CHAPTER 6

Graph Theory

6.1. Graphs

6.1.1. Graphs. Consider the following examples:

1. A road map, consisting of a number of towns connected with roads.

2. The representation of a binary relation defined on a given set. The relation of a given element x to another element y is represented with an arrow connecting x to y.

The former is an example of (undirected) *graph*. The latter is an example of a *directed graph* or *digraph*.

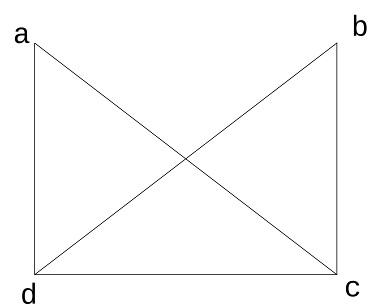

FIGURE 6.1. Undirected Graph.

In general a *graph* G consists of two things:

1. The *vertex set* V, whose elements are called *vertices*, *nodes* or *points*.

2. The *edge set* E or set of *edges* connecting pairs of vertices. If the edges are directed then they are also called *directed edges* or *arcs*. Each edge $e \in E$ is associated with a pair of vertices.

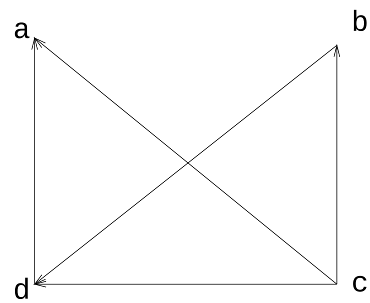

FIGURE 6.2. Directed Graph.

A graph is sometimes represented by the pair (V, E) (we assume V and E finite).

If the graph is undirected and there is a unique edge e connecting x and y we may write $e = \{x, y\}$, so E can be regarded as set of unordered pairs. In this context we may also write $e = (x, y)$, understanding that here (x, y) is not an ordered pair, but the name of an edge.

If the graph is directed and there is a unique edge e pointing from x to y, then we may write $e = (x, y)$, so E may be regarded as a set of ordered pairs. If $e = (x, y)$, the vertex x is called *origin*, *source* or *initial point* of the edge e, and y is called the *terminus*, *terminating vertex* or *terminal point*.

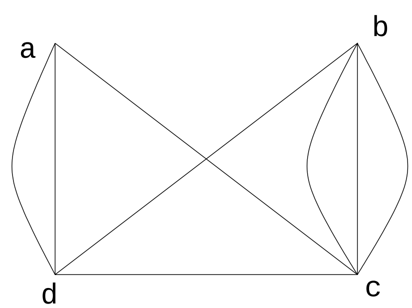

FIGURE 6.3. Graph with parallel edges.

Two vertices connected by an edge are called *adjacent*. They are also the *endpoints* of the edge, and the edge is said to be *incident* to each of its endpoints. If the graph is directed, an edge pointing from vertex x to vertex y is said to be *incident from* x and *incident to* y. An edge connecting a vertex to itself is called a *loop*. Two edges connecting the same pair of points are called *parallel*. A graph with neither loops nor parallel edges is called a *simple graph*.

The *degree* of a vertex v, represented $\delta(v)$, is the number of edges that contain it (loops are counted twice). A vertex of degree zero (not connected to any other vertex) is called *isolated*. A vertex of degree 1 is called *pendant*.

A *path* is a sequence of vertices (v_k) and edges (e_k) of the form $v_0, e_1, v_1, e_2, v_2, \ldots, e_n, v_n$, where each edge e_k connects v_{k-1} with v_k (and points from v_{k-1} to v_k if the graph is directed).

A *weighted graph* is a graph whose edges have been labeled with numbers. The *length* of a path in a weighted graph is the sum of the weights of the edges in the path.

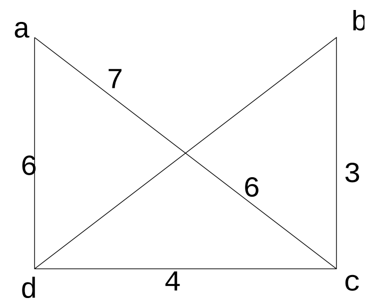

FIGURE 6.4. Weighted Graph.

6.1.2. Special Graphs.

Here we examine a few special graphs.

The n-cube: A graph with with 2^n vertices labeled $0, 1, \ldots, 2^n - 1$ so that two of them are connected with an edge if their binary representation differs in exactly one bit.

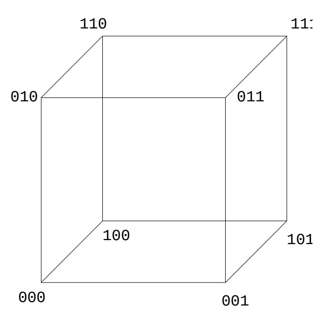

FIGURE 6.5. 3-cube.

Complete Graph: a simple undirected graph G such that every pair of distinct vertices in G are connected by an edge. The *complete graph* of n vertices is represented K_n (fig. 6.6). A *complete directed graph* is a simple directed graph $G = (V, E)$ such that every pair of distinct vertices in G are connected by exactly one edge—so, for each pair of distinct vertices, either (x, y) or (y, x) (but not both) is in E.

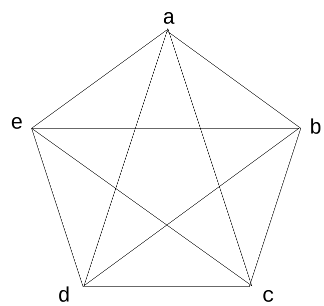

FIGURE 6.6. Complete graph K_5.

Bipartite Graph: a graph $G = (V, E)$ in which V can be partitioned into two subsets V_1 and V_2 so that each edge in G connects some vertex in V_1 to some vertex in V_2. A bipartite simple graph is called *complete* if each vertex in V_1 is connected to each vertex in V_2. If $|V_1| = m$ and $|V_2| = n$, the corresponding complete bipartite graph is represented $K_{m,n}$ (fig. 6.7).

A graph is bipartite iff its vertices can be colored with two colors so that every edge connects vertices of different color.

Question: Is the n-cube bipartite. Hint: color in red all vertices whose binary representation has an even number of 1's, color in blue the ones with an odd number of 1's.

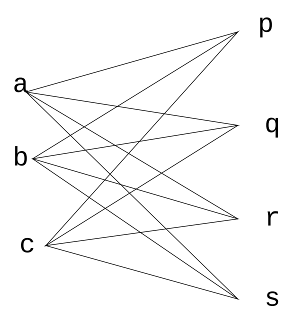

FIGURE 6.7. Complete bipartite graph $K_{3,4}$.

Regular Graph: a simple graph whose vertices have all the same degree. For instance, the n-cube is regular.

6.1.3. Subgraph. Given a graph $G = (V, E)$, a *subgraph* $G' = (V', E')$ of G is another graph such that $V' \subseteq V$ and $E' \subseteq E$. If $V' = V$ then G' is called a *spanning subgraph* of G.

Given a subset of vertices $U \subseteq V$, the subgraph of G *induced* by U, denoted $\langle U \rangle$, is the graph whose vertex set is U, and its edge set contains all edges from G connecting vertices in U.