1.5. Integration by Parts

The method of integration by parts is based on the product rule for differentiation:

$$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)\,,$$

which we can write like this:

$$f(x)g'(x) = [f(x)g(x)]' - f'(x)g(x)\,.$$

Integrating we get:

$$\int f(x)\,g'(x)\,dx = \int [f(x)g(x)]'\,dx - \int g(x)f'(x)\,dx\,,$$

i.e.:

$$\boxed{\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx}\,.$$

Writing $u = f(x)$, $v = g(x)$, we have $du = f'(x)\,dx$, $dv = g'(x)\,dx$, hence:

$$\boxed{\int u\,dv = uv - \int v\,du}\,.$$

Example: Integrate $\int xe^x\,dx$ by parts.

Answer: In integration by parts the key thing is to choose u and dv correctly. In this case the "right" choice is $u = x$, $dv = e^x\,dx$, so $du = dx$, $v = e^x$. We see that the choice is right because the new integral that we obtain after applying the formula of integration by parts is simpler than the original one:

$$\int \underbrace{x}_{u}\;\underbrace{e^x\,dx}_{dv} = \underbrace{x}_{u}\;\underbrace{e^x}_{v} - \int \underbrace{e^x}_{v}\;\underbrace{dx}_{du} = \boxed{xe^x - e^x + C}\,.$$

Usually it is a good idea to check the answer by differentiating it:

$$(xe^x - e^x + C)' = e^x + xe^x - e^x = xe^x\,.$$

A couple of additional typical examples:

Example: $\displaystyle\int x\sin x\,dx = \cdots$

$u = x$, $dv = \sin x\,dx$, so $du = dx$, $v = -\cos x$:

$$\cdots = \int \underbrace{x}_{u} \underbrace{\sin x\,dx}_{dv} = \underbrace{x}_{u} \underbrace{(-\cos x)}_{v} - \int \underbrace{(-\cos x)}_{v} \underbrace{dx}_{du}$$
$$= \boxed{-x\cos x + \sin x + C}.$$

Example: $\int \ln x\,dx = \cdots$

$u = \ln x$, $dv = dx$, so $du = \frac{1}{x}dx$, $v = x$:

$$\cdots = \int \underbrace{\ln x}_{u} \underbrace{dx}_{dv} = \underbrace{\ln x}_{u} \underbrace{x}_{v} - \int \underbrace{x}_{v} \underbrace{\frac{1}{x}\,dx}_{du}$$
$$= x\,\ln x - \int dx$$
$$= \boxed{x\,\ln x - x + C}.$$

Sometimes we need to use the formula more than once.

Example: $\int x^2 e^x\,dx = \ldots$

$u = x^2$, $dv = e^x\,dx$, so $du = 2x\,dx$, $v = e^x$:

$$\cdots = \int \underbrace{x^2}_{u} \underbrace{e^x\,dx}_{dv} = x^2 e^x - \int e^x 2x\,dx = \ldots$$

$u = 2x$, $dv = e^x\,dx$, so $du = 2dx$, $v = e^x$:

$$\cdots = x^2 e^x - \int \underbrace{2x}_{u} \underbrace{e^x\,dx}_{dv} = x^2 e^x - 2xe^x + \int 2e^x\,dx$$
$$= \boxed{x^2 e^x - 2xe^x + 2e^x + C}.$$

In the following example the formula of integration by parts does not yield a final answer, but an equation verified by the integral from which its value can be derived.

Example: $\int \sin x\, e^x\,dx = \ldots$

$u = \sin x$, $dv = e^x\,dx$, so $du = \cos x\,dx$, $v = e^x$:

$$\cdots = \int \underbrace{\sin x}_{u}\,\underbrace{e^x\,dx}_{dv} = \sin x \cdot e^x - \int e^x \cos x\,dx = \dots$$

$u = \cos x$, $dv = e^x\,dx$, so $du = -\sin x\,dx$, $v = e^x$:

$$\begin{aligned} \dots &= \sin x \cdot e^x - \int \underbrace{\cos x}_{u}\,\underbrace{e^x\,dx}_{dv} \\ &= \sin x \cdot e^x - \cos x \cdot e^x - \int e^x \sin x\,dx \end{aligned}$$

Hence the integral $I = \int \sin x\, e^x\,dx$ verifies

$$I = \sin x \cdot e^x - \cos x \cdot e^x - I\,,$$

i.e.,

$$2I = \sin x \cdot e^x - \cos x \cdot e^x\,,$$

hence

$$\boxed{I = \tfrac{1}{2}e^x(\sin x - \cos x) + C}\,.$$

1.5.1. Integration by parts for Definite Integrals. Combining the formula of integration by parts with the Evaluation Theorem we get:

$$\boxed{\int_a^b f(x)g'(x)\,dx = [f(x)g(x)]_a^b - \int_a^b g(x)f'(x)\,dx}\,.$$

Example: $\displaystyle\int_0^1 \tan^{-1} x\,dx = \cdots$

$u = \tan^{-1} x$, $dv = dx$, so $du = \dfrac{1}{1+x^2}\,dx$, $v = x$:

$$\begin{aligned} \cdots = \int_0^1 \underbrace{\tan^{-1} x}_{u}\ \underbrace{dx}_{dv} &= [\,\underbrace{\tan^{-1} x}_{u} \cdot \underbrace{x}_{v}\,]_0^1 - \int_0^1 \underbrace{x}_{v}\,\underbrace{\frac{1}{1+x^2}\,dx}_{du} \\ &= \left[\tan^{-1} 1 \cdot 1 - \tan^{-1} 0 \cdot 0\right] - \int_0^1 \frac{x}{1+x^2}\,dx \\ &= \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2}\,dx \end{aligned}$$

The last integral can be computed with the substitution $t = 1 + x^2$, $dt = 2x\,dx$:

$$\int_0^1 \frac{x}{1+x^2}\,dx = \frac{1}{2}\int_1^2 \frac{1}{t}\,dt = \frac{1}{2}\left[\ln t\right]_1^2 = \frac{\ln 2}{2}\,.$$

Hence the original integral is:

$$\int_0^1 \tan^{-1} x\,dx = \boxed{\frac{\pi}{4} - \frac{\ln 2}{2}}\,.$$

1.5.2. Reduction Formulas. Assume that we want to find the following integral for a given value of $n > 0$:

$$\int x^n e^x\,dx\,.$$

Using integration by parts with $u = x^n$ and $dv = e^x\,dx$, so $v = e^x$ and $du = nx^{n-1}\,dx$, we get:

$$\boxed{\int x^n e^x\,dx = x^n e^x - n\int x^{n-1}e^x\,dx}\,.$$

On the right hand side we get an integral similar to the original one but with x raised to $n-1$ instead of n. This kind of expression is called a *reduction formula*. Using this same formula several times, and taking into account that for $n = 0$ the integral becomes $\int e^x\,dx = e^x + C$, we can evaluate the original integral for any n. For instance:

$$\begin{aligned}
\int x^3 e^x\,dx &= x^3e^x - 3\int x^2e^x\,dx \\
&= x^3e^x - 3(x^2e^x - 2\int xe^x\,dx) \\
&= x^3e^x - 3(x^2e^x - 2(xe^x - \int e^x\,dx)) \\
&= x^3e^x - 3(x^2e^x - 2(xe^x - e^x)) + C \\
&= \boxed{x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C}\,.
\end{aligned}$$

Another example:

$$\begin{aligned}
\int \sin^n x\,dx &= \int \underbrace{\sin^{n-1} x}_{u}\,\underbrace{\sin x\,dx}_{dv} \\
&= -\sin^{n-1} x\cos x + (n-1)\int \underbrace{\cos^2 x}_{1-\sin^2 x}\sin^{n-2}\,dx \\
&= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2}\,dx \\
&\quad - (n-1)\int \sin^n x\,dx
\end{aligned}$$

Adding the last term to both sides and dividing by n we get the following reduction formula:

$$\boxed{\int \sin^n x\,dx = -\frac{\sin^{n-1} x\cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx}\,.$$