CHAPTER 4

Infinite Sequences and Series

4.1. Sequences

A *sequence* is an infinite ordered list of numbers, for example the sequence of odd positive integers:

$$1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29 \dots$$

Symbolically the *terms* of a sequence are represented with indexed letters:

$$a_1, a_2, a_3, a_4, a_5, a_6, a_7, \dots, a_n, \dots$$

Sometimes we start a sequence with a_0 (index zero) instead of a_1.

Notation: the sequence $a_1, a_2, a_3, \dots$ is also denoted by $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$.

Some sequences can be defined with a formula, for instance the sequence $1, 3, 5, 7, \dots$ of odd positive integers can be defined with the formula $a_n = 2n - 1$.

A *recursive definition* consists of defining the next term of a sequence as a function of previous terms. For instance the *Fibonacci sequence* starts with $f_1 = 1$, $f_2 = 1$, and then each subsequent term is the sum of the two previous ones: $f_n = f_{n-1} + f_{n-2}$; hence the sequence is:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \dots$$

4.1.1. Limits. The limit of a sequence is the value to which its terms approach indefinitely as n becomes large. We write that the limit of a sequence a_n is L in the following way:

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \text{ as } n \to \infty\,.$$

For instance

$$\lim_{n\to\infty} \frac{1}{n} = 0\,,$$

$$\lim_{n\to\infty} \frac{n+1}{n} = 1\,,$$

etc.

If a sequence has a (finite) limit the it is said to be *convergent*, otherwise it is *divergent*.

If the sequence becomes arbitrarily large then we write

$$\lim_{n\to\infty} a_n = \infty\,.$$

For instance

$$\lim_{n\to\infty} n^2 = \infty\,.$$

4.1.2. Theorem. Let f be a function defined in $[1,\infty]$. If $\lim_{x\to\infty} f(x) = L$ and $a_n = f(n)$ for integer $n \geq 1$ then $\lim_{n\to\infty} a_n = L$ (i.e., we can replace the limit of a sequence with that of a function.)

Example: Find $\displaystyle\lim_{n\to\infty} \frac{\ln n}{n}$.

Answer: According to the theorem that limit equals $\displaystyle\lim_{x\to\infty} \frac{\ln x}{x}$, where x represents a real (rather than integer) variable. But now we can use L'Hôpital's Rule:

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{(\ln x)'}{(x)'} = \lim_{x\to\infty} \frac{1/x}{1} = 0\,,$$

hence

$$\boxed{\lim_{n\to\infty} \frac{\ln n}{n} = 0}\,.$$

Example: Find $\displaystyle\lim_{n\to\infty} r^n$ $(r > 0)$.

Answer: This limit is the same as that of the exponential function r^x, hence

$$\boxed{\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } 0 < r < 1 \\ 1 & \text{if } r = 1 \\ \infty & \text{if } r > 1 \end{cases}}$$

4.1.3. Operations with Limits. If $a_n \to a$ and $b_n \to b$ then:

$(a_n + b_n) \to a + b$.

$(a_n - b_n) \to a - b$.

$ca_n \to ca$ for any constant c.

$a_n b_n \to ab$.

$\dfrac{a_n}{b_n} \to \dfrac{a}{b}$ if $b \neq 0$.

$(a_n)^p \to a^p$ if $p > 0$ and $a_n > 0$ for every n.

Example: Find $\displaystyle\lim_{n\to\infty} \frac{n^2 + n + 1}{2n^2 + 3}$.

Answer: We divide by n^2 on top and bottom and operate with limits inside the expression:

$$\lim_{n\to\infty} \frac{n^2 + n + 1}{2n^2 + 3} = \lim_{n\to\infty} \frac{1 + \frac{1}{n} + \frac{1}{n^2}}{2 + \frac{3}{n^2}} = \frac{1 + 0 + 0}{2 + 0} = \boxed{\frac{1}{2}}.$$

4.1.4. Squeeze Theorem. If $a_n \le b_n \le c_n$ for every $n \ge n_0$ and $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\displaystyle\lim_{n\to\infty} b_n = L$.

Consequence: If $\displaystyle\lim_{n\to\infty} |a_n| = 0$ then $\displaystyle\lim_{n\to\infty} a_n = 0$.

Example: Find $\displaystyle\lim_{n\to\infty} \frac{\cos n}{n}$.

Answer: We have $-\dfrac{1}{n} \le \dfrac{\cos n}{n} \le \dfrac{1}{n}$, and $\dfrac{1}{n} \to 0$ as $n \to \infty$, hence by the squeeze theorem

$$\boxed{\lim_{n\to\infty} \frac{\cos n}{n} = 0}.$$

4.1.5. Other definitions.

4.1.5.1. *Increasing, Decreasing, Monotonic.* A sequence is *increasing* if $a_{n+1} > a_n$ for every n. It is *decreasing* if $a_{n+1} < a_n$ for every n. It is called *monotonic* if it is either increasing or decreasing.

Example: Prove that the sequence $a_n = \dfrac{n+1}{n}$ is decreasing.

Answer: $a_{n+1} - a_n = \dfrac{n+2}{n+1} - \dfrac{n+1}{n} = \dfrac{-1}{n(n+1)} < 0$, hence $a_{n+1} < a_n$ for all positive n.

4.1.5.2. *Bounded.* A sequence is *bounded above* if there is a number M such that $a_n \le M$ for all n. It is *bounded below* if there is a number m such that $m \le a_n$ for all n. It is called just *bounded* if it is bounded above and below.

Example: Prove that the sequence $a_n = \dfrac{n+1}{n}$ is bounded.

Answer: It is in fact bounded below because all its terms are positive: $a_n > 0$. To prove that it is bounded above note that

$$a_n = \frac{n+1}{n} = 1 + \frac{1}{n} \le 2\,.$$

since $1/n \le 1$ for all positive integer n.

4.1.6. Monotonic Sequence Theorem. Every bounded monotonic sequence is convergent.

For instance, we proved that $a_n = \dfrac{n+1}{n}$ is bounded and monotonic, so it must be convergent (in fact $\frac{n+1}{n} \to 1$ as $n \to \infty$).

Next example shows that sometimes in order to find a limit you may need to make sure that the limits exists first.

Example: Prove that the following sequence has a limit. Find it:

$$\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \dots$$

Answer: The sequence can be defined recursively as $a_1 = \sqrt{2}$, $a_{n+1} = \sqrt{2 + a_n}$ for $n \ge 1$. First we will prove by induction that $0 < a_n < 2$, so the sequence is bounded.

We start (base of induction) by noticing that $0 < a_1 = \sqrt{2} < 2$. Next the induction step. Assume (induction hypothesis) that for a given value of n it is true that $0 < a_n < 2$. From here we must prove that the same is true for the next value of n, i.e. that $0 < a_{n+1} < 2$. In fact $(a_{n+1})^2 = 2 + (a_n) < 2 + 2 = 4$, hence $0 < a_{n+1} < \sqrt{4} = 2$, q.e.d. So by the induction principle all terms of the sequence verify that $0 < a_n < 2$.

Now we prove that a_n is increasing:

$$(a_{n+1})^2 = 2 + a_n > a_n + a_n = 2a_n > a_n \cdot a_n = (a_n)^2 \,,$$

hence $a_{n+1} > a_n$.

Finally, since the given sequence is bounded and increasing, by the monotonic sequence theorem it has a limit L. We can find it by taking limits in the recursive relation:

$$a_{n+1} = \sqrt{2 + a_n} \,.$$

Since $a_n \to L$ and $a_{n+1} \to L$ we have:

$$L = \sqrt{2 + L} \quad \Rightarrow \quad L^2 = 2 + L \quad \Rightarrow L^2 - L - 2 = 0 \,.$$

That equation has two solutions, -1 and 2, but since the sequence is positive the limit cannot be negative, hence $L = 2$.

Note that the trick works only when we know for sure that the limit exists. For instance if we try to use the same trick with the Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, \ldots$ ($f_1 = 1$, $f_2 = 1$, $f_n = f_{n-1} + f_{n-2}$), calling L the "limit" we get from the recursive relation that $L = L + L$, hence $L = 0$, so we "deduce" $\lim_{n\to\infty} f_n = 0$. But this is wrong, in fact the Fibonacci sequence is divergent.

4.2. Series

A *series* is an infinite sum:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

In order to define the value of this sum we start be defining its sequence of *partial sums*

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n \,.$$

Then, if $\lim_{n\to\infty} s_n = s$ exists the series is called *convergent* and its sum is that limit:

$$\sum_{n=1}^{\infty} a_n = s = \lim_{n\to\infty} s_n \,.$$

Otherwise the series is called *divergent*.

For instance, consider the following series:

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} + \cdots = \sum_{n=1}^{\infty} \frac{1}{2^n} \,.$$

Its partial sums are:

$$s_n = \sum_{i=1}^{n} \frac{1}{2^i} = \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n} \,.$$

Hence its sum is

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2^i} = \lim_{n\to\infty} \left(1 - \frac{1}{2^n}\right) = 1 + 0 = 1 \,.$$

4.2.1. Geometric Series. A series verifying $a_{n+1} = r a_n$, where r is a constant, is called *geometric series*. If the first term is $a \neq 0$ then the series is

$$a + ar + ar^2 + \cdots + ar^n + \cdots = \sum_{n=0}^{\infty} ar^n \,.$$

The partial sums are now:

$$s_n = \sum_{i=0}^{n} ar^i \,.$$

The nth partial sum can be found in the following way:

$$\begin{aligned} s_n &= a + ar + ar^2 + \cdots + ar^n \\ rs_n &= \qquad ar + ar^2 + \cdots + ar^n + ar^{n+1} \end{aligned}$$

hence

$$s_n - rs_n = a + \ 0\ +\ 0\ + \cdots +\ 0\ - ar^{n+1}\,,$$

so:

$$s_n = \frac{a(1-r^{n+1})}{1-r}\,.$$

If $|r| < 1$ we can rewrite the result like this:

$$s_n = \frac{a}{1-r} - \frac{a}{1-r} r^{n+1}\,,$$

and then get the limit as $n \to \infty$:

$$s = \lim_{n\to\infty} s_n = \frac{a}{1-r} - \frac{a}{1-r} \underbrace{\lim_{n\to\infty} r^{n+1}}_{\substack{\downarrow \\ 0}} = \frac{a}{1-r}$$

So for $|r| < 1$ the series is convergent and

$$\boxed{\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}}\,.$$

For $|r| \geq 1$ the series is divergent.

4.2.2. Telescopic Series. A telescopic series is a series whose terms can be rewritten so that most of them cancel out.

Example: Find $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

Answer: Note that $\displaystyle\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$. So the nth partial sum is

$$\begin{aligned} s_n &= \sum_{i=1}^{\infty} \left(\frac{1}{i} - \frac{1}{i+1} \right) \\ &= \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{n} - \frac{1}{n+1} \\ &= 1 - \frac{1}{n+1}\,. \end{aligned}$$

Hence, the sum of the series is

$$s = \lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = \boxed{1}.$$

4.2.3. Theorem. If the series $\sum_{n=0}^{\infty} a_n$ is convergent then $\lim_{n\to\infty} a_n = 0$.

Proof: If the series is convergent then the sequence of partial sums $s_n = \sum_{i=1}^{n} a_i$ have a limit s. On the other hand $a_n = s_n - s_{n-1}$, so taking limits we get $\lim_{n\to\infty} a_n = s - s = 0$.

The converse is not true in general. The harmonic series provides a counterexample.

4.2.4. The Harmonic Series. The following series is called *harmonic series*:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

The main fact about it is that it is *divergent*. In order to prove it we find

$$\begin{aligned}
s_1 &= 1\\
s_2 &= 1 + \tfrac{1}{2}\\
s_4 &= 1 + \tfrac{1}{2} + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) > 1 + \tfrac{1}{2} + \left(\tfrac{1}{4} + \tfrac{1}{4}\right) = 1 + \tfrac{1}{2} + \tfrac{1}{2} = 1 + \tfrac{2}{2}\\
s_8 &= 1 + \tfrac{1}{2} + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) + \left(\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8}\right)\\
&> 1 + \tfrac{1}{2} + \left(\tfrac{1}{4} + \tfrac{1}{4}\right) + \left(\tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8}\right) = 1 + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} = 1 + \tfrac{3}{2}
\end{aligned}$$

etc., so in general $s_{2^n} > 1 + \frac{n}{2}$, hence the sequence of partial sums grows without limit and the series diverges.

4.2.5. Test for Divergence. If $\lim\limits_{n\to\infty} a_n$ does not exist or if $\lim\limits_{n\to\infty} a_n \neq 0$ then $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

Example: Show that $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1}$ diverges.

Answer: We have $\displaystyle\lim_{n\to\infty} \frac{n}{n+1} = 1$. Since the nth term of the series does not tend to 0, the series diverges.

Example: Show that $\displaystyle\sum_{n=1}^{\infty} \sin n$ diverges.

Answer: All we need to show is that $\sin n$ does not tend to 0. If for some value of n, $\sin n \approx 0$, then $n \approx k\pi$ for some integer k, but then

$$\begin{aligned}\sin(n+1) &= \sin n \cos 1 + \cos n \sin 1 \\ &\approx \sin k\pi \cos 1 + \cos k\pi \sin 1 \\ &= 0 \pm \sin 1 \\ &= \pm 0.84 \cdots \neq 0\end{aligned}$$

So if a term $\sin n$ is close to zero, the next term $\sin(n+1)$ will be far from zero, so it is impossible for $\sin n$ to get permanently closer and closer to 0.

4.2.6. Operations with Series. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series and c is a constant then the following series are also convergent and:

(1) $\displaystyle\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$

(2) $\displaystyle\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

(3) $\displaystyle\sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$