CHAPTER 4

Induction, Recurences

4.1. Sequences and Strings

4.1.1. Sequences. A *sequence* is an (usually infinite) ordered list of elements. Examples:

1. The sequence of positive integers:
$$1, 2, 3, 4, \dots, n, \dots$$
2. The sequence of positive even integers:
$$2, 4, 6, 8, \dots, 2n, \dots$$
3. The sequence of powers of 2:
$$1, 2, 4, 8, 16, \dots, n^2, \dots$$
4. The sequence of Fibonacci numbers (each one is the sum of the two previous ones):
$$0, 1, 1, 2, 3, 5, 8, 13, \dots$$
5. The reciprocals of the positive integers:
$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots, \frac{1}{n}, \cdots$$

In general the elements of a sequence are represented with an indexed letter, say $s_1, s_2, s_3, \dots, s_n, \dots$. The sequence itself can be defined by giving a rule, e.g.: $s_n = 2n + 1$ is the sequence:
$$3, 5, 7, 9, \dots$$
Here we are assuming that the first element is s_1, but we can start at any value of the index that we want, for instance if we declare s_0 to be the first term, the previous sequence would become:
$$1, 3, 5, 7, 9, \dots$$
The sequence is symbolically represented $\{s_n\}$ or $\{s_n\}_{n=1}^{\infty}$.

If $s_n \leq s_{n+1}$ for every n the sequence is called *increasing*. If $s_n \geq s_{n+1}$ then it is called *decreasing*. For instance $s_n = 2n+1$ is increasing: $3, 5, 7, 9, \dots$, while $s_n = 1/n$ is decreasing: $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$.

If we remove elements from a sequence we obtain a *subsequence*. E.g., if we remove all odd numbers from the sequence of positive integers:

$$1, 2, 3, 4, 5 \dots ,$$

we get the subsequence consisting of the even positive integers:

$$2, 4, 6, 8, \dots$$

4.1.2. Sum (Sigma) and Product Notation. In order to abbreviate sums and products the following notations are used:

1. *Sum* (or *sigma*) notation:
$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + a_{m+2} + \cdots + a_n$$
2. *Product* notation:
$$\prod_{i=m}^{n} a_i = a_m \cdot a_{m+1} \cdot a_{m+2} \cdot \cdots \cdot a_n$$

For instance: assume $a_n = 2n+1$, then
$$\sum_{n=3}^{6} a_n = a_3 + a_4 + a_5 + a_6 = 7 + 9 + 11 + 13 = 40 \,.$$
$$\prod_{n=3}^{6} a_n = a_3 \cdot a_4 \cdot a_5 \cdot a_6 = 7 \cdot 9 \cdot 11 \cdot 13 = 9009 \,.$$

4.1.3. Strings. Given a set X, a *string over* X is a *finite* ordered list of elements of X.

Example: If X is the set $X = \{a, b, c\}$, then the following are examples of strings over X: *aba*, *aaaa*, *bba*, etc.

Repetitions can be specified with a superscripts, for instance: $a^2b^3ac^2a^3 = aabbbaccaaa$, $(ab)^3 = ababab$, etc.

The *length* of a string is its number of elements, e.g., $|abaccbab| = 8$, $|a^2b^7a^3c^6| = 18$.

The string with no elements is called *null string*, represented λ. Its length is, of course, zero: $|\lambda| = 0$.

The set of all strings over X is represented X^*. The set of no null strings over X (i.e., all strings over X except the null string) is represented X^+.

Given two strings α and β over X, the string consisting of α followed by β is called the *concatenation* of α and β. For instance if $\alpha = abac$ and $\beta = baaab$ then $\alpha\beta = abacbaaab$.