

2.5. Applications to Physics and Engineering

2.5.1. Work. Work is the energy produced by a force F pushing a body along a given distance d . If the force is constant, the work done is the product

$$W = F \cdot d.$$

The SI (international) unit of work is the joule (J), which is the work done by a force of one Newton (N) pushing a body along one meter (m). In the American system a unit of work is the foot-pound. Since $1 \text{ N} = 0.224809 \text{ lb}$ and $1 \text{ m} = 3.28084 \text{ ft}$, we have $1 \text{ J} = 0.737561 \text{ ft lb}$.

More generally, assume that the force is variable and depends on the position. Let $F(x)$ be the force function. Assume that the force pushes a body from a point $x = a$ to another point $x = b$. In order to find the total work done by the force we divide the interval $[a, b]$ into small subintervals $[x_{i-1}, x_i]$ so that the change of $F(x)$ is small along each subinterval. Then the work done by the force in moving the body from x_{i-1} to x_i is approximately:

$$\Delta W_i \approx F(x_i^*) \Delta x,$$

where $\Delta x = x_i - x_{i-1} = (b - a)/n$ and x_i^* is any point in $[x_{i-1}, x_i]$. So, the total work is

$$W = \sum_{i=1}^n \Delta W_i \approx \sum_{i=1}^n F(x_i^*) \Delta x.$$

As $n \rightarrow \infty$ the Riemann sum at the right converges to the following integral:

$$W = \int_a^b F(x) dx.$$

2.5.2. Elastic Springs. Consider a spring on the x -axis so that its right end is at $x = 0$ when the spring is at its rest position. According to *Hook's Law*, the force needed to stretch the spring from 0 to x is proportional to x , i.e.:

$$F(x) = kx,$$

where k is the so called *spring constant*.

The energy needed to stretch the spring from 0 to a is then the integral

$$W = \int_0^a kx dx = k \frac{a^2}{2}.$$

2.5.3. Work Done Against Gravity. According to Newton's Law, the force of gravity at a distance r from the center of the Earth is

$$F(r) = \frac{k}{r^2},$$

where k is some positive constant.

The energy needed to lift a body from a point at distance R_1 from the center of the Earth to another point at distance R_2 is given by the following integral

$$W = \int_{R_1}^{R_2} \frac{k}{r^2} dr = \left[-\frac{k}{r} \right]_{R_1}^{R_2} = k \left(\frac{1}{R_1} - \frac{1}{R_2} \right).$$

Example: Find the energy needed to lift 1000 Km a body whose weight is 1 N at the surface of Earth. The Earth radius is 6378 Km.

Answer: First we must determine the value of the constant k in this case. Since the weight of the body for $r = 6378$ Km is 1 N we have $k/6378^2 = 1$, so $k = 6378^2$. Next we have $R_1 = 6378$, $R_2 = 6378 + 1000 = 7378$, hence

$$W = 6378^2 \left(\frac{1}{6378} - \frac{1}{7378} \right) = 864.462 \text{ N Km}.$$

Since 1 Km = 1000 m, the final result in joule is

$$864.462 \text{ N Km} = 864.462 \text{ N} \times 1000 \text{ m} = 864462 \text{ J}.$$

2.5.4. Work Done Filling a Tank. Consider a tank whose bottom is at some height $y = a$ and its top is at $y = b$. Assume that the area of its cross section is $A(y)$. We fill the tank by lifting from the ground ($y = 0$) tiny layers of thickness dy each. Their weight is $dF = \rho A(y) dy$, where ρ is the density of the liquid that we are putting in the tank. The work needed to lift each layer is

$$dW = dF \cdot y = \rho y A(y) dy.$$

Hence, the work needed to fill the tank is

$$W = \int_a^b \rho y A(y) dy.$$

2.5.5. Emptying a Tank. Consider a tank like the one in the previous paragraph. Now we empty it by pumping its liquid to a fix height h . The analysis of the problem is similar to the previous paragraph, but now the work done to pump a tiny layer of thickness dy is

$$dW = dF \cdot (h - y) = \rho(h - y)A(y) dy.$$

Hence the total work needed to empty the tank is

$$W = \int_a^b \rho(h - y)A(y) dy.$$

2.5.6. Force Exerted by a Liquid Against a Vertical Wall.

The pressure p of an homogeneous liquid of density ρ at depth h is

$$p = \rho h.$$

When the pressure is constant, the force exerted by the liquid against a surface is the product of the pressure and the area of the surface. However the pressure against a vertical wall is not constant because it depends on the depth.

Assume that the surface of the liquid is at $y = c$ and we place a vertical plate of width $w(y)$ between $y = a$ and $y = b$. The force exerted at y (so at depth $h = c - y$) against a small horizontal strip of height dy and width $w(y)$ (area = $w(y) dy$) is

$$dF = \rho(c - y)w(y) dy.$$

hence the total force is

$$F = \int_a^b \rho(c - y)w(y) dy.$$

Example: A cylindrical tank of radius 1 m and full of water ($\rho = 9800 \text{ N/m}^3$) is lying on its side. What is the pressure exerted by the water on its (vertical) bottom?

Answer: We assume the center of the tank is at $y = 0$, so the top of the liquid is at $y = 1$, and its bottom is at $y = -1$. On the other hand we obtain geometrically $w(y) = 2\sqrt{1 - y^2}$. Hence the total force is:

$$F = \int_{-1}^1 9800 \cdot (1 - y)2\sqrt{1 - y^2} dy = 19800 \cdot \frac{\pi}{2} = 30787.6 \text{ N}.$$