PUTNAM TRAINING
PROBLEMS AND SOLUTIONS

October 31st, 2024

Putnam 2021-A1. A grasshopper starts at the origin in the coordinate plane and makes a sequence of hops. Each hop has length 5, and after each hop the grasshopper is at a point whose coordinates are both integers; thus, there are 12 possible locations for the grasshopper after the first hop. What is the smallest number of hops needed for the grasshopper to reach the point $(2021, 2021)$?

Solution. The answer is 578.

Each hop corresponds to adding one of the 12 vectors $(0, \pm 5)$, $(\pm 5, 0)$, $(\pm 3, \pm 4)$, $(\pm 4, \pm 3)$ to the position of the grasshopper. Since $(2021, 2021) = 288(3, 4) + 288(4, 3) + (0, 5) + (5, 0)$, the grasshopper can reach $(2021, 2021)$ in $288 + 288 + 1 + 1 = 578$ hops.

On the other hand, let $z = x+y$ denote the sum of the x and y coordinates of the grasshopper, so that it starts at $z = 0$ and ends at $z = 4042$. Each hop changes the sum of the x and y coordinates of the grasshopper by at most 7, and $4042 > 577 \times 7$; it follows immediately that the grasshopper must take more than 577 hops to get from $(0, 0)$ to $(2021, 2021)$.

Remark. This solution implicitly uses the distance function

$$d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$$

on the plane, variously called the *taxicab metric*, the *Manhattan metric*, or the L^1-*norm* (or ℓ_1-norm).

Putnam 2021-A3. Determine all positive integers N for which the sphere

$$x^2 + y^2 + z^2 = N$$

has an inscribed regular tetrahedron whose vertices have integer coordinates.

Solution. The integers N with this property are those of the form $3m^2$ for some positive integer m.

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In one direction, for $N = 3m^2$, the points

$$(m, m, m), (m, -m, -m), (-m, m, -m), (-m, -m, m)$$

verify $m^2 + m^2 + m^2 = 3m^2 = N$, and each of them is at a distance $\sqrt{0 + 4m^2 + 4m^2} = m\sqrt{8}$ from each other, so they form the vertices of a regular tetrahedron inscribed in the sphere $x^2 + y^2 + z^2 = N$.

Conversely, suppose that a sphere centered at the origin has an octahedron inscribed at rational points. Let r be the radius of the sphere, and let $\pm u_1$, $\pm u_2$, $\pm u_3$ be the vectors to the vertices of the octahedron. Then $u_1 \times u_2 = \pm r u_3$. The cross-product and u_3 are nonzero and have rational coordinates, so r must be rational as well.

Now suppose that a sphere of radius $\sqrt{N}$ has a regular tetrahedron inscribed at points with integer coordinates. Let v_1, v_2, v_3, and v_4 be the vectors from the center of the sphere to the vertices of the tetrahedron. The points $\pm v_1$, $\pm v_2$, $\pm v_3$, $\pm v_4$ form a cube inscribed in the sphere. The centers of the faces of this cube form an octahedron. All vertices of this octahedron are averages of points with integer coordinates and therefore have rational coordinates. Thus the sphere that circumscribes this octahedron has a rational radius r. By the Pythagorean theorem, $N = 3r^2$.

It remains to prove that r must be an integer. In fact, if $r = a/b$, where a, b are positive integers without common factors, then for $N = 3r^2 = 3a^2/b^2$ to be an integer we would need to have either $b = 1$, or $b^2 = 3$. The latest is impossible because 3 is not a perfect square, hence $b = 1$, and $r = a$, an integer.

Putnam 2021-B2. Determine the maximum value of the sum

$$S = \sum_{n=1}^{\infty} \frac{n}{2^n} (a_1 a_2 \cdots a_n)^{1/n}$$

over all sequences $a_1, a_2, a_3, \ldots$ of nonnegative real numbers satisfying

$$\sum_{k=1}^{\infty} a_k = 1.$$

Solution. The answer is $\frac{2}{3}$.

We have

$$\begin{aligned}
2^{n+1}(a_1 \cdots a_n)^{1/n} &= \left(4^{n(n+1)/2}\, a_1 \cdots a_n\right)^{1/n} \\
&= \left(4^{1+2+\cdots+n}\, a_1 \cdots a_n\right)^{1/n} \\
&= \big((4a_1)(4^2 a_2) \cdots (4^n a_n)\big)^{1/n} \\
&\leq \frac{\sum_{k=1}^{n} (4^k a_k)}{n}. \qquad \text{(AM-GM inequality)}
\end{aligned}$$

Hence

$$
\begin{aligned}
2S &= \sum_{n=1}^{\infty} \frac{2n}{2^n}(a_1a_2\cdots a_n)^{1/n} = \sum_{n=1}^{\infty} \frac{n}{4^n}\, \underbrace{2^{n+1}(a_1a_2\cdots a_n)^{1/n}}_{\leq \frac{\sum_{k=1}^{n}(4^k a_k)}{n} \text{ by } (*)} \\
&\leq \sum_{n=1}^{\infty} \frac{\sum_{k=1}^{n}(4^k a_k)}{4^n} = \sum_{n=1}^{\infty}\sum_{k=1}^{n}(4^{k-n}a_k) = \sum_{\substack{n,k\geq 1\\ n\geq k}} (4^{k-n}a_k) \\
&= \sum_{k=1}^{\infty}\sum_{n=k}^{\infty}(4^{\overset{\nearrow j}{k-n}}\, a_k) = \sum_{k=1}^{\infty}\sum_{j=0}^{\infty}(4^{-j}a_k) = \underbrace{\sum_{j=0}^{\infty} 4^{-j}}_{\frac{1}{1-1/4}=4/3}\ \underbrace{\sum_{k=1}^{\infty} a_k}_{1} = \frac{4}{3}\,,
\end{aligned}
$$

and $S \leq 2/3$. So, the minimum value of S cannot be larger than $2/3$. To prove that the minimum is fact $2/3$ we still have to show that $S = 2/3$ for some sequence. In fact, equality is achieved when $a_k = \frac{3}{4^k}$ for all k, since in this case $4a_1 = 4^2a_2 = \cdots = 4^n a_n$ for all n.

Putnam 2018-B2. Let n be a positive integer, and let $f_n(z) = n + (n-1)z + (n-2)z^2 + \cdots + z^{n-1}$. Prove that f_n has no roots in the closed unit disk $\{z \in \mathbb{C} \colon |z| \leq 1\}$.

Solution. First note that for $n = 1$ we have $f_1(z) = 1$, so f_1 has no roots at and we may look at the case $n \geq 2$ only. Then, note that $f_n(1) = n + (n-1) + \cdots + 1 > 0$, so 1 is not a root of f_n. Next, note that

$$
\begin{aligned}
(z-1)f_n(z) &= nz + (n-1)z^2 + (n-2)z^3 + \cdots + 2z^{n-1} + z^n \\
&\quad - n - (n-1)z - (n-2)z^2 - \cdots - z^{n-1} \\
&= z^n + \cdots + z - n\,.
\end{aligned}
$$

For $|z| \leq 1$, using the triangle inequality we have

$$
|z^n + \cdots + z| \leq |z^n| + \cdots + |z| = |z|^n + \cdots + |z| \leq \underbrace{1 + \cdots + 1}_{n \text{ terms}} = n\,,
$$

hence $|z^n + \cdots + z| \leq n$. Equality can only occur if $z, \ldots, z^n$ have norm 1 and the same argument, which implies $z = z^2 = \cdots = z^n$. This only happens for $z = 1$. Thus there can be no root of f_n with $|z| \leq 1$.

Putnam 2018-B3. Find all positive integers $n < 10^{100}$ for which simultaneously n divides 2^n, $n-1$ divides $2^n - 1$, and $n-2$ divides $2^n - 2$.

Solution. The values of n with this property are 2^{2^ℓ} for $\ell = 1, 2, 4, 8$.

First, note that n divides 2^n if and only if n is itself a power of 2; we may thus write $n = 2^m$ and note that if $n < 10^{100}$, then

$$2^m = n < 10^{100} < 10^{102} = (10^3)^{34} = 1000^{34} < 1024^{34} = (2^{10})^{34} = 2^{340}.$$

Moreover, the case $m = 0$ does not lead to a solution because for $n = 1$, $n - 1 = 0$ does not divide $2^n - 1 = 2^1 - 1 = 1$; hence we may thus assume $1 \leq m \leq 340$.

Next, using Euclidean division, write $n = am + b$ where a, b are nonnegative integers and $0 \leq b < m$. Then, we have

$$2^n - 1 = 2^{am+b} - 1 = 2^b(2^{ma} - 1) + 2^b - 1\,.$$

Since $2^{ma} - 1$ is divisible by $2^m - 1 = n - 1$ we have $2^n - 1 \equiv 2^b - 1 \pmod{n-1}$. We have $0 \leq b < m < 2^m = n$, hence $2^b - 1 \equiv 0 \pmod{n-1}$ precisely for $b = 0$, which implies $n = am$, i.e., m divides n. Since n is a power of 2 this happens if and only if m is a power of 2.

Next, write $m = 2^\ell$ and note that $2^\ell < 340 < 512$, so $\ell < 9$. The case $\ell = 0$ does not lead to a solution because for $n = 2$, $n - 2 = 0$ does not divide $2^n - 2 = 2$; we may thus assume $1 \leq \ell \leq 8$.

Finally, note that $n-2 = 2^m-2$ divides 2^n-2 if and only if $2^{m-1}-1$ divides $2^{n-1}-1$. By the same logic as the previous paragraph, this happens if and only if $m-1$ divides $n-1$, that is, if $2^\ell - 1$ divides $2^m - 1$. This in turn happens if and only if ℓ divides $m = 2^\ell$, which happens if and only if ℓ is a power of 2. The values allowed by the bound $\ell < 9$ are $\ell = 1, 2, 4, 8$; for these values, $m \leq 2^8 = 256$ and

$$n = 2^m \leq 2^{256} \leq (2^3)^{86} < 10^{86} < 10^{100},$$

so the solutions listed do satisfy the original inequality.

Arithmetic Mean-Geometric Mean (AM-GM) Inequality

The geometric mean of positive numbers is not greater than their arithmetic mean, i.e., if $x_1, x_2, \ldots, x_n > 0$*, then*

$$\left(\prod_{i=1}^{n} x_i\right)^{1/n} \leq \frac{1}{n} \sum_{i=1}^{n} x_i \, .$$

Equality happens only for $x_1 = \cdots = x_n$.