

MULTIDIMENSIONAL INTERLEAVINGS
AND APPLICATIONS TO TOPOLOGICAL INFERENCE

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Abstract

This thesis concerns the theoretical foundations of persistence-based topological data analysis. The primary focus of the work is on the development of theory of topological inference in the multidimensional persistence setting, where the set of available theoretical and algorithmic tools has remained comparatively underdeveloped, relative to the 1-D persistence setting. The thesis establishes a number of theoretical results centered around this theme, some of which are new and interesting even for 1-D persistent homology. In addition, this work presents theory of topological inference formulated directly at the (topological) level of filtrations, rather than only at the (algebraic) level of persistent homology modules.

The main mathematical objects of study in this work are **interleavings**. These are tools for quantifying the similarity between multidimensional filtrations and persistence modules. They were introduced for 1-D filtrations and persistence modules by Chazal et al. [8], where they were used to prove a strong and very useful generalization of the stability of persistence result of [14]; we generalize the definition of interleavings appearing in [8] in several directions and use these generalizations to define pseudometrics on multidimensional filtrations and multidimensional persistence modules called **interleaving distances**.

The first part of this thesis, adapted from the preprint [32], studies in detail the theory of interleavings and interleaving distances on multidimensional persistence modules. We present six main results about the interleaving distance.

First, we show that in the case of 1-D persistence, the interleaving distance is equal to the bottleneck distance on tame persistence modules.

Second, we prove a theorem which implies that the restriction of the interleaving

distance to finitely presented multidimensional persistence modules is a metric. The same theorem, together with our first result, also yields a converse to the algebraic stability theorem of [8]; this answers a question posed in that paper.

Third, we present an “extrinsic” characterization of interleaved pairs of multidimensional persistence modules which makes transparent the sense in which interleaved modules are algebraically similar. This characterization turns out to hold for a definition of interleavings of multidimensional persistence modules rather more general than that which we need to define the interleaving distance; the more general form of our result is an important ingredient in inferential theory we develop in the second part of this thesis.

Fourth, we observe that the interleaving distance is stable in four senses analogous to those in which the bottleneck distance is known to be stable.

Fifth, we introduce several notions of optimality of metrics on persistence modules and show that when the underlying field is \mathbb{Q} or a field of prime order, the interleaving distance is optimal with respect to one of these notions. This optimality result, which is new even for 1-D persistence, is the central result of the first part of this thesis. We also prove that a version of this result holds for ordinary persistence modules over any field, provided we restrict attention to a class of well behaved ordinary persistence modules containing the finitely presented ones.

Sixth, we show that the computation of the interleaving distance between two finitely presented multidimensional persistence modules M and N reduces to deciding the solvability of $O(\log m)$ systems of multivariate quadratic equations, each with $O(m^2)$ variables and $O(m^2)$ equations, where m is the total number of generators and relations in a minimal presentation for M and a minimal presentation for N .

In the second part of the thesis, we define interleavings and interleaving distances on multidimensional filtrations, and present a theoretical treatment of these parallel to our treatment of interleavings and interleaving distances on multidimensional persistence modules. We then use interleavings and interleaving distances on multidimensional filtrations to formulate and prove several analogues of a topological inference theorem of [11] in the multidimensional setting, and directly on the level of filtrations. In particular, we employ *localization of categories*, a standard construction

in homotopy theory, to define and study homotopy theoretic versions of interleavings and the interleaving distance on multidimensional filtrations, which we call **weak interleavings** and the **weak interleaving distance**. We formulate our main inference results using weak interleavings and the weak interleaving distance.

To describe these inference results in detail, let $\gamma : \mathbb{R}^m \rightarrow \mathbb{R}$ be a probability density function on \mathbb{R}^m , and for $z \in \mathbb{Z}_{\geq 0}$ let T_z be an i.i.d. sample of size z of a probability distribution with density γ . Let $F_z^{S\check{C}e}$ be the random Čech bifiltration with vertices T_z , filtered by the superlevelsets of $E(T_z)$, where E is a density estimator, and by the usual scale parameter for Čech complexes.

Our first main inference result is that under mild conditions on γ and E , $F_z^{S\check{C}e}$ converges in probability (with respect to the weak interleaving distance, and as $z \rightarrow \infty$) to a bifiltration constructed directly from γ , which we call the **superlevel-offset bifiltration**.

Our second main inference result is an analogue of the first result for Vietoris-Rips bifiltrations, filtered by the superlevelsets of $E(T_z)$ and by the usual scale parameter for Vietoris-Rips complexes.

We also present analogues for each of these results for probability density functions defined on Riemannian manifolds.

These inference results on the level of filtrations yield as corollaries analogous results on the level of persistent homology, formulated in terms of the interleaving distance and interleavings on persistence modules. Our extrinsic characterization of interleavings from the first part of our thesis yields concrete interpretations of these corollaries as statements about the similarity between presentations of persistent homology modules.

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Chapter 1

Introduction

In this introduction, we offer some context and motivation for the theory developed in this thesis and present an overview of our results. Section 1.1 introduces the broader context for the mathematics of this thesis and discusses the need for further development of the statistical foundations of topological data analysis; Section 1.2 introduces the problem of developing inferential theory for multidimensional filtrations, and presents background and motivation for the problem; Sections 1.3-1.5 present detailed overviews of the results of chapters 2-4 of this thesis; and Section 1.6 closes this introduction with discussion of our motivation for considering in this thesis persistence based inference directly at the level of filtrations.

We assume that the reader has some familiarity with some of the basic terminology and ideas of applied topology and topological data analysis. See the reviews [23, 27, 4] and the textbook [24] for treatment of the basics, and Sections 2.1, 3.1, and 4.1 for foundational definitions.

1.1 On The Need for Firm Statistical Foundations of Topological Data Analysis

To explain the mathematical context of this thesis, it will be useful for us to begin by formulating definitions of topological inference and topological data analysis.

Recall first that in statistics, we distinguish between *descriptive statistics* and *statistical inference*. Descriptive statistics, as the name suggests, is that part of statistics concerned with defining and studying descriptors of data. It involves no probability theory and aims simply to offer tools for describing, summarizing, and visualizing data. Statistical inference, on the other hand, concerns the more sophisticated enterprise of estimating descriptors of an unknown probability distribution from random samples of the distribution. The theory and methods of statistical inference are built on the tools of descriptive statistics: The estimators considered in statistical inference are of course, when stripped of their inferential interpretation, merely descriptors of data.

We define **descriptive topological data analysis (descriptive TDA)** to be the branch of descriptive statistics which uses topology to define and study qualitative descriptors of data sets.

We define **topological inference** to be the branch of statistical inference which

1. uses topology to define qualitative descriptors of probability distributions.
2. develops and studies estimators for inferring such descriptors from finite samples of the distributions.

We define **topological data analysis (TDA)** to refer collectively to descriptive TDA, topological inference, and the applications of these to science and engineering.

In the last 10 years the TDA community has introduced a number of novel tools for descriptive TDA [47, 7, 5, 42, 16, 1], and has begun developing applications of these in science and engineering. There is great interest in applying these tools to the study of random data, and a good deal of work has already been done in this direction [6, 43, 34, 11].

In the last few years, there also has been some important progress in topological inference [35, 11, 10]. However, the development of tools for descriptive TDA has, as a rule, outpaced the development of the theory of topological inference supporting the use of these tools in the study of random data. Indeed, statistical foundations for many of the most discussed tools in TDA—for example, Mapper [42], circle valued coordinatization [16], and persistent homology of Vietoris-Rips (multi-)filtrations

[7]—have either not been laid out at all, or have only partially (and recently) been laid out. In many cases, there is no theoretical framework in place for interpreting these descriptors of data as appropriately behaved estimators of descriptors of probability distributions. (An important exception is the work of Chazal et al. [11] on estimating the superlevelset persistent homology of density functions using Rips complexes; we’ll discuss this work in detail in Section 1.2.2.)

Additionally there is, to date, little to no discussion in the TDA literature on how to compute theoretically sound measures of confidence for any estimator of a topological descriptor. This last gap is an especially critical one in the statistical foundations of TDA: It is a basic principle of statistical inference that an estimate of some descriptor of a probability distribution is only meaningful insofar as we also have a good measure of confidence for that estimate.

In these senses, the statistical foundations of TDA remain quite underdeveloped; much further progress on the methodology and theory of topological inference is needed before the tools of TDA can sit comfortably amongst the more conventional tools in a statistician’s toolbox.

Carrying out this work is central to the program of fully realizing TDA as a data analysis paradigm of value in science and engineering. After all, if the statistical foundations of topological data analysis were firmer, the output of the tools of topological data analysis on random input would be more informative, and presumably statisticians, scientists, and engineers would be more receptive to the tools and more inclined to invest the time to develop new applications of them.

This perspective motivates the work of this thesis: This work is an effort to contribute to the program of fleshing out the statistical foundations of TDA. The program is a broad one, however, and the results of this thesis amount only to a narrow slice of what is needed to really put TDA on firm statistical footing. In particular, the thesis focuses primarily on theoretical foundations and devotes little direct attention to questions of data analysis methodology. Nevertheless, it is my hope that the results presented here can offer clarity on some basic issues in topological inference and TDA and, in so doing, open the door to progress in the development of methodology.

1.2 Overview: Inferential Theory for Multidimensional Filtrations

We focus in this thesis on the theoretical foundations of *persistence-based topological data analysis*, that central (though not all-encompassing) branch of topological data analysis that considers topological descriptors and random variables defined using filtrations and persistent homology. Our primary aims are, first, to develop inferential theory for multidimensional persistence, and, second, to develop inferential theory directly at the level of filtrations which strengthens the theory at the level of persistent homology modules.

1.2.1 Context and Motivation for Our Inference Results

The history of the problem of developing an inferential theory for multidimensional persistent homology dates back several years, to the original paper on multidimensional persistent homology [7]. Motivated by needs arising in their study of natural scene statistics [6], in [7] Carlsson and Zomorodian proposed the use of *Vietoris-Rips bifiltrations* to probe the qualitative structure of point cloud data of nonuniform density in exploratory data analysis applications. This proposal was motivated by the idea that when the point cloud data set T_z is obtained as an i.i.d. sample of size z of some probability distribution with density function $\gamma : \mathbb{R}^m \rightarrow \mathbb{R}$, the random Vietoris-Rips bifiltration with vertices T_z , filtered by superlevelsets of a density estimator E and by the usual scale parameter for a Vietoris-Rips complex, should encode topological information about γ .

A central aim of this thesis is to put this idea on firm mathematical footing.

Our approach builds in an essential way on recent work on persistence-based topological inference by Chazal, Guibas, Oudot, and Skraba [11]. In that work, the authors build on theory developed in their earlier papers [8, 12] to prove a result [11, Theorem 5.1] on the inference of the 1-D (superlevelset) persistent homology of a probability density function γ from a pair of filtered Vietoris-Rips complexes built on i.i.d. samples of a probability distribution with density γ . This result is a (loose)

analogue for persistent homology of the weak law of large numbers.

The primary aim of the paper [11] is to leverage persistent homology to introduce a clustering algorithm with good theoretical properties; the exposition there is such that the inference result [11, Theorem 5.1] plays a supporting role, serving as the engine for the development of theoretical guarantees on the clustering algorithm presented in that paper.

Nevertheless, the result is a very significant one in of itself: It proves (implicitly, at least), for the first time, the consistency of an estimator of the superlevelset persistent homology of a density function on a Euclidean space. Moreover, because that estimator is defined using only a pair of Vietoris-Rips complexes, it is simple and quite computable.

One disadvantage of the estimator introduced in [11], however, is that its construction depends on a choice of scale parameter. As noted in [11], a correct value for this parameter can be difficult to choose in practice. Moreover, in spite of the good asymptotic properties of the estimator studied in [11], for finite data sets having non-trivial topological features (in the usual TDA sense) across a range of density and length scales, it can be that for any single choice of scale parameter, this estimator is unable to fully detect the topological features of the data.

A main goal of this work is to adapt the topological inference result [11, Theorem 5.1] to the multidimensional persistence setting and directly to the level of filtrations. We present such adaptations Chapter 4 of this thesis. As we will see, the multidimensional setting has the advantage that in constructing estimators there analogous to that considered in [11], we do not need to choose a scale parameter.¹

¹For both the estimator considered in [11] and the estimators we consider here, it is necessary to chose a bandwidth parameter for a density estimator. In the Euclidean case, the need to select a bandwidth parameter is not an unmanageable difficulty, at least in low dimensions: For density functions on Euclidean spaces, the problem of optimally selecting a bandwidth parameter for a kernel density estimator has been well studied—see [39].

1.2.2 The Result of Chazal et al. on the Inference of Sub-levelset Persistent Homology Using Rips Complexes

To understand the context of our inference results, it is important to understand the result [11, Theorem 5.1]. We now present an asymptotic corollary of [11, Theorem 5.1] which will serve as our mathematical point of departure in our pursuit of our own inference results.

To prepare for the result, we review some basics about density estimation.

Density Estimation Preliminaries

Let $\mathcal{D}(\mathbb{R}^m)$ denote the set of probability density functions on \mathbb{R}^m . Define a density estimator E on a \mathbb{R}^m to be a sequence of functions $\{E_z : (\mathbb{R}^m)^z \rightarrow \mathcal{D}(\mathbb{R}^m)\}_{z \in \mathbb{N}}$ such that for each $z \in \mathbb{N}$, the restriction of E_z to any point in \mathbb{R}^m is a measurable function from $(\mathbb{R}^m)^z$ to \mathbb{R} . In formulating our results, we will consider pairs (γ, E) , where γ is a density function and E is a density estimator, for which one of the following two assumptions holds:

A1: E converges uniformly in probability to γ .

A2: E converges uniformly in probability to the convolution of γ with some kernel function K .

A1 is known to hold for kernel density estimators, for a wide class of kernels and density functions γ , provided the kernel width tends to 0 at an appropriate rate as z tends to infinity. A2 is known to hold for the kernel density estimator with kernel K (with fixed width as the number of samples z varies) for a wide class of kernels K and density functions γ . See Section 4.1.3 for more details and references.

Our Asymptotic Corollary of the Inference Result [11, Theorem 5.1]

The form of the result we quote here is different in several ways than that of [11, Theorem 5.1]. First, as we noted above, rather than recall the original form of [11, Theorem 5.1], we will present a tidier asymptotic corollary of it whose form is closer

to that of our main inference results. The proof of the asymptotic corollary, given the original form of the result, follows from an ϵ - δ argument very similar to that used in the proof of our Theorem 4.5.2. Second, because of the emphasis on clustering in [11], the result [11, Theorem 5.1] is stated only for 0th persistent homology, but as the authors note, the result adapts immediately to higher persistent homology via the results of [12]. We will present here a version of the asymptotic corollary which holds for i^{th} persistent homology, $i \in \mathbb{Z}_{\geq 0}$. Third, to minimize the technicalities in this introduction, we present the result for density functions defined on Euclidean space; the result adapts readily to Riemannian manifolds with sectional curvature bounded above and below.

Let $\gamma : \mathbb{R}^m \rightarrow \mathbb{R}$ be a c -Lipchitz density function such that for $i \in \mathbb{Z}_{\geq 0}$, the superlevelset persistent homology module $H_i(-\gamma)$ is tame²; let T_z be an i.i.d. random sample of a probability distribution with density γ of size z ; let d^p be the restriction of some L^p metric to T_z , for $1 \leq p \leq \infty$; and let E be a density estimator.

For $\delta > 0$, let $F^{SR}(T_z, d^p, -E(T_z), \delta)$ be a Rips filtration on (T_z, d^p) with fixed scale parameter δ , filtered by sublevelsets of $-E(T_z)$. For $i \in \mathbb{Z}_{\geq 0}$, let H_i denote the i^{th} persistent homology functor. The inclusion

$$F^{SR}(T_z, d^p, -E(T_z), \delta) \hookrightarrow F^{SR}(T_z, d^p, -E(T_z), 2\delta)$$

induces a homomorphism

$$j_{z,\delta} : H_i(F^{SR}(T_z, d^p, -E(T_z), \delta)) \rightarrow H_i(F^{SR}(T_z, d^p, -E(T_z), 2\delta))$$

of persistence modules. $\text{Im}(j_{z,\delta})$ is then itself a persistence module.

For a 1-D persistence module M , let M^+ denote the submodule of M generated by homogeneous vector space summands M_a of M with $a \geq 0$ and let $R_0(M) = M/M^+$.

Let d_B denote the bottleneck distance on tame persistence modules [8]. If $\{X_z\}_{z \in \mathbb{N}}$ is a sequence of random variables, and X is a random variable such that X_z converges

²In this thesis, a tame persistence module is one whose homogeneous vector space summands are each finite dimensional; see Section 2.1.3. The tameness condition ensures that the bottleneck distance in the statement of [11, Theorem 5.1] is well defined.

in probability to X , we write $X_z \xrightarrow{\mathcal{P}} X$.

Theorem 1.2.1 (Inference result of [11] (asymptotic form)).

(i) *If (γ, E) satisfies A1 then \exists a sequence $\{\delta_z\}_{z \in \mathbb{N}}$ such that*

$$d_B(R_0(\text{Im}(j_{\delta_z, z})), R_0(H_i(-\gamma))) \xrightarrow{\mathcal{P}} 0$$

(as $z \rightarrow \infty$).

(ii) *If (γ, E) satisfies A2 for some kernel K then \exists a sequence $\{\delta_z\}_{z \in \mathbb{N}}$ such that*

$$d_B(R_0(\text{Im}(j_{\delta_z, z})), R_0(H_i(-\gamma * K))) \xrightarrow{\mathcal{P}} 0.$$

1.2.3 Distances and Measures of Proximity on Filtrations and Persistence Modules

As we have noted above, one of the main goals of this work is to adapt the consistency result Theorem 1.2.1 to the multidimensional setting and to the level of filtrations. Since Theorem 1.2.1 is formulated in terms of the bottleneck distance, to carry out this adaptation we require analogues of this distance on multidimensional persistence modules and filtrations. The main obstacle to adapting Theorem 1.2.1 to the multidimensional setting is that the bottleneck distance of ordinary persistent homology does not admit a naive extension to the multidimensional setting. Similarly, the main obstacle to adapting the theorem to the level of filtrations is that the bottleneck distance does not admit a naive adaptation to a distance on filtrations, even in 1-D.

Most of the effort of this thesis is put not directly towards proving inference results, but rather towards introducing and studying generalizations of the bottleneck distance to multidimensional filtrations and persistence modules which we need to formulate our inference results. Indeed, one of the main themes of this work is that developing the theory of topological inference largely boils down to the problem of selecting and understanding distances (or more general measures of proximity, which strictly speaking, are not distances in any reasonable mathematical sense) on appropriate objects: Once we develop the mathematical vocabulary needed to properly formulate

our inference results in this thesis, the proofs of the results turn out to be reasonably straightforward, given existing ideas in the literature (and particularly those put forth [11]). The main mathematical challenge then is to make sense of the distances and measures of proximity between filtrations and between persistence modules with which we formulate the inference results, and to show that these measures are, in suitable senses, the right ones to use.

We formulate our inference results using **interleavings** and distances defined in terms of interleavings called **interleaving distances**. As noted in the abstract, interleavings are tools for quantifying the similarity between multidimensional filtrations and persistence modules. Interleaving distances are generalizations of the bottleneck distance to pseudometrics on multidimensional filtrations and on multidimensional persistence modules. The most basic type of interleavings, called ϵ -**interleavings**, were introduced for 1-D persistence filtrations and persistence modules by Chazal et al. [8]. This thesis introduces generalizations of the definition of ϵ -interleavings given in [8] to definitions of ϵ -interleavings on multidimensional filtrations and multidimensional persistence modules. We define interleaving distances in terms of these generalized ϵ -interleavings.

In fact, in this thesis we generalize the notion of ϵ -interleavings yet further to arrive at the definition of (J_1, J_2) -**interleavings**, which we also sometimes call **generalized interleavings**. We apply interleaving distances and (J_1, J_2) -interleavings to formulate results on multidimensional inference at the level of filtrations. In a sense that we will make precise later, (J_1, J_2) -interleavings allow us to quantify anisotropic and asymmetric similarities between filtrations and between persistence modules that cannot be completely described using interleaving distances. They turn out to be the right mathematical tools for interpreting random Vietoris-Rips bifiltrations as inferential objects.

1.3 Chapter 2: Interleavings and Interleaving Distances for Multidimensional Persistence Modules

In Chapter 2 of this thesis, we introduce interleavings and the interleaving distance d_I on multidimensional persistence modules, and treat the theory of these in detail. Chapter 2 of this thesis is adapted from the preprint [32]; the content of Chapter 2 is very close to that of [32]. The most important difference is that we define interleavings in greater generality and extend our extrinsic characterization of ϵ -interleavings in [32] to an extrinsic characterization of generalized interleavings. There are some other minor differences between Chapter 2 of the thesis and [32], but none of any great significance.

1.3.1 Results

In Chapter 2, we present six main results on interleavings and the interleaving distance. The first result, Theorem 2.4.2, shows that in the case of ordinary persistence, the interleaving distance is in fact equal to the bottleneck distance on tame persistence modules. Our proof relies on a generalization of the structure theorem [47] for finitely generated ordinary persistence modules to (discrete) tame persistence modules. This generalization is proven e.g. in [46].

Our second main result is Theorem 2.5.1, which tells us that if M and N are two finitely presented persistence modules and $d_I(M, N) = \epsilon$ then M and N are ϵ -interleaved. As an immediate consequence of this theorem, we have Corollary 2.5.2, which says that the interleaving distance restricts to a metric on finitely presented persistence modules. Theorems 2.4.2 and 2.5.1 together also yield Corollary 2.5.3, a converse to the algebraic stability theorem of [8]. The converse says, firstly, that if two tame 1-D persistence modules M and N are distance ϵ apart in the bottleneck distance, then they are $\epsilon + \delta$ -interleaved for any $\delta > 0$. Secondly, the converse says that if in addition M and N are each finitely presented (which is stronger than tameness), then M and N are in fact ϵ -interleaved. This result answers a question posed in [8].

Our third main result is Theorem 2.6.4, an “extrinsic” characterization of (J_1, J_2) -interleaved pairs of persistence modules; it expresses transparently the sense in which (J_1, J_2) -interleaved persistence modules are algebraically similar. Since ϵ -interleavings are a special type of (J_1, J_2) -interleaving, and since the interleaving distance is defined in terms of ϵ -interleavings, the result also yields an extrinsic characterization of the interleaving distance. The result is reminiscent of the extrinsic characterization of the Gromov-Hausdorff distance, which expresses the Gromov-Hausdorff distance between two compact metric spaces in terms of the Hausdorff distance between embeddings of two metric spaces into a third metric space. Roughly speaking, Theorem 2.6.4 characterizes ϵ -interleaved pairs of persistence modules in terms of a distance between embeddings of presentations of such modules into a free persistence module.

As noted above, Theorem 2.6.4 was presented in [32] only for the special case of ϵ -interleavings. The more general form of our result will be an important ingredient of the inferential theory we develop in the second part of this thesis.

Our fourth result is the observation that the interleaving distance is stable in four senses analogous to those in which the bottleneck distance is known to be stable. These stability results, while notable, require very little mathematical work; two of the stability results turn out to be trivial, the third follows from a minor modification of an argument given in [9], and the fourth admits a straightforward proof.

Our fifth main result, Corollary 2.10.2, is an optimality result for the interleaving distance. It tells us that when the underlying field is \mathbb{Q} or a field of prime order, the interleaving distance is stable in a sense analogous to that in which the bottleneck distance is shown to be stable in [14, 8], and further, that the interleaving distance is, in a uniform sense, the most sensitive of all stable pseudometrics. This “maximum sensitivity” property of the interleaving distance is equivalent to the property that, with respect to the interleaving distance, multidimensional persistent homology preserves the metric on source objects as faithfully as is possible for any choice of stable pseudometric on multidimensional persistence modules; see Remark 2.9.7 for a precise statement. Our optimality result is new even for 1-D persistence. In that case, it offers some mathematical justification, complementary to that of [14, 8], for the use of the bottleneck distance.

In fact, provided we restrict attention to a class of well behaved ordinary persistence modules containing the finitely presented ones, the assumption that the underlying field is \mathbb{Q} or a field of prime order is unnecessary; our Theorem 2.10.8 gives an analogue of our optimality result Corollary 2.10.2 for this class of modules, over arbitrary fields.

The main ingredient in the proof of Corollary 2.10.2 is our characterization result Theorem 2.6.4. Using that result, we present a constructive argument which shows that when the underlying field is \mathbb{Q} or a field of prime order, ϵ -interleavings can, in a suitable sense, be lifted to a category of \mathbb{R}^n -valued functions. Given this, our optimality result follows readily.

Given our first five main results, it is natural to ask if and how the interleaving distance can be computed. Our sixth main result speaks to this question. The result, which follows from Theorem 2.11.5 and Proposition 2.11.9, is that the computation of the interleaving distance between two finitely presented multidimensional persistence modules M and N reduces to deciding the solvability of $O(\log m)$ systems of multivariate quadratic equations, each with $O(m^2)$ variables and $O(m^2)$ equations, where m is the total number of generators and relations in a minimal presentation for M and a minimal presentation for N . This result is just a first step towards understanding the problem of computing the interleaving distance; we hope to address the problem more fully in future work.

A Note On Prior Work

After making the results of the first part of this thesis publicly available (in form of the preprint [32]), it was brought to our attention that in [22], d’Amico et al. proved an optimality result for the bottleneck distance similar to the optimality results given here, for the special case of 0-dimensional ordinary persistent homology. Our Theorem 2.10.8 generalizes a slight weakening of [22, Theorem 32]; see Remark 2.10.9.

1.4 Chapter 3: Strong and Weak Interleavings and Interleaving Distances for Multi-D Filtrations

In Chapter 3 we introduce and study interleavings and interleaving distances on multidimensional filtrations. The first goal of Chapter 3 is to present theory for interleavings on filtrations analogous to that which Chapter 2 presents for interleavings on multidimensional persistence modules; the second goal of the chapter is to establish the technical foundations needed to formulate and prove topological inference results directly at the level of filtrations.

We in fact introduce two types of interleavings on multidimensional filtrations, **strong interleavings** and **weak interleavings**; each induces a distance on multidimensional filtrations, the **strong interleaving distance** d_{SI} and the **weak interleaving distance** d_{WI} , respectively. We define strong interleavings in a way closely analogous to the way in which we define interleavings on persistence modules, and they share some of the favorable theoretical properties of such interleavings.

However, strong interleavings turn out to be too sensitive for the purpose of proving inference results analogous to Theorem 1.2.1 at the level of filtrations. The reason, put somewhat coarsely, is that strong interleavings are sensitive to the topology of the spaces in filtrations up to homeomorphism; in developing theory of topological inference, it turns out to be necessary to work with a notion of interleaving that is sensitive only to the homotopy type of the spaces in the filtrations. Weak interleavings have this property, in a certain sense. We use weak interleavings and the weak interleaving distance to formulate our main inference results on the level of filtrations.

Because our inference results at the level of filtrations are formulated using weak interleavings rather than strong interleavings, we are less interested in understanding strong interleavings and the strong interleaving distance than their weak counterparts. Nevertheless, the theory of strong interleavings is worth developing, if only as a bridge to developing the theory of weak interleavings.

1.4.1 Strong Interleavings and the Strong Interleaving Distance

Section 3.2 presents the definition and basic theory of strong interleavings. We present three main results on the theory of strong interleavings.

Our first main result, Theorem 3.2.7, is a characterization of strongly interleaved pairs of **filtrations of nested type**—filtrations of nested type are simply filtrations, each of whose transition maps is an injection; all filtrations we have occasion to consider in the development of our inferential theory in Chapter 4 are of nested type.

Our characterization of strongly interleaved pairs of **filtrations of nested type** is loosely analogous to our characterization Theorem 2.6.4 of interleaved pairs of persistence modules in Chapter 2. Whereas our characterization of interleaved pairs of persistence modules in Chapter 2 is given in terms of *free covers* of persistence modules (i.e. the 0^{th} modules in free resolutions of persistence modules) our characterization of strongly interleaved pairs of filtrations of nested type is given in terms of *colimits* of filtrations.

Theorems 3.2.10 and 3.2.11, our second and third main results on the theory of strong interleavings, are optimality results for d_{SI} , each analogous to our optimality result Corollary 2.10.2 for d_I . We prove these using Proposition 2.10.4 (which, as noted above, is the main step in our proof of Corollary 2.10.2, our optimality result for multidimensional persistence modules) and our characterization result Theorem 3.2.7.

1.4.2 Weak Interleavings and the Weak Interleaving Distance

In Section 3.3, we define and study weak interleavings and the weak interleaving distance. Our definition of weak (J_1, J_2) -interleavings, via which we formulate our second main inference result, Theorem 4.4.4, is perhaps the most interesting object of study of this thesis.

We'll now discuss at a high level the motivation for the definition of weak interleavings, the use of localization of categories in formulating the definition, and the theory of weak interleavings.

Motivation For Considering Weak Interleavings

As we have noted above, strong interleavings are too sensitive for our use in formulating our inference results. We begin by explaining the sense in which this is so: To adapt Theorem 1.2.1 to the level of filtrations, we want a notion of distance d between interleavings between filtrations such that if $f : X \rightarrow Y$ is a morphism of filtrations³ which is a *levelwise homotopy equivalence*, meaning that each component map $f_a : X_a \rightarrow Y_a$ is a homotopy equivalence, then $d(X, Y) = 0$. The reason we want a distance on filtrations with this property is simple: To adapt the proof of Theorem 1.2.1 to proofs of inference results at the level of filtrations, we need an adaptation of the persistent nerve lemma of [13] which holds on the level of filtrations rather than only on the level of persistent homology; such an adaptation can be formulated in terms of a distance d on filtrations with the above property.

When we give the definition of $d_{SI}(X, Y)$ in Section 3.2, it will be easy to see that it is not true that $d_{SI}(X, Y) = 0$ whenever there is a levelwise homotopy equivalence between X and Y ; see Remark 4.2.2 for a counterexample. On the other hand, our Proposition 4.2.1 shows that d_{WI} does have this property. In fact, we formulate the definitions of weak interleavings and d_{WI} so as to explicitly enforce the property.

Localization of Categories and Weak Interleavings

To define weak interleavings, we first modify the category of multidimensional filtrations by formally adjoining inverses of levelwise homotopy equivalences. The mathematical tool for this is *localization of categories*. Localization of categories is analogous to the localization of rings and modules in commutative algebra. In analogy with those versions of localization, it is characterized by a simple universal property. Localization of categories is intimately connected to homotopy theory and in particular to closed model categories and the axiomatic homotopy theory of Quillen [37]. We discuss this connection in Section 3.3.1.

We observe that we can define interleavings between filtrations in a localized category of multidimensional filtrations in much the same way that we define strong

³See Section 2.7.2 for the definition of a morphism of filtrations.

interleavings in the ordinary category of multidimensional filtrations. We define weak interleavings to be the interleavings in the localized category, and then define d_{WI} in terms of weak interleavings.

Properties of Weak Interleavings and the Weak Interleaving Distance

Our definitions are such that $d_{WI} \leq d_{SI}$. On the other hand, d_{WI} is sensitive enough that persistent homology functors defined over arbitrary commutative coefficient rings are stable, with respect to the weak interleaving distance on n -filtrations and the interleaving distance on persistence modules. This is the content of our Theorem 3.3.10. As we will see, this theorem is quite useful in passing from inference results on the level of filtrations to inference results on the level of persistent homology.

In this thesis, we do not prove an optimality result for d_{WI} analogous to those which we prove for d_I and d_{SI} . Such a result, if we had it, would offer a fuller picture of the sensitivity properties of d_{WI} . Also, we do not offer a geometrically transparent characterization of d_{WI} analogous either to the algebraic characterization of d_I given by Theorem 2.6.4 or to the geometric characterization of d_{SI} given by Theorem 3.2.7. Thus, in spite of our Theorem 3.3.10 and the close analogy between d_{WI} and both d_{SI} and d_I , for which this thesis presents transparent characterizations, at the conclusion of this work d_{WI} still remains somewhat of a mysterious object. An optimality result for d_{WI} and a geometrically transparent characterization of d_{WI} would do much to improve our understanding of d_{WI} . In Section 3.3.7, we discuss these problems further and offer a conjectural optimality result for d_{WI} .

1.5 Chapter 4: Approximation and Inference Results

In Chapter 4 we present our main inference results for multidimensional persistence. These adapt Theorem 1.2.1 to the multidimensional setting and to the level of filtrations. We present two main inference results, Theorems 4.4.2 and 4.4.4: The first concerns inference using Čech bifiltrations and the second concerns inference using

Rips bifiltrations.

We also present two additional results, one deterministic and one probabilistic, which are similar in spirit to our main results. These are our Theorems 4.2.10 and 4.5.2; we describe them at the end of this section.

1.5.1 Definitions of Filtrations

To state our first main inference result, we need to define a few types of filtrations. All definitions we present here also appear later in the thesis in fuller generality; see, in particular, Sections 2.7 and 3.1.

For $u, v \in (-\infty, \infty]$, a (u, v) -filtration X (of nested type) is a collection of topological spaces $\{X_{(a,b)}\}_{a < u, b < v}$ such that if $(a, b) \leq (a', b')$ (w.r.t. the usual partial ordering on \mathbb{R}^2) then $X_{(a,b)} \subset X_{(a',b')}$. We can define the (multidimensional) persistent homology functor of a (u, v) -filtration for any $(u, v) \in (-\infty, \infty]^2$ —see Section 3.1.2.

For an (∞, ∞) -filtration X and $u, v \in (-\infty, \infty]$, let $R_{(u,v)}(X)$ denote the (u, v) -filtration such that for $(a, b) < (u, v)$, $R_{(u,v)}(X)_{(a,b)} = X_{a,b}$.

Superlevelset-Offset Filtrations

We now introduce sublevelset-offset filtrations and superlevelset-offset filtrations. Informally, whereas the topology of a superlevelset filtration encodes only the height of topographical features of the graph of an \mathbb{R} -valued function, the topology of a superlevelset-offset filtration simultaneously encodes both the height and the width of those topographical features.

Superlevelset-offset filtrations are a natural common extension of superlevelset filtrations and *offset filtrations* [13], two types of 1-D filtrations which are standard objects of study in the computational topology literature.

If (Y, d) is a metric space, $X \subset Y$, and $\gamma : X \rightarrow \mathbb{R}$ is a function, $F^{SO}(X, Y, d, \gamma)$, the **sublevelset-offset filtration of γ (w.r.t the metric d)**, is the (∞, ∞) -filtration for which

$$F^{SO}(X, Y, d, \gamma)_{a,b} = \{y \in Y | d(y, \gamma^{-1}((\infty, a])) \leq b\}.$$

We write $F^{SO}(Y, Y, d, \gamma)$ simply as $F^{SO}(Y, d, \gamma)$.

Informally, $F^{SO}(X, Y, d, \gamma)_{a,b}$ is the thickening of the a -sublevelset of γ by offset parameter b . We call $F^{SO}(X, Y, d, -\gamma)$, the **superlevelset-offset filtration** of γ .

Superleveset-Čech Bifiltrations

Given X, Y , and d as above and $b \in \mathbb{R}$, let $\check{\text{Cech}}(X, Y, d, b)$, the (*closed*) Čech complex of (X, d) with parameter b , be the abstract simplicial complex with vertex set X such that for $l \geq 2$ and $x_1, x_2, \dots, x_l \in X$, $\check{\text{Cech}}(X, Y, d, b)$ contains the $(l-1)$ -simplex $[x_1, \dots, x_l]$ iff there is a point $y \in Y$ such that $d(y, x_i) \leq b$ for $1 \leq i \leq l$.

For any $\gamma : X \rightarrow \mathbb{R}$, let $F^{S\check{C}e}(X, Y, d, \gamma)$, the **sublevelset-Čech filtration** of γ (w.r.t the metric d), be the (∞, ∞) -filtration for which

$$F^{S\check{C}e}(X, Y, d, \gamma)_{a,b} = \check{\text{Cech}}(\gamma^{-1}((\infty, a]), Y, d, b).$$

We call $F^{S\check{C}e}(X, Y, d, -\gamma)$ the **superlevelset-Čech filtration** of γ .

Superlevelset-Rips Bifiltrations

Given a metric space (X, d) and $b \in \mathbb{R}$, let $\text{Rips}(X, d, b)$, the *Rips complex* of (X, d) with parameter b , be the maximal abstract simplicial complex with vertex set X such that for $x_1, x_2 \in X$, the 1-skeleton of $\text{Rips}(X, d, b)$ contains the edge $[x_1, x_2]$ iff $d(x_1, x_2) \leq 2b$.

For any $\gamma : X \rightarrow \mathbb{R}$, let $F^{SR}(X, d, \gamma)$, the **sublevelset-Rips filtration** of γ (w.r.t the metric d), be the (∞, ∞) -filtration for which

$$F^{SR}(X, d, \gamma)_{a,b} = \text{Rips}(\gamma^{-1}((\infty, a]), d, b).$$

We call $F^{S\check{C}e}(X, d, -\gamma)$ the **superlevelset-Rips filtration** of γ .

1.5.2 An Inferential Interpretation of Superlevelset-Čech Bi-filtrations

We now describe our first main inference result. For simplicity, we state the result here only for density functions on Euclidean space; it adapts readily to the case of density functions on Riemannian manifolds.

Let $\gamma : \mathbb{R}^m \rightarrow \mathbb{R}$ be a c -Lipchitz density function for some $c > 0$; for $z \in \mathbb{N}$, let T_z be an i.i.d. sample of size z of a random variable with density γ ; fix $p \in [1, \infty]$, let d^p denote both the L^p -metric on \mathbb{R}^m and (by slight abuse of notation) the restriction of the d^p to any subset of \mathbb{R}^m ; let E be a density estimator on \mathbb{R}^m .

Write $F_z^{SR} = F^{SR}(T_z, d^p, -E(T_z))$, $F_z^{S\check{C}e} = F^{S\check{C}e}(T_z, \mathbb{R}^m, d^p, -E(T_z))$, and $F^{SO} = F^{SO}(\mathbb{R}^m, \mathbb{R}^m, d^p, -\gamma)$). For a kernel K , write $F_K^{SO} = F^{SO}(\mathbb{R}^m, d^p, -\gamma * K)$.

Our first main result, Theorem 4.4.2, is the following.

Theorem.

(i) If (γ, E) satisfies A1 then

$$d_{WI}(R_{(0,\infty)}(F_z^{S\check{C}e}), R_{(0,\infty)}(F^{SO})) \xrightarrow{\mathcal{P}} 0.$$

(ii) If (γ, E) satisfies A2 for a kernel K then

$$d_{WI}(R_{(0,\infty)}(F_z^{S\check{C}e}), R_{(0,\infty)}(F_K^{SO})) \xrightarrow{\mathcal{P}} 0.$$

Note that unlike Theorem 1.2.1, the statement of this result does not involve a sequence $\{\delta_z\}_{z \in \mathbb{Z}}$ of scale parameters.

Let H_i denote the i^{th} persistent homology functor (the definition is given in Section 3.1.2). From our stability result for the weak interleaving distance, Theorem 3.3.10, we immediately obtain the following corollary, which is Corollary 4.4.3.

Corollary.

(i) If (γ, E) satisfies A1 then

$$d_I(H_i(R_{(0,\infty)}(F_z^{S\check{C}e})), H_i(R_{(0,\infty)}(F^{SO}))) \xrightarrow{\mathcal{P}} 0.$$

(ii) If (γ, E) satisfies A2 for a kernel K then

$$d_I(H_i(R_{(0,\infty)}(F_z^{S\check{C}e}))), H_i(R_{(0,\infty)}(F_K^{SO})) \xrightarrow{\mathcal{P}} 0.$$

Theorem 2.6.4, our extrinsic characterization of (J_1, J_2) -interleaved pairs of modules, offers a concrete interpretation of Corollary 4.4.3 as a statement about the similarity between presentations of persistent homology modules.

Theorem 4.4.2 and Corollary 4.4.3 also immediately give us corresponding results about Rips filtrations, in two special cases. First, it is easy to check that for any $T \subset \mathbb{R}^m$, and any scale parameter $a \geq 0$, $\text{Rips}(T, d^\infty, a) = \check{\text{Cech}}(T, \mathbb{R}^m, d^\infty, a)$. Thus

$$F_z^{S\check{C}e}(T_z, \mathbb{R}^m, d^\infty, -E(T_z)) = F_z^{SR}(T_z, d^\infty, -E(T_z))$$

and so when $p = \infty$, we may replace the filtrations $F_z^{S\check{C}e}$ with the filtrations F_z^{SR} everywhere in the statements of the above theorems.

Second, since the superlevelset-Rips filtrations and superlevelset-Čech filtrations always have equal 1-skeletons,

$$H_0(R_{(0,\infty)}(F_z^{S\check{C}e})) = H_0(R_{(0,\infty)}(F_z^{SR}))$$

in the the statement of Corollary 4.4.3, and so the corollary gives us in particular a consistency result for the the 0^{th} persistent homology of superlevelset-Rips filtrations.

1.5.3 An Inferential Interpretation of Superlevelset-Rips Bifiltrations

Our second main inference result, Theorem 4.4.4, is an analogue of our first main inference result for the random superlevelset-Rips bifiltrations F_z^{SR} . The result is formulated directly in terms of weak interleavings, rather than in terms of d_{WI} . Because of this, the precise statement of the result is technical and not readily presented before giving a careful treatment of weak (J_1, J_2) -interleavings. Thus we will describe the result here but defer its statement to Section 4.4.4.

The result quantifies the sense in which F_z^{SR} is topologically similar to F^{SO} in the asymptotic limit as $z \rightarrow \infty$, under the same assumptions as in our first main inference result. Roughly, the result says that in the limit as $z \rightarrow \infty$, F_z^{SR} and F^{SO} satisfy in the weak sense the same interleaving relationship that F_z^{SR} and $F_z^{S\check{C}e}$ satisfy in the strong sense for all z (with probability 1).

The result is not, in any reasonable sense, a consistency result. Indeed, $F_z^{S\check{C}e}$ and F_z^{SR} exhibit topological differences that do not become negligible with high probability as $z \rightarrow \infty$. Thus, given that under the assumptions of our first main inference result, the bifiltrations $F_z^{S\check{C}e}$ are (in a topological sense) consistent estimators of F^{SO} , we do not expect that under the same assumptions the bifiltrations F_z^{SR} would also be consistent estimators of F^{SO} . On the other hand, the topological differences between $F_z^{S\check{C}e}$ and F_z^{SR} are controlled by the well known inclusion relationships between Čech complexes and Rips complexes: Recall that for any metric space (Y, d) , $X \subset Y$ and any scale parameter $r \geq 0$, we have that

$$\check{\text{C}}\text{ech}(X, Y, d, r) \subset \text{Rips}(X, d, r) \subset \check{\text{C}}\text{ech}(X, Y, d, 2r). \quad (1.1)$$

Using the language of weak (J_1, J_2) -interleavings, we can encode the relationship between $F_z^{S\check{C}e}$ and F_z^{SR} induced by these inclusions. Then, via a simple triangle inequality type lemma for interleavings, we can quantify how that relationship, taken together with our first main inference result, controls the topological differences between F_z^{SR} and F^{SO} in the asymptotic limit. Our second main inference result does exactly this.

As in the case of our first main inference result, via the stability theorem 3.3.10 we obtain an analogue of our second main inference result for persistent homology modules, Corollary 4.4.5. Corollary 4.4.5 is formulated in terms of (J_1, J_2) -interleavings of persistence modules. As is the case for Corollary 4.4.3, Theorem 2.6.4 offers a concrete interpretation of Corollary 4.4.5 as a statement about the similarity between presentations of persistent homology modules.

1.5.4 Deterministic Approximation Results

In proving our main inference results, we follow a strategy analogous to that used by Chazal et. al to prove the inference result [11, Theorem 5.1]—namely we first prove deterministic approximation results which assume that the domain of the functions we consider is well sampled, and then use these deterministic results to obtain probabilistic results.

In specific, we first prove Theorem 4.2.3, a bound on the weak interleaving distance between the sublevelset-offset filtration of a function γ and the sublevelset-Čech filtration of an approximation of γ defined on a finite subset of the domain of γ . This bound implies an analogous result, Theorem 4.2.7, for sublevelset-Rips filtrations, formulated in terms of weak (J_1, J_2) -interleavings. These results are the content of Sections 4.2.2 and 4.2.3; they are analogues of [12, Theorem 3.1], and its extension [11, Theorem 4.5], but in the multidimensional setting and for filtrations rather than for persistent homology modules. In fact, our results treat the general situation that γ is an \mathbb{R}^n -valued function, $n \geq 1$, and that only a sublevelset of the domain of γ is well sampled.

The result [11, Theorem 4.5] for ordinary persistence is interesting and useful, independent of its application to statistical inference—see, for example, the application presented in [44]. We imagine that the deterministic bounds we present here may similarly be of independent interest.

Deterministic Approximation of Multidimensional Sublevelset Persistent Homology

Using the interleaving distance on multidimensional persistence modules, it also is possible to adapt the deterministic result [11, Theorem 4.5] to the multidimensional setting in a different—and more straightforward—way than that of Theorems 4.2.3 and 4.2.7: Our Theorem 4.2.10 generalizes the result [11, Theorem 4.5] to \mathbb{R}^n -valued functions, using sublevelset multifiltrations (as defined e.g. in Section 2.7.4), and sublevelset-Rips multifiltrations with a fixed scale parameter. The proof is essentially same as that of [11, Theorem 4.5] in the case of \mathbb{R} -valued functions.

This result does not lift to the level of filtrations, but an analogue of it for sublevelset-Čech filtrations rather than sublevelset-Rips filtrations does in fact lift to the level of filtrations. This is our Theorem 4.2.11.

1.5.5 An Inference Result for 1-D Filtrations

In addition to our main inference results, described above, in Section 4.5 we apply the weak interleaving distance and Theorem 4.2.11 to obtain an inference result, Theorem 4.5.2, for the superlevelset filtration of a probability density function. This is a variant of Theorem 1.2.1, the analogue of the weak law of large numbers for persistent homology of Chazal et al., holding at the level of filtrations. In general, the result holds only for filtered Čech complexes, not for filtered Rips complexes—Theorem 1.2.1, as stated above for filtered Rips complexes, does not lift to the level of filtrations. However, as for our main inference result for Čech bifiltrations, Theorem 4.5.2 can be interpreted as a result about Rips filtrations in the special case that we construct our filtrations using L^∞ metrics on \mathbb{R}^m .

1.6 Why Formulate Theory of Topological Inference Directly at the Level of Filtrations?

To close this introduction, we explain our choice to develop inferential theory directly at the level of filtrations rather than only at the level of persistent homology modules.

Our aim is to understand as deeply as possible the connection between the topological structure of discrete filtrations built on randomly sampled point cloud data and that of filtrations built directly from the probability distribution generating the random data; as we'll now argue, to do this, it is more natural to develop the inferential theory (and, in particular, the requisite notions of similarity of filtrations) directly at the level of filtrations than at the homology level.

We note first the trivial fact that singular homology induces an equivalence relation on topological spaces—namely, we can say that two topological spaces X and

X and Y are *homology equivalent*⁴ if $H_i(X)$ and $H_i(Y)$ are isomorphic for all i . However, such a notion of equivalence between topological spaces is not often adopted as a fundamental object of interest in algebraic topology. The reason is that there are stronger, more easily defined, and more geometrically transparent notions of equivalence of topological spaces, such as those of homotopy equivalence and weak homotopy equivalence, which exhibit the kinds of invariance properties one wants a notion of equivalence on spaces to have in algebraic topology and homotopy theory.⁵ Thus homotopy equivalence and weak homotopy equivalence are generally regarded as the fundamental notions of equivalence of topological spaces in algebraic topology, and homology serves primarily as a computational tool to understand spaces up to these notions of equivalence; homology is usually not taken as means of defining a notion of equivalence of topological spaces in of itself.

Now, for filtrations the analogue of homology is persistent homology, and we can define a notion of persistent homology equivalence of filtrations in a way analogous to the way in which we just defined homology equivalence of topological spaces. In fact, for ordinary persistence the bottleneck distance d_B on persistence modules affords us an *approximate* notion of persistent homology equivalence between two 1-D filtrations: we can interpret the statement that

$$\forall i \in \mathbb{Z}_{\geq 0}, d_B(H_i(X), H_i(Y)) \leq \epsilon$$

for two filtrations X and Y as saying that X and Y are *approximately persistent homology equivalent*, up to an error of ϵ . More generally, the interleaving distance studied in this thesis affords us in the same way an approximate notion of persistent homology equivalence between two multidimensional filtrations.

Given that for topological spaces it turns out to be most natural to regard homotopy equivalence, rather than homology equivalence, as the fundamental notion of

⁴Note: This is not standard terminology; we are introducing it here only in the service of the present discussion and only for use in this subsection.

⁵When we say that homotopy equivalence is a geometrically transparent notion of equivalence of topological spaces we are alluding specifically to the theorem which says that two spaces are homotopy equivalent if and only if they are each deformation retracts of some third space [30, Corollary 0.21].

equivalence between topological spaces, by analogy it is reasonable for us to ask for a notion of *approximate homotopy equivalence* of filtrations which has good invariance properties and which enjoys the same advantages over approximate homology equivalence that homotopy equivalence enjoys over homology equivalence for topological spaces. Further, if we were to have such a notion, then in light of its good properties and by analogy to classical algebraic topology, it would be natural for us to formulate persistence theory and in particular theory of topological inference directly at the level of filtrations in terms of that notion.

The weak interleaving distance affords us precisely such a well behaved notion of approximate homotopy equivalence of multidimensional filtrations, except that, as we noted in Section 1.4.2, it is not yet clear to what extent the weak interleaving distance is in fact geometrically transparent.

Thus, the present unavailability of a geometric characterization of the weak interleaving distance aside, it is natural to formulate our main inference results directly at the level of filtrations, in terms of the weak interleaving distance and the closely related notion of weak interleavings.

Chapter 2

Interleavings on Multidimensional Persistence Modules

In this chapter we introduce and study interleavings and interleaving distance on multidimensional persistence modules. See Section 1.3 for an overview of the chapter.

2.1 Algebraic Preliminaries

In this section we define persistence modules and review some (primarily) algebraic facts and definitions which we will need throughout the thesis.

2.1.1 First Definitions and Notation

Basic Notation

Let k be a field. Let \mathbb{N} denote the natural numbers and let $\mathbb{Z}_{\geq 0}$ denote the non-negative integers. Let $\hat{\mathbb{R}} = (-\infty, \infty]$.

We view \mathbb{R}^n as a partially ordered set, with $(a_1, \dots, a_n) \leq (b_1, \dots, b_n)$ iff $a_i \leq b_i$ for all i . Let \mathbf{e}_i denote the i^{th} standard basis vector in \mathbb{R}^n .

For $A \subset \mathbb{R}$ any subset, let $\bar{A} = A \cup \{-\infty, \infty\}$.

For $a = (a_1, \dots, a_n) \in \hat{\mathbb{R}}^n$ and $b \in \hat{\mathbb{R}}$, let $(a, b) = (a_1, \dots, a_n, b) \in \hat{\mathbb{R}}^{n+1}$.

For $f : X \rightarrow \mathbb{R}^n$, $a \in \mathbb{R}^n$, let $f_a = \{x \in X | f(x) \leq a\}$. We call f_a the *a-sublevelset of f* and we call $(-f)_a$ the *a-superlevelset of f*.

For $\epsilon \in \hat{\mathbb{R}}$, let $\vec{\epsilon}_n \in \mathbb{R}^n$ denote the vector whose components are each ϵ . We'll also often write $\vec{\epsilon}_n$ simply as $\vec{\epsilon}$ when n is understood. As a notational convenience, for $u \in \mathbb{R}^n$ and $\epsilon \in \mathbb{R}$, let $u + \epsilon$ denote $u + \vec{\epsilon}$.

Notation Related to Categories

For a category C , let $\text{obj}(C)$ denote the objects of C and let $\text{obj}^*(C)$ denote the isomorphism classes of objects of C . Let $\text{hom}(C)$ denote the morphisms in C , and for $X, Y \in \text{obj}(C)$ let $\text{hom}_C(X, Y)$ denote the morphisms from X to Y . When C is understood, we'll often write $\text{hom}_C(X, Y)$ simply as $\text{hom}(X, Y)$.

Metrics and Pseudometrics

Recall that a *pseudometric* on a set X is a function $d : X \times X \rightarrow [0, \infty]$ with the following three properties:

1. $d(x, x) = 0$ for all $x \in X$.
2. $d(x, y) = d(y, x)$ for all $x, y \in X$.
3. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We'll often use the term *distance* in this thesis as a synonym for pseudometric.

A *metric* is a pseudometric d with the additional property that $d(x, y) \neq 0$ whenever $x \neq y$.

Metrics on Categories

In this thesis we'll often have the occasion to define a pseudometric on $\text{obj}^*(C)$, for C some category. For d such a pseudometric, $M, N \in \text{obj}(C)$, and $[M], [N]$ the isomorphism classes of M and N , we'll always write $d(M, N)$ as shorthand for $d([M], [N])$.

2.1.2 Commutative Monoids and Monoid Rings

Monoid rings are generalizations of polynomial rings.

A *commutative monoid* is a pair $(G, +_G)$, where G is a set and $+_G$ is an associative, commutative binary operation on G with an identity element. Abelian groups are by definition commutative monoids with the additional property that each element has an inverse. We'll often denote the monoid $(G, +_G)$ simply as G . A submonoid of a monoid is defined in the obvious way, as is an isomorphism between two monoids.

Given a set S , let $k[S]$ denote the vector space of formal linear combinations of elements of S . If $\bar{G} = (G, +_G)$ is a monoid, then the operation $+_G$ induces a ring structure on $k[G]$, where multiplication is characterized by the property $(k_1 g_1)(k_2 g_2) = k_1 k_2 (g_1 +_G g_2)$ for $k_1, k_2 \in k$, $g_1, g_2 \in G$. We call the resulting ring the *monoid ring* generated by \bar{G} , and we denote it $k[\bar{G}]$.

Let \mathbf{A}_n denote $k[x_1, \dots, x_n]$, the polynomial ring in n variables with coefficients in k . For $n > 0$, $\mathbb{Z}_{\geq 0}^n$ is a monoid under the usual addition of vectors. It's easy to see that $k[\mathbb{Z}_{\geq 0}^n]$ is isomorphic to \mathbf{A}_n .

Similarly, $\mathbb{R}_{\geq 0}^n$ is a monoid under the usual addition of vectors. Let B_n denote the monoid ring $k[\mathbb{R}_{\geq 0}^n]$. We may think of B_n as an analogue of the usual polynomial ring in n -variables where exponents of the indeterminates are allowed to take on arbitrary non-negative real values rather than only non-negative integer values. With this interpretation in mind, we'll often write (r_1, \dots, r_n) as $x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n}$, for $(r_1, \dots, r_n) \in \mathbb{R}_{\geq 0}^n$.

2.1.3 Multidimensional Persistence Modules

We first review the definition of a multidimensional persistence module given in [7]. We then define analogues of these over the ring B_n .

In what follows, we'll often refer to multidimensional persistence modules simply as “persistence modules.”

A_n-Persistence Modules

Fix $n \in \mathbb{N}$. An **A_n-persistence module** is an \mathbf{A}_n -module \mathbf{M} with a direct sum decomposition as a k -vector space $\mathbf{M} \cong \bigoplus_{a \in \mathbb{Z}^n} \mathbf{M}_a$ such that the action of \mathbf{A}_n on \mathbf{M} satisfies $x_i(\mathbf{M}_a) \subset \mathbf{M}_{a+\mathbf{e}_i}$ for all $a \in \mathbb{Z}^n$. In other words, an \mathbf{A}_n -persistence module is simply an \mathbf{A}_n -module endowed with an n -graded structure.

For \mathbf{M} and \mathbf{N} \mathbf{A}_n -persistence modules, we define $\text{hom}(\mathbf{M}, \mathbf{N})$ to be the set of module homomorphisms $f : \mathbf{M} \rightarrow \mathbf{N}$ such that $f(\mathbf{M}_a) \subset \mathbf{N}_a$ for all $a \in \mathbb{Z}^n$. This defines a category whose objects are the \mathbf{A}_n -persistence modules. Let $\mathbf{A}_n\text{-mod}$ denote this category.

B_n-persistence modules

In close analogy with the definition of an n -graded \mathbf{A}_n -module, we define a **B_n-persistence module** to be a B_n -module M with a direct sum decomposition as a k -vector space $M \cong \bigoplus_{a \in \mathbb{R}^n} M_a$ such that the action of B_n on M satisfies $x_i^\alpha(M_a) \subset M_{a+\alpha\mathbf{e}_i}$ for all $a \in \mathbb{R}^n$, $\alpha \geq 0$.

For M and N B_n -persistence modules, we define $\text{hom}(M, N)$ to consist of module homomorphisms $f : M \rightarrow N$ such that $f(M_a) \subset N_a$ for all $a \in \mathbb{R}^n$. This defines a category whose objects are the B_n -persistence modules. Let $B_n\text{-mod}$ denote this category.

Our notational convention will be to use boldface to denote \mathbf{A}_n -persistence modules and italics to denote B_n -persistence modules. We'll often refer to \mathbf{A}_1 -persistence modules and B_1 -persistence modules as *ordinary* persistence modules.

On the Relationship Between A_n-persistence Modules and B_n-persistence Modules

Since \mathbf{A}_n is a subring of B_n , we can view B_n as an \mathbf{A}_n -module. If \mathbf{M} is an \mathbf{A}_n -persistence module then $\mathbf{M} \otimes_{\mathbf{A}_n} B_n$ is a B_n -module. Further, $\mathbf{M} \otimes_{\mathbf{A}_n} B_n$ inherits an n -grading from those on B_n and \mathbf{M} which gives $\mathbf{M} \otimes_{\mathbf{A}_n} B_n$ the structure of a B_n -persistence module.

In fact, $(\cdot) \otimes_{\mathbf{A}_n} B_n$ defines a functor from $\mathbf{A}_n\text{-mod}$ to $B_n\text{-mod}$. It can be checked

that this functor is fully faithful and descends to an injection on isomorphism classes of objects. Thus the functor induces an identification of $\mathbf{A}_n\text{-mod}$ with a subcategory of $B_n\text{-mod}$.

In light of this, we can think of B_n -persistence modules as generalizations of \mathbf{A}_n -persistence modules. Finitely presented B_n -persistence modules arise naturally in applications, as discussed in Section 2.7.4. In a sense that can be made precise using machinery mentioned in Remark 2.3.3, it is possible to view them as \mathbf{A}_n -persistence modules endowed with some additional data. However, this is awkward from the standpoint of constructing pseudometrics between B_n -persistence modules. We thus regard B_n -persistence modules as the fundamental objects of interest here, and use \mathbf{A}_n -persistence modules in this thesis only in the case $n = 1$ to translate results about \mathbf{A}_1 -persistence modules into analogous results about B_1 -persistence modules.

In the remainder of Section 2.1.3, we present some basic definitions related to B_n -persistence modules. All of these definitions have obvious analogues for \mathbf{A}_n -persistence modules; we'll use these analogues where needed without further comment.

Homogeneity

Let M be a B_n -persistence module. For $u \in \mathbb{R}^n$, we say that M_u is a *homogeneous summand* of M . We refer to an element $v \in M_u$ as a *homogeneous element* of grade u , and write $gr(v) = u$. A *homogeneous submodule* of a B_n -persistence module is a submodule generated by a set of homogeneous elements. The quotient of a B_n -persistence module M by a homogeneous submodule of M is itself a B_n -persistence module; the n -graded structure on the quotient is induced by that of M .

Tameness

Following [8] we'll call a B_n -persistence module *tame* if each homogeneous summand of the module is finite dimensional. Note that this is a more general notion of tameness than that which appears in the original paper on the stability of persistence [14].

Transition Maps

For M a B_n -persistence module and any $u \leq v \in \mathbb{R}^n$, the restriction to M_u of the action on M of the monomial $x_1^{v_1-u_1}x_2^{v_2-u_2}\cdots x_n^{v_n-u_n}$ defines a linear map with codomain M_v , which we call a *transition map*. Denote this map by $\varphi_M(u, v)$.

2.1.4 Shift Functors and Transition Morphisms

Shift Functors and Related Notation

For $v \in \mathbb{R}^n$, we define the **shift functor** $(\cdot)(v)$ as follows: For M a B_n -persistence module we let $M(v)$ be the B_n -persistence module such that for all $a \in \mathbb{R}^n$, $M(v)_a = M_{a+v}$. For $a \leq b \in \mathbb{R}^n$, we take $\varphi_{M(v)}(a, b) = \varphi_M(a+v, b+v)$. For $f \in \text{hom}(B_n\text{-mod})$ we let $f(v)_a = f_{a+v}$.

For $v \in \mathbb{R}^n$ and $f \in \text{hom}(B_n\text{-mod})$, we'll sometimes abuse notation and write $f(v)$ simply as f .

For $\epsilon \in \mathbb{R}$, let $M(\epsilon)$ denote $M(\vec{\epsilon})$. More generally, for any subset $Q \subset M$, let $Q(\epsilon) \subset M(\epsilon)$ denote the image of Q under the bijection between M and $M(\epsilon)$ induced by the identification of each summand $M(\epsilon)_u$ with $M_{u+\epsilon}$.

Transition Homomorphisms

For a B_n -persistence module M and $\epsilon \in \mathbb{R}_{\geq 0}$ let $S(M, \epsilon) : M \rightarrow M(\epsilon)$, the **(diagonal) ϵ -transition homomorphism**, be the homomorphism whose restriction to M_u is the linear map $\varphi_M(u, u + \epsilon)$ for all $u \in \mathbb{R}^n$.

2.1.5 ϵ -interleavings and the Interleaving Distance

For $\epsilon \geq 0$, we say that two B_n -persistence modules M and N are **ϵ -interleaved** if there exist homomorphisms $f : M \rightarrow N(\epsilon)$ and $g : N \rightarrow M(\epsilon)$ such that

$$\begin{aligned} g(\epsilon) \circ f &= S(M, 2\epsilon) \text{ and} \\ f(\epsilon) \circ g &= S(N, 2\epsilon); \end{aligned}$$

we refer to such f and g as ϵ -**interleaving** homomorphisms.

The definition of ϵ -interleaving homomorphisms was introduced for B_1 -persistence modules in [8].

Remark 2.1.1. It's easy to see that if $0 \leq \epsilon_1 \leq \epsilon_2$ and M and N are ϵ_1 -interleaved, then M and N are ϵ_2 -interleaved.

In Section 2.6.1 we will observe that the definition of ϵ -interleavings generalizes considerably, but for now we will not concern ourselves with the generalized form of the definition.

The Interleaving Distance on B_n -persistence modules

We define $d_I : \text{obj}^*(B_n\text{-mod}) \times \text{obj}^*(B_n\text{-mod}) \rightarrow [0, \infty]$, the **interleaving distance**, by taking

$$d_I(M, N) = \inf\{\epsilon \in \mathbb{R}_{\geq 0} \mid M \text{ and } N \text{ are } \epsilon\text{-interleaved}\}.$$

Note that d_I is pseudometric. However, the following example shows that d_I is not a metric.

Example 2.1.2. Let M be the B_1 -persistence module with $M_0 = k$ and $M_a = 0$ if $a \neq 0$. Let N be the trivial B_1 -persistence module. Then M and N are not isomorphic, and so are not 0-interleaved, but it is easy to check that M and N are ϵ -interleaved for any $\epsilon > 0$. Thus $d_I(M, N) = 0$.

2.1.6 Free B_n -persistence Modules, Presentations, and Related Algebraic Basics

In our study of B_n -persistence modules, we will make substantial use of free B_n -persistence modules and presentations of B_n -persistence modules; we define these objects here and present some basic results about them. We begin with some foundational definitions.

n-graded Sets

Define an *n*-graded set to be a pair $G = (\bar{G}, \iota_G)$ where \bar{G} is a set and $\iota_G : G \rightarrow \mathbb{R}^n$ is any function. When ι_G is clear from context, as it will usually be, we'll write $\iota_G(y)$ as $gr(y)$ for $y \in \bar{G}$. We'll sometimes abuse notation and write G to mean the set \bar{G} when no confusion is likely. The union of disjoint graded sets is defined in the obvious way. For $\epsilon \geq 0$ and $G = (\bar{G}, \iota_G)$ an *n*-graded set, let $G(\epsilon)$ be the *n*-graded set (\bar{G}, ι'_G) , where $\iota'_G(y) = \iota_G(y) - \epsilon$.

For G an *n*-graded set, define $gr(G) : \mathbb{R}^n \rightarrow \mathbb{Z}_{\geq 0} \cup \{\infty\}$ by taking $gr(G)(u)$ to be the number of elements $y \in G$ such that $gr(y) = u$. Note that for any B_n -persistence module M , a set $Y \subset M$ of homogeneous elements inherits the structure of an *n*-graded set from the graded structure on M , so that $gr(Y)$ is well defined.

Free B_n -persistence modules

The usual notion of a free module extends to the setting of B_n -persistence modules as follows: For G an *n*-graded set, let $\langle G \rangle = \bigoplus_{y \in \bar{G}} B_n(-gr(y))$. A *free B_n -persistence module* F is a B_n -persistence module such that for some *n*-graded set G , $F \cong \langle G \rangle$.

Equivalently, we can define a free B_n -persistence module as a B_n -persistence module which satisfies a certain universal property. Free A_n -persistence modules are defined via a universal property e.g. in [7, Section 4.2]. The definition for B_n -persistence modules is analogous; we refer the reader to [7] for details.

A *basis* for a free module F is a minimal set of homogeneous generators for F . For G any graded set, identifying $y \in G$ with the copy of $1(-gr(y))$ in the summand $B_n(-gr(y))$ of $\langle G \rangle$ corresponding to y gives an identification of G with a basis for $\langle G \rangle$. It can be checked that if B and B' are two bases for a free B_n -persistence module F then $gr(B) = gr(B')$. Clearly then, $gr(B)$ of an arbitrarily chosen basis B for F is an isomorphism invariant of F and determines F up to isomorphism.

For R a homogeneous subset of a free B_n -persistence module F , $\langle R \rangle$ will always denote the submodule of F generated by R . Since, as noted above, R can be viewed as an *n*-graded set, we emphasize that for such R , $\langle R \rangle$ **does not** denote $\bigoplus_{y \in R} B_n(-gr(y))$.

Free Covers and Lifts

For M a B_n -persistence module, define a *free cover* of M be a pair (F_M, ρ_M) , where F_M is a free B_n -persistence module and $\rho_M : F_M \rightarrow M$ a surjective morphism of B_n -persistence modules.

For M, N B_n -persistence modules, (F_M, ρ_M) and (F_N, ρ_N) free covers of M and N , and $f : M \rightarrow N$ a morphism, define a *lift* of f to be a morphism $\tilde{f} : F_M \rightarrow F_N$ such that the following diagram commutes.

$$\begin{array}{ccc} F_M & \xrightarrow{\tilde{f}} & F_N \\ \downarrow \rho_M & & \downarrow \rho_N \\ M & \xrightarrow{f} & N \end{array}$$

Lemma 2.1.3 (Existence and Uniqueness up to Homotopy of Lifts). *For any B_n -persistence modules M and N , free covers $(F_M, \rho_M), (F_N, \rho_N)$ of M, N , and a morphism $f : M \rightarrow N$, there exists a lift $\tilde{f} : F_M \rightarrow F_N$ of f . If $\tilde{f}' : F_M \rightarrow F_N$ is another lift of f , then $\text{im}(\tilde{f} - \tilde{f}') \subset \ker(\rho_N)$.*

Proof. This is just a specialization of the standard result on the existence and homotopy uniqueness of free resolutions [26, Eisenbud A3.13] to the 0th modules in free resolutions for M and N . The proof is straightforward. \square

Presentations of B_n -persistence Modules

A **presentation** of a B_n -persistence module M is a pair (G, R) where G is an n -graded set and $R \subset \langle G \rangle$ is a set of homogeneous elements such that $M \cong \langle G \rangle / \langle R \rangle$. We denote the presentation (G, R) as $\langle G | R \rangle$. For n -graded sets G_1, \dots, G_l and sets $R_1, \dots, R_m \subset \langle G_1 \cup \dots \cup G_l \rangle$, we'll let $\langle G_1, \dots, G_l | R_1, \dots, R_m \rangle$ denote $\langle G_1 \cup \dots \cup G_l | R_1 \cup \dots \cup R_m \rangle$.

If M is a B_n -persistence module such that there exists a presentation $\langle G | R \rangle$ for M with G and R finite, then we say M is *finitely presented*.

Minimal Presentations of B_n -persistence Modules

Let M be a B_n -persistence module. Define a presentation $\langle G|R \rangle$ of M to be *minimal* if

1. the quotient $\langle G \rangle \rightarrow \langle G \rangle / \langle R \rangle$ maps G to a minimal set of generators for $\langle G \rangle / \langle R \rangle$.
2. R is a minimal set of generators for $\langle R \rangle$.

It's clear that a minimal presentation for M exists.

Theorem 2.1.4. *If M is a finitely presented B_n -persistence module and $\langle G|R \rangle$ is a minimal presentation of M , then for any other presentation $\langle G'|R' \rangle$ of M , $gr(G) \leq gr(G')$ and $gr(R) \leq gr(R')$.*

Note that the theorem implies in particular that if $\langle G|R \rangle$ and $\langle G'|R' \rangle$ are two minimal presentations of M then $gr(G) = gr(G')$ and $gr(R) = gr(R')$.

We defer the proof of the theorem to Appendix A.2. The proof is an adaptation to our setting of a standard result [26, Theorem 20.2] about free resolutions of modules over a local ring. The main effort required in carrying out the adaptation is to prove that the ring B_n has a property known as *coherence*; we define coherence and prove that B_n is coherent in Appendix A.1.

2.2 Algebraic Preliminaries for 1-D Persistence

In this section, we review algebraic preliminaries and establish notation specific to 1-D persistent homology. This material will be used in Sections 2.3 and 2.4 to develop the machinery needed to prove Theorem 2.4.2.

Basic Notation

For S any subset of $\bar{\mathbb{R}}^2$, let $S_+ = \{(a, b) \in S | a < b\}$. For S a set and $f : S \rightarrow \mathbb{R}$ a function, let $\text{supp}(f) = \{s \in S | f(s) \neq 0\}$.

2.2.1 Structure Theorems For Tame \mathbf{A}_1 -Persistence Modules

The structure theorem for finitely generated \mathbf{A}_1 persistence modules [47] is well known in the applied topology community. In fact, this theorem generalizes to tame \mathbf{A}_1 -modules. The existence portion of the generalized theorem is given e.g. in [46]; the uniqueness is not mentioned there but is very easy to show; we do so below. To our knowledge, this generalization has not previously been discussed in the computational topology literature. We will use the more general theorem to show that the bottleneck distance is equal to the interleaving distance for ordinary persistence.

Before stating the results, we establish some notation. For $a < b \in \mathbb{Z}$, Let $\mathbf{C}(a, b)$ denote the module $(k[x]/(x^{b-a}))(-a)$. Let $\mathbf{C}(a, \infty) = k[x](-a)$. Note that for fixed b (possibly infinite), the set of modules $\{\mathbf{C}(a, b)\}_{a \in (-\infty, b)}$ has a natural directed system structure; let $\mathbf{C}(-\infty, b)$ denote the colimit of this directed system.

For M a module and $m \in \mathbb{Z}_{\geq 0}$, let M^m denote the direct sum of m copies of M .

Theorem 2.2.1 (Structure Theorem for finitely generated \mathbf{A}_1 -persistence modules [47]). *Let \mathbf{M} be a finitely generated \mathbf{A}_1 -module. Then there is a unique function $\mathcal{D}_{\mathbf{M}} : (\mathbb{Z} \times \bar{\mathbb{Z}})_+ \rightarrow \mathbb{Z}_{\geq 0}$ with finite support such that*

$$\mathbf{M} \cong \bigoplus_{(a,b) \in \text{supp}(\mathcal{D}_{\mathbf{M}})} \mathbf{C}(a, b)^{\mathcal{D}_{\mathbf{M}}(a,b)}.$$

Theorem 2.2.2 (Structure Theorem for tame \mathbf{A}_1 -persistence modules [46]). *Let \mathbf{M} be a tame \mathbf{A}_1 -module. Then there is a unique function $\mathcal{D}_{\mathbf{M}} : \bar{\mathbb{Z}}_+^2 \rightarrow \mathbb{Z}_{\geq 0}$ such that*

$$\mathbf{M} \cong \bigoplus_{(a,b) \in \text{supp}(\mathcal{D}_{\mathbf{M}})} \mathbf{C}(a, b)^{\mathcal{D}_{\mathbf{M}}(a,b)}.$$

The uniqueness part of Theorem 2.2.2 is an immediate consequence of the following lemma, upon noting that the right hand sides of the equations in the statement of the lemma do not depend on $\mathcal{D}_{\mathbf{M}}$.

Lemma 2.2.3. *Let \mathbf{M} be a tame \mathbf{A}_1 -module, and let $\mathcal{D}_{\mathbf{M}} : \bar{\mathbb{Z}}_+^2 \rightarrow \mathbb{Z}_{\geq 0}$ be a function such that $\mathbf{M} \cong \bigoplus_{(a,b) \in \text{supp}(\mathcal{D}_{\mathbf{M}})} \mathbf{C}(a, b)^{\mathcal{D}_{\mathbf{M}}(a,b)}$. Then*

(i) For $(a, b) \in \mathbb{Z}_+^2$,

$$\begin{aligned}\mathcal{D}_{\mathbf{M}}(a, b) &= \text{rank}(\varphi_{\mathbf{M}}(a, b - 1)) - \text{rank}(\varphi_{\mathbf{M}}(a, b)) \\ &\quad - \text{rank}(\varphi_{\mathbf{M}}(a - 1, b - 1)) + \text{rank}(\varphi_{\mathbf{M}}(a - 1, b)).\end{aligned}$$

(ii) For $b \in \mathbb{Z}$, $\mathcal{D}_{\mathbf{M}}(-\infty, b) = \lim_{a \rightarrow -\infty} \text{rank}(\varphi_{\mathbf{M}}(a, b - 1)) - \lim_{a \rightarrow -\infty} \text{rank}(\varphi_{\mathbf{M}}(a, b))$.

(iii) For $a \in \mathbb{Z}$, $\mathcal{D}_{\mathbf{M}}(a, \infty) = \lim_{b \rightarrow \infty} \text{rank}(\varphi_{\mathbf{M}}(a, b)) - \lim_{b \rightarrow \infty} \text{rank}(\varphi_{\mathbf{M}}(a - 1, b))$.

(iv) $\mathcal{D}_{\mathbf{M}}(-\infty, \infty) = \lim_{a \rightarrow -\infty} \lim_{b \rightarrow \infty} \text{rank}(\varphi_{\mathbf{M}}(a, b))$.

Proof. This is trivial. □

We call $\mathcal{D}_{\mathbf{M}}$ the **(discrete) persistence diagram** of \mathbf{M} .

In Section 2.3, we prove a structure theorem analogous to Theorem 2.2.2 for a subset of the tame B_1 -persistence modules which contains the finitely presented B_1 -persistence modules. We do not address the problem of generalizing this structure theorem to the full set of tame B_1 -persistence modules, but to echo a sentiment expressed in [8], it would be nice to have such a result.

2.2.2 Discrete Persistence Modules.

In order to define persistence diagrams of B_1 -persistence modules, we need a mild generalization of \mathbf{A}_1 -persistence modules.

Let $S \subset \mathbb{R}$ be a countably infinite set with no accumulation point. The authors of [8] define a **discrete persistence module** M_S to be a collection of vector spaces $\{M_s\}_{s \in S}$ indexed by S together with linear maps $\{\varphi_{M_S}(s_1, s_2)\}_{s_1 \leq s_2 \in S}$.

Define a **grid function** $t : \mathbb{Z} \rightarrow \mathbb{R}$ to be a strictly increasing function with no accumulation point.

Remark 2.2.4. Discrete persistence modules are of course closely related to \mathbf{A}_1 -persistence modules. A countably infinite subset of $S \subset \mathbb{R}$ with no accumulation point can be indexed by a grid function t with image S , and such a grid function is uniquely determined by the value of $t(0)$. Thus, pairs (M_S, s) , where M_S is a discrete persistence module and s is an element of S , are equivalent to pairs (\mathbf{M}', t) , where \mathbf{M}'

is an \mathbf{A}_1 -persistence module and t is a grid function; there is an equivalence sending each pair (M_S, s) to the pair (\mathbf{M}', t) , where t is a grid function with $\text{im}(t) = S$, $t(0) = s$, and \mathbf{M}' is the \mathbf{A}_1 -persistence module such that for $z \in \mathbb{Z}$, $\mathbf{M}'_z = M_{t(z)}$ and $\varphi_{\mathbf{M}'}(z_1, z_2) = \varphi_{M_S}(t(z_1), t(z_2))$.

As a matter of expository convenience, from now on we'll define discrete modules to be pairs (\mathbf{M}, t) where \mathbf{M} is an \mathbf{A}_1 -persistence module and t is a grid function. This in effect means we are carrying around the extra data of an element of S in our discrete persistent modules relative to those defined in [8], but this won't present a problem—in particular, the definition of the persistence diagram of a discrete persistence module that we present below is independent of the choice of this element, and is equivalent to that of [8].

2.2.3 Persistence Diagrams and the Bottleneck Distance

The definition of a persistence diagram that we present here differs in some cosmetic respects from that in [8]. Our choice in this regard is a matter of notational convenience; the reader may check that our definition of the bottleneck distance between tame B_1 -persistence modules is equivalent to that of [8].

For a grid function t , define $\bar{t} : \bar{\mathbb{Z}} \rightarrow \bar{\mathbb{R}}$ as

$$\bar{t}(z) = \begin{cases} t(z) & \text{if } z \in \mathbb{Z}, \\ -\infty & \text{if } z = -\infty, \\ \infty & \text{if } z = \infty. \end{cases}$$

Let $\bar{t} \times \bar{t} : \bar{\mathbb{Z}}_+^2 \rightarrow \bar{\mathbb{R}}_+^2$ be defined by $\bar{t} \times \bar{t}(a, b) = (\bar{t}(a), \bar{t}(b))$.

We define a **persistence diagram** to be a function $\mathcal{D} : \bar{\mathbb{R}}_+^2 \rightarrow \mathbb{Z}_{\geq 0}$.

For (\mathbf{M}, t) a discrete persistence module, define $\mathcal{D}_{(\mathbf{M}, t)}$, the **persistence diagram of** (\mathbf{M}, t) , to be the persistence diagram for which $\text{supp}(\mathcal{D}_{(\mathbf{M}, t)}) = \bar{t} \times \bar{t}(\text{supp}(\mathcal{D}_{\mathbf{M}}))$ and so that $\mathcal{D}_{(\mathbf{M}, t)}(\bar{t}(a), \bar{t}(b)) = \mathcal{D}_{\mathbf{M}}(a, b)$ for all $(a, b) \in \bar{\mathbb{Z}}_+^2$.

Bottleneck Metric

For $x \in \mathbb{R}$, define $x + \infty = \infty$ and $x - \infty = -\infty$. Then the usual definition of l^∞ norm on the plane extends to $\bar{\mathbb{R}}^2$; we denote it by $\|\cdot\|_\infty$.

Now define a **multibijection** between two persistence diagrams $\mathcal{D}_1, \mathcal{D}_2$ to be a function $\gamma : \text{supp}(\mathcal{D}_1) \times \text{supp}(\mathcal{D}_2) \rightarrow \mathbb{Z}_{\geq 0}$ such that

1. For each $x \in \text{supp}(\mathcal{D}_1)$, the set $\{y \in \text{supp}(\mathcal{D}_2) | (x, y) \in \text{supp}(\gamma)\}$ is finite and

$$\mathcal{D}_1(x) = \sum_{y \in \text{supp}(\mathcal{D}_2)} \gamma(x, y),$$

2. For each $y \in \text{supp}(\mathcal{D}_2)$, the set $\{x \in \text{supp}(\mathcal{D}_1) | (x, y) \in \text{supp}(\gamma)\}$ is finite and

$$\mathcal{D}_2(y) = \sum_{x \in \text{supp}(\mathcal{D}_1)} \gamma(x, y).$$

For persistence diagrams $\mathcal{D}_1, \mathcal{D}_2$, let $\mathcal{L}(\mathcal{D}_1, \mathcal{D}_2)$ denote the triples $(\mathcal{D}'_1, \mathcal{D}'_2, \gamma)$, where \mathcal{D}'_1 , and \mathcal{D}'_2 are persistence diagrams with $\mathcal{D}'_1 \leq \mathcal{D}_1, \mathcal{D}'_2 \leq \mathcal{D}_2$, and γ is a multibijection between \mathcal{D}'_1 and \mathcal{D}'_2 .

We define the **bottleneck metric** d_B between two persistence diagrams $\mathcal{D}_1, \mathcal{D}_2$ as

$$d_B(\mathcal{D}_1, \mathcal{D}_2) = \inf_{\substack{(\mathcal{D}'_1, \mathcal{D}'_2, \gamma) \\ \in \mathcal{L}(\mathcal{D}_1, \mathcal{D}_2)}} \max \left(\sup_{\substack{(a, b) \in \text{supp}(\mathcal{D}_1 - \mathcal{D}'_1) \\ \cup \text{supp}(\mathcal{D}_2 - \mathcal{D}'_2)}} \frac{1}{2}(b - a), \sup_{(x, y) \in \text{supp}(\gamma)} \|y - x\|_\infty \right).$$

Discretizations of B_1 -modules

Let t be a grid function. For M a B_1 -persistence module, we define the t -discretization of M to be the discrete persistence module $(\mathbf{P}_t(M), t)$ with $\mathbf{P}_t(M)$ defined as follows:

1. For $z \in \mathbb{Z}$, $\mathbf{P}_t(M)_z = M_{t(z)}$; let $\mathcal{I}_{M, t, z} : \mathbf{P}_t(M)_z \rightarrow M_{t(z)}$ denote this identification.
2. For $y, z \in \mathbb{Z}, y \leq z$, $\varphi_{\mathbf{P}_t(M)}(y, z) = \mathcal{I}_{M, t, z}^{-1} \circ \varphi_M(t(y), t(z)) \circ \mathcal{I}_{M, t, y}$.

Persistence diagrams of B_1 -persistence modules

We'll say a grid function t is an ϵ -cover if for any $a \in \mathbb{R}$, there exists $b \in \text{im}(t)$ with $|a - b| \leq \epsilon$. Now fix $\alpha \in \mathbb{R}$ and let $\{t_i\}_{i=1}^\infty$ be a sequence of grid functions with t_i a $1/2^i$ -cover.

It is asserted in [8] that for any tame B_1 -persistence module M the persistence diagrams $\mathcal{D}_{(\mathbf{P}_{t_i}(M), t_i)}$ converge in the bottleneck metric to a limiting persistence diagram \mathcal{D}_M and that \mathcal{D}_M is independent of the choice of the sequence $\{t_i\}$. We call \mathcal{D}_M the persistence diagram of M . For M and N tame B_1 -persistence modules, we define $d_B(M, N) = d_B(\mathcal{D}_M, \mathcal{D}_N)$.

Remark 2.2.5. Two non-isomorphic tame B_1 -persistence modules can have identical persistence diagrams. For example, take M and N to be the B_1 -persistence modules of Example 2.1.2. M and N are not isomorphic but it is easy to check that they have the same persistence diagram. Thus d_B defines a pseudometric (but not a metric) on isomorphism classes of tame B_1 -persistence modules.

2.3 A Structure Theorem for Well Behaved B_1 -persistence modules

In this section we prove an analogue of Theorem 2.2.2 for a certain subset of the tame B_1 -persistence modules which we call the *well behaved* persistence modules. The set of well behaved persistence modules contains the set of finitely presented B_1 -persistence modules. These modules are in a sense “essentially discrete.” Indeed, they are exactly the B_1 -persistence modules that are the images of tame \mathbf{A}_1 -persistence modules under a certain family of functors from $\mathbf{A}_1\text{-mod}$ to $B_1\text{-mod}$.

Our strategy for proving the structure theorem for well behaved persistence modules is to exploit Theorem 2.2.2, taking advantage of the functorial relationship between \mathbf{A}_1 -persistence modules and well behaved B_1 -persistence modules.

2.3.1 Well Behaved Persistence Modules

A *critical value* of a B_1 -persistence module M is a point $a \in \mathbb{R}$ such that for no $\epsilon \in \mathbb{R}_{\geq 0}$ is it true that for all $u \leq v \in [a - \epsilon, a + \epsilon]$, $\varphi_M(u, v)$ is an isomorphism.

We'll say a tame B_1 -persistence module M is **well behaved** if

1. The critical values of M are countable and have no accumulation point.
2. For each critical point a of M , there exists $\epsilon > 0$ such that $\varphi_M(a, y)$ is an isomorphism for all $y \in [a, a + \epsilon]$.

Proposition 2.3.1. *A finitely presented B_1 -persistence module is well behaved.*

Proof. Let M be a finitely presented B_1 -persistence module and let $U \subset \mathbb{R}$ be the set of grades of the generators and relations in a minimal presentation for M . (It follows from Theorem 2.1.4 that U is well defined). Lemma 2.5.4 below tells us that for any $a \leq b \in \mathbb{R}$ such that $(a, b] \cap U = \emptyset$, $\varphi_M(a, b)$ is an isomorphism. Since U is finite, the result follows immediately. \square

Let t be a grid function. Define $t^{-1} : \mathbb{R} \rightarrow \mathbb{Z}$ by $t^{-1}(y) = \max\{z \in \mathbb{Z} | t(z) \leq y\}$.

Define $\bar{t}^{-1} : \bar{\mathbb{R}} \rightarrow \bar{\mathbb{Z}}$ by

$$\bar{t}^{-1}(u) = \begin{cases} t^{-1}(u) & \text{if } u \in \mathbb{R}, \\ -\infty & \text{if } u = -\infty, \\ \infty & \text{if } u = \infty. \end{cases}$$

We'll now define a functor $E_t : \mathbf{A}_1\text{-mod} \rightarrow B_1\text{-mod}$ as follows:

1. Action of E_t on objects: For \mathbf{M} an \mathbf{A}_1 -persistence module and $u \in \mathbb{R}$, $E_t(\mathbf{M})_u = \mathbf{M}_{t^{-1}(u)}$; let $\mathcal{J}_{\mathbf{M}, t, u} : E_t(\mathbf{M})_u \rightarrow \mathbf{M}_{t^{-1}(u)}$ denote this identification. For $u, v \in \mathbb{R}$, $u \leq v$, let $\varphi_{E_t(\mathbf{M})}(u, v) = \mathcal{J}_{\mathbf{M}, t, v}^{-1} \circ \varphi_{\mathbf{M}}(t^{-1}(u), t^{-1}(v)) \circ \mathcal{J}_{\mathbf{M}, t, u}$.
2. Action of E_t on morphisms: For \mathbf{M} and \mathbf{N} \mathbf{A}_1 -persistence modules and $f \in \text{hom}(\mathbf{M}, \mathbf{N})$, define $E_t(f) : E_t(\mathbf{M}) \rightarrow E_t(\mathbf{N})$ by letting $E_t(f)_u = \mathcal{J}_{\mathbf{N}, t, u}^{-1} \circ f_{t^{-1}(u)} \circ \mathcal{J}_{\mathbf{M}, t, u}$ for all $u \in \mathbb{R}$.

We leave to the reader the easy verification that E_t is in fact a functor with target $B_1\text{-mod}$.

It's clear that if \mathbf{M} is a tame \mathbf{A}_1 -persistence module, then for any grid function t , $E_t(\mathbf{M})$ is tame. Moreover, it's easy to check that for any grid function t and any tame \mathbf{A}_1 -persistence module \mathbf{M} , $E_t(\mathbf{M})$ is well behaved.

Conversely, we have the following:

Proposition 2.3.2. *If M is a well behaved B_1 -persistence module, then there is some tame \mathbf{A}_1 -persistence-module \mathbf{M} and some grid function t such that $M \cong E_t(\mathbf{M})$.*

Proof. Let $t : \mathbb{Z} \rightarrow \mathbb{R}$ be a grid function whose image contains the critical points of M . Let (\mathbf{M}, t) denote the t -discretization of M , as defined in Section 2.2.3. \mathbf{M} clearly is tame. We'll show that $M \cong E_t(\mathbf{M})$.

For $u \in \mathbb{R}$, define $\sigma_u : E_t(\mathbf{M})_u \rightarrow M_u$ by $\sigma_u = \varphi_M(t \circ t^{-1}(u), u) \circ \mathcal{I}_{M,t,t^{-1}(u)} \circ \mathcal{J}_{\mathbf{M},t,u}$. By definition, $\mathcal{J}_{\mathbf{M},t,u}$ and $\mathcal{I}_{M,t,t^{-1}(u)}$ are isomorphisms. Moreover, a simple compactness argument shows that since M is well behaved, $\varphi_M(t \circ t^{-1}(u), u)$ is an isomorphism. Thus σ_u is an isomorphism.

We claim that the collection of maps $\{\sigma_u\}_{u \in \mathbb{R}}$ defines an isomorphism of modules. To see this, we need to show that for all $u, v \in \mathbb{R}$, $u \leq v$, $\sigma_v \circ \varphi_{E_t(\mathbf{M})}(u, v) = \varphi_M(u, v) \circ \sigma_u$:

$$\begin{aligned}
& \sigma_v \circ \varphi_{E_t(\mathbf{M})}(u, v) \\
&= \sigma_v \circ \mathcal{J}_{\mathbf{M},t,v}^{-1} \circ \varphi_{\mathbf{M}}(t^{-1}(u), t^{-1}(v)) \circ \mathcal{J}_{\mathbf{M},t,u} \\
&= \varphi_M(t \circ t^{-1}(v), v) \circ \mathcal{I}_{M,t,t^{-1}(v)} \circ \mathcal{J}_{\mathbf{M},t,v} \circ \mathcal{J}_{\mathbf{M},t,v}^{-1} \circ \varphi_{\mathbf{M}}(t^{-1}(u), t^{-1}(v)) \circ \mathcal{J}_{\mathbf{M},t,u} \\
&= \varphi_M(t \circ t^{-1}(v), v) \circ \mathcal{I}_{M,t,t^{-1}(v)} \circ \varphi_{\mathbf{M}}(t^{-1}(u), t^{-1}(v)) \circ \mathcal{J}_{\mathbf{M},t,u} \\
&= \varphi_M(t \circ t^{-1}(v), v) \circ \mathcal{I}_{M,t,t^{-1}(v)} \circ \mathcal{I}_{M,t,t^{-1}(v)}^{-1} \\
&\quad \circ \varphi_M(t \circ t^{-1}(u), t \circ t^{-1}(v)) \circ \mathcal{I}_{M,t,t^{-1}(u)} \circ \mathcal{J}_{\mathbf{M},t,u} \\
&= \varphi_M(t \circ t^{-1}(v), v) \circ \varphi_M(t \circ t^{-1}(u), t \circ t^{-1}(v)) \circ \mathcal{I}_{M,t,t^{-1}(u)} \circ \mathcal{J}_{\mathbf{M},t,u} \\
&= \varphi_M(t \circ t^{-1}(u), v) \circ \mathcal{I}_{M,t,t^{-1}(u)} \circ \mathcal{J}_{\mathbf{M},t,u} \\
&= \varphi_M(u, v) \circ \varphi_M(t \circ t^{-1}(u), u) \circ \mathcal{I}_{M,t,t^{-1}(u)} \circ \mathcal{J}_{\mathbf{M},t,u} \\
&= \varphi_M(u, v) \circ \sigma_u.
\end{aligned}$$

□

Remark 2.3.3. The material above can be adapted with only minor changes to the setting of B_n -persistence modules, where it sheds some light on the relationship between \mathbf{A}_n -persistence modules and B_n -persistence modules. Namely, the definitions of a well behaved persistence module, grid function, and the functors E_t generalize to the multidimensional setting, and analogues of Propositions 2.3.1 and 2.3.2 hold in that setting. It can be shown that the functor $(\cdot) \otimes_{\mathbf{A}_n} B_n$ mentioned in Section 2.1.3 is naturally isomorphic to a generalized functor E_t . The generalization of the above material also can be used to translate algebraic results about \mathbf{A}_n -persistence modules into analogous results about B_n -persistence modules. For example, it can be used to show that any finitely presented B_n -persistence module has a free resolution of length at most n —that is, an analogue of the Hilbert syzygy theorem holds for B_n -persistence modules.

However, as we have no immediate need for the generalization or its consequences in this thesis, we omit it.

2.3.2 The Structure Theorem

First, note that for any $a \in \mathbb{R}_{\geq 0}$, $k[[a, \infty)]$ (as defined in Section 2.1.2) is an ideal of B_1 .

For $a < b \in \mathbb{R}$, let $C(a, b)$ denote $(B_1/k[[b - a, \infty)))(-a)$; let $C(a, \infty)$ denote $B_1(-a)$. In analogy to the discrete case, for fixed b (possibly infinite), the set of modules $\{C(a, b) | a \in \mathbb{R}, a < b\}$ has a natural directed system structure; let $C(-\infty, b)$ denote the colimit of this directed system.

Lemma 2.3.4. *Let M be a well-behaved persistence module and let \mathcal{D} be a persistence diagram such that $M \cong \bigoplus_{(a,b) \in \text{supp}(\mathcal{D})} C(a, b)^{\mathcal{D}(a,b)}$. Then $\mathcal{D}_M = \mathcal{D}$.*

Proof. Let

$$\begin{aligned} A = & \{a \in \mathbb{R} | (a, b) \in \text{supp}(\mathcal{D}) \text{ for some } b \in \bar{\mathbb{R}}\} \\ & \cup \{b \in \mathbb{R} | (a, b) \in \text{supp}(\mathcal{D}) \text{ for some } a \in \bar{\mathbb{R}}\}. \end{aligned}$$

Let t be a grid function such that $A \subset \text{im}(t)$. We claim that $\mathcal{D}_{(\mathbf{P}_t(M), t)} = \mathcal{D}$. Since

$\text{supp}(\mathcal{D}) \in \text{im}(\bar{t} \times \bar{t})$, this is true if and only if $\mathcal{D}_{\mathbf{P}_t(M)}(y, z) = \mathcal{D}(\bar{t}(y), \bar{t}(z))$ for all $(y, z) \in \bar{\mathbb{Z}}_+^2$.

To show that $\mathcal{D}_{\mathbf{P}_t(M)}(y, z) = \mathcal{D}(\bar{t}(y), \bar{t}(z))$ for all $(y, z) \in \bar{\mathbb{Z}}_+^2$, we'll need the following analogue of Lemma 2.2.3.

Lemma 2.3.5. *Let M, \mathcal{D} , and t be as above.*

(i) *For $(y, z) \in \mathbb{Z}_+^2$,*

$$\begin{aligned} \mathcal{D}(t(y), t(z)) &= \text{rank}(\varphi_M(t(y), t(z-1))) - \text{rank}(\varphi_M(t(y), t(z))) \\ &\quad - \text{rank}(\varphi_M(t(y-1), t(z-1))) + \text{rank}(\varphi_M(t(y-1), t(z))). \end{aligned}$$

(ii) *For $z \in \mathbb{Z}$,*

$$\mathcal{D}(-\infty, t(z)) = \lim_{y \rightarrow -\infty} \text{rank}(\varphi_M(t(y), t(z-1))) - \lim_{y \rightarrow -\infty} \text{rank}(\varphi_M(t(y), t(z))).$$

(iii) *For $y \in \mathbb{Z}$,*

$$\mathcal{D}(t(y), \infty) = \lim_{z \rightarrow \infty} \text{rank}(\varphi_M(t(y), t(z))) - \lim_{z \rightarrow \infty} \text{rank}(\varphi_M(t(y-1), t(z))).$$

(iv)

$$\mathcal{D}(-\infty, \infty) = \lim_{y \rightarrow -\infty} \lim_{z \rightarrow \infty} \text{rank}(\varphi_M(t(y), t(z))).$$

Proof. The proof is straightforward; we omit it. \square

For $(y, z) \in \mathbb{Z}_+^2$ we have

$$\begin{aligned} \mathcal{D}_{\mathbf{P}_t(M)}(y, z) &= \text{rank}(\varphi_{\mathbf{P}_t(M)}(y, z-1)) - \text{rank}(\varphi_{\mathbf{P}_t(M)}(y, z)) \\ &\quad - \text{rank}(\varphi_{\mathbf{P}_t(M)}(y-1, z-1)) + \text{rank}(\varphi_{\mathbf{P}_t(M)}(y-1, z)) \\ &= \text{rank}(\varphi_M(t(y), t(z-1))) - \text{rank}(\varphi_M(t(y), t(z))) \\ &\quad - \text{rank}(\varphi_M(t(y-1), t(z-1))) + \text{rank}(\varphi_M(t(y-1), t(z))) \\ &= \mathcal{D}(t(y), t(z)), \end{aligned}$$

where the first equality follows from Lemma 2.2.3(i), and the last equality follows from Lemma 2.3.5(i).

Thus we have $\mathcal{D}_{\mathbf{P}_t(M)}(y, z) = \mathcal{D}(\bar{t}(y), \bar{t}(z))$ for all $(y, z) \in \mathbb{Z}_+^2$. Similar arguments using Lemma 2.2.3(ii)-(iv) and Lemma 2.3.5(ii)-(iv) in the cases where $y = -\infty$ or $z = \infty$ show that in fact this holds for $(y, z) \in \mathbb{Z}_+^2$. This proves the claim.

It follows easily from the fact that M is well behaved that A is equal to the set of critical values of M . There thus exists a sequence of grid functions $\{t_i\}_{i \in \mathbb{N}}$ such that t_i is a $1/2^i$ cover and $A \subset \text{im}(t_i)$ for each i . The lemma follows by writing \mathcal{D}_M as the limit of the persistence diagrams $\mathcal{D}_{(\mathbf{P}_{t_i}(M), t_i)}$. \square

Theorem 2.3.6. *Let M be a well behaved B_1 -persistence module. Let \mathcal{D}_M be the persistence diagram of M . Then*

$$M \cong \bigoplus_{(a,b) \in \text{supp}(\mathcal{D}_M)} C(a, b)^{\mathcal{D}_M(a,b)}.$$

This decomposition of M is unique in the sense that if \mathcal{D} is another persistence diagram such that $M \cong \bigoplus_{(a,b) \in \text{supp}(\mathcal{D})} C(a, b)^{\mathcal{D}(a,b)}$, then $\mathcal{D} = \mathcal{D}_M$.

Proof. By Lemma 2.3.4, it's enough show that there exists some persistence diagram \mathcal{D} such that $M \cong \bigoplus_{(a,b) \in \text{supp}(\mathcal{D})} C(a, b)^{\mathcal{D}(a,b)}$.

By Proposition 2.3.2, there exists a grid function t and a tame \mathbf{A}_1 -persistence module \mathbf{M} such that $E_t(\mathbf{M}) \cong M$. The structure theorem for tame \mathbf{A}_1 -persistence modules gives us that there's a persistence diagram $\mathcal{D}_\mathbf{M}$ supported in $\bar{\mathbb{Z}}_+^2$ such that we may take

$$\mathbf{M} = \bigoplus_{(a,b) \in \text{supp}(\mathcal{D}_\mathbf{M})} \mathbf{C}(a, b)^{\mathcal{D}_\mathbf{M}(a,b)}.$$

We'll show that

$$M \cong \bigoplus_{(a,b) \in \text{supp}(\mathcal{D}_\mathbf{M})} E_t(\mathbf{C}(a, b))^{\mathcal{D}_\mathbf{M}(a,b)}.$$

We have that $E_t(\mathbf{C}(a, b)) \cong C(\bar{t}(a), \bar{t}(b))$ for any $(a, b) \in \bar{\mathbb{Z}}_+^2$, so this gives the result.

To show that $M \cong \bigoplus_{(a,b) \in \text{supp}(\mathcal{D}_\mathbf{M})} E_t(\mathbf{C}(a, b))^{\mathcal{D}_\mathbf{M}(a,b)}$, we'll use the category theoretic characterization of direct sums of modules as coproducts [33]. Let A be a set. Recall that in an arbitrary category, an object X is a *coproduct* of objects $\{X^\alpha\}_{\alpha \in A}$ iff there exist morphisms $\{i^\alpha : X^\alpha \rightarrow X\}_{\alpha \in A}$, called canonical injections, with the

following universal property: for any object Y and morphisms $\{f^\alpha : X^\alpha \rightarrow Y\}_{\alpha \in A}$, there exists a unique morphism $f : X \rightarrow Y$ such that $f \circ i^\alpha = f^\alpha$ for each $\alpha \in A$. In a category of modules over a ring R , The coproduct of modules X^α is $\bigoplus_\alpha X^\alpha$; the canonical injections are just the usual inclusions $X^\alpha \hookrightarrow \bigoplus_\alpha X^\alpha$. The same is thus true for the module subcategories $\mathbf{A}_n\text{-mod}$ and $B_n\text{-mod}$.

Now let $\{\mathbf{M}^\alpha\}$ denote the indecomposable summands of \mathbf{M} in the direct sum decomposition $\mathbf{M} = \bigoplus_{(a,b) \in \text{supp}(\mathcal{D}_\mathbf{M})} \mathbf{C}(a,b)^{\mathcal{D}_\mathbf{M}(a,b)}$, so that each $\mathbf{M}^\alpha = \mathbf{C}(a,b)$ for some $(a,b) \in \bar{\mathbb{Z}}_+^2$. Let $\{i^\alpha : \mathbf{M}^\alpha \rightarrow \mathbf{M}\}$ denote the canonical injections.

We'll show that the maps $E_t(i^\alpha) : E_t(\mathbf{M}^\alpha) \rightarrow E_t(\mathbf{M})$ satisfy the universal property of a coproduct, so that $M \cong E_t(\mathbf{M}) \cong \bigoplus_\alpha E_t(\mathbf{M}^\alpha)$ as desired.

To show that the maps $E_t(i^\alpha) : E_t(\mathbf{M}^\alpha) \rightarrow E_t(\mathbf{M})$ satisfy the universal property of a coproduct, let Y be an arbitrary B_1 -persistence module and $\{f^\alpha : E_t(\mathbf{M}^\alpha) \rightarrow Y\}$ be homomorphisms.

For any $z \in \mathbb{Z}$, $\mathbf{M}_z \cong \bigoplus_\alpha \mathbf{M}_z^\alpha$. It follows from the definition of E_t that for any $r \in \mathbb{R}$,

$$E_t(\mathbf{M})_r \cong \bigoplus_\alpha E_t(\mathbf{M}^\alpha)_r$$

with the maps $E_t(i^\alpha)_r$ the canonical inclusions.

For each $r \in \mathbb{R}$, define $f_r : E_t(\mathbf{M})_r \rightarrow Y_r$ as $\bigoplus_\alpha f_r^\alpha$ (i.e. f_r is the map guaranteed to exist by the universal property of direct sums for vector spaces.) It's easy to check that the maps f_r commute with the transition maps in $E_t(\mathbf{M})$ and Y , so that they define a morphism $f : E_t(\mathbf{M}) \rightarrow Y$. We also have that $f \circ E_t(i^\alpha) = f^\alpha$ for each α . By the universal property of direct sums of vector spaces, for each r f_r is the unique linear transformation from $E_t(\mathbf{M})_r$ to Y_r such that for each α , $f_r \circ E_t(i^\alpha)_r = f_r^\alpha$. Therefore f must itself satisfy the desired uniqueness property. This completes the proof. \square

2.4 The Equality of the Interleaving and Bottleneck distances on Tame B_1 -persistence Modules

We show in this section that the restriction of the interleaving distance to tame B_1 -persistence modules is equal to the bottleneck distance. This shows that the interleaving distance is in fact a generalization of the bottleneck distance, as we want. The result is also instrumental in proving Corollary 2.5.3, our converse to the algebraic stability result of [8].

The Algebraic Stability of Persistence

The main result of [8], generalizing considerably the earlier result of [14], is the following:

Theorem 2.4.1 (Algebraic Stability of Persistence). *Let $\epsilon > 0$, and let M and N be two tame B_1 -persistence modules. If M and N are ϵ -interleaved, then $d_B(M, N) \leq \epsilon$.*

A Converse to the Algebraic Stability of Persistence?

In the conclusion of [8], the authors ask whether it's true that if M and N are tame B_1 -persistence modules with $d_B(M, N) = \epsilon$ then M and N are ϵ -interleaved. Example 2.1.2 shows that this is not true. However, Corollary 2.5.3 below, which follows immediately from Theorems 2.4.2 and 2.5.1, asserts that the result is true provided M and N are finitely presented. More generally, Corollary 2.5.3 tells us that if M and N are tame modules with $d_B(M, N) = \epsilon$ then M and N are $(\epsilon + \delta)$ -interleaved for any $\delta > 0$. In other words, the converse of Theorem 2.4.1 holds for tame modules to arbitrarily small error.

Theorem 2.4.2. *$d_B(M, N) = d_I(M, N)$ for any tame B_1 -persistence modules M and N .*

Proof. Theorem 2.4.1 tells us that $d_B(M, N) \leq d_I(M, N)$, so we just need to show

that $d_B(M, N) \geq d_I(M, N)$. It will follow from the structure theorem for well behaved persistence modules (Theorem 2.3.6) that $d_B(M', N') \geq d_I(M', N')$ for well behaved persistence modules M' and N' (Lemma 2.4.4 below). To extend this result to arbitrary tame modules, we will approximate the modules M and N up to arbitrarily small error in the interleaving distance by well behaved persistence modules (Lemma 2.4.5 below). The full result will follow readily from this approximation.

Lemma 2.4.3. *If $(a, b), (a', b') \in \bar{\mathbb{R}}_+^2$ with $\|(a, b) - (a', b')\|_\infty \leq \epsilon$, then $C(a, b)$ and $C(a', b')$ are ϵ -interleaved.*

Proof. This is easy to prove; we leave the details to the reader. \square

Lemma 2.4.4. *If M and N are two well behaved persistence modules and $d_B(M, N) = \epsilon$ then $d_I(M, N) = \epsilon$.*

Proof. By stability we just need to show that $d_I(M, N) \leq \epsilon$. By the structure theorem for well behaved persistence modules (Theorem 2.2.2), we have that

$$\begin{aligned} M &\cong \bigoplus_{(a,b) \in \text{supp}(\mathcal{D}_M)} C(a, b)^{\mathcal{D}_M(a,b)}, \\ N &\cong \bigoplus_{(a,b) \in \text{supp}(\mathcal{D}_N)} C(a, b)^{\mathcal{D}_N(a,b)}. \end{aligned}$$

Since $d_B(M', N') = \epsilon$, for any $\delta > 0$ there exist persistence diagrams D'_M and D'_N with $D'_M \leq D_M$, $D'_N \leq D_N$, and a multibijection γ between D'_M and D'_N such that

1. For any $(a, b) \in \text{supp}(D_M - D'_M) \cup \text{supp}(D_N - D'_N)$, $(b - a)/2 \leq \epsilon + \delta$,
2. For any $(x, y) \in \text{supp}(\gamma)$, $\|x - y\|_\infty \leq \epsilon + \delta$.

Fix such D'_M , D'_N , and γ . Now we can choose well behaved modules $M', M'' \subset M$ and $N', N'' \subset N$ such that $M = M' \oplus M''$, $N = N' \oplus N''$, $D_{M'} = D'_M$, $D_{N'} = D'_N$, $D_{M''} = D_M - D'_M$, and $D_{N''} = D_N - D'_N$.

If follows from Lemma 2.4.3 that for each $(a, b), (a', b') \in \text{supp}(\gamma)$, $C(a, b)$ and $C(a', b')$ are $(\epsilon + \delta)$ -interleaved. We may write

$$\begin{aligned} M' &\cong \bigoplus_{(a,b),(a',b') \in \text{supp}(\gamma)} C(a, b)^{\gamma((a,b),(a',b'))}, \\ N' &\cong \bigoplus_{(a,b),(a',b') \in \text{supp}(\gamma)} C(a', b')^{\gamma((a,b),(a',b'))}. \end{aligned}$$

It's clear from the form of these decompositions for M' and N' that a choice of a pair of $(\epsilon + \delta)$ -interleaving homomorphisms between $C(a, b)$ and $C(a', b')$ for each pair $(a, b), (a', b') \in \text{supp}(\gamma)$ induces a pair of $(\epsilon + \delta)$ -interleaving homomorphisms $\hat{f} : M' \rightarrow N'(\epsilon + \delta)$ and $\hat{g} : N' \rightarrow M'(\epsilon + \delta)$.

Now we extend this pair to a pair of homomorphisms $f : M \rightarrow N(\epsilon + \delta)$, $g : N \rightarrow M(\epsilon + \delta)$ by defining $f(y) = \hat{f}(y)$ for $y \in M'$, $f(M'') = 0$, $g(y) = \hat{g}(y)$ for $y \in N'$, and $g(M'') = 0$. Obviously, f and g restrict to $(\epsilon + \delta)$ -interleaving homomorphisms between M' and N' . Moreover, we have that $S(M'', 2\epsilon + \delta) = 0$ and $S(N'', 2\epsilon + \delta) = 0$, so f and g restrict to $(\epsilon + \delta)$ -interleaving homomorphisms between M'' and N'' . Thus by linearity, f and g are $(\epsilon + \delta)$ -interleaving homomorphisms between M and N . Since δ may be taken to be arbitrarily small, we must have $d_I(M, N) \leq \epsilon$, as we wanted to show. \square

Lemma 2.4.5. *For any tame B_1 -persistence module M and $\delta > 0$, there exists a well behaved persistence module M' with $d_I(M, M') \leq \delta$.*

Proof. Let t be an $\delta/2$ -cover of \mathbb{R} , as defined in Section 2.2.3. For any $r \in \mathbb{R}$, there exists $r' \in \text{im}(t)$, with $0 \leq r' - r \leq \delta$. Define a function $\lambda : \mathbb{R} \rightarrow \text{im}(t)$ such that $\lambda(r) = \min\{r' \geq r | r' \in \text{im}(t)\}$. Then $0 \leq \lambda(r) - r \leq \delta$ for all $r \in \mathbb{R}$.

Let $\mathbf{M} = \mathbf{P}_t(M)$ (as defined in Section 2.2.3) and let $M' = E_t(\mathbf{M})$. Then M' is well-behaved. We now show that M and M' are δ -interleaved, which implies that $d_I(M, M') \leq \delta$.

Define $f : M \rightarrow M'(\delta)$ to be the morphism for which

$$f_u : M_u \rightarrow M'_{u+\delta} = \varphi_{M'}(\lambda(u), u + \delta) \circ \mathcal{J}_{\mathbf{M}, t, \lambda(u)}^{-1} \circ \mathcal{I}_{M, t, t^{-1}(\lambda(u))}^{-1} \circ \varphi_M(u, \lambda(u)).$$

Define $g : M' \rightarrow M(\delta)$ to be the morphism for which

$$g_u : M'_u \rightarrow M_{u+\delta} = \varphi_M(\lambda(u), u + \delta) \circ \mathcal{I}_{M, t, t^{-1}(\lambda(u))} \circ \mathcal{J}_{\mathbf{M}, t, \lambda(u)} \circ \varphi_{M'}(u, \lambda(u)).$$

We need to check that f and g thus defined are in fact morphisms. We verify this for f ; the verification for g is similar; we omit it.

If $u \leq v \in \mathbb{R}$, we have

$$f_v \circ \varphi_M(u, v) = \varphi_{M'}(\lambda(v), v + \delta) \circ \mathcal{J}_{\mathbf{M}, t, \lambda(v)}^{-1} \circ \mathcal{I}_{M, t, t^{-1}(\lambda(v))}^{-1} \circ \varphi_M(v, \lambda(v)) \circ \varphi_M(u, v). \quad (2.1)$$

By definition, for any $u \leq v \in \mathbb{R}$, we have

$$\begin{aligned} \varphi_{M'}(u, v) &= \mathcal{J}_{\mathbf{M}, t, v}^{-1} \circ \varphi_{\mathbf{M}}(t^{-1}(u), t^{-1}(v)) \circ \mathcal{J}_{\mathbf{M}, t, u} \\ &= \mathcal{J}_{\mathbf{M}, t, v}^{-1} \circ \mathcal{I}_{M, t, t^{-1}(v)}^{-1} \circ \varphi_M(u, v) \circ \mathcal{I}_{M, t, t^{-1}(u)} \circ \mathcal{J}_{\mathbf{M}, t, u}. \end{aligned} \quad (2.2)$$

Using (2.2) to substitute for $\varphi_{M'}(\lambda(v), v + \delta)$ in (2.1) and simplifying gives us:

$$\begin{aligned} f_v \circ \varphi_M(u, v) &= \mathcal{J}_{\mathbf{M}, t, v+\delta}^{-1} \circ \mathcal{I}_{M, t, t^{-1}(v+\delta)}^{-1} \circ \varphi_M(\lambda(v), v + \delta) \circ \varphi_M(v, \lambda(v)) \circ \varphi_M(u, v) \\ &= \mathcal{J}_{\mathbf{M}, t, v+\delta}^{-1} \circ \mathcal{I}_{M, t, t^{-1}(v+\delta)}^{-1} \circ \varphi_M(u, v + \delta). \end{aligned}$$

On the other hand we have, using (2.2) again,

$$\begin{aligned} &\varphi_{M'}(u + \delta, v + \delta) \circ f_u \\ &= \varphi_{M'}(u + \delta, v + \delta) \circ \varphi_{M'}(\lambda(u), u + \delta) \circ \mathcal{J}_{\mathbf{M}, t, \lambda(u)}^{-1} \circ \mathcal{I}_{M, t, t^{-1}(\lambda(u))}^{-1} \circ \varphi_M(u, \lambda(u)) \\ &= \varphi_{M'}(\lambda(u), v + \delta) \circ \mathcal{J}_{\mathbf{M}, t, \lambda(u)}^{-1} \circ \mathcal{I}_{M, t, t^{-1}(\lambda(u))}^{-1} \circ \varphi_M(u, \lambda(u)) \\ &= \mathcal{J}_{\mathbf{M}, t, v+\delta}^{-1} \circ \mathcal{I}_{M, t, t^{-1}(v+\delta)}^{-1} \circ \varphi_M(\lambda(u), v + \delta) \circ \mathcal{I}_{M, t, t^{-1}(\lambda(u))} \\ &\quad \circ \mathcal{J}_{\mathbf{M}, t, \lambda(u)} \circ \mathcal{J}_{\mathbf{M}, t, \lambda(u)}^{-1} \circ \mathcal{I}_{M, t, t^{-1}(\lambda(u))}^{-1} \circ \varphi_M(u, \lambda(u)) \\ &= \mathcal{J}_{\mathbf{M}, t, v+\delta}^{-1} \circ \mathcal{I}_{M, t, t^{-1}(v+\delta)}^{-1} \circ \varphi_M(\lambda(u), v + \delta) \circ \varphi_M(u, \lambda(u)) \\ &= \mathcal{J}_{\mathbf{M}, t, v+\delta}^{-1} \circ \mathcal{I}_{M, t, t^{-1}(v+\delta)}^{-1} \circ \varphi_M(u, v + \delta). \end{aligned}$$

Thus $f_v \circ \varphi_M(u, v) = \varphi_{M'}(u + \delta, v + \delta) \circ f_u$, as we wanted to show.

Finally, we need to check that $g \circ f = S(M, 2\delta)$ and $f \circ g = S(M', 2\delta)$. We perform the first verification and omit the second, since the verifications are similar.

For $u \in \mathbb{R}$,

$$\begin{aligned}
& g_{u+\delta} \circ f_u \\
&= \varphi_M(\lambda(u + \delta), u + 2\delta) \circ \mathcal{I}_{M,t,t^{-1}(\lambda(u + \delta))} \circ \mathcal{J}_{\mathbf{M},t,\lambda(u + \delta)} \\
&\quad \circ \varphi_{M'}(u + \delta, \lambda(u + \delta)) \circ \varphi_{M'}(\lambda(u), u + \delta) \circ \mathcal{J}_{\mathbf{M},t,\lambda(u)}^{-1} \circ \mathcal{I}_{M,t,t^{-1}(\lambda(u))}^{-1} \circ \varphi_M(u, \lambda(u)) \\
&= \varphi_M(\lambda(u + \delta), u + 2\delta) \circ \mathcal{I}_{M,t,t^{-1}(\lambda(u + \delta))} \circ \mathcal{J}_{\mathbf{M},t,\lambda(u + \delta)} \\
&\quad \circ \varphi_{M'}(\lambda(u), \lambda(u + \delta)) \circ \mathcal{J}_{\mathbf{M},t,\lambda(u)}^{-1} \circ \mathcal{I}_{M,t,t^{-1}(\lambda(u))}^{-1} \circ \varphi_M(u, \lambda(u)).
\end{aligned}$$

Using (2.2) once again, we have that this last expression is equal to

$$\begin{aligned}
& \varphi_M(\lambda(u + \delta), u + 2\delta) \circ \mathcal{I}_{M,t,t^{-1}(\lambda(u + \delta))} \circ \mathcal{J}_{\mathbf{M},t,\lambda(u + \delta)} \circ \mathcal{J}_{\mathbf{M},t,\lambda(u + \delta)}^{-1} \\
&\quad \circ \mathcal{I}_{M,t,t^{-1}(\lambda(u + \delta))}^{-1} \circ \varphi_M(\lambda(u), \lambda(u + \delta)) \circ \mathcal{I}_{M,t,t^{-1}(\lambda(u))} \circ \mathcal{J}_{\mathbf{M},t,\lambda(u)} \\
&\quad \circ \mathcal{J}_{\mathbf{M},t,\lambda(u)}^{-1} \circ \mathcal{I}_{M,t,t^{-1}(\lambda(u))}^{-1} \circ \varphi_M(u, \lambda(u)) \\
&= \varphi_M(\lambda(u + \delta), u + 2\delta) \circ \varphi_M(\lambda(u), \lambda(u + \delta)) \circ \varphi_M(u, \lambda(u)) \\
&= \varphi_M(u, u + 2\delta).
\end{aligned}$$

□

Now we can complete the proof of Theorem 2.4.2. As mentioned above, by Theorem 2.4.1 it suffices to show $d_I(M, N) \leq d_B(M, N)$. Say $d_B(M, N) = \epsilon$. Choose $\delta > 0$. By Lemma 2.4.5, there exist well behaved modules M', N' with $d_I(M, M') \leq \delta$, $d_I(N, N') \leq \delta$. Then by Theorem 2.4.1, $d_B(M, M') \leq \delta$, $d_B(N, N') \leq \delta$, so by the triangle inequality, $d_B(M', N') \leq \epsilon + 2\delta$. By Lemma 2.4.4, $d_I(M', N') \leq \epsilon + 2\delta$. Applying the triangle inequality again, we get that $d_I(M, N) \leq \epsilon + 4\delta$. As δ may be taken to be arbitrarily small, we have $d_I(M, N) \leq \epsilon$, which completes the proof. □

2.5 If $d_I(M, N) = \epsilon$ and M, N are Finitely Presented then M and N are ϵ -interleaved

We now show that for finitely presented B_n -modules M and N , if $d_I(M, N) = \epsilon$ then M and N are ϵ -interleaved. This implies that the restriction of d_I to finitely presented persistence modules is a metric and, as noted in Section 2.4, yields a converse to

the algebraic stability of persistence for finitely presented B_1 -persistence modules. Theorem 2.5.1 will also be of some use to us in Section 2.11.

Theorem 2.5.1. *If M and N are finitely presented B_n -modules and $d_I(M, N) = \epsilon$ then M and N are ϵ -interleaved.*

Corollary 2.5.2. *d_I is a metric on finitely presented B_n -modules.*

Corollary 2.5.3 (Converse to Algebraic Stability).

- (i) *If M and N are finitely presented B_1 -persistence modules and $d_B(M, N) = \epsilon$ then M and N are ϵ -interleaved.*
- (ii) *If M and N are tame B_1 -persistence modules and $d_B(M, N) = \epsilon$ then M and N are $(\epsilon + \delta)$ -interleaved for any $\delta > 0$.*

Proof. (ii) follows directly from Theorem 2.4.2. (i) is immediate from that theorem and Theorem 2.5.1. \square

For a finitely presented B_n -persistence module M , let $U_M \subset \mathbb{R}^n$ be the set of grades of the generators and relations in a minimal presentation for M . Let $U_M^i \subset \mathbb{R}$ be the set of i^{th} coordinates of the elements of U_M .

Proof of Theorem 2.5.1.

Lemma 2.5.4. *If M is a finitely presented B_n -persistence module then for any $a \leq b \in \mathbb{R}^n$ such that $(a_i, b_i] \cap U_M^i = \emptyset$ for all i , $\varphi_M(a, b)$ is an isomorphism.*

Proof. This is straightforward; we omit the details. \square

Lemma 2.5.5. *If M is a finitely presented B_n -persistence module then for any $y \in \mathbb{R}^n$, there exists $r \in \mathbb{R}_{>0}$ such that $\varphi_M(y, y + r')$ is an isomorphism for all $0 \leq r' \leq r$.*

Proof. This is an immediate consequence of Lemma 2.5.4. \square

For a finitely presented B_n -persistence module M , let $fl_M : \mathbb{R}^n \rightarrow \prod_{i=1}^n \bar{U}_M^i$ be defined by $fl_M(a_1, \dots, a_n) = (a'_1, \dots, a'_n)$, where a'_i is the largest element of U_M^i such that $a'_i \leq a_i$, if such an element exists, and $a'_i = -\infty$ otherwise.

Lemma 2.5.6. *For any finitely presented B_n -module M and any $y \in \mathbb{R}^n$ with $fl_M(y) \in \mathbb{R}^n$, we have that $\varphi_M(fl_M(y), y)$ is an isomorphism.*

Proof. This too is an immediate consequence of Lemma 2.5.4. \square

Having stated these preliminary results, we proceed with the proof of Theorem 2.5.1. By Lemma 2.5.5 and the finiteness of U_M and U_N , there exists $\delta > 0$ such that for all $z \in U_M$, $\varphi_N(z + \epsilon, z + \epsilon + \delta)$ and $\varphi_M(z + 2\epsilon, z + 2\epsilon + 2\delta)$ are isomorphisms, and for all $z \in U_N$, $\varphi_M(z + \epsilon, z + \epsilon + \delta)$ and $\varphi_N(z + 2\epsilon, z + 2\epsilon + 2\delta)$ are isomorphisms.

By Remark 2.1.1, since $d_I(M, N) = \epsilon$, M and N are $(\epsilon + \delta)$ -interleaved.

Theorem 2.5.1 then follows from the following lemma, which will also be the key ingredient in the proof of Proposition 2.11.9.

Lemma 2.5.7. *Let M and N be finitely presented B_n -persistence modules and let $\epsilon \geq 0$ and $\delta > 0$ be such that*

1. *M and N are $\epsilon + \delta$ -interleaved,*
2. *for all $z \in U_M$, $\varphi_N(z + \epsilon, z + \epsilon + \delta)$ and $\varphi_M(z + 2\epsilon, z + 2\epsilon + 2\delta)$ are isomorphisms,*
3. *for all $z \in U_N$, $\varphi_M(z + \epsilon, z + \epsilon + \delta)$ and $\varphi_N(z + 2\epsilon, z + 2\epsilon + 2\delta)$ are isomorphisms.*

Then M and N are ϵ -interleaved.

Proof. Let $f : M \rightarrow N(\epsilon + \delta)$ and $g : N \rightarrow M(\epsilon + \delta)$ be interleaving homomorphisms.

We define ϵ -interleaving homomorphisms $\tilde{f} : M \rightarrow N(\epsilon)$ and $\tilde{g} : N \rightarrow M(\epsilon)$ via their action on homogeneous summands. First, for $z \in U_M$ define $\tilde{f}_z = \varphi_N^{-1}(z + \epsilon, z + \epsilon + \delta) \circ f_z$. Then for arbitrary $z \in \mathbb{R}^n$ such that $fl_M(z) \in \mathbb{R}^n$ define $\tilde{f}_z = \varphi_N(fl_M(z) + \epsilon, z + \epsilon) \circ \tilde{f}_{fl_M(z)} \circ \varphi_M^{-1}(fl_M(z), z)$. (Note that $\varphi_M^{-1}(fl_M(z), z)$ is well defined by Lemma 2.5.6.) Finally, for $z \in \mathbb{R}^n$ s.t. $fl_M(z) \notin \mathbb{R}^n$, define $\tilde{f}_z = 0$. (If $fl_M(z) \notin \mathbb{R}^n$ then $M_z = 0$, so this last part of the definition is reasonable.)

Symmetrically, for $z \in U_N$ define $\tilde{g}_z = \varphi_M^{-1}(z + \epsilon, z + \epsilon + \delta) \circ g_z$. For arbitrary $z \in \mathbb{R}^n$ such that $fl_N(z) \in \mathbb{R}^n$ define $\tilde{g}_z = \varphi_M(fl_N(z) + \epsilon, z + \epsilon) \circ \tilde{g}_{fl_N(z)} \circ \varphi_N^{-1}(fl_N(z), z)$. For $z \in \mathbb{R}^n$ s.t. $fl_N(z) \notin \mathbb{R}^n$, define $\tilde{g}_z = 0$.

We need to check that \tilde{f}, \tilde{g} as thus defined are in fact morphisms. We perform the check for \tilde{f} ; the check for \tilde{g} is the same.

If $y \in \mathbb{R}^n$ is such that $fl_M(y) \notin \mathbb{R}^n$, then since $M_y = 0$, it's clear that $\tilde{f}_z \circ \varphi_M(y, z) = \varphi_N(y + \epsilon, z + \epsilon) \circ \tilde{f}_y$.

For $y \leq z \in \mathbb{R}^n$ such that $fl_M(y) \in \mathbb{R}^n$,

$$\begin{aligned}\tilde{f}_z \circ \varphi_M(y, z) &= \varphi_N(fl_M(z) + \epsilon, z + \epsilon) \circ \tilde{f}_{fl_M(z)} \circ \varphi_M^{-1}(fl_M(z), z) \circ \varphi_M(y, z) \\ &= \varphi_N(fl_M(z) + \epsilon, z + \epsilon) \circ \varphi_N^{-1}(fl_M(z) + \epsilon, fl_M(z) + \epsilon + \delta) \circ f_{fl_M(z)} \\ &\quad \circ \varphi_M^{-1}(fl_M(z), z) \circ \varphi_M(y, z) \\ &= \varphi_N(fl_M(z) + \epsilon, z + \epsilon) \circ \varphi_N^{-1}(fl_M(z) + \epsilon, fl_M(z) + \epsilon + \delta) \circ f_{fl_M(z)} \\ &\quad \circ \varphi_M(fl_M(y), fl_M(z)) \circ \varphi_M^{-1}(fl_M(y), y) \\ &= \varphi_N(fl_M(z) + \epsilon, z + \epsilon) \circ \varphi_N^{-1}(fl_M(z) + \epsilon, fl_M(z) + \epsilon + \delta) \\ &\quad \circ \varphi_N(fl_M(y) + \epsilon + \delta, fl_M(z) + \epsilon + \delta) \circ f_{fl_M(y)} \circ \varphi_M^{-1}(fl_M(y), y) \\ &= \varphi_N(y + \epsilon, z + \epsilon) \circ \varphi_N(fl_M(y) + \epsilon, y + \epsilon) \\ &\quad \circ \varphi_N^{-1}(fl_M(y) + \epsilon, fl_M(y) + \epsilon + \delta) \circ f_{fl_M(y)} \circ \varphi_M^{-1}(fl_M(y), y) \\ &= \varphi_N(y + \epsilon, z + \epsilon) \circ \tilde{f}_y\end{aligned}$$

as desired.

To finish the proof, we need to check that $\tilde{g} \circ \tilde{f} = S(M, 2\epsilon)$ and $\tilde{f} \circ \tilde{g} = S(N, 2\epsilon)$. We perform the first check; the second check is the same.

For $z \in \mathbb{R}^n$, if $fl_M(z) \notin \mathbb{R}^n$ then since $M_z = 0$, $\tilde{g}_{z+\epsilon} \circ \tilde{f}_z = 0 = \varphi_M(z, z + 2\epsilon)$.

To show that the result also holds for z such that $fl_M(z) \in \mathbb{R}^n$, we'll begin by verifying the result for $z \in U_M$. We'll use this special case in proving the result for arbitrary $z \in \mathbb{R}^n$ such that $fl_M(z) \in \mathbb{R}^n$.

If $z \in U_M$ then, by assumption, $\varphi_M(z + 2\epsilon, z + 2\epsilon + 2\delta)$ is an isomorphism. Thus, to show that $\tilde{g}_{z+\epsilon} \circ \tilde{f}_z = \varphi_M(z, z + 2\epsilon)$, it suffices to show that $\varphi_M(z + 2\epsilon, z + 2\epsilon + 2\delta) \circ \tilde{g}_{z+\epsilon} \circ \tilde{f}_z = \varphi_M(z, z + 2\epsilon + 2\delta)$.

For $z \in U_M$, we have

$$\begin{aligned}
\tilde{g}_{z+\epsilon} \circ \tilde{f}_z &= \varphi_M(fl_N(z + \epsilon) + \epsilon, z + 2\epsilon) \circ \tilde{g}_{fl_N(z+\epsilon)} \circ \varphi_N^{-1}(fl_N(z + \epsilon), z + \epsilon) \circ \tilde{f}_z \\
&= \varphi_M(fl_N(z + \epsilon) + \epsilon, z + 2\epsilon) \circ \varphi_M^{-1}(fl_N(z + \epsilon) + \epsilon, fl_N(z + \epsilon) + \epsilon + \delta) \\
&\quad \circ g_{fl_N(z+\epsilon)} \circ \varphi_N^{-1}(fl_N(z + \epsilon), z + \epsilon) \circ \tilde{f}_z \\
&= \varphi_M(fl_N(z + \epsilon) + \epsilon, z + 2\epsilon) \circ \varphi_M^{-1}(fl_N(z + \epsilon) + \epsilon, fl_N(z + \epsilon) + \epsilon + \delta) \\
&\quad \circ g_{fl_N(z+\epsilon)} \circ \varphi_N^{-1}(fl_N(z + \epsilon), z + \epsilon) \circ \varphi_N^{-1}(z + \epsilon, z + \epsilon + \delta) \circ f_z.
\end{aligned}$$

Thus

$$\begin{aligned}
&\varphi_M(z + 2\epsilon, z + 2\epsilon + 2\delta) \circ \tilde{g}_{z+\epsilon} \circ \tilde{f}_z \\
&= \varphi_M(z + 2\epsilon, z + 2\epsilon + \delta) \circ \varphi_M(fl_N(z + \epsilon) + \epsilon, z + 2\epsilon) \\
&\quad \circ \varphi_M^{-1}(fl_N(z + \epsilon) + \epsilon, fl_N(z + \epsilon) + \epsilon + \delta) \circ g_{fl_N(z+\epsilon)} \\
&\quad \circ \varphi_N^{-1}(fl_N(z + \epsilon), z + \epsilon) \circ \varphi_N^{-1}(z + \epsilon, z + \epsilon + \delta) \circ f_z \\
&= \varphi_M(z + 2\epsilon + \delta, z + 2\epsilon + 2\delta) \circ \varphi_M(fl_N(z + \epsilon) + \epsilon + \delta, z + 2\epsilon + \delta) \\
&\quad \circ g_{fl_N(z+\epsilon)} \circ \varphi_N^{-1}(fl_N(z + \epsilon), z + \epsilon) \circ \varphi_N^{-1}(z + \epsilon, z + \epsilon + \delta) \circ f_z \\
&= \varphi_M(z + 2\epsilon + \delta, z + 2\epsilon + 2\delta) \circ g_{z+\epsilon} \circ \varphi_N(fl_N(z + \epsilon), z + \epsilon) \\
&\quad \circ \varphi_N^{-1}(fl_N(z + \epsilon), z + \epsilon) \circ \varphi_N^{-1}(z + \epsilon, z + \epsilon + \delta) \circ f_z \\
&= g_{z+\epsilon+\delta} \circ \varphi_N(z + \epsilon, z + \epsilon + \delta) \circ \varphi_N^{-1}(z + \epsilon, z + \epsilon + \delta) \circ f_z \\
&= g_{z+\epsilon+\delta} \circ f_z \\
&= \varphi(z, z + 2\epsilon + 2\delta)
\end{aligned}$$

as desired.

Finally, for arbitrary $z \in \mathbb{R}^n$ such that $fl_M(z) \notin \mathbb{R}^n$, we have, using that \tilde{g} is a

morphism,

$$\begin{aligned}
\tilde{g}_{z+\epsilon} \circ \tilde{f}_z &= \varphi_M(fl_N(z + \epsilon) + \epsilon, z + 2\epsilon) \circ \tilde{g}_{fl_N(z+\epsilon)} \\
&\quad \circ \varphi_N^{-1}(fl_N(z + \epsilon), z + \epsilon) \circ \varphi_N(fl_M(z) + \epsilon, z + \epsilon) \circ \tilde{f}_{fl_M(z)} \circ \varphi_M^{-1}(fl_M(z), z) \\
&= \tilde{g}_{z+\epsilon} \circ \varphi_N(fl_N(z + \epsilon), z + \epsilon) \circ \varphi_N^{-1}(fl_N(z + \epsilon), z + \epsilon) \\
&\quad \circ \varphi_N(fl_M(z) + \epsilon, z + \epsilon) \circ \tilde{f}_{fl_M(z)} \circ \varphi_M^{-1}(fl_M(z), z) \\
&= \tilde{g}_{z+\epsilon} \circ \varphi_N(fl_M(z) + \epsilon, z + \epsilon) \circ \tilde{f}_{fl_M(z)} \circ \varphi_M^{-1}(fl_M(z), z) \\
&= \varphi_M(fl_M(z) + 2\epsilon, z + 2\epsilon) \circ \tilde{g}_{fl_M(z)+\epsilon} \circ \tilde{f}_{fl_M(z)} \circ \varphi_M^{-1}(fl_M(z), z) \\
&= \varphi_M(fl_M(z) + 2\epsilon, z + 2\epsilon) \circ \varphi_M(fl_M(z), fl_M(z) + 2\epsilon) \circ \varphi_M^{-1}(fl_M(z), z) \\
&= \varphi_M(z, z + 2\epsilon)
\end{aligned}$$

as we wanted. \square

This completes the proof of Theorem 2.5.1. \square

Remark 2.5.8. As noted in Remark 2.3.3, the notion of a well behaved persistence module admits a generalization to the multi-dimensional setting. An interesting question is whether Theorem 2.5.1 generalizes to well behaved multidimensional persistence modules; if it does, then we obtain corresponding generalizations of Corollaries 2.5.2 and 2.5.3. Our proof of Theorem 2.5.1 does not generalize directly.

2.6 An Extrinsic Characterization of Interleaved Pairs of Multidimensional Persistence Modules

In this section, we introduce (J_1, J_2) -interleavings of pairs of B_n -persistence modules, a generalization of ϵ -interleavings of pairs of B_n -persistence modules. These generalized interleavings serve as a convenient language for expressing nuanced relationships between B_n -persistence modules which arise in our study of topological inference using Rips multifiltrations in Section 4.4.2.

After defining generalized interleavings, we present Theorem 2.6.4, our “extrinsic” characterization of (J_1, J_2) -interleaved pairs of persistence modules; as noted in the

introduction, this result expresses transparently the sense in which (J_1, J_2) -interleaved persistence modules are algebraically similar. In particular, Theorem 2.6.4 gives an algebraically transparent characterization of ϵ -interleaved pairs of modules, which we write down as Corollary 2.6.5. This characterization induces in an obvious way a corresponding characterization of the interleaving distance. It is also the most important step in our proof of our main optimality result Corollary 2.10.2.

2.6.1 (J_1, J_2) -Interleavings

In Section 2.1.4, we introduced shift functors and transition morphisms on B_n -persistence modules. To prepare for the definition of (J_1, J_2) -interleavings, we define generalized shift functors and generalized transition morphisms.

Generalized Shift Functors

We say that a bijection $J : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is **order-preserving** if $\forall a, b \in \mathbb{R}^n$, $a \leq b$ iff $J(a) \leq J(b)$. For any order-preserving map $J : \mathbb{R}^n \rightarrow \mathbb{R}^n$, we define the **generalized shift functor** $(\cdot)(J) : B_n\text{-mod} \rightarrow B_n\text{-mod}$, as follows:

1. Action of $(\cdot)(J)$ on objects: For $M \in \text{obj}(B_n\text{-mod})$, we define $M(J)$ by taking $M(J)_a = M_{J(a)}$ for all $a \in \mathbb{R}^n$. For all $a \leq b \in \mathbb{R}^n$ we take the transition map $\varphi_{M(J)}(a, b)$ to be the map $\varphi_M(J(a), J(b))$.
2. Action of $(\cdot)(J)$ on morphisms: For $M, N \in \text{obj}(B_n\text{-mod})$ and $f \in \text{hom}(M, N)$, we define $f(J) : M(J) \rightarrow N(J)$ to be the homomorphism for which $f(J)_a = f_{J(a)}$ for all $a \in \mathbb{R}^n$.

For $u \in \mathbb{R}^n$, let $J_u : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the map defined by $J_u(a) = a + u$, and for $\epsilon \in \mathbb{R}$, let J_ϵ denote $J_{\epsilon n}$. $(\cdot)(J_u)$ is equal to the shift functor $(\cdot)(u)$ introduced in Section 2.1.4 and $(\cdot)(J_\epsilon)$ is equal to the shift functor $(\cdot)(\epsilon)$ introduced in the same section.

As in the case of ordinary shift functors, for a morphism f and J an order-preserving map, we will sometimes abuse notation and write $f(J)$ simply as f .

Note the following contravariance property of generalized shift functors: If J_1 and J_2 are order-preserving, then

$$(\cdot)(J_2 \circ J_1) = (\cdot)(J_1) \circ (\cdot)(J_2).$$

Generalized Transition Homomorphisms of B_n -persistence Modules

Say an order-preserving map $J : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is **increasing** if $J(a) \geq a$ for all $a \in \mathbb{R}^n$. For a B_n -persistence module M and J an increasing map, let $S(M, J) : M \rightarrow M(J)$, the **J -transition homomorphism**, be the homomorphism whose restriction to M_a is the linear map $\varphi_M(a, J(a))$ for all $a \in \mathbb{R}^n$. Note that for $\epsilon \in \mathbb{R}_{\geq 0}$, $S(M, \epsilon) = S(M, J_\epsilon)$, where $S(M, \epsilon)$ is as defined in Section 2.1.4.

The following lemma gives three easy and useful identities for generalized transition morphisms. Of particular note is Lemma 2.6.1(i).

Lemma 2.6.1.

(i) For any $f : M \rightarrow N \in \text{hom}(B_n\text{-mod})$ and any $J : \mathbb{R}^n \rightarrow \mathbb{R}^n$ increasing,

$$S(N, J) \circ f = f(J) \circ S(M, J).$$

(ii) For any J, J' increasing and B_n -persistence module M ,

$$S(M, J')(J) \circ S(M, J) = S(M, J' \circ J).$$

(iii) For any J, J' increasing and B_n -persistence module M ,

$$S(M, J')(J) = S(M(J), J^{-1} \circ J' \circ J).$$

Proof. We'll prove (iii) and leave the proofs of (i) and (ii) to the reader. For any

$$a \in \mathbb{R}^n,$$

$$\begin{aligned} S(M, J')(J)_a &= S(M, J')_{J(a)} \\ &= \varphi_M(J(a), J' \circ J(a)) = \varphi_M(J(a), J \circ J^{-1} \circ J' \circ J(a)) \\ &= \varphi_{M(J)}(a, J^{-1} \circ J' \circ J(a)) = S(M(J), J^{-1} \circ J' \circ J)_a. \end{aligned} \quad \square$$

(J_1, J_2) -Interleavings of B_n -Persistence Modules

For $J_1, J_2 : \mathbb{R}^n \rightarrow \mathbb{R}^n$ increasing, we say an ordered pair of B_n -persistence modules (M, N) is (J_1, J_2) -**interleaved** if there exist homomorphisms $f : M \rightarrow N(J_1)$ and $g : N \rightarrow M(J_2)$ such that

$$\begin{aligned} g(J_1) \circ f &= S(M, J_2 \circ J_1) \text{ and} \\ f(J_2) \circ g &= S(N, J_1 \circ J_2); \end{aligned}$$

we say that (f, g) is a pair of (J_1, J_2) -**interleaving homomorphisms** for (M, N) .

Note the asymmetry of (J_1, J_2) -interleavings. If (M, N) is (J_1, J_2) -interleaved, it needn't be true that (M, N) is (J_2, J_1) -interleaved. It is however true that (N, M) is (J_2, J_1) -interleaved.

Remark 2.6.2. A note on terminology: When a pair (M, N) of B_n -persistence modules is (J_1, J_2) -interleaved, we will often say simply that M, N are (J_1, J_2) -interleaved, or that M and N are (J_1, J_2) -interleaved. Similarly, when (f, g) is a pair of (J_1, J_2) -**interleaving homomorphisms** for (M, N) , we will often say simply that f, g are (J_1, J_2) -interleaving homomorphisms for M, N .

Note that M and N are ϵ -interleaved if and only if the pair (M, N) is (J_ϵ, J_ϵ) -interleaved.

2.6.2 A Characterization of (J_1, J_2) -interleaved Pairs of Modules

We now present our characterization of pairs of (J_1, J_2) -interleaved B_n -persistence modules.

Notation

To state the theorem we need some notation.

If $G = (\bar{G}, \iota_G)$ an n -graded set, and $J : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is an order-preserving map, let $G(J)$ denote the n -graded set (\bar{G}, ι'_G) , where $\iota'_G(y) = J^{-1}(\iota_G(y))$. Note that if J_1 and J_2 are order-preserving, then $G(J_2 \circ J_1) = (G(J_2))(J_1)$.

Similarly, if M is a B_n -persistence module and $Q \subset M$ is a homogeneous subset, let $Q(J) \subset M(J)$ denote the image of Q under the bijection between M and $M(J)$ induced by the identification of each summand $M(J)_a$ with $M_{J(a)}$. Note that if J_1 and J_2 are order-preserving, then $Q(J_2 \circ J_1) = (Q(J_2))(J_1)$.

Remark 2.6.3. For any n -graded set G and $J : \mathbb{R}^n \rightarrow \mathbb{R}^n$ an increasing map, the homomorphism $S(\langle G(J^{-1}) \rangle, J) : \langle G(J^{-1}) \rangle \rightarrow \langle G \rangle$ is injective, and so gives an identification of $\langle G(J^{-1}) \rangle$ with a submodule of $\langle G \rangle$. More generally, if G_1 and G_2 are n -graded sets, we obtain in the obvious way an identification of $\langle G_1, G_2(J^{-1}) \rangle$ with a submodule of $\langle G_1, G_2 \rangle$. In particular, for any $\epsilon \geq 0$, we obtain an identification of $\langle G(-\epsilon) \rangle$ with a submodule of $\langle G \rangle$ and, more generally, of $\langle G_1, G_2(-\epsilon) \rangle$ with a submodule of $\langle G_1, G_2 \rangle$.

Theorem 2.6.4. *Let M and N be B_n -persistence modules. For any $J_1, J_2 : \mathbb{R}^n \rightarrow \mathbb{R}^n$ increasing, (M, N) is (J_1, J_2) -interleaved if and only if there exist n -graded sets $\mathcal{W}_1, \mathcal{W}_2$ and homogeneous sets $\mathcal{Y}_1, \mathcal{Y}_2 \subset \langle \mathcal{W}_1, \mathcal{W}_2 \rangle$ such that $\mathcal{Y}_1 \in \langle \mathcal{W}_1, \mathcal{W}_2(J_2^{-1}) \rangle$, $\mathcal{Y}_2 \in \langle \mathcal{W}_1(J_1^{-1}), \mathcal{W}_2 \rangle$, and*

$$\begin{aligned} M &\cong \langle \mathcal{W}_1, \mathcal{W}_2(J_2^{-1}) | \mathcal{Y}_1, \mathcal{Y}_2(J_2^{-1}) \rangle, \\ N &\cong \langle \mathcal{W}_1(J_1^{-1}), \mathcal{W}_2 | \mathcal{Y}_1(J_1^{-1}), \mathcal{Y}_2 \rangle. \end{aligned}$$

If M and N are finitely presented, then $\mathcal{W}_1, \mathcal{W}_2, \mathcal{Y}_1, \mathcal{Y}_2$ can be taken to be finite.

The following corollary is immediate.

Corollary 2.6.5. *Let M and N be B_n -persistence modules. For any $\epsilon \in \mathbb{R}_{\geq 0}$, M and N are ϵ -interleaved if and only if there exist n -graded sets $\mathcal{W}_1, \mathcal{W}_2$ and homogeneous*

sets $\mathcal{Y}_1, \mathcal{Y}_2 \subset \langle \mathcal{W}_1, \mathcal{W}_2 \rangle$ such that $\mathcal{Y}_1 \in \langle \mathcal{W}_1, \mathcal{W}_2(-\epsilon) \rangle$, $\mathcal{Y}_2 \in \langle \mathcal{W}_1(-\epsilon), \mathcal{W}_2 \rangle$,

$$\begin{aligned} M &\cong \langle \mathcal{W}_1, \mathcal{W}_2(-\epsilon) | \mathcal{Y}_1, \mathcal{Y}_2(-\epsilon) \rangle \\ N &\cong \langle \mathcal{W}_1(-\epsilon), \mathcal{W}_2 | \mathcal{Y}_1(-\epsilon), \mathcal{Y}_2 \rangle. \end{aligned}$$

If M and N are finitely presented, then $\mathcal{W}_1, \mathcal{W}_2, \mathcal{Y}_1, \mathcal{Y}_2$ can be taken to be finite.

Proof of Theorem 2.6.4. It's easy to see that if there exist n -graded sets $\mathcal{W}_1, \mathcal{W}_2$ and sets $\mathcal{Y}_1, \mathcal{Y}_2 \subset \langle \mathcal{W}_1, \mathcal{W}_2 \rangle$ as in the statement of the theorem then M, N are (J_1, J_2) -interleaved.

To prove the converse, we will first express M and N as isomorphic copies of two quotient persistent modules which are, in a suitable sense, algebraically similar. We will then construct the desired presentations by lifting the structure of these quotients to free covers.

The construction by which we express M and N as quotients is a generalization to (J_1, J_2) -interleaved pairs of B_n -persistence modules of one introduced for ϵ -interleaved pairs of B_1 -persistence modules in the proof of [8, Lemma 4.6].

Let f, g be (J_1, J_2) -interleaving homomorphisms for M, N .

Lemma 2.6.6. Let $\gamma_1 : M(J_1^{-1} \circ J_2^{-1}) \rightarrow M \oplus N(J_2^{-1})$ be given by $\gamma_1(y) = (S(M(J_1^{-1} \circ J_2^{-1}), J_2 \circ J_1)(y), -f(y))$. Let $\gamma_2 : N(J_2^{-1}) \rightarrow M \oplus N(J_2^{-1})$ be given by $\gamma_2(y) = (-g(y), y)$. Let $R \subset M \oplus N(J_2^{-1})$ be the submodule generated by $\text{im}(\gamma_1) \cup \text{im}(\gamma_2)$. Then

$$M \cong (M \oplus N(J_2^{-1})) / R.$$

Proof. Let $\iota : M \rightarrow M \oplus N(J_2^{-1})$ denote the inclusion, and let $\zeta : M \oplus N(J_2^{-1}) \rightarrow M \oplus N(J_2^{-1}) / R$ denote the quotient. We'll show that $\zeta \circ \iota$ is an isomorphism. For any $(y_M, y_N) \in M \oplus N(J_2^{-1})$, $(-g(y_N), y_N) \in \text{im}(\gamma_2) \subset R$, so $\zeta \circ \iota(g(y_N)) = (0, y_N) + R$. Therefore $\zeta \circ \iota(g(y_N) + y_M) = (y_M, y_N) + R$. Hence $\zeta \circ \iota$ is surjective.

$\zeta \circ \iota$ is injective iff $\iota(M) \cap R = 0$. It's clear that $\iota(M) \cap \text{im}(\gamma_2) = 0$. Thus to show that $\zeta \circ \iota$ is injective it's enough to show that $\text{im}(\gamma_1) \subset \text{im}(\gamma_2)$. If $y \in M(J_1^{-1} \circ J_2^{-1})$, then since $S(M(J_1^{-1} \circ J_2^{-1}), J_2 \circ J_1)(y) = g \circ f(y)$, $(S(M(J_1^{-1} \circ J_2^{-1}), J_2 \circ J_1)(y), -f(y)) = (g \circ f(y), -f(y)) = \gamma_2(-f(y))$. Thus $\text{im}(\gamma_1) \subset \text{im}(\gamma_2)$ and so $\zeta \circ \iota$ is injective.

Thus $\zeta \circ \iota$ is an isomorphism. \square

Now let $\langle G_M | R_M \rangle$ be a presentation for M and let $\langle G_N | R_N \rangle$ be a presentation for N . Without loss of generality we may assume $M = \langle G_M \rangle / \langle R_M \rangle$ and $N = \langle G_N \rangle / \langle R_N \rangle$. Let $\rho_M : \langle G_M \rangle \rightarrow M$, $\rho_N : \langle G_N \rangle \rightarrow N$ denote the quotient maps. Then $(\langle G_M \rangle, \rho_M)$ and $(\langle G_N \rangle, \rho_N)$ are free covers for M and N .

Let $\tilde{f} : \langle G_M \rangle \rightarrow \langle G_N(J_1) \rangle$ be a lift of f and let $\tilde{g} : \langle G_N \rangle \rightarrow \langle G_M(J_2) \rangle$ be a lift of g .

Let $R_{M,N} = \{y - \tilde{f}(y)\}_{y \in G_M(J_1^{-1})}$ and let $R_{N,M} = \{y - \tilde{g}(y)\}_{y \in G_N(J_2^{-1})}$. Note that $R_{M,N}$ is a homogeneous subset of $\langle G_M(J_1^{-1}), G_N \rangle$ and $R_{N,M}$ is a homogeneous subset of $\langle G_M, G_N(J_2^{-1}) \rangle$.

Let

$$\begin{aligned} P_M &= \langle G_M, G_N(J_2^{-1}) | R_M, R_N(J_2^{-1}), R_{M,N}(J_2^{-1}), R_{N,M} \rangle, \\ P_N &= \langle G_M(J_1^{-1}), G_N | R_M(J_1^{-1}), R_N, R_{M,N}, R_{N,M}(J_1^{-1}) \rangle. \end{aligned}$$

$R_{M,N}(J_2^{-1})$ lies in $\langle G_M(J_1^{-1} \circ J_2^{-1}), G_N(J_2^{-1}) \rangle$. By Remark 2.6.3, the map $S(G_M(J_1^{-1} \circ J_2^{-1}), J_2 \circ J_1)$ identifies $\langle G_M(J_1^{-1} \circ J_2^{-1}) \rangle$ with a subset of $\langle G_M \rangle$. Thus P_M is well defined. By an analogous observation, P_N is also well defined.

We claim that P_M is a presentation for M and P_N is a presentation for N . We'll prove that P_M is a presentation for M ; The proof that P_N is a presentation for N is identical.

Let

$$\begin{aligned} F &= \langle G_M, G_N(J_2^{-1}) \rangle, \\ K &= \langle R_M, R_N(J_2^{-1}), R_{M,N}(J_2^{-1}), R_{N,M} \rangle \\ K' &= \langle R_M, R_N(J_2^{-1}) \rangle. \end{aligned}$$

Let $p : F \rightarrow F/K'$ denote the quotient map. Clearly, we may identify F/K' with $M \oplus N(J_2^{-1})$ and p with (p_M, p_N) . We'll check that p maps $\langle R_{M,N}(J_2^{-1}) \rangle$ surjectively to $\text{im}(\gamma_1)$ and $\langle R_{N,M} \rangle$ surjectively to $\text{im}(\gamma_2)$, so that under the identification of F/K' with $M \oplus N(J_2^{-1})$, $K/K' = R$. Given this, it follows that P_M is a presentation for M .

by Lemma 2.6.6 and the third isomorphism theorem for modules [18].

We first check that $\langle p(R_{M,N}(J_2^{-1})) \rangle = \text{im}(\gamma_1)$. Viewing $R_{M,N}(J_2^{-1})$ as a subset of $\langle G_M, G_N(J_2^{-1}) \rangle$, $R_{M,N}(J_2^{-1}) = \{S(\langle G_M(J_1^{-1} \circ J_2^{-1}) \rangle, J_2 \circ J_1)(y) - \tilde{f}(y)\}_{y \in G_M(J_1^{-1} \circ J_2^{-1})}$. $S(\langle G_M(J_1^{-1} \circ J_2^{-1}) \rangle, J_2 \circ J_1)$ is a lift of $S(M(J_1^{-1} \circ J_2^{-1}), J_2 \circ J_1)$ and \tilde{f} is a lift of f , so for any $y \in G_M(J_1^{-1} \circ J_2^{-1})$,

$$\begin{aligned} & p(S(\langle G_M(J_1^{-1} \circ J_2^{-1}) \rangle, J_2 \circ J_1)(y) - \tilde{f}(y)) \\ &= (S(M(J_1^{-1} \circ J_2^{-1}), J_2 \circ J_1)(\rho_M(y)), -f(\rho_M(y))) \\ &= \gamma_1(\rho_M(y)). \end{aligned}$$

Thus $p(R_{M,N}(J_2^{-1})) \subset \text{im}(\gamma_1)$. Since G_M generates $\langle G_M \rangle$ and ρ_M is surjective, we have that $p(\langle R_{M,N}(J_2^{-1}) \rangle) = \text{im}(\gamma_1)$.

The check that $\langle p(R_{N,M}) \rangle = \text{im}(\gamma_2)$ is similar to the above verification that $\langle p(R_{M,N}(J_2^{-1})) \rangle = \text{im}(\gamma_1)$, but simpler. $R_{N,M} = \{y - \tilde{g}(y)\}_{y \in G_N(J_2^{-1})}$. \tilde{g} is a lift of g so for any $y \in G_N(J_2^{-1})$,

$$p(y - \tilde{g}(y)) = (-g(\rho_N(y)), \rho_N(y)) = \gamma_2(\rho_N(y)).$$

Thus $p(R_{N,M}) \subset \text{im}(\gamma_2)$. Since G_N generates $\langle G_N \rangle$ and ρ_N is surjective, we have that $p(\langle R_{N,M} \rangle) = \text{im}(\gamma_2)$.

This completes the verification that P_M is a presentation for M .

Now, taking $\mathcal{W}_1 = G_M$, $\mathcal{W}_2 = G_N$, $\mathcal{Y}_1 = R_M \cup R_{N,M}$, and $\mathcal{Y}_2 = R_N \cup R_{M,N}$ gives the first statement of Theorem 2.6.5. If M and N are finitely presented then $G_M, G_N, R_M, R_N, R_{M,N}$, and $R_{N,M}$ can all be taken to be finite; the second statement of Theorem 2.6.5 follows. \square

2.7 Geometric Preliminaries

In this third section of preliminaries, we present preliminaries of a geometric and topological nature which we will need in the remainder of the thesis.

One organizational note before proceeding: We define multidimensional filtrations and multidimensional persistent homology here, in Section 2.7.3. In our study of topological inference in Chapter 4, we will need a slightly more general definition of multidimensional filtrations and a correspondingly more general definition of multidimensional persistent homology. However, since these more general definitions are not needed in the remainder of this chapter, we will defer their introduction to Section 3.1.

2.7.1 CW-complexes and Cellular homology

Our proof of the optimality of the interleaving distance in Section 2.10 will involve the construction of CW-complexes and the computation of their cellular homology. We now briefly review finite dimensional CW-complexes and cellular homology.

Definition of a Finite-dimensional CW-complex

A CW-complex is a topological space X together with some additional data of *attaching maps* specifying how X is assembled as the union of open disks of various dimensions. We quote the procedural definition of a finite-dimensional CW-complex given in [30].

Let D^i denote the unit disk in \mathbb{R}^i ; for α contained in some indexing set (which will often be implicit in our notation) let D_α^i be a copy of D^i . When i is clear from context, we will sometimes denote D_α^i simply as D_α .

A finite-dimensional CW-complex is a space X constructed in the following way:

1. Start with a discrete set X^0 , the 0-cells of X .
2. Inductively, form the i -skeleton of X^i from X^{i-1} by attaching i -cells e_α^i via maps $\sigma_\alpha : S^{i-1} \rightarrow X^{i-1}$. This means that X^i is the quotient space of $X^{i-1} \coprod_\alpha D_\alpha^i$ under the identifications $x \sim \sigma_\alpha(x)$ for $x \in \delta D_\alpha^i$. The cell e_α^i is the homeomorphic image of $D_\alpha^i - \delta D_\alpha^i$ under the quotient map.
3. $X = X^r$ for some r . We call the smallest such r the *dimension* of X .

The *characteristic map* of the cell e_α^i is the map $\Phi_\alpha : D_\alpha^i \rightarrow X$ which is the composition

$D_\alpha^i \hookrightarrow X^{i-1} \amalg_\alpha D_\alpha^i \rightarrow X^i \hookrightarrow X$, where the middle map is the quotient map defining X^i .

A *subcomplex* of a CW-complex X is a closed subspace A of X which is a union of the cells of X ; those cells contained in A are taken to have the same attaching maps as they do in X .

Cellular Homology

We mention only what we need about cellular homology to prove our optimality result Theorem 2.10.1. For a more complete discussion and proofs of the results stated here, see e.g. [30] or [2].

For $i \in \mathbb{Z}_{\geq 0}$, we'll let H_i denote the i^{th} singular homology functor with coefficients in the field k .

For X a CW-complex and $i \in \mathbb{N}$, let $d_i^X : H_i(X^i, X^{i-1}) \rightarrow H_{i-1}(X^{i-1}, X^{i-2})$ denote the map induced by the boundary map in the long exact sequence of the pair (X^i, X^{i-1}) . It can be checked that the d_i^X give

$$\dots \xrightarrow{\delta_{i+1}^X} H_i(X^i, X^{i-1}) \xrightarrow{\delta_i^X} H_{i-1}(X^{i-1}, X^{i-2}) \xrightarrow{\delta_{i-1}^X} \dots \xrightarrow{\delta_1^X} H_0(X^0) \rightarrow 0$$

the structure of a chain complex, and that the i^{th} homology vector space of this chain complex, denoted $H_i^{CW}(X)$, is isomorphic to $H_i(X)$.

It can be shown that a choice of generator for $H_i(D^i, S^{i-1}) \cong \mathbb{Z}$ induces a choice of basis for $H_i(X^i, X^{i-1})$ whose elements correspond bijectively to the i -cells of X . We now fix a choice of generator $H_i(D^i, S^{i-1})$ for each $i \in \mathbb{N}$.¹ We can then think of $H_i(X^i, X^{i-1})$ as the k -vector space generated by the i -cells of X .

It follows from the equality $H_0^{CW}(X) = H_0(X)$ that in the case that X has a single 0-cell, $d_1^X = 0$.

For $i > 1$, the *cellular boundary formula* gives an explicit expression for d_i^X . To prepare for the formula, we note first that for $i \in \mathbb{N}$, the choice of generator for $H_i(D^i, S^{i-1})$ induces a choice of generator a_i for $H_i(D^i/S^{i-1})$ via the quotient map $D^i \rightarrow D^i/S^{i-1}$. Also, the choice of generator for $H_{i+1}(D^{i+1}, S^i)$ induces a choice of

¹Such a choice is induced e.g. by the standard orientation on D^i .

generator b_i for $H_i(S^i)$ via the boundary map in the long exact sequence of the pair (D^{i+1}, S^i) . For each $i \in \mathbb{N}$, choose $\rho^i : D^i/S^{i-1} \rightarrow S^i$ to be any homeomorphism such that $\rho_*^i : H_i(D^i/S^{i-1}) \rightarrow H_i(S^i)$ sends a_i to b_i .

For $i \in \mathbb{N}$ and an i -cell e_β^i of X , let $(e_\beta^i)^c$ denote the complement of e_β^i in X^i , and let $q_\beta : X^i \rightarrow X^i/(e_\beta^i)^c$ denote the quotient map. ρ^i and Φ_β induce an identification of $q_\beta(X^i)$ with S^i .

By a compactness argument [30, Section A.1], for any i -cell e_α^i the image of the attaching map σ_α of e_α^i meets only finitely many cells.

For $i > 1$, the cellular boundary formula states that

$$\delta_i^X(e_\alpha^i) = \sum_{\text{im}(\sigma_\alpha) \cup e_\beta^{i-1} \neq \emptyset} \deg(q_\beta \circ \sigma_\alpha) e_\beta^{i-1}.$$

Here, for any map $f : S^{i-1} \rightarrow S^{i-1}$, $\deg(f)$ denotes the field element $a \in k$ such that $f_* : H_{i-1}(S^{i-1}) \rightarrow H_{i-1}(S^{i-1})$ is multiplication by a .

We can endow the set of CW-complexes with the structure of a category by taking $\text{hom}(X, Y)$ for CW-complexes X, Y to be the set of continuous maps $f : X \rightarrow Y$ such that $f(X^i) \subset Y^i$ for all i . We call maps $f \in \text{hom}(X, Y)$ *cellular maps*. It can be shown that a cellular map f induces a map $H_i^{CW}(f) : H_i^{CW}(X) \rightarrow H_i^{CW}(Y)$ in such a way that H_i^{CW} becomes a functor.

Further, there exists a natural isomorphism [33] $\kappa : H_i^{CW} \rightarrow \bar{H}_i$, where \bar{H}_i is the restriction of H_i to the category of CW-complexes.

2.7.2 Multidimensional Filtrations

Fix $n \in \mathbb{N}$.

Define an **n -filtration** X to be a collection of topological spaces $\{X_a\}_{a \in \mathbb{R}^n}$, together with a collection of continuous maps $\{\phi_X(a, b) : X_a \rightarrow X_b\}_{a \leq b \in \mathbb{R}^n}$ such that if $a \leq b \leq c \in \mathbb{R}^n$ then

$$\phi_X(b, c) \circ \phi_X(a, b) = \phi_X(a, c).$$

Given two n -filtrations X and Y , we define a morphism f from X to Y to be a collection of continuous functions $\{f_a\}_{a \in \mathbb{R}^n} : X_a \rightarrow Y_a$ such that for all $a \leq b \in \mathbb{R}^n$,

$f_b \circ \phi_X(a, b) = \phi_Y(a, b) \circ f_a$. This definition of morphism gives the n -filtrations the structure of a category. Let $n\text{-filt}$ denote this category.

Said differently, $n\text{-filt}$ is the category of diagrams of topological spaces indexed by the poset category $(\mathbb{R}^n, <)$.

We can define the category of **cellular n -filtrations** analogously, as the category of diagrams of *CW*-complexes indexed by the poset category $(\mathbb{R}^n, <)$. In the same way, we can define the category of simplicial n -filtrations.

2.7.3 Multidimensional Persistent Homology

The multidimensional persistent homology functor H_i is a generalization of the ordinary homology functor with field coefficients to the setting where the source is an n -filtration and the target is a B_n -persistence module.

Singular Multidimensional Persistent Homology

For a topological space X and $j \in \mathbb{Z}_{\geq 0}$, let $C_j(X)$ denote the j^{th} singular chain module of X , with coefficients in k . For X, Y topological spaces and $f : X \rightarrow Y$ a continuous map, let $f^\# : C_j(X) \rightarrow C_j(Y)$ denote the map induced by f .

For X an n -filtration, define $C_j(X)$, the j^{th} singular chain module of X , as the B_n -persistence module for which $C_j(X)_u = C_j(X_u)$ for all $u \in \mathbb{R}^n$ and for which $\varphi_{C_j(X)}(u, v) = \phi_X^\#(u, v)$. Note that for any $j \in \mathbb{Z}_{\geq 0}$, the collection of boundary maps $\{\delta_j : C_j(X_u) \rightarrow C_{j-1}(X_u)\}_{u \in \mathbb{R}^n}$ induces a boundary map $\delta_j : C_j(X) \rightarrow C_{j-1}(X)$. These boundary maps give $\{C_j(X)\}_{j \geq 0}$ the structure of a chain complex. We define the $H_j(X)$, the j^{th} persistent homology module of X , to be the j^{th} homology module of this complex. For X and Y two n -filtrations, a morphism $f \in \text{hom}(X, Y)$ induces in the obvious way a morphism $H_j(f) : H_j(X) \rightarrow H_j(Y)$, making $H_j : n\text{-filt} \rightarrow B_n\text{-mod}$ a functor.

Cellular Multidimensional Persistent Homology

A construction analogous to the one above, with cellular chain complexes used in place of singular chain complexes, yields a definition of the cellular multidimensional

persistent homology of cellular n -filtrations. For a cellular filtration X , let $C_j^{CW}(X)$ denote the j^{th} cellular chain module of X , let $\delta_j^{CW} : C_j^{CW}(X) \rightarrow C_{j-1}^{CW}(X)$ denote the j^{th} cellular chain map of X , and let $H_j^{CW}(X)$ denote the j^{th} cellular multidimensional persistence module of X .

Remark 2.7.1. It follows from the naturality of the isomorphisms between singular and cellular homology that the singular and cellular multidimensional persistent homology modules of a cellular n -filtration are isomorphic.

2.7.4 Functors from Geometric Categories to Categories of n -filtrations

In applications of persistent homology one typically has some geometric² category of interest and a functor F from that category to n -filt for some $n \in \mathbb{N}$; one then studies and works with the functor $H_i \circ F$. These composite functors are also generally referred to as i^{th} persistent homology functors. There thus are a number of different such i^{th} persistent homology functors with different sources, each determined by a different choice of the functor F .

We next define several functors $F : C \rightarrow n\text{-filt}$, where C is some geometric category. We introduced special cases of some of the definitions presented here in Section 1.5.1. We'll use these functors to formulate and prove our inference results in Chapter 4.

In Section 2.8 we will prove stability results for $H_i \circ F$ for each of functors F introduced here.

In the following examples, we omit the specification of the action of these functors on morphisms. This should be clear from context.

Recall from Section 2.1.1 that for $a = (a_1, \dots, a_n) \in \hat{\mathbb{R}}^n$ and $b \in \hat{\mathbb{R}}$, we let $(a, b) = (a_1, \dots, a_n, b) \in \hat{\mathbb{R}}^{n+1}$.

Example 2.7.2. Sublevelset Filtrations

Let C^S be the category defined as follows:

²We are using the word “geometric” here in an informal (and rather broad) sense.

1. Objects of C^S are pairs (X, f) , where X is a topological space and $f : X \rightarrow \mathbb{R}^n$ is a function.
2. If $(X^1, f^1), (X^2, f^2) \in \text{obj}(C^S)$, then we define $\text{hom}_{C^S}((X^1, f^1), (X^2, f^2))$ to be the set of continuous functions $\gamma : X^1 \rightarrow X^2$ such that $f^1(x) \geq f^2(\gamma(x))$ for all $x \in X^1$.

Define the functor $F^S : C^S \rightarrow n\text{-filt}$ on objects by taking

$$F^S(X, f)_a = f_a$$

for all $a \in \mathbb{R}^n$ (and taking the map $\phi_{F^S(X, f)}(a, b)$ to be the inclusion for all $a \leq b \in \mathbb{R}^n$.) We refer to F^S as the **sublevelset filtration functor**, to $F^S(X, f)$ as the **sublevelset filtration of f** , and to $F^S(X, -f)$ as the **superlevelset filtration of f** .

Example 2.7.3. Sublevelset-Offset Filtrations

We define two variants of the sublevelset-offset filtration functor, the *closed* and *open* variants. It is the closed variant that is of greater interest to us and that we use in the statement of our theorems; we define the open variant purely as a technical tool for use in the proofs of the results of Chapter 4.

Let C^{SO} be the category defined as follows:

1. Objects of C^{SO} are quadruples (X, Y, d, f) , where (Y, d) is a metric space, $X \subset Y$, and $f : X \rightarrow \mathbb{R}^n$ is a function.
2. If $(X^1, Y^1, d^1, f^1), (X^2, Y^2, d^2, f^2) \in \text{obj}(C^{SO})$, then we define

$$\text{hom}_{C^{SO}}((X^1, Y^1, d^1, f^1), (X^2, Y^2, d^2, f^2))$$

to be the set of continuous functions $\gamma : Y^1 \rightarrow Y^2$ such that $\gamma(X^1) \subset X^2$, $f^1(x) \geq f^2(\gamma(x))$ for all $x \in X^1$, and $d^1(y_1, y_2) \geq d^2(\gamma(y_1), \gamma(y_2))$ for all $y_1, y_2 \in Y^1$.

For $(X, X, d, f) \in C^{SO}$, we will often write (X, d, f) as shorthand for (X, X, d, f) .

Define the functor $F^{SO} : C^{SO} \rightarrow (n+1)\text{-filt}$ on objects by taking

$$F^{SO}(X, Y, d, f)_{(a,b)} = \{y \in Y \mid d(y, f_a) \leq b\}$$

for all $(a, b) \in \mathbb{R}^n \times \mathbb{R}$. We refer to F^{SO} as the **(closed) sublevelset-offset filtration functor**.

Similarly, define the functor $F^{SO-Op} : C^{SO} \rightarrow (n+1)\text{-filt}$ on objects by taking

$$F^{SO-Op}(X, Y, d, f)_{(a,b)} = \{y \in Y \mid d(y, f_a) < b\}$$

for all $(a, b) \in \mathbb{R}^n \times \mathbb{R}$. We refer to F^{SO-Op} as the **open sublevelset-offset filtration functor**.

Example 2.7.4. Sublevelset-Čech Filtrations

As we did for sublevelset-offset filtrations, we define closed and open variants of the sublevelset-Čech filtration functor.

Let $C^{S\check{C}e}$ denote the full subcategory of F^{SO} whose objects are the quadruples (X, Y, d, f) with X finite.

Recall from Section 1.5.1 that given a metric space (Y, d) , a subset $X \subset Y$, and $b \in \mathbb{R}$, we let $\check{\text{Cech}}(X, Y, d, b)$, the *(closed) Čech complex* of (X, d) with parameter b , be the abstract simplicial complex with vertex set X such that for $l \geq 2$ and $x_1, x_2, \dots, x_l \in X$, $\check{\text{Cech}}(X, Y, d, b)$ contains the $(l-1)$ -simplex $[x_1, \dots, x_l]$ iff there is a point $y \in Y$ such that $d(y, x_i) \leq b$ for $1 \leq i \leq l$.

Let $\check{\text{Cech}}^\circ(X, Y, d, b)$, the *open Čech complex* of (X, d) with parameter b , be defined the same way as the closed Čech complex, except that we specify that $\check{\text{Cech}}^\circ(X, Y, d, b)$ contains the $(l-1)$ -simplex $[x_1, \dots, x_l]$ iff there is a point $y \in Y$ such that $d(y, x_i) < b$ for $1 \leq i \leq l$.

Define the functor $F^{S\check{C}e} : C^{S\check{C}e} \rightarrow (n+1)\text{-filt}$ on objects by taking

$$F^{S\check{C}e}(X, Y, d, f)_{(a,b)} = \check{\text{Cech}}(f_a, Y, d, b)$$

for all $(a, b) \in \mathbb{R}^n \times \mathbb{R}$. We refer to $F^{S\check{C}e}$ as the **(closed) sublevelset-offset filtration functor**.

Similarly, define the functor $F^{S\check{C}e-Op} : C^{S\check{C}e} \rightarrow (n+1)\text{-filt}$ on objects by taking

$$F^{S\check{C}e-Op}(X, Y, d, f)_{(a,b)} = \check{\text{Cech}}^\circ(f_a, Y, d, b)$$

for all $(a, b) \in \mathbb{R}^n \times \mathbb{R}$. We refer to $F^{S\check{C}e-Op}$ as the **open sublevelset-Čech filtration functor**.

Example 2.7.5. Sublevelset-Rips Filtrations

Let C^{SR} be the full subcategory of C^{SO} whose objects are quadruples (X, X, d, f) , where X is a finite metric space.

Recall from Section 1.5.1 that given a finite metric space (X, d) and $b \in \mathbb{R}$, we let $\text{Rips}(X, d, b)$, the *Rips complex* of (X, d) with parameter b , be the maximal abstract simplicial complex with vertex set X such that for $x_1, x_2 \in X$, the 1-skeleton of $\text{Rips}(X, d, b)$ contains the edge $[x_1, x_2]$ iff $d(x_1, x_2) \leq 2b$.

Define the functor $F^{SR} : C^{SR} \rightarrow (n+1)\text{-filt}$ on objects by taking

$$F^{SR}(X, d, f)_{(a,b)} = \text{Rips}(f_a, d, b)$$

for all $(a, b) \in \mathbb{R}^n \times \mathbb{R}$.

We refer to F^{SR} as the **sublevelset-Rips filtration functor**.

The functor F^{SR} has previously been considered (in the case $n = 1$) in [7].

2.7.5 Metrics on Geometric Categories

We now define metrics on the isomorphism classes of objects of each of the geometric categories we defined in the last section. We'll use these to formulate stability results for d_I .

Of these metrics, the first will be the most important to us, since we'll use it to formulate our main optimality results.

A Metric on $\text{obj}^*(C^S)$

We define a metric d^S on $\text{obj}^*(C^S)$ by taking

$$d^S((X, f_X), (Y, f_Y)) = \inf_{h \in \mathcal{H}} \sup_{x \in X} \|f_X(x) - f_Y(h(x))\|_\infty$$

where \mathcal{H} is the set of homeomorphisms between X and Y . d^S descends to a metric on $\text{obj}^*(C^S)$, which we also write as d^S . Note that $d^S((X, f_X), (Y, f_Y)) = \infty$ if X and Y are not homeomorphic.

When $n = 1$, d^S is known as the *natural pseudo-distance*; it features prominently in the work of Patrizio Frosini and his coauthors on persistent homology and size theory—see [22], for example.

Remark 2.7.6. d^S can be defined equivalently as an interleaving distance. To define an interleaving distance on $\text{obj}(C^S)$, we need to define ϵ -interleavings in the category C_S . To do this, it is enough to define an ϵ -shift functor $(\cdot)(\epsilon) : C^S \rightarrow C^S$ for each $\epsilon \in \mathbb{R}$, and a transition morphism $S((X, f_X), \epsilon) : (X, f_X) \rightarrow (X, f_X)(\epsilon)$ for each $(X, f_X) \in \text{obj}(C^S)$ and each $\epsilon \in [0, \infty)$.

We define $(\cdot)(\epsilon)$ on objects in C^S by taking $(X, f_X)(\epsilon) = (X, f'_X)$, where $f'_X(x) = f(x) - \epsilon$ for all $x \in X$; we define $(\cdot)(\epsilon)$ on morphisms in the obvious way; and we define $S((X, f_X), \epsilon)$ to be the identity map on X .

It is not hard to check that the interleaving distance induced by these choices is equal to d^S .

A Metric on $\text{obj}^*(C^{SO})$

We define a metric d^{SO} on $\text{obj}^*(C^{SO})$ by taking

$$\begin{aligned} d^{SO}((X^1, Y^1, d^1, f^1), (X^2, Y^2, d^2, f^2)) \\ = \inf_{h \in \mathcal{H}} \max\{ \sup_{x_1 \in X^1} \inf_{x_2 \in X^2} \max(d^2(h(x_1), x_2), \|f^1(x_1) - f^2(x_2)\|_\infty), \\ \sup_{x_2 \in X^2} \inf_{x_1 \in X^1} \max(d^2(h(x_1), x_2), \|f^1(x_1) - f^2(x_2)\|_\infty) \}, \end{aligned}$$

where \mathcal{H} is the set of isometries between (Y^1, d^1) and (Y^2, d^2) . d^{SO} is a sort of “function aware” variant of the Hausdorff distance. Note that

$$d^{SO}((X^1, Y^1, d^1, f^1), (X^2, Y^2, d^2, f^2)) = \infty$$

if (Y^1, d^1) and (Y^2, d^2) are not isometric.

Note also that since $\text{obj}(C^{S\check{C}e})$ is a full subcategory of C^{SO} , d^{SO} restricts to a metric on $\text{obj}^*(C^{S\check{C}e})$.

A Metric on $\text{obj}^*(C^{SR})$

Generalizing in a mild way a definition of [9], we define a metric d^{SR} on $\text{obj}^*(C^{SR})$. For $f_X, f_Y \equiv 0$, $d^{SR}((X, d_X, f_X), (Y, d_Y, f_Y))$ will be equal to the Gromov-Hausdorff metric [9]. (In fact, the definition extends to the subcategory of C^{SO} whose objects are the triplets (X, d_X, f_X) with X compact, but we won’t need the extra generality here.)

To define d^{SR} , we need some preliminary definitions and notation. Define a *correspondence* between two sets X and Y to be a subset $C \subseteq X \times Y$ such that $\forall x \in X$, $\exists y \in Y$ such $(x, y) \in C$, and $\forall y \in Y$, $\exists x \in X$ s.t. $(x, y) \in C$. Let $C(X, Y)$ denote the set of correspondences between X and Y .

For $(X, d_X, f_X), (Y, d_Y, f_Y) \in C^{SR}$, define $\Gamma_{X,Y} : X \times Y \times X \times Y \rightarrow \mathbb{R}_{\geq 0}$ by

$$\Gamma_{X,Y}(x, y, x', y') = |d_X(x, x') - d_Y(y, y')|.$$

For $C \in C(X, Y)$, define Γ_C as $\sup_{(x,y),(x',y') \in C} \Gamma_{X,Y}(x, y, x', y')$, and define $|f_X - f_Y|_C$ to be $\sup_{(x,y) \in C} \|f_X(x) - f_Y(y)\|_\infty$. Informally, Γ_C is the maximum distortion of the metrics under the correspondence C , and $|f_X - f_Y|_C$ is the maximum distortion of the functions under C .

Now define we define d^{SR} by taking

$$d^{SR}((X, d_X, f_X), (Y, d_Y, f_Y)) = \inf_{C \in C(X, Y)} \max\left(\frac{1}{2}\Gamma_C, |f_X - f_Y|_C\right).$$

2.7.6 Stability Results for Ordinary Persistence

There are two main geometric stability results for ordinary persistence in the literature. (Though see also the generalization [5]). Each is a consequence of the algebraic stability of persistence [8].

Theorem 2.7.7 (1-D Stability Result for C^S [8]). *For any $i \in \mathbb{Z}_{\geq 0}$, topological spaces X, Y , and functions $f_X : X \rightarrow \mathbb{R}$, $f_Y : Y \rightarrow \mathbb{R}$ such that $H_i \circ F^S(X, f_X)$ and $H_i \circ F^S(Y, f_Y)$ are tame,*

$$d_B(H_i \circ F^S(X, f_X), H_i \circ F^S(Y, f_Y)) \leq d^S((X, f_X), (Y, f_Y)).$$

For a 2-D filtration F , let $\text{diag}(F)$ denote the 1-D filtration for which $\text{diag}(F)_a = F_{(a,a)}$.

Theorem 2.7.8 (1-D Stability Result for C^{SR} [9]). *For any finite metric spaces $(X, d_X), (Y, d_Y)$ and functions $f_X : X \rightarrow \mathbb{R}$, $f_Y : Y \rightarrow \mathbb{R}$,*

$$\begin{aligned} & d_B(H_i \circ \text{diag}(F^{SR}(X, d_X, f_X)), H_i \circ \text{diag}(F^{SR}(Y, d_Y, f_Y))) \\ & \leq d^{SR}((X, d_X, f_X), (Y, d_Y, f_Y)). \end{aligned}$$

We'll see in Section 2.8 that both of these results admit generalizations to the setting of multidimensional persistence in terms of the interleaving distance.

2.8 Stability Properties of the Interleaving Distance

In this section, we observe that multidimensional persistent homology is stable with respect to the interleaving distance in four senses analogous to those in which ordinary persistent homology is known to be stable. As noted in the introduction, there is not much mathematical work to do here. Nevertheless, these observations are significant (if not particularly surprising) not only because they show that the interleaving distance is in several respects a well behaved distance, but also because

stability is closely related to the optimality of distances on persistence modules as we define it in Section 2.9; insofar as we wish to understand the optimality properties of the interleaving distance, the stability properties of the interleaving distance are of interest.

Each of the results here lifts readily to a stability result on the level of filtrations, formulated in terms of interleavings on filtrations; see Sections 3.2.2 and 3.3.3.

2.8.1 Stability of Sublevelset Multi-D Persistent Homology

Theorem 2.8.1. *For any $T_1, T_2 \in \text{obj}(C^S)$ and $i \in \mathbb{Z}_{\geq 0}$,*

$$d_I(H_i \circ F^S(T_1), H_i \circ F^S(T_2)) \leq d^S(T_1, T_2).$$

The case $n = 1$ is Theorem 2.7.7.

Proof. Let $d^S((X, f_X), (X, f_Y)) = \epsilon < \infty$. Then for any $\epsilon' > \epsilon$, there exists a homeomorphism $h : X \rightarrow Y$ such that $u \in \mathbb{R}^n$, $F^S(X, f_X)_u \subset F^S(X, f_Y \circ h)_{u+\epsilon'}$ and $F^S(X, f_Y \circ h)_u \subset F^S(X, f_X)_{u+\epsilon'}$. The images of these inclusions under the i^{th} singular homology functor define ϵ' -interleaving morphisms between $H_i \circ F^S(X, f_X)$ and $H_i \circ F^S(X, f_Y \circ h)$. $(X, f_Y \circ h)$ and (Y, f_Y) are isomorphic objects of C^S , so $H_i \circ F^S(X, f_Y \circ h)$ and $H_i \circ F^S(Y, f_Y)$ are 0-interleaved. Thus $H_i \circ F^S(X, f_X)$ and $H_i \circ F^S(Y, f_Y)$ are ϵ' -interleaved, and so $d_I(H_i \circ F^S(X, f_X), H_i \circ F^S(Y, f_Y)) \leq \epsilon$, as needed. \square

2.8.2 Stability of Sublevelset-Offset Multi-D Persistent Homology

Theorem 2.8.2. *For any $T_1, T_2 \in \text{obj}(C^{SO})$ we have, for any $i \in \mathbb{Z}_{\geq 0}$,*

$$d_I(H_i \circ F^{SO}(T_1), H_i \circ F^{SO}(T_2)) \leq d^{SO}(T_1, T_2).$$

Proof. This is similar to proof of the previous result. Let $d^{SO}(T_1, T_2) = \epsilon < \infty$ and write $T_1 = (X^1, Y^1, d^1, f^1)$, $T_2 = (X^2, Y^2, d^2, f^2)$. For any $\epsilon' > \epsilon > 0$, there exists an

isometry $h : X \rightarrow Y$ such that for any $u \in \mathbb{R}^n$ and $r \in \mathbb{R}$,

$$\begin{aligned} F^{SO}(T_1)_{(u,r)} &\subset F^{SO}(T_3)_{(u+\epsilon',r+\epsilon')} \text{ and} \\ F^{SO}(T_3)_{(u,r)} &\subset F^{SO}(T_1)_{(u+\epsilon',r+\epsilon')}, \end{aligned}$$

where $T_3 = (h^{-1}(X^2), Y^1, d^1, f^2 \circ h)$. The images of these inclusions under the i^{th} singular homology functor define ϵ' -interleaving morphisms between $H_i \circ F^{SO}(T_1)$ and $H_i \circ F^{SO}(T_3)$. T_2 and T_3 are isomorphic as elements of C^{SO} so $H_i \circ F^{SO}(T_2)$ and $H_i \circ F^{SO}(T_3)$ are 0-interleaved. The result now follows in the same way as in the previous proof. \square

2.8.3 Stability of Sublevelset-Čech Multi-D Persistent Homology

Theorem 2.8.3. *For any $T_1, T_2 \in \text{obj}(C^{S\check{C}e})$ and $i \in \mathbb{Z}_{\geq 0}$,*

$$d_I(H_i \circ F^{S\check{C}e}(T_1), H_i \circ F^{S\check{C}e}(T_2)) \leq d^{SO}(T_1, T_2).$$

Proof. Note that the definition of the action of $F^{S\check{C}e}$ on objects of $C^{S\check{C}e}$ extends from quadruples $(X, Y, d, f) \in \text{obj}(C^{S\check{C}e})$ to quadruples (\tilde{X}, Y, d, f) , where \tilde{X} is a finite multiset³ with underlying set X . Further, note that for any $(a, b) \in \mathbb{R}^n \times \mathbb{R}$ $F^{S\check{C}e}(X, Y, d, f)_{(a,b)}$ is a deformation retract of $F^{S\check{C}e}(\tilde{X}, Y, d, f)_{(a,b)}$, and the deformation retracts can be taken to commute with the inclusion maps in these filtrations. Thus

$$H_i \circ F^{S\check{C}e}(\tilde{X}, Y, d, f)_{(a,b)} \cong H_i \circ F^{S\check{C}e}(X, Y, d, f)_{(a,b)}.$$

Let $d^{SO}(T_1, T_2) = \epsilon < \infty$ and write $T_1 = (X^1, Y^1, d^1, f^1)$, $T_2 = (X^2, Y^2, d^2, f^2)$. Using an argument similar to that appearing in the proof above, it's straightforward to show that for all $\epsilon' > \epsilon$ there exist multisets \tilde{X}^1 and \tilde{X}^2 whose underlying sets are

³Informally, a multiset is a set whose elements are allowed to have multiplicity greater than or equal to one.

X^1 and X^2 respectively and a bijection $\lambda : \tilde{X}^1 \rightarrow \tilde{X}^2$ which induces a pair of strong ϵ -interleaving morphisms between $F^{S\check{C}e}(\tilde{X}^1, Y^1, d^1, f^1)$ and $F^{S\check{C}e}(\tilde{X}^2, Y^2, d^2, f^2)$.⁴ From this it follows that $H_i \circ F^{S\check{C}e}(\tilde{X}^1, Y^1, d^1, f^1)$ and $H_i \circ F^{S\check{C}e}(\tilde{X}^2, Y^2, d^2, f^2)$ are also ϵ' -interleaved. Thus $H_i \circ F^{S\check{C}e}(T_1)$ and $H_i \circ F^{S\check{C}e}(T_2)$ are ϵ' -interleaved; the result follows.

□

2.8.4 Stability of Sublevelset-Rips Multi-D Persistent Homology

Theorem 2.8.4. *For $T_1, T_2 \in \text{obj}(C^{SR})$ we have, for any $i \in \mathbb{Z}_{\geq 0}$,*

$$d_I(H_i \circ F^{SR}(T_1), H_i \circ F^{SR}(T_2)) \leq d^{SR}(T_1, T_2).$$

The proof of this is a very minor modification of the argument given in [9] to prove Theorem 2.7.8.

Note that Theorem 2.8.4 implies Theorem 2.7.8: When $n = 1$, if $H_i \circ F^{SR}(T_1)$ and $H_i \circ F^{SR}(T_2)$ are ϵ -interleaved, for $\epsilon \in \mathbb{R}_{\geq 0}$, then $H_i \circ \text{diag}(F^{SR}(T_1))$ and $H_i \circ \text{diag}(F^{SR}(T_2))$ are ϵ -interleaved.

2.9 Optimal Pseudometrics

In this section we introduce a relative notion of optimality of pseudometrics. This notion of optimality is rather general and specializes to a number of different definitions of interest in the context of multidimensional persistence.

We also present some basic theory about optimal pseudometrics. We'll make use of this theory in Section 2.10, where we prove our main optimality result for the interleaving distance and in Chapter 3, where we present optimality results for pseudometrics on filtrations.

⁴We give the definition of strong interleavings of filtrations in Section 3.2.1.

2.9.1 A General Definition of Optimal Pseudometrics

Let Y be a class.⁵ We define a **relative structure** on Y to be a triple $\mathcal{R} = (W, d_W, f_W)$, where (W, d_W) is a pseudometric space and $f_W : W \rightarrow Y$ is a function.

We say a pseudometric d on Y is **\mathcal{R} -stable** if for every $w_1, w_2 \in W$ we have $d(f_W(w_1), f_W(w_2)) \leq d_W(w_1, w_2)$.

We say a pseudometric d on Y is **\mathcal{R} -optimal** if d is \mathcal{R} -stable, and for every other \mathcal{R} -stable pseudometric d' on Y , we have $d'(y_1, y_2) \leq d(y_1, y_2)$ for all $y_1, y_2 \in \text{im}(f_W)$.

2.9.2 Basic Results About Optimal Pseudometrics

The following lemma is immediate, but important to understanding our definition of \mathcal{R} -optimality.

Lemma 2.9.1. *An \mathcal{R} -stable pseudometric d is \mathcal{R} -optimal iff for any \mathcal{R} -stable pseudometric d' and $w_1, w_2 \in W$,*

$$|d_W(w_1, w_2) - d(f_W(w_1), f_W(w_2))| \leq |d_W(w_1, w_2) - d'(f_W(w_1), f_W(w_2))|.$$

Our optimality proofs in this thesis will all make use of the following easy lemma.

Lemma 2.9.2. *Let (Y, d) be a pseudometric space and let $\mathcal{R} = (W, d_W, f_W)$ be a relative structure on Y such that d is \mathcal{R} -stable. Suppose that for each $y_1, y_2 \in \text{im}(f_W)$ and $\delta > 0$ there exists $w_1, w_2 \in W$ such that $f_W(w_1) = y_1$, $f_W(w_2) = y_2$, and $d_W(w_1, w_2) \leq d(y_1, y_2) + \delta$. Then d is \mathcal{R} -optimal.*

Proof. This follows immediately from the definitions. \square

When two relative structures \mathcal{R} and \mathcal{R}' are related, one might expect that there is a relationship between \mathcal{R} -optimality and \mathcal{R}' -optimality. The next two lemmas offer positive results in this direction; they will prove useful to us in Sections 2.10 and 3.2.5.

⁵As is common in category theoretic settings, in what follows we will sometimes work with proper classes.

Lemma 2.9.3. *Let (X, d_X) , (Y, d_Y) , (Z, d_Z) be pseudometric spaces and let $f_X : X \rightarrow Y$, $f_Y : Y \rightarrow Z$ be functions. If f_X is surjective, d_Y is (X, d_X, f_X) -optimal, and d_Z is (Y, d_Y, f_Y) -optimal then d_Z is also $(X, d_X, f_Y \circ f_X)$ -optimal.*

Proof. Clearly d_Z is $(X, d_X, f_Y \circ f_X)$ -stable. Let d'_Z be another $(X, d_X, f_Y \circ f_X)$ -stable pseudometric on Z . d'_Z induces a pseudometric d'_Y on Y given by $d'_Y(y_1, y_2) = d'_Z(f_Z(y_1), f_Z(y_2))$. Since d'_Z is $(X, d_X, f_Y \circ f_X)$ -stable, it follows that d'_Y is (X, d_X, f_X) -stable. Then, since f_X is surjective, $d'_Y \leq d_Y$. This implies that d'_Z is (Y, d_Y, f_Y) -stable. Then by the (Y, d_Y, f_Y) -optimality of d_Z , $d'_Z \geq d_Z$ on $\text{im}(f_Y) = \text{im}(f_Y \circ f_X)$ so d_Z is $(X, d_X, f_Y \circ f_X)$ -optimal, as desired. \square

Lemma 2.9.4. *Let (X, d_X) , (Y, d_Y) , (Z, d_Z) be metric spaces and let $f_X : X \rightarrow Y$, $f_Y : Y \rightarrow Z$ be functions. If $\text{im}(f_Y) = \text{im}(f_Y \circ f_X)$, d_Y is (X, d_X, f_X) -stable, d_Z is $(X, d_X, f_Y \circ f_X)$ -optimal, and d_Z is (Y, d_Y, f_Y) -stable then d_Z is also (Y, d_Y, f_Y) -optimal.*

Proof. Let d'_Z be another (Y, d_Y, f_Y) -stable pseudometric on Z . Then d'_Z is also $(X, d_X, f_Y \circ f_X)$ -stable. By the $(X, d_X, f_Y \circ f_X)$ -optimality of d_Z , $d'_Z \leq d_Z$ on $\text{im}(f_Y \circ f_X) = \text{im}(f_Y)$. Thus d_Z is (Y, d_Y, f_Y) -optimal. \square

2.9.3 Examples

Here we give examples of classes Y and relative structures (W, d_W, f_W) on Y for which it would be interesting or useful from the standpoint of the theory and applications of multidimensional persistent homology to have an explicit characterization of an \mathcal{R} -optimal pseudometric. In Section 2.10 we will focus exclusively on the relative structures of Examples 2.9.5 and 2.9.6; we leave it to future work to investigate in detail the optimality of pseudometrics with respect to the relative structures of Examples 2.9.9-2.9.13.

In chapter 3 of this thesis we'll introduce some more examples of relative structures relevant to our study of multidimensional persistence, in connection with our study of distances on multidimensional filtrations.

For examples 2.9.5-2.9.12, let $Y = \text{obj}^*(B_n\text{-mod})$.

Example 2.9.5. For $i \in \mathbb{Z}_{\geq 0}$, let $\mathcal{R}_{1,i} = (\text{obj}(C^S), d^S, H_i \circ F^S)$.

Example 2.9.6. Let $\mathcal{R}_1 = (W, d_W, f_W)$, where

$$W = \{(T, i) | T \in \text{obj}(C^S), i \in \mathbb{Z}_{\geq 0}\},$$

$$d_W((T_1, i_1), (T_2, i_2)) = \begin{cases} d^S(T_1, T_2) & \text{if } i_1 = i_2, \\ \infty & \text{otherwise,} \end{cases}$$

and

$$f_W(T, i) = H_i \circ F^S(T).$$

Then a pseudometric on Y is \mathcal{R}_1 -stable iff it is $\mathcal{R}_{1,i}$ -stable for all $i \in \mathbb{Z}_{\geq 0}$.

Remark 2.9.7. By Lemma 2.9.1, an \mathcal{R}_1 -stable pseudometric d is \mathcal{R}_1 -optimal iff for any \mathcal{R}_1 -stable pseudometric d' , $i \in \mathbb{Z}_{\geq 0}$, and $T_1, T_2 \in \text{obj}(C^S)$,

$$\begin{aligned} & |d^S(T_1, T_2) - d(H_i \circ F^S(T_1), H_i \circ F^S(T_2))| \\ & \leq |d^S(T_1, T_2) - d'(H_i \circ F^S(T_1), H_i \circ F^S(T_2))|. \end{aligned}$$

Remark 2.9.8. Since the definitions of $\mathcal{R}_{1,i}$ and \mathcal{R}_1 are similar, one might expect that there's a relationship between \mathcal{R}_1 -optimality and $\mathcal{R}_{1,i}$ -optimality. Corollary 2.10.2 shows that in the case that $i \in \mathbb{N}$ and either $k = \mathbb{Q}$ or $k = \mathbb{Z}/p\mathbb{Z}$ for some prime p , $\mathcal{R}_{1,i}$ -optimality implies \mathcal{R} -optimality.

Example 2.9.9. For any $i \in \mathbb{Z}_{\geq 0}$, $(\text{obj}(C^{SO}), d^{SO}, H_i \circ F^{SO})$ defines a relative structure on Y .

Example 2.9.10. Similarly, for any $i \in \mathbb{Z}_{\geq 0}$, $(\text{obj}(C^{SR}), d^{SR}, H_i \circ F^{SR})$ defines a relative structure on Y .

Example 2.9.11. For any $i \in \mathbb{Z}_{\geq 0}$, $(\text{obj}(C^{SCe}), d^{SCe}, H_i \circ F^{SCe})$ defines a relative structure on Y .

Example 2.9.12. We noted above that the relative structure \mathcal{R}_1 of Example 2.9.6 is an analogue of the relative structures $\mathcal{R}_{1,i}$ of Example 2.9.5 such that \mathcal{R}_1 -stability

is equivalent to $\mathcal{R}_{1,i}$ -stability for all $i \in \mathbb{Z}_{\geq 0}$. In the same way that we defined \mathcal{R}_1 , we can define analogues of the relative structures of each of Examples 2.9.9, 2.9.10, and 2.9.11 so that the same property holds.

Example 2.9.13. For any relative structure (W, d_W, f_W) on $\text{obj}^*(B_n\text{-mod})$, set Y , and function $\zeta : \text{obj}^*(B_n\text{-mod}) \rightarrow Y$, let $f' = \zeta \circ f_W$. Then (W, d_W, f') is a relative structure on W . For example, we can take ζ to be the rank invariant [7] and (W, d_W, f_W) to be the relative structure of Example 2.9.6.

2.10 Optimality of the Interleaving Distance (Relative to Sublevelset Persistence)

2.10.1 Optimality of Interleaving Distance and Geometric Lifts of Interleavings

The central result of this section is the following theorem:

Theorem 2.10.1. *For $k = \mathbb{Q}$ or $\mathbb{Z}/p\mathbb{Z}$ for some prime p , and $i \in \mathbb{N}$, d_I is $\mathcal{R}_{1,i}$ -optimal.*

This theorem also yields the following weaker optimality result, which has the aesthetic advantage of not depending in its formulation on a choice of homology dimension.

Corollary 2.10.2. *For $k = \mathbb{Q}$ or $\mathbb{Z}/p\mathbb{Z}$ for some prime p , d_I is \mathcal{R}_1 -optimal.*

We also present analogues of Theorems 2.10.1 and Corollary 2.10.2 for well behaved B_1 -persistence modules which hold for arbitrary fields k ; these are given as Theorem 2.10.8. Our analogue of Theorem 2.10.1 for well behaved B_1 -persistence modules holds for $i \in \mathbb{Z}_{\geq 0}$.

Remark 2.10.3. I suspect that theorem 2.10.1 holds, more generally, for arbitrary fields and for $i \in \mathbb{Z}_{\geq 0}$, though we do not prove that here. Such a generalization would subsume both Corollary 2.10.2 and Theorem 2.10.8.

Theorem 2.10.1 is an immediate consequence of the following proposition, by Lemma 2.9.2.

Proposition 2.10.4. *[Existence of Geometric Lifts of Interleavings] Let $k = \mathbb{Q}$ or $k = \mathbb{Z}/p\mathbb{Z}$ for some prime p . If $i \in \mathbb{N}$, and M and N are ϵ -interleaved B_n -modules then there exists a CW-complex X and continuous functions $\gamma_M, \gamma_N : X \rightarrow \mathbb{R}^n$ such that $H_i \circ F^S(X, \gamma_M) \cong M$, $H_i \circ F^S(X, \gamma_N) \cong N$, and $\|\gamma_M - \gamma_N\|_\infty = \epsilon$.*

The proposition tells us that interleavings on B_n -persistence modules lift to interleavings on objects of C^S , in the sense of Remark 2.7.6.

Another notable consequence of the proposition is this (compare [7, Theorem 2]):

Corollary 2.10.5. *If $k = \mathbb{Q}$ or $\mathbb{Z}/p\mathbb{Z}$ for some prime p then for every B_n -persistence module M and $i \in \mathbb{N}$, there exists a CW-complex X and a continuous function $\gamma : X \rightarrow \mathbb{R}^n$ such that $H_i \circ F^S(X, \gamma) \cong M$.*

Proof. This follows immediately from Proposition 2.10.4. □

We'll now deduce Corollary 2.10.2 from Theorem 2.10.1 and Corollary 2.10.5.

Proof of Corollary 2.10.2. Let $\mathcal{R}_1 = (W, d_W, f_W)$, as in Example 2.9.6. Clearly, d_I is \mathcal{R}_1 -stable. To show that d_I is \mathcal{R}_1 -optimal we'll apply Lemma 2.9.4.

For any $i \in \mathbb{Z}_{\geq 0}$, there is a map $\theta_i : \text{obj}(C^S) \rightarrow W$ which sends $T \in \text{obj}(C^S)$ to (T, i) . Then for any $T \in \text{obj}(C^S)$, $f_W \circ \theta_i(T) = H_i \circ F^S(T)$. Recall from Example 2.9.5 that

$$\mathcal{R}_{1,i} = (\text{obj}(C^S), d^S, H_i \circ F^S) = (\text{obj}(C^S), d^S, f_W \circ \theta_i).$$

Since the restriction of d_W to $\text{im}(\theta_i)$ is equal to d^S , d_W is clearly $(\text{obj}(C^S), d^S, \theta_i)$ -stable. Now fix some $i \in \mathbb{N}$. By Corollary 2.10.5, $\text{im}(f_W \circ \theta_i) = \text{im}(f_W) = \text{obj}^*(B_n\text{-mod})$. Theorem 2.10.1 tells us that d_I is $\mathcal{R}_{1,i}$ -optimal. Thus Lemma 2.9.4 applies to give us that d_I is \mathcal{R}_1 -optimal as well. □

Note that the extension of Theorem 2.10.1 to the case $i = 0$ would follow from Lemma 2.9.2, given the following conjectural extension of Proposition 2.10.4 to the case $i = 0$.

Conjecture 2.10.6. Let $k = \mathbb{Q}$ or $k = \mathbb{Z}/p\mathbb{Z}$ for some prime p . Let M and N be ϵ -interleaved B_n -modules such that for some topological spaces X_M, X_N and functions $f_M : X_M \rightarrow \mathbb{R}^n, f_N : X_N \rightarrow \mathbb{R}^n, H_0 \circ F^S(X_M, f_M) \cong M$ and $H_0 \circ F^S(X_N, f_N) \cong N$. Then there exists a CW-complex X and continuous functions $\gamma_M, \gamma_N : X \rightarrow \mathbb{R}^n$ such that $H_0 \circ F^S(X, \gamma_M) \cong M, H_0 \circ F^S(X, \gamma_N) \cong N$, and $\|\gamma_M - \gamma_N\|_\infty = \epsilon$.

Sections 2.10.3-2.10.6 are devoted to the proof of Proposition 2.10.4.

However, before proceeding with the proof of Proposition 2.10.4 we present optimality results for the bottleneck distance.

2.10.2 Optimality of the Bottleneck Distance

Let Y denote the set of isomorphism classes of tame B_1 -persistence modules. Since the bottleneck distance d_B is only defined between elements of Y , formulating statements about the optimality of d_B requires that we consider a relative structure on Y rather than on $\text{obj}^*(B_1\text{-mod})$. We can (in the obvious way) define restrictions \mathcal{R}_2 and $\mathcal{R}_{2,i}$ of the relative structures \mathcal{R}_1 and $\mathcal{R}_{1,i}$ to relative structures on Y .

Then given the existence of geometric lifts of interleavings, as established in Proposition 2.10.4, the proofs of Theorem 2.10.1 and Corollary 2.10.2 adapt to give that for $k = \mathbb{Q}$ or $k = \mathbb{Z}/p\mathbb{Z}$ for p a prime, d_B is \mathcal{R}_2 -optimal, and for any $i \in \mathbb{N}$, d_B is also $\mathcal{R}_{2,i}$ -optimal.

Moreover, if we further restrict our attention to well behaved B_1 -persistence modules then we can prove an optimality result for arbitrary fields k and persistent homology of any dimension.

The key is the following variant of Proposition 2.10.4:

Proposition 2.10.7. [Existence of Geometric Lifts of Interleavings for Well Behaved B_1 -persistence Modules] Let k be any field and let $i \in \mathbb{Z}_{\geq 0}$. Let M and N be well behaved B_1 -persistence modules with $d_B(M, N) = \epsilon$. If $i = 0$, assume further that for some topological spaces X_M, X_N and functions $f_M : X_M \rightarrow \mathbb{R}, f_N : X_N \rightarrow \mathbb{R}, H_0 \circ F^S(X_M, f_M) \cong M$ and $H_0 \circ F^S(X_N, f_N) \cong N$. Then for any $\delta > 0$, there exists a CW-complex X and continuous functions $\gamma_M, \gamma_N : X \rightarrow \mathbb{R}$ such that $H_i \circ F^S(X, \gamma_M) \cong M, H_i \circ F^S(X, \gamma_N) \cong N$, and $\|\gamma_M - \gamma_N\|_\infty \leq \epsilon + \delta$.

Proof. An easy constructive proof, similar on a high level to our proof of Proposition 2.10.4 below but much simpler, follows from the definition of d_B and the structure theorem for well behaved persistence modules (Theorem 2.3.6). We leave the details to the reader. \square

Now let Y' denote the set of isomorphism classes of well behaved B_1 -persistence modules. Let \mathcal{R}_3 and $\mathcal{R}_{3,i}$ denote the restrictions of the relative structures \mathcal{R}_1 and $\mathcal{R}_{1,i}$ to relative structures on Y' .

Our optimality result for the bottleneck distance on well behaved persistence modules is the following:

Theorem 2.10.8. *For any field k and $i \in \mathbb{Z}_{\geq 0}$, d_B is $\mathcal{R}_{3,i}$ -optimal. Further, d_B is \mathcal{R}_3 -optimal.*

Proof. Using Proposition 2.10.7 in place of Proposition 2.10.4, the proofs of Theorem 2.10.1 and Corollary 2.10.2 carry over with only minor modifications to give the result. \square

Remark 2.10.9. In the case of 0-D ordinary persistence, our Theorem 2.10.8 implies a slight weakening of the optimality result of [22]. It is easy to check that in the geometric context considered in [22], the persistent homology modules obtained must satisfy property 2 in the definition of a well behaved B_1 -persistence module, but they needn't satisfy property 1. To strengthen our optimality result to a full generalization of the result of [22], we'd want a generalization of Theorem 2.3.6 to all B_1 -persistence modules satisfying property 2. We presume that this can be obtained via a slight strengthening of Theorem 2.2.2, but we do not pursue the details of this here.

2.10.3 Proof of Existence of Geometric Lifts of Interleavings, Part 1: Constructing the CW-complex

The rest of Section 2.10 is devoted to the proof of Proposition 2.10.4.

Corollary 2.6.5 gives us n -graded sets $\mathcal{W}_1, \mathcal{W}_2$ and homogeneous sets $\mathcal{Y}_1, \mathcal{Y}_2 \subset$

$\langle \mathcal{W}_1, \mathcal{W}_2 \rangle$ such that $\mathcal{Y}_1 \in \langle \mathcal{W}_1, \mathcal{W}_2(-\epsilon) \rangle$, $\mathcal{Y}_2 \in \langle \mathcal{W}_1(-\epsilon), \mathcal{W}_2 \rangle$, and

$$\begin{aligned} M &\cong \langle \mathcal{W}_1, \mathcal{W}_2(-\epsilon) | \mathcal{Y}_1, \mathcal{Y}_2(-\epsilon) \rangle, \\ N &\cong \langle \mathcal{W}_1(-\epsilon), \mathcal{W}_2 | \mathcal{Y}_1(-\epsilon), \mathcal{Y}_2 \rangle. \end{aligned}$$

Given such $\mathcal{W}_1, \mathcal{W}_2, \mathcal{Y}_1, \mathcal{Y}_2$, we proceed with the proof of Proposition 2.10.4, beginning with the construction of the CW-complex X whose existence is posited by the proposition.

Write $\mathcal{W} = \mathcal{W}_1 \cup \mathcal{W}_2$ and $\mathcal{Y} = \mathcal{Y}_1 \cup \mathcal{Y}_2$. If $M = N = 0$, Proposition 2.10.4 clearly holds, so we may assume without loss of generality that M and N are not both trivial. Under this assumption, $\mathcal{W} \neq \emptyset$.

For an n -graded set S , let $GCD(S) = (v_1, \dots, v_n)$, where $v_i = \inf_{s \in S} gr(s)_i$. In general, some of the components of $GCD(\mathcal{W})$ may be equal to $-\infty$. To simplify our exposition, we first present the remainder of the proof of Proposition 2.10.4 in the special case that $GCD(\mathcal{W}) \in \mathbb{R}^n$. The adaptation of our proof to the general case is reasonably straightforward; we outline this adaptation in Section 2.10.6.

We'll define X so that

1. X has a single 0-cell B .
2. X has an i -cell e_w^i for each $w \in \mathcal{W}$.
3. X has an $(i+1)$ -cell e_y^{i+1} for each $y \in \mathcal{Y}$.

For such X , the attaching map for each i -cell e_w^i must be the constant map to B . To define X , then, we need only to specify the attaching map $\sigma_y : S^i \rightarrow X^i$ for each $y \in \mathcal{Y}$.

We do this for $k = \mathbb{Q}$, and leave to the reader the easy adaptation of the construction (and its use in the remainder of the proof of Proposition 2.10.4) to the case $k = \mathbb{Z}/p\mathbb{Z}$.

For any $y \in \mathcal{Y}$, we may choose a finite set $\mathcal{W}_y \subset \mathcal{W}$ such that $gr(w) \leq gr(y)$ for each $w \in \mathcal{W}_y$, and

$$y = \sum_{w \in \mathcal{W}_y} a'_{wy} \varphi_{\langle \mathcal{W} \rangle}(gr(w), gr(y))(w) \tag{2.3}$$

for some $a'_{wy} \in \mathbb{Q}$. There's an integer z such that for each a'_{wy} in the sum, $za'_{wy} \in \mathbb{Z}$. Let $a_{wy} = za'_{wy}$. For $w \notin W_y$, define $a_{wy} = 0$.

Lemma 2.10.10. *There exists a choice of attaching map $\sigma_y : S^i \rightarrow X^i$ for each $y \in \mathcal{Y}$ such that the CW-complex X constructed via these attaching maps has $\delta_{i+1}^X(e_y^{i+1}) = \sum_{w \in \mathcal{W}} a_{wy} e_w^i$ for all $y \in \mathcal{Y}$.*

Proof. Let ρ^i be as defined in Section 2.7.1. For each $w \in \mathcal{W}$, ρ^i and the characteristic map Φ_w induce an identification of $\text{im}(\Phi_w)$ with an i -sphere S_w^i . We have that $(X^i, B) = \wedge_{w \in \mathcal{W}} (S_w^i, B)$. Choose a basepoint $o \in S^i$ and for each $w \in \mathcal{W}_y$, let $\sigma_{wy} : (S^i, o) \rightarrow (S_w^i, B)$ be a based map of degree a_{wy} . $[\sigma_{wy}] \in \pi_i(X^i, B)$, where $\pi_i(X^i, B)$ denotes the i^{th} homotopy group of X^i with basepoint B .

Order the elements of \mathcal{W}_y arbitrarily and call them w_1, \dots, w_l . Let $\sigma_y : (S^i, o) \rightarrow (X^i, B)$ be a map in $[\sigma_{w_1y}] \cdot [\sigma_{w_2y}] \cdot \dots \cdot [\sigma_{w_ly}] \in \pi_i(X^i, B)$. Then for any $w \in \mathcal{W}$, $q_w \circ \sigma_y$ is a map of degree a_{wy} . (See Section 2.7.1 for the definition of q_w). By the definition of δ_{i+1}^X given in Section 2.7.1, the lemma now follows. \square

We choose the attaching maps σ_y so that δ_{i+1}^X has the property specified in Lemma 2.10.10.

2.10.4 Proof of Existence of Geometric Lifts of Interleavings, Part 2: Defining γ_M and γ_N

Having defined the CW-complex X , we next define $\gamma_M, \gamma_N : X \rightarrow \mathbb{R}^n$.

Let $\tilde{X} = \{B\} \amalg_{w \in \mathcal{W}} D_w^i \amalg_{y \in \mathcal{Y}} D_y^{i+1}$.

X is the quotient of \tilde{X} under the equivalence relation generated by the attaching maps of the cells of X . Let $\pi : \tilde{X} \rightarrow X$ denote the quotient map. For a topological space A , let $\mathfrak{C}(A, \mathbb{R}^n)$ denote the space of continuous functions from A to \mathbb{R}^n . The map $\tilde{\cdot} : \mathfrak{C}(X, \mathbb{R}^n) \rightarrow \mathfrak{C}(\tilde{X}, \mathbb{R}^n)$ defined by $\tilde{f}(x) = f(\pi(x))$ is a bijective correspondence between elements of $\mathfrak{C}(X, \mathbb{R}^n)$ and elements of $\mathfrak{C}(\tilde{X}, \mathbb{R}^n)$ which are constant on equivalence classes.

In what follows, we'll define γ_M and γ_N by specifying their lifts $\tilde{\gamma}_M, \tilde{\gamma}_N$.

We'll take each of our functions $\tilde{\gamma}_M, \tilde{\gamma}_N$ to have the property that for each disk of \tilde{X} , the restriction of the function to any *radial line segment* (i.e. a line segment from the origin of the disk to the boundary of the disk) is linear. Given this assumption, to specify each function it is enough to specify its values on the origins of each disk of \tilde{X} . We now do this.

For any $i \in \mathbb{N}$ and any unit disk D in \mathbb{R}^i , let $O(D)$ denote the origin of D .

- $\tilde{\gamma}_M(B) = \tilde{\gamma}_N(B) = GCD(\mathcal{W})$;
- For $x \in \mathcal{W}_1 \cup \mathcal{Y}_1$, $\tilde{\gamma}_M(O(D_x)) = gr(x)$;
- For $x \in \mathcal{W}_2 \cup \mathcal{Y}_2$, $\tilde{\gamma}_M(O(D_x)) = gr(x(-\epsilon))$.
- For $x \in \mathcal{W}_1 \cup \mathcal{Y}_1$, $\tilde{\gamma}_N(O(D_x)) = gr(x(-\epsilon))$;
- For $x \in \mathcal{W}_2 \cup \mathcal{Y}_2$, $\tilde{\gamma}_N(O(D_x)) = gr(x)$.

Lemma 2.10.11. $\|\gamma_M - \gamma_N\|_\infty = \epsilon$.

Proof. Assume that for a disk D of \tilde{X} , $|\tilde{\gamma}_M(a) - \tilde{\gamma}_N(a)| \leq \epsilon$ for all $a \in \delta D$, and that $|\tilde{\gamma}_M(O(D)) - \tilde{\gamma}_N(O(D))| = \epsilon$. We'll show that then $|\tilde{\gamma}_M(a) - \tilde{\gamma}_N(a)| \leq \epsilon$ for all $a \in D$. Applying this result once gives that the lemma holds on the restriction of γ_M, γ_N to X^i . Applying the result a second time gives that the lemma holds on all of X .

To show that $|\tilde{\gamma}_M(a) - \tilde{\gamma}_N(a)| \leq \epsilon$, let x be a point in D and write $a = tO(D) + (1-t)b$ for some $b \in \delta D$, and $0 \leq t \leq 1$. Since the restrictions of $\tilde{\gamma}_M$ and $\tilde{\gamma}_N$ to any radial line segment from $O(D)$ to δD are linear, we have that $\tilde{\gamma}_M(a) = t\tilde{\gamma}_M(O(D)) + (1-t)\tilde{\gamma}_M(b)$, and $\tilde{\gamma}_N(a) = t\tilde{\gamma}_N(O(D)) + (1-t)\tilde{\gamma}_N(b)$. Thus $|\tilde{\gamma}_M(a) - \tilde{\gamma}_N(a)| \leq t|\tilde{\gamma}_M(O(D)) - \tilde{\gamma}_N(O(D))| + (1-t)|\tilde{\gamma}_M(b) - \tilde{\gamma}_N(b)| \leq t\epsilon + (1-t)\epsilon = \epsilon$ as needed. \square

2.10.5 Proof of Existence of Geometric Lifts of Interleavings, Part 3: Showing that $H_i \circ F^S(X, \gamma_M) \cong M$, $H_i \circ F^S(X, \gamma_N) \cong N$

Now it remains to show that $H_i \circ F^S(X, \gamma_M) \cong M$, $H_i \circ F^S(X, \gamma_N) \cong N$. We'll show that $H_i \circ F^S(X, \gamma_M) \cong M$; the argument that $H_i \circ F^S(X, \gamma_N) \cong N$ is essentially same.

For $a \in \mathbb{R}^n$, let \mathcal{F}_a denote the subcomplex of X consisting of only those cells e such that $\gamma_M(O(D(e))) \leq a$, where in this expression $D(e)$ is the disk of \tilde{X} whose interior maps to e under π . $\{\mathcal{F}_a\}_{a \in \mathbb{R}^n}$ defines a cellular filtration, which we'll denote \mathcal{F} . Let $X_a = F^S(X, \gamma_N)_a$. It's easy to see that \mathcal{F}_a is a deformation retract of X_a . Further, the inclusions of each $\mathcal{F}_a \hookrightarrow X_a$ define a morphism χ of filtrations; this morphism of filtrations maps under H_i to a morphism $H_i(\chi) : H_i(\mathcal{F}) \rightarrow H_i(F^S(X, \gamma_M))$ of B_n -persistence modules whose maps $H_i(\chi)_a : H_i(\mathcal{F}_a) \rightarrow H_i(X_a)$ are isomorphisms. Any homomorphism of B_n -persistence modules whose action on each homogeneous summand is a vector space isomorphism must be an isomorphism of B_n -persistence modules, so $H_i(\chi)$ is an isomorphism. Thus, to prove that $H_i \circ F^S(X, \gamma_M) \cong M$, it's enough to show that $H_i(\mathcal{F}) \cong M$.

By Remark 2.7.1, $H_i(\mathcal{F}) \cong H_i^{CW}(\mathcal{F})$.

Note that \mathcal{F} has the property that each cell e of X has a unique minimal grade of appearance $gr_{\mathcal{F}}(e)$ in \mathcal{F} . Since each cell has a unique minimal grade of appearance, for any $j \in \mathbb{Z}_{\geq 0}$, $C_j^{CW}(\mathcal{F})$ is free:

$$C_j^{CW}(\mathcal{F}) = \bigoplus_{e^j \subset X \text{ a } j\text{-cell}} B_n(-gr_{\mathcal{F}}(e^j)).$$

The usual identification of j -cells of X with a basis for $C_j^{CW}(X)$ extends in the obvious way to an identification of the j -cells of X with a basis for $C_j^{CW}(\mathcal{F})$.

Moreover, the boundary homomorphism $\delta_{i+1}^X : C_{i+1}^{CW}(X) \rightarrow C_i^{CW}(X)$ and the boundary homomorphism $\delta_{i+1}^{\mathcal{F}} : C_{i+1}^{CW}(\mathcal{F}) \rightarrow C_i^{CW}(\mathcal{F})$ are related in a simple way:

Lemma 2.10.12. $\delta_{i+1}^{\mathcal{F}}(e_y^{i+1}) = \sum_{w \in \mathcal{W}} a_{wy} \varphi_{C_i^{CW}(\mathcal{F})}(gr_{\mathcal{F}}(e_w^i), gr_{\mathcal{F}}(e_y^{i+1}))(e_w^i)$.

Proof. Recall that we constructed X in such a way that for $y \in \mathcal{Y}$, $\delta_{i+1}^X(e_y^{i+1}) = \sum_{w \in \mathcal{W}} a_{wy} e_w^i$. The result follows in a routine way from this expression for $\delta_{i+1}^X(e_y^{i+1})$ and the definition of the boundary map $\delta_{i+1}^{\mathcal{F}}$. \square

Now note that we have $\delta_i^{\mathcal{F}} = 0$. If $i \neq 1$ this follows from the fact that \mathcal{F} has no $i-1$ cells. If $i = 1$, it is still true because of the isomorphism between cellular and singular persistent homology: we must have $C_0^{CW}(\mathcal{F}) \cong B_n(-gr_{\mathcal{F}}(B)) \cong H_0(\mathcal{F}) \cong H_0^{CW}(\mathcal{F})$, so $\delta_1 = 0$.

Therefore $H_i^{CW}(\mathcal{F}) = C_i^{CW}(\mathcal{F})/\text{im}(\delta_{i+1}^{\mathcal{F}})$.

The bijection which sends $w \in \mathcal{W}$ to the cell e_w^i induces an isomorphism $\Lambda : \langle \mathcal{W}_1 \cup \mathcal{W}_2(-\epsilon) \rangle \rightarrow C_i^{CW}(\mathcal{F})$.

By the expression (2.3) for y in terms of a'_{wy} given in Section 2.10.3, for $y \in \mathcal{Y}_1 \cup \mathcal{Y}_2(-\epsilon)$,

$$\Lambda(y) = \sum_{w \in \mathcal{W}} a'_{wy} \varphi_{C_i^{CW}(\mathcal{F})}(gr_{\mathcal{F}}(e_w^i), gr_{\mathcal{F}}(e_y^{i+1}))(e_w^i).$$

Thus

$$\begin{aligned} & \Lambda(\langle \mathcal{Y}_1 \cup \mathcal{Y}_2(-\epsilon) \rangle) \\ &= \langle \{ \sum_{w \in \mathcal{W}} a'_{wy} \varphi_{C_i^{CW}(\mathcal{F})}(gr_{\mathcal{F}}(e_w^i), gr_{\mathcal{F}}(e_y^{i+1}))(e_w^i) \}_{y \in \mathcal{Y}} \rangle \\ &= \langle \{ \sum_{w \in \mathcal{W}} a_{wy} \varphi_{C_i^{CW}(\mathcal{F})}(gr_{\mathcal{F}}(e_w^i), gr_{\mathcal{F}}(e_y^{i+1}))(e_w^i) \}_{y \in \mathcal{Y}} \rangle \\ &= \text{im}(\delta_{i+1}^{\mathcal{F}}) \end{aligned}$$

by Lemma 2.10.12.

Λ therefore descends to an isomorphism between $C_i^{CW}(\mathcal{F})/\text{im}(\delta_{i+1}^{\mathcal{F}})$ and

$$\langle \mathcal{W}_1, \mathcal{W}_2(-\epsilon) \rangle / \langle \mathcal{Y}_1, \mathcal{Y}_2(-\epsilon) \rangle.$$

This shows that $H_i^{CW}(\mathcal{F}) = M$ and thus completes the proof of Proposition 2.10.4 in the special case that $GCD(\mathcal{W}) \in \mathbb{R}^n$.

2.10.6 Proof of Existence of Geometric Lifts of Interleavings, Part 4: Adapting the Proof to the Case That $GCD(\mathcal{W}) \notin \mathbb{R}^n$

As noted above, the proof of Proposition 2.10.4 we have given for the special case that $GCD(\mathcal{W}) \in \mathbb{R}^n$ adapts readily to a proof for the general case. We now outline this adaptation, leaving to the reader the straightforward details.

Let X' be the standard CW-complex structure on \mathbb{R} . That is, we take each $z \in \mathbb{Z}$

to be a 0-cell in X' , and for each $z \in \mathbb{Z}$, we take the interval $(z, z + 1)$ to be a 1-cell in X' .

To carry out the proof for the general case, we first present a modified version of the construction of the CW-complex X given in Section 2.10.3. In this modified version, we construct the CW-complex X so that

1. X' is a subcomplex of X .
2. X has an i -cell e_w^i for each $w \in \mathcal{W}$.
3. X has an $(i + 1)$ -cell e_y^{i+1} for each $y \in \mathcal{Y}$.
4. As a set, $X = X' \amalg \{e_w^i\}_{w \in \mathcal{W}} \amalg \{e_y^{i+1}\}_{y \in \mathcal{Y}}$.

For $r = (r_1, \dots, r_n) \in \mathbb{R}^n$, let $\lfloor r \rfloor = \max\{z \in \mathbb{Z} | z \leq r_j \text{ for } 1 \leq j \leq n\}$. For all $w \in \mathcal{W}$, let the attaching map of e_w^i be the constant map to the 0-cell $\lfloor gr(w) \rfloor \subset X'$.

This defines X^i . To complete the construction of X , it remains only to specify the attaching map $\sigma_y : S^i \rightarrow X^i$ for each $y \in \mathcal{Y}$.

Let a_{wy} be defined as in Section 2.10.3. An analogue of Lemma 2.10.10 holds in our present setting and admits a similar proof. Invoking this result, for each $y \in \mathcal{Y}$ we choose $\sigma_y : S^i \rightarrow X^i$ such that the CW-complex X constructed via these attaching maps satisfies $\delta_{i+1}^X(e_y^{i+1}) = \sum_{w \in \mathcal{W}} a_{wy} e_w^i$ for each $y \in \mathcal{Y}$.

For $r \in X' = \mathbb{R}$, we define $\gamma_M(r) = \gamma_N(r) = r\vec{1}$. As in the special case, we define the values of $\tilde{\gamma}_M$ and $\tilde{\gamma}_N$ at the origin of the disks D_w^i and D_y^{i+1} as follows:

- For $x \in \mathcal{W}_1 \cup \mathcal{Y}_1$, $\tilde{\gamma}_M(O(D_x)) = gr(x)$;
- For $x \in \mathcal{W}_2 \cup \mathcal{Y}_2$, $\tilde{\gamma}_M(O(D_x)) = gr(x(-\epsilon))$.
- For $x \in \mathcal{W}_1 \cup \mathcal{Y}_1$, $\tilde{\gamma}_N(O(D_x)) = gr(x(-\epsilon))$;
- For $x \in \mathcal{W}_2 \cup \mathcal{Y}_2$, $\tilde{\gamma}_N(O(D_x)) = gr(x)$.

As in the special case, we require the restriction of $\tilde{\gamma}_M$ and $\tilde{\gamma}_N$ to radial line segments of the disks D_w^i and D_y^{i+1} to be linear. This completes the specification of γ_M and γ_N .

The argument of Lemma 2.10.11 shows that $\|\gamma_M - \gamma_N\|_\infty = \epsilon$, and the argument of Section 2.10.5 adapts in a straightforward way to show that $H_i \circ F^S(X, \gamma_M) \cong M$ and $H_i \circ F^S(X, \gamma_N) \cong N$. \square

2.11 Reducing the Evaluation of d_I to Deciding Solvability of Quadratics

Let $\mathcal{MQ}(k)$ denote the set of multivariate systems of quadratic equations over the field k .

Fix $n \in \mathbb{N}$ and Let M and N be finitely presented B_n -persistence modules. Let q be the total number of generators and relations in a minimal presentation for M and in a minimal presentation for N . We show in this section that given minimal presentations for M and N , for any $\epsilon > 0$ deciding whether M and N are ϵ -interleaved is equivalent to deciding the solvability of an instance of $\mathcal{MQ}(k)$ with $O(q^2)$ unknowns and $O(q^2)$ equations.

We also show that d_I must be equal to one of the elements of an order $O(q^2)$ subset of $\mathbb{R}_{\geq 0}$ defined in terms of the grades of generators and relations of M and N . Thus, by searching through these values, we can compute d_I by deciding whether M and N are ϵ -interleaved for $O(\log q)$ values of ϵ . That is, we can compute $d_I(M, N)$ by deciding the solvability of $O(\log q)$ instances of $\mathcal{MQ}(k)$.

If e.g. k is a field of prime order, a standard algorithm based on Gröbner bases determines the solvability of systems in $\mathcal{MQ}(k)$. $\mathcal{MQ}(k)$ is NP-complete, however, and this algorithm is for general instances of $\mathcal{MQ}(k)$ prohibitively inefficient. We leave it to future work to investigate the complexity and tractability in practice of deciding the solvability of systems in $\mathcal{MQ}(k)$ arising from our reduction.

In practice, we are interested in computing the interleaving distance between the simplicial persistent homology modules of two simplicial n -filtrations. To apply the reduction presented here to this problem, we need a way of computing a presentation of the multidimensional persistent homology module of a simplicial n -filtration; strictly speaking, our reduction does not require that the presentations of our modules be minimal. However, to minimize the number and size of the quadratic systems we need to consider in computing the interleaving distance via this reduction, we do want the presentations we compute to be minimal.

We hope to address the problem of computing a minimal presentation of the simplicial persistent homology module of a simplicial n -filtration in future work.

2.11.1 Linear Algebraic Representations of Homogeneous Elements and Morphisms of Free B_n -persistence Modules

Representing Homogeneous Elements of Free B_n -persistence Modules as Vectors

Given a finitely generated free B_n -persistence module F and an (ordered) basis $B = b_1, \dots, b_l$ for F , we can represent a homogeneous element $v \in F$ as a pair $([v, B], gr(v))$ where $[v, B] \in k^l$ is a vector: if $v = \sum_{i: gr(v) \geq gr(b_i)} a_i \varphi_F(gr(b_i), gr(v))(b_i)$, with each $a_i \in k$, then for $1 \leq i \leq l$ we define

$$[v, B]_i = \begin{cases} a_i & \text{if } gr(v) \geq gr(b_i) \\ 0 & \text{otherwise.} \end{cases}$$

Remark 2.11.1. Note that for $1 \leq i \leq l$, $[b_i, B] = \mathbf{e}_i$, where \mathbf{e}_i denotes the i^{th} standard basis vector in k^l .

If $V \subset F$ is a homogeneous set, we define $[V, B] = \{[v, B] | v \in V\}$.

Representing Morphisms of B_n -persistence Modules as Matrices

Given finitely generated B_n -persistence modules F and F' and (ordered) bases $B = b_1, \dots, b_l$ and $B' = b'_1, \dots, b'_m$ for F and F' respectively, let $Mat_k(B, B')$ denote the set of $m \times l$ matrices A with entries in k such that $A_{ij} = 0$ whenever $gr(b_j) < gr(b'_i)$.

We can represent any $f \in \hom(F, F')$ as a matrix $[f, B, B'] \in Mat_k(B, B')$, where if $f(b_j) = \sum_{i: gr(b_j) \geq gr(b'_i)} a_{ij} \varphi_{F'}(gr(b'_i), gr(b_j))(b'_i)$, with each $a_{ij} \in k$, then

$$[f, B, B']_{ij} = \begin{cases} a_{ij} & \text{if } gr(b_j) \geq gr(b'_i) \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2.11.2. *The map $[\cdot, B, B'] : \hom(F, F') \rightarrow Mat_k(B, B')$ is a bijection.*

Proof. The proof is straightforward. \square

Note also the following additional properties of these matrix representations of morphisms between free B_n -modules:

Lemma 2.11.3. *Let F, F', F'' be free B_n -persistence modules with ordered bases B, B', B'' .*

- (i) *If $f_1, f_2 \in \text{hom}(F, F')$ then $[f_1 + f_2, B, B'] = [f_1, B, B'] + [f_2, B, B']$,*
- (ii) *If $f_1 \in \text{hom}(F, F')$, $f_2 \in \text{hom}(F', F'')$ then $[f_2 \circ f_1, B, B''] = [f_2, B', B''][f_1, B, B']$,*
- (iii) *For any $\epsilon \geq 0$, $[S(F, \epsilon), B, B(\epsilon)] = I_{|B|}$, where for $m \in \mathbb{N}$, I_m denotes the $m \times m$ identity matrix.*

Proof. The proof of each of these results is straightforward. \square

For a graded set W and $u \in \mathbb{R}^n$, let $W^u = \{y \in W | gr(y) \leq u\}$.

Lemma 2.11.4. *If F_1, F_2 are free B_n -persistence modules with bases B_1, B_2 and $W_1 \subset F_1, W_2 \subset F_2$ are sets of homogeneous elements then a morphism $f : F_1 \rightarrow F_2$ maps $\langle W_1 \rangle$ into $\langle W_2 \rangle$ iff $[f, B_1, B_2][w, B_1] \in \text{span}[W_2^{gr(w)}, B_2]$ for every $w \in W_1$.*

Proof. This is straightforward. \square

2.11.2 Deciding Whether Two B_n -persistence Modules are ϵ -interleaved is Equivalent to Deciding the Solvability of a System in $\mathcal{MQ}(k)$

Let $\langle G_M | R_M \rangle, \langle G_N | R_N \rangle$ be presentations for finitely presented B_n -modules M and N , and assume the elements of each of the sets G_M, G_N, R_M, R_N are endowed with a total order, which may be chosen arbitrarily. For a finite ordered set T and $1 \leq i \leq |T|$, let $T_{i,i}$ denote the i^{th} element of T .

We now define six matrices of variables, each with some of the variables constrained to be 0. Namely, we define

- **A**, a $|G_N| \times |G_M|$ matrix of variables with $\mathbf{A}_{ij} = 0$ if $gr(G_{M,j}) < gr(G_{N,i}) + \epsilon$.
- **B**, a $|G_M| \times |G_N|$ matrix of variables with $\mathbf{B}_{ij} = 0$ if $gr(G_{N,j}) < gr(G_{M,i}) + \epsilon$.

- \mathbf{C} , an $|R_N| \times |R_M|$ matrix of variables with $C_{ij} = 0$ if $gr(R_{M,j}) < gr(R_{N,i}) + \epsilon$.
- \mathbf{D} , an $|R_M| \times |R_N|$ matrix of variables with $D_{ij} = 0$ if $gr(R_{N,j}) < gr(R_{M,i}) + \epsilon$.
- \mathbf{E} , an $|R_M| \times |G_M|$ matrix of variables with $E_{ij} = 0$ if $gr(G_{M,j}) < gr(R_{M,i}) + 2\epsilon$.
- \mathbf{F} , an $|R_N| \times |G_N|$ matrix of variables with $F_{ij} = 0$ if $gr(G_{N,j}) < gr(R_{N,i}) + 2\epsilon$.

Let T_M denote the $|G_M| \times |R_M|$ matrix whose i^{th} column is $[R_{M,i}, G_M]$ and let T_N denote the $|G_N| \times |R_N|$ matrix whose i^{th} column is $[R_{N,i}, G_N]$.

Theorem 2.11.5. *M and N are ϵ -interleaved iff the multivariate system of quadratic equations*

$$\begin{aligned}\mathbf{A}T_M &= T_N\mathbf{C} \\ \mathbf{B}T_N &= T_M\mathbf{D} \\ \mathbf{B}\mathbf{A} - I_{|G_M|} &= T_M\mathbf{E} \\ \mathbf{A}\mathbf{B} - I_{|G_N|} &= T_N\mathbf{F}\end{aligned}$$

has a solution.

Proof. To prove the result, we proceed in three steps. First, we observe that for any free covers (F_M, ρ_M) and (F_N, ρ_N) of M and N , the existence of ϵ -interleaving morphisms between M and N is equivalent to the existence of a pair of morphisms between F_M and F_N having certain properties. We then note that the existence of such maps is equivalent to the existence of two matrices, one in $Mat_k(G_M, G_N)$ and the other in $Mat_k(G_N, G_M)$, having certain properties. Finally, we observe that the existence of such matrices is equivalent to the existence of a solution to the above multivariate system of quadratics.

Let (F_M, ρ_M) and (F_N, ρ_N) be free covers of M and N .

Lemma 2.11.6. *M and N are ϵ -interleaved iff there exist morphisms $\tilde{f} : F_M \rightarrow F_N(\epsilon)$ and $\tilde{g} : F_N \rightarrow F_M(\epsilon)$ such that*

1. $\tilde{f}(\ker(\rho_M)) \subset (\ker(\rho_N))(\epsilon)$,
2. $\tilde{g}(\ker(\rho_N)) \subset (\ker(\rho_M))(\epsilon)$,
3. $\tilde{g} \circ \tilde{f} - S(F_M, 2\epsilon) \subset (\ker(\rho_M))(2\epsilon)$,

$$4. \tilde{f} \circ \tilde{g} - S(F_N, 2\epsilon) \subset (\ker(\rho_N))(2\epsilon).$$

We'll call morphisms \tilde{f}, \tilde{g} satisfying the above properties **ϵ -interleaved lifts** of the free covers (F_M, ρ_M) and (F_N, ρ_N) .

Proof. Let $f : M \rightarrow N(\epsilon)$ and $g : N \rightarrow M(\epsilon)$ be interleaving morphisms. Then by Lemma 2.1.3 there exist lifts $\tilde{f} : F_M \rightarrow F_N(\epsilon)$ and $\tilde{g} : F_N \rightarrow F_M(\epsilon)$ of f and g . By the definition of a lift, \tilde{f} and \tilde{g} satisfy properties 1 and 2 in the statement of the lemma. $\tilde{g} \circ \tilde{f}$ is a lift of $g \circ f = S(M, 2\epsilon)$. $S(F_M, 2\epsilon)$ is also a lift of $S(M, 2\epsilon)$, so by the uniqueness up to homotopy of lifts (Lemma 2.1.3), \tilde{f} and \tilde{g} satisfy property 3. The same argument shows that \tilde{f} and \tilde{g} satisfy property 4.

The converse direction is straightforward; we omit the details. \square

Now let $F_M = \langle G_M \rangle$, $F_N = \langle G_N \rangle$, and let $\rho_M : F_M \rightarrow F_M/\langle R_M \rangle$, $\rho_N : F_N \rightarrow F_N/\langle R_N \rangle$ be the quotient maps. Since the interleaving distance between two modules is an isomorphism invariant of the modules, we may assume without loss of generality that $F_M/\langle R_M \rangle = M$ and $F_N/\langle R_N \rangle = N$. Then (F_M, ρ_M) and (F_N, ρ_N) are free covers of M and N .

Lemma 2.11.7. *M and N are ϵ -interleaved iff there exist matrices $A \in \text{Mat}_k(G_M, G_N)$ and $B \in \text{Mat}_k(G_N, G_M)$ such that*

1. $A[w, G_M] \in \text{span}[R_N^{gr(w)+\epsilon}, G_N]$ for all $w \in R_M$,
2. $B[w, G_N] \in \text{span}[R_M^{gr(w)+\epsilon}, G_M]$ for all $w \in R_N$,
3. $(BA - I_{|G_M|})(\mathbf{e}_i) \in \text{span}[R_M^{gr(G_{M,i})+2\epsilon}, G_M]$ for $1 \leq i \leq |G_M|$,
4. $(AB - I_{|G_N|})(\mathbf{e}_i) \in \text{span}[R_N^{gr(G_{N,i})+2\epsilon}, G_N]$ for $1 \leq i \leq |G_N|$.

Proof. By Lemma 2.11.6, M and N are ϵ -interleaved iff there exists ϵ -interleaved lifts $\tilde{f} : F_M \rightarrow F_N$ and $\tilde{g} : F_N \rightarrow F_M$ of the free covers (F_M, ρ_M) and (F_N, ρ_N) .

By Lemma 2.11.4, morphisms $\tilde{f} : F_M \rightarrow F_N$ and $\tilde{g} : F_N \rightarrow F_M$, are ϵ -interleaved lifts iff

1. $[\tilde{f}, G_M, G_N(\epsilon)][w, G_M] \in \text{span}[R_N(\epsilon)^{gr(w)}, G_N(\epsilon)]$ for all $w \in R_M$,
2. $[\tilde{g}, G_N, G_M(\epsilon)][w, G_N] \in \text{span}[R_M(\epsilon)^{gw(w)}, G_M(\epsilon)]$ for all $w \in R_N$,

3. $[\tilde{g} \circ \tilde{f} - S(F_M, 2\epsilon), G_M, G_M(2\epsilon)][w, G_M] \in \text{span}[R_M(2\epsilon)^{gr(w)}, G_M(2\epsilon)]$ for all $w \in G_M$,
4. $[\tilde{f} \circ \tilde{g} - S(F_N, 2\epsilon), G_N, G_N(2\epsilon)][w, G_N] \in \text{span}[R_N(2\epsilon)^{gr(w)}, G_N(2\epsilon)]$ for all $w \in G_N$.

By Lemma 2.11.3,

$$[\tilde{g} \circ \tilde{f} - S(F_M, 2\epsilon), G_M, G_M(2\epsilon)] = [\tilde{g}, G_N, G_M(\epsilon)][\tilde{f}, G_M, G_N(\epsilon)] - I_{|G_M|}$$

and

$$[\tilde{f} \circ \tilde{g} - S(F_N, 2\epsilon), G_N, G_N(2\epsilon)] = [\tilde{f}, G_M, G_N(\epsilon)][\tilde{g}, G_N, G_M(\epsilon)] - I_{|G_N|}.$$

Also, by Remark 2.11.1, for $1 \leq i \leq |G_M|$, $[G_{M,i}, G_M] = \mathbf{e}_i$, where \mathbf{e}_i is the i^{th} standard basis vector in $k^{|G_M|}$. Similarly, for $1 \leq i \leq |G_N|$, $[G_{N,i}, G_N] = \mathbf{e}_i$, where \mathbf{e}_i is the i^{th} standard basis vector in $k^{|G_N|}$.

Finally, note that we have that

$$\begin{aligned} [R_N(\epsilon)^{gr(w)}, G_N(\epsilon)] &= [R_N^{gr(w)+\epsilon}, G_N] \text{ for all } w \in R_M, \\ [R_M(\epsilon)^{gw(w)}, G_M(\epsilon)] &= [R_M^{gr(w)+\epsilon}, G_M] \text{ for all } w \in R_N, \\ [R_M(2\epsilon)^{gr(w)}, G_M(2\epsilon)] &= [R_M^{gr(w)+2\epsilon}, G_M] \text{ for all } w \in G_M, \\ [R_N(2\epsilon)^{gr(w)}, G_N(2\epsilon)] &= [R_N^{gr(w)+2\epsilon}, G_N] \text{ for all } w \in G_N. \end{aligned}$$

Using all of these observations, Lemma 2.11.7 now follows from Lemma 2.11.2. \square

Finally, Theorem 2.11.5 follows from Lemma 2.11.7 by way of elementary matrix algebra and, in particular, the basic fact that for $l, m \in \mathbb{N}$ and vectors v, v_1, \dots, v_l in k^m , $v \in \text{span}(v_1, \dots, v_l)$ iff there exists a vector $w \in k^l$ such that $v = Vw$, where V is the $m \times l$ matrix whose i^{th} column is v_i . \square

Remark 2.11.8. Note that the size of the system of quadratic equations in the statement of Theorem 2.11.5 is $O(q^2)$, where q is the total number of generators and

relations in the presentations for M and N . For any $\epsilon \geq 0$, the system of quadratics has as few variables and equations as possible when the presentations for M and N are minimal.

2.11.3 Determining Possible Values for $d_I(M, N)$

Let M and N be finitely presented B_n -modules, and let U_M^i , U_N^i , U_M , and U_N be as defined at the beginning of Section 2.5. Let

$$U_{M,N} = \bigcup_i \left(\{|x - y|\}_{x \in U_M^i, y \in U_N^i} \cup \left\{ \frac{1}{2}|x - y| \right\}_{x, y \in U_M^i} \cup \left\{ \frac{1}{2}|x - y| \right\}_{x, y \in U_N^i} \right) \cup \{0, \infty\}.$$

Note that $|U_{M,N}| = O(q^2)$, where as above q is the total number of generators and relations in a minimal presentation for M and a minimal presentation for N .

Proposition 2.11.9. $d_I(M, N) \in U_{M,N}$.

Proof. Assume that for some $\epsilon' > 0$, $\epsilon' \notin U_{M,N}$, M and N are ϵ' -interleaved. Let ϵ be the largest element of $U_{M,N}$ such that $\epsilon' > \epsilon$, and let $\delta = \epsilon' - \epsilon$.

We'll check that M, N, ϵ and δ satisfy the hypotheses of Lemma 2.5.7. The lemma then implies that M and N are ϵ -interleaved. The result follows.

By assumption, M and N are $(\epsilon + \delta)$ -interleaved, so the first hypothesis of Lemma 2.5.7 is satisfied. We'll show that the second hypothesis is satisfied; the proof that the third hypothesis is satisfied is the same as that for the second hypothesis.

If $z \in U_M$ then for no i , $1 \leq i \leq n$, can an element of U_N^i lie in $(z_i + \epsilon, z_i + \epsilon + \delta]$; if, to the contrary, for some i there were an element $u \in U_N^i \cap (z_i + \epsilon, z_i + \epsilon + \delta]$, then we would have $|u - z_i| \in U_{M,N}$, and $\epsilon < |u - z_i| \leq \epsilon + \delta$, which contradicts the way we chose ϵ and δ . Thus by Lemma 2.5.4, $\varphi_N(z + \epsilon, z + \epsilon + \delta)$ is an isomorphism.

Similarly, for no i , $1 \leq i \leq n$, can an element of U_M^i lie in $(z_i + 2\epsilon, z_i + 2\epsilon + 2\delta]$; if, to the contrary, for some i there were an element $u \in U_M^i \cap (z_i + 2\epsilon, z_i + 2\epsilon + 2\delta]$, then we would have $\frac{1}{2}|u - z_i| \in U_{M,N}$, and $\epsilon < \frac{1}{2}|u - z_i| \leq \epsilon + \delta$, which again contradicts the way we chose ϵ and δ . By Lemma 2.5.4, $\varphi_M(z + 2\epsilon, z + 2\epsilon + 2\delta)$ is an isomorphism.

Thus the second hypothesis of Lemma 2.5.7 is satisfied by our M, N, ϵ, δ , as we wanted to show. \square

2.12 Discussion of Future Work on the Interleaving Distance

Theorem 2.4.2, Corollary 2.5.2, and Corollary 2.10.2 show that the interleaving distance is a natural generalization of the bottleneck distance to the setting of multidimensional persistence.

Insofar as the interleaving distance is in fact a good choice of distance on multidimensional persistence modules, the question of how to compute it is interesting and, it seems to us, potentially important from the standpoint of applications. The results of Section 2.11 suggest a path towards the development of a theory of computation of the interleaving distance. As noted in Section 2.11, to exploit the connection with multivariate quadratics in the development of such a theory in practice, we need in particular a way of computing minimal presentations of simplicial homology modules of simplicial n -filtrations. We hope to address this problem in future work.

Corollary 2.10.2 demonstrates that the interleaving distance is optimal in the sense of Example 2.9.6 when $k = \mathbb{Q}$ or $\mathbb{Z}/p\mathbb{Z}$. However, our discussion of optimality of pseudometrics in Section 2.9 raises many more questions than it answers. Some of the more interesting questions are:

1. Can we extend the result of Theorem 2.10.1 to arbitrary ground fields?
2. Can we extend the result of Theorem 2.10.1 to the case $i = 0$?
3. Can we prove that the interleaving distance is \mathcal{R} -optimal for \mathcal{R} any of the relative structures on $\text{obj}^*(B_n\text{-mod})$ defined in Examples 2.9.9-2.9.12? The case of Example 2.9.10 is of particular interest to us.
4. Can we obtain analogous results about the optimality of pseudometrics on more general types of persistent homology modules? For instance, can we prove a result analogous to Theorem 2.10.1 for levelset zigzag persistence [5]?

An interesting question related to question 4 above is whether there is a way of algebraically reformulating the bottleneck distance for zigzag persistence modules as an analogue of the interleaving distance in such way that the definition generalizes to a larger classes of quiver representations [17].

It seems quite likely that Theorem 2.4.1, the algebraic stability result of [8], generalizes to a theorem which quantifies the similarity between the persistence diagrams of a pair of tame (J_1, J_2) -interleaved B_1 -persistence modules. It would be nice to have a generalized algebraic stability theorem of this kind.

Finally, we mention again that it would be nice to have an extension of Theorem 2.3.6 to a structure theorem for arbitrary tame B_1 -persistence modules, and an extension of Corollary 2.5.2 to well behaved tame B_n -persistence modules.

Chapter 3

Strong and Weak Interleavings on Multidimensional Filtrations

In this chapter we introduce and study weak and strong interleavings and interleaving distances on multidimensional filtrations. See Section 1.4 for an overview of the chapter.

3.1 u -filtrations and their Persistent Homology

In this section we define u -filtrations, a mild generalization of the n -filtrations introduced in Section 2.7.2 which allows for the coordinates of points in the index set of the spaces in a multidimensional filtration to have finite upper bounds. We also generalize the multidimensional persistent homology functor, introduced in Section 2.7.3, to a category whose objects are u -filtrations. These generalizations will be useful in formulating our inference results in Chapter 4.

3.1.1 u -Filtrations

Fix $n \in \mathbb{N}$. For $u \in \mathbb{R}^n$, we now define the category $u\text{-filt}$ of u -filtrations.

Define a **u -filtration** X to be a collection of topological spaces $\{X_a\}_{a < u}$, together with a collection of continuous functions $\{\phi_X(a, b) : X_a \rightarrow X_b\}_{a \leq b < u}$ such that if

$a \leq b \leq c < u$ then $\phi_X(b, c) \circ \phi_X(a, b) = \phi_X(a, c)$. We call the functions $\{\phi_X(a, b) : X_a \rightarrow X_b\}_{a \leq b < u}$ *transition maps*.

Note that an ∞_n -filtration is the same as an n -filtration, as defined in Section 2.7.2. To avoid ambiguity, we stipulate that in this thesis, whenever $n \in \mathbb{N}$, by n -filtration we will always mean an ∞_n -filtration, and not a u -filtration with $u = n$.

Given two u -filtrations X and Y , we define a morphism f from X to Y to be a collection of continuous functions $\{f_a\}_{a < u} : X_a \rightarrow Y_a$ such that for all $a \leq b < u$, $f_b \circ \phi_X(a, b) = \phi_Y(a, b) \circ f_a$. This definition of morphism gives the u -filtrations the structure of a category. Let $u\text{-filt}$ denote this category.

For all $u' \leq u$, we can define the **restriction functor** $R_{u'} : u\text{-filt} \rightarrow u'\text{-filt}$ in the obvious way. (The dependence of $R_{u'}$ on u is implicit in our notation.) We'll use these functors heavily in what follows.

3.1.2 The Multidimensional Persistent Homology Functor on u -filtrations

Here we define the multidimensional persistent homology functor $H_i : u\text{-filt} \rightarrow B_n\text{-mod}$ for any $u \in \hat{\mathbb{R}}^n$, extending the definition introduced in Section 2.7.3 for $u = \infty_n$.

As in Section 2.7.3, for a topological space X and $i \in \mathbb{Z}_{\geq 0}$, let $C_i(X)$ denote the i^{th} singular chain module of X , with coefficients in k ; for X, Y topological spaces and $f : X \rightarrow Y$ a continuous map, let $f^\# : C_i(X) \rightarrow C_i(Y)$ denote the map induced by f .

For X a u -filtration, define $C_i(X)$, the i^{th} singular chain module of X , to be the B_n -persistence module for which

- $C_i(X)_a = C_i(X_a)$ for all $a < u$,
- $C_i(X)_a = 0$ for all $a \not< u$,
- $\varphi_{C_i(X)}(a, b) = \phi_X^\#(a, b)$ for $b < u$,
- $\varphi_{C_i(X)}(a, b) = 0$ for $b \not< u$.

Note that for any $i \in \mathbb{Z}_{\geq 0}$, the collection of boundary maps $\{\delta_i : C_i(X_a) \rightarrow C_{i-1}(X_a)\}_{a < u}$ induces a boundary map $\delta_i : C_i(X) \rightarrow C_{i-1}(X)$. These boundary maps give $\{C_i(X)\}_{i \in \mathbb{Z}_{\geq 0}}$ the structure of a chain complex. We define the $H_i(X)$, the i^{th}

persistent homology module of X , to be the i^{th} homology module of this complex. For X and Y two u -filtrations, a morphism $f \in \text{hom}(X, Y)$ induces in the obvious way a morphism $H_i(f) : H_i(X) \rightarrow H_i(Y)$, making $H_i : u\text{-filt} \rightarrow B_n\text{-mod}$ a functor.

Compatibility of Multidimensional Persistent Homology With Restriction Functors

For all $u \in \hat{\mathbb{R}}^n$, we define the **restriction functor** $R_u : B_n\text{-mod} \rightarrow B_n\text{-mod}$ by taking $R_u(M)$ of a B_n -persistence module M to be such that

- $R_u(M)_a = M_a$ for all $a < u$,
- $R_u(M)_a = 0$ for all $a \not< u$,
- $\varphi_{R_u(M)}(a, b) = \varphi_M(a, b)$ for $b < u$,
- $\varphi_{R_u(M)}(a, b) = 0$ for $b \not< u$.

We define the action of R_u on morphisms in the obvious way.

Remark 3.1.1. Note that if M is a B_n -persistence module having a finite presentation $\langle G|R \rangle$ [32], then for any $u \in \hat{\mathbb{R}}^n$, $R_u(M)$ has a presentation consisting of at most $|G|$ generators and $|R| + n|G|$ relations.

Lemma 3.1.2 (Compatibility of persistent homology with restriction functors). *For any $u' \leq u \in \mathbb{R}^n$,*

$$H_i \circ R_{u'} = R_{u'} \circ H_i.$$

Proof. The proof is easy; we omit it. □

3.2 Strong Interleavings and the Strong Interleaving Distance on u -Filtrations

Overview

In this section, we introduce strong interleavings and the strong interleaving distance on u -filtrations, and develop their basic theory; in Section 3.3 we will introduce and

study our homotopy theoretic variants of these, weak interleavings and the weak interleaving distance.

As we noted in the introduction, our inference results in Chapter 4 are formulated using weak interleavings and the weak interleaving distance. In this sense, weak interleavings and the weak interleaving distance are of more interest to us than strong interleavings and the strong interleaving distance. Nevertheless, because the program of understanding weak interleavings is related to the simpler program of understanding strong interleavings, we choose to devote some effort here to writing down some basic results about strong interleavings and the strong interleaving distance.

To begin this section, we define shift functors and transition morphisms for categories of u -filtrations. Using these, we define strong (J_1, J_2) -interleavings of u -filtrations and the strong interleaving distance; the definitions are similar to those we introduced for B_n -persistence modules in Chapter 2. Whereas we first introduced ϵ -interleavings of B_n -persistence modules in Section 2.1.5 and then presented the more general definition of strong (J_1, J_2) -interleavings of B_n -persistence modules in Section 2.6.1, we proceed here directly with the general definition of strong interleavings of u -filtrations.

We present a characterization result for strongly (J_1, J_2) -interleaved pairs of u -filtrations, for an important class of u -filtrations. We also present two optimality results for the strong interleaving distance. These are analogues of our optimality result Theorem 2.10.1 for the interleaving distance on B_n -persistence modules.

3.2.1 Definitions of Strong Interleavings and the Strong Interleaving Distance

Shift Functors

Fix $u \in \hat{\mathbb{R}}^n$ and let $J : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be an order-preserving map. We define the J -shift functor $(\cdot)(J) : u\text{-filt} \rightarrow J^{-1}(u)\text{-filt}$ in a way analogous to the way we did for B_n -persistence modules:

1. Action of $(\cdot)(J)$ on objects: For $X \in \text{obj}(u\text{-filt})$, we define $X(J)$ by taking

$X(J)_a = X_{J(a)}$ for all $a < J^{-1}(u)$. We take the maps $\{\phi_{X(J)}(a, b)\}_{a \leq b < J^{-1}(u)}$ to be induced by those of X in the obvious way.

2. Action of $(\cdot)(J)$ on morphisms: For $X, Y \in \text{obj}(u\text{-filt})$ and $f \in \text{hom}(X, Y)$, we define $f(J) : X(J) \rightarrow Y(J)$ to be the morphism for which $f(J)_a = f_{J(a)}$ for all $a < J^{-1}(u)$.

Note that the dependence of the definition of the functor $(\cdot)(J)$ on the choice of u is implicit in our notation.

As in the case of B_n -persistence modules, for $\epsilon \in \mathbb{R}$ we let $(\cdot)(\epsilon)$ denote the functor $(\cdot)(J_\epsilon)$.

Transition Morphisms

For a u -filtration X and $J : \mathbb{R}^n \rightarrow \mathbb{R}^n$ increasing, let $S(X, J) : R_{J^{-1}(u)}(X) \rightarrow X(J)$, the **J -transition morphism**, be the morphism whose restriction to X_a is the map $\phi_X(a, J(a))$ for all $a < J^{-1}(u)$.

For $\epsilon \in \mathbb{R}_{\geq 0}$, let $S(X, \epsilon) = S(X, J_\epsilon)$.

The expected analogue of Lemma 2.6.1 holds, with the same proof.

Lemma 3.2.1.

(i) For any $f : X \rightarrow Y \in \text{hom}(u\text{-filt})$ and any $J : \mathbb{R}^n \rightarrow \mathbb{R}^n$ increasing,

$$S(Y, J) \circ f = f(J) \circ S(X, J).$$

(ii) For any J, J' increasing and u -filtration X ,

$$S(X, J')(J) \circ S(X, J) = S(X, J' \circ J).$$

(iii) For any J, J' increasing and u -filtration X ,

$$S(X, J')(J) = S(X(J), J^{-1} \circ J' \circ J).$$

Strong (J_1, J_2) -Interleavings of u -filtrations

For $J_1, J_2 : \mathbb{R}^n \rightarrow \mathbb{R}^n$ increasing, we say that a pair (X, Y) of u -filtrations is **strongly (J_1, J_2) -interleaved** (or, more colloquially, that X and Y are strongly (J_1, J_2) -interleaved) if there exist morphisms $f : R_{J_1^{-1}(u)}(X) \rightarrow Y(J_1)$ and $g : R_{J_2^{-1}(u)}(Y) \rightarrow X(J_2)$ such that

$$\begin{aligned} g(J_1) \circ R_{J_1^{-1} \circ J_2^{-1}(u)}(f) &= S(X, J_2 \circ J_1) \text{ and} \\ f(J_2) \circ R_{J_2^{-1} \circ J_1^{-1}(u)}(g) &= S(Y, J_1 \circ J_2); \end{aligned}$$

we say (f, g) is a pair of **strong (J_1, J_2) -interleaving morphisms** for (X, Y) .

Remark 3.2.2. When discussing (J_1, J_2) -interleavings and (J_1, J_2) -interleaving morphisms of u -filtrations, we will follow terminological conventions analogous to those introduced for (J_1, J_2) -interleaved B_n -persistence modules in Remark 2.6.2.

If for $\epsilon \geq 0$, a pair of u -filtrations is strongly (J_ϵ, J_ϵ) -interleaved, we say that they are strongly ϵ -interleaved.

Since the definition of strong (J_1, J_2) -interleavings is a bit complex, the reader may find it illuminating to see the definition in the special case of ϵ -interleavings: For $\epsilon \in [0, \infty)$, two u -filtrations X and Y are strongly ϵ -interleaved if and only if there exist morphisms $f : R_{u-\epsilon}(X) \rightarrow Y(\epsilon)$ and $g : R_{u-\epsilon}(Y) \rightarrow X(\epsilon)$ such that

$$\begin{aligned} g(\epsilon) \circ R_{u-2\epsilon}(f) &= S(X, 2\epsilon) \text{ and} \\ f(\epsilon) \circ R_{u-2\epsilon}(g) &= S(Y, 2\epsilon). \end{aligned}$$

When $u = \vec{\infty}_n$ the definition of ϵ -interleavings simplifies yet further to give a definition closely analogous to that of ϵ -interleavings of B_n -persistence modules: For $\epsilon \in [0, \infty)$, two n -filtrations X and Y are strongly ϵ -interleaved if and only if there exist morphisms $f : X \rightarrow Y(\epsilon)$ and $g : Y \rightarrow X(\epsilon)$ such that

$$\begin{aligned} g(\epsilon) \circ f &= S(X, 2\epsilon) \text{ and} \\ f(\epsilon) \circ g &= S(Y, 2\epsilon). \end{aligned}$$

The Strong Interleaving Distance

We define $d_{SI} : \text{obj}^*(u\text{-filt}) \times \text{obj}^*(u\text{-filt}) \rightarrow [0, \infty]$, the **strong interleaving distance**, by taking

$$d_{SI}(X, Y) = \inf\{\epsilon \in \mathbb{R}_{\geq 0} \mid X \text{ and } Y \text{ are strongly } \epsilon\text{-interleaved}\}.$$

3.2.2 Stability Results for the Strong Interleaving distance

d_{SI} satisfies stability properties with respect to the functors F^S , F^{SO} , $F^{S\check{C}e}$ analogous to the stability properties Theorems 2.8.1-2.8.3 satisfied by the interleaving distance on B_n -persistence modules with respect to the functors $H_i \circ F^S$, $H_i \circ F^{SO}$, and $H_i \circ F^{S\check{C}e}$. The formulation and proofs of these results are easy modifications of those of Section 2.8.

3.2.3 First Examples of (J_1, J_2) -interleaved Filtrations

Example 3.2.3.

- (i) For any $\delta > 0$ and $(X, Y, d, f) \in \text{obj}(C^{SO})$, $F^{SO}(X, Y, d, f)$, $F^{SO-Op}(X, Y, d, f)$ are strongly δ -interleaved.
- (ii) For any $\delta > 0$ and $(X, Y, d, f) \in \text{obj}(C^{S\check{C}e})$, $F^{S\check{C}e}(X, Y, d, f)$, $F^{S\check{C}e-Op}(X, Y, d, f)$ are strongly δ -interleaved.

Example 3.2.4. For $n \in \mathbb{N}$, let $\text{id}_n : \mathbb{R}^n \rightarrow \mathbb{R}^n$ denote the identity map.

Let $\mathcal{J}_1 : \mathbb{R} \rightarrow \mathbb{R}$ be given by

$$\mathcal{J}_1(y) = \begin{cases} 2y & \text{if } y \geq 0, \\ y & \text{if } y < 0. \end{cases}$$

If (Y, d) is any metric space, $X \subset Y$, $F^R(X, d)$ is the usual Rips filtration on (X, d) , and $F^{\check{C}e}(X, Y, d)$ is the usual Čechfiltration on (X, Y, d) , then the inclusions (1.1) in Section 1.5.3 immediately imply that $F^R(X, d)$, $F^{\check{C}e}(X, Y, d)$ are $(\mathcal{J}_1, \text{id}_1)$ -interleaved.

To generalize this example, for $n \in \mathbb{N}$, let $\mathcal{J}_{n+1} : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$ be given by

$$\mathcal{J}_{n+1}(y_1, \dots, y_{n+1}) = \begin{cases} (y_1, \dots, y_n, 2y_{n+1}) & \text{if } y_{n+1} \geq 0, \\ (y_1, \dots, y_{n+1}) & \text{if } y < 0. \end{cases}$$

Then it again follows immediately from the inclusions (1.1) in Section 1.5.3 that for any $(X, Y, d, \gamma) \in \text{obj}(C^{SCe})$, $F^{SR}(X, d, \gamma)$, $F^{S\check{C}e}(X, Y, d, \gamma)$ are $(\mathcal{J}_{n+1}, \text{id}_{n+1})$ -interleaved.

3.2.4 A Characterization of Strongly Interleaved Pairs of Modules of Nested Type

We would like to have a transparent characterization of strongly (J_1, J_2) -interleaved pairs of u -filtrations, as Theorem 2.6.4 gives us for (J_1, J_2) -interleaved pairs of B_n -persistence modules.

It is not clear how obtain such a characterization in full generality; the obstacle to adapting the approach of Theorem 2.6.4 is that, at least naively, in the category of u -filt there is no reasonably behaved notion of free covers, as we have in the category $B_n\text{-mod}$.

However, we present a characterization result, Theorem 3.2.7, for strongly (J_1, J_2) -interleaved pairs u -filtrations, provided we restrict attention to the u -filtrations in a full subcategory **u -nest** of u -filt which is large enough to contain all of the examples of u -filtrations that we have occasion to consider in this thesis. In the special case that $u = \vec{\infty}_n$, this characterization is very simple and transparent (Corollary 3.2.9). For general u , the characterization is not as transparent, but it does imply a transparent necessary condition for the existence of a strong (J_1, J_2) -interleaving between two u -filtrations of nested type, Corollary 3.2.8, similar to our necessary and sufficient condition for n -filtrations.

u -filtrations of Nested Type

We define u -nest to be the full subcategory of u -filt whose objects are filtrations X such that for all $a \leq b < u$, the transition map $\phi_X(a, b)$ is injective. We refer to the

objects of u -filt as **u -filtrations of nested type**.¹

As we will now see, we can regard u -filtrations of nested type as an n -parameter family of nested subspaces of an ambient topological space; our characterization of strongly (J_1, J_2) -interleaved u -filtrations follows from the adoption of this viewpoint.

Colimits of u -filtrations

Let **Top** denote the category of topological spaces.

Define the **diagonal functor** $\Delta : \mathbf{Top} \rightarrow u\text{-filt}$ to act on objects W in **Top** by taking $\Delta(W)_a = W$ for all $a < u$ and $\phi_{\Delta(W)}(a, b) = \text{id}_W$ for all $a \leq b < u$; we define the action of Δ on $\text{hom}(\mathbf{Top})$ in the obvious way.

It is well known that the usual category of topological spaces has all small colimits [33, 21]. In particular, we can define the colimit of a u -filtration X . This is a topological space $\text{colim}(X)$ together with a morphism $s^X : X \rightarrow \Delta(\text{colim}(X))$ satisfying the following universal property: If W is a topological space and $s' : X \rightarrow \Delta(W)$ is a morphism, then there is a unique map $t : \text{colim}(X) \rightarrow W$ such that $s' = \Delta(t) \circ s^X$.

For a u -filtration X $\text{colim}(X)$ is, as a set, the quotient of $\coprod_{a < u} X_a$ by the relation \sim generated by taking $x \sim \phi_X(a, b)(x)$, for all $a \leq b < u$ and $x \in X_a$. We define s^X so that for each $a < u$, s_a^X takes each element of X_a to its equivalence class in $\text{colim}(X)$. We take $\text{colim}(X)$ to be endowed with the *final topology*. This means that a set $U \in \text{colim}(X)$ is open if and only if $(s_a^X)^{-1}(U)$ is open for all $a < u$.

We can, in the obvious way, also define an action of $\text{colim}(\cdot)$ on $\text{hom}(u\text{-filt})$ to obtain a functor $\text{colim} : u\text{-filt} \rightarrow \mathbf{Top}$.

Lemma 3.2.5. *If X is a u -filtration of nested type then s_a^X is injective for all $a < u$.*

Proof. The proof is straightforward; we leave it to the reader. \square

Colimit Representations of u -Filtrations of Nested Type

Let $\text{Sub}(u) = \{y | y < u\}$. Let A_u denote the set of subsets S of $\text{Sub}(u)$ such that if $y \in S$ and $y \leq y' < u$ then $y' \in S$.

¹Note that in the literature on persistent homology, filtrations of nested type are typically the only kinds of filtrations defined and studied; there is not a huge loss of generality in restricting attention to these.

Let CL_u denote the category whose objects are pairs (W, f) , where W is a topological space and $f : W \rightarrow A_u$ is a function. For $(W_1, f_1), (W_2, f_2) \in \text{obj}(CL_u)$, let $\text{hom}((W_1, f_1), (W_2, f_2))$ denote continuous maps $g : W_1 \rightarrow W_2$ such that for all $y \in W_1$, $f_1(y) \subset f_2(g(y))$.

We'll define a functor $\mathcal{A} : u\text{-filt} \rightarrow CL_u$ such that \mathcal{A} restricts to an equivalence between $u\text{-nest}$ and its image under \mathcal{A} .

First, for X a u -filtration, we define a function $ROA_X : \text{colim}(X) \rightarrow A_u$ by

$$ROA_X(x) = \{y | y < u \text{ and } x \text{ lies in the image of } s_y^X\}.$$

We say that $ROA_X(x)$ is the **region of appearance of x in X** . We define \mathcal{A} by taking $\mathcal{A}(X) = (\text{colim}(X), ROA_X)$ for any u -filtration X and $\mathcal{A}(f) = \text{colim}(f)$ for all $f \in \text{hom}(u\text{-filt})$.

We can define a functor $\text{filt} : CL_u \rightarrow u\text{-nest}$ by taking $\text{filt}((W, f))_a = \{y \in W | a \in f(y)\}$ and endowing $\text{filt}((W, f))_a$ with the subspace topology. The action of filt on $\text{hom}(CL_u)$ is defined in the obvious way.

Lemma 3.2.6. *The restriction of \mathcal{A} to $u\text{-nest}$ is equivalence of categories between $u\text{-nest}$ and the full subcategory CL'_u of CL_u whose set of objects is the image of $\text{obj}(u\text{-nest})$ under \mathcal{A} ; the restriction of filt to CL'_u is an inverse of the restriction of \mathcal{A} to $u\text{-nest}$.*

Proof. This is straightforward; we leave the details to the reader. \square

Characterization of Interleaved pairs of Filtrations of Nested Type

Now we are ready to offer our characterization of (J_1, J_2) -interleaved pairs of u -filtrations of nested type. Roughly, it says that u -filtrations X, Y are (J_1, J_2) -interleaved if and only if there is a pair of inverse homeomorphisms between “large” subspaces of $\text{colim}(X)$ and $\text{colim}(Y)$ which is compatible with the data of ROA_X and ROA_Y in an appropriate sense, and additionally these homeomorphisms admit certain extensions.

Note that if $u \in \hat{\mathbb{R}}^n$, $u' \leq u$, and X is a u -filtration, then $\text{colim}(R_{u'}(X))$ can be identified with a subspace of $\text{colim}(X)$.

Theorem 3.2.7. *For J_1, J_2 increasing, a pair (X, Y) of u -filtrations of nested type is (J_1, J_2) -interleaved if and only if there are maps*

$$\begin{aligned} f : \text{colim}(R_{J_1^{-1}(u)}(X)) &\rightarrow \text{colim}(Y) \text{ and} \\ g : \text{colim}(R_{J_2^{-1}(u)}(Y)) &\rightarrow \text{colim}(X) \end{aligned}$$

such that

1. f and g restrict to inverse homeomorphisms between

$$\begin{aligned} \text{colim}(R_{J_1^{-1} \circ J_2^{-1}(u)}(X)) \cup g(\text{colim}(R_{J_2^{-1} \circ J_1^{-1}(u)}(Y))) &\text{ and} \\ \text{colim}(R_{J_2^{-1} \circ J_1^{-1}(u)}(Y)) \cup f(\text{colim}(R_{J_1^{-1} \circ J_2^{-1}(u)}(X))), \end{aligned}$$

2. $J_1(ROA_X(y)) \subset ROA_Y(f(y))$ for all $y \in \text{colim}(R_{J_1^{-1}(u)}(X))$, and
3. $J_2(ROA_Y(y)) \subset ROA_X(g(y))$ for all $y \in \text{colim}(R_{J_2^{-1}(u)}(Y))$.

Proof. The proof is a straightforward application of Lemma 3.2.6; we leave the unwinding of definitions to the reader. \square

The following immediate consequence of the last theorem gives our transparent necessary condition for the existence of a (J_1, J_2) -interleaving between two u -filtrations.

Corollary 3.2.8. *If a pair (X, Y) of u -filtrations of nested type is (J_1, J_2) -interleaved then there are sets D_X, D_Y with*

$$\begin{aligned} \text{colim}(R_{J_1^{-1} \circ J_2^{-1}(u)}(X)) \subset D_X \subset \text{colim}(X), \\ \text{colim}(R_{J_2^{-1} \circ J_1^{-1}(u)}(Y)) \subset D_Y \subset \text{colim}(Y), \end{aligned}$$

and a homeomorphism $f : D_X \rightarrow D_Y$ such that

1. $J_1(ROA_X(y)) \subset ROA_Y(f(y))$ for all $y \in D_X$, and
2. $J_2(ROA_Y(y)) \subset ROA_X(f^{-1}(y))$ for all $y \in D_Y$.

When $u = \vec{\infty}_n$, Theorem 3.2.7 reduces to the following:

Corollary 3.2.9. *A pair (X, Y) of n -filtrations of nested type is (J_1, J_2) -interleaved if and only if there is a homeomorphism $f : \text{colim}(X) \rightarrow \text{colim}(Y)$, such that*

1. $J_1(\text{ROA}_X(y)) \subset \text{ROA}_Y(f(y))$ for all $y \in \text{colim}(X)$, and
2. $J_2(\text{ROA}_Y(y)) \subset \text{ROA}_X(f^{-1}(y))$ for all $y \in \text{colim}(Y)$.

3.2.5 Optimality Properties of the Strong Interleaving Distance

Using Corollary 3.2.9, we now observe that d_{SI} is optimal in a sense analogous to that in which we have proven d_I to be optimal in Section 2.10. We also show that d_I is \mathcal{R} -optimal, for \mathcal{R} a relative structure defined in terms of d_{SI} .

The results here, while perhaps not especially surprising in light of the optimality results of Section 2.10, are of significance in that they show that optimality theory for d_{SI} behaves as one would hope, given those results.

The reader may find it helpful to now review the notation and terminology of Section 2.9.2.

Theorem 3.2.10. *Let $\mathcal{R}_4 = (\text{obj}(C^S), d^S, F^S)$, so that \mathcal{R}_4 is a relative structure on $\text{obj}^*(n\text{-filt})$. Then d_{SI} is \mathcal{R}_4 -optimal.*

Proof. We have observed above that d_{SI} is \mathcal{R}_4 -stable. By Lemma 2.9.2, to prove the result it's enough to show that if two n -filtrations $X, Y \in \text{im}(F^S)$ are strongly ϵ -interleaved then there exists a topological space T and functions $\gamma_X, \gamma_Y : T \rightarrow \mathbb{R}^n$ such that $F^S(T, \gamma_X) \cong X$, $F^S(T, \gamma_Y) \cong Y$, and $\sup_{y \in T} \|f_1(y) - f_2(y)\|_\infty \leq \epsilon$. This follows from Corollary 3.2.9. \square

Theorem 3.2.11. *For $i \in \mathbb{N}$, let $\mathcal{R}_{5,i} = (\text{obj}^*(n\text{-filt}), d_{SI}, H_i(X))$, so that \mathcal{R}_4 is a relative structure on $\text{obj}^*(B_n\text{-mod})$. If $k = \mathbb{Q}$ or $k = \mathbb{Z}/p\mathbb{Z}$ for some prime p then d_I is $\mathcal{R}_{5,i}$ -optimal.*

Proof. Clearly d_I is $\mathcal{R}_{5,i}$ -stable. The result then follows from Theorem 2.10.1 and Lemma 2.9.4. \square

For the reader trying to understand the relationship between the various optimality results presented here, it may be helpful to note that by Lemma 2.9.2, Theorems 3.2.10 and 3.2.11 together imply Theorem 2.10.1.

3.3 Weak Interleavings and the Weak Interelaving Distance on u -Filtrations

Overview

We now turn to the definition and basic theory of weak interelavings and the weak interleaving distance.

To begin, we prepare for the definition of weak interleavings by reviewing localization of categories and defining the localization of the category of u -filtrations with respect to *levelwise homotopy equivalences*. We observe that the shift and restriction functors descend to the localized category. Using the descents of these functors, we define weak (J_1, J_2) -interleavings and the weak interleaving distance.

After proving some basic facts about weak interleavings, we show that persistent homology is stable with respect to the weak (J_1, J_2) -interleavings on u -filtrations and (J_1, J_2) -interleavings on B_n -persistence modules. This result will be very useful to us for passing from our results about u -filtrations to corresponding results about persistent homology modules.

We finish the section by posing some questions about the weak interleaving distance which we hope to answer in future work.

3.3.1 Localization of Categories

Definition

Let C be a category and let $W \subset \text{hom}(C)$ be a class of morphisms. A *localization* of C with respect to W is a category $C[W^{-1}]$ together with a functor $\Gamma : C \rightarrow C[W^{-1}]$ such that

- (i) $\Gamma(f)$ is an isomorphism for each $f \in W$.

- (ii) [universality] Whenever $G : C \rightarrow D$ is a functor carrying elements of W to isomorphisms, there exists a unique functor $G' : C[W^{-1}] \rightarrow D$ such that $G' \circ \Gamma = G$.

Condition (ii) guarantees that if $\Gamma : C \rightarrow D$ and $\Gamma' : C \rightarrow D'$ are two localizations of C with respect to W , then there is a unique isomorphism of categories $\Theta : D \rightarrow D'$ such that $\theta \circ \Gamma = \Gamma'$. We will often denote $C[W^{-1}]$ as $Ho(C)$, suppressing the dependence of this category on W .

For an arbitrary category C and arbitrary $W \subset \text{hom}(C)$, we can construct the localization of C with respect to W —see [19, Section 42.7]. However, a localization of a locally small category needn’t itself be locally small, and we may need to take some care in setting up set-theoretic foundations in order to rigorously interpret a general localization as a set-theoretic object [19, 40]. The localizations we consider in this thesis, though, will always be locally small.

From the form of the construction of the localization of a category given in [19, Section 42.7], we obtain the following lemma.

Lemma 3.3.1. *For any category C , $W \subset \text{hom}(C)$ there is a localization $\Gamma : C \rightarrow C[W^{-1}]$ such that $\text{obj}(C[W^{-1}]) = \text{obj}(C)$ and $\text{hom}(C[W^{-1}])$ is generated under composition by the images under Γ of elements of $\text{hom}(C)$ together with the inverses of the images under Γ of elements of W .*

The Connection Between Localization and Closed Model Categories

Let $W \subset \text{hom}(\mathbf{Top})$ denote the homotopy equivalences. Then it can be shown that $\mathbf{Top}[W^{-1}]$ is isomorphic to the usual homotopy category of topological spaces (i.e. the category having the same objects as \mathbf{Top} and homotopy classes of continuous maps as morphisms) [21].

In fact, there is a very general setting in which a localization can be constructed in a homotopy theoretic fashion—this is the setting of *closed model categories*, as introduced by the 1967 monograph of Daniel Quillen [37]. A closed model category is a category together with certain extra structure that allows for the axiomatic development of homotopy theory in that category. In more detail, a closed model

category C is a category together with three distinguished classes of morphisms called *weak equivalences*, *fibrations*, and *cofibrations*. The category is required to have the property that all finite limits and colimits exist [21], and the weak equivalences, fibrations, and cofibrations are required to satisfy certain conditions. Let W denote the weak equivalences of C . Quillen showed that the closed model structure on C allows for a homotopy theoretic construction of $C[W^{-1}]$ generalizing the construction of the usual homotopy category of topological spaces. From this construction, it follows in particular that $C[W^{-1}]$ is locally small. For more details, see the expository articles [21, 19].

We won't appeal to the theory of closed model categories in this thesis, and indeed it is not clear that the particular localization we consider in our definition of weak interleavings admits an interpretation in terms of closed model categories. However, a very closely related notion of localization does admit such an interpretation; see Remark 3.3.2 below. In any case, it may be useful for the reader to keep in mind the connection between localization and homotopy theory provided by Quillen's theory, even if we do not take advantage of it here; it is quite plausible that closed model categories and axiomatic homotopy theory will be useful in the further study of weak interleavings and topological inference on the filtration level.

Localization of u -filt with Respect to Levelwise Homotopy Equivalences

For any $u \in \hat{\mathbb{R}}^n$, let $W \subset \text{hom}(u\text{-filt})$ denote the *levelwise homotopy equivalences*, the morphisms f for which f_a is a homotopy equivalence for all $a < u$. In what follows, we'll work only with localizations of u -filt with respect to W . Thus from now on, $\text{Ho}(u\text{-filt})$ will always denote $u\text{-filt}[W^{-1}]$, and $\Gamma : u\text{-filt} \rightarrow \text{Ho}(u\text{-filt})$ will denote the localization functor; the dependence of Γ on u will be implicit.

The category $u\text{-filt}[W^{-1}]$ is locally small, by a result in [20]. Thanks to Mike Shulman for explaining this to me.

Remark 3.3.2. I am not aware of any way of endowing u -filt with the structure of a closed model category having W the class of weak equivalences. However, let $W' \subset \text{hom}(u\text{-filt})$ denote the *levelwise weak homotopy equivalences*, the morphisms f

for which f_a is a *weak homotopy equivalence* [30] for all $a < u$. It is known that u -filt can be given the structure of a closed model category having W' as the class of weak equivalences; see the discussion at the end of [21, Section 10] and the references given there.

3.3.2 Definitions of Weak Interleavings and the Weak Interleaving Distance

Shift Functors and Restriction Functors in the Homotopy Category

By the universal property of the localization of a category, for any $u \in \hat{\mathbb{R}}^n$ and $J : \mathbb{R}^n \rightarrow \mathbb{R}^n$ order-preserving, the functor $(\cdot)(J) : u\text{-filt} \rightarrow J^{-1}(u)\text{-filt}$ descends to a functor from $\text{Ho}(u\text{-filt})$ to $\text{Ho}(J^{-1}(u)\text{-filt})$. By abuse of notation, we also write this functor as $(\cdot)(J)$. Note that

$$\Gamma \circ (\cdot)(J) = (\cdot)(J) \circ \Gamma.$$

Similarly, by the universal property of the localization of a category, for any $u' \leq u \in \hat{\mathbb{R}}^n$ the functor $R_{u'} : u\text{-filt} \rightarrow u'\text{-filt}$ descends to a functor from $\text{Ho}(u\text{-filt})$ to $\text{Ho}(u'\text{-filt})$. By abuse of notation, we also write this functor as $R_{u'}$. Note that

$$\Gamma \circ R_{u'} = R_{u'} \circ \Gamma.$$

Commutativity in the Localized Category of Transition Morphisms with Arbitrary Morphisms

The following is an analogue of Lemma 3.2.1(i) in the category $\text{Ho}(u\text{-filt})$. It is the key step in our proofs of Lemmas 3.3.7 and 3.3.8 below.

Lemma 3.3.3. *For any $u \in \hat{\mathbb{R}}^n$, u -filtrations X and Y , $f \in \text{hom}_{\text{Ho}(u\text{-filt})}(X, Y)$, and J increasing, we have that*

$$f(J) \circ \Gamma(S(X, J)) = \Gamma(S(Y, J)) \circ f.$$

Proof. By Lemma 3.3.1 we can write $f = f_1 \circ \dots \circ f_l$ where for each i , either $f_i = \Gamma(\tilde{f})$ for some $\tilde{f} \in \text{hom}(C)$ or $f_i = \Gamma(\tilde{f})^{-1}$ for some $\tilde{f} \in W$.

Note that $f(J) = f_1(J) \circ \dots \circ f_l(J)$. Thus to prove the result it's enough to check that for $\tilde{f} \in W$,

$$\Gamma(\tilde{f})^{-1}(J) \circ S(X, J) = S(Y, J) \circ \Gamma(\tilde{f})^{-1}.$$

From the equation

$$\tilde{f}(J) \circ S(X, J) = S(Y, J) \circ \tilde{f},$$

we have that

$$\Gamma(\tilde{f}(J)) \circ \Gamma(S(X, J)) = \Gamma(S(Y, J)) \circ \Gamma(\tilde{f}).$$

Composing both sides of the equation on the left by $\Gamma(\tilde{f})^{-1}(J)$ and on the right by $\Gamma(\tilde{f})^{-1}$, we obtain that

$$\Gamma(\tilde{f})^{-1}(J) \circ \Gamma(S(X, J)) = \Gamma(S(Y, J)) \circ \Gamma(\tilde{f})^{-1}$$

as desired. \square

Weak (J_1, J_2) -interleavings of u -filtrations

For $J_1, J_2 : \mathbb{R}^n \rightarrow \mathbb{R}^n$ increasing, we say that an ordered pair of u -filtrations (X, Y) is **weakly (J_1, J_2) -interleaved** if there exist morphisms $f : \Gamma(R_{J_1^{-1}(u)}(X)) \rightarrow \Gamma(Y(J_1))$ and $g : \Gamma(R_{J_2^{-1}(u)}(Y)) \rightarrow \Gamma(X(J_2))$ such that

$$\begin{aligned} g(J_1) \circ R_{J_1^{-1} \circ J_2^{-1}(u)}(f) &= \Gamma(S(X, J_2 \circ J_1)) \\ f(J_2) \circ R_{J_2^{-1} \circ J_1^{-1}(u)}(g) &= \Gamma(S(Y, J_1 \circ J_2)); \end{aligned}$$

we say (f, g) is a pair of weak (J_1, J_2) -interleaving morphisms for (X, Y) .

For $\epsilon \geq 0$, we define weak ϵ -interleavings in the expected way.

Remark 3.3.4. To build on Remark 3.3.2, we could just as well work with the category $u\text{-filt}[W'^{-1}]$ rather than the category $u\text{-filt}[W^{-1}]$ in formulating the definition of weak (J_1, J_2) -interleavings and in developing the theory of the remainder of this

thesis. By the universal property of localization, a weak (J_1, J_2) -interleaving, as defined in terms of $u\text{-filt}[W^{-1}]$, induces one defined in terms of $u\text{-filt}[W'^{-1}]$; in this sense, the definition of weak (J_1, J_2) -interleaving given in terms of the category $u\text{-filt}[W^{-1}]$ is the stronger one. For this reason, I have have a slight preference in formulating the theory here in terms of $u\text{-filt}[W^{-1}]$.

The Weak Interleaving Distance on u -filtrations

We define $d_{WI} : \text{obj}^*(u\text{-filt}) \times \text{obj}^*(u\text{-filt}) \rightarrow [0, \infty]$, the **weak interleaving distance**, by taking

$$d_{WI}(X, Y) = \inf\{\epsilon \in \mathbb{R}_{\geq 0} \mid X \text{ and } Y \text{ are weakly } \epsilon\text{-interleaved}\}.$$

It follows from Lemma 3.3.8(ii) below that d_{WI} is a pseudometric on $\text{obj}^*(u\text{-filt})$. It is easily checked that this pseudometric is not a metric.

3.3.3 Stability Results for the Weak Interleaving Distance

d_{WI} satisfies stability properties with respect to the functors F^S , F^{SO} , F^{SR} , and $F^{S\check{C}e}$ analogous to the stability properties Theorems 2.8.1-2.8.4 satisfied by the interleaving distance on B_n -persistence modules with respect to the functors $H_i \circ F^S$, $H_i \circ F^{SO}$, $H_i \circ F^{SR}$, and $H_i \circ F^{S\check{C}e}$. The formulation and proofs of these results are easy modifications of those of Theorems 2.8.1-2.8.4; analogues of Theorems 2.8.1-2.8.3 follow via Lemma 3.3.5 below from the analogues of these theorems for d_{SI} mentioned in Section 3.2.1. The analogue of Theorem 2.8.4 for d_{WI} is proved in essentially the same way as Theorem 2.8.4, but requires Proposition 4.2.1(i), a lift of the persistent nerve theorem of [13] to the level of filtrations.

3.3.4 Basic Results about Weak Interleavings

Lemma 3.3.5. *If u -filtrations X, Y are strongly (J_1, J_2) -interleaved then X, Y are weakly (J_1, J_2) -interleaved.*

Proof. It's easy to check that if f, g are strong (J_1, J_2) -interleaving morphisms for X, Y then $\Gamma(f), \Gamma(g)$ are weak (J_1, J_2) -interleaving morphisms for X, Y . We leave the details to the reader. \square

Lemma 3.3.6. *For any $u' \leq u \in \hat{\mathbb{R}}^n$ and $(J_1, J_2) \in \mathbb{R}^n$, if u -filtrations X, Y are strongly (weakly) (J_1, J_2) -interleaved, then $R_{u'}(X), R_{u'}(Y)$ are also strongly (weakly) (J_1, J_2) -interleaved.*

Proof. It's easy to check that if f, g are strong (weak) (J_1, J_2) -interleaving morphisms for X, Y then $R_{u'}(f), R_{u'}(g)$ are strong (weak) (J_1, J_2) -interleavings for $R_{u'}(X), R_{u'}(Y)$. \square

To prove our consistency results, we will require the following lemma.

Lemma 3.3.7. *For any $u' \leq u \in \hat{\mathbb{R}}^n$ and $\epsilon \geq 0$, if X, Y are u -filtrations such that $R_{u'}(X), R_{u'}(Y)$ are strongly (weakly) ϵ -interleaved then X, Y are strongly (weakly) $(J_{\epsilon+u-u'}, J_{\epsilon+u-u'})$ -interleaved.*

Proof. Let $f : R_{u'-\epsilon}(X) \rightarrow R_{u'}(Y)(\epsilon)$ and $g : R_{u'-\epsilon}(Y) \rightarrow R_{u'}(X)(\epsilon)$ be strong ϵ -interleaving homomorphisms for $R_{u'}(X), R_{u'}(Y)$. Note that $R_{u'}(X)(\epsilon) = R_{u'-\epsilon}(X(\epsilon))$ and $R_{u'}(Y)(\epsilon) = R_{u'-\epsilon}(Y(\epsilon))$. Thus the homomorphisms $S(Y(\epsilon), J_{u-u'}) \circ f$ and $S(X(\epsilon), J_{u-u'}) \circ g$ are well defined, and it's easy to check that these are strong $(J_{\epsilon+u-u'}, J_{\epsilon+u-u'})$ -interleaving homomorphisms for X, Y .

Invoking Lemma 3.3.3, the same argument also gives the result for weak interleavings. \square

The next lemma will prove very useful to us. For $u = (u_1, \dots, u_n), u' = (u'_1, \dots, u'_n) \in \hat{\mathbb{R}}^n$, let $\gcd(u, u') = (\min(u_1, u'_1), \dots, \min(u_n, u'_n))$.

Lemma 3.3.8 (Triangle Inequality for (J_1, J_2) -interleavings).

- (i) Suppose we are given $J_1, J_2, J_3, J_4 : \mathbb{R}^n \rightarrow \mathbb{R}^n$ each increasing, $u \in \hat{\mathbb{R}}^n$, $u_1, u_2 \leq u$, and u -filtrations X_1, X_2, X_3 such that $R_{u_1}(X_1), R_{u_1}(X_2)$ are strongly (weakly) (J_1, J_2) -interleaved and $R_{u_2}(X_2), R_{u_2}(X_3)$ are strongly (weakly) (J_3, J_4) -interleaved. Then $R_{\gcd(u_1, u_2)}(X_1), R_{\gcd(u_1, u_2)}(X_3)$ are strongly (weakly) $(J_3 \circ J_1, J_2 \circ J_4)$ -interleaved.

- (ii) In particular, if we have $\epsilon_1, \epsilon_2 \geq 0$ such that X_1, X_2 are strongly (weakly) ϵ_1 -interleaved and X_2, X_3 are strongly (weakly) ϵ_2 -interleaved, then X_1, X_3 are strongly (weakly) $(\epsilon_1 + \epsilon_2)$ -interleaved.

Proof. By Lemma 3.3.6, it suffices to assume that $u_1 = u_2 = u$. It's easy to check that if f_1, g_1 are strong (J_1, J_2) -interleaving homomorphisms for X_1, X_2 and f_2, g_2 are strong (J_3, J_4) -interleaving homomorphisms for X_2, X_3 then $f_2(J_1) \circ f_1, g_1(J_4) \circ g_2$ are strong $(J_3 \circ J_1, J_2 \circ J_4)$ -interleaving homomorphisms for X_1, X_3 . This gives (i) for strong interleavings. The case of weak interleavings follows via the same argument, using Lemma 3.3.3. \square

We noted in Remark 2.1.1 that if $\epsilon < \epsilon' \in \mathbb{R}$ and two B_n -persistence modules are ϵ -interleaved, then they are ϵ' -interleaved. Here is the analogous statement for (J_1, J_2) -interleavings, on the level of filtrations. (The same proof gives the corresponding result for (J_1, J_2) -interleaved B_n -persistence modules.)

For $J, J' : \mathbb{R}^n \rightarrow \mathbb{R}^n$ bijections, say that $J \leq J'$ if $J(a) \leq J'(a)$ for all $a \in \mathbb{R}^n$.

Lemma 3.3.9. *Let J_1, J'_1, J_2, J'_2 be increasing and $J_1 \leq J'_1, J_2 \leq J'_2$. If u -filtrations X, Y are strongly (weakly) (J_1, J_2) -interleaved then X, Y are strongly (weakly) (J'_1, J'_2) -interleaved.*

Proof. Since J_1 , and J'_1 are increasing, there is an increasing map $J''_1 : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $J'_1 = J''_1 \circ J_1$. Similarly, there is an increasing map $J''_2 : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $J'_2 = J''_2 \circ J_2$. Let f, g be strong (J_1, J_2) -interleaving morphisms for X, Y . We show that $S(Y, J''_1)(J_1) \circ f, S(X, J''_2)(J_2) \circ g$ are strong (J'_1, J'_2) -interleaving momorphisms

for X, Y . By Lemma 3.2.1,

$$\begin{aligned}
& S(X, J''_2)(J_2)(J'_1) \circ g(J'_1) \circ S(Y, J''_1)(J_1) \circ f \\
&= S(X, J''_2)(J_2 \circ J'_1) \circ g(J'_1) \circ S(Y(J_1), J_1^{-1} \circ J'_1) \circ f \\
&= S(X, J''_2)(J_2 \circ J'_1) \circ g(J'_1) \circ f(J_1^{-1} \circ J'_1) \circ S(X, J_1^{-1} \circ J'_1) \\
&= S(X, J''_2)(J_2 \circ J'_1) \circ S(X, J_2 \circ J_1)(J_1^{-1} \circ J'_1) \circ S(X, J_1^{-1} \circ J'_1) \\
&= S(X, J''_2)(J_2 \circ J'_1) \circ S(X, J_2 \circ J_1 \circ J_1^{-1} \circ J'_1) \\
&= S(X, J''_2 \circ J_2 \circ J'_1) \\
&= S(X, J'_2 \circ J'_1).
\end{aligned}$$

The symmetric argument shows that

$$S(Y, J''_1)(J_1)(J'_2) \circ f(J'_2) \circ S(X, J''_2)(J_2) \circ g = S(Y, J'_1 \circ J'_2).$$

This gives the result for strong interleavings; the result follows for weak interleavings by the same argument, using Lemma 3.3.3. \square

3.3.5 Stability of Persistent Homology with Respect to Interleavings

In this section, we prove that persistent homology is stable with respect to weak interleavings on u -filtrations and interleavings on B_n -persistence modules.

Theorem 3.3.10. *For any $i \in \mathbb{Z}_{\geq 0}$, $u \in \hat{\mathbb{R}}^n$, and $J_1, J_2 : \mathbb{R}^n \rightarrow \mathbb{R}^n$ increasing, if u -filtrations X, Y are weakly (J_1, J_2) -interleaved then $H_i(X), H_i(Y)$ are (J_1, J_2) -interleaved.*

Corollary 3.3.11. *For any $i \in \mathbb{Z}_{\geq 0}$, $u \in \hat{\mathbb{R}}^n$, and u -filtrations X and Y ,*

$$d_I(H_i(X), H_i(Y)) \leq d_{WI}(X, Y).$$

Proof of Theorem 3.3.10. We begin with a couple of lemmas.

Lemma 3.3.12. *For any $i \in \mathbb{Z}_{\geq 0}$ and $J : \mathbb{R}^n \rightarrow \mathbb{R}^n$ order-preserving,*

$$H_i \circ (\cdot)(J) = (\cdot)(J) \circ H_i.$$

Proof. We leave the straightforward proof to the reader. \square

Note that if $f \in \text{hom}(B_n\text{-mod})$ is such that f_a is an isomorphism for all $a \in \mathbb{R}^n$, then f is an isomorphism. Thus, since the singular homology functor maps homotopy equivalences to isomorphisms, H_i takes elements of W to isomorphisms in $B_n\text{-mod}$. Then by the universal property of localizations, for any $u \in \hat{\mathbb{R}}^n$, there's a functor $\tilde{H}_i : \text{Ho}(u\text{-filt}) \rightarrow B_n\text{-mod}$ such that

$$H_i = \tilde{H}_i \circ \Gamma.$$

Lemma 3.3.13. *For any $i \in \mathbb{Z}_{\geq 0}$ and J increasing,*

$$\tilde{H}_i \circ (\cdot)(J) = (\cdot)(J) \circ \tilde{H}_i.$$

Proof. $H_i \circ (\cdot)(J)$ takes elements of W to isomorphisms in $B_n\text{-mod}$. Thus by the universal property of localization there exists a unique functor $G : \text{Ho}(u\text{-filt}) \rightarrow B_n\text{-mod}$ such that $G \circ \Gamma = H_i \circ (\cdot)(J)$. We have that

$$\tilde{H}_i \circ (\cdot)(J) \circ \Gamma = \tilde{H}_i \circ \Gamma \circ (\cdot)(J) = H_i \circ (\cdot)(J) = (\cdot)(J) \circ H_i = (\cdot)(J) \circ \tilde{H}_i \circ \Gamma.$$

Thus by the uniqueness property of G , $\tilde{H}_i \circ (\cdot)(J) = G = (\cdot)(J) \circ \tilde{H}_i$. \square

Now let $f : \Gamma(R_{J_1^{-1}(u)}(X)) \rightarrow \Gamma(Y(J_1))$ and $g : \Gamma(R_{J_2^{-1}(u)}(Y)) \rightarrow \Gamma(X(J_2))$ be weak (J_1, J_2) -interleaving morphisms for X, Y .

The homomorphisms

$$\begin{aligned} \tilde{H}_i(f) &: H_i(R_{J_1^{-1}(u)}(X)) \rightarrow H_i(Y)(J_1), \\ \tilde{H}_i(g) &: H_i(R_{J_2^{-1}(u)}(Y)) \rightarrow H_i(X)(J_2) \end{aligned}$$

extend to homomorphisms $f_E : H_i(X) \rightarrow H_i(Y)(J_1)$ and $g_E : H_i(Y) \rightarrow H_i(X)(J_2)$ by

taking f_E to be identically zero on all homogenous summands $H_i(X)_a$ of $H_i(X)$ such that $a \not\prec J_1^{-1}(u)$, and taking g_E to be identically zero on all homogenous summands $H_i(Y)_a$ of $H_i(Y)$ such that $a \not\prec J_2^{-1}(u)$.

We'll now show that f_E, g_E are (J_1, J_2) -interleaving homomorphisms for $H_i(X)$, $H_i(Y)$. We have that

$$\begin{aligned} \tilde{H}_i(g)(J_1) \circ \tilde{H}_i(R_{J_1^{-1} \circ J_2^{-1}(u)}(f)) &= \tilde{H}_i(g(J_1)) \circ \tilde{H}_i(R_{J_1^{-1} \circ J_2^{-1}(u)}(f)) \\ &= \tilde{H}_i(g(J_1)) \circ R_{J_1^{-1} \circ J_2^{-1}(u)}(f) = \tilde{H}_i \circ \Gamma(S(X, J_2 \circ J_1)) \\ &= H_i(S(X, J_2 \circ J_1)) = S(H_i(X), J_2 \circ J_1). \end{aligned}$$

This implies that for $a < J_1^{-1} \circ J_2^{-1}(u)$,

$$(g_E(J_1) \circ f_E)_a = (\tilde{H}_i(g)(J_1) \circ \tilde{H}_i(R_{J_1^{-1} \circ J_2^{-1}(u)}(f)))_a = S(H_i(X), J_2 \circ J_1)_a.$$

On the other hand, for $a \not\prec J_1^{-1} \circ J_2^{-1}(u)$,

$$(g_E(J_1) \circ f_E)_a = 0 = S(H_i(X), J_2 \circ J_1)_a.$$

Thus, $g_E(J_1) \circ f_E = S(H_i(X), J_2 \circ J_1)$. The symmetric argument shows that $f_E(J_2) \circ g_E = S(H_i(Y), J_1 \circ J_2)$. This shows that f_E, g_E are (J_1, J_2) -interleaving homomorphisms for $H_i(X), H_i(Y)$, as desired. \square

Example 3.3.14. By Example 3.2.4 and Theorem 3.3.10, we have that for any $i \geq 0$ and $(X, Y, d, \gamma) \in \text{obj}(C^{SCE})$, $H_i \circ F^R(X, d)$, $H_i \circ F^{\check{C}e}(X, Y, d)$ are $(\mathcal{J}_1, \text{id}_1)$ -interleaved, and $H_i \circ F^{SR}(X, d, \gamma)$, $H_i \circ F^{S\check{C}e}(X, Y, d, \gamma)$ are $(\mathcal{J}_{n+1}, \text{id}_{n+1})$ -interleaved.

3.3.6 On Our Choice of Definition of Weak Interleavings

To define weak interleavings, we have considered the localization of u -filt with respect to levelwise homotopy equivalences. An alternative approach, and one worth mentioning because it is very simple, is to define weak interleavings by passing to a category of filtrations over the homotopy category of topological spaces. We take a moment here to explain this alternative approach to defining weak interleavings and

the weak interleaving distance and to discuss the relationship between the distance thus obtained and d_{WI} .

For $u \in \hat{\mathbb{R}}^n$, let $u\text{-filt}^*$ be the category whose objects are collections of topological spaces $\{X_a\}_{a < u}$ together with morphisms $\{\phi_X(a, b) \in \text{hom}_{Ho(\mathbf{Top})}(X_a, X_b)\}_{a \leq b < u}$ such that if $a \leq b \leq c < u$ then $\phi_X(b, c) \circ \phi_X(a, b) = \phi_X(a, c)$.

Given two objects X and Y of $u\text{-filt}^*$, we define a morphism $f \in \text{hom}_{u\text{-filt}^*}(X, Y)$ to be a collection of morphisms $\{f_a \in \text{hom}_{Ho(\mathbf{Top})}(X_a, Y_a)\}_{a < u}$ such that for all $a \leq b < u$, $f_b \circ \phi_X(a, b) = \phi_Y(a, b) \circ f_a$.

The usual functor $\Gamma' : \mathbf{Top} \rightarrow Ho(\mathbf{Top})$ induces a functor $\Gamma' : u\text{-filt} \rightarrow u\text{-filt}^*$. This functor takes levelwise weak equivalences to isomorphisms. Further, we can define shift and restriction functors on the categories $u\text{-filt}^*$ just as we have for the categories $u\text{-filt}$. Thus for any $u \in \hat{\mathbb{R}}^n$ we have all the structure we need to formulate an alternative definition of weak (J_1, J_2) -interleavings of u -filtrations via the functor Γ' analogous to the one we have formulated via the localization functor Γ . We'll call the weak interleavings thus defined *A*-weak interleavings. The definition of *A*-weak ϵ -interleavings induces a definition of an *A*-weak interleaving distance on u -filtrations which we denote d_{AWI} .

By the universal property of localization, we have a functor $\Theta : Ho(u\text{-filt}) \rightarrow u\text{-filt}^*$ such that $\Gamma' = \Theta \circ \Gamma$. It follows that if two filtrations are weakly (J_1, J_2) -interleaved, then they are *A*-weakly (J_1, J_2) -interleaved, and in particular, $d_{AWI} \leq d_{WI}$. This implies that all of our inference results in this thesis which are formulated in terms of weak interleavings are still true if instead formulated in terms of *A*-weak interleavings.

We would like the distance with which we develop our inferential theory to be as sensitive as possible, subject to the conditions that it has reasonable stability properties (such as those mentioned in Section 3.3.2) and that the distance between filtrations X and Y is 0 whenever there exists a levelwise homotopy equivalence $f : X \rightarrow Y$. Thus the fact that $d_{AWI} \leq d_{WI}$ offers some justification for our working with d_{WI} rather than d_{AWI} .

I do not yet know if it is in fact true that $d_{AWI} = d_{WI}$, though I suspect that this equality does not hold in general. More generally, we can ask the following: Under what circumstances does the existence of an *A*-weak (J_1, J_2) -interleaving between

two u -filtrations imply the existence of a weak (J_1, J_2) -interleaving between the two filtrations? This question is closely related to the problem of understanding the relationship between a homotopy category of diagrams of topological spaces of a given shape and the category of diagrams of the same shape over the homotopy category of topological spaces. It is well known that these two categories are usually not equivalent [21, Remark 10.3], and in fact there are results in the homotopy theory literature which quantify the difference between the two categories [15]. We leave it to the future to study how those results bear on the question of when $d_{AWI} = d_{WI}$.

3.3.7 Weak Interleavings and the Optimality of d_{WI} : Open Questions

To close this section, we present some questions regarding theoretical properties of weak interleavings and the interleaving distance. Answers to these questions would, it seems, offer us a satisfactory understanding of weak interleavings and the weak interleaving distance.

1. Can we use homotopy colimits to give an analogue for weak interleavings of our characterization theorem 3.2.7 for strong interleavings?
2. If u -filtrations X, Y are weakly (J_1, J_2) -interleaved, do there exist strongly (J_1, J_2) -interleaved u -filtrations X', Y' with $d_{WI}(X, X') = d_{WI}(Y, Y') = 0$?
3. Is d_{WI} optimal in the sense that $d \leq d_{WI}$ for any pseudometric d on $\text{obj}(u\text{-filt})$ such that $d \leq d_{SI}$ and $d(X, Y) = 0$ whenever X, Y are weakly 0-interleaved? A positive answer to this question would provide strong justification for the use of weak interleavings in the development of the theory of topological inference. Note that a positive answer to the last question would imply a positive answer to this question.
4. A question already raised in Section 3.3.6: Under what circumstances does the existence of an A -weak (J_1, J_2) -interleaving for a pair of u -filtrations imply the existence of a weak (J_1, J_2) -interleaving for the pair?

Chapter 4

Approximation and Inference Results for Multidimensional Filtrations

In this chapter we apply the interleaving machinery introduced and studied in the previous two chapters to formulate and prove topological inference results for multi-dimensional filtrations. See Section 1.5 for an overview of the chapter.

4.1 Inference Preliminaries

4.1.1 Basic Notation

Thoughtout this chapter, fix $p \in [1, \infty]$ and $m \in \mathbb{N}$. Let d^p denote the L^p metric on \mathbb{R}^m .

For (X, d) a metric space and $X' \subset X$, we'll often abuse notation slightly and let d also denote the restriction of d to X' .

If (X, d) is a metric space, $x \in X$, and $r \in \mathbb{R}_{\geq 0}$, we let $B_d(x, r)$ denote the closed metric ball of radius r centered at x . That is, $B_d(x, r) = \{x' \in X | d(x, x') \leq r\}$.

Let (X, d) be a metric space, and $S \subset X$. We say a set $L \subset X$ is an ϵ -sample of S (w.r.t d) if for any $s \in S$, there exists some $l \in L$ such that $d(s, l) \leq \epsilon$.

If (X, d_X) and (Y, d_Y) are metric spaces, a function $\gamma : X \rightarrow Y$ is said to be c -*Lipschitz* if for all $x_1, x_2 \in X$, $d_Y(\gamma(x_1), \gamma(x_2)) \leq c \cdot d_X(x_1, x_2)$. In this thesis, the codomain (Y, d_Y) of a Lipschitz function will always be (\mathbb{R}^n, d^∞) for some n . When we want to make explicit the metric d on the domain of a Lipchitz function f , we will refer to f as a c -Lipchitz function (w.r.t. d).

4.1.2 Riemannian Manifolds and Probability Density Functions

In this thesis Riemannian manifolds will always be understood to be manifolds with boundary.

Let M be a Riemannian manifold of dimension l . The Riemannian structure on M induces a metric d^M on M , the *geodesic metric*. In turn, the geodesic metric induces a measure \mathcal{H}_M^l on (M, \mathcal{B}_M) , the l -dimensional *Hausdorff measure* [3]. (Here \mathcal{B}_M is the Borel σ -algebra of M). From now on we'll write \mathcal{H}_M^l as \mathcal{H}_M ; l will be implicit in this notation.

When $M = \mathbb{R}^n$, endowed with the standard Euclidean metric, \mathcal{H}_M is the usual Lebesgue measure on \mathbb{R}^n .

For $y \in M$ and $r > 0$, we say a ball $B \equiv B_{d^M}(y, r)$ is *strongly convex* if for every pair of points y', y'' in the closure of B , there exists a unique shortest path in M between y' and y'' , and the interior of this path is included in B . Define $\rho(M)$, the strong convexity radius of M , by

$$\rho(M) = \inf_{y \in M} \sup_{r > 0} \{r \mid B_{d^M}(y, r) \text{ is strongly convex}\}$$

As noted in [11], $\rho(M)$ is positive when M is compact. When M is a Euclidean space, $\rho(M) = \infty$.

A density function on M is a \mathcal{B}_M -measurable function $\gamma : M \rightarrow [0, \infty)$ such that $\int_M \gamma d\mathcal{H}_M = 1$. A density function γ defines a probability measure \mathcal{P}_γ on M with the property that for any $A \in \mathcal{B}_M$, $\mathcal{P}_\gamma(A) = \int_A \gamma d\mathcal{H}_M$.

4.1.3 Density Estimators

As we explained in the introduction, the superlevelset-Rips filtrations we consider in the formation of our consistency results are filtered by density estimators. We review here some basic concepts and results related to density estimation that we will need in what follows. Some of the material here also appeared in Section 1.2.2; we include that material again here for readability's sake.

Let M be a Riemannian manifold and let $\mathcal{D}(M)$ be the set of density functions on M . We'll define a density estimator E on M to be a sequence of functions $\{E_z : M^z \rightarrow \mathcal{D}(M)\}_{z \in \mathbb{N}}$ such that for any $y \in M$, the function $E_{z,y} : M^z \rightarrow \mathbb{R}$ defined by $E_{z,y}(T) = (E_z(T))(y)$ is measurable. By slight abuse of terminology, we'll also refer to the individual functions E_z as density estimators.

We formulate our results in terms of pairs (E, γ) of density estimators and density functions satisfying one of two properties. Let T_z be a random sample of \mathcal{P}_γ of size z . The first property is:

A1: $E_z(T_z)$ converges uniformly in probability to γ .

In stating the second property we'll assume that (E, γ) is defined on \mathbb{R}^m for some $m \in \mathbb{N}$. The property is:

A2: $E_z(T_z)$ converges uniformly in probability to the convolution of γ with some kernel function K .

A1 is known to hold for kernel density estimators on Euclidean spaces, for a wide class of kernels and density functions γ , provided the kernel width tends to 0 at an appropriate rate as $z \rightarrow \infty$ [28]. Further, Pelletier has shown that the notion of kernel density estimators extends to Riemannian manifolds [36], and a recent article by Henry and Rodriguez [31] shows that under mild assumptions on (E, γ) and a similar condition on the rate at which the bandwidth of the kernel tends to 0 as $z \rightarrow \infty$, assumption A1 holds for the estimators defined by Pelletier. In fact, each of the cited results gives a.s. uniform convergence of $E_z(T_z)$ to γ .

A2 also is known to hold for kernel density estimators E with kernel K on Euclidean spaces, for a wide class of kernels K and density functions γ , when the bandwidth of the estimator E_z is held fixed as z varies [38, Proposition 9].

Let $\gamma : M \rightarrow \mathbb{R}$ be a density function, $\tilde{\gamma} : M \rightarrow \mathbb{R}$ be another density function, $q \in (0, 1]$, $C > 0$, $z \in \mathbb{N}$, and $E_z : M^z \rightarrow \mathcal{D}(M)$ be a density estimator. We'll say that E_z is a (q, C) -density estimator of $\tilde{\gamma}$ w.r.t. \mathcal{P}_γ if for T_z a finite i.i.d. random sample of \mathcal{P}_γ of size z ,

$$P\left(\sup_{l \in T_z} |E_z(T_z)(l) - \tilde{\gamma}(l)| > C\right) \leq q.$$

When E_z is a (q, C) -density estimator of $\tilde{\gamma}$ w.r.t. \mathcal{P}_γ , we'll say simply that E_z is a (q, C) -density estimator of γ .

The two cases of interest to us will be where, in the above definition, $\tilde{\gamma} = \gamma$ and, in the special case that $M = \mathbb{R}^m$, $\tilde{\gamma} = \gamma * K$ for a kernel K .

Lemma 4.1.1.

- (i) If M is a Riemannian manifold, $\gamma : M \rightarrow \mathbb{R}$ is a density function, and E is density estimator on M such that (E, γ) satisfies A1, then for any $C > 0$ and $q \in (0, 1]$ there exists z_0 such that for all $z \geq z_0$, E_z is a (q, C) -density estimator of γ .
- (ii) If $\gamma : \mathbb{R}^m \rightarrow \mathbb{R}$ is a density function and E is density estimator on \mathbb{R}^m such that (E, γ) satisfies A2 for some kernel K , then for any $C \in (0, \infty)$ and $q \in (0, 1]$ there exists z_0 such that for all $z \geq z_0$, E_z is a (q, C) -density estimator of $\gamma * K$ w.r.t. \mathcal{P}_γ .

Proof. This is immediate from the definitions. \square

4.2 Deterministic Approximation of Multidimensional Filtrations via Discrete Filtrations

Overview

In this section, we prove two theorems concerning the deterministic topological approximation of sublevelset-offset filtrations via discrete filtrations. The first theorem

concerns approximation by sublevelset-Čech filtrations; the second theorem concerns approximation by sublevelset-Rips filtrations.

These results are multidimensional analogues of [12, Theorem 3.1] and its extension [11, Theorem 4.5] to good samplings of sublevelsets of γ . They hold on the level of filtrations rather than merely on the level of persistent homology modules. By Theorem 3.3.10, we also obtain analogues of these results on the level of persistent homology modules.

In addition to our results on the topological approximation of sublevelset-offset filtrations, we present results on the deterministic approximation of multidimensional sublevelset filtrations and their persistent homology via Čech and Rips complexes with fixed scale parameter.

4.2.1 Weak 0-interleavings of Open Sublevelset-offset and Open Sublevelset-Čech Filtrations

To prove our approximation results, we need the following proposition, which offers some motivation for our definition of weak interleavings of u -filtrations; see Remark 4.2.2 below.

Proposition 4.2.1.

- (i) For any $X \subset \mathbb{R}^m$ and $\gamma : X \rightarrow \mathbb{R}^n$, $R_{\infty_{n+1}} \circ F^{SO-Op}(X, \mathbb{R}^m, d^p, \gamma)$ and $R_{\infty_{n+1}} \circ F^{SCe-Op}(X, \mathbb{R}^m, d^p, \gamma)$ are weakly 0-interleaved.
- (ii) Let M be a Riemannian manifold. For any $X \subset M$ and $\gamma : X \rightarrow \mathbb{R}^n$, $R_{(\infty_n, \rho(M))} \circ F^{SO-Op}(X, M, d^M, \gamma)$ and $R_{(\infty_n, \rho(M))} \circ F^{SCe-Op}(X, M, d^M, \gamma)$ are weakly 0-interleaved.

Note that in (ii), if $\rho(M) = 0$ then all spaces of each of the two filtrations are the empty set, so in this case (ii) holds vacuously.

Proof. We give the proof for (ii); the proof of (i) is the same.

The same argument that Chazal and Oudot use to prove the persistent nerve lemma [13] gives us that there exists an $(\infty_n, \rho(M))$ -filtration Z (a “Mayer-Vietoris

blowup filtration,” to borrow the terminology of [48]) such that there are level-wise homotopy equivalences $f : Z \rightarrow R_{(\vec{\infty}_n, \rho(M))} \circ F^{SO-Op}(X, M, d^M, \gamma)$, and $g : R_{(\vec{\infty}_n, \rho(M))} \circ Z \rightarrow F^{SCe-Op}(X, M, d^M, \gamma)$ in $\text{hom}((\vec{\infty}_n, \rho(M))\text{-filt})$. Thus $\Gamma \circ R_{(\vec{\infty}_n, \rho(M))} \circ F^{SO-Op}(X, M, d^M, \gamma)$ and $\Gamma \circ R_{(\vec{\infty}_n, \rho(M))} \circ F^{SCe-Op}(X, M, d^M, \gamma)$ are isomorphic in $\text{Ho}((\vec{\infty}_n, \rho(M))\text{-filt})$, and hence are weakly 0-interleaved. \square

Remark 4.2.2. Note that for M, X , and γ defined as in the statement of Proposition 4.2.1(ii), it needn’t be true that $R_{(\vec{\infty}_n, \rho(M))} \circ F^{SO-Op}(X, M, d^M, \gamma)$ and $R_{(\vec{\infty}_n, \rho(M))} \circ F^{SCe-Op}(X, M, d^M, \gamma)$ are strongly 0-interleaved. For example, Let $M = [0, 1]$ (endowed with the Euclidean metric), $X = \{0, 1\}$, and define $\gamma : \{0, 1\} \rightarrow \mathbb{R}$ by $\gamma(0) = \gamma(1) = 0$. Then $\rho(M) = \infty$. It’s straightforward to check that there can be no pair of strong 0-interleaving homomorphisms between $F^{SO-Op}(X, M, d^M, \gamma)$ and $F^{SCe-Op}(X, M, d^M, \gamma)$.

Nevertheless, it is natural to think of the filtrations $R_{(\vec{\infty}_n, \rho(M))} \circ F^{SO-Op}(X, M, d^M, \gamma)$ and $R_{(\vec{\infty}_n, \rho(M))} \circ F^{SCe-Op}(X, M, d^M, \gamma)$ as being “topologically equivalent.” This motivates a choice of pseudometric on u -filtrations with respect to which u -filtrations which are isomorphic in $\text{Ho}(u\text{-filt})$ are distance 0 from one another.

4.2.2 Topological Approximation of Sublevelset-Offset Filtrations via Sublevelset-Čech Filtrations

We state our first approximation result, Theorem 4.2.3, in two parts. It is easily seen that each is a special case of a more general result, but the fully general form of the result is not particularly interesting and it seems more expedient to just state the two special cases of interest separately.

Theorem 4.2.3(i) says that for any $W \subset \mathbb{R}^m$, Lipschitz function $\gamma : W \rightarrow \mathbb{R}^n$, $T \subset W$ and $\tilde{\gamma} : T \rightarrow \mathbb{R}^n$, $F^{S\check{C}e}(T, \mathbb{R}^m, d^p, \tilde{\gamma})$ gives a good topological approximation to $F^{SO}(W, \mathbb{R}^m, d^p, \gamma)$ when the Hausdorff distance between T and W is small and $\sup_{l \in T} \|\tilde{\gamma}(l) - \gamma(l)\|$ is also small.

In fact, Theorem 4.2.3(i) says more generally that for any $u \in \hat{\mathbb{R}}^n$, $R_{u,\infty} \circ F^{S\check{C}e}(T, \mathbb{R}^m, d^p, \tilde{\gamma})$ gives a good topological approximation to $R_{u,\infty} \circ F^{SO}(W, \mathbb{R}^m, d^p, \gamma)$

when the Hausdorff distance between $T \cap \gamma_u$ and γ_u is small and $\sup_{l \in T} \|\tilde{\gamma}(l) - \gamma(l)\|$ is also small.

Theorem 4.2.3(ii) gives a variant of this result for Riemannian manifolds.

Theorem 4.2.3.

- (i) Let W be a subset of \mathbb{R}^m , let $\gamma : W \rightarrow \mathbb{R}^n$ be a c -Lipschitz function (w.r.t d^p) for some $c > 0$, let $c' = \max(1, c)$, and let $u \in \hat{\mathbb{R}}^n$. Let $T \subset W$ be a $\frac{\epsilon}{c'}$ -sample of γ_u (w.r.t d^p). Let $C \in \mathbb{R}_{\geq 0}$ and $\tilde{\gamma} : T \rightarrow \mathbb{R}^n$ be a function such that $\|\tilde{\gamma}(l) - \gamma(l)\|_\infty \leq C$ for all $l \in T$. Then

$$d_{WI}(R_{(u,\infty)} \circ F^{SO}(W, \mathbb{R}^m, d^p, \gamma), R_{(u,\infty)} \circ F^{S\check{C}e}(T, \mathbb{R}^m, d^p, \tilde{\gamma})) \leq \epsilon + C.$$

- (ii) Let M be a Riemannian manifold and let $\gamma : M \rightarrow \mathbb{R}^n$ be a c -Lipschitz function (w.r.t d^M) for some $c > 0$, let $c' = \max(1, c)$, and let $u \in \hat{\mathbb{R}}^n$. Let $T \subset M$ be a $\frac{\epsilon}{c'}$ -sample of γ_u (w.r.t d^M). Let $C \in \mathbb{R}_{\geq 0}$ and $\tilde{\gamma} : T \rightarrow \mathbb{R}^n$ be a function such that $\|\tilde{\gamma}(l) - \gamma(l)\|_\infty \leq C$ for all $l \in T$. Then,

$$d_{WI}(R_{(u,\rho(M))} \circ F^{SO}(M, d^M, \gamma), R_{(u,\rho(M))} \circ F^{S\check{C}e}(T, M, d^M, \tilde{\gamma})) \leq \epsilon + C.$$

Proof. We present the proof of (i); the proof of (ii) is essentially the same as the proof of (i), using Proposition 4.2.1(ii) in place of Proposition 4.2.1(i).

We begin with a lemma. Let $\gamma_T : T \rightarrow \mathbb{R}^n$ denote the restriction of γ to T .

Lemma 4.2.4. $R_{(u,\infty)} \circ F^{SO}(W, \mathbb{R}^m, d^p, \gamma)$ and $R_{(u,\infty)} \circ F^{SO}(T, \mathbb{R}^m, d^p, \gamma_T)$ are strongly ϵ -interleaved.

Proof. For any $(a, b) \in \mathbb{R}^n \times \mathbb{R}$, let $\gamma_{a,b}$ denote $F^{SO}(W, \mathbb{R}^m, d^p, \gamma)_{(a,b)}$ and let $\lambda_{a,b}$ denote $F^{SO}(T, \mathbb{R}^m, d^p, \gamma_T)_{(a,b)}$. Consider some $(a, b) \leq (u, \infty)$ and $p \in \gamma_{a,b}$. There is a point $p' \in \gamma_a$ with $d^p(p, p') \leq b$. Since T is an $\frac{\epsilon}{c'}$ -sample of γ_a , there is a point $p'' \in T$ with $d^p(p'', p') \leq \frac{\epsilon}{c'}$. Thus $d^p(p'', p) \leq \frac{\epsilon}{c'} + b$. Since γ is c -Lipschitz, $\gamma(p'') \leq a + \frac{c\epsilon}{c'} \leq a + \epsilon$. Hence $p \in \lambda_{a+\epsilon, b+\frac{\epsilon}{c'}}$. Therefore $\gamma_{a,b} \subset \lambda_{a+\epsilon, b+\frac{\epsilon}{c'}} \subset \lambda_{a+\epsilon, b+\epsilon}$. The inclusions thus define a morphism $f : R_{(u,\infty)} \circ F^{SO}(W, \mathbb{R}^m, d^p, \gamma) \rightarrow R_{(u,\infty)} \circ F^{SO}(T, \mathbb{R}^m, d^p, \gamma_T)(\epsilon)$.

Now let $p \in \lambda_{a,b}$. Then there is a point $p' \in T$ with $d^p(p, p') \leq b$ and $\gamma(p') \leq a$. Thus $p \in \gamma_{a,b}$. Therefore $\lambda_{a,b} \subset \gamma_{a,b} \subset \gamma_{a+\epsilon, b+\epsilon}$. The inclusions thus define a morphism $g : R_{(u,\infty)} \circ F^{SO}(T, \mathbb{R}^m, d^p, \gamma_T) \rightarrow R_{(u,\infty)} \circ F^{SO}(W, \mathbb{R}^m, d^p, \gamma)(\epsilon)$. Clearly, f and g are a pair of strong interleaving ϵ -interleaving homomorphisms, so $R_{(u,\infty)} \circ F^{SO}(W, \mathbb{R}^m, d^p, \gamma)$ and $R_{(u,\infty)} \circ F^{SO}(T, \mathbb{R}^m, d^p, \gamma_T)$ are strongly ϵ -interleaved, as desired. \square

We now observe that we have a chain of interleaving relationships between filtrations:

- By Lemma 4.2.4, $R_{(u,\infty)} \circ F^{SO}(W, \mathbb{R}^m, d^p, \gamma)$ and $R_{(u,\infty)} \circ F^{SO}(T, \mathbb{R}^m, d^p, \gamma_T)$ are strongly ϵ -interleaved.
- By Example 3.2.3(i), $F^{SO}(T, \mathbb{R}^m, d^p, \gamma_T)$ and $F^{SO-Op}(T, \mathbb{R}^m, d^p, \gamma_T)$ are strongly δ -interleaved for all $\delta > 0$.
- By Proposition 4.2.1(i), $F^{SO-Op}(T, \mathbb{R}^m, d^p, \gamma_T)$ and $F^{SCe-Op}(T, \mathbb{R}^m, d^p, \gamma_T)$ are weakly 0-interleaved.
- By Example 3.2.3(ii), $F^{SCe-Op}(T, \mathbb{R}^m, d^p, \gamma_T)$ and $F^{SCe}(T, \mathbb{R}^m, d^p, \gamma_T)$ are also strongly δ -interleaved.
- $F^{SCe}(T, \mathbb{R}^m, d^p, \gamma_T)$ and $F^{SCe}(T, \mathbb{R}^m, d^p, \tilde{\gamma})$ are strongly C -interleaved.

Applying Lemmas 3.3.5 and 3.3.8 several times gives that $R_{(u,\infty)} \circ F^{SO}(W, \mathbb{R}^m, d^p, \gamma)$ and $R_{(u,\infty)} \circ F^{SCe}(T, \mathbb{R}^m, d^p, \tilde{\gamma})$ are $(\epsilon + C + 2\delta)$ -interleaved. Since this holds for all $\delta > 0$,

$$d_{WI}(R_{(u,\infty)} \circ F^{SO}(W, \mathbb{R}^m, d^p, \gamma), R_{(u,\infty)} \circ F^{SR}(T, \mathbb{R}^m, d^p, \tilde{\gamma})) \leq \epsilon + C,$$

as we wanted to show. \square

Remark 4.2.5. Theorem 4.2.3 can be strengthened somewhat using the language of (J_1, J_2) -interleavings, but the strengthening is not especially interesting, so we choose to frame the result using the simpler language of ϵ -interleavings.

Corollary 4.2.6. *For any $i \geq 0$,*

(i) Under the same assumptions as in Theorem 4.2.3(i),

$$d_I(H_i \circ R_{(u,\infty)} \circ F^{SO}(W, \mathbb{R}^m, d^p, \gamma), H_i \circ R_{(u,\infty)} \circ F^{S\check{C}e}(T, \mathbb{R}^m, d^p, \tilde{\gamma})) \leq \epsilon + C.$$

(ii) Under the same assumptions as in Theorem 4.2.3(ii),

$$d_I(H_i \circ R_{(u,\rho(M))} \circ F^{SO}(M, d^M, \gamma), H_i \circ R_{(u,\rho(M))} \circ F^{S\check{C}e}(T, M, d^M, \tilde{\gamma})) \leq \epsilon + C.$$

Proof. This follows immediately from Theorem 3.3.10. \square

4.2.3 Topological Approximation of Sublevelset-Offset Filtrations via Sublevelset-Rips Filtrations

Our second main result of this section, Theorem 4.2.7, is an analogue of Theorem 4.2.3 for sublevelset-Rips filtrations rather than sublevelset-Čech filtrations. We formulate the result using weak (J_1, J_2) -interleavings.

For (J_1, J_2) increasing, we'll say that two u -filtrations X and Y are strongly (weakly) **almost (J_1, J_2) -interleaved** if for all $\epsilon > 0$, X and Y are strongly (weakly) $(J_\epsilon \circ J_1, J_\epsilon \circ J_2)$ -interleaved. Similarly, we'll say that two B_n -persistence modules M and N are **almost (J_1, J_2) -interleaved** if for all $\epsilon > 0$, M and N are $(J_\epsilon \circ J_1, J_\epsilon \circ J_2)$ -interleaved.

For the statement theorem, recall that in Example 3.2.4 we defined the map $\mathcal{J}_{n+1} : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$ by $\mathcal{J}_{n+1}((a, b)) = (a, 2b)$ for $a \in \mathbb{R}^n$, $b \in \mathbb{R}$.

Theorem 4.2.7.

- (i) Under the same assumptions as in Theorem 4.2.3(i), $R_{(u,\infty)} \circ F^{SO}(W, \mathbb{R}^m, d^p, \gamma)$ and $R_{(u,\infty)} \circ F^{SR}(T, d^p, \tilde{\gamma})$ are almost weakly $(J_{\epsilon+C}, \mathcal{J}_{n+1} \circ J_{\epsilon+C})$ -interleaved.
- (ii) Under the same assumptions as in Theorem 4.2.3(ii), $R_{(u,\rho(M))} \circ F^{SO}(M, d^M, \gamma)$ and $R_{(u,\rho(M))} \circ F^{SR}(T, d^M, \tilde{\gamma})$ are almost weakly $(J_{\epsilon+C}, \mathcal{J}_{n+1} \circ J_{\epsilon+C})$ -interleaved.

Proof. We present the proof of (i); the proof of (ii) is essentially the same as the proof of (i), using Theorem 4.2.3(ii) in place of Theorem 4.2.3(i).

By Theorem 4.2.3(i), for any $\delta > 0$ $R_{(u,\infty)} \circ F^{SO}(W, \mathbb{R}^m, d^p, \gamma)$ and $R_{(u,\infty)} \circ F^{S\check{C}e}(T, \mathbb{R}^m, d^p, \tilde{\gamma})$ are $(\epsilon + C + \delta)$ -interleaved. By Example 3.2.4, $F^{S\check{C}e}(T, \mathbb{R}^m, d^p, \tilde{\gamma})$ and $F^{SR}(T, d^p, \tilde{\gamma})$ are $(\text{id}_{n+1}, \mathcal{J}_{n+1})$ -interleaved. Thus by Lemmas 3.3.6 and 3.3.8, $R_{(u,\infty)} \circ F^{SO}(W, \mathbb{R}^m, d^p, \gamma)$ and $R_{(u,\infty)} \circ F^{SR}(T, d^p, \tilde{\gamma})$ are $(J_{\epsilon+C+\delta}, \mathcal{J}_{n+1} \circ J_{\epsilon+C+\delta})$ -interleaved. Thus by Lemma 3.3.9 they are also $(J_{2\delta} \circ J_{\epsilon+C}, J_{2\delta} \circ \mathcal{J}_{n+1} \circ J_{\epsilon+C})$ -interleaved. The result follows. \square

Remark 4.2.8. Theorem 4.2.7 can be tightened by tightening the result of Theorem 4.2.3 on which it depends; see Remark 4.2.5.

Corollary 4.2.9. *For any $i \geq 0$,*

- (i) *Under the same assumptions as in Theorem 4.2.3(i), $H_i \circ R_{(u,\infty)} \circ F^{SO}(W, \mathbb{R}^m, d^p, \gamma)$ and $H_i \circ R_{(u,\infty)} \circ F^{SR}(T, d^p, \tilde{\gamma})$ are almost $(J_{\epsilon+C}, \mathcal{J}_{n+1} \circ J_{\epsilon+C})$ -interleaved.*
- (ii) *Under the same assumptions as in Theorem 4.2.3(ii), $H_i \circ R_{(u,\rho(M))} \circ F^{SO}(M, d^M, \gamma)$ and $H_i \circ R_{(u,\rho(M))} \circ F^{S\check{C}e}(T, d^M, \tilde{\gamma})$ are almost $(J_{\epsilon+C}, \mathcal{J}_{n+1} \circ J_{\epsilon+C})$ -interleaved.*

4.2.4 Approximating Multi-D Sublevelset Persistence via Discrete Filtrations with Fixed Scale Parameter

We now observe that the deterministic result [12, Theorem 3.7], on the approximation of sublevelset persistence of \mathbb{R} -valued functions via filtered Rips complexes with fixed scale parameter, admits a straightforward generalization \mathbb{R}^n -valued functions, $n \geq 1$. This generalization, Theorem 4.2.10, is proven in the same way as [12, Theorem 3.7], using the interleaving distance in place of the bottleneck distance.

Unlike the results of the previous sections, this result gives an approximation only on the level of persistent homology and not one on the level of filtrations. However, we prove a variant of the result, Theorem 4.2.11, formulated using filtered Čech complexes with fixed scale parameter instead of filtered Rips complexes with fixed scale parameter, which does hold on the level of filtrations. This latter result is new

even for 1-D filtrations; we will use it in Section 4.5 to present a result, Theorem 4.5.2, on the topological inference of the superlevelset filtration of a density function γ via a filtered Čech complex with fixed scale parameter built on i.i.d. samples of \mathcal{P}_γ .

Given a finite metric space $(X, d, f) \in C^{SR}$, and $\delta > 0$, let $F^{SR}(X, d, f, \delta)$ denote the n -filtration obtained by fixing the last parameter in the $(n + 1)$ -filtration $F^{SR}(X, d, f)$ to be δ . For $u \in \mathbb{R}^n$, $i \in \mathbb{Z}_{\geq 0}$, and $\delta_2 \geq \delta_1 > 0$, let

$$H_i^u(X, d, f, \delta_1, \delta_2) \subset H_i \circ R_u \circ F^{SR}(X, d, f, \delta_2)$$

be the image of the map $H_i(j)$, where

$$j : R_u \circ F^{SR}(X, d, f, \delta_1) \hookrightarrow R_u \circ F^{SR}(X, d, f, \delta_2)$$

is the inclusion.

[11, Theorem 5.1] was stated for geodesic metrics on Riemannian manifolds, but the analogous result holds for L^p metrics on \mathbb{R}^m , $1 \leq p \leq \infty$. Similarly, we state our generalization for geodesic metrics on Riemannian manifolds, but an analogue holds for L^p metrics as well.

Theorem 4.2.10. *Let M be a Riemannian manifold and let $\gamma : M \rightarrow \mathbb{R}^n$ be a c -Lipschitz function (w.r.t d^M). For $u \in \hat{\mathbb{R}}^n$, let T be an ϵ -sample of γ_u (w.r.t. d^M). Let $\tilde{\gamma} : T \rightarrow \mathbb{R}^n$ be such that $|\tilde{\gamma}(l) - \gamma(l)| \leq C$ for all $l \in T$. If $\epsilon < \frac{\rho(M)}{4}$ then for any $i \in \mathbb{Z}_{\geq 0}$ and $\delta \in [2\epsilon, \frac{\rho(M)}{2}]$,*

$$d_I(H_i^u(T, d^M, \tilde{\gamma}, \delta, 2\delta), H_i \circ R_u \circ F^S(M, \gamma)) \leq 2c\delta + C.$$

Proof. As noted above, the proof of [12, Theorem 3.7] adapts directly. The one difference is that [12, Theorem 3.7] is formulated in terms of “open” variants of the sublevelset filtration and the sublevelset-Rips filtration, analogous to the open variant of the sublevelset-offset filtration presented in Section 2.7.4. Thus to adapt the proof of [12, Theorem 3.7] to our setting we need to appeal to example 3.2.3.

We note also that using our module-theoretic definition of interleavings, the proof of [12, Theorem 3.7] may be written down in terms of module homomorphisms rather

than in terms of maps of vector spaces, as in [12]. This makes the proof somewhat less cumbersome. \square

Now we now formulate our variant of Theorem 4.2.10 for Čech filtrations. As for Theorem 4.2.10 we state the result for geodesic metrics on Riemannian manifolds, but an analogue holds for the L^p metric as well.

For $(X, Y, d, f) \in C^{SCe}$ and $\delta > 0$, let $F^{S\check{C}e}(X, Y, d, f, \delta)$ denote the n -filtration obtained by fixing the last parameter in the $(n + 1)$ -filtration $F^{S\check{C}e}(X, Y, d, f)$ to be δ .

Theorem 4.2.11. *Let M be a compact Riemannian manifold and $\gamma : M \rightarrow \mathbb{R}^n$ be a c -Lipschitz function (w.r.t d^M). For $u \in \hat{\mathbb{R}}^n$, let T be an ϵ -sample of γ_u (w.r.t. d^M). Let $\tilde{\gamma} : T \rightarrow \mathbb{R}^n$ be such that $|\tilde{\gamma}(l) - \gamma(l)| \leq C$ for all $l \in T$. Then for any $\delta \geq \epsilon$,*

$$d_{WI}(R_u \circ F^{S\check{C}e}(T, M, d^M, \tilde{\gamma}, \delta), R_u \circ F^S(M, \gamma)) \leq c\delta + C.$$

Proof. The proof is similar to the proof of Theorem 4.2.3. We leave the details to the reader. \square

4.3 Bounds on the Probability that an I.I.D. Sample of a Manifold is an ϵ -sample of a Superlevelset

Overview

[11, Section 4.2] presents a bound on the probability that an i.i.d. sample of a Riemannian manifold M with density γ is an ϵ -sample (w.r.t d^M) of a superlevelset of γ .

Here we recall that bound and apply it to obtain a bound on the probability that an i.i.d. sample of a submanifold M of \mathbb{R}^m with density γ is an ϵ -sample (w.r.t d^p) of a superlevelset of γ .

In the next section, we will use these bounds to prove our main inference results.

Definitions and Results

Let M be a Riemannian manifold. Let A be a subset of M . For $r \in \mathbb{R}$, let

$$\mathcal{V}(A, r) = \inf_{x \in A} \mathcal{H}(B_{d^M}(x, r)).$$

Let $\mathcal{N}(A, r) \in \mathbb{N} \cup \{\infty\}$ be the r -covering number of A —that is, $\mathcal{N}(A, r)$ is the minimum number of closed d -balls of the same radius r needed to cover A (the balls do not need to be centered in A).

The following lemma is [11, Lemma 4.3].

Lemma 4.3.1. *Let M be a Riemannian manifold, $\gamma : M \rightarrow \mathbb{R}$ be a c -Lipschitz probability density function (w.r.t d^M), and $c' = \max(1, c)$. Let T_z be an i.i.d. sample of \mathcal{P}_γ of size z . Then for any $\epsilon > 0$ and $\alpha > \frac{c\epsilon}{c'}$, T_z is an $\frac{\epsilon}{c'}$ -sample of $-\gamma_{-\alpha}$ (w.r.t. to d^M) with probability at least $1 - \mathcal{N}(-\gamma_{-\alpha}, \frac{\epsilon}{2c'}) e^{-z(\alpha - \frac{c\epsilon}{c'}) \mathcal{V}(-\gamma_{-\alpha}, \frac{\epsilon}{2c'})}$.*

Remark 4.3.2. As noted in [11], Lemma 4.3.1 tells us that if $\mathcal{N}(-\gamma_{-\alpha}, \frac{\epsilon}{2c'})$ is positive and $\mathcal{V}(-\gamma_{-\alpha}, \frac{\epsilon}{2c'})$ is finite for every $\epsilon > 0$ then the Hausdorff distance between $T_z \cap -\gamma_{-\alpha}$ and $-\gamma_{-\alpha}$ converges in probability to 0 as $z \rightarrow \infty$.

Remark 4.3.3. It is observed in [11] that $\mathcal{N}(-\gamma_{-\alpha}, \epsilon)$ is positive and $\mathcal{V}(-\gamma_{-\alpha}, \epsilon)$ is finite for any $\alpha > 0$ and $\epsilon > 0$, provided M is compact or, more generally, the sectional curvature of M is bounded above and below.

Now we extend Lemma 4.3.1 to the case of L^p metrics on submanifolds of \mathbb{R}^m . For $1 \leq p \leq \infty$, let

$$K(p) = \begin{cases} \sqrt{m} & \text{if } p \in [1, 2), \\ 1 & \text{if } p \in [2, \infty]. \end{cases}$$

Lemma 4.3.4. *Let M be a submanifold of \mathbb{R}^m , $\gamma : M \rightarrow \mathbb{R}$ be a c -Lipschitz probability density function (w.r.t d^p), let $c' = \max(1, c)$, and let T_z be an i.i.d. sample of \mathcal{P}_γ of size z . Then for any $\epsilon > 0$ and $\alpha > \frac{c\epsilon}{c'}$, T_z is an $\frac{\epsilon}{c'}$ -sample of $-\gamma_{-\alpha}$ (w.r.t. d^p) with probability at least $1 - \mathcal{N}(-\gamma_{-\alpha}, \frac{\epsilon}{2c'K(p)}) e^{-z(\alpha - \frac{c\epsilon}{c'}) \mathcal{V}(-\gamma_{-\alpha}, \frac{\epsilon}{2c'K(p)})}$.*

Proof. For any $y_1, y_2 \in M$,

$$d^p(y_1, y_2) \leq K(p) d^2(y_1, y_2) \leq K(p) d^M(y_1, y_2). \quad (4.1)$$

Thus a c -Lipschitz function w.r.t. d^p is a $cK(p)$ -Lipschitz density function (w.r.t d^M).

Let $c'' = \max(1, cK(p))$. Lemma 4.3.1 tells us that for $\gamma : M \rightarrow \mathbb{R}$ a $cK(p)$ -Lipschitz probability density function (w.r.t d^M), T_z an i.i.d. sample of \mathcal{P}_γ of size z , $\epsilon' > 0$, and $\alpha > \frac{cK(p)\epsilon'}{c''}$, T_z is an $\frac{\epsilon'}{c''}$ -sample of $-\gamma_{-\alpha}$ (w.r.t. d^M) with probability at least $1 - \mathcal{N}(-\gamma_{-\alpha}, \frac{\epsilon'}{2c''})e^{-z(\alpha - \frac{cK(p)\epsilon'}{c''})\mathcal{V}(-\gamma_{-\alpha}, \frac{\epsilon'}{2c''})}$.

Now substituting $\epsilon' = \frac{c''\epsilon}{c'K(p)}$ in the above statement, we obtain for any $\epsilon > 0$, $\alpha > \frac{c\epsilon}{c'}$, T_z is an $\frac{\epsilon}{c'K(p)}$ -sample of M (w.r.t. to d^M) with probability at least $1 - \mathcal{N}(-\gamma_{-\alpha}, \frac{\epsilon}{2c'K(p)})e^{-z(\alpha - \frac{c\epsilon}{c'})\mathcal{V}(-\gamma_{-\alpha}, \frac{\epsilon}{2c'K(p)})}$. Thus by (4.1), with at least the same probability T_z is an $\frac{\epsilon}{c}$ -sample of M (w.r.t. d^p), as we wanted to show. \square

Note that remark 4.3.2 adapts immediately to the setting of Lemma 4.3.4.

4.4 Inference of the Superlevelset-offset Bifiltration of a Density Function via Discrete Bifiltrations Built on I.I.D. Samples

We're now ready to present our results on the inference of the superlevelset-offset filtration of a density function γ on a Riemannian manifold from an i.i.d. sample of \mathcal{P}_γ . These are the culmination of the theory we have developed so far in this thesis.

We first present results for inference using superlevelset-Čech filtrations. Theorem 4.4.1 shows that with high probability, superlevelset-Čech filtrations built on sufficiently large i.i.d. samples of \mathcal{P}_γ (and filtered by the superlevelsets of a well behaved density estimator) give good approximations to the superlevelset-offset filtration of γ , with respect to d_{WI} . Theorem 4.4.2, an asymptotic form of this result, then tells us that such superlevelset-Čech filtrations are *consistent* estimators (w.r.t. d_{WI}) of the superlevelset-offset filtration of γ .

After presenting our results for superlevelset-Čech filtrations, we present analogues of each these results for superlevelset-Rips filtrations. The main result is Theorem 4.4.4, which shows that in the large sample limit, the superlevelset-Rips filtration built on an i.i.d. sample of \mathcal{P}_γ (and filtered by the superlevels of a well behaved density estimator) and the superlevelset-offset filtration of γ are weakly $(\mathcal{J}_{n+1}, \text{id}_{n+1})$ -interleaved. Since as observed in Example 3.2.4, superlevelset-Rips and superlevelset-Čech filtration functors built on the same input data are always strongly $(\mathcal{J}_{n+1}, \text{id}_{n+1})$ -interleaved, Theorem 4.4.4 says that in the large sample limit, the superlevelset-Rips and superlevelset-Offset filtrations satisfy in the weak sense the same interleaving relationship that superlevelset-Rips and superlevelset-Čech filtrations always satisfy in the strong sense.

4.4.1 Inference Results for Superlevelset-Čech Bifiltrations

Theorem 4.4.1 (Inference of superlevelset-offset filtrations via superlevelset-Čech filtrations (finite sample case)). *Let M be a Riemannian manifold, $\gamma : M \rightarrow \mathbb{R}$ be a density function, T_z be a finite i.i.d. random sample of \mathcal{P}_γ of size z , and E_z be a (q, C) -density estimator of another density function $\tilde{\gamma} : M \rightarrow \mathbb{R}$ for some $C > 0$ and $q \in [0, 1]$.*

- (i) *If M is an embedded submanifold of \mathbb{R}^m , γ is c -Lipschitz w.r.t. d^p for some $c > 0$, and $c' = \max(1, c)$ then for any $\epsilon > 0$ and $\alpha > \frac{c\epsilon}{c'}$,*

$$d_{WI}(R_{(-\alpha, \infty)} \circ F^{S\check{C}e}(T_z, \mathbb{R}^m, d^p, -E_z(T_z)), R_{(-\alpha, \infty)} \circ F^{SO}(M, \mathbb{R}^m, d^p, -\tilde{\gamma})) \leq \epsilon + C$$

with probability at least $1 - \mathcal{N}(-\gamma_{-\alpha}, \frac{\epsilon}{2c'K(p)})e^{-z(\alpha - ce/c')\mathcal{V}(-\gamma_{-\alpha}, \frac{\epsilon}{2c'K(p)})} - q$.

- (ii) *If γ is c -Lipschitz w.r.t. d^M for some $c > 0$, and $c' = \max(1, c)$, then for any $\epsilon > 0$ and $\alpha > \frac{c\epsilon}{c'}$,*

$$d_{WI}(R_{(-\alpha, \rho(M))} \circ F^{S\check{C}e}(T_z, M, d^M, -E_z(T_z)), R_{(-\alpha, \rho(M))} \circ F^{SO}(M, d^M, -\tilde{\gamma})) \leq \epsilon + C$$

with probability at least $1 - \mathcal{N}(-\gamma_{-\alpha}, \frac{\epsilon}{2c'})e^{-z(\alpha - ce/c')\mathcal{V}(-\gamma_{-\alpha}, \frac{\epsilon}{2c'})} - q$.

Proof. To prove (i), first note that in the event that

$$\{\sup_{l \in T_z} |\tilde{\gamma}(l) - E_z(T_z)(l)| \leq C\} \cap \{T_z \text{ is an } \frac{\epsilon}{c'}\text{-sample of } M \text{ (w.r.t. } d^p)\},$$

Theorem 4.2.3(i) applies to give that

$$d_{WI}(R_{(-\alpha, \infty)} \circ F^{S\check{C}e}(T_z, \mathbb{R}, d^p, -E_z(T_z)), R_{(-\alpha, \infty)} \circ F^{SO}(M, \mathbb{R}^m, d^p, -\tilde{\gamma})) \leq \epsilon + C.$$

By Lemma 4.3.4, the definition of a (q, C) -density estimator, and the union bound, the probability of this event is at least $1 - \mathcal{N}(-\gamma_{-\alpha}, \frac{\epsilon}{2c'}) e^{-z(\alpha - c\epsilon/c')\mathcal{V}(-\gamma_{-\alpha}, \frac{\epsilon}{2c'})} - q$. This gives (i).

The proof of (ii) is the same as that of (i), using Lemma 4.3.1 in place of Lemma 4.3.4 and Theorem 4.2.3(ii) in place of Theorem 4.2.3(i). \square

Theorem 4.4.2 (Consistent estimation of superlevelset-offset filtration via superlevelset-Čech filtrations). *Let M be Riemannian manifold with sectional curvature bounded above and below, let $\gamma : M \rightarrow \mathbb{R}$ be a density function, and let T_z be an i.i.d. sample of \mathcal{P}_γ of size z .*

(i) *If M is a submanifold of \mathbb{R}^m , γ is c -Lipchitz (w.r.t. d^p) for some $c > 0$ and E is a density estimator on M such that (E, γ) satisfies A1 then*

$$d_{WI}(R_{(0, \infty)} \circ F^{S\check{C}e}(T_z, \mathbb{R}^m, d^p, -E_z(T_z)), R_{(0, \infty)} \circ F^{SO}(M, \mathbb{R}^m, d^p, -\gamma)) \xrightarrow{P} 0.$$

(ii) *If $M = \mathbb{R}^m$, γ is c -Lipchitz (w.r.t. d^p) for some $c > 0$ and E is a density estimator on \mathbb{R}^m such that (E, γ) satisfies A2 for some kernel K then*

$$d_{WI}(R_{(0, \infty)} \circ F^{S\check{C}e}(T_z, \mathbb{R}^m, d^p, -E_z(T_z)), R_{(0, \infty)} \circ F^{SO}(\mathbb{R}^m, d^p, -\gamma * K)) \xrightarrow{P} 0.$$

(iii) *If $\gamma : M \rightarrow \mathbb{R}$ is c -Lipschitz (w.r.t. d^M) for some $c > 0$ and E is a density estimator such that (E, γ) satisfies A1 then*

$$d_{WI}(R_{(0, \rho(M))} \circ F^{S\check{C}e}(T_z, M, d^M, -E_z(T_z)), R_{(0, \rho(M))} \circ F^{SO}(M, d^M, -\gamma)) \xrightarrow{P} 0.$$

Proof. To prove (i), we need to show that for any $\epsilon > 0$ and $q \in (0, 1]$, there exists some $z_1 \in \mathbb{N}$ such that for all $z \geq z_1$,

$$d_{WI}(R_{(0,\infty)} \circ F^{S\check{C}e}(T_z, \mathbb{R}^m, d^p, -E_z(T_z)), R_{(0,\infty)} \circ F^{SO}(M, \mathbb{R}^m, d^p, -\gamma)) \leq \epsilon$$

with probability at least $1 - q$. Since (E, γ) satisfies A1, Lemma 4.1.1(i) tells us that for any $q \in (0, 1]$ and $\epsilon > 0$, there exists $z_0 \in \mathbb{N}$ such that for all $z \geq z_0$, E_z is a $(\frac{q}{2}, \frac{\epsilon}{4})$ -density estimator of γ .

Choose $\epsilon' \in (0, \frac{\epsilon}{4})$. Then $\frac{\epsilon}{2} > \frac{c\epsilon'}{c'}$. Since M has bounded absolute sectional curvature, it follows from Remark 4.3.3 that there exists some $z_1 \in \mathbb{N}$ with $z_1 > z_0$ such that for all $z \geq z_1$, $\mathcal{N}(-\gamma_{-\frac{\epsilon}{2}}, \frac{\epsilon'}{2c'}) e^{-z(\frac{\epsilon}{2}-c\epsilon'/c')\mathcal{V}(-\gamma_{-\frac{\epsilon}{2}}, \frac{\epsilon'}{2c'})} < \frac{q}{2}$.

Invoking Theorem 4.4.1(i) with $\alpha = \frac{\epsilon}{2}$, for all $z \geq z_1$

$$d_{WI}(R_{(-\frac{\epsilon}{2}, \infty)} \circ F^{S\check{C}e}(T_z, \mathbb{R}^m, d^p, -E_z(T_z)), R_{(-\frac{\epsilon}{2}, \infty)} \circ F^{SO}(M, \mathbb{R}^m, d^p, -\gamma)) \leq \epsilon' + \frac{\epsilon}{4} \leq \frac{\epsilon}{2}$$

with probability at least $1 - q$.

Then by Lemma 3.3.7, for all $z > z_1$,

$$d_{WI}(R_{(0,\infty)} \circ F^{S\check{C}e}(T_z, \mathbb{R}^m, d^p, -E_z(T_z)), R_{(0,\infty)} \circ F^{SO}(M, \mathbb{R}^m, d^p, -\gamma)) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

with probability at least $1 - q$. This completes the proof of (i).

The proof of (ii) is essentially the same as that of statement (i), using Lemma 4.1.1(ii) in place of Lemma 4.1.1(i). The proof of (iii) is also essentially the same as that of statement (i), using Theorem 4.4.1(ii) in place of Theorem 4.4.1(i). \square

Corollary 4.4.3. *For any $i \in \mathbb{Z}_{\geq 0}$,*

(i) *Under the same assumptions as in the statement of Theorem 4.4.2(i),*

$$d_I(H_i \circ R_{(0,\infty)} \circ F^{S\check{C}e}(T_z, \mathbb{R}^m, d^p, -E_z(T_z)), H_i \circ R_{(0,\infty)} \circ F^{SO}(M, \mathbb{R}^m, d^p, -\gamma)) \xrightarrow{P} 0.$$

(ii) *Under the same assumptions as in the statement of Theorem 4.4.2(ii),*

$$d_I(H_i \circ R_{(0,\infty)} \circ F^{S\check{C}e}(T_z, \mathbb{R}^m, d^p, -E_z(T_z)), H_i \circ R_{(0,\infty)} \circ F^{SO}(\mathbb{R}^m, d^p, -\gamma * K)) \xrightarrow{P} 0.$$

(iii) Under the same assumptions as in the statement of Theorem 4.4.2(iii),

$$\begin{aligned} d_I(H_i \circ R_{(0,\rho(M))} \circ F^{S\check{C}e}(T_z, M, d^M, -E_z(T_z)), \\ H_i \circ R_{(0,\rho(M))} \circ F^{SO}(M, \mathbb{R}^m, d^M, -\gamma)) \xrightarrow{P} 0. \end{aligned}$$

Proof. This follows immediately from Theorems 4.4.2 and 3.3.10. \square

4.4.2 Inference Results for Superlevelset-Rips Bifiltrations

Convergence in Probability up to Interleavings

To describe our asymptotic results for superlevelset-Rips filtrations it will be convenient for us to introduce some terminology. Let $J_1, J_2 : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be increasing maps and let $u \in \hat{\mathbb{R}}^n$. We say that a sequence $\{X_z\}_{z \in \mathbb{N}}$ of random u -filtrations converges in probability to a random u -filtration X up to weak (J_1, J_2) -interleaving, and we write

$$X_z \xrightarrow[z \rightarrow \infty]{P, J_1, J_2} X$$

if for all $\epsilon > 0$

$$\lim_{z \rightarrow \infty} P(X_z \text{ and } X \text{ are weakly } (J_\epsilon \circ J_1, J_\epsilon \circ J_2)\text{-interleaved}) = 1.$$

Note that $X_n \xrightarrow{P, \text{id}_n, \text{id}_n} X$ if and only if $d_{WI}(X_n, X) \xrightarrow{P} 0$.

Similarly, we say that a sequence $\{M_z\}_{z \in \mathbb{N}}$ of random B_n -persistence modules converges in probability to a random B_n -persistence module M up to weak (J_1, J_2) -interleaving, and we write

$$M_z \xrightarrow{P, J_1, J_2} M$$

if for all $\epsilon > 0$

$$\lim_{z \rightarrow \infty} P(M_z \text{ and } M \text{ are weakly } (J_\epsilon \circ J_1, J_\epsilon \circ J_2)\text{-interleaved}) = 1.$$

$M_n \xrightarrow{P, \text{id}_n, \text{id}_n} M$ if and only if $d_I(M_n, M) \xrightarrow{P} 0$.

Theorem 4.4.4 (Asymptotics of estimation of superlevelset-offset filtration via superlevelset-Rips filtrations).

(i) Under the same assumptions as in Theorem 4.4.2(i),

$$R_{(0,\infty)} \circ F^{SR}(T_z, d^p, -E_z(T_z)) \xrightarrow{P, \mathcal{J}_{n+1}, \text{id}_{n+1}} R_{(0,\infty)} \circ F^{SO}(M, \mathbb{R}^m, d^p, -\gamma).$$

(ii) Under the same assumptions as in Theorem 4.4.2(ii),

$$R_{(0,\infty)} \circ F^{SR}(T_z, d^p, -E_z(T_z)) \xrightarrow{P, \mathcal{J}_{n+1}, \text{id}_{n+1}} R_{(0,\infty)} \circ F^{SO}(\mathbb{R}^m, d^p, -\gamma * K).$$

(iii) Under the same assumptions as in Theorem 4.4.2(iii),

$$R_{(0,\rho(M))} \circ F^{SR}(T_z, d^M, -E_z(T_z)) \xrightarrow{P, \mathcal{J}_{n+1}, \text{id}_{n+1}} R_{(0,\rho(M))} \circ F^{SO}(M, d^M, -\gamma).$$

Proof. To prove (i), we need to show that under the same assumptions as in Theorem 4.4.2(i), for any $\epsilon > 0$ and $q \in (0, 1]$, there exists some $z' \in \mathbb{N}$ such that for all $z \geq z'$, $R_{(0,\infty)} \circ F^{SR}(T_z, d^p, -E_z(T_z))$ and $R_{(0,\infty)} \circ F^{SO}(M, \mathbb{R}^m, d^p, -\gamma)$ are $(J_\epsilon \circ \mathcal{J}_{n+1}, J_\epsilon \circ \text{id}_{n+1})$ -interleaved with probability at least $1 - q$.

By Example 3.2.4, $F^{SR}(T, d^p, \tilde{\gamma})$ and $F^{S\check{C}e}(T, \mathbb{R}^m, d^p, \tilde{\gamma})$ are $(\mathcal{J}_{n+1}, \text{id}_{n+1})$ -interleaved.

Theorem 4.4.2 tells us that there exists some $z'' > 0$ such that for all $z > z''$, $R_{(0,\infty)} \circ F^{S\check{C}e}(T_z, \mathbb{R}^m, d^p, -E_z(T_z))$ and $R_{(0,\infty)} \circ F^{SO}(M, \mathbb{R}^m, d^p, -\gamma)$ are ϵ -interleaved with probability at least $1 - q$.

By Lemma 3.3.8 then, for all $z > z''$, $R_{(0,\infty)} \circ F^{SR}(T_z, d^p, -E_z(T_z))$ and $R_{(0,\infty)} \circ F^{SO}(M, \mathbb{R}^m, d^p, -\gamma)$ are $(J_\epsilon \circ \mathcal{J}_{n+1}, J_\epsilon)$ -interleaved with probability at least $1 - q$. Thus taking $z' = z''$ gives the result.

The proofs of (ii) and (iii) are essentially the same as that of (i), using Theorems 4.4.2(ii) and 4.4.2(iii) in place of Theorem 4.4.2(i). \square

Corollary 4.4.5. For any $i \in \mathbb{Z}_{\geq 0}$,

(i) Under the same assumptions as in the statement of Theorem 4.4.2(i),

$$H_i \circ R_{(0,\infty)} \circ F^{SR}(T_z, d^p, -E_z(T_z)) \xrightarrow{P, \mathcal{J}_{n+1}, \text{id}_{n+1}} H_i \circ R_{(0,\infty)} \circ F^{SO}(M, \mathbb{R}^m, d^p, -\gamma).$$

(ii) Under the same assumptions as in the statement of Theorem 4.4.2(ii),

$$H_i \circ R_{(0,\infty)} \circ F^{SR}(T_z, d^p, -E_z(T_z)) \xrightarrow{P, \mathcal{J}_{n+1}, \text{id}_{n+1}} H_i \circ R_{(0,\infty)} \circ F^{SO}(\mathbb{R}^m, d^p, -\gamma * K).$$

(iii) Under the same assumptions as in the statement of Theorem 4.4.2(iii),

$$H_i \circ R_{(0,\rho(M))} \circ F^{SR}(T_z, d^M, -E_z(T_z)) \xrightarrow{P, \mathcal{J}_{n+1}, \text{id}_{n+1}} R_{(0,\rho(M))} \circ F^{SO}(M, d^M, -\gamma).$$

Proof. This follows immediately from Theorem 4.4.4 and Theorem 3.3.10. □

4.5 Inference of the Superlevelset Filtration of a Probability Density via Filtered Čech Complexes

In this section we apply Theorem 4.2.11 to obtain a version of the inference theorem of [11, Theorem 5.1] which holds on the level of filtrations rather than only on the level of persistent homology modules. Unlike [11, Theorem 5.1], this result holds for only estimators defined using Čech filtrations, not for estimators defined using Rips filtrations; the result of [11, Theorem 5.1] for estimators defined using Rips filtrations does not in general lift to the level of filtrations.

We state the result for geodesic metrics on Riemannian manifolds; an analogous result holds for L^p -metrics on submanifolds of \mathbb{R}^m .

Theorem 4.5.1 (Inference of superlevelset filtrations). *Let M be a Riemannian manifold, let $\gamma : M \rightarrow \mathbb{R}$ be a c -Lipschitz density function w.r.t. d^M for some $c > 0$. Let T_z be a finite i.i.d. random sample of \mathcal{P}_γ of size z . Let E_z be a (q, C) -density estimator of another density function $\tilde{\gamma} : \mathbb{R}^m \rightarrow \mathbb{R}$ for some $C > 0$ and $q \in [0, 1]$. Then for*

any $\epsilon > 0$, $\delta \geq \epsilon$, and $\alpha > c\epsilon$.

$$d_{WI}(R_{(-\alpha, \rho(M))} \circ F^{S\check{C}e}(T_z, M, d^M, -E_z(T_z), \delta), R_{(-\alpha, \rho(M))} \circ F^S(M, -\tilde{\gamma})) \leq c\delta + C$$

with probability at least $1 - \mathcal{N}(-\gamma_{-\alpha}, \frac{\epsilon}{2})e^{-z(\alpha-c\epsilon)\mathcal{V}(-\gamma_{-\alpha}, \frac{\epsilon}{2})} - q$.

Proof. The proof is the essentially same as the proof of Theorem 4.4.1, using Theorem 4.2.11 in place of Theorem 4.2.3. \square

Theorem 4.5.2 (convergence in probability of superlevelset-Čech filtrations with fixed scale parameter to superlevelset filtration of density function). *Let M be a Riemannian manifold with bounded absolute sectional curvature, let $\gamma : M \rightarrow \mathbb{R}$ be a c -Lipschitz density function (w.r.t. d^M) for some $c > 0$, and let T_z be an i.i.d. sample of \mathcal{P}_γ of size z .*

- (i) *If $M = \mathbb{R}^m$ and E is a density estimator on \mathbb{R}^m such that (E, γ) satisfies A1 then there exists a sequence of positive real numbers $\{\delta_z\}_{z \in \mathbb{N}}$ such that*

$$d_{WI}(R_{(0, \infty)} \circ F^{S\check{C}e}(T_z, \mathbb{R}^m, d^p, -E_z(T_z), \delta_z), R_{(0, \infty)} \circ F^S(\mathbb{R}^m, -\gamma)) \xrightarrow{P} 0.$$

- (ii) *If $M = \mathbb{R}^m$ and E is a density estimator on \mathbb{R}^m such that (E, γ) satisfies A2 for some kernel K then there exists a sequence of positive real numbers $\{\delta_z\}_{z \in \mathbb{N}}$ such that*

$$d_{WI}(R_{(0, \infty)} \circ F^{S\check{C}e}(T_z, \mathbb{R}^m, d^p, -E_z(T_z), \delta_z), R_{(0, \infty)} \circ F^S(\mathbb{R}^m, -\gamma * K)) \xrightarrow{P} 0.$$

- (iii) *If E is a density estimator such that (E, γ) satisfies A1 then there exists a sequence of positive real numbers $\{\delta_z\}_{z \in \mathbb{N}}$ such that*

$$d_{WI}(R_{(0, \rho(M))} \circ F^{S\check{C}e}(T_z, M, d^M, -E_z(T_z), \delta_z), R_{(0, \rho(M))} \circ F^S(M, -\gamma)) \xrightarrow{P} 0.$$

Proof. We'll prove (iii). The proofs of (i) and (ii) are essentially the same, using Lemma 4.1.1(ii) in place of Lemma 4.1.1(i) and an analogue of Theorem 4.5.1 for L^p metrics on \mathbb{R}^m .

To prove (i), we need to find a sequence of positive real numbers $\{\delta_z\}_{z \in \mathbb{N}}$ such that for any $\epsilon > 0$ and $p \in (0, 1]$, there exists some $z_1 \in \mathbb{N}$ such that for all $z \geq z_1$,

$$d_{WI}(R_{(0,\rho(M))} \circ F^{S\check{C}e}(T_z, M, d^M, -E_z(T_z), \delta_z), R_{(0,\rho(M))} \circ F^S(M, -\gamma)) \leq \epsilon$$

with probability at least $1 - p$. To construct our sequence $\{\delta_z\}_{z \in \mathbb{N}}$, we first choose a monotonically decreasing sequence $\xi : \mathbb{N} \rightarrow (0, 1]$ such that $\lim_{z \rightarrow \infty} \xi(z) = 0$. Since (E, γ) satisfies A1, Lemma 4.1.1(i) tells us that there exists a strictly increasing sequence of natural numbers $j : \mathbb{N} \rightarrow \mathbb{N}$ such that for any $y \in \mathbb{N}$ and any $z \geq j(y)$, E_z is a $(\frac{\xi(y)}{2}, \frac{\xi(y)}{4})$ -density estimator of γ .

Choose a second monotonically decreasing sequence $\xi' : \mathbb{N} \rightarrow (0, 1]$ such that for each $z \in \mathbb{N}$, $\xi'(z) < \frac{\xi(z)}{4}$. Let $c' = \max(1, c)$. For each $z \in \mathbb{N}$, $\frac{\xi(z)}{2} > \frac{c\xi'(z)}{c'}$. Since M has bounded absolute sectional curvature, it follows from Remark 4.3.3 that there exists a strictly increasing sequence of natural numbers $l : \mathbb{N} \rightarrow \mathbb{N}$ with $l(z) \geq j(z)$ for all $z \in \mathbb{N}$ such that for $y \in \mathbb{N}$ and all $z \geq l(y)$,

$$\mathcal{N}(-\gamma_{-\frac{\xi(y)}{2}}, \frac{\xi'(y)}{2c'}) e^{-z(\frac{\xi(y)}{2} - c\xi'(y)/c')\mathcal{V}(-\gamma_{-\frac{\xi(y)}{2}}, \frac{\xi'(y)}{2c'})} < \frac{\xi(y)}{2}.$$

Invoking Theorem 4.5.1, with the variables $(\alpha, \epsilon, \delta)$ in the statement of that Theorem set equal to $(\frac{\xi(y)}{2}, \frac{\xi'(y)}{c'}, \frac{\xi'(y)}{c'})$, we obtain the following result, which we state as a lemma.

Lemma 4.5.3. *For all $y \in \mathbb{N}$, $\xi(y)$, $\xi'(y)$, $l(y)$ defined as above, and all $z \geq l(y)$*

$$\begin{aligned} & d_{WI}(R_{(-\frac{\xi(y)}{2}, \rho(M))} \circ F^{S\check{C}e}(T_z, M, d^M, -E_z(T_z), \xi'(y)), R_{(-\frac{\xi(y)}{2}, \rho(M))} \circ F^S(M, -\gamma)) \\ & \leq \xi'(y) + \frac{\xi(y)}{4} \leq \frac{\xi(y)}{2} \end{aligned}$$

with probability at least $1 - \xi(y)$.

Now define $l^{-1} : \mathbb{N} \rightarrow \mathbb{N}$ by

$$l^{-1}(z) = \begin{cases} \max\{z' \in \mathbb{N} | l(z') \leq z\} & \text{if } l(1) \leq z, \\ 1 & \text{otherwise.} \end{cases}$$

Define $\{\delta_z\}_{z \in \mathbb{N}}$ by taking $\delta_z = \xi'(l^{-1}(z))$ for all $z \in \mathbb{N}$.

Now choose $\epsilon > 0$ and $p \in (0, 1]$. There's some $y \in \mathbb{N}$ such that $\xi(y) < \min(\epsilon, p)$. We claim that for all $z \geq l(y)$,

$$d_{WI}(R_{(-\frac{\xi(y)}{2}, \rho(M))} \circ F^{S\check{C}e}(T_z, M, d^M, -E_z(T_z), \delta_z), R_{(-\frac{\xi(y)}{2}, \rho(M))} \circ F^{SO}(M, -\gamma)) \leq \frac{\xi(y)}{2}$$

with probability at least $1 - \xi(y)$.

Then, given the claim, we set $z_1 = l(y)$. By Lemma 3.3.7, for all $z \geq z_1$,

$$\begin{aligned} & d_{WI}(R_{(0, \rho(M))} \circ F^{S\check{C}e}(T_z, M, d^M, -E_z(T_z), \delta_y), R_{(0, \rho(M))} \circ F^S(M, -\gamma)) \\ & \leq \frac{\xi(y)}{2} + \frac{\xi(y)}{2} = \xi(y) \leq \epsilon \end{aligned}$$

with probability at least $1 - \xi(y) \geq 1 - p$, which completes the proof of (i).

To prove the claim, we plug in $y = l^{-1}(z)$ into Lemma 4.5.3, writing

$$F = R_{(-\frac{\xi(l^{-1}(z))}{2}, \rho(M))} \circ F^{S\check{C}e}(T_z, M, d^M, -E_z(T_z), \xi'(l^{-1}(z)))$$

to obtain that

$$d_{WI}(F, R_{(-\frac{\xi(l^{-1}(z))}{2}, \rho(M))} \circ F^S(M, -\gamma)) \leq \xi'(l^{-1}(z)) + \frac{\xi(l^{-1}(z))}{4} \leq \frac{\xi(l^{-1}(z))}{2}$$

with probability at least $1 - \xi(l^{-1}(z))$.

Since l is strictly increasing, when $z \geq l(y)$, $l^{-1}(z) \geq l^{-1}(l(y)) = y$. Since the sequences ξ and ξ' are monotonically decreasing, we then have that $\xi(l^{-1}(z)) \leq \xi(y)$ and $\xi'(l^{-1}(z)) \leq \xi'(y)$. Thus,

$$d_{WI}(F, R_{(-\frac{\xi(l^{-1}(z))}{2}, \rho(M))} \circ F^S(M, -\gamma)) \leq \frac{\xi(y)}{2}$$

with probability at least $1 - \xi(y)$. Applying Lemma 3.3.6 and using the definition of δ_z , we obtain that

$$\begin{aligned} d_{WI}(R_{(-\frac{\xi(y)}{2}, \rho(M))} \circ F^{S\check{C}e}(T_z, M, d^M, -E_z(T_z), \delta_z), R_{(-\frac{\xi(y)}{2}, \rho(M))} \circ F^S(M, -\gamma)) \\ \leq \xi'(y) + \frac{\xi(y)}{4} \leq \frac{\xi(y)}{2} \end{aligned}$$

with probability at least $1 - \xi(y)$. This proves the claim and completes the proof of (i). \square

4.6 Future Work on Persistence-Based Topological Inference

We close this chapter with a discussion of directions for future work on the statistical aspects of persistence-based topological inference.

Almost Sure Convergence Results

Our asymptotic results are formulated in terms of convergence in probability and a variant of it which we have called *convergence in probability up to interleavings*. It would be interesting to know whether our results can be strengthened to almost sure convergence results.

Bootstrap Confidence Balls for Persistence

The natural next step in the development of the theory of persistence-based topological inference is to develop a theory of confidence regions for the consistent estimators studied in this chapter. This problem has hitherto not been addressed in any published work, for even for 1-D persistent homology.

In the setting of Theorem 4.4.1, it seems that it should be possible to employ the bootstrap methodology [25] to compute approximate confidence balls (in the pseudometric space of multidimensional filtrations given by the weak interleaving

interleaving distance) for a superlevelset-Čech estimator X of the superlevelset-offset filtration Y of a probability density function.

For $a \in (0, 1)$, to compute an approximate a -confidence ball for X , a simple approach would be to

1. Construct bootstrap replicates X_1, \dots, X_l of the filtration X by taking superlevelset-Čech filtrations of bootstrap replicates of the point cloud data on which X was built.
2. Compute the smallest value of y such that a fraction a of the bootstrap replicates X_1, \dots, X_l lie within weak interleaving distance y of X .

We expect that, under mild hypotheses, the ball of radius y centered at X would then be an approximate a -confidence ball for X , and that an analogous result would hold on the level of persistent homology modules.

A well developed theory along these lines of bootstrap confidence balls for persistent homology estimates, together with an efficient computational pipeline for computing the radii of confidence balls, could add in a significant way to power of persistent homology as a tool for understanding the qualitative structure of random data.

Testing Persistence-Based Hypotheses

Another important problem in topological inference is to develop a theory of hypothesis testing for hypotheses formulated using the language of persistence. Given the close relationship between hypothesis testing and computation of confidence regions, this problem is much in the same spirit as the problem of computing approximate confidence balls for the estimators of this thesis.

The problem of testing hypotheses formulated using persistence has hitherto not yet been treated by the topological data analysis community, yet seems to be of basic importance, particularly in the case of 0^{th} persistent homology. To explain, on the one hand 0^{th} persistent homology offers an elegant and user-friendly language for describing the multi-modality of functions in a way that is sensitive to the “size” of the modes. On the other hand, mode detection of densities and regression functions is an old and well studied problem in statistics, and one of fundamental interest—see,

for example [41] and [45, Chapter 20]. Classical statistics has developed approaches to testing for multi-modality based on smoothing ideas [41], but as the classical language for describing multi-modality is impoverished relative to that offered by persistent homology, there seems little doubt that persistent homology would have something significant to offer to the problem of testing multi-modality hypotheses.

Inference of Persistent Homology at the Cycle Level

It is possible to geometrically represent the persistent homology module of a filtration, at the chain level, by choosing a presentation for the persistent homology module and then choosing representative cycles and boundaries for the generators and relations in the presentation. There are of course many different ways to make such choices, most of which are not geometrically interesting. However, for filtrations whose topological spaces are equipped with a sufficient amount of geometric structure we can in principle define notions of optimality of chain-level representations of persistence modules in such a way that the optimal choice will be geometrically meaningful. Then, using such notions, we can pose persistence-based inference problems directly on the chain level. The results on clustering of [11] are loosely in this spirit, for 0^{th} persistent homology. It would be very interesting to understand if and how those results adapt to higher persistent homology and to the multidimensional setting.

The problem of pursuing inference at the chain level is an extremely important one, at least to the extent that we are interested in the use of persistent homology for topological inference: A descriptor of a probability distribution defined only using the isomorphism class of a persistent homology module gives us only indirect information about the qualitative structure of the probability distribution; ultimately we want to understand how the algebraic features of the persistent homology descriptor correspond to specific geometric features of the probability distribution. For that we need to consider representatives of those algebraic features on the chain level.

Chapter 5

Conclusion

In this thesis, we have introduced and studied (J_1, J_2) -interleavings on multidimensional persistence modules and strong and weak (J_1, J_2) -interleavings on multidimensional filtrations. We have undertaken a careful study of such interleavings and of the interleaving distances d_I , d_{SI} , and d_{WI} defined in terms of them. We have applied interleavings and interleaving distances to adapt the persistence-based topological inference result [11, Theorem 5.1] to the multidimensional setting and directly to the level of filtrations. The culmination of our efforts is a pair of results, Theorems 4.4.2 and 4.4.4, describing the topological asymptotics of random sublevelset-Čechand sublevelset-Rips bifiltrations (filtered by the superlevelsets of a density estimator) in the large sample limit. These theorems put on firm mathematical footing the idea, first put forth in [7], that such filtrations should encode topological information about the probability density function of a probability distribution generating the point cloud data on which the bifiltrations are built.

One of the central themes of this thesis has been that

1. To formulate results in the theory of persistence-based topological inference, one first needs to select notions of similarly between filtrations and persistence modules.
2. Much of the substantive theoretical work to be done in the study of topological persistence lies in understanding what the right choices of those notions of similarity are.

Indeed, we have spent the bulk of our effort in this thesis developing the theory of interleavings and interleaving distances. The theory we have developed makes a strong, if still incomplete, case that interleavings are natural tools with which to frame the theory of topological inference of multifiltrations.

Our discussions of directions for future work in Sections 2.12, 3.3.7, and 4.6 make it clear that there is much more work to be done on the theory of interleavings of multidimensional persistence modules, on the theory of weak interleavings of multi-dimensional filtrations, and on the statistical aspects of persistence-based topological inference. I believe that progress in these directions, seen as a part of the larger program of fleshing out the statistical foundations of topological data analysis, stands a good chance of contributing something fundamental to the way scientists, engineers, and statisticians think about and perform the qualitative analysis of data.

Appendix A

Minimal Presentations of Multidimensional Persistence Modules

A.1 The Coherence of B_n

A.1.1 Coherence: Basic Definitions and Results

$k[x_1, \dots, x_n]$ is well known to be a Noetherian ring. Finitely generated modules over Noetherian rings have some very nice algebraic properties. Here we define a standard weakening of the Noetherian property called coherence. Analogues of many of the same nice algebraic properties that hold for finitely generated modules over Noetherian rings hold for finitely presented modules over coherent rings. In particular, we have Corollary A.1.4, which we will use in Appendix A.2 to prove Theorem 2.1.4.

Definition. *For R a ring, we say an R -module M is **coherent** if M is finitely generated and every finitely generated submodule of M is finitely presented. We say a ring R is **coherent** if it is a coherent module over itself.*

Coherent commutative rings and coherent modules are well studied; the following results are standard. The reader may refer to [29] for the proofs.

Proposition A.1.1. *If R is a Noetherian ring then R is coherent.*

Theorem A.1.2. *If R is a coherent ring then every finitely presented R -module is coherent.*

Theorem A.1.3. *If $f : M \rightarrow N$ is a morphism between coherent R -modules M and N then $\ker(f)$, $\text{im}(f)$, and $\text{coker}(f)$ are coherent R -modules.*

Combining these last two theorems immediately gives

Corollary A.1.4. *If R is a coherent ring and $f : M \rightarrow N$ is a morphism between finitely presented R -modules M and N , then $\ker(f)$, $\text{im}(f)$, and $\text{coker}(f)$ are finitely presented.*

A.1.2 The Ring B_n is Coherent

Theorem A.1.5. *For any $n \in \mathbb{N}$, B_n is coherent.*

Proof. The key to the proof is the following theorem:

Theorem A.1.6 ([29, Theorem 2.3.3]). *Let $\{R_\alpha\}_{\alpha \in S}$ be a directed system of rings and let $R = \lim_{\rightarrow} R_\alpha$. Suppose that for $\alpha \leq \beta$, R_β is a flat R_α module and that R_α is coherent for every α . Then R is a coherent ring.*

First, recall that \mathbb{R} is a vector space over \mathbb{Q} . We'll say that $a_1, \dots, a_l \in \mathbb{R}_{\geq 0}$ are *rationally independent* if they are linearly independent as vectors in \mathbb{R} over the field \mathbb{Q} .

We next extend this definition to vectors in $\mathbb{R}_{\geq 0}^n$: We say a finite set $V \subset \mathbb{R}_{\geq 0}^n$ is *rationally independent* if

1. V is the union of sets V_1, \dots, V_n , where each element of V_i has a non-zero i^{th} coordinate and all other coordinates are equal to zero.
2. For any i , if a_1, \dots, a_l are the non-zero coordinates of the elements of V_i (listed with multiplicity), then a_1, \dots, a_l are rationally independent in the sense defined above.

We define an *n-grid* to be a monoid generated by some rationally independent set $V \subset \mathbb{R}_{\geq 0}^n$. Denote the *n-grid* generated by the rationally independent set V as $\Gamma(V)$. $\Gamma(V)$ is a submonoid of $\mathbb{R}_{\geq 0}^n$.

Lemma A.1.7. *If V is a rationally independent set, then the *n-grid* generated by V is isomorphic to $\mathbb{Z}_{\geq 0}^{|V|}$.*

Proof. The proof is straightforward; we omit it. \square

As noted in Section 2.1.2, for any $m \in \mathbb{N}$, $k[\mathbb{Z}_{\geq 0}^m] \cong k[x_1, \dots, x_m]$. As the latter ring is Noetherian, it is coherent by proposition A.1.1. Thus if G is an *n-grid*, $k[G]$ is coherent.

Lemma A.1.8. *For any finite set $A \subset \mathbb{R}_{\geq 0}$, there's a rationally independent set $B \subset \mathbb{R}_{\geq 0}$ such that A lies in the monoid generated by B .*

Proof. We proceed by induction on the number of elements l in the set A . The base case is trivial. Now assume the result holds for sets of order $l - 1$. Write $A = \{a_1, \dots, a_l\}$. By the induction hypothesis there exists a finite rationally independent set $A' = \{a'_1, \dots, a'_m\}$ such that $\{a_1, \dots, a_{l-1}\}$ lies in $\Gamma(A')$. If $A' \cup a_l$ is rationally independent, take $B = A' \cup a_l$. Otherwise $a_l = q_1 a'_1 + \dots + q_{m-1} a'_{m-1}$ for some $q_1, \dots, q_{l-1} \in \mathbb{Q}$; we may take $B = \{q'_1 a'_1, \dots, q'_{l-1} a'_{m-1}\}$, where $q'_i = 1/b_i$ for some $b_i \in \mathbb{N}$ such that $q_i = a/b_i$ for some $a \in \mathbb{Z}_{\geq 0}$. \square

Lemma A.1.9. *The set of *n*-grids forms a directed system under inclusion with direct limit $\mathbb{R}_{\geq 0}^n$.*

Proof. To show that the set of *n*-grids forms a directed system, we need that given two *n*-grids G_1 and G_2 , there's an *n-grid* G_3 such that $G_1 \subset G_3$ and $G_2 \subset G_3$. This follows readily from Lemma A.1.8; we leave the details to the reader. Any element of $\mathbb{R}_{\geq 0}^n$ lies in an *n-grid*, so $\mathbb{R}_{\geq 0}^n$ must be the colimit of the directed system. \square

For a monoid A and a submonoid $A' \subset A$, we have $k[A'] \subset k[A]$. This implies the following:

Lemma A.1.10. *The set of rings $\{k[G] \mid G \text{ is an } n\text{-grid}\}$ has the structure of a directed system induced by the directed system structure on the set of n -grids, and B_n is the direct limit of this directed system.*

Proposition A.1.11. *Given two positive n -grids G', G with $G' \subset G$, $k[G]$ is a free $k[G']$ module.*

Proof. We begin by establishing a couple of lemmas.

Lemma A.1.12. *For any rationally independent set V' and n -grid A containing $\Gamma(V')$, there is a rationally independent set V such that $A = \Gamma(V)$ and such that for each $a \in V'$, V contains an element of the form a/b for some $b \in \mathbb{N}$.*

We call V an *extension* of V' .

Proof. The proof of Lemma A.1.12 is similar to the proof of Lemma A.1.8; we omit it. \square

Let \mathcal{S} denote the set of maximal sets of the form $g + G' \equiv \{g + g' \mid g' \in G'\}$ for some $g \in G$.

Lemma A.1.13. *The sets \mathcal{S} form a partition of G .*

Proof. It's enough to show that if $g_1 + G', g_2 + G' \in \mathcal{S}$ and $g_1 + G' \cap g_2 + G' \neq \emptyset$, then $g_1 + G' = g_2 + G'$.

Let V' be a rationally independent set with $\Gamma(V') = G'$, and let V be an extension of V' with $\Gamma(V) = G$. Write $V' = \{v_1, \dots, v_l\}$ and $V = \{v_1/b_1, \dots, v_l/b_l, v_{l+1}, \dots, v_m\}$ for some $b_1, \dots, b_l \in \mathbb{N}$.

Assume there exist $g'_1, g'_2 \in G'$ such that $g_1 + g'_1 = g_2 + g'_2$. We'll show that there then exists an element $g_3 \in G$ such that $g_1, g_2 \in g_3 + G$. By the maximality of $g_1 + G'$ and $g_2 + G'$, this implies $g_1 + G' = g_2 + G'$, as needed. We write

$$\begin{aligned} g_1 &= y_1 v_1/b_1 + \cdots + y_l v_l/b_l + y_{l+1} v_{l+1} + \cdots + y_m v_m, \\ g_2 &= z_1 v_1/b_1 + \cdots + z_l v_l/b_l + z_{l+1} v_{l+1} + \cdots + z_m v_m, \\ g'_1 &= y'_1 v_1 + \cdots + y'_l v_l, \\ g'_2 &= z'_1 v_1 + \cdots + z'_l v_l. \end{aligned}$$

for some $y_1, \dots, y_m, z_1, \dots, z_m, y'_1, \dots, y'_l, z'_1, \dots, z'_l \in \mathbb{Z}$. By the rational independence of V and the fact that $g_1 + g'_1 = g_2 + g'_2$, we have that $y_i = z_i$ for $l+1 \leq i \leq m$.

Define

$$\begin{aligned} g_3 &= \min(y_1, z_1)v_1/b_1 + \dots + \min(y_l, z_l)v_l/b_1 + y_{l+1}v_{l+1} + \dots + y_mv_m, \\ g''_1 &= (y_1 - \min(y_1, z_1))v_1/b_1 + \dots + (y_l - \min(y_l, z_l))v_l/b_l, \\ g''_2 &= (z_1 - \min(y_1, z_1))v_1/b_1 + \dots + (z_l - \min(y_l, z_l))v_l/b_l. \end{aligned}$$

$g_3 + g''_1 = g_1$ and $g_3 + g''_2 = g_2$, so if we can show that $g''_1, g''_2 \in G'$ we are done.

By the rational independence of V and the fact that $g_1 + g'_1 = g_2 + g'_2$, for $1 \leq i \leq l$ we have that $\min(y_i, z_i)/b_i + \max(y'_i, z'_i) = y_i/b_i + y'_i$. This implies that $\max(y'_i, z'_i) - y'_i = (y_i - \min(y_i, z_i))/b_i$. In particular, the term on the right hand side lies in $\mathbb{Z}_{\geq 0}$. Thus $g''_1 \in G'$. The same argument shows $g''_2 \in G'$. \square

Now we are ready to complete the proof of Proposition A.1.11. It's easy to see that for any $s \in \mathcal{S}$, the natural action of G' on s extends to give $k[s]$ the structure of a free $k[G']$ module of rank 1. It follows from Lemma A.1.13 that the sets $\{k[s]\}_{s \in \mathcal{S}}$ have trivial intersection as $k[G']$ -submodules of $k[G]$. We then have that as a $k[G']$ module, $k[G] = \bigoplus_{s \in \mathcal{S}} k[s]$, and so in particular $k[G]$ is a free $k[G']$ -module, as we wanted to show. \square

Given Lemma A.1.10 and Proposition A.1.11, Theorem A.1.6 applies to give that B_n is coherent, since free modules are flat [26]. \square

A.2 Minimal Presentations of B_n -persistence Modules

This section is devoted to the proof of Theorem 2.1.4.

A.2.1 Free Hulls

We first observe that some standard results about resolutions and minimal resolutions of modules over local rings adapt to B_n -persistence modules. We'll only be interested in the specialization of such results to the 0th modules in a free resolution, and for the sake of simplicity we phrase the results only for this special case. However, the results discussed here do extend to statements about free resolutions of finitely presented B_n -persistence modules.

Let \mathfrak{m} denote the ideal of B_n generated by the set

$$\{v \in B_n \mid v \text{ is homogeneous and } gr(v) > 0\}.$$

Define a *free hull* of M to be a free cover (F_M, ρ_M) such that $\ker(\rho_M) \subset \mathfrak{m}F_M$.

Nakayama's lemma [26] is a key ingredient in the proofs of the results about free resolutions over local rings that we would like to adapt to our setting. To adapt these proofs, we need an n -graded version of Nakayama's lemma.

Lemma A.2.1 (Nakayama's Lemma for Persistence Modules). *Let M be a finitely generated B_n -persistence module. If $y_1, \dots, y_m \in M$ have images in $M/\mathfrak{m}M$ that generate the quotient, then y_1, \dots, y_m generate M .*

Proof. The usual Proof of Nakayama's lemma [26] carries over with only minor changes. \square

Lemma A.2.2. *A free cover (F_M, ρ_M) of a finitely generated B_n -persistence module M is a free hull iff a basis for F_M maps under ρ_M to a minimal set of generators for M .*

Proof. Given the adaptation Lemma A.2.1 of Nakayama's lemma to our setting, the proof of [26, Lemma 19.4] gives the result. \square

It follows easily from Lemma A.2.2 that a free hull exists for any finitely generated B_n -persistence module M . Corollary A.2.4 below gives a uniqueness result for free hulls.

Theorem A.2.3. *If (F_M, ρ_M) is a free hull of a finitely presented B_n -persistence module M and (F'_M, ρ'_M) is any free cover of M , then F_M includes as a direct summand of F'_M in such a way that $F'_M \cong F_M \oplus F''_M$ for some free module F''_M , and $\ker(\rho'_M) = \ker(\rho_M) \oplus F''_M \subset F_M \oplus F''_M$.*

Sketch of Proof. The statement of the theorem is the specialization to 0^{th} modules in the free resolutions of M of an adaptation of [26, Theorem 20.2] to our B_n -persistence setting. To modify Eisenbud's proof of [26, Theorem 20.2] to obtain a proof of Theorem A.2.3, one needs to invoke the coherence of B_n and use Corollary A.1.4 to show that $\ker(\rho_M)$ is finitely generated. Given this, the strategy of proof adapts in a straightforward way. \square

Corollary A.2.4 (Uniqueness of free hulls). *If M is a finitely presented B_n -persistence module, and (F_M, ρ_M) , (F'_M, ρ'_M) are two free hulls of M , then there is an isomorphism from F_M to F'_M which is a lift of the identity map of M .*

Proof. By Theorem A.2.3, we can identify F_M with a submodule of F'_M in such a way that $F'_M = F_M \oplus F''_M$ for some free module F''_M and $\ker(\rho'_M) = \ker(\rho_M) \oplus F''_M \subset F_M \oplus F''_M$. Since F'_M is a free hull, we must have $\ker(\rho'_M) \in \mathfrak{m}F'_M$, which implies $F''_M = 0$. The result follows. \square

Corollary A.2.5. *If M is a finitely presented B_n -persistence module and B, B' are two minimal sets of generators for M , then $gr(B) = gr(B')$.*

Proof. This follows from Corollary A.2.4 and Lemma A.2.2. \square

A.2.2 Proof of Theorem 2.1.4

Recall that a minimal presentation $\langle G|R \rangle$ of a B_n -persistence module M is one such that

1. the quotient $\langle G \rangle \rightarrow \langle G \rangle / \langle R \rangle$ maps G to a minimal set of generators for $\langle G \rangle / \langle R \rangle$.
2. R is a minimal set of generators for $\langle R \rangle$.

Let M be a finitely presented B_n -persistence module. Let $\langle G|R \rangle$ be a minimal presentation of M . We need to show that for any other presentation $\langle G'|R' \rangle$ of M , $gr(G) \leq gr(G')$ and $gr(R) \leq gr(R')$.

Let $\psi : \langle G \rangle / \langle R \rangle \rightarrow M$ and $\psi' : \langle G' \rangle / \langle R' \rangle \rightarrow M$ be isomorphisms, let $\pi : \langle G \rangle \rightarrow \langle G \rangle / \langle R \rangle$ and $\pi' : \langle G' \rangle \rightarrow \langle G' \rangle / \langle R' \rangle$ be the quotient homomorphisms, let $\rho = \psi \circ \pi$, and let $\rho' = \psi' \circ \pi'$. Then by Lemma A.2.2, $(\langle G \rangle, \rho)$ is a free hull of M , and $(\langle G' \rangle, \rho')$ is a free cover of M .

By Theorem A.2.3, $\langle G \rangle$ includes as a direct summand of $\langle G' \rangle$. The image of G under this inclusion can be extended to a basis for $\langle G' \rangle$. Recall that if B and B' are two bases for a free B_n -persistence module F , then $gr(B) = gr(B')$. We thus have that $gr(G) \leq gr(G')$.

Theorem A.2.3 also implies that $\langle R' \rangle \cong \langle R \rangle \oplus F$ for some free B_n -persistence module F . Let B be a basis for F . Then $R \cup B$ is a minimal set of generators for $\langle R \rangle \oplus F$. Let R'' denote the image of R' under an isomorphism from $\langle R' \rangle$ to $\langle R \rangle \oplus F$ and let $p : \langle R \rangle \oplus F \rightarrow \langle R \rangle$ denote projection onto the first summand. Since p is surjective, $p(R'')$ is a set of homogeneous generators for $\langle R \rangle$.

Since $\langle G \rangle$ and M are finitely presented, by Corollary A.1.4 $\ker(\rho) = \langle R \rangle$ is also finitely presented. Then by Corollary A.2.5, $gr(R) \leq gr(p(R''))$. Since $gr(p(R'')) \leq gr(R'') = gr(R')$ we have that $gr(R) \leq gr(R')$. \square

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