

1.4. The Substitution Rule

1.4.1. The Substitution Rule. The substitution rule is a trick for evaluating integrals. It is based on the following identity between differentials (where u is a function of x):

$$du = u' dx .$$

Hence we can write:

$$\boxed{\int f(u) u' dx = \int f(u) du}$$

or using a slightly different notation:

$$\boxed{\int f(g(x)) g'(x) dx = \int f(u) du}$$

where $u = g(x)$.

Example: Find $\int \sqrt{1+x^2} 2x dx$.

Answer: Using the substitution $u = 1 + x^2$ we get

$$\begin{aligned} \int \sqrt{1+x^2} 2x dx &= \int \sqrt{u} u' dx \\ &= \int \sqrt{u} du = \frac{2}{3} u^{3/2} + C \\ &= \boxed{\frac{2}{3} (1+x^2)^{3/2} + C} . \end{aligned}$$

Most of the time the only problem in using this method of integration is finding the right substitution.

Example: Find $\int \cos 2x dx$.

Answer: We want to write the integral as $\int \cos u du$, so $\cos u = \cos 2x \Rightarrow u = 2x$, $u' = 2$. Since we do not see any factor 2 inside the

integral we write it, taking care of dividing by 2 outside the integral:

$$\begin{aligned}\int \cos 2x \, dx &= \frac{1}{2} \int \cos 2x \, 2 \, dx \\ &= \frac{1}{2} \int \cos u \, u' \, dx \\ &= \frac{1}{2} \int \cos u \, du \\ &= \frac{1}{2} \sin u + C\end{aligned}$$

(always remember to undo the substitution)

$$= \boxed{\frac{1}{2} \sin 2x + C}.$$

In general we need to look at the integrand as a function of some expression (which we will later identify with u) multiplied by the derivative of that expression.

Example: Find $\int e^{-x^2} x \, dx$.

Answer: We see that x is “almost”, the derivative of $-x^2$, so we use the substitution $u = -x^2$, $u' = -2x$, hence in order to get u' inside the integral we do the following:

$$\begin{aligned}\int e^{-x^2} x \, dx &= -\frac{1}{2} \int \underbrace{e^{-x^2}}_{e^u} \underbrace{(-2x) \, dx}_{du} \\ &= -\frac{1}{2} \int e^u \, du = -\frac{1}{2} e^u + C = \boxed{-\frac{1}{2} e^{-x^2} + C}.\end{aligned}$$

Sometimes the substitution is hard to see until we make some ingenious transformation in the integrand.

Example: Find $\int \tan x \, dx$.

Answer: Here the idea is to write $\tan x = \frac{\sin x}{\cos x}$ and use that $(\cos x)' = -\sin x$, so we make the substitution $u = \cos x$, $u' = -\sin x$:

$$\begin{aligned}\int \tan x \, dx &= \int \frac{\sin x}{\cos x} \, dx = - \int \frac{u'}{u} \, dx = - \int \frac{1}{u} \, du \\ &= -\ln |u| + C = \boxed{-\ln |\cos x| + C}.\end{aligned}$$

In general we need to identify inside the integral some expression of the form $f(u) u'$, where f is some function with a known antiderivative.

Example: Find $\int \frac{e^x}{e^{2x} + 1} \, dx$.

Answer: Let's write

$$\int \frac{e^x}{e^{2x} + 1} \, dx = k \int f(u) u' \, dx$$

(where k is some constant to be determined later) and try to identify the function f , the argument u and its derivative u' . Since $(e^x)' = e^x$ it seems natural to choose $u = e^x$, $u' = e^x$, so $e^{2x} = u^2$ and

$$\begin{aligned}\int \frac{e^x}{e^{2x} + 1} \, dx &= \int \frac{u'}{u^2 + 1} \, dx = \int \frac{1}{u^2 + 1} \, du \\ &= \tan^{-1} u + C = \boxed{\tan^{-1}(e^x) + C}.\end{aligned}$$

There is no much more that can be said in general, the way to learn more is just to practice.

1.4.2. Other Changes of Variable. Sometimes rather than making a substitution of the form $u = \text{function of } x$, we may try a change of variable of the form $x = \text{function of some other variable such as } t$, and write $dx = x'(t) \, dt$, where x' = derivative of x respect to t .

Example: Find $\int \sqrt{1 - x^2} \, dx$.

Answer: Here we write $x = \sin t$, so $dx = \cos t \, dt$, $1 - x^2 = 1 - \sin^2 t = \cos^2 t$, and

$$\int \sqrt{1 - x^2} \, dx = \int \underbrace{\cos t}_x \underbrace{\cos t \, dt}_{dx} = \int \cos^2 t \, dt.$$

Since we do not know yet how to integrate $\cos^2 t$ we leave it like this and will be back to it later (after we study integrals of trigonometric functions).

1.4.3. The Substitution Rule for Definite Integrals. When computing a definite integral using the substitution rule there are two possibilities:

- (1) Compute the definite integral first, then use the evaluation theorem:

$$\begin{aligned}\int f(u) u' dx &= F(x); \\ \int_a^b f(u) u' dx &= F(b) - F(a).\end{aligned}$$

- (2) Use the substitution rule for definite integrals:

$$\int_a^b f(u) u' dx = \int_{u(a)}^{u(b)} f(u) du.$$

The advantage of the second method is that we do not need to undo the substitution.

Example: Find $\int_0^4 \sqrt{2x+1} dx$.

Answer: Using the first method first we compute the indefinite integral:

$$\begin{aligned}\int \sqrt{2x+1} dx &= \frac{1}{2} \int \sqrt{2x+1} 2x dx \quad (u = 2x+1) \\ &= \frac{1}{2} \int \sqrt{u} du \\ &= \frac{1}{3} u^{3/2} + C \\ &= \frac{1}{3} (2x+1)^{3/2} + C.\end{aligned}$$

Then we use it for computing the definite integral:

$$\int_0^4 \sqrt{2x+1} dx = \left[\frac{1}{3} (2x+1)^{3/2} \right]_0^4 = \frac{1}{3} 9^{3/2} - \frac{1}{3} 1^{3/2} = \frac{27}{3} - \frac{1}{3} = \boxed{\frac{26}{3}}.$$

In the second method we compute the definite integral directly adjusting the limits of integration after the substitution:

$$\begin{aligned}\int_0^4 \sqrt{2x+1} \, dx &= \frac{1}{2} \int_0^4 \sqrt{2x+1} \, 2x \, dx \quad (u = 2x+1; \, u' = 2) \\ &= \frac{1}{2} \int_1^9 \sqrt{u} \, du\end{aligned}$$

(note the change in the limits of integration to $u(0) = 1$ and $u(4) = 9$)

$$\begin{aligned}&= \left[\frac{1}{3} u^{3/2} \right]_1^9 \\ &= \frac{1}{3} 9^{3/2} - \frac{1}{3} 1^{3/2} \\ &= \frac{27}{3} - \frac{1}{3} = \boxed{\frac{26}{3}}.\end{aligned}$$