AMAT/TMAT 118

Solutions to Exercises on Sets

Exercise 1. Is it true that $0 \in \{\{0\}\}$?

Answer: No, because $\{\{0\}\}$ is the set whose only element is $\{0\}$. And $0 \neq \{0\}$.

Exercise 2. If $S = \{\{a\}, \{b\}\}$, and $T = \{\{a\}, b\}$, is it true that $S = T$?

Answer: No, because the sets have different elements.

Exercise 3. If $S = \{A, B\}$, what is $S \cup \emptyset$?

Answer: $S \cup \emptyset = S$. Many people wrote $S = \{A, B, \{\}\}$, but this is wrong; $\{\}$ is not an element of either S or $\{\}$.

Exercise 4. For S as in the previous exercise, what is $S \cup S$?

Answer: S.

Exercise 5. If $S = \{A, B\}$, what is $S \cap \emptyset$?

Answer: $\emptyset$.

Exercise 6. If $S = \{A, B\}$, what is $S \cap S$?

Answer: S.

Exercise 7. Is it true that if $S \subset T$ and $T \subset S$, then $S = T$?

Answer: Yes.

Exercise 8. Find all the subsets of $\{\{A, B\}, \{\}\}$.

Answer: $\{\{A, B\}, \{\}\}$, $\{\{A, B\}\}$, $\{\{\}\}$, $\{\}$.

Exercise 9. For $S = \{1, 2\}$, what is $S \times S$?

Answer: $\{(1, 1), (1, 2), (2, 1), (2, 2)\}$

Exercise 10. What is the range of the identity function $\mathrm{Id} : S \to S$?

Answer: S. (Some students assumed that $S = \{A, B\}$. Because of where the question was placed, this was reasonable, though not what I intended, so I also accepted the answer $\{A, B\}$.)

Exercise 11. For $S = \{A, B\}$, $T = \{A, B, C\}$, how many functions $f : S \to T$ are there with the range of f equal to the range of the inclusion $j : S \to T$?

Answer: Most students got this wrong, at least in part. There are two functions. One is given by $f = j$. The other is given by $f(A) = B$, $f(B) = A$.

Exercise 12 (bonus)**.** The example above is a bit special because the right hand side of the equation is 0. Generalizing the above, explain how the solution to the equation $f(x) = g(x)$ can be interpreted as a subset of $\mathbb{Z}$ for any functions $f : \mathbb{Z} \to \mathbb{Z}$, $g : \mathbb{Z} \to \mathbb{Z}$.

Answer: the solutions are the set $\{z \in \mathbb{Z} \mid f(z) = g(z)\}$.

Exercise 13. Let $S = \{1, 2\}$ and $T = \{A, B, C\}$.

(i) How many functions are there from S to T?

 Answer: 9, because there are 3 options for where to map 1 to, and 3options for where to map 2 to. $3 \times 3 = 9$.

(ii) How many 1-1 functions are there from S to T?

 Answer: 6. Of the 9 functions from (i), 3 are not 1-1.

(iii) How many onto functions are there from S to T?

 Answer: 0

(iv) How many bijections are there from S to T?

 Answer: 0

Important Fact 0.1.

(i) A map $f : S \to T$ is a bijection if and only if f has an inverse.

(ii) If an inverse of f exists, it is unique.

Exercise 14 (bonus)**.** Proof part (ii) of the important fact.

Answer: We need to show that if $g : T \to S$ and $h : T \to S$ are both inverses of f, then $g = h$. The equation $g = h$ means $g(t) = h(t)$ for all $t \in T$. Since f has an inverse, it is surjective, so for all all $t \in T$ there is an element $s \in S$ with $f(s) = t$. Using that both g and h are inverses of f, we then have

$$g(t) = g \circ f(s) = \mathrm{Id}_S = h \circ f(s) = h(t).$$

Exercise 15. Find the inverse of the function f from **??**. (HINT: See the first couple of sentences of the above proof.)

Answer: The inverse is the function $g : T \to S$ given by $g(A) = 1$, $g(B) = 2$.

Exercise 16. For which integers a does the function $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = x + a$ have an inverse? For each such a, what is the inverse?

Answer: All integers a. The inverse $g : \mathbb{Z} \to \mathbb{Z}$ is given by $g(z) = z - a$.

Exercise 17. For which integers a does the function $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = ax$ have an inverse? For each such a, what is the inverse?

Answer: $a \in \{1, -1\}$. The inverse $g : \mathbb{Z} \to \mathbb{Z}$ is given by $g(z) = z/a$.

Exercise 18. Is $\{1, 2, 3\}$ a partition of $\{1, 2, 3\}$?

Answer: No. It is not even a set of subsets of $\{1, 2, 3\}$.

Exercise 19. Find all partitions of $\{1, 2, 3\}$.

Answer:
$\{\{1, 2, 3\}\}$,
$\{\{1\}, \{2\}, \{3\}\}$,
$\{\{1, 2\}, \{3\}\}$,
$\{\{1\}, \{2, 3\}\}$,
$\{\{2\}, \{1, 3\}\}$.