Last lecture:

$$\boxed{\gamma_{opt} = \inf_{Q_c \in \mathcal{H}_\infty} \| T_1 - T_2 Q_c \|_\infty} \quad ✪$$

✪ : One block $\mathcal{H}_\infty$ control problem or Model Matching problem

Given $T_1, T_2 \in \mathcal{H}_\infty$ find γ_{opt} and the corresponding $Q_{c,opt} \in \mathcal{H}_\infty$.
In this problem Q_c is the free parameter appearing in the parameterization of all stabilizing controllers;
T_1 and T_2 come from the specific problem
i.e. Sensitivity minimization (nominal performance) or
Robust stability (robusness optimization).

Inner-Outer factorization of T_2: $\quad T_2(s) = T_{2i}(s)\, T_{2o}(s)$

Assumption: $T_{2o}^{-1} \in \mathcal{H}_\infty$ i.e. T_{2o} is bi-proper

Then
$$\gamma_{opt} = \inf_{Q \in \mathcal{H}_\infty} \| T_1 - T_{2i} Q \|_\infty$$

$$Q = T_{2o} Q_c \quad \longleftrightarrow \quad Q_c = T_{2o}^{-1} Q$$

i.e. there is an invertible relation between Q_c and Q in $\mathcal{H}_\infty$

Definition: $\quad T_1 = W \qquad T_{2i} = M$

The model matching problem is $\quad \gamma_{opt} = \inf_{Q \in \mathcal{H}_\infty} \| W - MQ \|_\infty$

given $W(s)$, $M(s)$ find γ_{opt} and corresponding $Q_{opt} \in \mathcal{H}_\infty$

$W(s)$ is stable, can be rational or irrational transfer function
$M(s)$ is inner stable can be rational (finite dimensional)
or irrational (infinite dimensional)

We will see alternative solutions of this problem
for diferent cases.

Model Matching Problem:

$$\gamma_{opt} = \inf_{Q \in \mathcal{H}_\infty} \|W - MQ\|_\infty$$

Example: $M(s) = \left(\frac{s-a}{s+a}\right)$ $\quad a > 0$

Claim: $\gamma_{opt} = |W(a)|$

Proof:

$$\|W - MQ\|_\infty = \sup_{Re(s)\geq 0} \left|W(s) - \left(\frac{s-a}{s+a}\right)Q(s)\right| \geq |W(a)| \quad \forall Q \in \mathcal{H}_\infty$$

So, $\gamma_{opt} \geq |W(a)|$

Conversely, let $Q_{opt}(s) = \dfrac{W(s) - W(a)}{\left(\frac{s-a}{s+a}\right)} \in \mathcal{H}_\infty$

Then $W - MQ_{opt} = W(a)$ $\qquad \|W - MQ_{opt}\|_\infty = |W(a)|$

Hence $\gamma_{opt} \leq |W(a)|$ .... conclusion $\gamma_{opt} = |W(a)|$ is achieved by Q_{opt} given above. $\square$

Application to Robustness Optimization (Possible Exam Question):

Given $P(s) = \dfrac{1}{s-a}$ $a>0$, $W_2(s) = \delta(s+0.01)$ $\delta > 0$ find the largest $\delta > 0$ such that there exists a robustly stabilizing controller for all plants in the form ✓ $P_\Delta(s) = P(s)\left(1 + \Delta_m(s)\right)$ where $|\Delta_m(j\omega)| < |W_2(j\omega)|$ $\forall \omega$ and $P_\Delta(s)$ has one pole in $\mathbb{C}_+$.

Solution: In previous lecture we found a parameterization of all stabilizing controllers for ✓ P: $\quad C = \dfrac{X + DQ_c}{Y - NQ_c}$, $Q_c \in \mathcal{H}_\infty$

with $X = (a+b)$, $b > 0$ $\quad Y = 1$ $\quad N = \dfrac{1}{s+b}$ $\quad D(s) = \dfrac{s-a}{s+b}$ $\Big\}$ let $b = a$ for simplification of inner-outer factorizations below

Robust Stability condition:

$$\|W_2 T\|_\infty \leq 1 \iff \|W_2 N(X + DQ_c)\|_\infty \leq 1 \quad \text{RS1}$$

RS1 holds $\iff$ $\left\| \frac{2a(s+0.01)}{(s+a)} + \left(\frac{s-a}{s+a}\right)\left(\frac{s+0.01}{s+a}\right) Q_c \right\|_\infty \leq \frac{1}{\delta}$

$W(s) = \frac{2a(s+0.01)}{(s+a)}$ $\quad M(s) = \left(\frac{s-a}{s+a}\right)$ $\quad Q(s) = -\left(\frac{s+0.01}{s+a}\right) Q_c(s)$

The largest allowable $\delta > 0$ is $\frac{1}{\gamma_{opt}}$ where $\gamma_{opt} = |W(a)|$

i.e.

$\delta_{max} = \left(\frac{1}{a+0.01}\right)$; Corresponding $Q_{opt} = \frac{W(s) - W(a)}{\left(\frac{s-a}{s+a}\right)}$

$$Q_{opt}(s) = \frac{\frac{2a(s+0.01)}{s+a} - (a+0.01)}{\left(\frac{s-a}{s+a}\right)} = \frac{2a(s+0.01) - (s+a)(a+0.01)}{(s-a)}$$

$$= \frac{(a-0.01)s + 2a\,0.01 - a(a+0.01)}{(s-a)} = \frac{(a-0.01)s - a(a-0.01)}{(s-a)} = (a-0.01)$$

$$Q_{c,opt} = -\left(\frac{s+a}{s+0.01}\right)(a-0.01)$$

$$C_{opt} = \left(\frac{2a + \left(\frac{s-a}{s+a}\right) Q_{c,opt}(s)}{1 - \left(\frac{1}{s+a}\right) Q_{c,opt}(s)}\right) = \frac{2a - \frac{(a-0.01)}{s+0.01}(s-a)}{1 + \frac{(a-0.01)}{s+0.01}} = \frac{2a(s+0.01) - (a-0.01)(s-a)}{s+0.01+a-0.01}$$

$$= \frac{(a+0.01)s + 0.02a + a(a-0.01)}{(s+a)} = \frac{(a+0.01)(s+a)}{(s+a)} = (a+0.01)$$

Complementary sensitivity: $T = \frac{PC}{1+PC} = \frac{\frac{a+0.01}{s-a}}{1 + \frac{a+0.01}{s-a}} = \left(\frac{a+0.01}{s+0.01}\right)$

$W_2 T = \delta (a+0.01)$

$\|W_2 T\|_\infty \leq 1 \iff \delta \leq \frac{1}{a+0.01} = \delta_{max}$

Summary: When $M(s) = \left(\frac{s-a}{s+a}\right)$ $\quad a > 0$

$\gamma_{opt} = |W(a)| \quad \rightarrow \quad Q_{opt} = \frac{W(s) - W(a)}{\left(\frac{s-a}{s+a}\right)}$

What happens when $M(s) = \prod_{i=1}^{n} \left(\frac{s-\alpha_i}{s+\alpha_i}\right)$ with $Re(\alpha_i) > 0$

Solution will be given by the Nevanlinna-Pick interpolation

Nevanlinna-Pick interpolation Problem

Let $M(s) = \prod_{i=1}^{n} \left(\frac{s-\alpha_i}{s+\bar{\alpha}_i} \right)$ where $\alpha_1, \dots, \alpha_n$ are distinct with $Re(\alpha_i) > 0$

$$\gamma_{opt} = \inf_{Q \in \mathcal{H}_\infty} \| W(s) - M(s) Q(s) \|_\infty$$

Clearly $\gamma_{opt} \geq \max_i |W(\alpha_i)|$ ... but this lower bound may not be achieved

Define $F(s) = W(s) - M(s) Q(s)$, $Q \in \mathcal{H}_\infty$

Note that $F(\alpha_i) = W(\alpha_i) =: \beta_i \quad \forall i$, for all $Q \in \mathcal{H}_\infty$

Our objective is to find $F \in \mathcal{H}_\infty$ such that

$$F(\alpha_i) = \beta_i \qquad i = 1, \dots, n$$

and $\|F\|_\infty \leq \gamma$ for the smallest possible γ

This is the Nevanlinna-Pick interpolation problem.

Its solution is given as follows.
(Section 2.4.1 of OGKY2018 book)

Fact 1 : γ_{opt} (smallest possible γ for which the Nevanlinna-Pick problem is solvable) is

$$\gamma_{opt} = \sqrt{\lambda_{max}(A^{-1}B)}$$

where $\lambda_{max}(\cdot)$ denotes the largest eigenvalue

and $[A]_{ij} = \left(\frac{1}{\bar{\alpha}_i + \alpha_j} \right)$ $\quad [B]_{ij} = \frac{\bar{\beta}_i \beta_j}{\bar{\alpha}_i + \alpha_j}$

$\longrightarrow$ To find γ_{opt} construct $n \times n$ matrices A, B

Compute the largest eigenvalue of $A^{-1}B$ and take its square root

$\rightarrow$ What is the optimal interpolant satisfying

$F_{opt} \in \mathcal{H}_\infty$, $F_{opt}(\alpha_i) = \beta_i$ and $\|F_{opt}\|_\infty = \gamma_{opt}$?

$$F_{opt}(s) = \lambda \left(\frac{\phi_{n-1} s^{n-1} - \phi_{n-2} s^{n-2} + \cdots + (-1)^{n-1}\phi_0}{\phi_{n-1} s^{n-1} + \phi_{n-2} s^{n-2} + \cdots + \phi_0} \right)$$

where $\lambda = \pm \gamma_{opt}$, i.e. $|\lambda| = \gamma_{opt}$

$\Phi = \begin{bmatrix} \phi_{n-1} \\ \vdots \\ \phi_0 \end{bmatrix}$ is determined from a set of linear equations given below.

$$F_{opt} = \lambda \frac{[s^{n-1} \; s^{n-2} \; \cdots \; s^0] \, J \Phi}{[s^{n-1} \; s^{n-2} \; \cdots \; s^0] \, \Phi} \qquad J = \begin{bmatrix} 1 & & & 0 \\ & -1 & & \\ & & \ddots & \\ 0 & & & (-1)^{n-1} \end{bmatrix}_{n \times n}$$

$$F_{opt}(\alpha_i) = \beta_i \iff$$

i^{th} equation: $\lambda [\alpha_i^{n-1} \cdots \alpha_i^0] J \Phi = \beta_i [\alpha_i^{n-1} \cdots \alpha_i^0] \Phi$

define: $V_\alpha = \begin{bmatrix} \alpha_1^{n-1} & \cdots & \alpha_1^0 \\ \vdots & & \\ \alpha_n^{n-1} & \cdots & \alpha_n^0 \end{bmatrix}$ Vandermonde matrix; $D_\beta = \begin{bmatrix} \beta_1 & & 0 \\ & \ddots & \\ 0 & & \beta_n \end{bmatrix}$

n equations: $\left(\lambda V_\alpha J - D_\beta V_\alpha \right) \Phi = 0$

$$\iff \left(\lambda I - J V_\alpha^{-1} D_\beta V_\alpha \right) \Phi = 0$$

Thus, $\Phi \neq \underline{0}$ is the eigenvector corresponding to the eigenvalue $\lambda = \pm \gamma_{opt}$

<u>Conclusion</u>: Given $\begin{Bmatrix} \alpha_1, \ldots, \alpha_n \\ \beta_1, \ldots, \beta_n \end{Bmatrix}$ construct $A, B, J, V_\alpha, D_\beta$

find $\gamma_{opt} = \sqrt{\lambda_{max}(A^{-1}B)}$

find the eigenvalues of $(J V_\alpha^{-1} D_\beta V_\alpha)$ determine which one is $\lambda = \pm \gamma_{opt}$; find its eigenvector Φ.

Then $F_{opt}(s) = \lambda \dfrac{[s^{n-1} \cdots s^0] \, J \Phi}{[s^{n-1} \cdots s^0] \, \Phi}$.