3.2. Directional Fields and Euler's Method

Here we study a graphical method (direction fields) and a numerical method (Euler's method) to solve differential equations.

3.2.1. Slope Fields. Consider a differential equation of the form

$$\frac{dy}{dx} = F(x,y)\,.$$

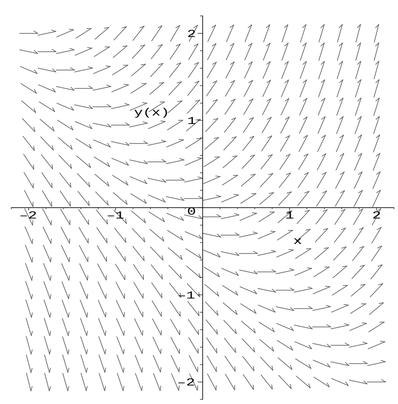
If $y(x)$ is a solution, the slope of its graph at each point $(x, y(x))$ should be equal to $F(x,y)$. So the right hand side of the equation can be interpreted as a *slope field* in the xy-plane. The graph of a solution is called a *solution curve* for the slope field. Each solution curve is a *particular solution* of the slope field. A point (x_0, y_0) in the xy-plane plays the role of an *initial condition*, and the solution curve that passes through that point corresponds to the solution of the differential equation satisfying the corresponding initial condition $y(x_0) = y_0$.

3.2.2. Direction Fields. This method consists of interpreting the differential equation as a slope field and sketch solutions just by following the field.

Example: The direction field for the differential equation

$$y' = x + y$$

looks like this:



3.2.3. Euler's Method. Euler's method consists of approximating solutions to a differential equation with polygonal lines made of short straight lines each with slope equal to y' at their initial point. So assume that we want to find approximate values of a solution for an initial-value problem

$$\begin{cases} y' = F(x, y) \\ y(x_0) = y_0 \end{cases}$$

at equally spaced points x_0, $x_1 = x_0 + h$, $x_2 = x_1 + h$, $\dots$, where h is called the *step size*. We take (x_0, y_0) as the initial point of the solution. The slope at (x_0, y_0) is $y_0' = F(x_0, y_0)$, hence next point will be (x_1, y_1) so that $(y_1 - y_0)/h = F(x_0, y_0)$, i.e., $y_1 = y_0 + hF(x_0, y_0)$. Proceeding in the same way we get in general:

$$\begin{aligned} y_1 &= y_0 + hF(x_0, y_0) \\ y_2 &= y_1 + hF(x_1, y_1) \\ &\dots \\ y_n &= y_{n-1} + hF(x_{n-1}, y_{n-1}) \\ &\dots \end{aligned}$$

Example: Use Euler's method with step size 0.1 to find approximate values of the solution to the initial value problem

$$\begin{cases} y' = x + y \\ y(0) = 1 \end{cases}$$

Answer: We have:

$$\begin{aligned} &y(0) = y_0 = 1 \\ &y(0.1) \approx y_1 = y_0 + hF(x_0, y_0) = 1 + 0.1(0 + 1) = 1.1 \\ &y(0.2) \approx y_2 = y_1 + hF(x_1, y_1) = 1.1 + 0.1(0.1 + 1.1) = 1.22 \\ &y(0.3) \approx y_3 = y_2 + hF(x_2, y_2) = 1.22 + 0.1(0.2 + 1.22) = 1.362 \\ &\qquad \dots \end{aligned}$$