

CS 310 (sec 20) - Spring 2005 - Midterm Exam (solutions)

SOLUTIONS

1. (Logic)

- (1) Show that $\neg(p \leftrightarrow q)$ and $(\neg p) \leftrightarrow q$ are logically equivalent.
- (2) Determine the truth value of each of these statements if the universe of discourse of each variable consists of all integers.
 - (a) $\forall n(n^2 \geq 0)$ (b) $\exists n(n^2 = 2)$ (c) $\forall n(n^2 \geq n)$ (d) $\exists n(n^2 < 0)$
- (3) Determine the truth value of each of these statements if the universe of discourse of each variable consists of all real numbers.
 - (a) $\exists x(x^2 = 2)$ (b) $\exists x(x^2 = -1)$ (c) $\forall x(x^2 + 2 \geq 1)$ (d) $\forall x(x^2 \neq x)$

Solution:

- (1) Using a truth table:

p	q	$\neg p$	$p \leftrightarrow q$	$\neg(p \leftrightarrow q)$	$(\neg p) \leftrightarrow q$
T	T	F	T	F	F
T	F	F	F	T	T
F	T	T	F	T	T
F	F	T	T	F	F

They have the same truth values

- (2)
 - (a) TRUE (n^2 is always in fact non-negative).
 - (b) FALSE ($\sqrt{2}$ is not an integer.)
 - (c) TRUE (it can be false only for $0 < n < 1$, but there are no integers between 0 and 1.)
 - (d) FALSE (it is the negation of (a)).
- (3)
 - (a) TRUE ($\sqrt{2}$ is a real number.)
 - (b) FALSE ($i = \sqrt{-1}$ is not real.)
 - (c) TRUE (since $x^2 \geq 0$).
 - (d) FALSE (counterexamples: $x = 0$ and $x = 1$.)

2. (Sets)

1. Let $A_i = \{1, 2, 3, \dots, i\}$ for $i = 1, 2, 3, \dots$. Find

(a) $\bigcup_{i=1}^n A_i$

(b) $\bigcap_{i=1}^n A_i$

2. Let $A_i = \{\dots, -2, -1, 0, 1, 2, 3, \dots, i\}$ for $i = 1, 2, 3, \dots$. Find

(a) $\bigcup_{i=1}^n A_i$

(b) $\bigcap_{i=1}^n A_i$

3. Let A_i be the set of all nonempty bit strings (that is, bit strings over $\{0, 1\}$ of length at least one) of length not exceeding i . Find

(a) $\bigcup_{i=1}^n A_i$

(b) $\bigcap_{i=1}^n A_i$

Solution:

1. (a) $\bigcup_{i=1}^n A_i = A_n = \{1, 2, 3, \dots, n\}$

(b) $\bigcap_{i=1}^n A_i = A_1 = \{1\}$

2. (a) $\bigcup_{i=1}^n A_i = A_n = \{\dots, -2, -1, 0, 1, 2, 3, \dots, n\}$

(b) $\bigcap_{i=1}^n A_i = A_1 = \{\dots, -2, -1, 0, 1\}$

3. (a) $\bigcup_{i=1}^n A_i = A_n = \{s \in \{0, 1\}^* \mid 1 \leq |s| \leq n\}$

(b) $\bigcap_{i=1}^n A_i = A_1 = \{0, 1\}$

3. (Relations) For each of the following relations defined on the set of positive integers \mathbb{Z}^+ , determine which ones are partial orders, and which ones are equivalence relations (some may be both or none).

- (a) $x \mathcal{R} y \Leftrightarrow x \leq y$.
- (b) $x \mathcal{R} y \Leftrightarrow x < y$.
- (c) $x \mathcal{R} y \Leftrightarrow x \nless y$.
- (d) $x \mathcal{R} y \Leftrightarrow$ for every prime number p , if p divides x then p divides y .
- (e) $x \mathcal{R} y \Leftrightarrow$ for every prime number p , p divides x if and only if p divides y .
- (f) $x \mathcal{R} y \Leftrightarrow x = y$.

Solution:

- (a) Partial order.
- (b) None (it is not reflexive).
- (c) Partial order (it is the same as $x \geq y$.)
- (d) None. It is not an order relation because it is not antisymmetric (e.g., $2 \mathcal{R} 4$ and $4 \mathcal{R} 2$, but $2 \neq 4$.) It is not an equivalence relation because it is not symmetric (e.g. $2 \mathcal{R} 6$, but 6 is not related to 2.)
- (e) Equivalence— x and y are related precisely when they are divided by the same set of primes. It is not a partial order because it is not antisymmetric (e.g., $12 \mathcal{R} 18$ and $18 \mathcal{R} 12$, because both are divided precisely by the primes 2 and 3, but $12 \neq 18$.)
- (f) Both, it is an *equivalence* relation (obviously), and it is also a *partial order* relation. Note that it verifies the antisymmetric property, because $x \mathcal{R} y$ and $y \mathcal{R} x$ means $x = y$ and $y = x$, which in fact implies $x = y$.

4. (Algorithms) Find a Θ notation for the value $f(n)$ returned by the following recursive algorithm:

```
1: procedure f(n)
2:   if n ≤ 1 then
3:     return 1
4:   else
5:     return (2 * f(n-2))
6: end f
```

Solution:

(Note that the question is about the *value* of the function, not about the complexity of the algorithm.)

We have that if $n \leq 0$ then $f(n) = 1$, otherwise

$$f(n) = 2f(n-2) = 4f(n-4) = 8f(n-6) = 16f(n-8) = 32f(n-10) = \dots$$

If n is even the last term is $2^{n/2}f(0) = 2^{n/2}$.

If n is odd then we get $2^{(n-1)/2}f(1) = 2^{(n-1)/2} = \frac{1}{\sqrt{2}}2^{n/2}$.

Hence $\frac{1}{\sqrt{2}}2^{n/2} \leq f(n) \leq 2^{n/2}$, so the answer is

$$\boxed{f(n) = \Theta(2^{n/2})}$$

5. (Integers)

- (a) Remember that $\phi(m)$ = number of positive integers not greater than m and relatively prime to m . Find $\phi(11)$.
- (b) Find $3^{98765432} \bmod 11$ = the remainder when $3^{98765432}$ is divided by 11. In other words, if $3^{98765432} = 11q + r$ where q and r are integers, and $0 \leq r < 11$, what is r ?

Solution:

- (a) Since 11 is prime we have $\phi(11) = 11 - 1 = \boxed{10}$.
- (b) We have $98765432 \equiv 2 \pmod{10}$, hence (by Euler's theorem) $3^{98765432} \equiv 3^2 \equiv 9 \pmod{11}$. Hence $3^{98765432} \bmod 11 = \boxed{9}$.

6. (Counting) How many ways are there to select four pieces of fruit from a bowl containing apples, oranges, and pears if the order in which the pieces are selected does not matter, only the type of fruit and not the individual piece matters, and there are at least four pieces of each type of fruit in the bowl?

Solution:

This is equivalent to finding the number of non-negative solutions to the following equation:

$$x_1 + x_2 + x_3 = 4.$$

The answer is $C(4 + 3 - 1, 4) = C(6, 4) = \frac{6!}{4!2!} = \boxed{15}$.