4.7. Taylor and MacLaurin Series

4.7.1. Polynomial Approximations. Assume that we have a function f for which we can easily compute its value $f(a)$ at some point a, but we do not know how to find $f(x)$ at other points x close to a. For instance, we know that $\sin 0 = 0$, but what is $\sin 0.1$? One way to deal with the problem is to find an *approximate* value of $f(x)$. If we look at the graph of $f(x)$ and its tangent line at $(a, f(a))$, we see that the points of the tangent line are close to the graph, so the y-coordinates of those points are possible approximations for $f(x)$.

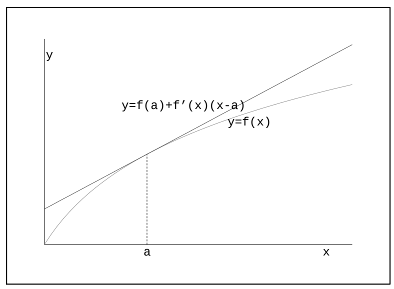

FIGURE 4.7.1. Linear approximation of $f(x)$.

The equation of the tangent line to $y = f(x)$ at $x = a$ is

$$y = f(a) + f'(a)(x - a)\,,$$

hence

$$f(x) \approx f(a) + f'(a)(x - a)\,,$$

for x close to a. For instance:

$$\sin(x) \approx \sin a + \cos a\,(x - a)\,.$$

For $a = 0$ we get:

$$\sin(x) \approx \sin 0 + \cos 0 \cdot (x - 0) = x\,,$$

so $\sin(0.1) \approx 0.1$. In fact $\sin(0.1) = 0.099833416\ldots$, which is close to 0.1.

The tangent line is the graph of the first degree polynomial

$$T_1(x) = f(a) + f'(a)(x - a)\,.$$

This polynomial agrees with the value and the first derivative of $f(x)$ at $x = a$:

$$T_1(a) = f(a)$$
$$T_1'(a) = f'(a)$$

We can extend the idea to higher degree polynomials in the hope of obtaining closer approximations to the function. For instance, we may try a second degree polynomial of the from:

$$T_2(x) = c_0 + c_1(x-a) + c_2(x-a)^2 \,,$$

with the following conditions:

$$\begin{aligned} T_2(a) &= f(a) \\ T_2'(a) &= f'(a) \\ T_2''(a) &= f''(a) \end{aligned}$$

i.e.:

$$\begin{cases} c_0 = f(a) \\ c_1 = f'(a) \\ 2c_2 = f''(a) \end{cases}$$

After solving the system of equations obtained we get:

$$\begin{aligned} c_0 &= f(a) \\ c_1 &= f'(a) \\ c_2 &= \frac{f''(a)}{2} \end{aligned}$$

hence:

$$T_2(x) = f(a) + f'(a)x + \frac{f''(a)}{2}x^2 \,.$$

In general the nth polynomial approximation of $f(x)$ at $x = a$ is an nth degree polynomial

$$T_n(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n$$

verifying

$$\begin{aligned} T_n(a) &= f(a) \\ T_n'(a) &= f'(a) \\ T_n''(a) &= f''(a) \\ &\dots \\ T_n^{(n)}(a) &= f^{(n)}(a) \end{aligned}$$

From here we get a system of $n+1$ equations with the following solution:

$$\begin{aligned}
c_0 &= f(a) \\
c_1 &= f'(a) \\
c_2 &= \frac{f''(a)}{2!} \\
&\dots \\
c_n &= \frac{f^{(n)}(a)}{n!}
\end{aligned}$$

hence:

$$\begin{aligned}
T_n(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \\
&= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \,.
\end{aligned}$$

That polynomial is the so called *nth-degree Taylor polynomial of $f(x)$ at $x = a$.*

Example: The third-degree Taylor polynomial of $f(x) = \sin x$ at $x = a$ is

$$T_3(x) = \sin a + \cos a \cdot (x-a)^2 - \frac{\sin a}{2}(x-a)^2 - \frac{\cos a}{3!}(x-a)^3 \,.$$

For $a = 0$ we have $\sin 0 = 0$ and $\cos 0 = 1$, hence:

$$T_3(x) = x - \frac{x^3}{6} \,.$$

So in particular

$$\sin 0.1 \approx 0.1 - \frac{0.1^3}{6} = 0.09983333\dots \,.$$

The actual value of $\sin 0.1$ is

$$\sin 0.1 = 0.099833416 \,,$$

which agrees with the value obtained from the Taylor polynomial up to the sixth decimal place.

4.7.2. Taylor's Inequality. The difference between the value of a function and its Taylor approximation is called *remainder*:

$$R_n(x) = f(x) - T_n(x) \,.$$

The Taylor's inequality states the following: If $|f^{(n+1)}(x)| \leq M$ for $|x-a| \leq d$ then the reminder satisfies the inequality:

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \qquad \text{for } |x-a| \leq d\,.$$

Example: Find the third degree Taylor approximation for $\sin x$ at $x = 0$, use it to find an approximate value for $\sin 0.1$ and estimate its difference from the actual value of the function.

Answer: We already found

$$T_3(x) = x - \frac{x^3}{6}\,,$$

and

$$T_3(0.1) = 0.99833333\ldots$$

Now we have $f^{(4)}(x) = \sin x$ and $|\sin x| \leq 1$, hence

$$|R_3(0.1)| =\leq \frac{1}{4!}0.1^4 = 0.0000041666\cdots < 0.0000042 = 4.2 \cdot 10^{-6}\,.$$

In fact the estimation is correct, the approximate value differs from the actual value in

$$|T_3(0.1) - \sin 0.1| = 0.000000083313\cdots < 8.34 \cdot 10^{-8}\,.$$

4.7.3. Taylor Series. If the given function has derivatives of all orders and $R_n(x) \to 0$ as $n \to \infty$, then we can write

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \\ &\quad \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots \end{aligned}$$

The infinite series to the right is called *Taylor series* of $f(x)$ at $x = a$. If $a = 0$ then the Taylor series is called *Maclaurin series*.

Example: The Taylor series of $f(x) = e^x$ at $x = 0$ is:

$$1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

For $|x| < d$ the remainder can be estimated taking into account that $f^{(n)}(x) = e^x$ and $|e^x| < e^d$, hence

$$|R_n(x)| < \frac{e^d}{(n+1)!}|x|^{n+1}.$$

We know that $\lim_{n\to\infty} x^n/n! = 0$, so

$$\lim_{n\to\infty} \frac{e^d}{(n+1)!}|x|^{n+1} = 0$$

hence $R_n(x) \to 0$ as $n \to \infty$. Consequently we can write:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

For $x = 1$ this formula provides a way of computing number e:

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots = 2.718281828459\ldots$$

The following are Maclaurin series of some common functions:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\ln(1+x) = -\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$(1+x)^\alpha = \sum_{n=0} \binom{\alpha}{n} x^n = 1 + \alpha x + \binom{\alpha}{2} x^2 + \binom{\alpha}{3} x^3 + \cdots$$

where $\displaystyle\binom{\alpha}{n} = \frac{\alpha(\alpha-1)(\alpha-2)\ldots(\alpha-n+1)}{n!}$.

$$\frac{1}{1+x} = (1+x)^{-1} = \sum_{n=0} (-1)^n x^n = 1 - x + x^2 - x^3 + \cdots$$

$$\tan^{-1} x = \sum_{n=0} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

Remark: By letting $x = 1$ in the Taylor series for $\tan^{-1} x$ we get the beautiful expression:

$$\boxed{\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}}.$$

Unfortunately that series converges too slowly for being of practical use in computing π. Since the series for $\tan^{-1} x$ converges more quickly for small values of x, it is more convenient to express π as a combination of inverse tangents with small argument like the following one:

$$\boxed{\frac{\pi}{4} = 4 \tan^{-1} \frac{1}{5} - \tan^{-1} \frac{1}{239}}.$$

That identity can be checked with plain trigonometry. Then the inverse tangents can be computed using the Maclaurin series for $\tan^{-1} x$, and from them an approximate value for π can be found.

4.7.4. Finding Limits with Taylor Series. The following example shows an application of Taylor series to the computation of limits:

Example: Find $\displaystyle\lim_{x \to 0} \frac{e^x - 1 - x}{x^2}$.

Answer: Replacing e^x with its Taylor series:

$$\begin{aligned}
\lim_{x \to 0} \frac{e^x - 1 - x}{x^2} &= \lim_{x \to 0} \frac{(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots) - 1 - x}{x^2} \\
&= \lim_{x \to 0} \frac{\frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots}{x^2} \\
&= \lim_{x \to 0} \left\{ \frac{1}{2} + \frac{x}{6} + \frac{x^2}{24} + \dots \right\} = \boxed{\frac{1}{2}}.
\end{aligned}$$