Extending the Four Color Theorem to Infinite Planar Maps

Miguel Lerma

October 20, 2025

Abstract

The classical Four Color Theorem asserts that every finite planar map can be colored with four colors so that adjacent regions receive different colors. In this note we present three arguments showing that the result extends to certain infinite planar maps: (1) a purely graph–theoretic proof based on König's Infinity Lemma, (2) a topological compactness argument using Tychonoff's theorem, and (3) a model–theoretic compactness proof. We also discuss the role of local finiteness and comment on the algorithmic and computability aspects of these extensions.

1 The setting

A *planar graph* is *locally finite* if each vertex has finite degree. A (possibly infinite) planar map corresponds to such a graph if every region touches only finitely many others. We wish to show that the Four Color Theorem (4CT) for finite graphs implies that every locally finite planar graph admits a proper 4-coloring.

2 Graph–theoretic proof via König's Lemma

Theorem 1. *Every locally finite planar graph is* 4*-colorable.*

Proof. Let $G = (V, E)$ be a planar graph in which every vertex has finite degree. Assume G is connected. Fix a root vertex $v_0 \in V$, and for each integer $n \geq 0$ define

$$S_n = \{\, v \in V : \operatorname{dist}(v_0, v) \leq n \,\}.$$

Because G is locally finite, each S_n is finite. Let G_n denote the finite induced subgraph on S_n.

By the finite Four Color Theorem, each G_n admits at least one proper 4-coloring. Define the *coloring tree* T whose level L_n consists of all proper 4-colorings of G_n, and in which $c_{n+1} \in L_{n+1}$ is adjacent to $c_n \in L_n$ when $c_{n+1}\restriction_{S_n} = c_n$. Each L_n is nonempty, so T is infinite, and each node has finitely many children (because the extension to S_{n+1} involves finitely many vertices). Thus T is an infinite, finitely branching tree.

By *König's Infinity Lemma*, such a tree has an infinite branch

$$c_0 \prec c_1 \prec c_2 \prec \cdots, \qquad c_{n+1}\restriction_{S_n} = c_n.$$

Define $c(v) = c_n(v)$ for any n with $v \in S_n$. This is well defined and yields a proper 4-coloring of G, since every edge lies within some finite G_n.

If G is disconnected, color each component independently. $\square$

Remark. *Local finiteness is essential: without it, some vertex might have infinitely many neighbors, and the coloring tree could branch infinitely, invalidating König's lemma.*

3 Topological compactness proof

Let $\mathcal{C} = \{1, 2, 3, 4\}^V$ be the set of all colorings of V. Endow $\{1, 2, 3, 4\}$ with the discrete topology and $\mathcal{C}$ with the product topology. By Tychonoff's theorem, $\mathcal{C}$ is compact.

For each edge $uv \in E$ define

$$A_{uv} = \{\, f \in \mathcal{C} : f(u) \neq f(v) \,\}.$$

Each A_{uv} is clopen, hence closed. If $F \subseteq E$ is finite, then the subgraph spanned by the endpoints of F is finite and planar, so $\bigcap_{uv \in F} A_{uv} \neq \varnothing$ (by the finite Four Color Theorem). The family $\{A_{uv}\}_{uv \in E}$ thus has the finite intersection property, and compactness gives

$$\bigcap_{uv \in E} A_{uv} \neq \varnothing.$$

Any f in this intersection is a proper 4-coloring of G.

4 Model–theoretic compactness proof

In a first–order language with constants for vertices and unary predicates $C_1, \ldots, C_4$ representing the four colors, include axioms stating: each vertex has exactly one color, and adjacent vertices have distinct colors. Every finite subset of this theory corresponds to a finite planar graph, hence has a model by the finite 4CT. By the *Compactness Theorem* of first–order logic, the entire theory has a model, yielding a 4-coloring of G.

5 Algorithmic and computability aspects

Among the three arguments, the proof based on König's Infinity Lemma carries the most constructive information. If one can systematically enumerate the layers S_n and effectively compute at least one proper 4-coloring of each finite G_n (as the finite 4CT is constructive in principle), then the branch guaranteed by König's lemma can be generated step by step.

In particular:

- Given a computable planar embedding and adjacency structure, one can compute $S_0, S_1, \ldots$ by breadth–first search.

- At each stage, one can compute all 4-colorings of G_n or select one by any deterministic extension rule compatible with some future extension (backtracking if necessary).
- If G is infinite but countable and locally finite, this yields a *computable infinite sequence* of compatible partial colorings whose union is a total coloring.

However, the process is not *uniformly* computable in general: there is no recursive bound on the stage at which conflicts might appear, and for non–locally finite graphs or uncountable embeddings the construction may fail. Thus the theorem is not algorithmic in the strict sense, but it does provide a constructive schema that can color any specific computably presented infinite planar graph.

6 Historical note

König's Infinity Lemma first appeared in 1927 as a bridge between finite and infinite combinatorial reasoning. Its influence extended throughout infinite graph theory and compactness methods in logic. The idea of extending finite graph colorings to the infinite case through compactness was developed independently by de Bruijn and Erdős (1951), who proved that if every finite subgraph of an infinite graph is k-colorable, then the entire graph is k-colorable.

The finite Four Color Theorem was finally proved by Appel and Haken (1977), following over a century of effort, and later confirmed and simplified by Robertson, Sanders, Seymour, and Thomas (1997). Although their proofs are computer-assisted, the logical structure of the argument ensures that each finite planar graph (and therefore each finite subgraph of a planar map) is effectively 4-colorable.

The convergence of these ideas, finitary combinatorics (König), logical compactness (de Bruijn–Erdős), and computer verification (Appel–Haken, Robertson–Thomas), illustrates a remarkable unity: the finitary Four Color Theorem entails its infinite analogue through the general principles of compactness and the existence of consistent extensions of partial colorings.

6.1 Summary

- Appel–Haken (1977) and Robertson–Seymour–Thomas (1997) are the definitive proofs of the finite Four Color Theorem.
- König (1927) is the original source of the Infinity Lemma.
- de Bruijn–Erdős (1951) provides the compactness-style extension to infinite graphs.
- Tychonoff (1930) is for the compactness argument in topology.
- Chang–Keisler (1973) gives the logical compactness theorem reference.
- Diestel (2024) (or 5th ed., 2017) gives modern graph-theoretic context and proofs.
- Halin (1964) is a classical reference for infinite graph theory.
- Soifer (2009) is a readable historical/expository source connecting all these strands.

References

[1] K. Appel and W. Haken, *Every planar map is four colorable. Part I: Discharging*, Illinois J. Math. **21** (1977), 429–490.

[2] K. Appel, W. Haken, and J. Koch, *Every planar map is four colorable. Part II: Reducibility*, Illinois J. Math. **21** (1977), 491–567.

[3] N. Robertson, D. P. Sanders, P. D. Seymour, and R. Thomas, *The four-colour theorem*, J. Combin. Theory Ser. B **70** (1997), 2–44.

[4] D. König, *Über eine Schlussweise aus dem Endlichen ins Unendliche*, Acta Sci. Math. (Szeged) **3** (1927), 121–130.

[5] N. G. de Bruijn and P. Erdős, *A colour problem for infinite graphs and a problem in the theory of relations*, Nederl. Akad. Wetensch. Proc. Ser. A **54** (1951), 371–373.

[6] A. Tychonoff, *Über die topologische Erweiterung von Räumen*, Math. Ann. **102** (1930), 544–561.

[7] C. C. Chang and H. J. Keisler, *Model Theory*, North–Holland, Amsterdam, 1973.

[8] R. Diestel, *Graph Theory*, 6th edition, Springer, Berlin, 2024.

[9] R. Halin, *Über unendliche Wege in Graphen*, Math. Ann. **157** (1964), 125–137.

[10] A. Soifer, *The Mathematical Coloring Book: Mathematics of Coloring and the Colorful Life of Its Creators*, Springer, New York, 2009.