MATH 214-2 (41) - Fall 2001 - Second Midterm (solutions)

SOLUTIONS

1. Differentiate the following functions:

(a) $f(x) = \log_{10}(\log_{10} x)$.

(b) $f(x) = \pi^{x^3}$.

(c) $f(x) = \ln\left(\dfrac{\sqrt[3]{x}\, 2^{1+x^2}}{e^{\sqrt{x}}}\right)$ (simplify first).

Solution:

(a) $f'(x) = \dfrac{(\log_{10} x)'}{\ln 10\, \log_{10} x} = \dfrac{\frac{1}{x \ln 10}}{\ln 10\, \log_{10} x} = \boxed{\dfrac{1}{(\ln 10)^2\, x\, \log_{10} x}}$

Another solution, using $\log_{10} u = \ln u / \ln 10$:

$f(x) = \dfrac{\ln(\ln x/\ln 10)}{\ln 10} = \dfrac{1}{\ln 10}(\ln\ln x - \ln\ln 10) \quad \Rightarrow$

$$\boxed{f'(x) = \frac{1}{\ln 10}\left(\frac{1}{x\, \ln x}\right)}$$

(b) $f'(x) = \boxed{3x^2\, \pi^{x^3}\, \ln \pi}$

(c) $f(x) = \dfrac{1}{3}\ln x + (1+x^2)\ln 2 - \sqrt{x} \quad \Rightarrow \quad \boxed{f'(x) = \dfrac{1}{3x} + 2x\ln 2 - \dfrac{1}{2\sqrt{x}}}$

2. Find the following indefinite integral:

$$\int x\,10^{-x^2}\,dx =$$

Solution:

Substituting $u = -x^2$, $du = -2x\,dx$, we get:

$$\int x\,10^{-x^2}\,dx = -\frac{1}{2}\int 10^u\,du = -\frac{1}{2}\frac{10^u}{\ln 10} + C = \boxed{-\frac{10^{-x^2}}{2\ln 10} + C}$$

3. Zembla had a population of 1.5 million in 1990. Assume that this country's population is growing continuously at a 4% annual rate and that Zembla absorbs 50000 newcomers per year. What will its population be in the year 2010? (you do not need to perform all the computations, just simplify the answer).

Solution:

We measure time in years starting at $t = 0$ in the year 1990, so in 2010 we have $t = 20$.

- *Method 1.* Using the formula of population growth with immigration:

$$P(t) = P_0 e^{kt} + \frac{I}{k}(e^{kt} - 1)$$

with $P_0 = 1500000$, $k = 4/100$, $I = 50000$, $t = 20$, we get:

$$P(20) = 1500000\, e^{4/5} + 1250000\, (e^{4/5} - 1) = \boxed{2750000\, e^{4/5} - 1250000}$$

i.e., $4,870,238$ people.

- *Method 2.* We pose the differential equation of population growth with immigration with $k = 4/100$, $I = 50000$:

$$\frac{dP}{dt} = kP + I = \frac{1}{25}P + 50000\,,$$

and solve it by separation of variables:

$$\frac{dP}{P + 1250000} = \frac{1}{25}\,dt \quad \Rightarrow \quad \int \frac{dP}{P + 1250000} = \frac{1}{25}\int dt \quad \Rightarrow$$

$$\ln(P + 1250000) = \frac{t}{25} + C\,.$$

Using the initial condition $P(0) = 1500000$ for $t = 0$ we get $C = \ln(2750000)$, so:

$$P(t) = 2750000\, e^{t/25} - 1250000\,,$$

hence

$$\boxed{P(20) = 2750000\, e^{4/5} - 1250000}$$

4. Find the following integrals using appropriate inverse trigonometric functions:

(a) $\displaystyle\int \frac{x\,dx}{1+x^4}$.

(b) $\displaystyle\int \frac{dx}{\sqrt{1-4x^2}}$.

Solution:

(a) $u = x^2$, $du = 2x\,dx$;

$$\int \frac{x\,dx}{1+x^4} = \frac{1}{2}\int \frac{du}{1+u^2} = \frac{1}{2}\tan^{-1}u + C = \boxed{\frac{1}{2}\tan^{-1}x^2 + C}$$

(b) $u = 2x$, $du = 2\,dx$;

$$\int \frac{dx}{\sqrt{1-4x^2}} = \frac{1}{2}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{2}\sin^{-1}u = \boxed{\frac{1}{2}\sin^{-1}2x}$$

5. Find the following limit using l'Hôpital:

$$L = \lim_{x \to 0} \frac{e^x - 1 - x - x^2/2}{x^3}$$

Solution:

$$L = \lim_{x \to 0} \frac{e^x - 1 - x}{3x^2} = \lim_{x \to 0} \frac{e^x - 1}{6x} = \lim_{x \to 0} \frac{e^x}{6} = \boxed{\frac{1}{6}}$$