Alternating Series with Logarithm. Determine for which values of $p > 0$ the following series is absolutely convergent, conditionally convergent, and divergent:

$$S = \sum_{n=2}^{\infty} \log(1 + (-1)^n n^{-p}) .$$

Answer. The series is absolutely convergent for $p > 1$, conditionally convergent for $1/2 < p \leq 1$, and divergent for $0 < p \leq 1/2$.

- *Proof.* The second order Taylor expansion of $\log(1 + x)$ with Lagrange's residue is

$$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3(1 + \xi)^3} ,$$

where ξ is some real number between 0 and x. Hence

$$\begin{aligned} \log(1 + (-1)^n n^{-p}) &= (-1)^n n^{-p} - \frac{n^{-2p}}{2} + \frac{(-1)^n n^{-3p}}{3(1 + \xi_{p,n})^3} \\ &= (-1)^n n^{-p} - \frac{n^{-2p}}{2} \left(1 - \frac{(-1)^n n^{-p}}{3(1 + \xi_{p,n})^3}\right) , \end{aligned}$$

where $\xi_{p,n}$ is between 0 and $(-1)^n n^{-p}$. Now, the series S can be written as the term-wise sum of series $S = S_1 + (-S_2)$, where

$$\begin{aligned} S_1 &= \sum_{n=2}^{\infty} (-1)^n n^{-p} , \\ S_2 &= \sum_{n=2}^{\infty} \frac{n^{-2p}}{2} \left(1 - \frac{(-1)^n n^{-p}}{3(1 + \xi_{p,n})^3}\right) . \end{aligned}$$

Then, the result follows from the following lemmas.

Lemma 1. S_1 is absolutely convergent for $p > 1$, and conditionally convergent for $0 < p \leq 1$.

- *Proof.* Apply p-test and alternating series test respectively to S_1. $\square$

Lemma 2. S_2 is absolutely convergent for $p > 1/2$, and divergent for $0 < p \leq 1/2$.

- *Proof.* We have

$$\left| \frac{(-1)^n n^{-p}}{3(1 + \xi_{p,n})^3} \right| < \frac{n^{-p}}{3(1 - 2^{-p})^3} \underset{n\to\infty}{\longrightarrow} 0 ,$$

hence

$$\left(1 - \frac{(-1)^n n^{-p}}{3(1 + \xi_{p,n})^3}\right) \underset{n\to\infty}{\longrightarrow} 1 .$$

Then, by the comparison test, convergence/divergence of S_2 is the same as that of $\sum_{n=2}^{\infty} \frac{n^{-2p}}{2}$. Finally, applying the p-test the result follows. $\square$

So:

1. If $p > 1$ then S_1 and S_2 are absolutely convergent, hence, S is absolutely convergent.
2. If $1/2 < p \leq 1$ then S_1 is conditionally convergent and S_2 is absolutely convergent, hence S is conditionally convergent.
3. If $0 < p \leq 1/2$ then S_1 is (conditionally) convergent and S_2 is divergent, hence S is divergent.

This completes the proof.

$\square$

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