

2.4. Average Value of a Function (Mean Value Theorem)

2.4.1. Average Value of a Function. The average value of finitely many numbers y_1, y_2, \dots, y_n is defined as

$$y_{\text{ave}} = \frac{y_1 + y_2 + \cdots + y_n}{n}.$$

The average value has the property that if each of the numbers y_1, y_2, \dots, y_n is replaced by y_{ave} , their sum remains the same:

$$y_1 + y_2 + \cdots + y_n = \overbrace{y_{\text{ave}} + y_{\text{ave}} + \cdots + y_{\text{ave}}}^{(n \text{ times})}$$

Analogously, the average value of a function $y = f(x)$ in the interval $[a, b]$ can be defined as the value of a constant f_{ave} whose integral over $[a, b]$ equals the integral of $f(x)$:

$$\int_a^b f(x) dx = \int_a^b f_{\text{ave}} dx = (b - a) f_{\text{ave}}.$$

Hence:

$$\boxed{f_{\text{ave}} = \frac{1}{b - a} \int_a^b f(x) dx}.$$

2.4.2. The Mean Value Theorem for Integrals. If f is continuous on $[a, b]$, then there exists a number c in $[a, b]$ such that

$$f(c) = f_{\text{ave}} = \frac{1}{b - a} \int_a^b f(x) dx,$$

i.e.,

$$\int_a^b f(x) dx = f(c)(b - a).$$

Example: Assume that in a certain city the temperature (in °F) t hours after 9 A.M. is represented by the function

$$T(t) = 50 + 14 \sin \frac{\pi t}{12}.$$

Find the average temperature in that city during the period from 9 A.M. to 9 P.M.

Answer:

$$\begin{aligned} T_{\text{ave}} &= \frac{1}{12 - 0} \int_0^{12} \left(50 + 14 \sin \frac{\pi t}{12} \right) dt \\ &= \frac{1}{12} \left[50t - \frac{14 \cdot 12}{\pi} \cos \frac{\pi t}{12} \right]_0^{12} \\ &= \frac{1}{12} \left\{ \left(50 \cdot 12 - \frac{168}{\pi} \cos \frac{12\pi}{12} \right) - \left(50 \cdot 0 - \frac{168}{\pi} \cos 0 \right) \right\} \\ &= 50 + \frac{28}{\pi} \approx 58.9. \end{aligned}$$