CHAPTER 9

Boolean Algebras

9.1. Combinatorial Circuits

9.1.1. Introduction. At their lowest level digital computers handle only binary signals, represented with the symbols 0 and 1. The most elementary circuits that combine those signals are called *gates*. Figure 9.1 shows three gates: OR, AND and NOT.

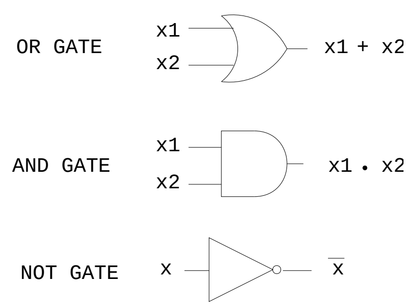

FIGURE 9.1. Gates.

Their outputs can be expressed as a function of their inputs by the following *logic tables*:

x_1	x_2	$x_1 + x_2$
1	1	1
1	0	1
0	1	1
0	0	0
OR GATE		

x_1	x_2	$x_1 \cdot x_2$
1	1	1
1	0	0
0	1	0
0	0	0
AND GATE		

x	$\overline{x}$
1	0
0	1
NOT GATE	

These are examples of *combinatorial circuits*. A combinatorial circuit is a circuit whose output is uniquely defined by its inputs. They do not have memory, previous inputs do not affect their outputs. Some combinations of gates can be used to make more complicated combinatorial circuits. For instance figure 9.2 is combinatorial circuit with the logic table shown below, representing the values of the *Boolean expression* $y = \overline{(x_1 + x_2) \cdot x_3}$.

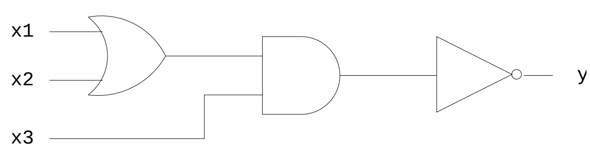

FIGURE 9.2. A combinatorial circuit.

x_1	x_2	x_3	$y = \overline{(x_1 + x_2) \cdot x_3}$
1	1	1	0
1	1	0	1
1	0	1	0
1	0	0	1
0	1	1	0
0	1	0	1
0	0	1	1
0	0	0	1

However the circuit in figure 9.3 is *not* a combinatorial circuit. If $x_1 = 1$ and $x_2 = 0$ then y can be 0 or 1. Assume that at a given time $y = 0$. If we input a signal $x_2 = 1$, the output becomes $y = 1$, and stays so even after x_2 goes back to its original value 0. That way we can store a bit. We can "delete" it by switching input x_1 to 0.

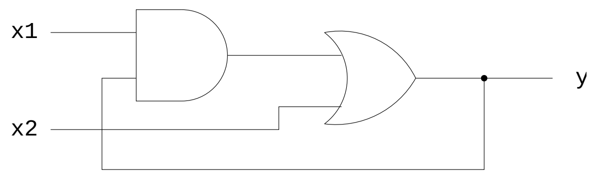

FIGURE 9.3. Not a combinatorial circuit.

9.1.2. Properties of Combinatorial Circuits. Here $\mathbb{Z}_2 = \{0, 1\}$ represents the set of signals handled by combinatorial circuits, and the operations performed on those signals by AND, OR and NOT gates are represented by the symbols $\cdot$, $+$ and $^{-}$ respectively. Then their properties are the following (a, b, c are elements of $\mathbb{Z}_2$, i.e., each represents either 0 or 1):

1. Associative
$$(a + b) + c = a + (b + c)$$
$$(a \cdot b) + c = a \cdot (b \cdot c)$$
2. Commutative
$$a + b = b + a$$
$$a \cdot b = b \cdot a$$
3. Distributive
$$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$$
$$a + (b \cdot c) = (a + b) \cdot (a + c)$$
4. Identity
$$a + 0 = a$$
$$a \cdot 1 = a$$
5. Complement
$$a + \overline{a} = 1$$
$$a \cdot \overline{a} = 0$$

A system satisfying those properties is called a *Boolean algebra*.

Two Boolean expressions are defined to be *equal* is they have the same values for all possible assignments of values to their literals. *Example*: $\overline{x + y} = \overline{x} \cdot \overline{y}$, as shown in the following table:

x	y	$\overline{x+y}$	$\overline{x}\cdot\overline{y}$
1	1	0	0
1	0	0	0
0	1	0	0
0	0	1	1

9.1.3. Abstract Boolean Algebras. Here we deal with general Boolean algebras; combinatorial circuits are an example, but there are others.

A Boolean algebra $B = (S, \vee, \wedge, \overline{\ }, 0, 1)$ is a set S containing two distinguished elements 0 and 1, two binary operators $\vee$ and $\wedge$ on S, and a unary operator $\overline{\ }$ on S, satisfying the following properties (x, y, z are elements of S):

1. Associative
$$(x \vee y) \vee z = x \vee (y \vee z)$$
$$(x \wedge y) \vee z = x \wedge (y \wedge z)$$
2. Commutative
$$x \vee y = y \vee x$$
$$x \wedge y = y \wedge x$$
3. Distributive
$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$
$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$
4. Identity
$$x \vee 0 = x$$
$$x \wedge 1 = x$$
5. Complement
$$x \vee \overline{x} = 1$$
$$x \wedge \overline{x} = 0$$

Example: $(\mathbb{Z}_2, +, \cdot, \overline{\ }, 0, 1)$ is a Boolean algebra.

Example: If U is a universal set and $\mathcal{P}(U)$= the power set of S (collection of subsets of S) then $(\mathcal{P}(U), \cup, \cap, \overline{\ }, \emptyset, U)$. is a Boolean algebra.

9.1.4. Other Properties of Boolean Algebras. The properties mentioned above define a Boolean algebra, but Boolean algebras also have other properties:

1. Idempotent
$$x \vee x = x$$
$$x \wedge x = x$$
2. Bound
$$x \vee 1 = 1$$
$$x \wedge 0 = 0$$
3. Absorption
$$x \vee xy = x$$
$$x \wedge (x \vee y) = x$$
4. Involution
$$\overline{\overline{x}} = x$$
5. 0 and 1
$$\overline{0} = 1$$
$$\overline{1} = 0$$
6. De Morgan's
$$\overline{x \vee y} = \overline{x} \wedge \overline{y}$$
$$\overline{x \wedge y} = \overline{x} \vee \overline{y}$$

For instance the first idempotent law can be proved like this: $x = x \vee 0 = x \vee x \wedge \overline{x} = (x \vee x) \wedge (x \vee \overline{x}) = (x \vee x) \wedge 1 = x \vee x$.