CS 310-0
Homework Assignment No. 5
Due Fri 5/12/2000

1. Find the number of integer solutions to the following equation
$$x_1 + x_2 + x_3 + x_4 = 12$$
with each one of the following restrictions:
(a) $x_1, x_2, x_3, x_4 \geq 0$.
(b) $x_1, x_2, x_3, x_4 > 0$.
(c) $1 \leq x_1$, $2 \leq x_2$, $3 \leq x_3$, $4 \leq x_4$.

2. How many integers divisible by 3, 5 or 7 are there in the interval $[1, 630]$?

3. Let Σ be a finite set, and let Σ^n be the set of n-strings of elements of Σ. For instance, if $\Sigma = \{0, 1, 2\}$ and $n = 8$ then Σ^8 = set of all 8-digit numbers in base 3 (including those with leading zeroes such as 00120010). The *Hamming distance* d_H between two elements of Σ^n is the number of places where they differ. For instance, $d_H(11201001, 11020020) = 5$, because they differ in exactly 5 places (3rd, 4th, 5th, 7th and 8th). Prove that the Hamming distance verifies the following *triangle inequality*:
$$d_H(s, t) \leq d_H(s, u) + d_H(u, t)$$
for every $s, t, u \in \Sigma^n$. (Hint: $d_H(s, t) = |A_{s,t}|$, where $A_{s,t}$ = set of places where s and t differ. Also: $A_{s,t} \subseteq A_{s,u} \cup A_{u,t}$.)

4. Let A be the set of all 8-digit numbers in base 3, including those with leading zeroes.
(a) How many elements are there in A?
(b) Given a non-negative integer r and an element $a \in A$, the set
$$S_r(a) = \{b \in A \mid d_H(a, b) = r\}$$
of all elements of A whose Hamming distance to a is precisely r is called *sphere* of radius r and center a. Find the size (number of elements) of one such a sphere as a function of r.
(c) Similarly, the set
$$B_r(a) = \{b \in A \mid d_H(a, b) \leq r\}$$
of all elements of A whose Hamming distance to a is at most r is called *ball* of radius r and center a. Find the size (number of elements) of a ball of radius 3 in A.
(d) Prove that it is impossible to have 12 pairwise disjoint balls of radius 3 in A.
(e) Prove that given 12 elements from A, two of them coincide in at least 2 places.

5. We define recursively a sequence x_n in the following way:
$$x_0 = 9\,; \qquad x_{n+1} = \frac{x_n^2 + 9}{2\,x_n} \quad (n \geq 0)\,.$$
Prove by induction: $0 < x_n - 3 < 1/2^{2^{n-2}}$ for every $n \geq 2$.