CHAPTER 4

Counting

4.1. Basic Principles

4.1.1. The Rule of Sum. If a task can be performed in m ways, while another task can be performed in n ways, and the two tasks cannot be performed simultaneously, then performing either task can be accomplished in $m + n$ ways.

Set theoretical version of the rule of sum: If A and B are disjoint sets ($A \cap B = \emptyset$) then

$$|A \cup B| = |A| + |B| .$$

More generally, if the sets $A_1, A_2, \dots, A_n$ are pairwise disjoint, then:

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n| .$$

For instance, if a class has 30 male students and 25 female students, then the class has $30 + 25 = 45$ students.

4.1.2. The Rule of Product. If a task can be performed in m ways and another independent task can be performed in n ways, then the combination of both tasks can be performed in mn ways.

Set theoretical version of the rule of product: Let $A \times B$ be the Cartesian product of sets A and B. Then:

$$|A \times B| = |A| \cdot |B| .$$

More generally:

$$|A_1 \times A_2 \times \cdots \times A_n| = |A_1| \cdot |A_2| \cdots |A_n| .$$

For instance, assume that a license plate contains two letters followed by three digits. How many different license plates can be printed? *Answer*: each letter can be printed in 26 ways, and each digit can be printed in 10 ways, so $26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 676000$ different plates can be printed.

Exercise: Given a set A with m elements and a set B with n elements, find the number of functions from A to B.

4.1.3. The Inclusion-Exclusion Principle. The *inclusion-exclusion principle* generalizes the rule of sum to non-disjoint sets.

In general, for arbitrary (but finite) sets A, B:

$$|A \cup B| = |A| + |B| - |A \cap B| .$$

Example: Assume that in a university with 1000 students, 200 students are taking a course in mathematics, 300 are taking a course in physics, and 50 students are taking both. How many students are taking at least one of those courses?

Answer: If U = total set of students in the university, M = set of students taking Mathematics, P = set of students taking Physics, then:

$$|M \cup P| = |M| + |P| - |M \cap P| = 300 + 200 - 50 = 450$$

students are taking Mathematics or Physics.

For three sets the following formula applies:

$$|A \cup B \cup C| =$$
$$|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| ,$$

and for an arbitrary union of sets:

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = s_1 - s_2 + s_3 - s_4 + \cdots \pm s_n ,$$

where s_k = sum of the cardinalities of all possible k-fold intersections of the given sets.

4.2. Combinatorics

4.2.1. Permutations. Assume that we have n objects. Any arrangement of any k of these objects in a given order is called a *permutation* of size k. If $k = n$ then we call it just a *permutation* of the n objects. For instance, the permutations of the letters a, b, c are the following: *abc*, *acb*, *bac*, *bca*, *cab*, *cba*. The permutations of size 2 of the letters a, b, c, d are: *ab*, *ac*, *ad*, *ba*, *bc*, *bd*, *ca*, *cb*, *cd*, *da*, *db*, *dc*.

Note that the order is important. Given two permutations, they are considered equal if they have the same elements arranged in the same order.

We find the number $P(n, k)$ of permutations of size k of n given objects in the following way: The first object in an arrangement can be chosen in n ways, the second one in $n - 1$ ways, the third one in $n - 2$ ways, and so on, hence:

$$P(n,k) = n \times (n-1) \times \overset{(k \text{ factors})}{\cdots} \times (n-k+1) = \frac{n!}{(n-k)!}\,,$$

where $n! = 1 \times 2 \times 3 \times \overset{(n \text{ factors})}{\cdots} \times n$ is called "n *factorial*".

The number $P(n, k)$ of permutations of n objects is

$$P(n,n) = n!\,.$$

By convention $0! = 1$.

For instance, there are $3! = 6$ permutations of the 3 letters a, b, c. The number of permutations of size 2 of the 4 letters a, b, c, d is $P(4, 2) = 4 \times 3 = 12$.

Exercise: Given a set A with m elements and a set B with n elements, find the number of one-to-one functions from A to B.

4.2.2. Combinations. Assume that we have a set A with n objects. Any subset of A of size r is called a *combination of n elements taken r at a time*. For instance, the combinations of the letters a, b, c, d, e taken 3 at a time are: *abc*, *abd*, *abe*, *acd*, *ace*, *ade*, *bcd*, *bce*, *bde*, *cde*, where two combinations are considered identical if they have the same elements regardless of their order.

The number of subsets of size r in a set A with n elements is:

$$C(n,r) = \frac{n!}{r!\,(n-r)!}\,.$$

The symbol $\binom{n}{r}$ (read "n choose r") is often used instead of $C(n,r)$.

One way to derive the formula for $C(n,r)$ is the following. Let A be a set with n objects. In order to generate all possible permutations of size r of the elements of A we 1) take all possible subsets of size r in the set A, and 2) permute the k elements in each subset in all possible ways. Task 1) can be performed in $C(n,r)$ ways, and task 2) can be performed in $P(r,r)$ ways. By the product rule we have $P(n,r) = C(n,r) \times P(r,r)$, hence

$$C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{n!}{r!\,(n-r)!}\,.$$