

## 4.2. Series

A *series* is an infinite sum:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

In order to define the value of this sum we start by defining its sequence of *partial sums*

$$s_n = \sum_{i=1}^n a_i = a_1 + a_2 + \cdots + a_n.$$

Then, if  $\lim_{n \rightarrow \infty} s_n = s$  exists the series is called *convergent* and its sum is that limit:

$$\sum_{n=1}^{\infty} a_n = s = \lim_{n \rightarrow \infty} s_n.$$

Otherwise the series is called *divergent*.

For instance, consider the following series:

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} + \cdots = \sum_{n=1}^{\infty} \frac{1}{2^n}.$$

Its partial sums are:

$$s_n = \sum_{i=1}^n \frac{1}{2^i} = \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}.$$

Hence its sum is

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{2^i} = \lim_{n \rightarrow \infty} \left( 1 - \frac{1}{2^n} \right) = 1 + 0 = 1.$$

**4.2.1. Geometric Series.** A series verifying  $a_{n+1} = ra_n$ , where  $r$  is a constant, is called *geometric series*. If the first term is  $a \neq 0$  then the series is

$$a + ar + ar^2 + \cdots + ar^n + \cdots = \sum_{n=0}^{\infty} ar^n.$$

The partial sums are now:

$$s_n = \sum_{i=0}^n ar^i.$$

The  $n$ th partial sum can be found in the following way:

$$\begin{aligned}s_n &= a + ar + ar^2 + \cdots + ar^n \\ rs_n &= ar + ar^2 + \cdots + ar^n + ar^{n+1}\end{aligned}$$

hence

$$s_n - rs_n = a + 0 + 0 + \cdots + 0 - ar^{n+1},$$

so:

$$s_n = \frac{a(1 - r^{n+1})}{1 - r}.$$

If  $|r| < 1$  we can rewrite the result like this:

$$s_n = \frac{a}{1 - r} - \frac{a}{1 - r} r^{n+1},$$

and then get the limit as  $n \rightarrow \infty$ :

$$s = \lim_{n \rightarrow \infty} s_n = \frac{a}{1 - r} - \frac{a}{1 - r} \underbrace{\lim_{n \rightarrow \infty} r^{n+1}}_{\downarrow 0} = \frac{a}{1 - r}$$

So for  $|r| < 1$  the series is convergent and

$$\boxed{\sum_{n=0}^{\infty} ar^n = \frac{a}{1 - r}}.$$

For  $|r| \geq 1$  the series is divergent.

**4.2.2. Telescopic Series.** A telescopic series is a series whose terms can be rewritten so that most of them cancel out.

*Example:* Find  $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ .

*Answer:* Note that  $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$ . So the  $n$ th partial sum is

$$\begin{aligned}s_n &= \sum_{i=1}^n \left( \frac{1}{i} - \frac{1}{i+1} \right) \\ &= \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{n} - \frac{1}{n+1} \\ &= 1 - \frac{1}{n+1}.\end{aligned}$$

Hence, the sum of the series is

$$s = \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n+1}\right) = \boxed{1}.$$

**4.2.3. Theorem.** If the series  $\sum_{n=0}^{\infty} a_n$  is convergent then  $\lim_{n \rightarrow \infty} a_n = 0$ .

*Proof:* If the series is convergent then the sequence of partial sums  $s_n = \sum_{i=1}^n a_i$  have a limit  $s$ . On the other hand  $a_n = s_n - s_{n-1}$ , so taking limits we get  $\lim_{n \rightarrow \infty} a_n = s - s = 0$ .

The converse is not true in general. The harmonic series provides a counterexample.

**4.2.4. The Harmonic Series.** The following series is called *harmonic series*:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

The main fact about it is that it is *divergent*. In order to prove it we find

$$s_1 = 1$$

$$s_2 = 1 + \frac{1}{2}$$

$$s_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{1}{2} + \frac{1}{2} = 1 + \frac{2}{2}$$

$$s_8 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)$$

$$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + \frac{3}{2}$$

etc., so in general  $s_{2^n} > 1 + \frac{n}{2}$ , hence the sequence of partial sums grows without limit and the series diverges.

**4.2.5. Test for Divergence.** If  $\lim_{n \rightarrow \infty} a_n$  does not exist or if  $\lim_{n \rightarrow \infty} a_n \neq 0$  then  $\sum_{n=1}^{\infty} a_n$  diverges.

*Example:* Show that  $\sum_{n=1}^{\infty} \frac{n}{n+1}$  diverges.

*Answer:* We have  $\lim_{n \rightarrow \infty} \frac{n}{n+1} = 1$ . Since the  $n$ th term of the series does not tend to 0, the series diverges.

*Example:* Show that  $\sum_{n=1}^{\infty} \sin n$  diverges.

*Answer:* All we need to show is that  $\sin n$  does not tend to 0. If for some value of  $n$ ,  $\sin n \approx 0$ , then  $n \approx k\pi$  for some integer  $k$ , but then

$$\begin{aligned} \sin(n+1) &= \sin n \cos 1 + \cos n \sin 1 \\ &\approx \sin k\pi \cos 1 + \cos k\pi \sin 1 \\ &= 0 \pm \sin 1 \\ &= \pm 0.84 \cdots \neq 0 \end{aligned}$$

So if a term  $\sin n$  is close to zero, the next term  $\sin(n+1)$  will be far from zero, so it is impossible for  $\sin n$  to get permanently closer and closer to 0.

**4.2.6. Operations with Series.** If  $\sum_{n=1}^{\infty} a_n$  and  $\sum_{n=1}^{\infty} b_n$  are convergent series and  $c$  is a constant then the following series are also convergent and:

$$\begin{aligned} (1) \quad & \sum_{n=1}^{\infty} ca_n = c \sum_{n=1}^{\infty} a_n \\ (2) \quad & \sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n \\ (3) \quad & \sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n \end{aligned}$$