4.5. Power Series

A *power series* is a series of the form

$$\sum_{n=0}^{\infty} c_0 x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$$

where x is a variable of indeterminate. It can be interpreted as an infinite polynomial. The c_n's are the *coefficients* of the series. The sum of the series is a function

$$f(x) = \sum_{n=0}^{\infty} c_0 x^n$$

For instance the following series converges to the function shown for $-1 < x < 1$:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots = \frac{1}{1-x}\,.$$

More generally given a fix number a, a *power series in* $(x-a)$, or *centered in* a, or *about* a, is a series of the form

$$\sum_{n=0}^{\infty} c_0 (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots$$

4.5.1. Convergence of Power Series.

For a given power series $\sum_{n=1}^{\infty} c_n(x-a)^n$ there are only three possibilities:

(1) The series converges only for $x = a$.
(2) The series converges for all x.
(3) There is a number R, called *radius of convergence*, such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

The *interval of convergence* is the set of values of x for which the series converges.

Example: Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(x-3)^n}{n}\,.$$

Answer: We use the Ratio Test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(x-3)^{n+1}/(n+1)}{(x-3)^n/n} = (x-3)\frac{n}{n+1} \underset{n\to\infty}{\longrightarrow} x-3\,,$$

So the power series converges if $|x-3| < 1$ and diverges if $|x-3| > 1$. Consequently, the radius of convergence is $R = 1$. On the other hand, we know that the series converges inside the interval $(2,4)$, but it remains to test the endpoints of that interval. For $x = 4$ the series becomes

$$\sum_{n=0}^{\infty} \frac{1}{n}\,,$$

i.e., the harmonic series, which we know diverges. For $x = 2$ the series is

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n}\,,$$

i.e., the alternating harmonic series, which converges. So the interval of convergence is $[2,4)$.

4.6. Representation of Functions as Power Series

We have already seen that a power series is a particular kind of function. A slightly different matter is that sometimes a given function can be written as a power series. We already know the example

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n \qquad (|x| < 1)$$

Replacing x with other expressions we may write other functions in the same way, for instance by replacing x with $-2x^2$ we get:

$$\frac{1}{1+2x^2} = 1 - 2x^2 + 4x^4 - 8x^6 + \cdots + (-1)^n 2^n x^{2n} + \cdots = \sum_{n=0}^{\infty} (-1)^n 2^n x^{2n}\,,$$

which converges for $|-2x^2| < 1$, i.e., $|x| < 1/\sqrt{2}$.

4.6.1. Differentiation and Integration of Power Series.

Since the sum of a power series is a function we can differentiate it and integrate it. The result is another function that can also be represented with another power series. The main related result is that the derivative or integral of a power series can be computed by *term-by-term differentiation and integration*:

4.6.1.1. *Term-By-Term Differentiation and Integration.* If the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has radius of convergence $R > 0$ then the function

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable on the interval $(a-R, a+R)$ and

(1) $$f'(x) = \sum_{n=0}^{\infty} \{c_n(x-a)^n\}' = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$$

(2) $$\int f(x)\,dx = \sum_{n=0}^{\infty} \int c_n(x-a)^n\,dx = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

The radii of convergence of the series in the above equations is R.

Example: Find a power series representation for the function

$$f(x) = \frac{1}{(1-x)^2}\,.$$

Answer: We have

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\frac{1}{1-x}$$

and

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \,,$$

hence

$$\begin{aligned}\frac{1}{(1-x)^2} &= \frac{d}{dx}\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty}\frac{d}{dx}x^n = \sum_{n=1}^{\infty} nx^{n-1} \\ &= 1 + 2x + 3x^2 + 4x^3 + \cdots = \boxed{\sum_{n=0}^{\infty}(n+1)x^n} \quad \text{(re-indexed)}\end{aligned}$$

The radius of convergence is $R = 1$.

Example: Find a power series representation for $\tan^{-1} x$.

Answer: That function is the antiderivative of $1/(1+x^2)$, hence:

$$\begin{aligned}\tan^{-1} x &= \int \frac{1}{1+x^2}\,dx \\ &= \int \sum_{n=0}^{\infty}(-1)^n x^{2n}\,dx \\ &= \sum_{n=0}^{\infty}\int (-1)^n x^{2n}\,dx \\ &= C + \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1} \\ &= C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots\end{aligned}$$

Since $\tan^{-1} 0 = 0$ then $C = 0$, hence

$$\boxed{\tan^{-1} x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots}$$

The radius of convergence is $R = 1$.

Example: Find a power series representation for $\ln(1+x)$.

Answer: The derivative of that function is $1/(1+x)$, hence

$$\begin{aligned}
\ln(1+x) &= \int \frac{1}{1+x}\,dx \\
&= \int \sum_{n=0}^{\infty} (-1)^n x^n \,dx \\
&= C + \sum_{n=0}^{\infty} \int (-1)^n \frac{x^{n+1}}{n+1}\,dx \\
&= C + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots
\end{aligned}$$

Since $\ln 1 = 0$ then $C = 0$, so

$$\boxed{\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}}$$

The radius of convergence is $R = 1$.