

EEE444 Homework #2

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PART I

We can write δ as:

$$\delta = P_\delta - P = \frac{e^{-hs}}{\tau s - 1} - \frac{1}{0.2s - 1}$$

After this, we can plot $|\delta(j\omega)|$ for the possible values of $\tau \in [0.2, 0.25]$ and $h \in [0, 0.05]$.

$|\delta(j\omega)|$

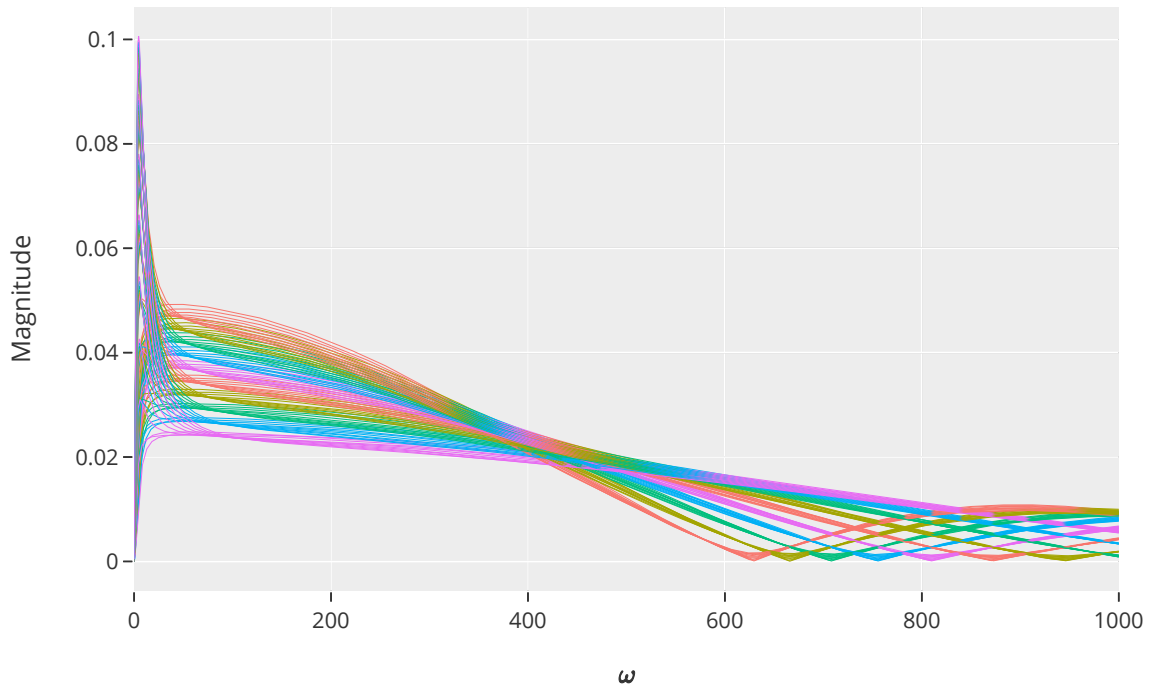


Figure 1: $|\delta|$ vs ω

By numerical trial/error I can also plot

$$|W_a(j\omega)| = \left| \frac{a_1 j\omega}{(a_2 j\omega + 1)^2} \right|$$

for different values of $a_1, a_2 > 0$ and check if $|W_a| > |\delta|$ for all ω and also $|W_a|$ is minimal. I have found this to happen at $a_1 = 0.045$ and $a_2 = 0.066$. Below is the result:

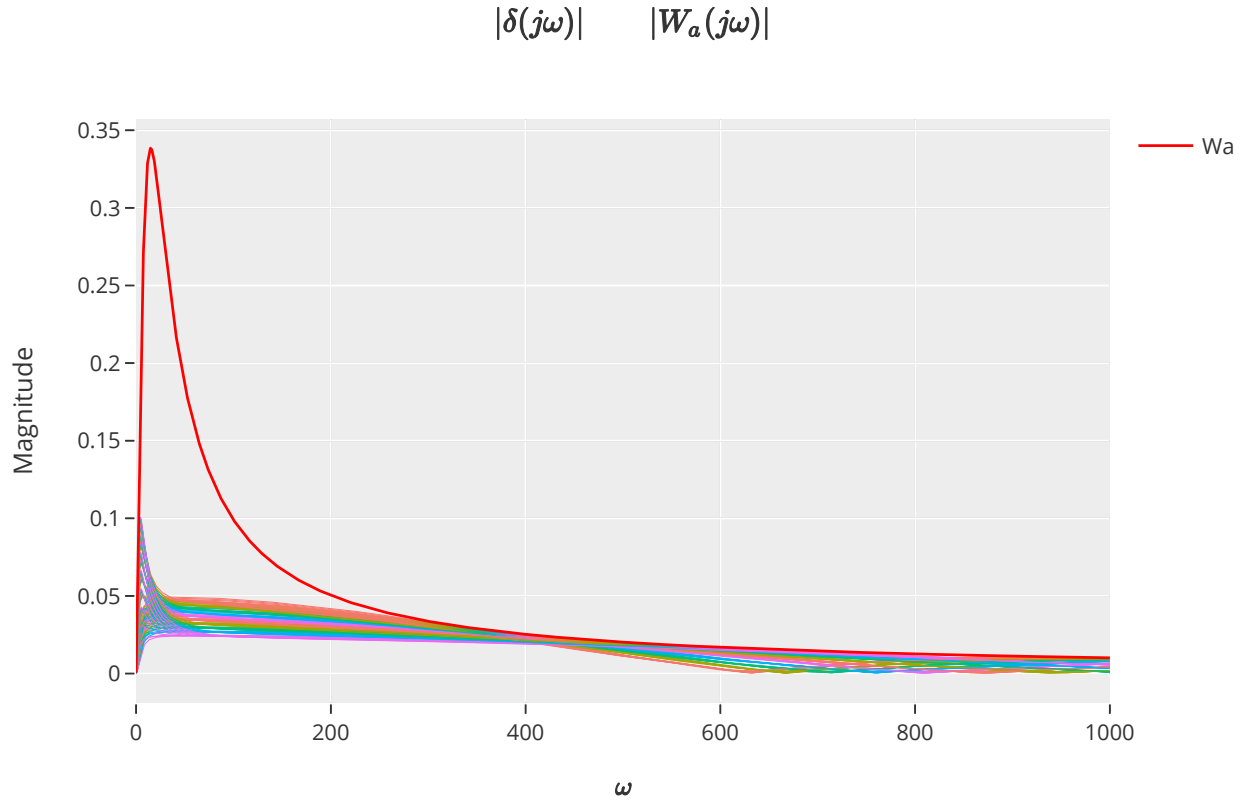


Figure 2: $|\delta|$ and $|W_a|$

And here is the plot when the x axis is logarithmic:

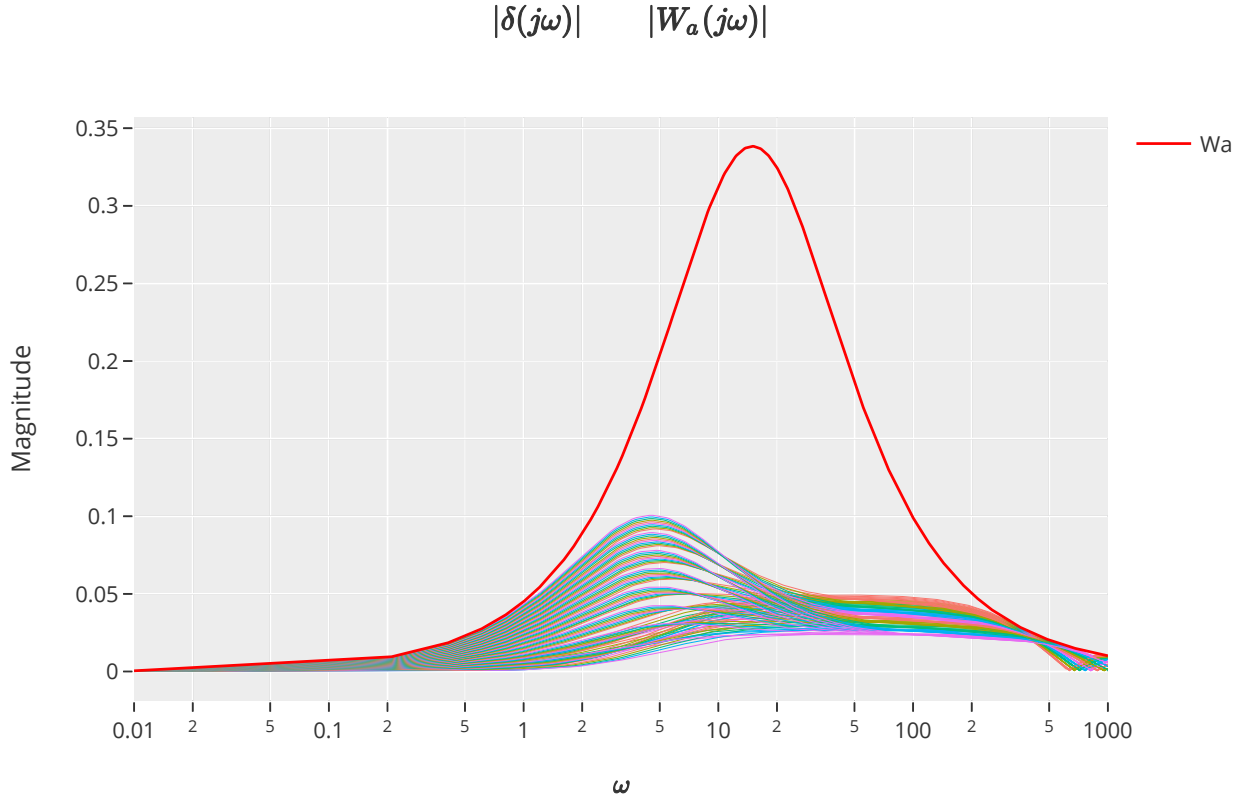


Figure 3: $|\delta|$ and $|W_a|$ with logarithmic x axis.

PART II

For the controller

$$C(s) = \frac{15s + 1}{\beta s}$$

to be robustly stabilizing (C, P_Δ) for all $P_\Delta \in P$: it has to satisfy the following conditions:

1. (C, P) must be stable
2. $|WCS| < 1$ for all ω .

For the first condition we have:

$$P(s) = \frac{1}{0.2s - 1}, \quad C(s) = \frac{15s + 1}{\beta s}$$

$$G(s) = \frac{PC}{1 + PC} = \frac{15s + 1}{Bs(0.2s - 1) + 15s + 1}$$

$$D(s) = 0.2Bs^2 + (15.0 - 1.0B)s + 1.0$$

Constructing the Routh-Hurwitz array for $D(s)$:

$$\begin{bmatrix} 0.2B & 1.0 \\ 15.0 - 1.0B & 0 \\ 1 & 0 \end{bmatrix}$$

For the first column to be greater than 0, we have the condition

$$0 < \beta < 15$$

For the second robustness condition, we can construct $|WCS|$ as:

$$W = \frac{a_1 j\omega}{(a_2 j\omega + 1)^2} = \frac{0.045j\omega}{(0.066j\omega + 1)^2}$$

$$P = \frac{1}{0.2j\omega - 1}, \quad C = \frac{15j\omega + 1}{\beta s}$$

$$S = (1 + PC)^{-1} = \frac{\beta w (0.2jw - 1)}{\beta w (0.2jw - 1) + 15w - i}$$

$$|WCS| = \frac{0.045w \sqrt{9.0w^4 + 225.04w^2 + 1} \sqrt{\frac{1}{0.04\beta^2 w^4 + \beta^2 w^2 - 30.4\beta w^2 + 225w^2 + 1}}}{\sqrt{1.8974736 \cdot 10^{-5} w^4 + 0.008712w^2 + 1}}$$

By numerically plugging in different values for β , we can see when the condition $|WCS| < 1$ is satisfied.

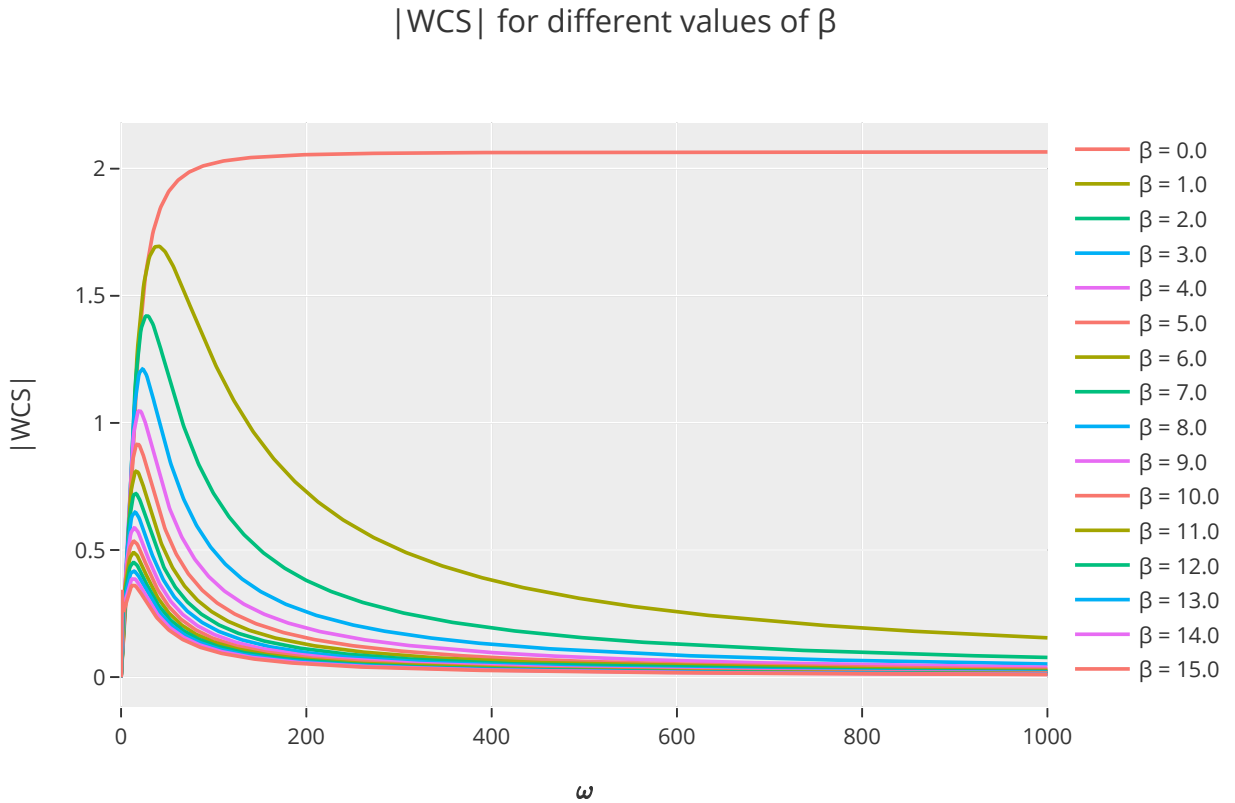


Figure 4: $|WCS|$ for different values of β

As can be seen, somewhere between $\beta = 4$ and 5 the magnitude seems to start getting lower than 1. More precisely, I have found this value to be $\beta = 4.261$.

Combining both conditions, for the controller to be robustly stabilizing (C, P_Δ) for all $P_\Delta \in P$, I have found that $4.261 < \beta < 15$.

PART III

Noting that P_1 is a time-delayed plant, we can deconstruct it as:

$$P_1 = \frac{e^{-hs}}{0.25s - 1} = P_0 P_d$$

where

$$P_0 = \frac{1}{0.25s - 1}, \quad P_d = e^{-hs}$$

The term P_d is non-polynomial, and thus has to be approximated by a polynomial using *Padé approximation*. Just to be on the safe side, let us use the 10th order approximation:

$$P_d = e^{-hs} \approx P_a$$

where

$$P_a = \frac{\frac{h^{10}s^{10}}{670442572800} - \frac{h^9s^9}{6094932480} + \frac{h^8s^8}{112869120} - \frac{h^7s^7}{3255840} + \frac{7h^6s^6}{930240} - \frac{7h^5s^5}{51680} + \frac{7h^4s^4}{3876} - \frac{h^3s^3}{57} + \frac{9h^2s^2}{76} - \frac{hs}{2} + 1}{\frac{h^{10}s^{10}}{670442572800} + \frac{h^9s^9}{6094932480} + \frac{h^8s^8}{112869120} + \frac{h^7s^7}{3255840} + \frac{7h^6s^6}{930240} + \frac{7h^5s^5}{51680} + \frac{7h^4s^4}{3876} + \frac{h^3s^3}{57} + \frac{9h^2s^2}{76} + \frac{hs}{2} + 1}$$

Choosing $\beta = 4.5$, we can construct the system with :

$$C(s) = \frac{15s + 1}{4.5s}, \quad P(s) = P_0 P_a, \quad G = \frac{PC}{1 + PC}$$

The denominator polynomial $D(s)$ of the system is:

$$\begin{aligned} D(s) = & 3.64148451940312 \cdot 10^{-26} s^{12} + 8.04525313153463 \cdot 10^{-23} s^{11} + \\ & 8.51330851197038 \cdot 10^{-20} s^{10} + 6.07958237746093 \cdot 10^{-17} s^9 + \\ & 2.83545747745511 \cdot 10^{-14} s^8 + 1.08562400350004 \cdot 10^{-11} s^7 + \\ & 2.63845814580706 \cdot 10^{-9} s^6 + 5.74573497162023 \cdot 10^{-7} s^5 + \\ & 6.45127422313955 \cdot 10^{-5} s^4 + 0.00694030214424951 s^3 + 0.141732456140351 \\ & s^2 + 2.32777777777778 s + 0.22222222222222 \end{aligned}$$

Constructing the Routh-Hurwitz array for $D(s)$:

$$\begin{bmatrix} 3.64148451940312 \cdot 10^{-26} & 8.51330851197038 \cdot 10^{-20} & 2.83545747745511 & \dots \\ 8.04525313153463 \cdot 10^{-23} & 6.07958237746093 \cdot 10^{-17} & 1.08562400350004 \cdot 10^{-11} & \dots \\ 5.76153616248595 \cdot 10^{-20} & 2.3440766671805 \cdot 10^{-14} & 2.37839168662087 & \dots \\ 2.80637717396288 \cdot 10^{-17} & 7.53511826726918 \cdot 10^{-12} & 4.8887615452316 & \dots \\ 4.23708643387235 \cdot 10^{-10} & 2.8995689473794 \cdot 10^{-5} & 0.129019148023105 & \dots \\ 1.37114900321653 \cdot 10^{-7} & 0.00544473196071004 & 2.32527158375522 & \dots \\ 1.21705305212096 \cdot 10^{-5} & 0.121833658583602 & 0.22222222222222 & \dots \\ 0.00407213691044493 & 2.32276799709594 & 0 & \dots \\ 0.114891524901938 & 0.22222222222222 & 0 & \dots \\ 2.31489170409517 & 0 & 0 & \dots \\ 0.22222222222222 & 0 & 0 & \dots \end{bmatrix}$$

As can be seen, all of the elements of the first column are > 0 , albeit barely, as the first entry is in the order of 10^{-26} . By the Routh-Hurwitz theorem, this proves (C, P) is stable where:

$$C(s) = \frac{15s + 1}{4.5s}, \quad P(s) = \frac{e^{-hs}}{0.25s - 1}$$

PART IV

By plugging in the values the robust performance condition at $\gamma_r = 10$:

$$\left| \frac{W_r}{\gamma_r} S \right| + |WCS| \leq 1 \quad \forall \omega$$

the condition becomes:

$$\frac{(\beta \sqrt{7.59 \cdot 10^{-7} w^6 + 0.0003 w^4 + 0.05 w^2 + 1} + 0.45 w \sqrt{9.0 w^4 + 225 w^2 + 1}) \sqrt{\frac{1}{0.04 \beta^2 w^4 + \beta^2 w^2 - 30.4 \beta w^2 + 225 w^2 + 1}}}{10 \sqrt{1.9 \cdot 10^{-5} w^4 + 0.008 w^2 + 1}} \leq 1$$

As done in the previous parts, we can plot this magnitude using different values of β :

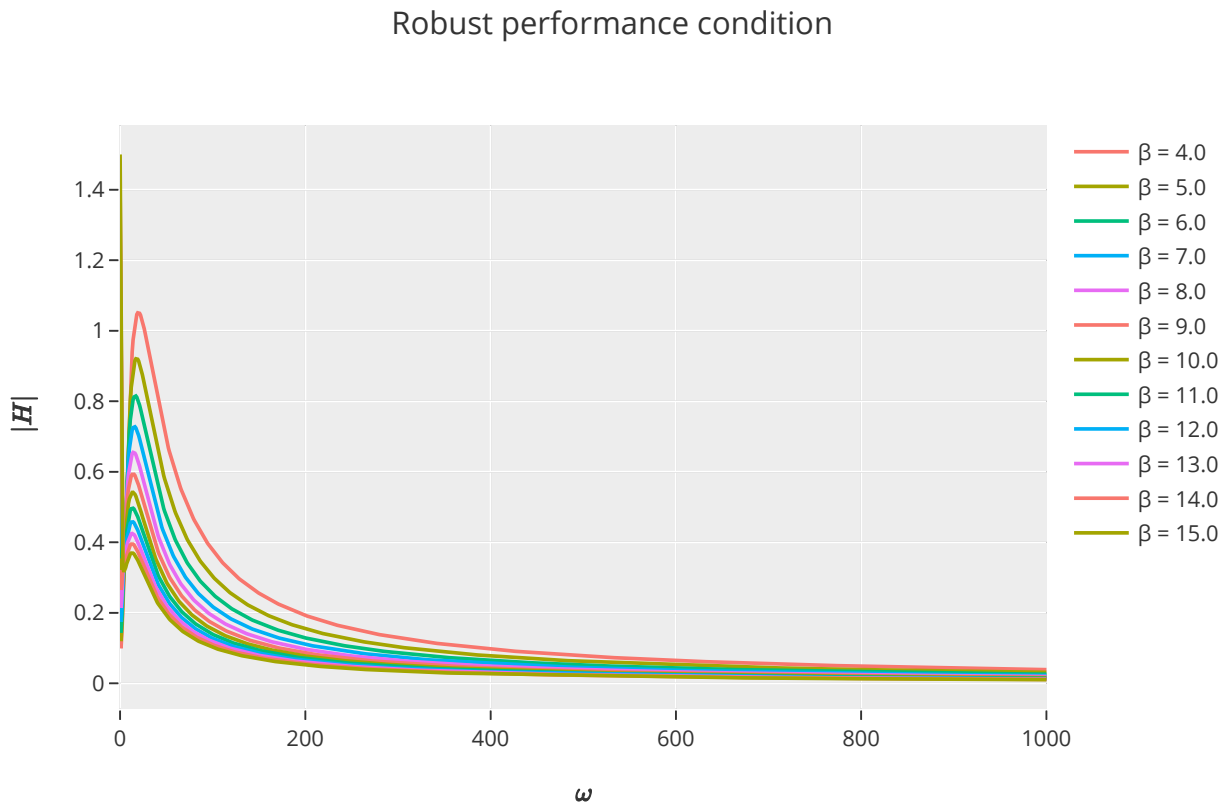


Figure 5: Robust performance condition for different values of β .

Zooming in further, we can see that at the higher values of β , the magnitude stays high at very low frequencies so they do not satisfy the performance condition.

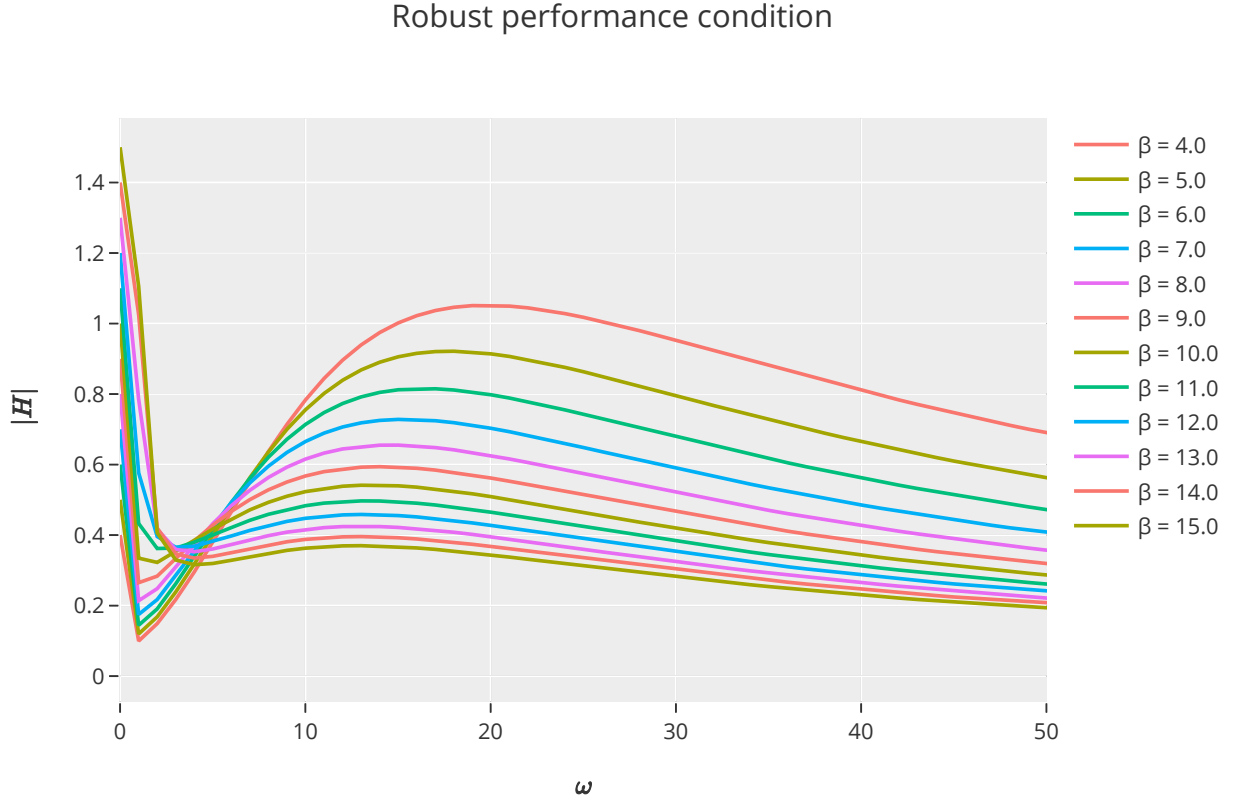


Figure 6: Robust performance condition, zoomed in.

More specifically the range of acceptable values for β while $\gamma_r = 10$ is $4.44 < \beta < 9.88$.

I have also found that for $\gamma_r = 4.445$ this range reduces to $4.42 < \beta < 4.45$. So the smallest value of γ_r the corresponding value of β for which there exists a β satisfying the robust performance condition are:

$$\gamma_r = 4.445, \quad \beta = 4.43$$

Notes

All related files can be found at:

<https://github.com/mlg556/EEE444>