

**CS 310-0**  
**Homework Assignment No. 4**  
Due Tue 2/6/2001

1. Let  $f, g : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$  be the functions defined by  $f(x, y) = x + y$  and  $g(x, y) = x - y$ . Find  $f^{-1}(\{2, 3\}) \cap g^{-1}(\{-1, 0, 1\})$  (intersection of *preimages*).
2. Let  $A_1, A_2, B_1, B_2$  be non-empty sets such that  $A = A_1 \cap A_2 = \emptyset$  and  $B_1 \cap B_2 = \emptyset$ . Let  $f_1 : A_1 \rightarrow B_1, f_2 : A_2 \rightarrow B_2$  two given functions. Define a function  $f : A_1 \cup A_2 \rightarrow B_1 \cup B_2$  by cases in the following way:

$$f(x) = \begin{cases} f_1(x) & \text{if } x \in A_1 \\ f_2(x) & \text{if } x \in A_2 \end{cases}$$

Prove:

- (a)  $f$  is one-to-one if and only if  $f_1$  and  $f_2$  are one-to-one.
  - (b)  $f$  is onto if and only if  $f_1$  and  $f_2$  are onto.
  - (c)  $f$  is a one-to-one correspondence if and only if  $f_1$  and  $f_2$  are one-to-one correspondences. In this case, find  $f^{-1}$  in terms of  $f_1^{-1}$  and  $f_2^{-1}$ .
3. For each one of the following functions, determine if it is one-to-one (but not onto), onto (but not one-to-one), or a one-to-one correspondence. If it is a one-to-one correspondence, find its inverse.
    - (a)  $f : \mathbb{N} \rightarrow \mathbb{N}, f(x) = 2x^2 + 1$ .
    - (b)  $f : \mathbb{N} \rightarrow \mathbb{N}, f(x) = \lfloor (x+2)/3 \rfloor$ , where  $\lfloor x \rfloor =$  “greatest integer less than or equal to  $x$ ” (*floor* function.)
    - (c)  $f : \mathbb{Z} \rightarrow \mathbb{Z}$ , defined by cases in the following way:

$$f(x) = \begin{cases} x - 3 & \text{if } x \text{ is even} \\ x + 1 & \text{if } x \text{ is odd} \end{cases}$$

4. Find the properties (commutative, associative, existence of identity element, existence of inverse) verified by the following operations:
  - (a)  $x \circ y = x + y - a$  on  $\mathbb{Z}$ , where  $a$  is a fix integer.
  - (b)  $x * y = x + y - xy$  on  $\mathbb{Z}$ .

Is there any value of  $a$  for which the operations  $*$  and  $\circ$  defined above verify that  $*$  is distributive respect to  $\circ$ ?

5. Let  $\mathbb{Q}[\sqrt{2}]$  be the set  $\mathbb{Q}[\sqrt{2}] = \{x + y\sqrt{2} \mid x, y \in \mathbb{Q}\}$ .
  - (a) Prove that  $(\mathbb{Q}[\sqrt{2}], +)$  is a commutative group.
  - (b) Prove that  $(\mathbb{Q}[\sqrt{2}], +, \cdot)$  is a commutative ring with unity.
  - (c) Is  $(\mathbb{Q}[\sqrt{2}], +, \cdot)$  a field? Prove it or disprove it.
  - (d) Prove that the function  $f : (\mathbb{Q}[\sqrt{2}], +, \cdot) \rightarrow (\mathbb{Q}[\sqrt{2}], +, \cdot)$  defined

$$f(x + y\sqrt{2}) = x - y\sqrt{2}$$

is a ring-homomorphism.