}{n^2}$ converges or diverges.

*Answer*: We have

$$0 < \frac{\cos^2 n}{n^2} \leq \frac{1}{n^2} \quad \text{for all } n \geq 1$$

and we know that the series $p$-series $\sum_{n=1}^\infty \frac{1}{n^2}$ converges. Hence by the comparison test, the given series also converges (incidentally, its sum is $\frac{1}{2} - \frac{\pi}{2} + \frac{\pi^2}{6} = 0.5736380465\ldots$, although we cannot prove it here).

**4.3.4. The Limit Comparison Test.** Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n \to \infty} \frac{a_n}{b_n} = c \,,$$

where $c$ is a finite strictly positive number, then either both series converge or both diverge.

*Example*: Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{1 + 4n^2}}$ converges or diverges.

*Answer*: We will use the limit comparison test with the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$. We have

$$\lim_{n \to \infty} \frac{1/n}{1/\sqrt{1 + 4n^2}} = \lim_{n \to \infty} \frac{\sqrt{1 + 4n^2}}{n}$$

$$= \lim_{n \to \infty} \sqrt{\frac{1 + 4n^2}{n^2}}$$

$$= \lim_{n \to \infty} \sqrt{\frac{1}{n^2} + 4} \quad = \sqrt{4} = 2 \,,$$

so the given series has the same behavior as the harmonic series. Since the harmonic series diverges, so does the given series.

**4.3.5. Remainder Estimate for the Integral Test.** The difference between the sum $s = \sum_{n=1}^{\infty} a_n$ of a convergent series and its $n$th partial sum $s_n = \sum_{i=1}^{n} a_i$ is the *remainder*:

$$R_n = s - s_n = \sum_{i=n+1}^{\infty} a_i \,.$$

The same graphic used to see why the integral test works allows us to estimate that remainder. Namely: If $\sum a_n$ converges by the Integral Test and $R_n = s - s_n$, then

$$\boxed{\int_{n+1}^{\infty} f(x) \, dx \leq R_n \leq \int_{n}^{\infty} f(x) \, dx}$$

Equivalently (adding $s_n$):

$$s_n + \int_{n+1}^{\infty} f(x)\, dx \leq s \leq s_n + \int_{n}^{\infty} f(x)\, dx$$

*Example*: Estimate $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$ to the third decimal place.

*Answer*: We need to reduce the remainder below 0.0005, i.e., we need to find some $n$ such that

$$\int_{n}^{\infty} \frac{1}{x^4}\, dx < 0.0005\,.$$

We have

$$\int_{n}^{\infty} \frac{1}{x^4}\, dx = \left[-\frac{1}{3x^3}\right]_{n}^{\infty} = \frac{1}{3n^3}\,,$$

hence

$$\frac{1}{3n^3} < 0.0005 \quad \Rightarrow \quad n > \sqrt[3]{\frac{3}{0.0005}} = 18.17\ldots\,,$$

so we can take $n = 19$. So the sum of the 15 first terms of the given series coincides with the sum of the whole series up to the third decimal place:

$$\sum_{i=1}^{19} \frac{1}{i^4} = 1.082278338\ldots$$

From here we deduce that the actual sum $s$ of the series is between $1.08227\ldots-0.0005 = 1.08177\ldots$ and $1.08227\ldots+0.0005 = 1.08277\ldots$, so we can claim $s \approx 1.082$. (The actual sum of the series is $\frac{\pi^4}{90} = 1.0823232337\ldots$.)

## 4.4. Other Convergence Tests

**4.4.1. Alternating Series.** An *alternating series* is a series whose terms are alternately positive and negative., for instance

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}.$$

4.4.1.1. *The Alternating Series Test.* If the sequence of positive terms $b_n$ verifies

(1) $b_n$ is decreasing.
(2) $\lim_{n\to\infty} b_n = 0$

then the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n = b_1 - b_2 + b_3 - b_4 + \cdots$$

converges.

*Example*: The alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

converges because $1/n \to 0$. (Its sum is $\ln 2 = 0.6931471806\ldots$.)

4.4.1.2. *Alternating Series Estimation Theorem.* If $s = \sum_{n=1}^{\infty} (-1)^n b_n$ is the sum of and alternating series verifying that $b_n$ is decreasing and $b_n \to 0$, then the remainder of the series verifies:

$$|R_n| = |s - s_n| \le b_{n+1}.$$

**4.4.2. Absolute Convergence.** A series $\sum_{n=1}^{\infty} a_n$ is called *absolutely convergent* if the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ converges.

Absolute convergence implies convergence, i.e., if a series $\sum a_n$ is absolutely convergent, then it is convergent.

The converse is not true in general. For instance, the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is convergent but it is not absolutely convergent.

*Example*: Determine whether the series

$$\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$$

is convergent or divergent.

*Answer*: We see that the series of absolute values $\sum_{n=1}^{\infty} \frac{|\cos n|}{n^2}$ is convergent by comparison with $\sum_{n=1}^{\infty} \frac{1}{n^2}$, hence the given series is absolutely convergent, therefore it is convergent (its sum turns out to be $1/4 - \pi/2 + \pi^2/6 = 0.324137741\ldots$, but the proof of this is beyond the scope of this notes).

### 4.4.3. The Ratio Test.

(1) If $\displaystyle\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$ then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(2) If $\displaystyle\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$ (including $L = \infty$) then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent.

(3) If $\displaystyle\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$ then the test is inconclusive (we do not know whether the series converges or diverges).

*Example*: Test the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{n!}{n^n}$$

for absolute convergence.

*Answer*: We have:

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{(n+1)!/(n+1)^{n+1}}{n!/n^n} = \frac{n^n}{(n+1)^n} = \frac{1}{\left(1 + \frac{1}{n}\right)^n} \xrightarrow[n\to\infty]{} e^{-1} < 1,$$

hence by the Ratio Test the series is absolutely convergent.