CHAPTER 3

Differential Equations

3.1. Differential Equations and Separable Equations

3.1.1. Population Growth. The growth of a population is usually modeled with an equation of the form

$$\frac{dP}{dt} = kP\,,$$

where P represents the number of individuals an a given time t. This model assumes that the rate of growth of population is proportional to the population size.

A solution to this equation is the exponential function:

$$P(t) = Ce^{kt}\,.$$

Check: $P'(t) = kCe^{kt} = kP(t)$.

A more realistic model takes into account that any environment has a limited *carrying capacity* K, so if P reaches K the population stops growing. The model in this case is the following:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right)\,.$$

This is called the *logistic differential equation.*

3.1.2. Motion of a Spring. Consider an object of mass m at the end of a vertical spring. According to Hook's law the restoring force of a spring stretched (or compressed) a distance x from its natural length is

$$F = -kx\,,$$

where k is a positive constant (the *spring constant*) and the negative sign expresses that the sense of the force is opposite to the sense of the stretching. By Newton's Second Law (force equals mass times acceleration):

$$m\frac{d^2x}{dt^2} = -kx\,,$$

or equivalently:

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x\,.$$

This is an example of a *second order differential equation* because it involves second order derivatives.

3.1.3. General Differential Equations. A *differential equation* is an equation that contains one or more unknown functions and one or more of its derivatives. The *order* of the differential equation is the order of the highest derivative that occurs in the equation.

3.1.4. First-order Differential Equations. A *first-order differential equation* is an equation of the form

$$\frac{dy}{dx} = F(x,y)\,,$$

where $F(x,y)$ is a function of x and y. A *solution* of the differential equation is a function $y(x)$ such that $y'(x) = F(x, y(x))$ for all x in some appropriate interval.

Example: Consider the following differential equation:

$$\frac{dy}{dx} = \frac{2y}{x}\,.$$

A possible solution for that equation is, for instance, $y = x^2$, because

$$\frac{dy}{dx} = y'(x) = (x^2)' = 2x\,,$$

and

$$2\,\frac{y}{x} = 2\,\frac{x^2}{x} = 2x\,,$$

hence $y'(x) = \dfrac{2\,y(x)}{x}$ for all $x \neq 0$.

3.1.5. Separable Differential Equations. A differential equation is said to be *separable* if it can be written in the form

$$f(y)\,dy = g(x)\,dx\,,$$

so that the left hand side depends on y only and the right hand side depends on x only. In particular this is true if the equation is of the form

$$\frac{dy}{dx} = g(x)\,\phi(y)\,,$$

where the right hand side is a product of a function of x and a function of y. In this case we get:

$$\frac{1}{\phi(y)}\,dy = g(x)\,dx\,.$$

Given the equation

$$f(y)\,dy = g(x)\,dx\,,$$

we can solve it by integrating both sides. Since the antiderivatives of a function differ in a constant, we get:

$$\int f(y)\,dy = \int g(x)\,dx + C\,,$$

If $F(y) = \int f(y)\,dy$ and $G(x) = \int g(x)\,dx$ then the solution takes the form

$$F(y) = G(x) + C\,.$$

Next we will try to solve this equation algebraically in order to either write y as a function of x, or x as a function of y.

Example: Consider the equation

$$\frac{dy}{dx} = y^2\,x\,.$$

The right hand side is the product of a function of x and a function of y, so it is separable:

$$\frac{1}{y^2}\,dy = x\,dx\,.$$

Integrating both sides we get:

$$-\frac{1}{y} = \frac{x^2}{2} + C\,,$$

hence

$$y = -\frac{2}{x^2+2C} = -\frac{2}{x^2+C'}\,,$$

where C' is a new constant equal to $2C$.

3.1.6. Initial Value Problems. A differential equation together with an initial condition

$$\begin{cases} \dfrac{dy}{dx} = F(x, y) \\ y(x_0) = y_0 \end{cases}$$

is called an *initial value problem.*

The initial condition can be used to determine the value of the constant in the solution of the equation.

Example: Solve the following initial value problem:

$$\begin{cases} \dfrac{dy}{dx} = y^2\, x \\ y(0) = 1 \end{cases}$$

Solution: We already found the general solution to the differential equation:

$$y = -\frac{2}{x^2 + C}\,.$$

Next we let $x = 0$ and $y = 1$, and solve for C:

$$1 = -\frac{2}{C} \quad \Longrightarrow \quad C = -2\,.$$

So the solution is

$$y = -\frac{2}{x^2 - 2} = \frac{2}{2 - x^2}\,.$$