2.5. Applications to Physics and Engineering

2.5.1. Work. Work is the energy produced by a force F pushing a body along a given distance d. If the force is constant, the work done is the product

$$W = F \cdot d\,.$$

The SI (international) unit of work is the joule (J), which is the work done by a force of one Newton (N) pushing a body along one meter (m). In the American system a unit of work is the foot-pound. Since 1 N = 0.224809 lb and 1 m = 3.28084 ft, we have 1 J = 0.737561 ft lb.

More generally, assume that the force is variable and depends on the position. Let $F(x)$ be the force function. Assume that the force pushes a body from a point $x = a$ to another point $x = b$. In order to find the total work done by the force we divide the interval $[a, b]$ into small subintervals $[x_{i-1}, x_i]$ so that the change of $F(x)$ is small along each subinterval. Then the work done by the force in moving the body from x_{i-1} to x_i is approximately:

$$\Delta W_i \approx F(x_i^*)\,\Delta x\,,$$

where $\Delta x = x_i - x_{i-1} = (b - a)/n$ and x_i^* is any point in $[x_{i-1}, x_i]$. So, the total work is

$$W = \sum_{i=1}^{n} \Delta W_i \approx \sum_{i=1}^{n} F(x_i^*)\,\Delta x\,.$$

As $n \to \infty$ the Riemann sum at the right converges to the following integral:

$$W = \int_a^b F(x)\,dx\,.$$

2.5.2. Elastic Springs. Consider a spring on the x-axis so that its right end is at $x = 0$ when the spring is at its rest position. According to *Hook's Law*, the force needed to stretch the spring from 0 to x is proportional to x, i.e.:

$$F(x) = kx\,,$$

where k is the so called *spring constant*.

The energy needed to stretch the spring from 0 to a is then the integral

$$W = \int_0^a kx\,dx = k\frac{a^2}{2}\,.$$

2.5.3. Work Done Against Gravity. According to Newton's Law, the force of gravity at a distance r from the center of the Earth is

$$F(r) = \frac{k}{r^2}\,,$$

where k is some positive constant.

The energy needed to lift a body from a point at distance R_1 from the center of the Earth to another point at distance R_2 is given by the following integral

$$W = \int_{R_1}^{R_2} \frac{k}{r^2}\,dr = \left[-\frac{k}{r}\right]_{R_1}^{R_2} = k\left(\frac{1}{R_1} - \frac{1}{R_2}\right).$$

Example: Find the energy needed to lift 1000 Km a body whose weight is 1 N at the surface of Earth. The Earth radius is 6378 Km.

Answer: First we must determine the value of the constant k in this case. Since the weight of the body for $r = 6378$ Km is 1 N we have $k/6378^2 = 1$, so $k = 6378^2$. Next we have $R_1 = 6378$, $R_2 = 6378 + 1000 = 7378$, hence

$$W = 6378^2\left(\frac{1}{6378} - \frac{1}{7378}\right) = 864.462 \text{ N Km}\,.$$

Since 1 Km = 1000 m, the final result in joule is

$$864.462 \text{ N Km} = 864.462 \text{ N} \times 1000 \text{ m} = 864462 \text{ J}\,.$$

2.5.4. Work Done Filling a Tank. Consider a tank whose bottom is at some height $y = a$ and its top is at $y = b$. Assume that the area of its cross section is $A(y)$. We fill the tank by lifting from the ground ($y = 0$) tiny layers of thickness dy each. Their weight is $dF = \rho A(y)\,dy$, where ρ is the density of the liquid that we are putting in the tank. The work needed to lift each layer is

$$dW = dF \cdot y = \rho y A(y)\,dy\,.$$

Hence, the work needed to fill the tank is

$$W = \int_a^b \rho y A(y)\,dy\,.$$

2.5.5. Emptying a Tank. Consider a tank like the one in the previous paragraph. Now we empty it by pumping its liquid to a fix height h. The analysis of the problem is similar to the previous paragraph, but now the work done to pump a tiny layer of thickness dy is

$$dW = dF \cdot (h - y) = \rho\,(h - y) A(y)\,dy\,.$$

Hence the total work needed to empty the tank is

$$W = \int_a^b \rho\,(h - y) A(y)\,dy\,.$$

2.5.6. Force Exerted by a Liquid Against a Vertical Wall. The pressure p of an homogeneous liquid of density ρ at depth h is

$$p = \rho h\,.$$

When the pressure is constant, the force exerted by the liquid against a surface is the product of the pressure and the area of the surface. However the pressure against a vertical wall is not constant because it depends on the depth.

Assume that the surface of the liquid is at $y = c$ and we place a vertical plate of width $w(y)$ between $y = a$ and $y = b$. The force exerted at y (so at depth $h = c - y$) against a small horizontal strip of height dy and width $w(t)$ (area $= w(t)\,dy$) is

$$dF = \rho\,(c - y) w(y)\,dy\,.$$

hence the total force is

$$F = \int_a^b \rho\,(c - y) w(y)\,dy\,.$$

Example: A cylindrical tank of radius 1 m and full of water ($\rho = 9800$ N/m^3) is lying on its side. What is the pressure exerted by the water on its (vertical) bottom?

Answer: We assume the center of the tank is at $y = 0$, so the top of the liquid is at $y = 1$, and its bottom is at $y = -1$. On the other hand we obtain geometrically $w(y) = 2\sqrt{1 - y^2}$. Hence the total force is:

$$F = \int_{-1}^{1} 9800 \cdot (1 - y) 2\sqrt{1 - y^2}\,dy = 19800 \cdot \frac{\pi}{2} = 30787.6 \text{ N}\,.$$

2.5.7. Moments, Center of Mass. Here we will find a point in a solid in which it can be balanced. That point is called *center of mass.*

In the particular case when the solid consists of just two particles, we can use the Law of the Lever, discovered by Archimedes. If two mases m_1 and m_2 are attached to a rod of negligible mass on opposite sides of a fulcrum at distance d_1, d_2 from the fulcrum, the rod will balance if

$$m_1 d_1 = m_2 d_2 \,.$$

If the particles lie along the x axis at coordinates x_1 and x_2 respectively (with $x_1 < x_2$), and $\overline{x}$ is the coordinate of the center of mas, then $d_1 = \overline{x} - x_1$, $d_2 = x_2 - \overline{x}$, so

$$\begin{aligned} m_1(\overline{x} - x_1) &= m_2(x_2 - \overline{x}) \\ m_1\overline{x} + m_2\overline{x} &= m_1 x_1 + m_2 x_2 \\ \overline{x} &= \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \,. \end{aligned}$$

The numbers $m_1 x_1$ and $m_2 x_2$ are called *moments* of the masses m_1 and m_2. So the center of masses is found by dividing the total moment by the total mass. More generally, for a solid made up of n particles of masses m_i placed on the x-axis at points x_i:

$$\overline{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} = \frac{M}{m} \,,$$

where $M = \sum_{i=1}^{n} m_i x_i$ is the total moment of the solid, and $m = \sum_{i=1}^{n} m_i$ is its total mass.

If we now consider particles distributed in a plane with coordinates (x_i, y_i), then the moment of the system about the y-axis is

$$M_y = \sum_{i=1}^{n} m_i x_i \,,$$

and the moment of the system about the x-axis is

$$M_x = \sum_{i=1}^{n} m_i y_i \,.$$

M_y measures the tendency of the solid to rotate about the y-axis, and M_x measures the tendency of the solid to rotate about the x-axis.

The coordinates of the center of mass $(\overline{x}, \overline{y})$ are now

$$\overline{x} = \frac{M_y}{m}, \quad \overline{y} = \frac{M_x}{m}.$$

Now consider a flat plate of uniform density ρ occupying the plane region under a curve $y = f(x)$ between $x = a$ and $x = b$. Slices parallel to the y axis of width Δx and length $f(x)$ have mass $\rho f(\overline{x})\,\Delta x$, and their moment about the y-axis will be $x\rho f(x)\,\Delta x$. The sum of their moments in the limit as $\Delta x \to 0$ is the integral:

$$M_y = \rho \int_a^b x f(x)\,dx\,.$$

The moment about the x-axis of a slice can be found taking into account that by symmetry its center of mass is at distance $f(x)/2$ from the x-axis, so the moment is $\frac{1}{2}f(x)\rho f(\overline{x})\,\Delta x = \rho\frac{1}{2}[f(x)]^2\,\Delta x$. Adding and taking the limit we get the integral

$$M_x = \rho \int_a^b \tfrac{1}{2}[f(x)]^2\,dx\,.$$

The total mass of the plate is its density times its area:

$$m = \rho \int_a^b f(x)\,dx\,.$$

Hence, the coordinates of its center of mass are:

$$\overline{x} = \frac{M_y}{m} = \frac{\rho\int_a^b x f(x)\,dx}{\rho\int_a^b f(x)\,dx} = \frac{\int_a^b x f(x)\,dx}{\int_a^b f(x)\,dx}$$

$$\overline{y} = \frac{M_x}{m} = \frac{\rho\int_a^b \frac{1}{2}[f(x)]^2\,dx}{\rho\int_a^b f(x)\,dx} = \frac{\int_a^b \frac{1}{2}[f(x)]^2\,dx}{\int_a^b f(x)\,dx}\,.$$

Example: Find the center of mass of a semicircular plate of radius r.

Answer: We use coordinates so that the plate occupies the region under the graph of $y = \sqrt{r^2 - x^2}$, $-r \leq x \leq r$. The area of the semicircle is $A = \pi r^2/2$. By symmetry $\overline{x} = 0$, so we only need to find $\overline{y}$:

$$\begin{aligned}\overline{y} &= \frac{1}{A}\int_{-r}^{r} \tfrac{1}{2}\{f(x)\}^2\,dx = \frac{1}{\pi r^2/2}\cdot\tfrac{1}{2}\int_{-r}^{r}(\sqrt{r^2-x^2})^2\,dx \\ &= \frac{2}{\pi r^2}\int_0^r (r^2-x^2)\,dx = \frac{2}{\pi r^2}\left[r^2x-\frac{x^3}{3}\right]_0^r\,dx \\ &= \frac{2}{\pi r^2}\left(r^3-\frac{r^3}{3}\right) = \frac{2}{\pi r^2}\frac{2r^3}{3} = \boxed{\frac{4r}{3\pi}}.\end{aligned}$$