CS 310-0
Homework Assignment No. 2
Due Tue 1/23/2001

1. (This is a problem in both Logic and Set Theory.) We want to define the sets below using the set-builder notation. All the definitions are of the form $\{x \in \mathcal{U} \mid \dots\}$, where $\mathcal{U}$ is the universe of discourse. Complete the definitions by replacing the dots with a property containing **only** variables, parentheses, logical connectives, quantifiers, the equality sign $(=)$ and symbols for arithmetical operations ($+$ and $\cdot$):
(a) set of even integers $= \{x \in \mathbb{Z} \mid \dots\}$
(b) $\{0\} = \{x \in \mathbb{R} \mid \dots\}$
(c) $\{0, 1\} = \{x \in \mathbb{R} \mid \dots\}$
(d) $\mathbb{N}^* = \{x \in \mathbb{N} \mid \dots\}$
(e) $[0, \infty) = \{x \in \mathbb{R} \mid \dots\}$
For instance: set of even integers $= \{x \in \mathbb{Z} \mid \exists y\,(y + y = x)\}$.[1]

2. Using the Principle of Extension, prove the following ($\triangle$ is *symmetric difference*):
(a) $A \triangle (B \triangle C) = (A \triangle B) \triangle C$.
(b) $A \cap (B \triangle C) = (A \cap B) \triangle (A \cap C)$.
(Hint: use known properties of *conjunction* and *exclusive or* such as the ones proven in problem 2 of homework assignment No. 1.)

3. For any integers m, n such that $m \le n$ let $[m, n]$ be the interval of real numbers x such that $m \le x \le n$. Find the following: (a) $\bigcup_{n=m}^{\infty} [m, n]$, (b) $\bigcap_{m=1}^{\infty} \Big(\bigcup_{n=m}^{\infty} [m, n] \Big)$.

4. Assume that for $A_1, A_2, A_3, \dots$ and $B_1, B_2, B_3, \dots$ are two infinite collections of sets.
(a) Prove that
$$\bigcup_{n=1}^{\infty} (A_n \cap B_n) \subseteq \Big(\bigcup_{n=1}^{\infty} A_n \Big) \cap \Big(\bigcup_{n=1}^{\infty} B_n \Big).$$
(b) Prove (with a counterexample) that the containment in the opposite direction is not true in general.
(c) Find and prove analogous results concerning the sets
$$\bigcap_{n=1}^{\infty} (A_n \cup B_n) \quad \text{and} \quad \Big(\bigcap_{n=1}^{\infty} A_n \Big) \cup \Big(\bigcap_{n=1}^{\infty} B_n \Big).$$

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[1]Note that the (otherwise correct) definition $\{x \in \mathbb{Z} \mid \exists y\,(2 \cdot y = x)\}$ would not be a valid answer in this particular problem because the property "$\exists y\,(2 \cdot y = x)$" uses the symbol "2", which is not allowed here. For the same reason the following answers are not acceptable: $\{0\} = \{x \in \mathbb{R} \mid x = 0\}$, $\{0, 1\} = \{x \in \mathbb{R} \mid (x = 0) \vee (x = 1)\}$. We are not allowed to use symbols for inequality either, so the following definition is not a valid answer here: $\mathbb{N}^* = \{x \in \mathbb{N} \mid \exists y\,(x > y)\}$.

5. Use a Venn diagram and arrows to represent the relation $x \mathcal{R} y \Leftrightarrow 3 \mid (x - y)$ (i.e., x is related to y iff 3 divides $x - y$) on the set $\{1, 2, 3, 4, 5, 6, 7\}$.

6. Find the properties (reflexive, transitive, symmetric, antisymmetric) verified by the following relations:
   (a) Strict inequality of integers: $x \mathcal{R} y \Leftrightarrow x < y$.
   (b) Set disjointness: $A \mathcal{R} B \Leftrightarrow A \cap B = \emptyset$.
   (c) The following relation on $\mathbb{R}$: $x \mathcal{R} y \Leftrightarrow x - y \in \mathbb{Z}$.
   (d) The following relation on $\mathbb{R}$: $x \mathcal{R} y \Leftrightarrow x - y \in \mathbb{N}$.
   (e) The following relation on $\mathbb{R}^+$: $x \mathcal{R} y \Leftrightarrow x/y \in \mathbb{Q}$.
   (f) The following relation on $\mathbb{R}^+$: $x \mathcal{R} y \Leftrightarrow x/y \in \mathbb{Z}$.