2.4. Average Value of a Function (Mean Value Theorem)

2.4.1. Average Value of a Function. The average value of finitely many numbers $y_1, y_2, \dots, y_n$ is defined as

$$y_{\mathrm{ave}} = \frac{y_1 + y_2 + \cdots + y_n}{n}\,.$$

The average value has the property that if each of the numbers $y_1, y_2, \dots, y_n$ is replaced by y_{ave}, their sum remains the same:

$$y_1 + y_2 + \cdots + y_n = \overbrace{y_{\mathrm{ave}} + y_{\mathrm{ave}} + \cdots + y_{\mathrm{ave}}}^{(n \text{ times})}$$

Analogously, the average value of a function $y = f(x)$ in the interval $[a, b]$ can be defined as the value of a constant f_{ave} whose integral over $[a, b]$ equals the integral of $f(x)$:

$$\int_a^b f(x)\,dx = \int_a^b f_{\mathrm{ave}}\,dx = (b - a)\,f_{\mathrm{ave}}\,.$$

Hence:

$$\boxed{f_{\mathrm{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx}\,.$$

2.4.2. The Mean Value Theorem for Integrals. If f is continuous on $[a, b]$, then there exists a number c in $[a, b]$ such that

$$f(c) = f_{\mathrm{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx\,,$$

i.e.,

$$\int_a^b f(x)\,dx = f(c)(b - a)\,.$$

Example: Assume that in a certain city the temperature (in °F) t hours after 9 A.M. is represented by the function

$$T(t) = 50 + 14\sin\frac{\pi t}{12}\,.$$

Find the average temperature in that city during the period from 9 A.M. to 9 P.M.

Answer:

$$\begin{aligned}
T_{\text{ave}} &= \frac{1}{12-0}\int_0^{12}\left(50+14\sin\frac{\pi t}{12}\right)\,dt \\
&= \frac{1}{12}\left[50t-\frac{14\cdot 12}{\pi}\cos\frac{\pi t}{12}\right]_0^{12} \\
&= \frac{1}{12}\left\{\left(50\cdot 12-\frac{168}{\pi}\cos\frac{12\pi}{12}\right)-\left(50\cdot 0-\frac{168}{\pi}\cos 0\right)\right\} \\
&= 50+\frac{28}{\pi}\approx 58.9\,.
\end{aligned}$$