1.4. The Substitution Rule

1.4.1. The Substitution Rule. The substitution rule is a trick for evaluating integrals. It is based on the following identity between differentials (where u is a function of x):

$$du = u'\,dx\,.$$

Hence we can write:

$$\boxed{\int f(u)\,u'\,dx = \int f(u)\,du}$$

or using a slightly different notation:

$$\boxed{\int f(g(x))\,g'(x)\,dx = \int f(u)\,du}$$

where $u = g(x)$.

Example: Find $\displaystyle\int \sqrt{1+x^2}\,2x\,dx$.

Answer: Using the substitution $u = 1 + x^2$ we get

$$\begin{aligned}\int \sqrt{1+x^2}\,2x\,dx &= \int \sqrt{u}\,u'\,dx \\ &= \int \sqrt{u}\,du = \tfrac{2}{3}u^{3/2} + C \\ &= \boxed{\tfrac{2}{3}(1+x^2)^{3/2} + C}\,.\end{aligned}$$

Most of the time the only problem in using this method of integration is finding the right substitution.

Example: Find $\displaystyle\int \cos 2x\,dx$.

Answer: We want to write the integral as $\int \cos u\,du$, so $\cos u = \cos 2x \Rightarrow u = 2x$, $u' = 2$. Since we do not see any factor 2 inside the integral we write it, taking care of dividing by 2 outside the integral:

$$\begin{aligned}\int \cos 2x\,dx &= \frac{1}{2}\int \cos 2x\,2\,dx \\ &= \frac{1}{2}\int \cos u\,u'\,dx \\ &= \frac{1}{2}\int \cos u\,du \\ &= \frac{1}{2}\sin u + C\end{aligned}$$

(always remember to undo the substitution)

$$= \boxed{\frac{1}{2}\sin 2x + C}\,.$$

In general we need to look at the integrand as a function of some expression (which we will later identify with u) multiplied by the derivative of that expression.

Example: Find $\displaystyle\int e^{-x^2}\,x\,dx$.

Answer: We see that x is "almost", the derivative of $-x^2$, so we use the substitution $u = -x^2$, $u' = -2x$, hence in order to get u' inside the integral we do the following:

$$\begin{aligned}\int e^{-x^2}\,x\,dx &= -\frac{1}{2}\int \underbrace{e^{-x^2}}_{e^u}\,\underbrace{(-2x)\,dx}_{du} \\ &= -\frac{1}{2}\int e^u\,du = -\tfrac{1}{2}e^u + C = \boxed{-\tfrac{1}{2}e^{-x^2} + C}\,.\end{aligned}$$

Sometimes the substitution is hard to see until we make some ingenious transformation in the integrand.

Example: Find $\displaystyle\int \tan x\,dx$.

Answer: Here the idea is to write $\tan x = \dfrac{\sin x}{\cos x}$ and use that $(\cos x)' = -\sin x$, so we make the substitution $u = \cos x$, $u' = -\sin x$:

$$\begin{aligned}\int \tan x\,dx &= \int \frac{\sin x}{\cos x}\,dx = -\int \frac{u'}{u}\,dx = -\int \frac{1}{u}\,du \\ &= -\ln|u| + C = \boxed{-\ln|\cos x| + C}\,.\end{aligned}$$

In general we need to identify inside the integral some expression of the form $f(u)\,u'$, where f is some function with a known antiderivative.

Example: Find $\displaystyle\int \frac{e^x}{e^{2x}+1}\,dx$.

Answer: Let's write

$$\int \frac{e^x}{e^{2x}+1}\,dx = k\int f(u)\,u'\,dx$$

(where k is some constant to be determined later) and try to identify the function f, the argument u and its derivative u'. Since $(e^x)' = e^x$ it seems natural to chose $u = e^x$, $u' = e^x$, so $e^{2x} = u^2$ and

$$\begin{aligned}\int \frac{e^x}{e^{2x}+1}\,dx &= \int \frac{u'}{u^2+1}\,dx = \int \frac{1}{u^2+1}\,du \\ &= \tan^{-1} u + C = \boxed{\tan^{-1}(e^x) + C}\,.\end{aligned}$$

There is no much more that can be said in general, the way to learn more is just to practice.

1.4.2. Other Changes of Variable. Sometimes rather than making a substitution of the form u = function of x, we may try a change of variable of the form x = function of some other variable such as t, and write $dx = x'(t)\,dt$, where x' = derivative of x respect to t.

Example: Find $\displaystyle\int \sqrt{1-x^2}\,dx$.

Answer: Here we write $x = \sin t$, so $dx = \cos t\,dt$, $1 - x^2 = 1 - \sin^2 t = \cos^2 t$, and

$$\int \sqrt{1-x^2}\,dx = \int \underbrace{\cos t}_{x}\ \underbrace{\cos t\,dt}_{dx} = \int \cos^2 t\,dt\,.$$

Since we do not know yet how to integrate $\cos^2 t$ we leave it like this and will be back to it later (after we study integrals of trigonometric functions).

1.4.3. The Substitution Rule for Definite Integrals. When computing a definite integral using the substitution rule there are two possibilities:

(1) Compute the indefinite integral first, then use the evaluation theorem:

$$\int f(u)\,u'\,dx = F(x)\,;$$
$$\int_a^b f(u)\,u'\,dx = F(b) - F(a)\,.$$

(2) Use the substitution rule for definite integrals:

$$\int_a^b f(u)\,u'\,dx = \int_{u(a)}^{u(b)} f(u)\,du\,.$$

The advantage of the second method is that we do not need to undo the substitution.

Example: Find $\displaystyle\int_0^4 \sqrt{2x+1}\,dx$.

Answer: Using the first method first we compute the indefinite integral:

$$\begin{aligned}\int \sqrt{2x+1}\,dx &= \frac{1}{2}\int \sqrt{2x+1}\,2\,dx \quad (u = 2x+1)\\ &= \frac{1}{2}\int \sqrt{u}\,du\\ &= \frac{1}{3}u^{3/2} + C\\ &= \frac{1}{3}(2x+1)^{3/2} + C\,.\end{aligned}$$

Then we use it for computing the definite integral:

$$\int_0^4 \sqrt{2x+1}\,dx = \left[\frac{1}{3}(2x+1)^{3/2}\right]_0^4 = \frac{1}{3}9^{3/2} - \frac{1}{3}1^{3/2} = \frac{27}{3} - \frac{1}{3} = \boxed{\frac{26}{3}}\,.$$

In the second method we compute the definite integral directly adjusting the limits of integration after the substitution:

$$\int_0^4 \sqrt{2x+1}\,dx = \frac{1}{2}\int_0^4 \sqrt{2x+1}\,2\,dx \qquad (u = 2x+1;\ u' = 2)$$
$$= \frac{1}{2}\int_1^9 \sqrt{u}\,du$$

(note the change in the limits of integration to $u(0) = 1$ and $u(4) = 9$)

$$= \left[\frac{1}{3}u^{3/2}\right]_1^9$$
$$= \frac{1}{3}9^{3/2} - \frac{1}{3}1^{3/2}$$
$$= \frac{27}{3} - \frac{1}{3} = \boxed{\frac{26}{3}}.$$