Mixed Sensitivity Minimization: $\gamma_{opt} = \inf_{(C,P)\,\text{stable}} \left\| \begin{bmatrix} W_1 S \\ W_2 T \end{bmatrix} \right\|_\infty$ $\quad S = (1+PC)^{-1}$, $T = PC(1+PC)^{-1}$

given $W_1, W_2, P \Rightarrow$ find γ_{opt} and the corresponding C_{opt}

Step 1: Controller Parameterization

$$C = \frac{X + DQ_c}{Y - NQ_c}, \quad Q_c \in \mathcal{H}_\infty; \quad N, D, X, Y \in \mathcal{H}_\infty, \quad P = N/D \quad \text{and} \quad NX + DY = 1$$

This leads to $S = (1+PC)^{-1} = D(Y - NQ_c)$; $\quad T = N(X + DQ_c)$

$$\gamma_{opt} = \inf_{Q_c \in \mathcal{H}_\infty} \left\| \begin{bmatrix} W_1 D(Y - NQ_c) \\ W_2 N(X + DQ_c) \end{bmatrix} \right\|_\infty$$

recall that for $A, B \in \mathcal{H}_\infty$ $\quad \left\| \begin{bmatrix} A \\ B \end{bmatrix} \right\|_\infty = \sup_\omega \left(|A(j\omega)|^2 + |B(j\omega)|^2 \right)^{1/2}$

Step 2: Spectral and Inner-Outer Factorizations

Fact: let $L(j\omega)$ be a 2×2 unitary matrix $\forall \omega$

i.e. $L(j\omega)^* L(j\omega) = L(j\omega) L(j\omega)^* = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$L(j\omega)^* = \overline{L(j\omega)}^T$ (complex conjugate transpose)

Then for any $A, B \in \mathcal{H}_\infty$ we have $\left\| L \begin{bmatrix} A \\ B \end{bmatrix} \right\|_\infty = \left\| \begin{bmatrix} A \\ B \end{bmatrix} \right\|_\infty$.

$$\gamma_{opt} = \inf_{Q_c \in \mathcal{H}_\infty} \left\| \begin{bmatrix} W_1 DY \\ W_2 NX \end{bmatrix} - \begin{bmatrix} W_1 \\ -W_2 \end{bmatrix} ND\, Q_c \right\|_\infty$$

(*) Spectral factorization (1)

$$\begin{cases} W_1^* W_1 + W_2^* W_2 = G^* G \\ G^{-1} \in \mathcal{H}_\infty \quad \ldots \text{ find } G \end{cases}$$

Notation: $W_1^* \equiv W_1(-s)$

find G,

Example: $\quad W_1(s) = \dfrac{1}{s+1} \qquad W_2(s) = ks$

$$G^*G = \frac{1}{1-s^2} - k^2 s^2 = \left(\frac{k^2 s^4 - k^2 s^2 + 1}{1-s^2}\right) = k^2\left(\frac{s^4 - s^2 + 1/k^2}{1-s^2}\right)$$

$$s^4 - s^2 + 1/k^2 = (s^2 + as + b)(s^2 - as + b) = s^4 - (a^2 - 2b)s^2 + b^2$$

$$\left.\begin{aligned} a^2 - 2b &= 1 \\ b^2 &= 1/k^2 \end{aligned}\right\} \quad \begin{aligned} a &= \sqrt{1 + 2/k} \\ b &= 1/k \end{aligned} \implies G(s) = \frac{k\left(s^2 + \sqrt{1+2/k}\, s + 1/k\right)}{(s+1)}$$

$$G^{-1}(s) = \left(\frac{s+1}{ks^2 + k\sqrt{1+2/k}\, s + 1}\right) \in \mathcal{H}_\infty \quad \checkmark$$

Generalization: $\quad W_1(-s)W_1(s) + W_2(-s)W_2(s) = \dfrac{nW_1(-s)\, nW_1(s)}{dW_1(-s)\, dW_1(s)} + \dfrac{nW_2(-s)\, nW_2(s)}{dW_2(-s)\, dW_2(s)}$

$$\frac{nW_1(-s)\, dW_2(-s)\, nW_1(s)\, dW_2(s) + nW_2(-s)\, dW_1(-s)\, nW_2(s)\, dW_1(s)}{dW_1(-s)\, dW_1(s)\, dW_2(-s)\, dW_2(s)} = \frac{nG(-s)\, nG(s)}{dG(-s)\, dG(s)}$$

$dG(s) = dW_1(s)\, dW_2(s)$

find the symmetric roots of $nG(-s)\, nG(s)$ collect roots in $\mathbb{C}_-$ as the roots of $nG(s)$; roots in $\mathbb{C}_+$ as roots of $nG(-s)$

Now define

$$G^{-*} = G^{-1}(-s) \qquad L = \begin{bmatrix} W_1^* G^{-*} & -W_2^* G^{-*} \\ W_2 G^{-1} & W_1 G^{-1} \end{bmatrix} \qquad L^* = \begin{bmatrix} W_1 G^{-1} & W_2^* G^{-*} \\ -W_2 G^{-1} & W_1^* G^{-*} \end{bmatrix}$$

$$LL^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = L^*L$$

$$\gamma_{opt} = \inf_{Q_c \in \mathcal{H}_\infty} \left\| \underbrace{\begin{bmatrix} W_1^* G^{-*} & -W_2^* G^{-*} \\ W_2 G^{-1} & W_1 G^{-1} \end{bmatrix} \begin{bmatrix} W_1 D Y \\ W_2 N X \end{bmatrix}}_{} - \underbrace{\begin{bmatrix} G \\ 0 \end{bmatrix}}_{} N D Q_c \right\|_\infty$$

$$\gamma_{opt} = \inf_{Q_c \in \mathcal{H}_\infty} \left\| \begin{bmatrix} W_1^* G^{-*} W_1 D Y - W_2^* G^{-*} W_2 N X \\ W_2 G^{-1} W_1 D Y + W_1 G^{-1} W_2 N X \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} N D G Q_c \right\|_\infty$$

$N = N_i N_o$ $\quad$ $D = D_i D_o$ $\quad$ Recall $\quad NX + DY = 1$

$$\gamma_{opt} = \inf_{Q_c \in \mathcal{H}_\infty} \left\| \begin{bmatrix} W_1^* W_1 G^{-*} - GNX - NDGQ_c \\ W_1 W_2 G^{-1} \end{bmatrix} \right\|_\infty$$

$$\boxed{\gamma_{opt} = \inf_{Q \in \mathcal{H}_\infty} \left\| \begin{bmatrix} W_1^* W_1 G^{-*} - G N_i N_o X - N_i D_i \underbrace{(N_o D_o G Q_c)}_{Q} \\ W_1 W_2 G^{-1} \end{bmatrix} \right\|_\infty}$$

$Q = \underbrace{N_o D_o G}_{\text{outer}} Q_c$ $\qquad$ $Q_c = (N_o D_o G)^{-1} Q$

$$L_1 = \begin{bmatrix} N_i^* D_i^* & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \gamma_{opt} = \inf_{Q \in \mathcal{H}_\infty} \left\| \begin{bmatrix} \overbrace{N_i^* D_i^* W_1^* G^{-*} W_1 - D_i^* G N_o X}^{R} - Q \\ \underbrace{W_1 W_2 G^{-1}}_{V} \end{bmatrix} \right\|_\infty$$

$R = N_i^* D_i^* W_1^* G^{-*} W_1 - D_i^* G N_o X$ = in $\mathcal{L}_\infty$, has poles in $\mathbb{C}_+$, $\mathbb{C}_-$

$V = W_1 W_2 G^{-1} \in \mathcal{H}_\infty$

$$\boxed{\gamma_{opt} = \inf_{Q \in \mathcal{H}_\infty} \left\| \begin{bmatrix} R - Q \\ V \end{bmatrix} \right\|_\infty}$$

→ the two block $\mathcal{H}_\infty$ problem

$$\left\| \begin{bmatrix} R - Q \\ V \end{bmatrix} \right\|_\infty \le \gamma \Leftrightarrow |R-Q|^2 + |V|^2 \le \gamma^2$$

for all ω

$|V|^2 = V(j\omega)^* V(j\omega) \equiv V(-s)V(s)$

$$|R-Q|^2 \le (\gamma^2 - |V|^2)$$

$\gamma > \|V\|_\infty$ → a lower bound for γ_{opt}

Spectral factorization (2)

$V_\gamma^* V_\gamma = (\gamma^2 - V^* V)$; $V_\gamma, V_\gamma^{-1} \in \mathcal{H}_\infty$

find ~~V_γ~~ V_γ

$$\circledast \iff (R-Q)^*(R-Q) \le V_\gamma^* V_\gamma \iff \| V_\gamma^{-1}R - V_\gamma^{-1}Q \|_\infty \le 1$$ one block problem

stabsep

$$V_\gamma^{-1}R = R_{s\gamma} + R_{u\gamma} \qquad Q_1 = V_\gamma^{-1}Q - R_{s\gamma} \iff Q = (Q_1 + R_{s\gamma})V_\gamma$$

$$\circledast \iff \| R_{u\gamma} - Q_1 \|_\infty \le 1$$

Step 3: Bi-section Search

Last inequality $\Rightarrow$ γ_{opt} is the smallest γ for which the Nehari problem defined by $R_{u\gamma}$ has the best achievable norm ≤ 1.

Computation of γ_{opt}: pick a feasible interval γ_{min}, γ_{max}

$\|V\|_\infty = \gamma_{min}$, γ_{max} very large

let $\gamma = \dfrac{\gamma_{min} + \gamma_{max}}{2}$ compute $R_{u\gamma}$ as above

test point γ

Check if the best achievable norm of the associated Nehari problem (or Nevanlinna-Pick problem) is ≤ 1

→ Yes define new $\gamma_{max} \leftarrow \gamma$

→ No define new $\gamma_{min} \leftarrow \gamma$

Repeat the process until $|\gamma_{max} - \gamma_{min}| \le \varepsilon$

compute the optimal solution from the last γ value which is $\approx \gamma_{opt}$. → obtain $Q_{\gamma_{opt}} \in \mathcal{H}_\infty$ from this last γ value

$Q_{opt} \rightarrow Q_{c,opt} \rightarrow C_{opt}$.

Matlab command >> mixsyn