7.2. Representations of Graphs

7.2.1. Adjacency matrix. The *adjacency matrix* of a graph is a matrix with rows and columns labeled by the vertices and such that its entry in row i, column j, $i \neq j$, is the number of edges incident on i and j. For instance the following is the adjacency matrix of the graph of figure 7.8:[1]

$$\begin{array}{cc} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

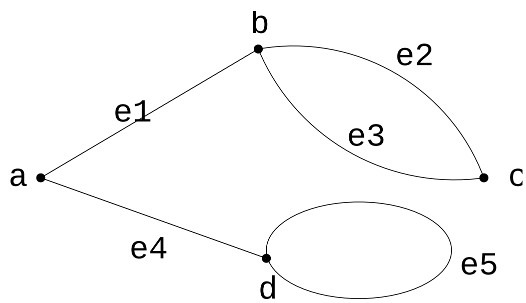

FIGURE 7.8

One of the uses of the adjacency matrix A of a simple graph G is to compute the number of paths between two vertices, namely entry (i, j) of A^n is the number of paths of length n from i to j.

7.2.2. Incidence matrix. The incidence matrix of a graph G is a matrix with rows labeled by vertices and columns labeled by edges, so that entry for row v column e is 1 if e is incident on v, and 0 otherwise. As an example, the following is the incidence matrix of graph of figure 7.8:

[1]For some authors if $i = j$ then the entry is twice the number of loops incident on i; so in the example of figure 7.8 entry (d, d) would be 2 instead of 1.

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

7.2.3. Graph Isomorphism. Two graphs $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$, are called *isomorphic* if there is a bijection $f : V_1 \to V_2$ and a bijection $g : E_1 \to E_2$ such that an edge e is adjacent to vertices v and w if and only if $g(e)$ is adjacent to $f(v)$ and $f(w)$ (fig. 7.9).

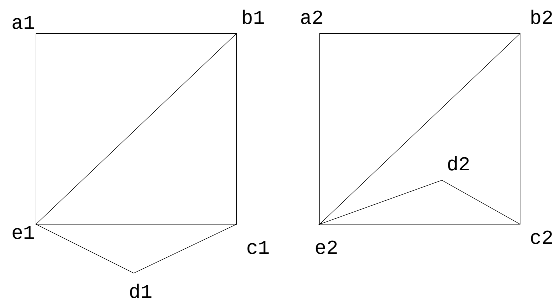

FIGURE 7.9. Two isomorphic graphs.

Two graphs are isomorphic if and only if for some ordering of their vertices their adjacency matrices are equal.

An *invariant* is a property such that if a graph has it then all graphs isomorphic to it also have it. Examples of invariants are their number of vertices, their number of edges, "has a vertex of degree k", "has a simple cycle of length l", etc. It is possible to prove that two graphs are not isomorphic by showing an invariant property that one has and the other one does not have. For instance the graphs in figure 7.10 cannot be isomorphic because one has a vertex of degree 2 and the other one doesn't.

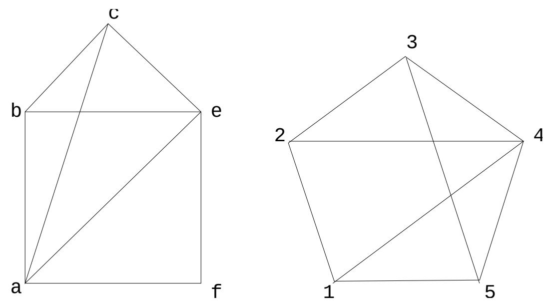

FIGURE 7.10. Non isomorphic graphs.