Time Series Analysis
Assignment 3

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July 5, 2024

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1 Explorative Data Analysis

1.1 Make plots of the data

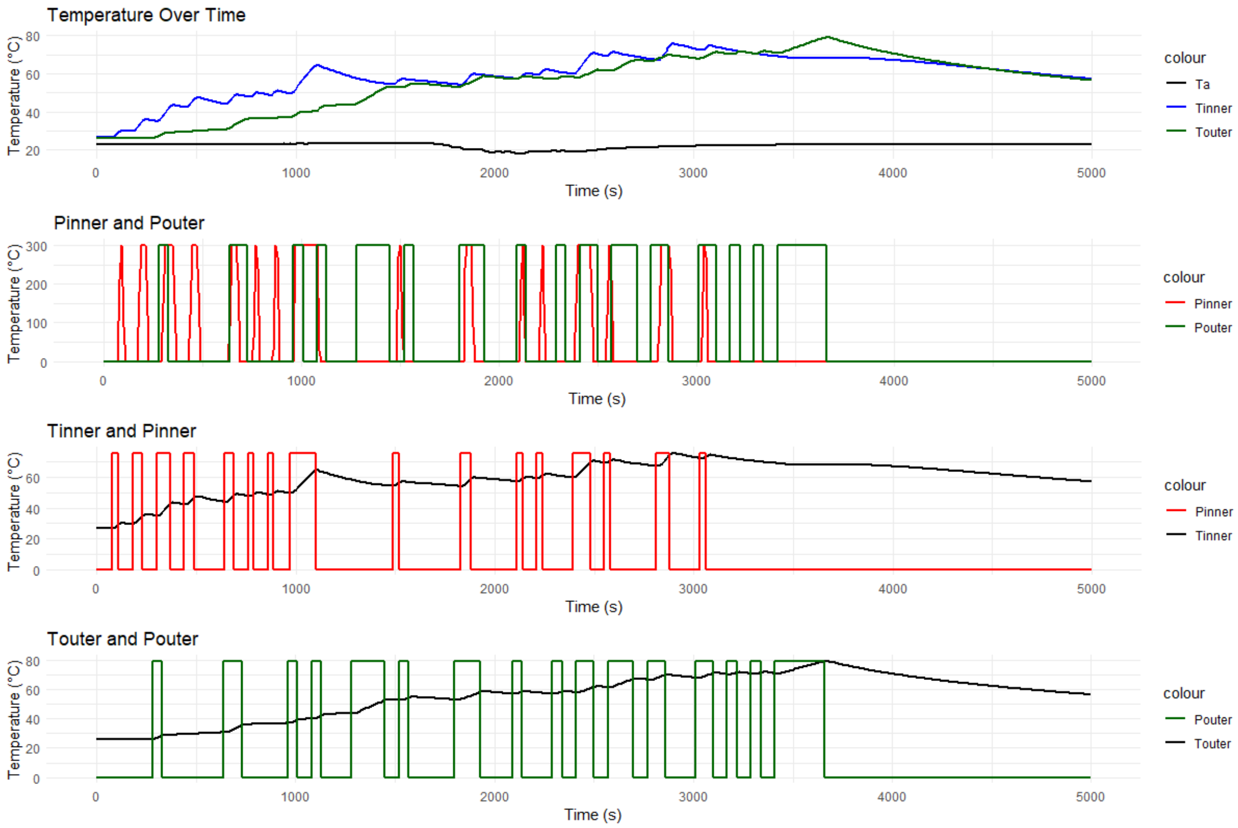

Figure 1: Initial Plots

1.2 From the plots describe the experiment

In the system, heaters are responsible for the rising temperature as power is applied. This is evident from figure 1.

Looking at the plots of figure 1, we can see the relation between the different actors of the experiment: in the windows of time in which the outer/inner heater is turned on, we see a constant increase of the pot/glass temperature. The temperature of these two elements is also correlated: since they share a surface, heat exchange will occur, generating the pattern we see in the first plot. The whole system will converge towards equilibrium (same temperature in the pot and the drinking glass). The greater the temperature difference, the faster the sharper the convergence. On the other hand, the room temperature seems mostly independent of both heat inputs.

Regarding causalities, the independent variables are the power inputs (Pinner and Pouter), which are controlled by the experimenter and can be set to desired levels. The dependent variables are the temperatures (T_a, T_{inner}, T_{outer}), which change in response to the power inputs and $|T_{inner} - T_{outer}|$.

The system's temperature response to the power input seems consistent: the both inner and outer temperatures change in the same fashion whenever the power is turned on independently of time, suggesting that it is indeed a LTI system.

1.3 Investigate the relations between the variables using the cross-correlation function

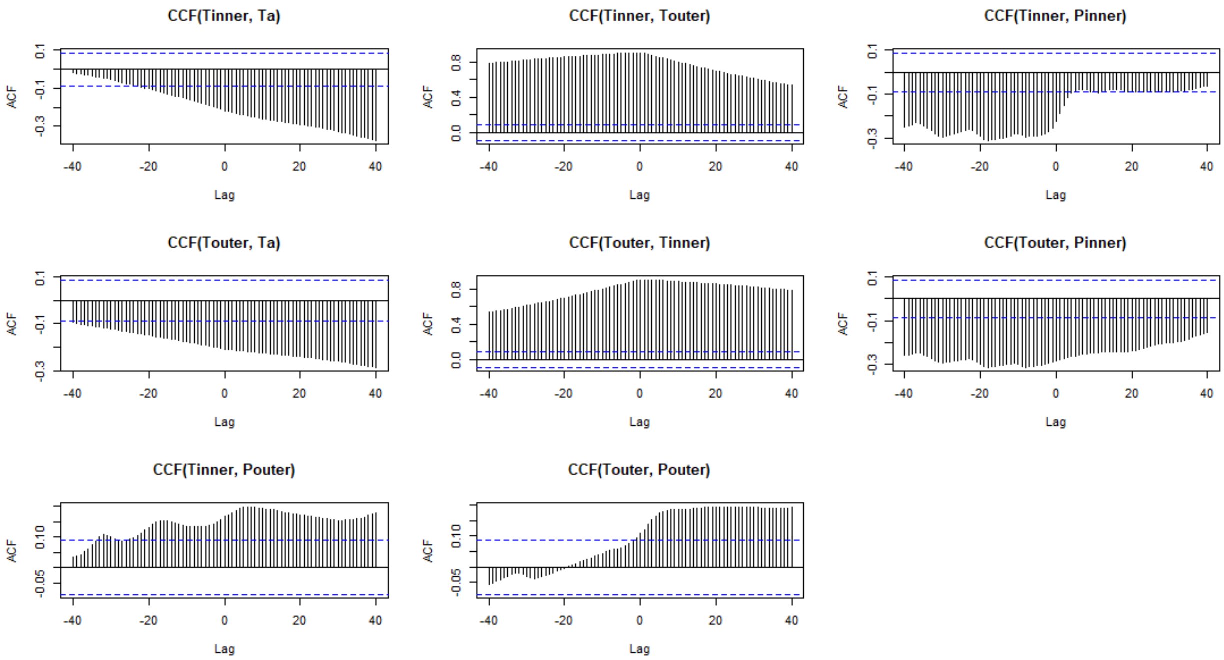

Figure 2: Cross-correlation functions

CCF(Tinner, Ta) & CCF(Touter, Ta): The correlation values remain below zero across different lags without crossing into the positive range, which suggests a persistent inverse relationship through the observed lags. The plots, also, indicate that the temperature of the ambient air surrounding the system influences the temperature of the water in the pot and of the water in the glass.

CCF(Tinner, Touter) & CCF(Touter, Tinner): The correlation is significant and positive across all lags, suggesting a very strong positive correlation between Tinner and Touter with Tinner influencing Touter.

CCF(Tinner, Pinner) & CCF(Touter, Pinner): The correlation is significant and negative across all lags, except for the positive lags in the CCF plot for Tinner and Pinner, where there is no significant correlation. This suggests a very strong negative correlation between Tinner and Touter with Pinner. The Power input of the heater in the glass is leading the temperature of the water in the pot and in the glasss.

CCF(Tinner, Pouter) & CCF(Touter, Pouter): The correlation is positive and significant across all lags except the first few. This indicates a significant influence of the temperature of the water in the glass on the power input of the heater in the pot. By observing the CCF plot of Touter and Pouter we make the same conclusions, but in the negative lags we have non significant negative and positive correlations.

CCF(Pouter, Pinner) & CCF(Pinner, Pouter): Pouter and Pinner are independent series and their CCF plot should reflect this lack of relationship. The cross-correlation coefficients at various lags would be close to zero.

In this case, the CCF plots are not that useful as they just confirm the patterns that are already clearly visible in the regular plots of the time series.

2 ARMAX model of the glass water temperature

2.1 Implement the model in Equation (2) as a linear regression model and estimate the parameters with the least squares method

The mentioned equation is:

$$\phi(B)T_{i,t} = \omega_a(B)T_{a,t} + \epsilon_t \tag{1}$$

where ϵ_t is assumed i.i.d. The ARX model of order 1 is:

$$(1 + \phi_1 B)T_{i,t} = \omega_{a,1}Ta_{t-1} + \epsilon_t \tag{2}$$

Which is equivalent to:

$$T_{i,t} = -\phi_1 T_{i,t-1} + \omega_{a,1}Ta_{t-1} + \epsilon_t \tag{3}$$

```
Call:
lm(formula = ARX("Tinner", c("Ta"), 1), data = X)

Residuals:
     Min       1Q   Median       3Q      Max 
-0.48409 -0.19806 -0.14309 -0.04731  1.52702 

Coefficients:
          Estimate Std. Error t value Pr(>|t|)    
Tinner.l1 0.993750   0.001647  603.51  < 2e-16 ***
Ta.l1     0.019165   0.004467    4.29 2.14e-05 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 0.4514 on 498 degrees of freedom
  (1 observation deleted due to missingness)
Multiple R-squared:  0.9999,	Adjusted R-squared:  0.9999 
F-statistic: 4.468e+06 on 2 and 498 DF,  p-value: < 2.2e-16
```

Figure 3: Summary of the ARX model of order 1

From Figure 3 we observe that the estimates for both $T_{a,t-1}$ and $T_{inner,t-1}$ are statistically significant. It is also possible to see that the weight attributed to $T_{a,t-1}$ is much less than the one of $T_{inner,t-1}$, which confirm our initial observations: the ambient temperature of the room does not have much impact on the temperature of the inner pot, and the latter is much closely related to the previous temperature of said inner pot.

2.2 Make a model validation with the plot of residuals, ACF, and PACF, as well as CCFs from the inputs to the residuals

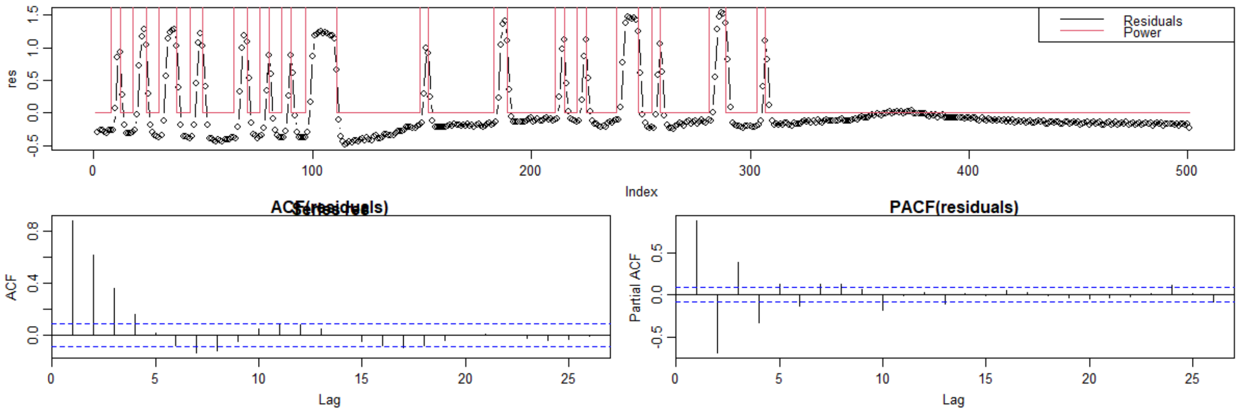

Figure 4: Plot of the residuals, ACF and PACF

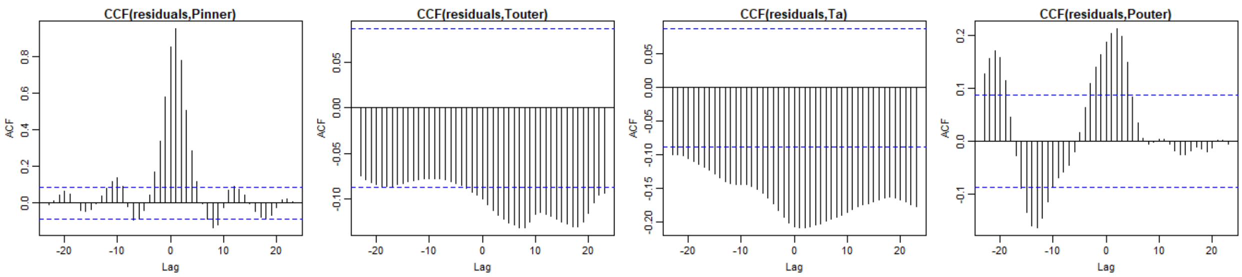

Figure 5: CCFs from the inputs to the residuals

From Figures 4 and 5 we can see that ARX of order 1 model is not a suitable choice since we can observe that the residuals don't behave like white noise, which is an assumption for a well-fitted time series model. In these ACF plots we can see spikes in the correlation beyond lag 0, implying not independent noise terms. On the other hand, the CCF plots suggests correlation between the residuals of the model and other variables, which implies that there are factors that the ARX(1) model is not taking into account and therefore, is insufficient.

2.3 Identify a suitable ARX model for $T_{i,t}$

Initially, we begin by dropping the factor Ta since the $Tinner$ isn't affected by the temperature of the ambient air surrounding the system. Our model now has the factors $Tinner$, $Pinner$, $Touter$, and $Pouter$ because they are the rest of the variables of the system that have an impact in the temperature of the inner pot.

Order	1	2	3	4	5	6	7	8	9	10
AIC	-74	-2148	-2244	-2277	-2325	-2327	-2350	-2357	-2356	-2356

Table 1: AIC for ARX models of order 1, 2, 3, to 10

As we can see in the table above, most if the AIC's are very similar, independently of the lag. The best AIC's (albeit only marginally) belong to lags 5 to 10. In order to reduce computational cost and avoid overfitting we deem better to reduce the number of variables. In conclusion, we think that the best model would use lag 5 or maybe 6.

```
Call:
lm(formula = ARX("Tinner", c("Touter", "Pinner", "Pouter"), 1:5),
    data = X)

Residuals:
      Min        1Q    Median        3Q       Max
-0.099838 -0.013128  0.002617  0.014666  0.074536

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
Tinner.l1  1.451e+00  4.432e-02  32.747  < 2e-16 ***
Tinner.l2 -4.798e-01  7.892e-02  -6.079 2.48e-09 ***
Tinner.l3  3.367e-01  8.059e-02   4.179 3.49e-05 ***
Tinner.l4 -4.072e-01  5.902e-02  -6.900 1.67e-11 ***
Tinner.l5  9.189e-02  1.310e-02   7.015 7.93e-12 ***
Touter.l1  1.517e-01  4.620e-02   3.284  0.00110 **
Touter.l2 -3.288e-02  7.629e-02  -0.431  0.66670
Touter.l3 -1.135e-01  7.650e-02  -1.483  0.13865
Touter.l4  7.679e-03  6.402e-02   0.120  0.90458
Touter.l5 -5.989e-03  1.850e-02  -0.324  0.74632
Pinner.l1  1.620e 03  2.075e 05  78.046  < 2e 16 ***
Pinner.l2  2.000e-03  7.549e-05  26.489  < 2e-16 ***
Pinner.l3 -3.983e-04  1.148e-04  -3.469  0.00057 ***
Pinner.l4 -6.910e-04  1.137e-04  -6.077 2.51e-09 ***
Pinner.l5 -7.459e-04  1.086e-04  -6.869 2.03e-11 ***
Pouter.l1 -7.684e-06  1.931e-05  -0.398  0.69090
Pouter.l2 -5.317e-05  3.467e-05  -1.534  0.12576
Pouter.l3 -1.866e-04  5.647e-05  -3.304  0.00102 **
Pouter.l4 -1.804e-04  5.714e-05  -3.158  0.00169 **
Pouter.l5 -9.461e-05  5.642e-05  -1.677  0.09421 .
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

```
Call:
lm(formula = ARX("Tinner", c("Touter", "Pinner", "Pouter"), 1:6),
    data = X)

Residuals:
      Min        1Q    Median        3Q       Max
-0.095188 -0.013336  0.002017  0.015347  0.076500

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
Tinner.l1  1.403e+00  4.647e-02  30.186  < 2e-16 ***
Tinner.l2 -4.409e-01  8.039e-02  -5.485 6.76e-08 ***
Tinner.l3  3.310e-01  8.141e-02   4.066 5.60e-05 ***
Tinner.l4  2.573e 01  8.146e 02   3.159 0.001685 **
Tinner.l5 -6.862e-02  6.163e-02  -1.114 0.266054
Tinner.l6  2.660e-02  1.372e-02   1.939 0.053040 .
Touter.l1  1.244e 01  4.851e 02   2.565 0.010630 *
Touter.l2 -1.882e-02  7.803e-02  -0.241 0.809496
Touter.l3 -9.068e-02  7.683e-02  -1.180 0.238493
Touter.l4  5.074e 02  7.736e 02   0.656 0.512189
Touter.l5 -7.141e-02  6.656e-02  -1.073 0.283909
Touter.l6  1.249e-02  1.906e-02   0.655 0.512572
Pinner.l1  1.628e 03  2.069e 05  78.673  < 2e 16 ***
Pinner.l2  2.062e-03  7.803e-05  26.425  < 2e-16 ***
Pinner.l3 -2.256e-04  1.227e-04  -1.839 0.066513 .
Pinner.l4  6.395e 04  1.160e 04   5.511 5.90e 08 ***
Pinner.l5 -8.545e-04  1.180e-04  -7.241 1.83e-12 ***
Pinner.l6 -3.769e-04  1.134e-04  -3.324 0.000955 ***
Pouter.l1  5.571e 06  1.947e 05   0.286 0.774881
Pouter.l2 -4.870e-05  3.498e-05  -1.392 0.164581
Pouter.l3 -1.450e-04  5.975e-05  -2.427 0.015587 *
Pouter.l4 -1.617e-04  5.700e-05  -2.836 0.004760 **
Pouter.l5 -1.017e-04  5.892e-05  -1.726 0.084925 .
Pouter.l6 -7.257e-05  5.866e-05  -1.237 0.216639
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Figure 6: Summary of the ARX model of order 5(left) and 6(right)

In Figure 6 we can see that the introduction of the sixth lag does not substantially increase the standard error of the estimates, and the coefficients remain robust. However, we can see that for the variable $Pouter$, there is not much significance in many of the lags and the coefficients are very small, which suggests that said variable does not have much impact in the model. That is why we will repeat the same selection process without this variable.

Order	1	2	3	4	5	6	7	8	9	10
AIC	-76	-2146	-2224	-2247	-2312	-2319	-2350	-2359	-2360	-2354

Table 2: AIC for ARX using *Tinner*, *Pinner* and *Toutter* models of order 1 to 10

```
Call:
lm(formula = ARX("Tinner", c("Touter", "Pinner"), 1:7), data = X)

Residuals:
      Min        1Q    Median        3Q       Max 
-0.098616 -0.014029  0.001788  0.015258  0.074642 

Coefficients:
             Estimate Std. Error t value Pr(>|t|)    
Tinner.l1  1.406e+00  4.497e-02  31.261  < 2e-16 ***
Tinner.l2 -4.846e-01  7.877e-02  -6.152 1.63e-09 ***
Tinner.l3  4.072e-01  8.111e-02   5.021 7.31e-07 ***
Tinner.l4 -3.186e-01  8.092e-02  -3.937 9.48e-05 ***
Tinner.l5  2.367e-01  8.035e-02   2.946  0.00338 ** 
Tinner.l6 -3.282e-01  5.981e-02  -5.487 6.66e-08 ***
Tinner.l7  8.009e-02  1.327e-02   6.037 3.17e-09 ***
Touter.l1  3.461e-03  1.518e-02   0.228  0.81970    
Touter.l2  2.671e-02  4.278e-02   0.624  0.53277    
Touter.l3 -5.902e-02  5.802e-02  -1.017  0.30951    
Touter.l4  4.389e-02  6.054e-02   0.725  0.46883    
Touter.l5  2.125e-02  5.793e-02   0.367  0.71386    
Touter.l6 -4.970e-02  4.273e-02  -1.163  0.24543    
Touter.l7  1.487e-02  1.506e-02   0.987  0.32396    
Pinner.l1  1.628e-03  2.011e-05  80.963  < 2e-16 ***
Pinner.l2  2.064e-03  7.597e-05  27.172  < 2e-16 ***
Pinner.l3 -1.792e-04  1.191e-04  -1.505  0.13298    
Pinner.l4 -5.749e-04  1.144e-04  -5.028 7.05e-07 ***
Pinner.l5 -9.116e-04  1.112e-04  -8.199 2.29e-15 ***
Pinner.l6 -7.445e-04  1.190e-04  -6.257 8.80e-10 ***
Pinner.l7 -6.386e-04  1.109e-04  -5.760 1.51e-08 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 0.02193 on 473 degrees of freedom
  (7 observations deleted due to missingness)
Multiple R-squared:      1,	Adjusted R-squared:      1 
F-statistic: 1.798e+08 on 21 and 473 DF,  p-value: < 2.2e-16
```

Figure 7: Summary of the ARX model using *Tinner*, *Pinner* and *Toutter* for lag 7

We can see that lag 7 has one of the best AIC's. The summary of the model allows us to see the same phenomenon again: *Toutter* does not have have a lot of impact in the model. Therefore, we repeat the process one last time with only *Tinner* and *Pinner*.

Order	1	2	3	4	5	6	7	8	9	10
AIC	-68	-1249	-2080	-2177	-2269	-2276	-2292	-2288	-2279	-2272

Table 3: AIC for ARX using *Tinner* and *Pinner* of order 1 to 10

```
Call:
lm(formula = ARX("Tinner", c("Pinner"), 1:7), data = X)

Residuals:
      Min        1Q    Median        3Q       Max
-0.109013 -0.015077  0.002233  0.015559  0.070260

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
Tinner.l1  1.518e+00  4.523e-02  33.554  < 2e-16 ***
Tinner.l2 -5.666e-01  8.311e-02  -6.817 2.79e-11 ***
Tinner.l3  3.920e-01  8.597e-02   4.560 6.51e-06 ***
Tinner.l4 -3.716e-01  8.593e-02  -4.324 1.86e-05 ***
Tinner.l5  2.259e-01  8.546e-02   2.644  0.00847 **
Tinner.l6 -2.627e-01  6.113e-02  -4.298 2.09e-05 ***
Tinner.l7  6.510e-02  1.332e-02   4.887 1.40e-06 ***
Pinner.l1  1.623e-03  2.101e-05  77.278  < 2e-16 ***
Pinner.l2  1.883e-03  7.718e-05  24.403  < 2e-16 ***
Pinner.l3 -5.304e-04  1.144e-04  -4.637 4.57e-06 ***
Pinner.l4 -7.672e-04  1.124e-04  -6.824 2.68e-11 ***
Pinner.l5 -9.091e-04  1.070e-04  -8.494 2.53e-16 ***
Pinner.l6 -5.805e-04  1.138e-04  -5.101 4.88e-07 ***
Pinner.l7 -4.763e-04  1.105e-04  -4.312 1.97e-05 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 0.0234 on 480 degrees of freedom
  (7 observations deleted due to missingness)
Multiple R-squared:      1,     Adjusted R-squared:      1
F-statistic: 2.369e+08 on 14 and 480 DF,  p-value: < 2.2e-16
```

Figure 8: Summary of the ARX model using *Tinner* and *Pinner* for lag 7

The AIC's for this set of models has not decreased significantly but as we can see on the summary, now we have significance on every variable of the model while decreasing the complexity of the model.

While it might be concerning that we have left out the impact of the outer pot (Touter, Pouter) in the inner pot, the information provided proves that its impact is negligible and it is worth it to discard it in favor of a more stable and less complex model.

2.4 Present the validation of the selected model

First, we will display all the relevant plots concerning the white noise structure of the residuals.

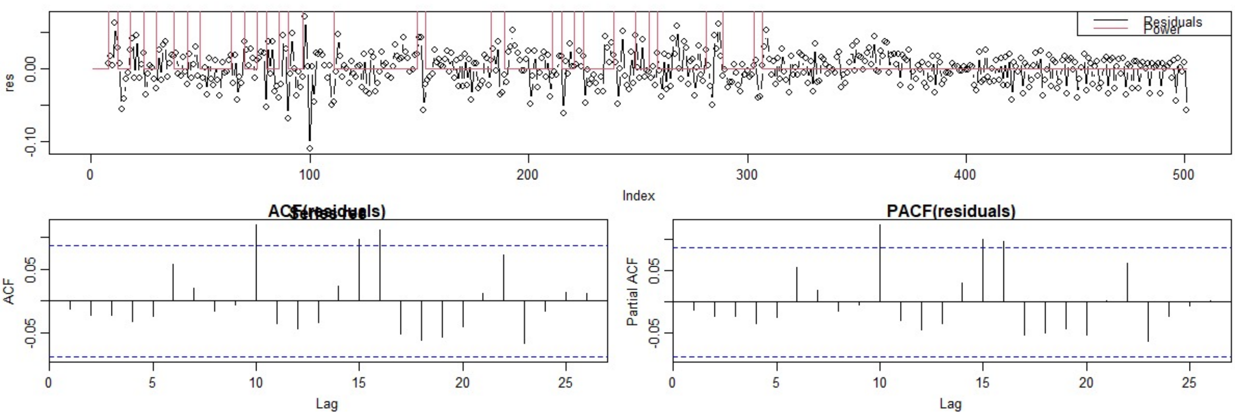

Figure 9: Plot of the residuals, ACF and PACF

From the first plot we can see the the residuals appear to follow a mean 0 distribution. The second and third (ACF and PACF) plots appear to have very few and barely significant correlations between lags of residuals suggesting the i.i.d random variables.

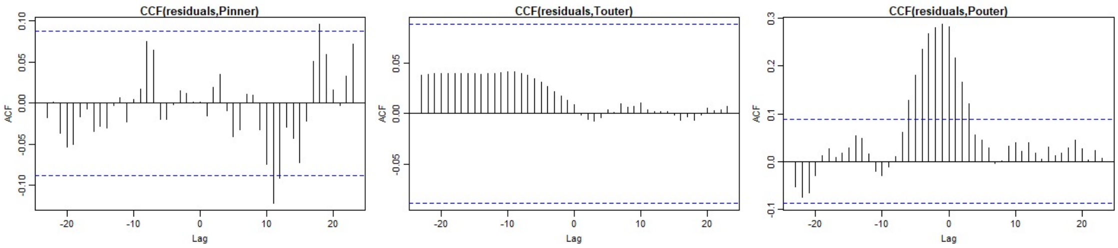

Figure 10: Validation plots of the ARX model of order 7

The CCF plots between residuals and inputs tell us how much of the inputs our models capture. This is because no correlation between residuals and inputs mean that they are not sources of error. Our model's residuals do not correlate with $Pinner$ and $Touter$: they have no impact in how wrong our model is. However, when we consider $Pouter$, we can conclude that the pulses generated by this input are not being taken into account by our model and are causing the error to increase. We take this into account and in section 2.5 we explain why this causes no concern.

2.5 Discuss the results

Generally speaking, selecting the inputs required an extensive search through out many possible combinations of input variables and lags.

At first we made use of common sense: every possible input variable could have an impact in the temperature of the inner pot. However, it has been proven in the previous section that they actually do not hold that much significance, reaching the conclusion that the actually important factors are the previous temperature of the pot and the power of the corresponding heater. However, when validating the models we can clearly see that there is positive significant correlation between the residuals and the power of the outer heater. This tells us that there are aspects that the model it is not taking into account. This is not currently a problem since the AIC's tell us that the error is not increasing. In conclusion: even if the chosen model does not take into account the impact of the outer heater/temperature, the current behaviour/data suggests that said impact is negligible. However, even if we gain in simplicity, this model may not be very robust towards radical changes in pattern of activation of the outer heater.

Lastly, we also choose to not include the variable Ta since the temperature of the room remains constant for the most part and the change that occurs does not appear to cause any kind of change.

3 ARMAX model of the glass water temperature

3.1 Use the marima package to estimate the ARX order 1 model and check that the residuals are equal (or very very close)

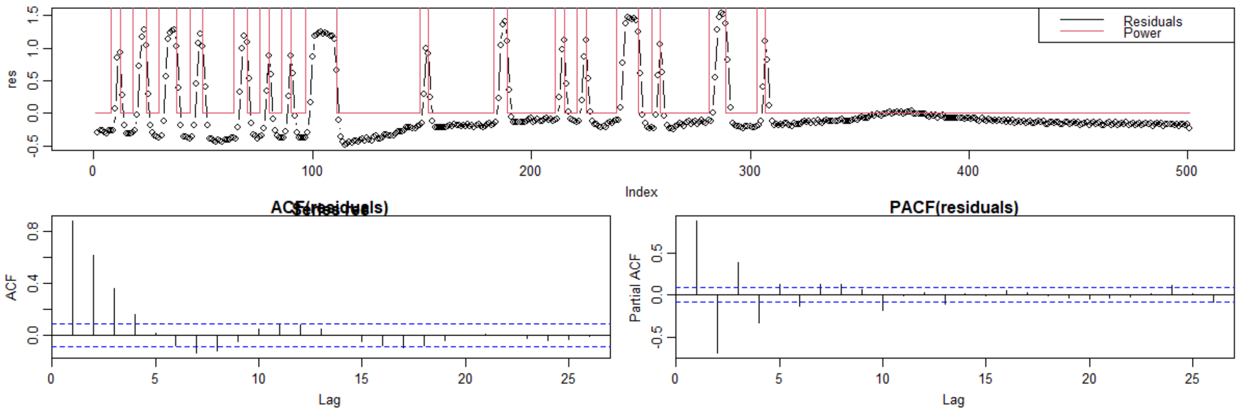

Figure 11: Validation plots from the marima package ARX (equation 2)

The command we use for the computation of the residuals is:

```
fit <- marima("Tinner ~ AR(1) + Ta(1)", data=X)
```

Indeed, as we can see from Figure 12 and Figure 4 the residuals from the marima2 package seem to be very close to the residuals from the ARX.

3.2 Use marima to estimate an ARMAX model order 1 with all relevant inputs and compare the difference to the similar ARX

Figures 12 and 13 show substantial differences in the ACF and PACF plots of the two models.

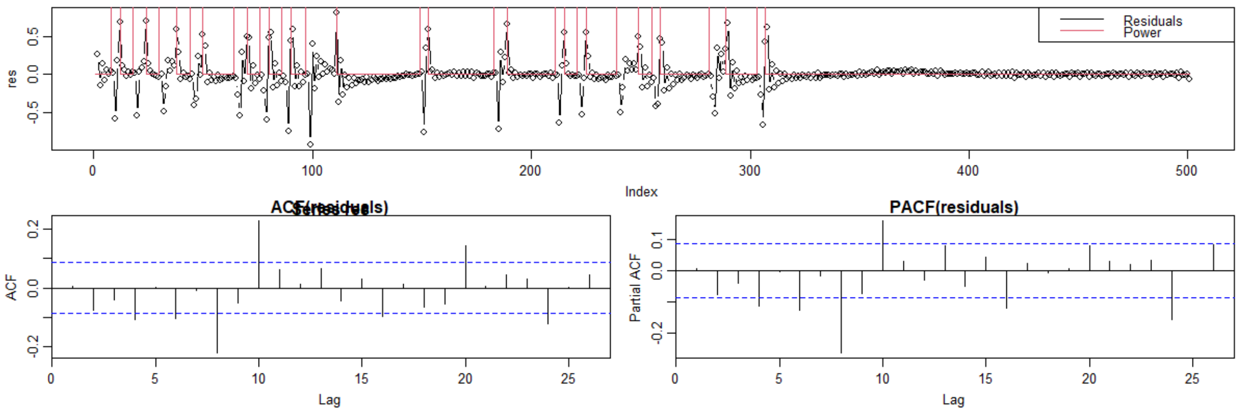

Figure 12: Validation plots ARMAX order 1

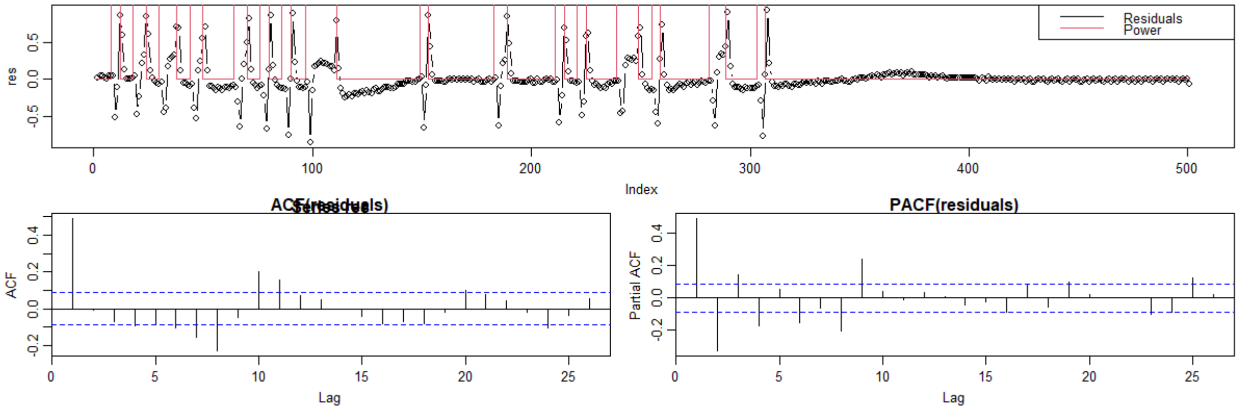

Figure 13: Validation plots ARX order 1

The inclusion of the MA component seems to be helping our model explain the behaviour of the system. One possible explanation for this fact is the thermodinamical nature of the system: heat tends to spread and therefore temperature changes do not usually happen suddenly. This is consistent with an MA component, which tends to dampen the abrupt changes that might be generated by the AR part of the model.

Specifically, here we can see that at the ACF and PACF plots of the residuals of the ARX model at lag 1 we have a significant correlation. Adding the MA part attenuates this correlation completely. It seems to capture the autocorrelation structure that is left unexplained by the AR part, resulting in cleaner residuals that behave more like white noise, which is an assumption for a well-fitted time series model.

3.3 Identify a suitable ARMAX

Our strategy to identify a suitable ARMAX model is to start the examination of our models from the summary tables. For all the models the parameter 'penalty = 2' which aims to balance model fit with complexity by penalizing the inclusion of additional parameters using AIC for model selection.

Initially, we explore the ARMAX model containing the same factors as in the previous exercise: *Pinner*, and *Pouter*.

```
fit <- marima("Tinner ~ AR(i) + Pinner(i) + MA(i), data=X, penalty=2)
```

```
fit <- marima("Tinner ~ AR(i) + Pinner(i) + Ta(i) + MA(i), data=X, penalty
   =2)
```

$Tinner

	Name	Estimate	Pval	Stars
1	AR.l1	-2.2491343915	0.000000e+00	***
2	AR.l2	1.5347409982	4.574071e-248	***
3	AR.l3	-0.2855683584	1.583609e-149	***
4	Pinner.l1	-0.0015980000	7.394882e-274	***
5	Pinner.l2	-0.0007618657	2.908225e-53	***
6	Pinner.l3	0.0022485621	4.532218e-184	***
7	MA.l1	-0.7196643713	3.276455e-40	***
8	MA.l2	-0.1408895220	2.693976e-03	**
9	MA.l3	0.2143583370	1.926746e-08	***

$Tinner

	Name	Estimate	Pval	Stars
1	AR.l1	-2.0885694992	6.078321e-185	***
2	AR.l2	1.6621163231	8.226244e-147	***
3	AR.l3	-1.0460134011	1.012157e-23	***
4	AR.l4	0.4891352046	8.976595e-12	***
5	AR.l5	0.0000000000	0.000000e+00	***
6	AR.l6	-0.0166251387	7.478897e-04	***
7	Pinner.l1	-0.0016228457	6.947620e-274	***
8	Pinner.l2	-0.0009607008	5.681748e-28	***
9	Pinner.l3	0.0012367226	4.446645e-15	***
10	Pinner.l4	0.0000000000	0.000000e+00	***
11	Pinner.l5	0.0009057719	1.577850e-08	***
12	Pinner.l6	0.0003110592	2.265625e-07	***
13	MA.l1	-0.5868183947	1.055183e-18	***
14	MA.l2	0.2209820268	1.127440e-03	**
15	MA.l4	-0.2340720715	1.328499e-06	***
16	MA.l5	0.1894627348	1.091193e-04	***

Figure 14: Summary table ARMAX order 3(left) and order 6(right)

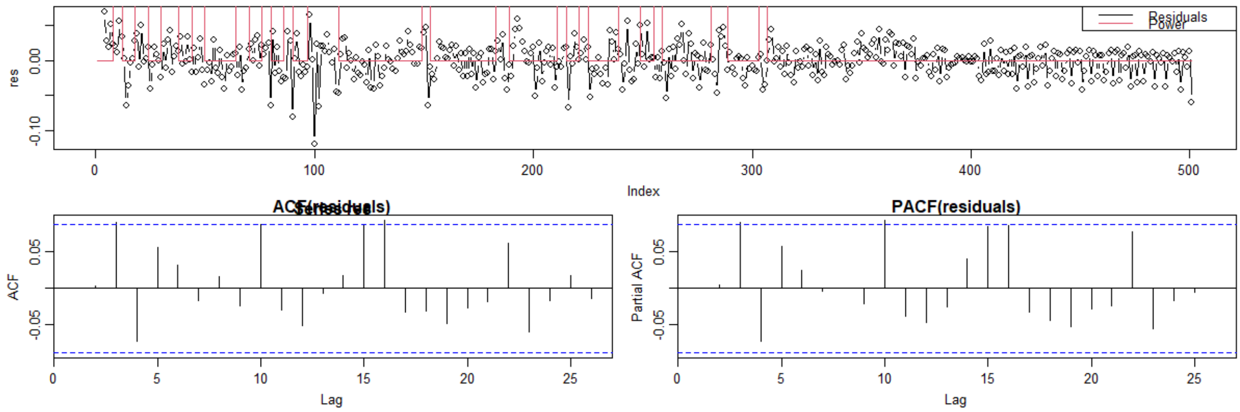

Figure 15: Validation plots ARMAX order 3

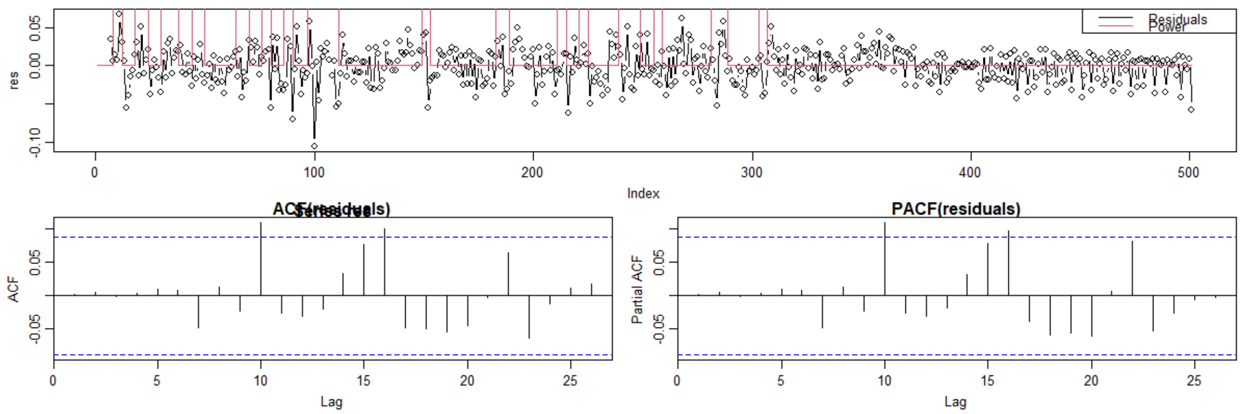

Figure 16: Validation plots ARMAX order 6

After plotting and examining the validation plots from lag 1 to 10, while observing to identify the most noise-resembling behavior of the residuals, we conclude that the ARMAX model of order 3 is a well-fitted time series model for our system with low complexity. Specifically, all of lags included in the summary appear statistically significant and in both ACF and PACF plots of the residuals, there are only 2 significant spikes, while in models with higher complexity we can observe the same or more, except the model of order 10 which is expected. Lastly, the plot of the residuals is also indicative of a noise-like behavior with a mean around zero.

In terms of the inputs, we can see once again that Ta is not significant in the results even up until the model of order 7. (19)

```
> fit <- marima("Tinner ~ AR(1:7) + Ta(1:7) + Pinner(1:7) + MA(1:7)", data=X, penalty=2)
All cases in data,  1  to  501  accepted for completeness.
501 3  = MARIMA - dimension of data
> summary(fit)
$Tinner
        Name     Estimate          Pval Stars
1      AR.l1 -0.859286876 1.137500e-111   ***
2      AR.l2  0.000000000  0.000000e+00   ***
3      AR.l3 -0.516949911  6.491040e-10   ***
4      AR.l4  0.000000000  0.000000e+00   ***
5      AR.l5 -0.172800484  1.875606e-01
6      AR.l6  0.738818167  1.691412e-13   ***
7      AR.l7 -0.189610331  3.033485e-21   ***
8      Ta.l1  0.000000000  0.000000e+00   ***
9      Ta.l2  0.000000000  0.000000e+00   ***
10     Ta.l3  0.000000000  0.000000e+00   ***
11     Ta.l4  0.000000000  0.000000e+00   ***
12     Ta.l5  0.000000000  0.000000e+00   ***
13     Ta.l6  0.000000000  0.000000e+00   ***
14     Ta.l7  0.000000000  0.000000e+00   ***
15 Pinner.l1 -0.001604695 2.124560e-262   ***
16 Pinner.l2 -0.002985417 1.066623e-225   ***
17 Pinner.l3 -0.001377813  6.346201e-20   ***
18 Pinner.l4  0.000000000  0.000000e+00   ***
19 Pinner.l5  0.001567173  1.017739e-12   ***
20 Pinner.l6  0.002264162  8.232945e-70   ***
21 Pinner.l7  0.001639651  1.108603e-25   ***
22     MA.l1  0.620003460  1.671796e-29   ***
23     MA.l2  0.316923303  1.868655e-07   ***
24     MA.l4 -0.195118664  1.546549e-02     *
25     MA.l5 -0.314757947  1.409152e-05   ***
26     MA.l6  0.185998262  1.064005e-04   ***
27     MA.l7  0.088319309  2.403576e-02     *
```

Figure 17: Summary of ARMAX model of order 7, with the added Ta input parameter

3.4 Discuss the results, did you get the same results or is there some difference between the results for ARX and ARMAX

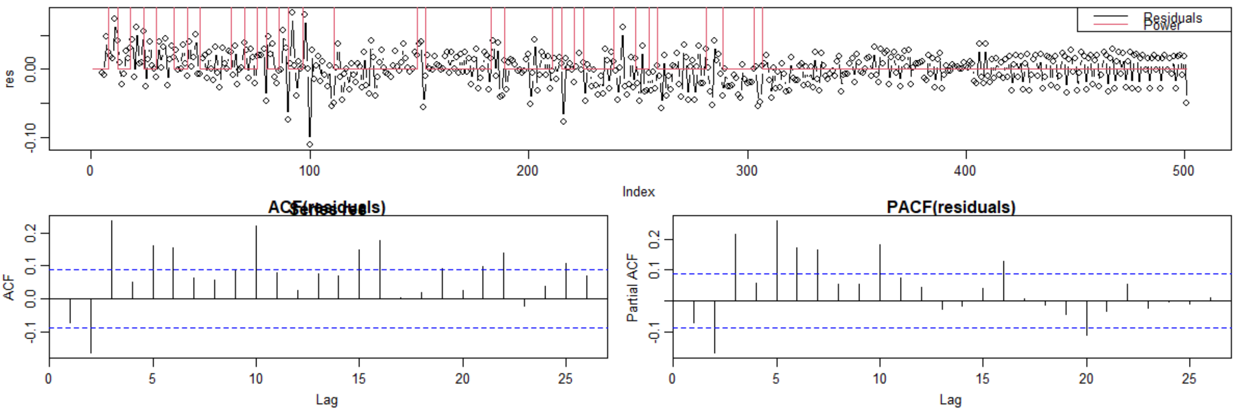

Figure 18: Validation plots ARX order 4

From Figures 14 and 15 we can conclude that the MA part efficiently nullifies most of the spikes, except lag 10 and lag 16, with the latter being persistent in all orders of ARMAX. In other words, the MA component in the ARX model addresses some of the autocorrelation that the ARX model fails to capture, possibly moving closer to the correct complexity needed to model the specific problem. All in all, there are significant differences between the results of the two models.

3.5 Make a multi-step prediction with the selected model

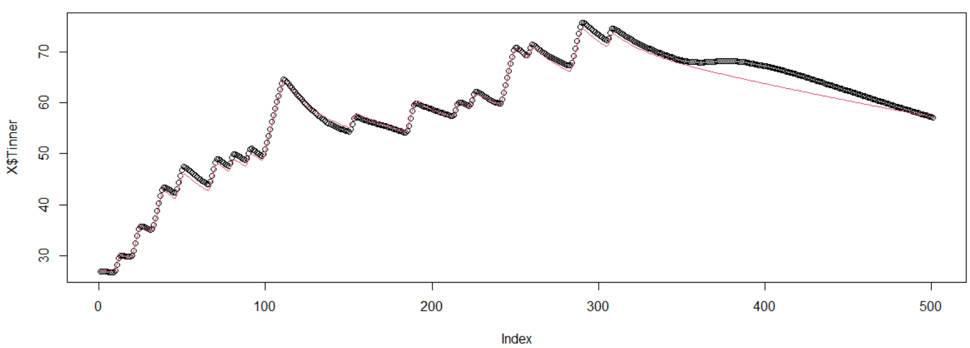

Figure 19: Multi-step prediction ARMAX order

As we can see in Figure 19, the model successfully predict the temperature of *Tinner* throughout the experiment.

3.6 Make some simulated 'step responses' of the temperature with a step from 0 to 100 W on the heat input

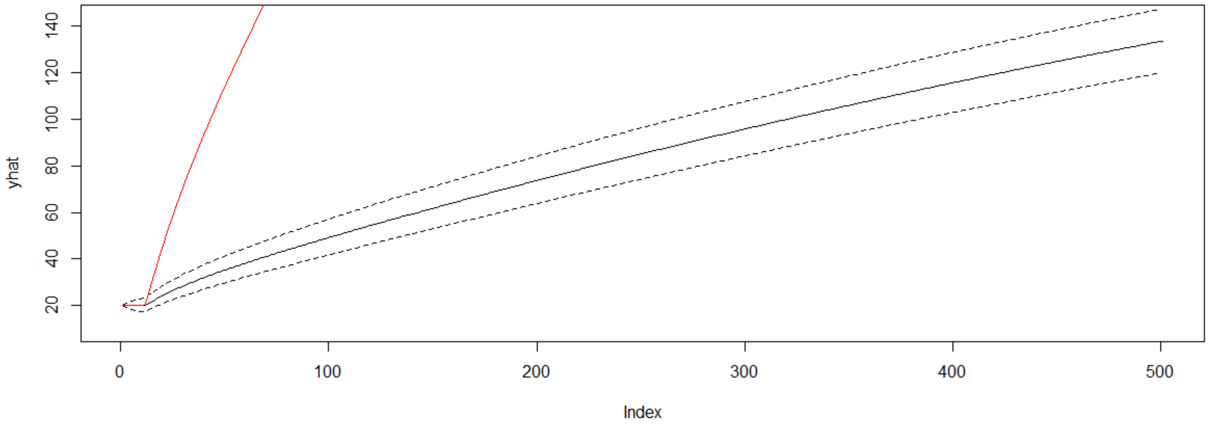

Figure 20: Simulated 'Step responses'

In this plot we can observe how the system behaves given the change in the data. In black, the simulated steps of 100W, and in red, more extreme ones of 500W. We can clearly see how after a little lag the temperature increases dramatically in a not very realistic way: the water reaches a 100 degrees with a third of the power of a regular kettle, which in turn suggests that the model overestimates the impact of the heater and we can only see it when the power is amplified. This might stem from an incorrect assumption of a linear relationship: transfer of energy from a heat source is not linear as transference rate decreases when temperature difference between the source and the system is reduced.

Another source of error related to the linearity assumption of the model is the physical nature of the model: it is clear that the temperature of a system made out of liquid water will not reach a temperature higher than a 100 degrees. Moreover, the temperature increase of such system will no increase linearly: the molecules that reach enough energy for evaporation will evaporate, leaving the system. Therefore the heating curve of water is not a line, but a curve with a lower inclination when approaching the mark of 100 degrees.

Finally, as water is evaporating, temperature cannot surpass said mark and when all the water is gone, temperature measured should drop to ambient temperature as all liquid water is gone. This is a behaviour that our model has not seen and therefore is ill prepared to predict.

4 ARMAX model of the pot water temperature

4.1 Identify an ARMAX model for the water temperature of the pot

We will start our analysis using the following ARMAX model that includes all the available inputs to our system:

```
fit <- marima("Touter ~ AR(1) + Tinner(1) + Pinner(1) + Pouter(1) + Ta(1) +
   MA(1)", data=X, penalty)\\
```

Again, our strategy to identify a suitable ARMAX model is to examine our models from the summary tables. Initially, we try to fit the models both with penalties 1 and 2 to review the best for this occasion.

After close inspection of the ACF and PACF charts, we conclude that the model with order 6 provides the best combination of performance and low complexity to use. In 21, it can be observed that even though the model using penalty 1 is slightly better, is also much more complex and has less significant parameters. Specifically, it uses 28 parameters while the model using penalty 2 only uses 20. Thus, we will continue using penalty 2. Additionally, it is also evident that *Pinner* is 0 for our chosen model and thus we will make another iteration of our model excluding this parameter.

```
> fit <- marima("Touter ~ AR(1:6) + Tinner(1:6) + Pinner(1:6) +
All cases in data,  1  to  501  accepted for completeness.
501 5  = MARIMA - dimension of data
> summary(fit)
$Touter
        Name      Estimate          Pval Stars
1      AR.l1 -1.324046e+00 1.557489e-280   ***
2      AR.l2  0.000000e+00  0.000000e+00   ***
3      AR.l3  0.000000e+00  0.000000e+00   ***
4      AR.l4  2.929898e-01  1.292012e-09   ***
5      AR.l5  8.570599e-02  1.312683e-01
6      AR.l6 -5.410361e-02  1.382465e-03    **
7  Tinner.l1 -3.002219e-02  1.029521e-01
8  Tinner.l2  6.452177e-02  7.725400e-02     .
9  Tinner.l3 -6.410937e-02  2.958210e-02     *
10 Tinner.l4  3.175554e-02  1.566512e-02     *
11 Tinner.l5  0.000000e+00  0.000000e+00   ***
12 Tinner.l6 -2.624565e-03  2.469973e-01
13 Pinner.l1 -2.815261e-05  1.075339e-01
14 Pinner.l2  8.313274e-05  8.760739e-02     .
15 Pinner.l3  0.000000e+00  0.000000e+00   ***
16 Pinner.l4  0.000000e+00  0.000000e+00   ***
17 Pinner.l5  0.000000e+00  0.000000e+00   ***
18 Pinner.l6  0.000000e+00  0.000000e+00   ***
19 Pouter.l1 -3.477997e-04  3.886245e-67   ***
20 Pouter.l2 -9.653317e-04 2.096044e-126   ***
21 Pouter.l3  0.000000e+00  0.000000e+00   ***
22 Pouter.l4  5.544506e-04  2.872398e-25   ***
23 Pouter.l5  5.030271e-04  3.222445e-38   ***
24 Pouter.l6  1.985723e-04  5.863812e-05   ***
25     Ta.l1 -3.130721e-02  1.335692e-01
26     Ta.l2  7.475704e-02  1.080256e-01
27     Ta.l3 -9.249825e-02  6.461355e-02     .
28     Ta.l4  6.100954e-02  4.627346e-02     *
29     Ta.l5  0.000000e+00  0.000000e+00   ***
30     Ta.l6 -1.201733e-02  1.754558e-01
31     MA.l1 -1.944577e-01  8.418246e-05   ***
32     MA.l2 -4.043202e-01  3.402288e-15   ***
33     MA.l3 -1.539170e-01  2.123301e-03    **
34     MA.l5 -1.030834e-01  3.357064e-02     *
```

```
> fit <- marima("Touter ~ AR(1:6) + Tinner(1:6) + Pinner(1:6) +
All cases in data,  1  to  501  accepted for completeness.
501 5  = MARIMA - dimension of data
> summary(fit)
$Touter
        Name      Estimate          Pval Stars
1      AR.l1 -1.3246376673 4.242314e-284   ***
2      AR.l2  0.0000000000  0.000000e+00   ***
3      AR.l3  0.0000000000  0.000000e+00   ***
4      AR.l4  0.2911378815  1.143814e-09   ***
5      AR.l5  0.0873371601  1.184583e-01
6      AR.l6 -0.0535014214  1.312792e-03    **
7  Tinner.l1  0.0000000000  0.000000e+00   ***
8  Tinner.l2  0.0000000000  0.000000e+00   ***
9  Tinner.l3 -0.0096125650  5.614413e-06   ***
10 Tinner.l4  0.0093801744  1.760201e-05   ***
11 Tinner.l5  0.0000000000  0.000000e+00   ***
12 Tinner.l6  0.0000000000  0.000000e+00   ***
13 Pinner.l1  0.0000000000  0.000000e+00   ***
14 Pinner.l2  0.0000000000  0.000000e+00   ***
15 Pinner.l3  0.0000000000  0.000000e+00   ***
16 Pinner.l4  0.0000000000  0.000000e+00   ***
17 Pinner.l5  0.0000000000  0.000000e+00   ***
18 Pinner.l6  0.0000000000  0.000000e+00   ***
19 Pouter.l1 -0.0003489350  1.484044e-68   ***
20 Pouter.l2 -0.0009667499 3.211909e-128   ***
21 Pouter.l3  0.0000000000  0.000000e+00   ***
22 Pouter.l4  0.0005570319  9.288082e-26   ***
23 Pouter.l5  0.0005058096  1.995285e-39   ***
24 Pouter.l6  0.0002037126  2.623790e-05   ***
25     Ta.l1 -0.0340364896  9.636368e-02     .
26     Ta.l2  0.0807320655  7.553178e-02     .
27     Ta.l3 -0.1000918932  4.079044e-02     *
28     Ta.l4  0.0672778855  2.542371e-02     *
29     Ta.l5  0.0000000000  0.000000e+00   ***
30     Ta.l6 -0.0140150130  1.112393e-01
31     MA.l1 -0.2016788409  4.173698e-05   ***
32     MA.l2 -0.3980978577  4.556511e-15   ***
33     MA.l3 -0.1498372608  2.541741e-03    **
34     MA.l5 -0.0951505500  4.703594e-02     *
```

Figure 21: Left: Summary of 6th order model using penalty=1, Right: Summary of 6th order model using penalty=2.

The new model showcases a similar performance with less complexity, specifically by using order 5 which can be seen in 22. The relevant summary is presented in 23.

```
fit <- marima("Touter ~ AR(1) + Pouter(1) + Ta(1) + MA(1)", data=X, penalty
    =2)
```

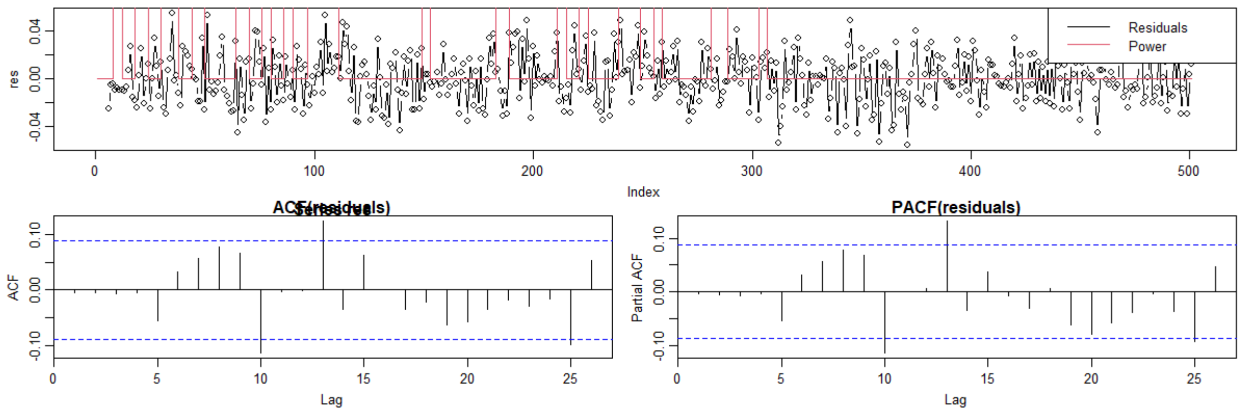

Figure 22: PACF, ACF and Residuals for the updated model of order 5.

```
> fit <- marima("Touter ~ AR(1:5) + Pouter(1:5) + Ta(1:5) + MA(1:5)", data=X, penalty=2)
All cases in data,  1  to  501  accepted for completeness.
501 3   = MARIMA - dimension of data
> summary(fit)
$Touter
        Name      Estimate         Pval Stars
1      AR.l1 -1.6437557211 6.074991e-71   ***
2      AR.l2  0.4260004681 2.281289e-04   ***
3      AR.l3  0.0000000000 0.000000e+00   ***
4      AR.l4  0.3231583758 8.799219e-09   ***
5      AR.l5 -0.1052719326 2.642306e-09   ***
6  Pouter.l1 -0.0003502872 4.155604e-63   ***
7  Pouter.l2 -0.0008615432 6.083656e-70   ***
8  Pouter.l3  0.0003216954 2.347675e-05   ***
9  Pouter.l4  0.0005354102 5.059182e-33   ***
10 Pouter.l5  0.0003261693 9.108505e-08   ***
11     Ta.l1 -0.0474573656 3.399418e-02     *
12     Ta.l2  0.1121718714 3.147476e-02     *
13     Ta.l3 -0.1428690898 2.233299e-02     *
14     Ta.l4  0.1181885952 2.407974e-02     *
15     Ta.l5 -0.0403199842 7.376393e-02     .
16     MA.l1 -0.3907089228 1.685917e-05   ***
17     MA.l2 -0.2733711819 2.020856e-07   ***
18     MA.l4  0.0912778203 8.165063e-02     .
19     MA.l5 -0.0664187620 6.051458e-02     .
```

Figure 23: Summary for model of order 5.

Additionally, after noticing the relatively low significance of the *Tinner* parameter, we tried excluding it from the same model and found out that the performance is the same. In other words, we could keep that iteration of our model which performs equally.

4.2 Compare the two identified ARMAX models

In the case of the inner temperature, we have chosen an ARMAX model with inputs $Tinner$ and $Pinner$ and an MA component all of them with lag 3.

In the case of the outer temperature, we have chosen an ARMAX model with inputs $Touter$, $Pouter$, Ta and an MA component all of them with lag 5.

Interestingly enough, the model for the outer pot benefits from a greater complexity and seems to be more susceptible to factors like ambient temperature, possibly because of a greater surface, making it more unstable.

On the other hand, the model for $Tinner$ seems to give more weight to the previous temperature of the inner pot, while the model for $Touter$ seems to distribute those weights more amongst different parameters even if these parameters do not hold that much significance. Both models share the similarity of giving the greatest weight of the model to the previous temperature. This makes sense since the specific heat capacity of water is especially high and therefore there are few abrupt changes in it.

5 ARMAX multi-output model

5.1 Use marima to estimate an ARMAX multi-output model of the coupled system.

Having chosen the best models of ARMAX both for *Tinner* and for *Touter* we can now create the multi-output model combining those. The inputs and order of the specific models have been combined as shown below:

```
fit <- marima("Tinner ~ AR(1:7) + Ta(1:7) + Pinner(1:7) + MA(1:7)",
"Touter ~ AR(1:5) + Pouter(1:5) + Ta(1:5) + MA(1:5)", data=X,penalty=2)
```

$Tinner

	Name	Estimate	Pval	Stars
1	AR.l1	-1.1931828338	0.000000e+00	***
2	AR.l2	0.0000000000	0.000000e+00	***
3	AR.l3	0.0000000000	0.000000e+00	***
4	AR.l4	0.0000000000	0.000000e+00	***
5	AR.l5	0.0000000000	0.000000e+00	***
6	AR.l6	0.2692409961	3.476318e-157	***
7	AR.l7	-0.0759426269	7.605626e-39	***
8	Ta.l1	0.0000000000	0.000000e+00	***
9	Ta.l2	0.0000000000	0.000000e+00	***
10	Ta.l3	0.0000000000	0.000000e+00	***
11	Ta.l4	0.0000000000	0.000000e+00	***
12	Ta.l5	0.0000000000	0.000000e+00	***
13	Ta.l6	0.0000000000	0.000000e+00	***
14	Ta.l7	0.0000000000	0.000000e+00	***
15	Pinner.l1	-0.0016348518	5.227057e-290	***
16	Pinner.l2	-0.0023792926	1.850631e-317	***
17	Pinner.l3	0.0000000000	0.000000e+00	***
18	Pinner.l4	0.0009412624	6.324872e-105	***
19	Pinner.l5	0.0010297426	5.880428e-118	***
20	Pinner.l6	0.0010816744	2.024517e-101	***
21	Pinner.l7	0.0006312351	3.705995e-57	***
22	MA.l1	0.3089210648	8.047766e-11	***
23	MA.l2	-0.0904089663	5.076269e-02	.
24	MA.l3	0.0668799626	1.518284e-01	
25	MA.l4	-0.1032454863	2.694343e-02	*
26	MA.l5	-0.0914362760	4.967793e-02	*
27	MA.l6	0.1253173573	7.747595e-03	**

$Touter

	Name	Estimate	Pval	Stars
1	AR.l1	-1.6536359317	6.249226e-73	***
2	AR.l2	0.4403167101	1.084982e-04	***
3	AR.l3	0.0000000000	0.000000e+00	***
4	AR.l4	0.3167076904	9.550184e-09	***
5	AR.l5	-0.1032705683	3.436061e-09	***
6	Ta.l1	-0.0492982034	2.805234e-02	*
7	Ta.l2	0.1162788136	2.629364e-02	*
8	Ta.l3	-0.1469364829	1.925517e-02	*
9	Ta.l4	0.1212972096	2.111438e-02	*
10	Ta.l5	-0.0416110854	6.600688e-02	.
11	Pouter.l1	-0.0003469753	1.195131e-61	***
12	Pouter.l2	-0.0008609712	1.053396e-70	***
13	Pouter.l3	0.0003293734	1.247546e-05	***
14	Pouter.l4	0.0005357433	3.718258e-33	***
15	Pouter.l5	0.0003209384	9.806198e-08	***
16	MA.l1	-0.3986881941	8.876325e-06	***
17	MA.l2	-0.2676117864	2.014657e-07	***
18	MA.l4	0.0815623621	1.181241e-01	
19	MA.l5	-0.1230160235	9.523001e-03	**

Figure 24: Summary of the Multi-Output model, on the left is the part of the *Tinner*, and on the right is the part of the *Touter*.

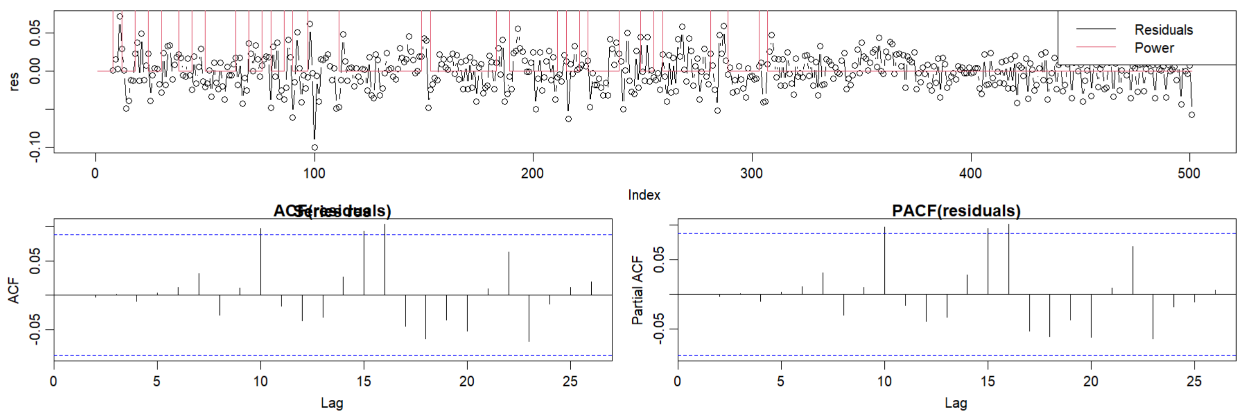

Figure 25: The ACF, PACF and Residuals are presented here.

5.2 Identify the order.

After decreasing and increasing the order equally for the 2 outputs of the model, we intend to identify what model could provide even better results. The available pairs of orders are (4,2), (5,3), (6,4), (7,5), (8,6). From the results, we can see that most of those options provide very good noise-resembling performance and relatively high significance of the parameters. Thus, we ended up choosing the (5,3) pair for *Touter* and *Tinner* correspondingly, which has even less complexity than our initial model and performs even better.

```
fit <- marima("Tinner ~ AR(1:5) + Ta(1:5) + Pinner(1:5) + MA(1:5)",
"Touter ~ AR(1:3) + Pouter(1:3) + Ta(1:3) + MA(1:3)", data=X,penalty=2)
```

$Tinner

	Name	Estimate	Pval	Stars
1	AR.l1	-1.6719268394	1.954881e-84	***
2	AR.l2	0.4980707751	4.703367e-06	***
3	AR.l3	0.0000000000	0.000000e+00	***
4	AR.l4	0.2444475477	1.461995e-06	***
5	AR.l5	-0.0705424061	3.524545e-08	***
6	Ta.l1	0.0000000000	0.000000e+00	***
7	Ta.l2	0.0000000000	0.000000e+00	***
8	Ta.l3	0.0000000000	0.000000e+00	***
9	Ta.l4	0.0000000000	0.000000e+00	***
10	Ta.l5	0.0000000000	0.000000e+00	***
11	Pinner.l1	-0.0016230089	6.408156e-277	***
12	Pinner.l2	-0.0016355394	7.761332e-37	***
13	Pinner.l3	0.0013140212	4.832100e-22	***
14	Pinner.l4	0.0012159878	1.469007e-24	***
15	Pinner.l5	0.0005826049	2.126724e-06	***
16	MA.l1	-0.1700062231	4.220542e-02	*
17	MA.l2	-0.3159187972	3.664779e-11	***
18	MA.l5	0.1020542683	8.669368e-03	**

$Touter

	Name	Estimate	Pval	Stars
1	AR.l1	-2.2786013229	2.220109e-172	***
2	AR.l2	1.6232568797	3.770806e-76	***
3	AR.l3	-0.3442761921	4.939330e-42	***
4	Ta.l1	-0.0205347010	1.588946e-01	
5	Ta.l2	0.0198221431	1.724335e-01	
6	Ta.l3	0.0000000000	0.000000e+00	***
7	Pouter.l1	-0.0003606911	8.568041e-60	***
8	Pouter.l2	-0.0006286457	2.225991e-52	***
9	Pouter.l3	0.0008704698	4.330071e-49	***
10	MA.l1	-0.9367269481	6.060709e-36	***
11	MA.l2	0.0986740260	3.549929e-02	*
12	MA.l3	0.2794590239	3.123872e-09	***

Figure 26: Summary of the Multi-Output model, on the left is the part of the *Tinner*, and on the right is the part of the *Touter*.

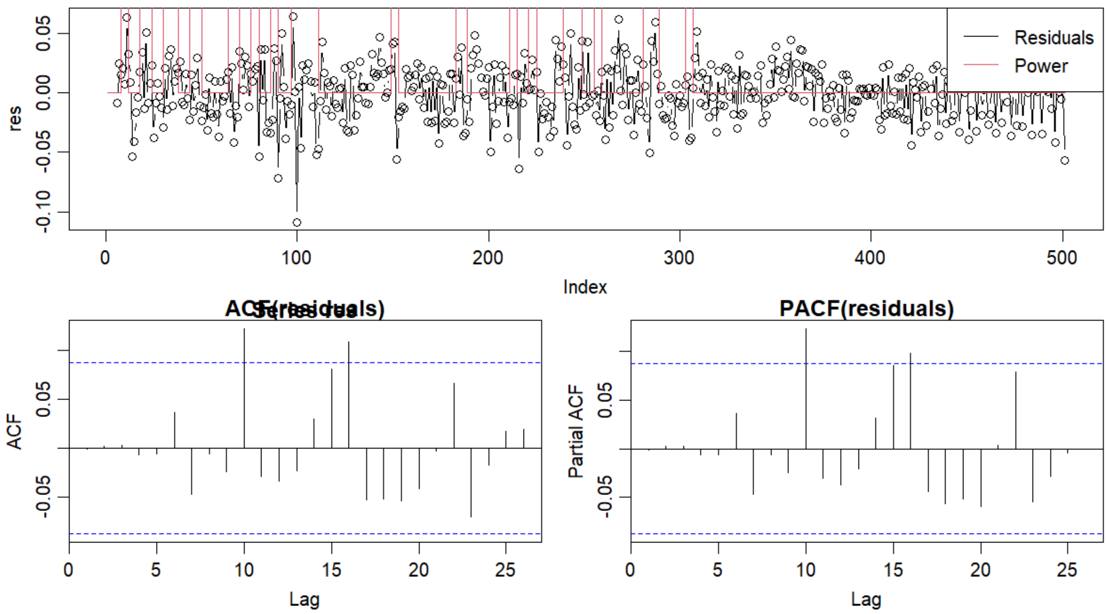

Figure 27: The ACF, PACF, and residuals for the model of order (5,3) are presented here.

5.3 Make a multi-step prediction with the selected model. Can it predict both temperatures throughout the experiment?

After choosing the multi-output model of orders (5,3) as our best one, we will do a multi-step prediction using it.

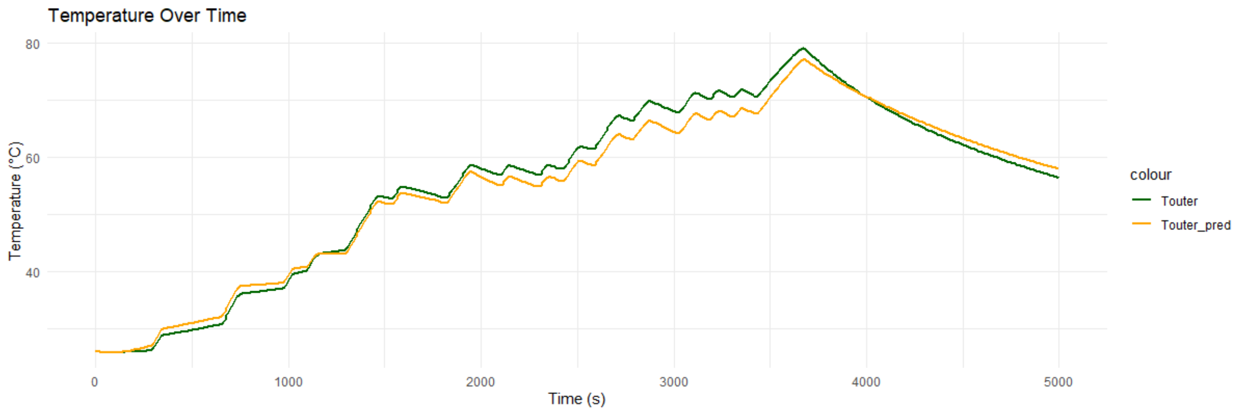

Figure 28: The plot depicts the prediction of our chosen model for $Touter$.

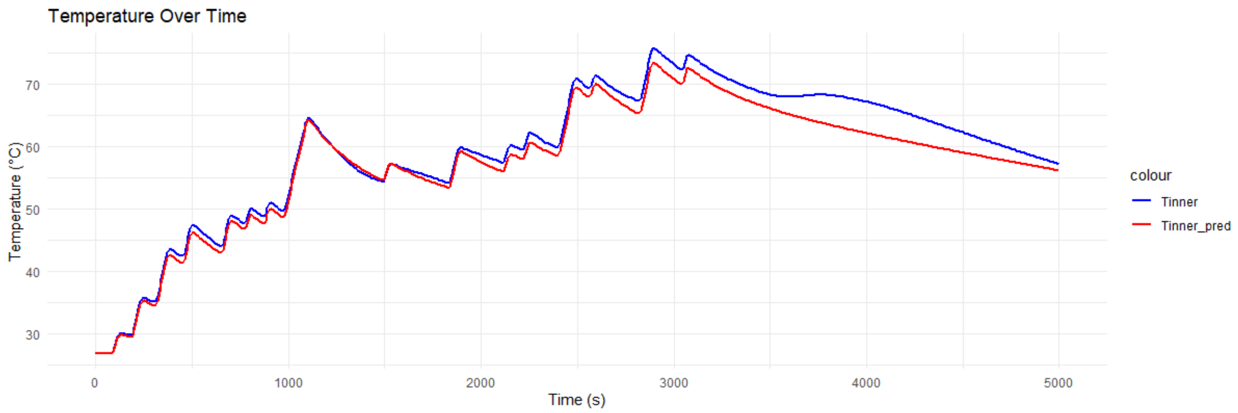

Figure 29: The plot depicts the prediction of our chosen model for $Tinner$.

By observing the two plots, we can see that the prediction lines for Touter and Tinner seem to follow the actual temperature quite closely. The chosen model estimates well the temperature from both sensors during the experiment. The predicted temperatures align well with the real measurements, but there are a few minor mismatches. For instance, the model is a bit slow to catch up with the highest point in the Touter readings, and at the last points where the actual Tinner temperature shows a quick increase or decrease, the model's predicted line might show a more gradual change.

5.4 Simulation of step response from one input to both temperatures simultaneously. Present three interesting simulations.

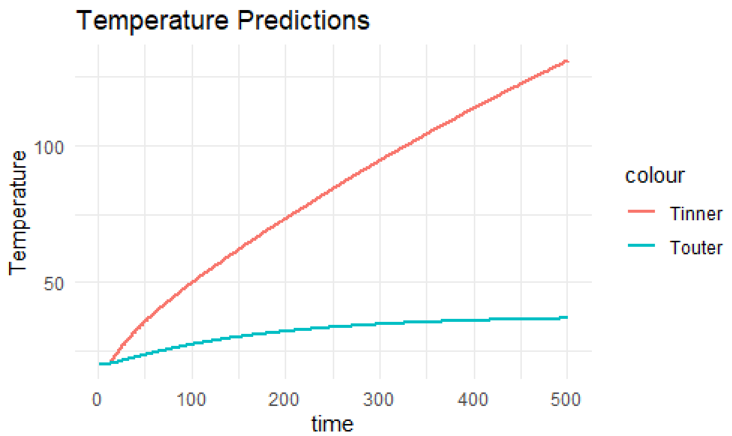

Figure 30

In the first simulation, Figure 30, Tinner climbs steadily. On the other hand, *Touter* seems to have increased slightly a sign that our model captures the interactive effects between the inner and outer systems.

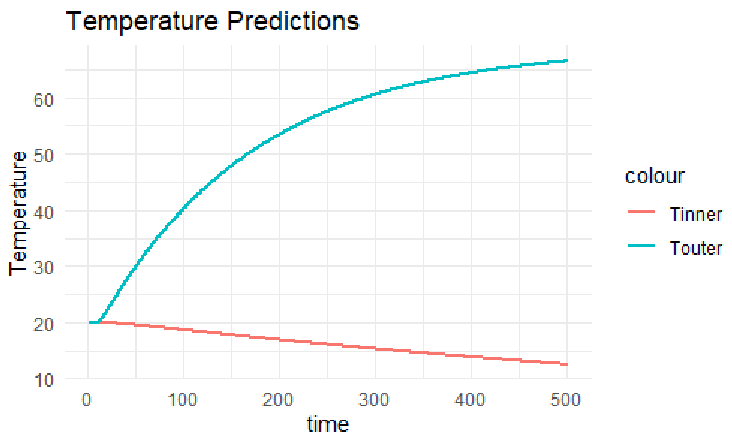

Figure 31

In the second simulation, *Touter* quickly goes up and levels off, showing that this part reacts fast to heat changes and then holds steady. On the other hand, *Tinner* decreases, which is something we expect since in our model we don't consider *Pouter* and *Touter* as factors that affect *Tinner*. So it's logical to expect *Tinner* to increase only if *Pinner* is on.

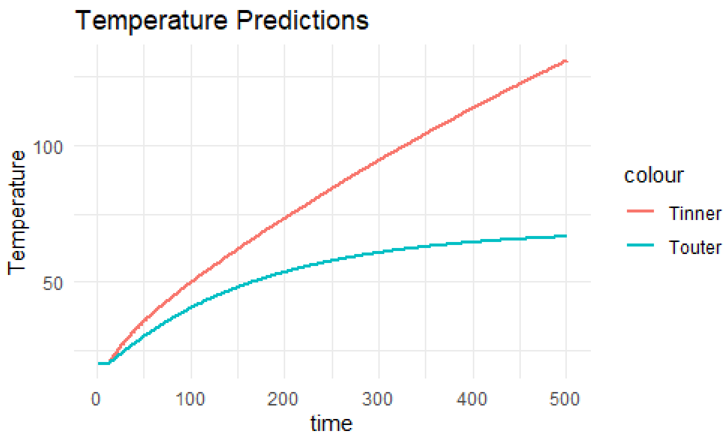

Figure 32

In the third simulation, both $Tinner$ and $Touter$ increase, with the first showing a sharp and constant increase, while the latter has a much more gradual rate. This is due to the interaction of $Touter$ with Ta.

5.5 Discuss the pros and cons of the coupled model over the two independent models.

Our individual models previously fell short in capturing the interactive effects between the inner and outer systems, an interaction we know exists based on the physical properties of the system. On the other hand, the multi-output model designed to analyze both systems concurrently is better at capturing the relationship between the inner and outer temperature systems, which could lead to better predictions than looking at them separately.

Furthermore, the multi-output approach helps to reduce noise that was previously misinterpreted by our models as part of the system's behavior, leading to overfitting through the inclusion of excessive lags. By accurately capturing these subtleties, the coupled model requires fewer lags, reducing the model's complexity and consequently the likelihood of overfitting.

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