1.4. Mathematical Induction

Many properties of positive integers can be proved by mathematical induction.

1.4.1. Principle of Mathematical Induction. Let P be a property of positive integers such that:

1. *Basis Step*: $P(1)$ is true, and
2. *Inductive Step*: if $P(n)$ is true, then $P(n+1)$ is true.

Then $P(n)$ is true for all positive integers.

Remark: The premise $P(n)$ in the inductive step is called *Induction Hypothesis*.

The validity of the Principle of Mathematical Induction is obvious. The basis step states that $P(1)$ is true. Then the inductive step implies that $P(2)$ is also true. By the inductive step again we see that $P(3)$ is true, and so on. Consequently the property is true for all positive integers.

Remark: In the basis step we may replace 1 with some other integer m. Then the conclusion is that the property is true for every integer n greater than or equal to m.

Example: Prove that the sum of the n first odd positive integers is n^2, i.e., $1 + 3 + 5 + \cdots + (2n-1) = n^2$.

Answer: Let $S(n) = 1 + 3 + 5 + \cdots + (2n-1)$. We want to prove by induction that for every positive integer n, $S(n) = n^2$.

1. *Basis Step*: If $n = 1$ we have $S(1) = 1 = 1^2$, so the property is true for 1.
2. *Inductive Step*: Assume (*Induction Hypothesis*) that the property is true for some positive integer n, i.e.: $S(n) = n^2$. We must prove that it is also true for $n+1$, i.e., $S(n+1) = (n+1)^2$. In fact:

   $$S(n+1) = 1 + 3 + 5 + \cdots + (2n+1) = S(n) + 2n + 1\,.$$

But by induction hypothesis, $S(n) = n^2$, hence:

$$S(n+1) = n^2 + 2n + 1 = (n+1)^2\,.$$

This completes the induction, and shows that the property is true for all positive integers.

Example: Prove that $2n + 1 \leq 2^n$ for $n \geq 3$.

Answer: This is an example in which the property is not true for all positive integers but only for integers greater than or equal to 3.

1. *Basis Step*: If $n = 3$ we have $2n + 1 = 2 \cdot 3 + 1 = 7$ and $2^n = 2^3 = 8$, so the property is true in this case.

2. *Inductive Step*: Assume (*Induction Hypothesis*) that the property is true for some positive integer n, i.e.: $2n + 1 \leq 2^n$. We must prove that it is also true for $n+1$, i.e., $2(n+1)+1 \leq 2^{n+1}$. By the induction hypothesis we know that $2n \leq 2^n$, and we also have that $3 \leq 2^n$ if $n \geq 3$, hence

$$2(n+1) + 1 = 2n + 3 \leq 2^n + 2^n = 2^{n+1}\,.$$

This completes the induction, and shows that the property is true for all $n \geq 3$.

Exercise: Prove the following identities by induction:

- $1 + 2 + 3 + \cdots + n = \dfrac{n\,(n+1)}{2}$.
- $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n\,(n+1)\,(2n+1)}{6}$.
- $1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$.

1.4.2. Strong Form of Mathematical Induction. Let P be a property of positive integers such that:

1. *Basis Step*: $P(1)$ is true, and

2. *Inductive Step*: if $P(k)$ is true for all $1 \leq k \leq n$ then $P(n+1)$ is true.

Then $P(n)$ is true for all positive integers.

Example: Prove that every integer $n \geq 2$ is prime or a product of primes. *Answer*:

1. *Basis Step*: 2 is a prime number, so the property holds for $n = 2$.

2. *Inductive Step*: Assume that if $2 \leq k \leq n$, then k is a prime number or a product of primes. Now, either $n+1$ is a prime number or it is not. If it is a prime number then it verifies the property. If it is not a prime number, then it can be written as the product of two positive integers, $n+1 = k_1\,k_2$, such that $1 < k_1, k_2 < n+1$. By induction hypothesis each of k_1 and k_2 must be a prime or a product of primes, hence $n+1$ is a product of primes.

This completes the proof.

1.4.3. The Well-Ordering Principle. Every nonempty set of positive integers has a smallest element.

Example: Prove that $\sqrt{2}$ is irrational (i.e., $\sqrt{2}$ cannot be written as a quotient of two positive integers) using the well-ordering principle. *Answer*: Assume that $\sqrt{2}$ is rational, i.e., $\sqrt{2} = a/b$, where a and b are integers. Note that since $\sqrt{2} > 1$ then $a > b$. Now we have $2 = a^2/b^2$, hence $2\,b^2 = a^2$. Since the left hand side is even, then a^2 is even, but this implies that a itself is even, so $a = 2\,a'$. Hence: $2\,b^2 = 4\,a'^2$, and simplifying: $b^2 = 2\,a'^2$. From here we see that $\sqrt{2} = b/a'$. Hence starting with a fractional representation of $\sqrt{2} = a/b$ we end up with another fractional representation $\sqrt{2} = b/a'$ with a smaller numerator $b < a$. Repeating the same argument with the fraction b/a' we get another fraction with an even smaller numerator, and so on. So the set of possible numerators of a fraction representing $\sqrt{2}$ cannot have a smallest element, contradicting the well-ordering principle. Consequently, our assumption that $\sqrt{2}$ is rational has to be false.