CS 310-0
Homework Assignment No. 3
Due Tue 1/30/2001

1. Find (if they exist) the *greatest element*, the *least element*, the *least upper bound* and the *greatest lower bound* for each of the following subsets of $(\mathbb{R}, \leq)$:
   (a) $A = \{(-1)^n + 1/n \mid n \in \mathbb{Z}^+\}$.
   (b) $B = \{x \in \mathbb{R} \mid x^2 \leq 5\}$.
   (c) $C = \{x \in \mathbb{Q} \mid x^2 \leq 5\}$.
   (d) $D = \{x \in \mathbb{Z} \mid x^2 \leq 5\}$.

2. Let $P = \{a\omega + b \mid a, b \in \mathbb{N}\}$ be the set of expressions of the form $a\omega + b$, where a and b are natural numbers and ω is a symbol.[1] On P we define the relation

   $$a\omega + b \leq a'\omega + b' \quad \text{iff} \quad a < a', \text{ or } a = a' \text{ and } b \leq b' .$$

   For instance, $5\omega + 7 \leq 6\omega + 3$ because $5 < 6$. On the other hand, $6\omega + 3 \leq 6\omega + 7$ because $6 = 6$ and $3 \leq 7$.
   1. Prove that "$\leq$" is a *total order* on P.[2] Is it a *well order*?[3]
   2. For each of the following elements of P find a *successor* an *immediate successor*, a *predecessor* and an *immediate predecessor*, or show that there is none:

      $$3\omega + 1,\ 2\omega,\ 7,\ 0 .$$

   3. Show that every element of P has an *immediate successor*, but some have no *immediate predecessor*. Characterize the elements with no *immediate predecessor*.
   4. An element in P is said to be *infinite* if it is greater than any natural number, otherwise it is called *finite*. Prove that ω is the least infinite element in P.

3. Let X be the set $X = \{a, b, c\}$. Draw the *Hasse diagram* for the poset $(\mathcal{P}(X), \subseteq)$, where "$\mathcal{P}(X)$" is the set of subsets of X, and "$\subseteq$" is the containment relation. Find the *minimal* and *maximal* elements in $S = \mathcal{P}(X) - \{\emptyset, X\}$.

4. Let $P = \{ax + b \mid a, b \in \mathbb{N}\}$ be the set of polynomials of degree at most 1 with natural coefficients. On P we define the relation

   $$ax + b \ \mathcal{R}\ a'x + b' \quad \text{iff} \quad a = a' .$$

   Prove that $\mathcal{R}$ is an equivalence relation. Describe the equivalence classes.[4]

5. Prove that the following is an *equivalence relation* on $\mathbb{R}^2 - \{(0,0)\}$:

   $$(x, y)\, \mathcal{R}\, (x', y') \quad \text{iff} \quad \exists \lambda \in \mathbb{R}^*,\ (x', y') = (\lambda x, \lambda y) .$$

   Let F be the set $F = \{(x, y) \mid (x^2 + y^2 = 1) \wedge (-1 < x \leq 1) \wedge (0 \leq y)\}$. Prove that F contains exactly one representative from each equivalence class.

[1] When a or b are zero we write $0\omega + b = b$, $a\omega + 0 = a\omega$, $0\omega + 0 = 0$
[2] You have to prove two things: that it is an order and it is total.
[3] Remember that $(\mathbb{N}, \leq)$ is well ordered, i.e., every non-empty subset of $\mathbb{N}$ has a least element.
[4] I.e., each class is of the form $\{ax + b \in P \mid \dots\}$ (replace the dots with an appropriate statement.)