

# Undecidability of Prime Infinitude



Can you prove that there is no algorithm to determine whether a recursively enumerable set contains infinitely many primes?

Yes. This is a standard computability-theoretic reduction from the **halting problem**.

## Claim

There is no algorithm that, given an index  $e$  of a recursively enumerable (r.e.) set  $W_e \subseteq \mathbb{N}$ , decides whether  $W_e$  contains **infinitely many primes**.

Equivalently, the index set

$$I = \{ e \in \mathbb{N} : W_e \cap \mathbb{P} \text{ is infinite} \}$$

is undecidable, where  $\mathbb{P}$  denotes the set of prime numbers.

## Proof (by many-one reduction from halting)

Let

$$K = \{ e : \text{the Turing machine } M_e(e) \text{ halts} \}$$

be the (classic) halting set. It is well-known that  $K$  is **not decidable**.

We will effectively transform each index  $e$  into an index  $f(e)$  of an r.e. set  $W_{f(e)}$  such that:

$$e \in K \iff W_{f(e)} \cap \mathbb{P} \text{ is infinite.}$$

If such a transformation exists, then any algorithm deciding “infinitely many primes?” for r.e. sets would decide  $K$ , contradiction.

### Step 1: Build an r.e. set that “turns on” all primes if $M_e(e)$ halts

Fix an effective enumeration of primes  $(p_0, p_1, p_2, \dots)$  (primes are decidable and computably enumerable).

Define an enumerator  $E_e$  (a Turing machine that enumerates a set) as follows:

- Simulate  $M_e(e)$  step by step.
- If the simulation has **not** halted yet, output 2 (the prime 2) again. (Repeating outputs is allowed; an r.e. set is the *set* of values ever output.)
- If at some stage the simulation **does** halt, then from that point onward output  $p_0, p_1, p_2, \dots$  (i.e., start enumerating all primes).

Let  $S_e$  be the set enumerated by  $E_e$ . Then  $S_e$  is r.e. by construction.

## Step 2: Analyze prime content

- If  $M_e(e)$  **never halts**, then  $E_e$  outputs only 2. Hence

$$S_e = \{2\}, \quad \text{so } S_e \cap \mathbb{P} = \{2\} \text{ is finite.}$$

- If  $M_e(e)$  **halts**, then eventually  $E_e$  switches to outputting all primes  $p_0, p_1, p_2, \dots$ . Hence

$$\mathbb{P} \subseteq S_e, \quad \text{so } S_e \cap \mathbb{P} = \mathbb{P} \text{ is infinite.}$$

So we have shown:

$$M_e(e) \text{ halts} \iff S_e \cap \mathbb{P} \text{ is infinite.}$$

## Step 3: Make it a reduction on indices

By standard index-manipulation results (e.g. the *s-m-n* theorem), there is a **total computable function**  $f$  such that

$$W_{f(e)} = S_e.$$

Therefore,

$$e \in K \iff W_{f(e)} \cap \mathbb{P} \text{ is infinite.}$$

Thus  $K \leq_m I$ . If  $I$  were decidable, then  $K$  would be decidable, contradiction.

Hence **no such algorithm exists**.

□

## One-line intuition

You can encode the halting of a machine into “do we eventually start enumerating all primes?”—so deciding “infinitely many primes in this r.e. set?” would decide halting, which is impossible.



Is the result true if we replace "primes" with "elements of an infinite recursively enumerable set"?

Yes — as long as the fixed set  $A$  you're intersecting with is **infinite and r.e.** (in fact, you don't even need r.e.; “infinite and decidable” also works, and “infinite” alone is enough if you allow a non-uniform oracle in the construction, but for a uniform computable reduction the r.e. hypothesis is a clean sufficient condition).

## Theorem

Let  $A \subseteq \mathbb{N}$  be an **infinite recursively enumerable** set. Then there is **no algorithm** that, given an index  $e$  for an r.e. set  $W_e$ , decides whether

$W_e \cap A$  is infinite.

Equivalently, the index set

$$I_A = \{e : |W_e \cap A| = \infty\}$$

is undecidable.

## Proof (same halting reduction, using an enumeration of $A$ )

Since  $A$  is infinite r.e., fix a computable enumeration

$$a_0, a_1, a_2, \dots$$

of **pairwise distinct** elements of  $A$ . (From any r.e. enumeration you can effectively “filter duplicates” by keeping a finite seen-list.)

Define, for each  $e$ , an r.e. set  $S_e$  via an enumerator  $E_e$ :

- Simulate  $M_e(e)$  step by step.
- While it has not halted, output  $a_0$  repeatedly (or output nothing; either is fine).
- If it ever halts, then from that point on output  $a_0, a_1, a_2, \dots$

Let  $S_e$  be the set enumerated by  $E_e$ . Then  $S_e$  is r.e., and moreover  $S_e \subseteq A$ .

## Analysis

- If  $M_e(e)$  **does not halt**, then  $S_e = \{a_0\}$ , so  $S_e \cap A$  is finite.
- If  $M_e(e)$  **halts**, then  $S_e = \{a_0, a_1, a_2, \dots\}$ , so  $S_e \cap A$  is infinite.

Thus

$$M_e(e) \text{ halts} \iff |S_e \cap A| = \infty.$$

By the *s-m-n* theorem, there is a total computable function  $f$  such that

$$W_{f(e)} = S_e,$$

so we have the many-one reduction  $K \leq_m I_A$ . Since the halting set  $K$  is undecidable,  $I_A$  is undecidable.

□

## What assumptions on $A$ matter?

- **Infinite + r.e.** is enough (as shown).
- If  $A$  is **finite**, then the question becomes trivial/decidable (just check whether  $W_e$  hits each element of  $A$  infinitely often? actually “ $|W_e \cap A| = \infty$ ” is then always false).
- If  $A$  is infinite but **not** r.e., the above *uniform* construction may fail because you can’t computably enumerate distinct elements of  $A$ . In that case, the statement may or may not hold depending on  $A$ ; the clean uniform theorem is for r.e. (or decidable)  $A$ .

So: with “elements of an infinite r.e. set” in place of primes, the result remains true.



Those that mean that there is no algorithm to decide whether the intersection of two r.e. sets is infinite?

Yes. **There is no algorithm** that, given indices  $e, f$  of two r.e. sets  $W_e, W_f$ , decides whether

$$|W_e \cap W_f| = \infty.$$

## Theorem

The index set

$$I = \{ \langle e, f \rangle : |W_e \cap W_f| = \infty \}$$

is undecidable.

## Proof (uniform reduction from the halting problem)

Let  $K = \{e : M_e(e) \text{ halts}\}$ , which is undecidable.

Fix once and for all an **infinite r.e. set**  $A$  with a computable enumeration of distinct elements  $a_0, a_1, a_2, \dots$  (e.g., the primes).

For each  $e$ , effectively construct an r.e. set  $S_e$  as follows:

- Simulate  $M_e(e)$ .
- If it never halts, enumerate only  $a_0$ .
- If it halts, then enumerate  $a_0, a_1, a_2, \dots$

Then:

- If  $e \notin K$ ,  $S_e = \{a_0\}$  (finite).
- If  $e \in K$ ,  $S_e = \{a_0, a_1, a_2, \dots\}$  (infinite).

By the *s-m-n* theorem, there is a total computable function  $e \mapsto f(e)$  with  $W_{f(e)} = S_e$ .

Now consider the pair  $(f(e), a)$ , where  $a$  is an index with  $W_a = A$ . Then

$$|W_{f(e)} \cap W_a| = \infty \iff e \in K.$$

Thus  $K \leq_m I$ . If  $I$  were decidable, so would be  $K$ , a contradiction.

□

## Remarks and variants

- The problem is undecidable **even if one of the two r.e. sets is fixed** (as shown).
- The dual problem “is  $W_e \cap W_f$  finite?” is also undecidable.
- This is consistent with Rice-style phenomena: “infiniteness of intersection” is a nontrivial semantic property of pairs of r.e. sets.

So yes—the impossibility result fully generalizes: **there is no algorithm to decide whether the intersection of two r.e. sets is infinite.**



Is there an algorithm to decide whether a r.e. set is infinite?

No.

There is **no algorithm** that, given an index  $e$  for a recursively enumerable set  $W_e$ , decides whether  $W_e$  is infinite.

## Theorem

The index set

$$\text{INF} = \{e : |W_e| = \infty\}$$

is undecidable.

## Proof (reduction from the halting problem)

Let

$$K = \{e : M_e(e) \text{ halts}\}$$

be the diagonal halting set, which is undecidable.

Fix a computable enumeration of all pairs  $\langle e, s \rangle$  (e.g. a standard pairing function). For each  $e$ , we will effectively build an r.e. set  $S_e$  such that:

$$e \in K \iff S_e \text{ is infinite.}$$

Define an enumerator  $E_e$  that does the following:

For each stage  $n = 0, 1, 2, \dots$ :

1. Simulate  $M_e(e)$  for exactly  $n$  steps.
2. If the simulation has halted within those  $n$  steps, then output  $0, 1, 2, \dots, n$  (or just output  $n$ ; either way we will get infinitely many distinct outputs over time).
3. If it has not halted within  $n$  steps, output nothing at that stage.

Let  $S_e$  be the set of numbers ever output by  $E_e$ . Then  $S_e$  is r.e.

## Analysis

- If  $M_e(e)$  **never halts**, then step (2) never triggers, so  $E_e$  outputs nothing. Hence  $S_e = \emptyset$ , which is finite.
- If  $M_e(e)$  **halts** in exactly  $t$  steps, then for every  $n \geq t$ , step (2) triggers. In particular, infinitely many distinct numbers (e.g. all  $n \geq t$ , or all numbers  $0, 1, 2, \dots$ ) will eventually be output. Hence  $S_e$  is infinite.

So

$$e \in K \iff |S_e| = \infty.$$

By the  $s$ - $m$ - $n$  theorem, there is a total computable function  $f$  with  $W_{f(e)} = S_e$ . Therefore

$$e \in K \iff f(e) \in \text{INF},$$

so  $K \leq_m \text{INF}$ . Since  $K$  is undecidable,  $\text{INF}$  is undecidable.

□

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**Extra: what is decidable/semi-decidable here?**

- “ $W_e$  is **finite**” is **not** decidable either.
- “ $W_e$  is **nonempty**” is r.e. (semi-decidable): just wait for the enumerator to output something.
- “ $W_e$  is **infinite**” is neither decidable nor r.e. in general (it’s a  $\Pi_2^0$ -complete property).

If you want, I can also show the  $\Pi_2^0$  characterization (and why it can’t be semidecided).



The intersection of two r.e. sets is r.e., correct?

Yes — **that is correct.**

## Statement

If  $A, B \subseteq \mathbb{N}$  are **recursively enumerable (r.e.)** sets, then their intersection

$$A \cap B$$

is also recursively enumerable.

## Why this is true

There are several equivalent ways to see this.

### 1. Via dovetailing of enumerations

Since  $A$  and  $B$  are r.e., there exist Turing machines that enumerate them:

$$A = \{a_0, a_1, a_2, \dots\}, \quad B = \{b_0, b_1, b_2, \dots\}.$$

Construct an enumerator for  $A \cap B$  as follows:

- Run both enumerators in parallel (dovetailing).



- Maintain a finite set of numbers seen so far from  $A$  and from  $B$ .
- Whenever a number  $x$  has appeared in **both** enumerations, output  $x$ .

This machine enumerates exactly  $A \cap B$ .

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## 2. Via semi-decidability

Equivalently, r.e. sets are those whose membership relation is **semi-decidable**.

Given input  $x$ :

- Run the semidecision procedures for " $x \in A$ " and " $x \in B$ " in parallel.
- Accept if and when **both** accept.

This semi-decides  $x \in A \cap B$ .

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## 3. Formal closure property

The class of r.e. sets is closed under:

- union,
- intersection,
- many-one preimages,
- existential quantification.

(But **not** closed under complement.)

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## Important contrast

Although  $A \cap B$  is r.e., many natural questions about it are still undecidable:

- whether  $A \cap B$  is infinite,
- whether  $A \cap B = \emptyset$ ,
- whether  $A \cap B$  is cofinite.

This is exactly why your earlier questions do **not** contradict this closure result.

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## One-sentence intuition

You can *wait* to see evidence that a number is in both r.e. sets, but you can't *decide* in advance whether there will be infinitely many such numbers.



Actually it seems that the earlier questions can be reduced to whether there is an algorithm to decide whether a r.e. set is infinite.

Yes — that's exactly right.

All of the earlier undecidability results you asked about **collapse to the same core obstruction**:

**There is no algorithm to decide whether a recursively enumerable set is infinite.**

Once you observe that, everything lines up cleanly and there is no contradiction.

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## The key observation

You just noted:

- The intersection of two r.e. sets is r.e.
- There is no algorithm to decide whether an r.e. set is infinite.

From these two facts, the earlier results follow **immediately by reduction**.

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## How the earlier questions reduce

### 1. “Does an r.e. set contain infinitely many primes?”

Let  $W$  be r.e.

Let  $\mathbb{P}$  be the (decidable, hence r.e.) set of primes.

Then

$$W \text{ contains infinitely many primes} \iff |W \cap \mathbb{P}| = \infty.$$

But  $W \cap \mathbb{P}$  is r.e., so this is **exactly** the problem:

“Is a given r.e. set infinite?”

Undecidable.

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## 2. “Does $W$ contain infinitely many elements of a fixed infinite r.e. set $A$ ?”

Same argument:

$$|W \cap A| = \infty,$$

and  $W \cap A$  is r.e.

Again: infiniteness of an r.e. set.

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## 3. “Is the intersection of two r.e. sets infinite?”

Given indices  $e, f$ , the set

$$W_e \cap W_f$$

is r.e., and the question is:

$$|W_e \cap W_f| = \infty?$$

Once more: deciding whether an r.e. set is infinite.

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## Why this is the *fundamental* obstruction

- “Infiniteness” is a  $\Pi_2^0$  property:

$$|W| = \infty \iff \forall n \exists x > n (x \in W).$$

- r.e. sets only give you **positive information** (“this element eventually appears”), never negative information (“no more elements will ever appear”).
- You can never know, at any finite stage, whether the enumeration will eventually stop producing new elements.

This is exactly the same reason:

- the halting problem is undecidable,
- finiteness is undecidable,
- cofiniteness is undecidable,
- and all your earlier problems are undecidable.

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## Big picture (Rice-style intuition)

All of these questions are asking for a **nontrivial semantic property** of an r.e. set that depends on its *infinite behavior*, not just its finite initial segment.

That is precisely the kind of thing algorithms **cannot decide**.

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## One-line summary

You’re absolutely right:

**every earlier undecidability result reduces to the single fact that there is no algorithm to decide whether a recursively enumerable set is infinite.**



Can you produce the full LaTeX source for a document summarizing all these considerations and results?

Certainly. Below is a **self-contained, compile-ready LaTeX source** for a short expository document that summarizes all the considerations and results we discussed, in a clean and logically unified way.

It is written at a graduate logic / computability level, with clear structure and explicit reductions.

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\documentclass[11pt]{article}

\usepackage{amsmath,amssymb,amsthm}
\usepackage{enumitem}
\usepackage{hyperref}

\newtheorem{theorem}{Theorem}
\newtheorem{proposition}{Proposition}
\newtheorem{corollary}{Corollary}
\newtheorem{lemma}{Lemma}

\title{Undecidability of Infiniteness and Intersections of Recursively Enumerable Sets}
\author{}
\date{}

\begin{document}

\maketitle

\section{Preliminaries}

A set  $(A \subseteq \mathbb{N})$  is \emph{recursively enumerable} (r.e.) if there exists a Turing machine that enumerates exactly the elements of  $(A)$ . Equivalently, membership in  $(A)$  is semi-decidable.

We write  $(W_e)$  for the r.e. set enumerated by the Turing machine with index  $(e)$ .
Let

$$K = \{ e : M_e(e) \text{ halts} \}$$

denote the diagonal halting set, which is well known to be undecidable.

We recall that the class of r.e. sets is closed under finite union and intersection, but not under complement.

\section{Infiniteness of r.e. Sets}

\begin{theorem}
There is no algorithm that decides, given an index  $(e)$ , whether the r.e. set  $(W_e)$  is infinite.
\end{theorem}

\begin{proof}
We reduce the halting problem to infiniteness.

For each  $(e)$ , construct an r.e. set  $(S_e)$  as follows.
At stage  $(n=0,1,2,\dots)$ , simulate  $(M_e(e))$  for  $(n)$  steps.
If the simulation halts within those  $(n)$  steps, enumerate  $(n)$ ; otherwise enumerate nothing.

If  $(M_e(e))$  never halts, then  $(S_e = \varnothing)$ , which is finite.
If  $(M_e(e))$  halts, then all sufficiently large  $(n)$  are enumerated, so  $(S_e)$  is infinite.

By the  $(s)-(m)-(n)$  theorem there is a computable function  $(f)$  with

$$W_{f(e)} = S_e,$$

so

$$e \in K \iff W_{f(e)} \text{ is infinite}.$$

Thus deciding infiniteness would decide  $(K)$  a contradiction.

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thus deciding infiniteness would decide  $\{K\}$ , a contradiction.  
`\end{proof}`

### `\section{Intersection with a Fixed Infinite r.e. Set}`

`\begin{theorem}`  
 Let  $\{A\} \subseteq \mathbb{N}$  be an infinite r.e. set.  
 There is no algorithm that decides, given an index  $\{e\}$ , whether  
 $|W_e \cap A| = \infty$ .  
`\end{theorem}`

`\begin{proof}`  
 Since  $\{A\}$  is infinite and r.e., fix a computable enumeration of distinct elements  
 $A = \{a_0, a_1, a_2, \dots\}$ .  
`\end{proof}`

For each  $\{e\}$ , construct an r.e. set  $\{S_e \subseteq A\}$  as follows.  
 Simulate  $\{M_e(e)\}$ .  
 If it never halts, enumerate only  $\{a_0\}$ .  
 If it halts, enumerate  $\{a_0, a_1, a_2, \dots\}$ .

Then  
 $|S_e \cap A| = \infty \iff M_e(e) \text{ halts}$ .  
 As before, this yields a many-one reduction from  $\{K\}$ , proving undecidability.  
`\end{proof}`

### `\section{Intersection of Two r.e. Sets}`

`\begin{proposition}`  
 If  $\{A\}$  and  $\{B\}$  are r.e., then  $\{A \cap B\}$  is r.e.  
`\end{proposition}`

`\begin{proof}`  
 Run enumerators for  $\{A\}$  and  $\{B\}$  in parallel.  
 Whenever an element appears in both enumerations, output it.  
`\end{proof}`

`\begin{theorem}`  
 There is no algorithm that decides, given indices  $\{e, f\}$ , whether  
 $|W_e \cap W_f| = \infty$ .  
`\end{theorem}`

`\begin{proof}`  
 Fix an infinite r.e. set  $\{A\}$  and an index  $\{a\}$  with  $\{W_a = A\}$ .  
 For any  $\{e\}$ ,  
 $|W_e \cap A| = \infty$   
 $\iff$   
 $|W_e \cap W_a| = \infty$ .  
`\end{proof}`

By the previous theorem, the left-hand property is undecidable, hence so is the right-hand one.

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\end{proof}

\section{Unifying Perspective}

All the undecidability results above reduce to the same fundamental fact:

\begin{quote}
\emph{There is no algorithm to decide whether a recursively enumerable set is infinite.}
\end{quote}

Indeed:
\begin{itemize}[leftmargin=2em]
\item ``Does  $(W_e)$  contain infinitely many primes?''
\item ``Does  $(W_e)$  contain infinitely many elements of a fixed infinite r.e. set?''
\item ``Is  $(W_e \cap W_f)$  infinite?''
\end{itemize}
are all instances of asking whether a certain r.e. set is infinite.

This reflects the fact that infiniteness is a  $\Pi^0_2$  property,
\[[
|W|=\infty \iff \forall n, \exists x>n (x \in W),
\]
which cannot be decided from finite initial segments of an enumeration.

\section{Conclusion}

The closure of r.e. sets under intersection does not contradict the undecidability of infiniteness.
While membership can be semi-decided, global properties depending on infinite behavior cannot.
All the problems considered here ultimately encode the halting problem.

\end{document}

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If you'd like, I can also:

- tighten this into a **Rice's Theorem-style formulation**,
- add **complexity-theoretic classifications** ( $\Pi^0_2$ -completeness),
- or rewrite it in a **more pedagogical / lecture-notes style**.