EEE444 Homework #1

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PART A

We first deconstruct the plant transfer function as $P = \frac{N_P}{D_P}$ where:

$$N_P = q_0 s + q1, \qquad D_P = r_0 s^6 + r_1 s^5 + r_2 s^4 + r_3 s^3 + r_4 s^2 + r_5 s + r_6$$

The four *Kharitonov polynomials* for N_P are:

$$N_1 = q_0^- s + q_1^-, \quad N_2 = q_0^+ s + q_1^+, \quad N_3 = q_0^+ s + q_1^-, \quad N_4 = q_0^- s + q_1^+$$

Similarly, the four *Kharitonov polynomials* for D_P are:

$$\begin{aligned} D_1 &= r_0^+ s^6 + r_1^- s^5 + r_2^- s^4 + r_3^+ s^3 + r_4^+ s^2 + r_5^- s + r_6^- \\ D_2 &= r_0^- s^6 + r_1^+ s^5 + r_2^+ s^4 + r_3^- s^3 + r_4^- s^2 + r_5^+ s + r_6^+ \\ D_3 &= r_0^+ s^6 + r_1^+ s^5 + r_2^- s^4 + r_3^- s^3 + r_4^+ s^2 + r_5^+ s + r_6^- \\ D_4 &= r_0^- s^6 + r_1^- s^5 + r_2^+ s^4 + r_3^+ s^3 + r_4^- s^2 + r_5^- s + r_6^+ \end{aligned}$$

We then deconstruct the controller transfer function as $C = \frac{N_C}{D_C}$ where $N_C = K - 4s$ and $D_C = s$.

Since the controller is of 1^{st} order, we can utilize the *16-plant theorem* to write the *16-plant polynomials* e_{1-16} as:

$$e_i = N_C N_{i_1} + D_C D_{i_2}$$

where $i_1 \in \{1, 2, 3, 4\}$ and $i_2 \in \{1, 2, 3, 4\}$.

For example, the first polynomial is:

$$\begin{gathered} e_1(s) = N_C N_1 + D_C D_1 \\ = (K - 4s)(q_0^- s + q_1^-) + s(r_0^+ s^6 + r_1^- s^5 + r_2^- s^4 + r_3^+ s^3 + r_4^+ s^2 + r_5^- s + r_6^-) \\ = s^7 + 9s^6 + 50s^5 + 150s^4 + 200s^3 + 116.2s^2 + (0.95K - \delta - 1.4)\, s + 0.35K \end{gathered}$$

The *16-plant theorem* states that for the system to be *robustly stable* all 16 polynomials should be stable. We can construct the *Routh arrays* for each polynomial to check for stability (all the elements of the first column should be greater than zero). For example, the first *Routh array* is:

$$\begin{bmatrix} 1 & 50 & 2 \cdot 10^2 & 0.95K - 1\delta - 1.4 \\ 9 & 1.5 \cdot 10^2 & 1.2 \cdot 10^2 & 0.35K \\ 33 & 1.9 \cdot 10^2 & 0.91K - 1\delta - 1.4 & 0 \\ 1 \cdot 10^2 & -0.25K + 0.27\delta + 1.2 \cdot 10^2 & 0.35K & 0 \\ 082K - 09\delta + 1.5 \cdot 10^2 & 0.79K - 1\delta - 1.4 & 0 & 0 \\ \frac{-79K + 1 \cdot 10^2 \delta + \left(-0.25K + 0.27\delta + 1.2 \cdot 10^2\right)\left(082K - 09\delta + 1.5 \cdot 10^2\right) + 1.4 \cdot 10^2}{082K - 09\delta + 1.5 \cdot 10^2} & 0.35K & 0 & 0 \\ \frac{018K^3 - 061K^2\delta + 93K^2 + 067K\delta^2 - 2.2 \cdot 10^2 K\delta - 6.3 \cdot 10^3 K - 024\delta^3 + 1.3 \cdot 10^2 \delta^2 + 1.8 \cdot 10^4 \delta + 2.4 \cdot 10^4}{02K^2 - 044K\delta + 1.1 \cdot 10^2 K + 024\delta^2 - 1.3 \cdot 10^2 \delta - 1.7 \cdot 10^4} & 0 & 0 & 0 \\ 0.35K & 0 & 0 & 0 \end{bmatrix}$$

We have to construct all 16 arrays and solve for all the first columns to be greater than zero. I was able to achieve this with the combination of `sympy` and `Mathematica`. However, since we cannot solve the inequalities with both of the two unknowns K and δ, an iterative approach was taken where we find the interval K should be in for given values of δ.

For example, starting with $\delta = 0$, I found that $6.53978 < K < 47.1073$. However $\delta = 5$ yielded no possible solutions. After a number of trials, I have reached the maximum value of $\delta = 3.156$, where the range is $30.5957 < K < 30.653$. So the values $\delta = 3.156$ and $K = 30.6$ satisfy the stability conditions.

PART B

Using `MATLAB`, I can plot the of the feedback system $G(s)$ using the controller given $C(s)$ and plant $P(s)$, plugging in the value $\delta_{max} = 3.156$ found in **PART A**.

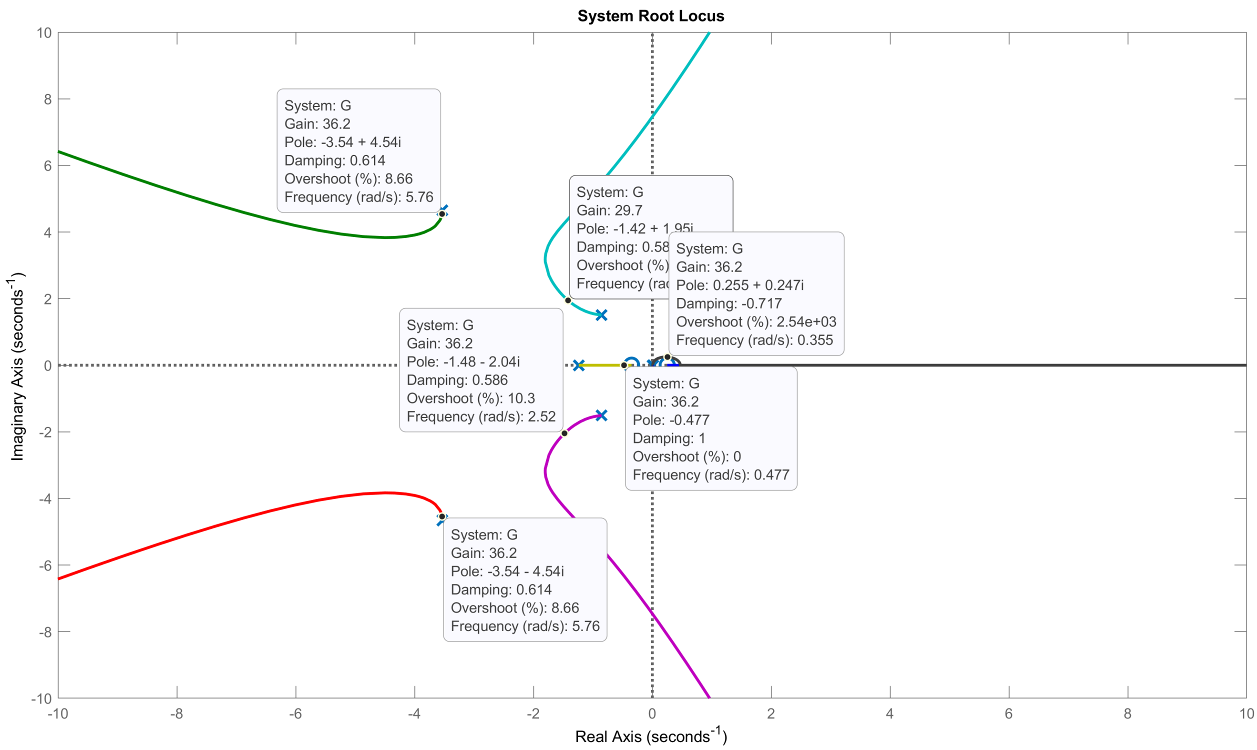

Figure 1: The root locus of the system.

PART C

Using the values $\delta = 3.156$ and $K = 30.6$, the system transfer function becomes:

$$G(s) = \frac{-4s^2 + 29.2s + 10.71}{s^7 + 10s^6 + 60s^5 + 140s^4 + 200s^3 + 125s^2 - 1.578s}$$

To find the stability margins, I have plotted the Bode diagram for the system using `MATLAB`.

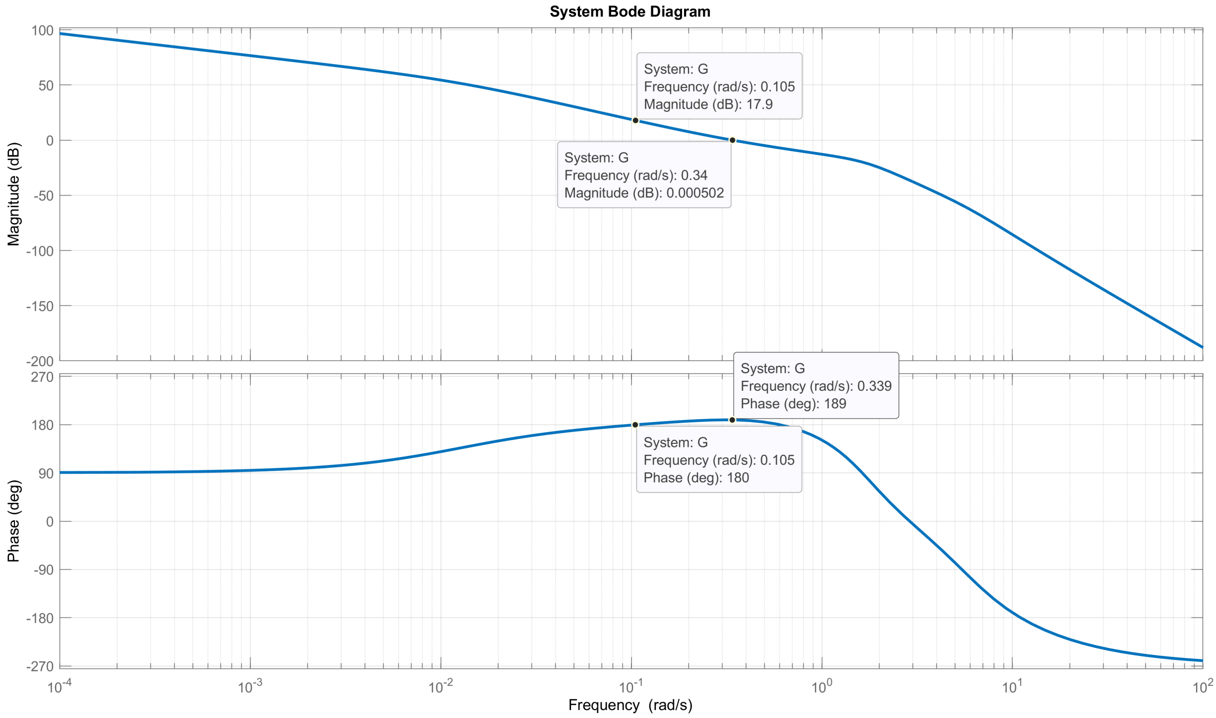

Figure 2: The Bode plot of the system.

The gain margin can be calculated by first finding the frequency ω_g where the phase is 180° and finding 0 plus the gain at that frequency. As can be seen in *Figure 2*, this happens at $\omega_c = 0.105$, where the gain is $17.9dB$: this means the gain margin is $GM = 17.9dB$.

For the phase margin, I first find the frequency ω_p where the gain is 0 dB, and find the phase at that frequency. *Figure 2* shows that this happens at $\omega_p = 0.34$, where the phase is $-189°$, meaning the phase margin is $PM = -189 + 180 = 9°$.

For the delay margin, I need to find what this phase shift corresponds to in the time domain. It is given by the expression $\frac{PM}{\omega_p} = \frac{\text{deg2rad}(9)}{0.34} = 0.462s$.

Given the system transfer function $G(s)$, the *sensitivity function* is defined as:

$$S(s) = \frac{1}{1 + G(s)}$$

Using the values $\delta = 3.156$ and $K = 30.6$, this corresponds to:

$$S(s) = \frac{s^7 + 10s^6 + 60s^5 + 140s^4 + 200s^3 + 125s^2 - 1.578s}{s^7 + 10s^6 + 60s^5 + 140s^4 + 200s^3 + 121s^2 + 27.62s + 10.71}$$

Substituting $s = j\omega$ and plotting the complex magnitude depending on the frequency ω using `MATLAB` produces the following figure:

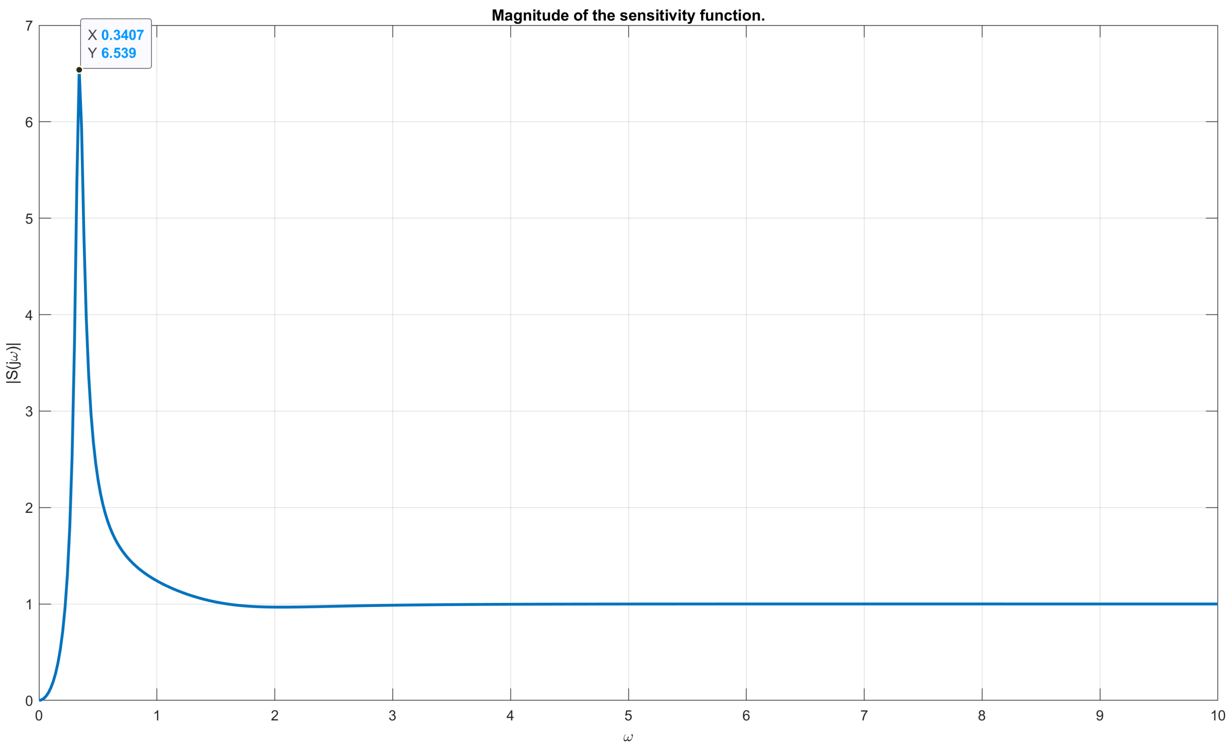

Figure 3: The magnitude of the sensitivity function.

As can be seen, the sensitivity function $S(s)$ reaches it's peak value of 6.539 at $\omega = 0.3407$.