PUTNAM TRAINING

November 18, 2025

Putnam Problems

Putnam 1986-A3. Evaluate $\sum_{n=0}^{\infty} \operatorname{arccot}(n^2+n+1)$, where $\operatorname{arccot}(t)$ for $t \geq 0$ denotes the number θ in the interval $0 < \theta \leq \pi/2$ with $\cot\theta = t$.

Solution. The answer is $\dfrac{\pi}{2}$.

In fact, we have

$$S = \sum_{n=0}^{\infty} \operatorname{arccot}(n^2+n+1).$$

Using $t = \cot\theta = 1/\tan\theta$ for $\theta \in (0, \pi/2]$, so $\tan\theta = 1/\cot\theta = 1/t$, and we get

$$\theta = \operatorname{arccot}(t) = \arctan\left(\frac{1}{t}\right)$$

for $t > 0$, hence

$$\operatorname{arccot}(n^2+n+1) = \arctan\left(\frac{1}{n^2+n+1}\right).$$

Next let $\theta_n = \operatorname{arccot} n = \arctan\frac{1}{n}$, $\tan\theta_n = \frac{1}{n}$. Using the tangent addition/subtraction formula we have

$$\tan(\theta_n - \theta_{n+1}) = \frac{\tan\theta_n - \tan\theta_{n+1}}{1+\tan\theta_n \tan\theta_{n+1}} \quad \Rightarrow \quad \theta_n - \theta_{n+1} = \arctan\left(\frac{\tan\theta_n - \tan\theta_{n+1}}{1+\tan\theta_n \tan\theta_{n+1}}\right),$$

so for $n \geq 1$,

$$\arctan\left(\frac{1}{n}\right) - \arctan\left(\frac{1}{n+1}\right) = \arctan\left(\frac{\frac{1}{n} - \frac{1}{n+1}}{1 + \frac{1}{n(n+1)}}\right) = \arctan\left(\frac{1}{n^2+n+1}\right).$$

Hence,

$$\operatorname{arccot}(n^2+n+1) = \arctan\left(\frac{1}{n}\right) - \arctan\left(\frac{1}{n+1}\right) \quad \text{for } n \geq 1.$$

Training session conducted by Miguel A. Lerma.

The first term of the sum is $\operatorname{arccot}(1) = \frac{\pi}{4}$, and the rest of the series telescopes:

$$\begin{aligned}
S &= \arctan(1) + \sum_{n=1}^{\infty} \operatorname{arccot}(n^2+n+1) \\
&= \frac{\pi}{4} + \sum_{n=1}^{\infty} \left[\arctan\left(\frac{1}{n}\right) - \arctan\left(\frac{1}{n+1}\right)\right] \\
&= \frac{\pi}{4} + \left(\arctan(1) - \lim_{N\to\infty} \arctan\frac{1}{N+1}\right) \\
&= \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}.
\end{aligned}$$

Putnam 1984-B1. Let n be a positive integer, and define

$$f(n) = 1! + 2! + \cdots + n!\,.$$

Find polynomials $P(x)$ and $Q(x)$ such that

$$f(n+2) = P(n)f(n+1) + Q(n)f(n)$$

for all $n \geq 1$.

Solution. We have

$$f(n+2) - f(n+1) = (n+2)! = (n+2)(n+1)! = (n+2)(f(n+1) - f(n))\,,$$

hence

$$\begin{aligned}
f(n+2) &= (n+2)(f(n+1) - f(n)) + f(n+1) \\
&= (n+3)f(n+1) - (n+2)f(n)\,,
\end{aligned}$$

and we can take $P(x) = x+3$, $Q(x) = -x-2$.

Putnam 2010-A1. Given a positive integer n, what is the largest k such that the numbers $1, 2, \ldots, n$ can be put into k boxes so that the sum of the numbers in each box is the same? [When $n = 8$, the example $\{1,2,3,6\}, \{4,8\}, \{5,7\}$ shows that the largest k is *at least* 3.]

Solution. The answer is $k = \lfloor \frac{n+1}{2} \rfloor = \lceil \frac{n}{2} \rceil$.

To get many boxes we need to put as few elements as possible in each box. Since n itself must be in one of the boxes, the common sum cannot be less than n, hence $k \leq (1+\cdots+n)/n = (n+1)/2$. The largest k with that property is k is $\lfloor \frac{n+1}{2} \rfloor$. I remains to show that the bound is tight. In fact, for n even, this value is achieved by the partition

$$\{1, n\}, \{2, n-1\}, \ldots,$$

and for n odd, it is achieved by the partition

$$\{n\}, \{1, n-1\}, \{2, n-2\}, \ldots$$

Putnam 2010-B1. Is there an infinite sequence of real numbers $a_1, a_2, a_3, \ldots$ such that
$$a_1^m + a_2^m + a_3^m + \cdots = m$$
for every positive integer m?

Solution. No such sequence exists. We provide two proofs of this claim.

- *First proof:* If it did, then the Cauchy-Schwartz inequality would imply
$$\begin{aligned} 8 = 2 \cdot 4 &= (a_1^2 + a_2^2 + \cdots)(a_1^4 + a_2^4 + \cdots) \\ &\geq (a_1^3 + a_2^3 + \cdots)^2 = 9, \end{aligned}$$
contradiction.

- *Second proof:* Suppose that such a sequence exists. If $a_k^2 \in [0, 1]$ for all k, then $a_k^4 \leq a_k^2$ for all k, and so
$$4 = a_1^4 + a_2^4 + \cdots \leq a_1^2 + a_2^2 + \cdots = 2,$$
contradiction. There thus exists a positive integer k for which $a_k^2 > 1$. However, in this case, for m large, $a_k^{2m} > 2m$ and so $a_1^{2m} + a_2^{2m} + \cdots \neq 2m$.

Problems from other competitions

Spanish Mathematical Olympiad 1971, Problem No. 4. Prove that in a triangle with sides a, b, c and opposite angles A, B, C (in radians) the following relation holds:
$$\frac{a\,A + b\,B + c\,C}{a + b + c} \geq \frac{\pi}{3}\,.$$
You may assume $a \geq b \geq c \;\Rightarrow\; A \geq B \geq C$.

Solution. Actually more is true:
$$\frac{\pi}{3} \leq \frac{a\,A + b\,B + c\,C}{a + b + c} < \frac{\pi}{2}.$$

For the first inequality, assume $a \geq b \geq c$, $A \geq B \geq C$. Then
$$\begin{aligned} 0 &\leq (a - b)(A - B) + (a - c)(A - C) + (b - c)(B - C) \\ &= 3(aA + bB + cC) - (a + b + c)(A + B + C)\,. \end{aligned}$$
Using $A + B + C = \pi$ and dividing by $3(a + b + c)$ we get the desired result. Equality holds precisely when $a = b = c$, and $A = B = C$. [Note: This could also be proved by using *Chebyshev's Inequality.*]

For the second inequality, use the triangle inequalities
$$a + b + c > 2a, \quad a + b + c \;\; > 2b, \quad a + b + c > 2c.$$

Multiplying by A, B and C and adding we get

$$(a+b+c)(A+B+C) > 2(aA+bB+cC),$$

hence

$$\frac{a\,A+b\,B+c\,C}{a+b+c} < \frac{\pi}{2}.$$

Addendum. The problem allows to assume that $a \ge b \ge c \implies A \ge B \ge C$, but the statement has a proof, as shown below.

Proof. We prove that $a \ge b \ge c \Rightarrow A \ge B \ge C$ using the Law of Cosines.

Since $\cos x$ is strictly decreasing on $(0, \pi)$, to show $A \ge B$ it suffices to show

$$\cos A \le \cos B.$$

Using the Law of Cosines,

$$\cos A = \frac{b^2+c^2-a^2}{2bc}, \qquad \cos B = \frac{c^2+a^2-b^2}{2ca}.$$

Compute the difference:

$$\cos B - \cos A = \frac{c^2+a^2-b^2}{2ca} - \frac{b^2+c^2-a^2}{2bc}.$$

Multiplying by the positive quantity $2abc$,

$$2abc\,(\cos B - \cos A) = b(c^2+a^2-b^2) - a(b^2+c^2-a^2).$$

Expanding and regrouping yields

$$2abc\,(\cos B - \cos A) = (b-a)\,(c^2+a^2-b^2).$$

Under the assumption $a \ge b \ge c$, we have

$$b-a \le 0, \qquad c^2+a^2-b^2 > 0.$$

Thus

$$\cos B - \cos A \le 0 \quad \implies \quad \cos A \le \cos B \quad \implies \quad A \ge B.$$

The proof that $B \ge C$ is analogous. Hence

$$a \ge b \ge c \implies A \ge B \ge C.$$

$\square$

Killer Problem 1. Find all functions $F(x) : \mathbb{R} \to \mathbb{R}$ having the property that for any x_1, x_2 the following inequality holds:

$$F(x_1) - F(x_2) \leq (x_1 - x_2)^2.$$

Solution. Since the inequality must still hold after swapping x_1 and x_2 we actually have $|F(x_1) - F(x_2)| \leq (x_1 - x_2)^2$. Next, for any x and $h \neq 0$ let $x_2 = x$, $x_1 = x + h$. Then, the given inequality implies

$$\frac{|F(x+h) - F(x)|}{|h|} = \frac{|F(x_1) - F(x_2)|}{|x_1 - x_2|} \leq |x_1 - x_2| = |h| \longrightarrow 0$$

as $h \to 0$. So F is differentiable and $F'(x) = 0$, hence F is constant. Since all constant functions verify the inequality, then F verifies the inequality if and only if F is constant.

Challenges

Challenge. Compute the last two digits of Graham's number. Graham's number is an astronomically large power-tower of the form

$$G = 3^{3^{3^{3^{\cdot^{\cdot^{\cdot}}}}}}.$$

Solution. The answer is 87.

Partial Proof. Here we prove the simpler statement that the last digit of G is 7.

Claim. The last digit of Graham's number is 7.

Compute the last digit of the first few powers of 3:

$$3^1 = 3, \quad 3^2 = 9, \quad 3^3 = 27, \quad 3^4 = 81.$$

Thus, modulo 10,

$$3, 9, 7, 1, 3, 9, 7, 1, \ldots$$

so the last digit of 3^n depends only on $n \bmod 4$. That is,

$$3^n \equiv \begin{cases} 3 & \text{if } n \equiv 1 \pmod 4, \\ 9 & \text{if } n \equiv 2 \pmod 4, \\ 7 & \text{if } n \equiv 3 \pmod 4, \\ 1 & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

Graham's number is a power tower of 3's, hence $n = 3^m$. The sequence of powers 3^m modulo 4 are

$$3^1 \equiv 3, \quad 3^2 = 9 \equiv 1, \quad 3^3 = 27 \equiv 3, \quad \ldots \qquad \pmod 4$$

repeating with period 2, hence $3^m \bmod 4$ depends only on the parity of m.

In Graham's number m is again a power of 3, hence odd. Therefore $n = 3^m \equiv 3 \pmod 4$, and $3^n \equiv 7 \pmod{10}$.

Hence, the last digit of Graham's number is 7. $\square$

Next, the full proof.

Full proof. The last two digits of G are $3^E \bmod 100$, where E is an extremely large power tower of 3's as defined in the construction of Graham's number.

Since $100 = 4 \times 25$ and $\gcd(4, 25) = 1$, we can use the Chinese Remainder Theorem (CRT), that allows us to find $3^E \bmod 100$ by computing $3^E \bmod 4$ and $3^E \bmod 25$ separately. Here we do not really need the full force of the CRT, but we will use that the problem of finding $3^E \bmod 100$ can be separated into the two subproblems $3^E \bmod 4$ and $3^E \bmod 25$.

- Modulo 4:

We have that the order of 3 modulo 4 is 2 because

$$3^1 \equiv 3, \quad 3^2 \equiv 1 \pmod 4,$$

hence

$$3^E \equiv 3 \pmod 4,$$

because E is odd.

- Modulo 25:

We could do the same and compute successive powers of 3 modulo 25 until the sequence repeats, but we can take a shortcut by using Euler's theorem

$$a^{\varphi(n)} \equiv 1 \pmod n \quad \text{if} \quad \gcd(a, n) = 1,$$

where $\varphi(n)$ is Euler's totient function. If p is prime then $\varphi(p^k) = p^{k-1}(p-1)$, so for $n = 25 = 5^2$ we have $\varphi(5^2) = 5 \times 4 = 20$, hence

$$3^{\varphi(25)} \equiv 3^{20} \equiv 1 \pmod{25},$$

i.e., $3^{20k+r} \equiv 3^r \pmod{25}$, so it suffices to find $E \bmod 20$. (*Note*: this does not necessarily means that the order of 3 modulo 25 is 20, it could be shorter, but $3^{20} \equiv 1 \pmod{25}$ is enough for our purposes.)

Next, we compute $E \pmod{20}$. Let $E = 3^m$ for some very large m. The order of 3 modulo 20 is 4 because

$$3^1 \equiv 3, \quad 3^2 \equiv 9, \quad 3^3 \equiv 7, \quad 3^4 \equiv 1 \pmod{20}.$$

Hence

$$3^m \bmod 20 \text{ depends only on } m \bmod 4.$$

Since m is odd, $m \equiv 3 \pmod 4$, so

$$E \equiv 3^3 \equiv 27 \equiv 7 \pmod{20}.$$

Therefore,

$$3^E \equiv 3^7 \pmod{25}.$$

- Modulo 100:

So we have

$$G \equiv 3^E \equiv 3 \pmod 4,$$
$$G \equiv 3^E \equiv 3^7 \pmod{25}.$$

Since 7 is odd we have $3^7 \equiv 3 \pmod 4$, hence $G \equiv 3^7 \pmod 4$ and $G \equiv 3^7 \pmod{25}$, in other words, $4 \mid G - 3^7$ and $25 \mid G - 3^7$, hence $100 \mid G - 3^7$, i.e., $G \equiv 3^7 \pmod{100}$.

So, compute:

$$3^4 = 81, \quad 3^5 \equiv 43, \quad 3^6 \equiv 29, \quad 3^7 \equiv 87 \pmod{100}.$$

Hence, he last two digits of Graham's number are 87 as claimed. $\square$

USEFUL RESULTS

The Chinese Remainder Theorem. Let $m_1, m_2, \dots, m_k$ be pairwise coprime positive integers, and let

$$M = m_1 m_2 \cdots m_k.$$

For any integers $a_1, a_2, \dots, a_k$, the system of congruences

$$x \equiv a_i \pmod{m_i}, \qquad i = 1, \dots, k,$$

has a solution, and this solution is unique modulo M.

The solution can be found using the following formula.

Define

$$M_i = \frac{M}{m_i},$$

and let N_i be the modular inverse of M_i modulo m_i, that is,

$$M_i N_i \equiv 1 \pmod{m_i}.$$

Then the solution to the system is given by

$$x \equiv \sum_{i=1}^{k} a_i \, M_i \, N_i \pmod{M}.$$

Euler's theorem. Let $\varphi(n)$ be Euler's totient function, defined as the number of positive integers less that or equal n that are relatively prime to n. Then, if $\gcd(a, n) = 1$:

$$a^{\varphi(n)} \equiv 1 \pmod{n}.$$

Euler's totient function can be computed for $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ (where the p_i are distinct primes) as

$$\varphi(n) = \prod_{i=1}^{k} p_i^{e_i - 1}(p_i - 1).$$

Cauchy-Schwarz Inequality (dot product form). For any vectors $\mathbf{u} = (u_1, \dots, u_n)$, $\mathbf{v} = (v_1, \dots, v_n)$ in $\mathbb{R}^n$,

$$|\mathbf{u} \cdot \mathbf{v}| \ \le \ \|\mathbf{u}\| \, \|\mathbf{v}\|,$$

where

$$\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{n} u_i v_i, \qquad \|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}}.$$

Equivalently:

$$\left(\sum_{i=1}^{n} u_i v_i \right)^2 \le \left(\sum_{i=1}^{n} u_i^2 \right) \left(\sum_{i=1}^{n} v_i^2 \right).$$

Equality holds if and only if $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent.

The inequality also applies to infinite series such that $\sum_{i=1}^{\infty} |u_i|^2 < \infty$ and $\sum_{i=1}^{\infty} |v_i|^2 < \infty$.

Chebyshev Inequality. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be sequences of real numbers which are monotonic in the same direction (we have $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$, or we could reverse all inequalities.) Then

$$\frac{1}{n}\sum_{i=1}^{n} a_i b_i \geq \left(\frac{1}{n}\sum_{i=1}^{n} a_i\right)\left(\frac{1}{n}\sum_{i=1}^{n} b_i\right).$$