**CS 310-0**
**Homework Assignment No. 2**
Due Fri 4/14/2000

1. Using the Principle of Extension, prove the following ($\triangle$ is *symmetric difference*):
   (a) $A \triangle (B \triangle C) = (A \triangle B) \triangle C$.
   (b) $A \cap (B \triangle C) = (A \cap B) \triangle (A \cap C)$.
   You may use properties stated in the Notes or proven in previous homework assignments.

2. Prove that the following statements are equivalent:
   (a) $A \subseteq B$.
   (b) $A \cap B = A$.
   (c) $A \cup B = B$.
   (d) $A - B = \emptyset$.
   (One way to prove the equivalence is to prove the chain of implications: (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (a).)

3. Prove the following:
   (a) $A \triangle B = \emptyset \Leftrightarrow A = B$.
   (b) $A \triangle B \subseteq S \Leftrightarrow A - S = B - S$.

4. For $n \geq 1$ let $A_n$ be the following interval: $A_n = [-2, (-1)^n(1 + 1/n)]$. Find the following ($k \in \mathbb{Z}^+$): (a) $\displaystyle\bigcap_{n=k}^{\infty} A_n$, (b) $\displaystyle\bigcup_{n=k}^{\infty} A_n$, (c) $\displaystyle\bigcup_{k=1}^{\infty}\Big(\bigcap_{n=k}^{\infty} A_n\Big)$, (d) $\displaystyle\bigcap_{k=1}^{\infty}\Big(\bigcup_{n=k}^{\infty} A_n\Big)$.

5. Find the properties (reflexive, transitive, symmetric, antisymmetric) verified by the following relations:
   (a) Strict inequality of integers: $x \,\mathcal{R}\, y \Leftrightarrow x < y$.
   (b) Set disjointness: $A \,\mathcal{R}\, B \Leftrightarrow A \cap B = \emptyset$.
   (c) The following relation on $\mathbb{Q}$: $x \,\mathcal{R}\, y \Leftrightarrow x - y \in \mathbb{Z}$.
   (d) The following relation on $\mathbb{Q}$: $x \,\mathcal{R}\, y \Leftrightarrow x - y \in \mathbb{N}$.

6. We define the following relation on $\mathbb{N}$: $x \preccurlyeq y$ if and only if
   (a) $x$ is even and $y$ is odd, or
   (b) $x$ and $y$ have the same parity and $x \leq y$.
   For instance: $2 \preccurlyeq 6$, $3 \preccurlyeq 7$, $2 \preccurlyeq 5$, $24 \preccurlyeq 3$.
   1. Prove that $\preccurlyeq$ is a total order.
   2. For each of the following numbers find a *successor* an *immediate successor*, a *predecessor* and an *immediate predecessor*, or show that there is none: 3, 2, 1, 0.

7. Prove that the following is an *equivalence relation* on $\mathbb{N}^2$:
$$(a, b) \,\mathcal{R}\, (a', b') \Leftrightarrow a + b' = a' + b\,.$$

8. Let $\mathcal{U}$ be a nonempty set, and let $\mathcal{P}(\mathcal{U})$ be its power set. Let $S \in \mathcal{P}(\mathcal{U})$ be a subset of $\mathcal{U}$. In $\mathcal{P}(\mathcal{U})$ we define the following relation: $A \,\mathcal{R}\, B \Leftrightarrow A \triangle B \subseteq S$.
   (a) Prove that $\mathcal{R}$ is an *equivalence relation*.
   (b) Prove that all *equivalence classes* have the form $\mathcal{C}_A = \{A \cup S' \mid S' \in \mathcal{P}(S)\}$ for $A \in \mathcal{P}(\overline{S})$.

9. Find (if they exist) the *greatest element*, the *least element*, the *least upper bound* and the *greatest lower bound* for each of the following subsets of $(\mathbb{R}, \leq)$:
   (a) $A = \{(-1)^n + 1/n \mid n \in \mathbb{Z}^+\}$.
   (b) $B = \{x \in \mathbb{R} \mid x^2 < 5\}$.
   (c) $C = \{x \in \mathbb{Q} \mid x^2 < 5\}$.
   (d) $D = \{x \in \mathbb{Z} \mid x^2 < 5\}$.

10. Draw the *Hasse diagram* for the poset $P = (\{2, 3, 4, 5, 6, 7, 8, 9, 10\}, |)$, where "|" represents divisibility. Find the *minimal* and *maximal* elements in $P$.