PUTNAM TRAINING

November 11, 2025

Putnam Problems

Putnam 2003-A1. Let n be a fixed positive integer. How many ways are there to write n as a sum of positive integers, $n = a_1 + a_2 + \cdots + a_k$, with k an arbitrary positive integer and $a_1 \leq a_2 \leq \cdots \leq a_k \leq a_1 + 1$? For example, with $n = 4$ there are four ways: 4, 2+2, 1+1+2, 1+1+1+1.

Solution. The answer is n. More precisely, there is exactly one such sum with k terms for each of $k = 1, \ldots, n$.

In fact, since all parts differ by at most 1, there exist integers $m \geq 1$ and r such that the terms of the sum consist of $k - r$ copies of m and r copies of $m + 1$, with $0 \leq r < k$. Then

$$n = (k - r)m + r(m + 1) = km + r.$$

Dividing by k we get

$$\frac{n}{k} = m + \frac{r}{k}.$$

Since m is an integer and $0 \leq \frac{r}{k} < 1$ we have

$$m = \left\lfloor \frac{n}{k} \right\rfloor, \qquad r = n - km = n - k \left\lfloor \frac{n}{k} \right\rfloor.$$

Taking into account that $m = \lfloor \frac{n}{k} \rfloor$ and $m + 1 = \lceil \frac{n}{k} \rceil$, the corresponding partition is uniquely determined as

$$m = \underbrace{\left\lfloor \frac{n}{k} \right\rfloor + \cdots + \left\lfloor \frac{n}{k} \right\rfloor}_{k-r \text{ times}} + \underbrace{\left\lceil \frac{n}{k} \right\rceil + \cdots + \left\lceil \frac{n}{k} \right\rceil}_{r \text{ times}}.$$

For every integer k with $1 \leq k \leq n$, the construction above yields exactly one valid partition. Therefore, the total number of such partitions is n.

Putnam 2008-A1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that $f(x, y) + f(y, z) + f(z, x) = 0$ for all real numbers x, y, and z. Prove that there exists a function $g : \mathbb{R} \to \mathbb{R}$ such that $f(x, y) = g(x) - g(y)$ for all real numbers x and y.

Training session conducted by Miguel A. Lerma.

Solution. The function $g(x) = f(x, 0)$ works.

Substituting $(x, y, z) = (0, 0, 0)$ into the given functional equation yields $f(0, 0) = 0$.

Substituting $(x, y, z) = (x, 0, 0)$ yields $f(x, 0) + f(0, x) = 0$.

Finally, substituting $(x, y, z) = (x, y, 0)$ yields

$$f(x, y) = -f(y, 0) - f(0, x) = -f(y, 0) + f(x, 0) = g(x) - g(y).$$

Remark: A similar argument shows that the possible functions g are precisely those of the form $f(x, 0) + c$ for some c.

Putnam 1986-A2. What is the units (i.e., rightmost) digit of

$$\left\lfloor \frac{10^{20000}}{10^{100} + 3} \right\rfloor ?$$

Solution. The answer is 3.

We use the following identity:

$$x^{2n} - y^{2n} = (x + 1)(x^{2n-1} - x^{2n-2}y + x^{2n-3}y^2 - x^{2n-4}y^3 + \cdots + xy^{2n-2} - y^{2n-1}).$$

Letting $x = 10^{100}$ and $y = 3$ we get

$$I = \frac{10^{20000} - 3^{200}}{10^{100} + 3} = \frac{x^{200} - y^{200}}{x + y} = xM - y^{199} = 10^{100}M - 3^{199},$$

where M is an integer. So I is an integer.

On the other hand, since

$$0 < \frac{10^{20000}}{10^{100} + 3} - I = \frac{3^{200}}{10^{100} + 3} < 1$$

we have

$$\left\lfloor \frac{10^{20000}}{10^{100} + 3} \right\rfloor = I = 10^{100}M - 3^{199}.$$

Finally the rightmost digit of I can be found as the 1-digit number congruent to -3^{199} (mod 10). The sequence 3^n (mod 10) $= 3, 9, 7, 1, 3, 9, 7, 1, 3, \dots$ has period 4 and $199 = 3 + 4 \cdot 49$, hence $-3^{199} \bmod 10 = -3^3 \bmod 10 = -27 \bmod 10 = 3$. Hence the units digit of I is 3.

Pending Issues: A Challenge and a Discussion

Challenge. As an exercise in modular arithmetic I challenged the students to compute the last two digits of Graham's number, due by the next training session. Graham's number is an astronomically large power-tower of the form

$$G = 3^{3^{3^{3^{\cdot^{\cdot^{\cdot}}}}}}.$$

Discussion. Putnam problem 2008-A1 asks the following:

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that $f(x, y) + f(y, z) + f(z, x) = 0$ for all real numbers x, y, and z. Prove that there exists a function $g : \mathbb{R} \to \mathbb{R}$ such that $f(x, y) = g(x) - g(y)$ for all real numbers x and y.

Question. Is the following argument valid?

> By way of contradiction, assume $f(x, y) \neq g(x) - g(y)$. Then
> $$f(x, y) + f(y, z) + f(z, x) \neq g(x) - g(y) + g(y) - g(z) + g(z) - g(x) = 0,$$
> contradiction.

Answer. Here is a detailed analysis of the logic behind it and why the argument is not valid.

We want to prove that

$$\exists g : \mathbb{R} \to \mathbb{R} \text{ such that } f(x, y) = g(x) - g(y) \quad \forall x, y \in \mathbb{R}.$$

Its logical negation is

$$\forall g : \mathbb{R} \to \mathbb{R},\ \exists x, y \in \mathbb{R} \text{ such that } f(x, y) \neq g(x) - g(y).$$

In the argument one assumes the existence of such a function g and then manipulates expressions involving it. However, under the negation we have no such g available: we only know that *no* function g satisfies the required property. Hence, it is illegitimate to continue the argument using a g that we do not know to exist.

Formally, the mistake is confusing

$$\forall g \; \big(P(g) \Rightarrow Q(g)\big) \quad \text{with} \quad \exists g \; \big(P(g) \Rightarrow Q(g)\big),$$

and the first statement does not imply the second unless such a g actually exists.

An analogy is the vacuous truth of a universal statement over the empty set:

$$\forall x \in \emptyset,\ P(x)$$

is true for any predicate P, but this does not entail the existence of any x satisfying $P(x)$. Likewise, a statement of the form $\forall g\,(\dots)$ does not guarantee that there exists a g for which the condition holds. Therefore, we cannot proceed by assuming a function g whose existence has not yet been established.