EEE444 Homework #2

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PART I

We can write δ as:

$$\delta = P_\delta - P = \frac{e^{-hs}}{\tau s - 1} - \frac{1}{0.2s - 1}$$

After this, we can plot $|\delta(j\omega)|$ for the possible values of $\tau \in [0.2, 0.25]$ and $h \in [0, 0.05]$.

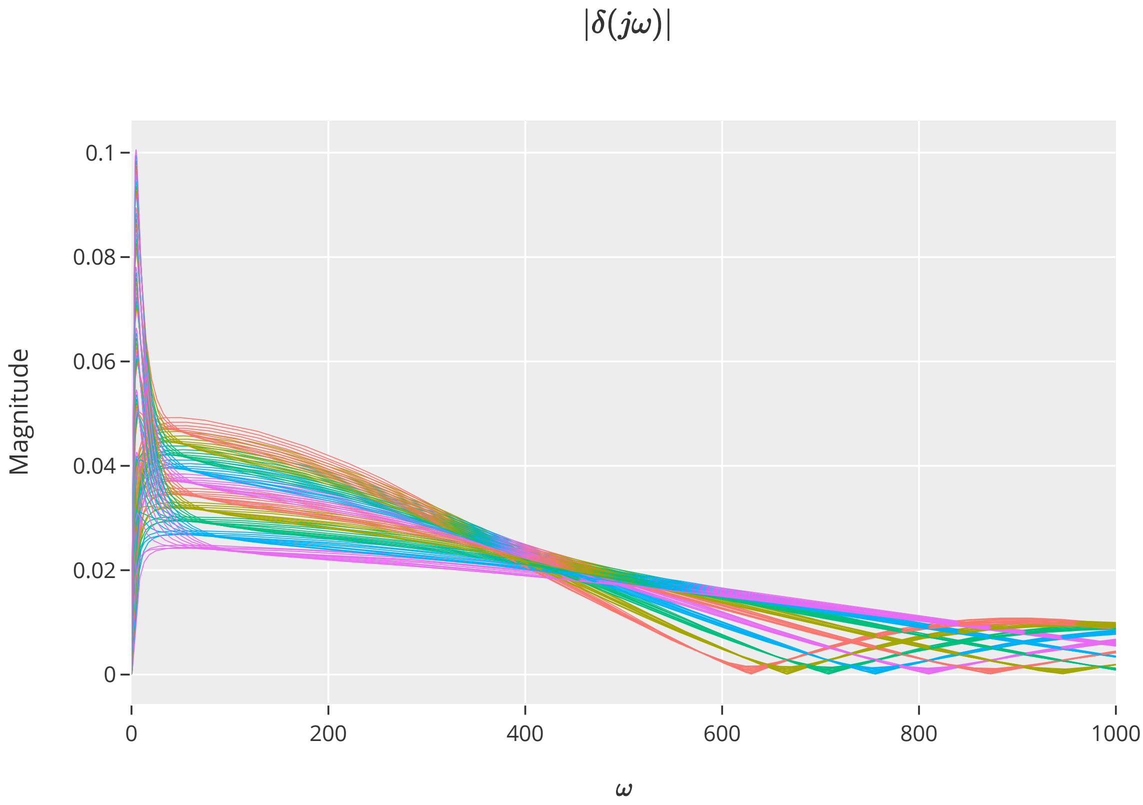

Figure 1: $|\delta|$ vs ω

By numerical trial/error I can also plot

$$|W_a(j\omega)| = \left| \frac{a_1 j\omega}{(a_2 j\omega + 1)^2} \right|$$

for different values of $a_1, a_2 > 0$ and check if $|W_a| > |\delta|$ for all ω and also $|W_a|$ is minimal. I have found this to happen at $a_1 = 0.045$ and $a2 = 0.066$. Below is the result:

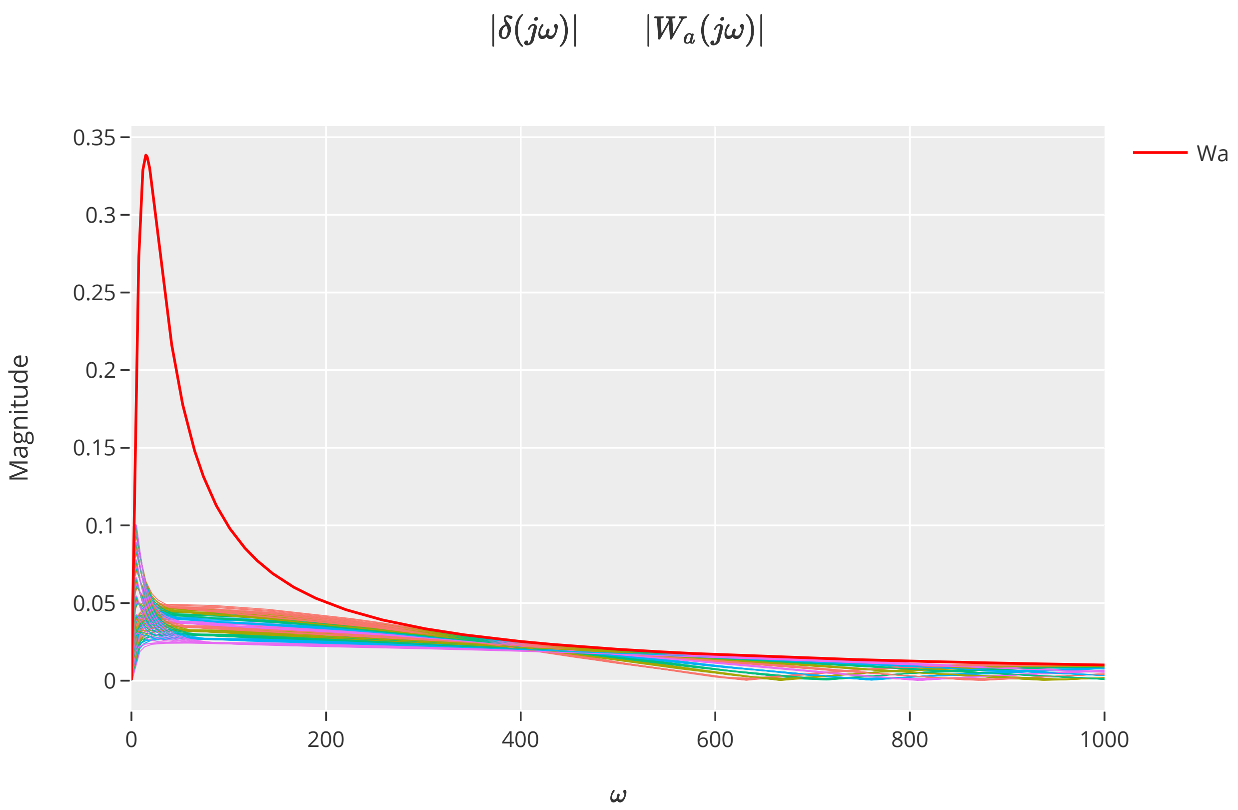

Figure 2: $|\delta|$ and $|W_a|$

And here is the plot when the x axis is logarithmic:

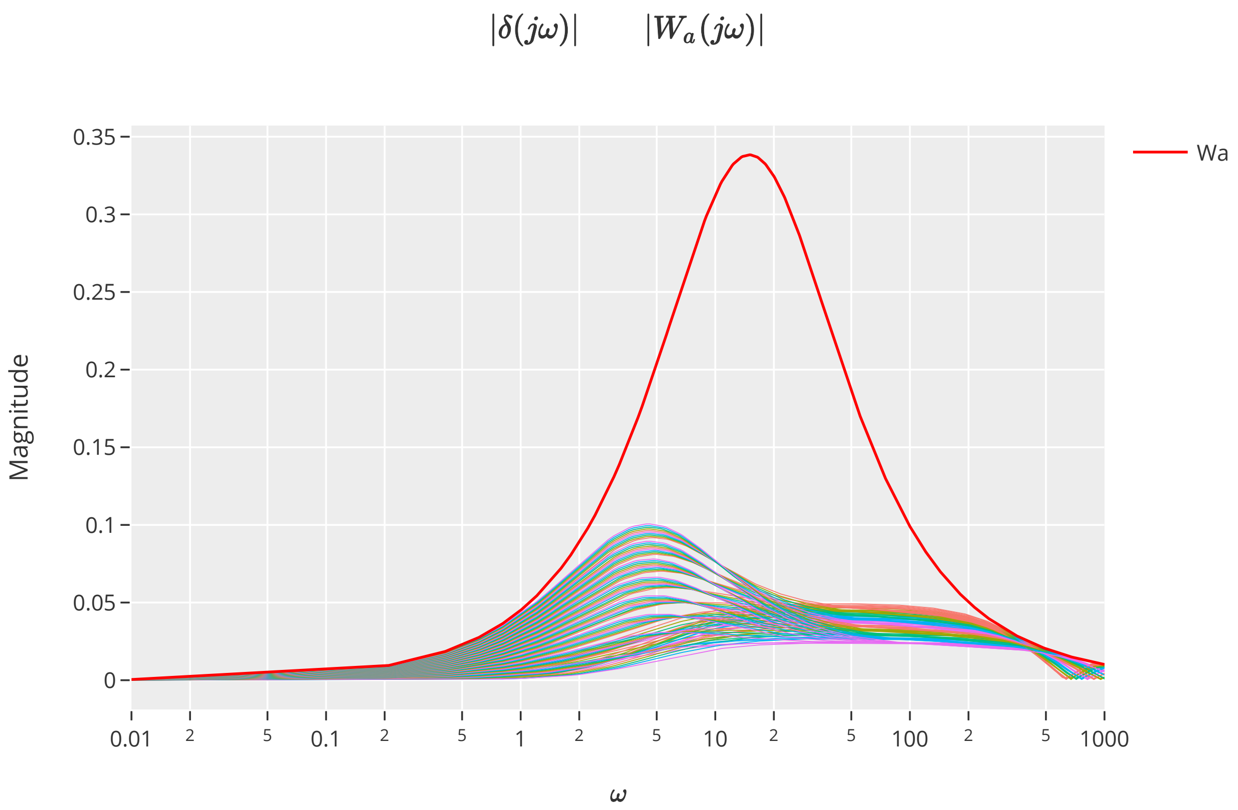

Figure 3: $|\delta|$ and $|W_a|$ with logarithmic x axis.

PART II

For the controller

$$C(s) = \frac{15s+1}{\beta s}$$

to be robustly stabilizing (C, P_Δ) for all $P_\Delta \in P$: it has to satisfy the following conditions:

1. (C, P) must be stable
2. $|WCS| < 1$ for all ω.

For the first condition we have:

$$P(s) = \frac{1}{0.2s-1}, \quad C(s) = \frac{15s+1}{\beta s}$$

$$G(s) = \frac{PC}{1+PC} = \frac{15s+1}{Bs\,(0.2s-1)+15s+1}$$

$$D(s) = 0.2Bs^2 + (15.0 - 1.0B)\,s + 1.0$$

Constructing the Routh-Hurwitz array for $D(s)$:

$$\begin{bmatrix} 0.2B & 1.0 \\ 15.0-1.0B & 0 \\ 1 & 0 \end{bmatrix}$$

For the first column to be greater than 0, we have the condition

$$0 < \beta < 15$$

For the second robustness condition, we can construct $|WCS|$ as:

$$W = \frac{a_1 j\omega}{(a_2 j\omega + 1)^2} = \frac{0.045jw}{(0.066jw+1)^2}$$
$$P = \frac{1}{0.2j\omega - 1}, \quad C = \frac{15j\omega + 1}{\beta s}$$
$$S = (1+PC)^{-1} = \frac{\beta w\,(0.2jw-1)}{\beta w\,(0.2jw-1) + 15w - i}$$
$$|WCS| = \frac{0.045w\sqrt{9.0w^4 + 225.04w^2 + 1}\sqrt{\frac{1}{0.04\beta^2w^4 + \beta^2w^2 - 30.4\beta w^2 + 225w^2 + 1}}}{\sqrt{1.8974736 \cdot 10^{-5}w^4 + 0.008712w^2 + 1}}$$

By numerically plugging in different values for β, we can see when the condition $|WCS| < 1$ is satisfied.

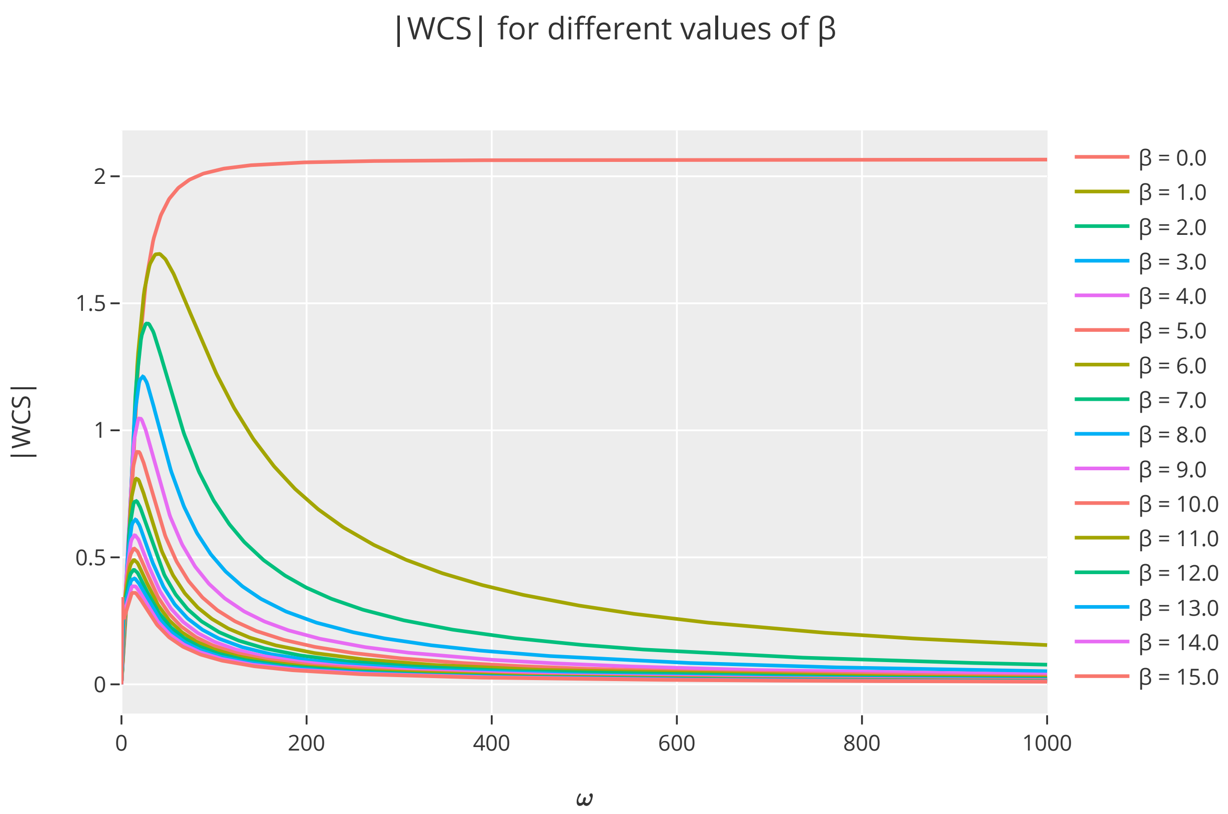

Figure 4: |WCS| for different values of β

As can be seen, somewhere between $\beta = 4$ and 5 the magnitude seems to start getting lower than 1. More precisely, I have found this value to be $\beta = 4.261$.

Combining both conditions, for the controller to be robustly stabilizing (C, P_Δ) for all $P_\Delta \in P$, I have found that $4.261 < \beta < 15$.

PART III

Padé Approximation Approach

Noting that P_1 is a time-delayed plant, we can deconstruct it as:

$$P_1 = \frac{e^{-hs}}{0.25s - 1} = P_0 P_d$$

where

$$P_0 = \frac{1}{0.25s - 1}, \qquad P_d = e^{-hs}$$

The term P_d is non-polynomial, and thus has to be approximated by a polynomial using *Padé approximation.* Just to be on the safe side, let us use the 10th order approximation:

$$P_d = e^{-hs} \approx P_a$$

where

$$P_a = \frac{\frac{h^{10}s^{10}}{670442572800} - \frac{h^9s^9}{6094932480} + \frac{h^8s^8}{112869120} - \frac{h^7s^7}{3255840} + \frac{7h^6s^6}{930240} - \frac{7h^5s^5}{51680} + \frac{7h^4s^4}{3876} - \frac{h^3s^3}{57} + \frac{9h^2s^2}{76} - \frac{hs}{2} + 1}{\frac{h^{10}s^{10}}{670442572800} + \frac{h^9s^9}{6094932480} + \frac{h^8s^8}{112869120} + \frac{h^7s^7}{3255840} + \frac{7h^6s^6}{930240} + \frac{7h^5s^5}{51680} + \frac{7h^4s^4}{3876} + \frac{h^3s^3}{57} + \frac{9h^2s^2}{76} + \frac{hs}{2} + 1}$$

Choosing $\beta = 4.5$, we can construct the system with :

$$C(s) = \frac{15s + 1}{4.5s}, \qquad P(s) = P_0 P_a, \qquad G = \frac{PC}{1 + PC}$$

The denominator polynomial $D(s)$ of the system is:

$$\begin{gathered} D(s) = 3.64148451940312 \cdot 10^{-26} s^{12} + 8.04525313153463 \cdot 10^{-23} s^{11} + \\ 8.51330851197038 \cdot 10^{-20} s^{10} + 6.07958237746093 \cdot 10^{-17} s^9 + \\ 2.83545747745511 \cdot 10^{-14} s^8 + 1.08562400350004 \cdot 10^{-11} s^7 + \\ 2.63845814580706 \cdot 10^{-9} s^6 + 5.74573497162023 \cdot 10^{-7} s^5 + \\ 6.45127422313955 \cdot 10^{-5} s^4 + 0.00694030214424951 s^3 + 0.141732456140351 \\ s^2 + 2.32777777777778 s + 0.222222222222222 \end{gathered}$$

Constructing the Routh-Hurwitz array for $D(s)$:

$$\begin{bmatrix} 3.64148451940312 \cdot 10^{-26} & 8.51330851197038 \cdot 10^{-20} & 2.83545747745511 & \dots \\ 8.04525313153463 \cdot 10^{-23} & 6.07958237746093 \cdot 10^{-17} & 1.08562400350004 \cdot 10^{-11} & \dots \\ 5.76153616248595 \cdot 10^{-20} & 2.3440766671805 \cdot 10^{-14} & 2.37839168662087 & \dots \\ 2.80637717396288 \cdot 10^{-17} & 7.53511826726918 \cdot 10^{-12} & 4.8887615452316 & \dots \\ 4.23708643387235 \cdot 10^{-10} & 2.8995689473794 \cdot 10^{-5} & 0.129019148023105 & \dots \\ 1.37114900321653 \cdot 10^{-7} & 0.00544473196071004 & 2.32527158375522 & \dots \\ 1.21705305212096 \cdot 10^{-5} & 0.121833658583602 & 0.222222222222222 & \dots \\ 0.00407213691044493 & 2.32276799709594 & 0 & \dots \\ 0.114891524901938 & 0.222222222222222 & 0 & \dots \\ 2.31489170409517 & 0 & 0 & \dots \\ 0.222222222222222 & 0 & 0 & \dots \end{bmatrix}$$

As can be seen, all of the elements of the first column are > 0, albeit barely, as the first entry is in the order of 10^{-26}. By the Routh-Hurwitz theorem, this proves (C, P) is stable where:

$$C(s) = \frac{15s + 1}{4.5s}, \qquad P(s) = \frac{e^{-hs}}{0.25s - 1}$$

Bode Plot Approach

As an alternative approach, I directly constructed and analyzed the Bode plot of the system to judge stability of the system where:

$$C(s) = \frac{15s+1}{4.5s}, \qquad P(s) = \frac{e^{-hs}}{0.25s-1}$$

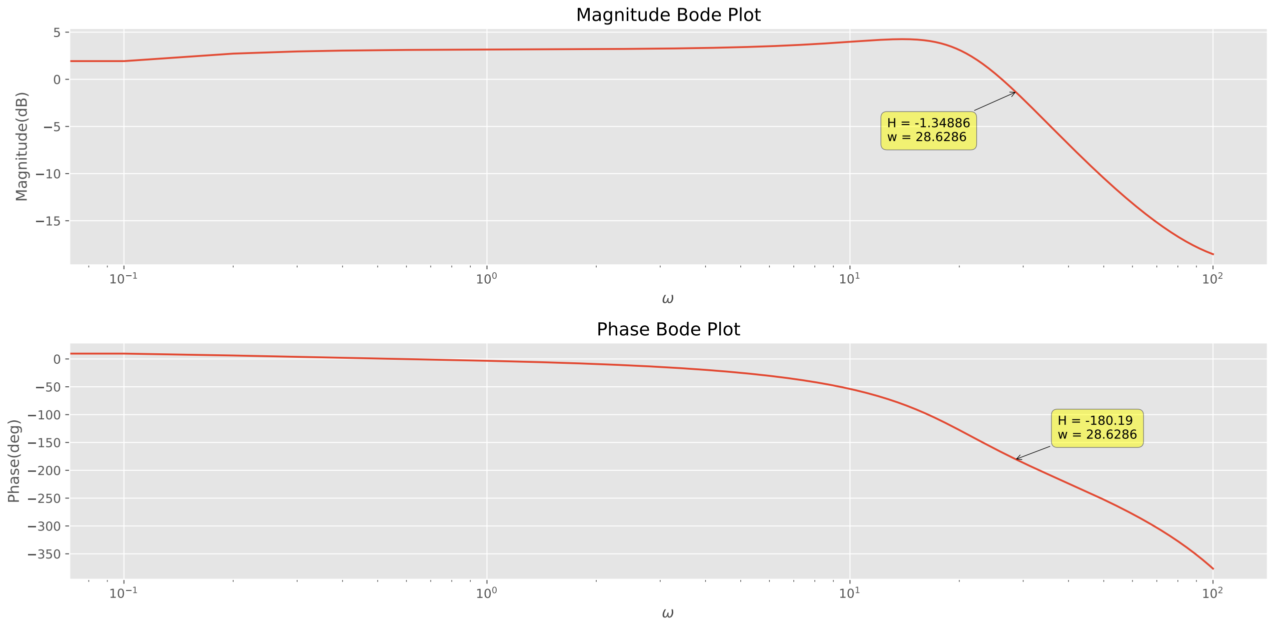

Figure 5: Bode plot of the system.

Defining the phase crossover frequency ω_{pc} as the frequency where phase shift is equal to -180, The *Bode Stability Criterion* states that if at $|H(j\omega_{pc})| < 0dB$, then the system is stable.

Figure 5 shows that this is indeed the case where $\omega_{pc} = 28.6286$ and $|H(j\omega_{pc})| = -1.34dB$.

Both approaches agree that the system is robustly stable.

PART IV

By plugging in the values the robust performance condition at $\gamma_r = 10$:

$$\left|\frac{W_r}{\gamma_r} S\right| + |WCS| \leq 1 \quad \forall \omega$$

the condition becomes:

$$\frac{\left(\beta\sqrt{7.59 \cdot 10^{-7} w^6 + 0.0003 w^4 + 0.05 w^2 + 1} + 0.45 w \sqrt{9.0 w^4 + 225 w^2 + 1}\right) \sqrt{\frac{1}{0.04\beta^2 w^4 + \beta^2 w^2 - 30.4\beta w^2 + 225 w^2 + 1}}}{10\sqrt{1.9 \cdot 10^{-5} w^4 + 0.008 w^2 + 1}} \leq 1$$

As done in the previous parts, we can plot this magnitude using different values of β:

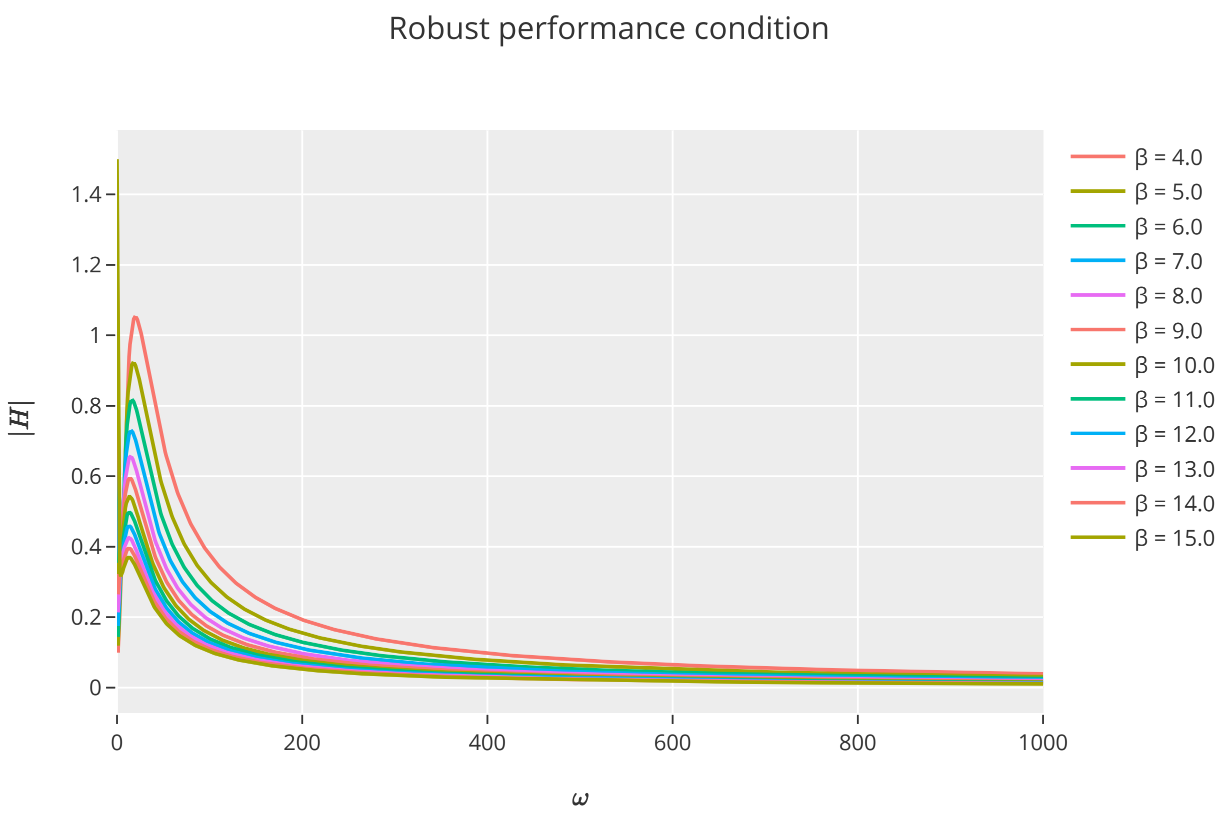

Figure 6: Robust performance condition for different values of β.

Zooming in further, we can see that at the higher values of β, the magnitude stays high at very low frequencies so they do not satisfy the performance condition.

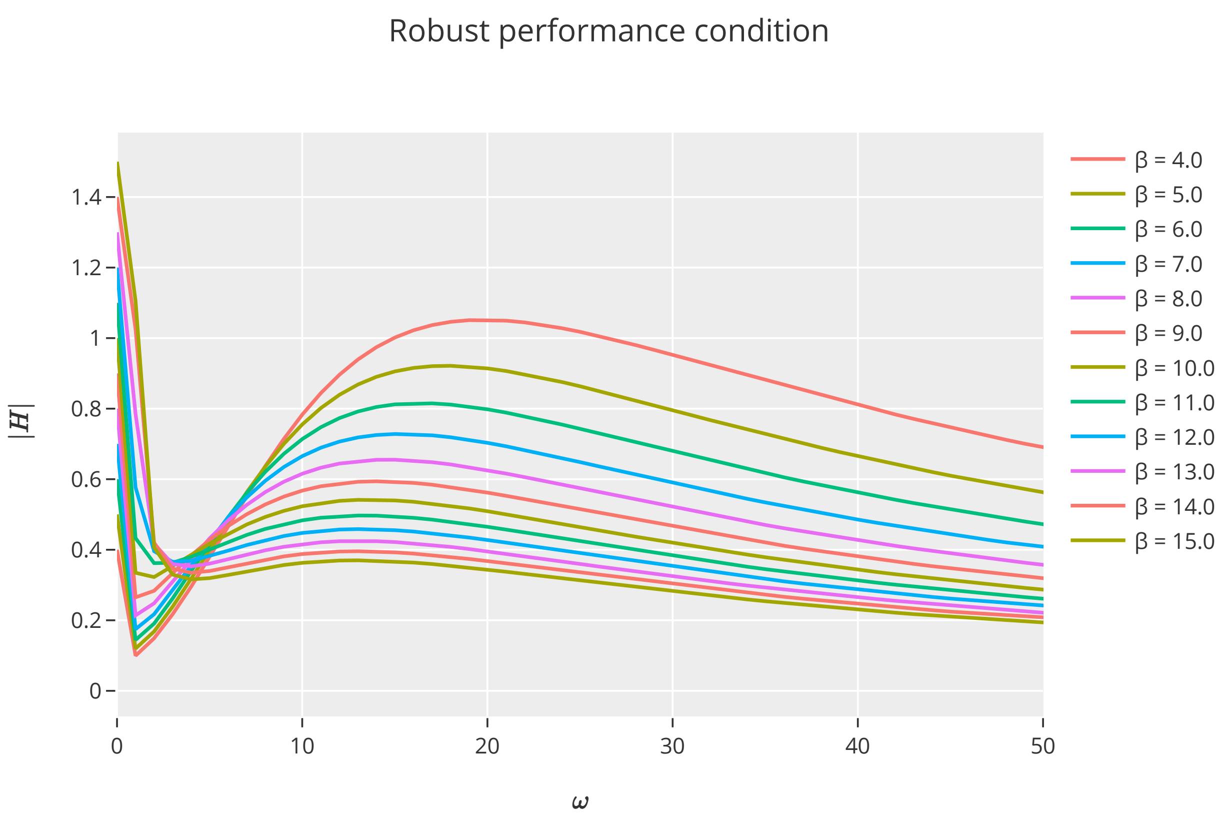

Figure 7: Robust performance condition, zoomed in.

More specifically the range of acceptable values for β while $\gamma_r = 10$ is $4.44 < \beta < 9.88$.

I have also found that for $\gamma_r = 4.445$ this range reduces to $4.42 < \beta < 4.45$. So the smallest value of γ_r the corresponding value of β for which there exists a β satisfying the robust performance condition are:

$$\gamma_r = 4.445, \quad \beta = 4.43$$

Notes

All related files can be found at:

https://github.com/mlg556/EEE444