Expanding Graphs

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Abstract

In this essay we shall explore methods used for eigenvalue estimates leading to the definition of Ramanujan graphs, and gain a first insight into explicit constructions of families of expanding graphs.

Introduction

It appears not to be easy to find out for which purpose or by whom expanders were first introduced; in the literature this aspect seems to be confined to the note that M. Pinsker was the first to prove existence of families of expanders in 1973 (cf [5]). Instead of a historical placement of expanders we will therefore try to intuitively understand their definition starting from the frequently cited assertion that expanders are related to problems of network design[1].

A network is understood as a number of vertices sending and receiving units of data via edges connecting them. Edges all have the same capacity, and additionally they are made of rather expensive wire. What kind of properties should a good network have? Among possible graphs on n vertices we are looking for graphs with low diameter (as transferring data takes longer the longer the way it needs to travel) and restricted degree (taking account of the fact that wire is expensive). Furthermore, since capacity is restricted, it matters how well disjoint subsets of vertices are connected, i.e. how many edges or disjoint paths connect them. If two large disjoint sets are connected by only few edges, this will not give a good "connection". How, however, can we measure this kind of "connectivity"? One possibility is to consider a ratio of what amount of data can leave an arbitrary subset S of the set of vertices compared to the amount of data inside S, i.e. the ratio of the number of edges leaving S (or the order of the neighbourhood of S in G) to the order of S. To obtain useful information it is necessary to somewhat restrict the set of sets S in consideration since for large S we will get a small ratio just because there are not enough possible edges or neighbours left. With these thoughts of what we would like to measure, we will now give the definitions of the isoperimetric and expansion constant: for $F \subset V$ let ∂F denote the set $e(F, V \setminus F)$. Then the isoperimetric constant $h(X)$, as defined in Davidoff, Sarnak & Valette [5], is given by

$$h(X) = \inf \left\{ \frac{|\partial F|}{\min(|F|, |V \setminus F|)} \mid F \subset V, 0 < |F| < \infty \right\} .$$

Following the definion of expander Alon and Spencer [2] give one can define the expansion constant to be the supremum of all $c \in \mathbb{R}^+$ such that

$$|N(F)| \geq c\,|F|$$

[1] See the introductory chapter of [5] for a similar account.

for all $F \subset V$ with $|F| \leq \frac{1}{2}|V|$. A d-regular graph of order n is called an (n, d, c)-*expander* if its expansion constant is at least c.

As every connected d-regular graph is an expander for some c chosen small enough, it is more valuable to consider *families of expanders*, i.e. families $(G_n)_{n\in\mathbb{N}}$ of d-regular graphs with $|G_n| \to \infty$ as $n \to \infty$ and expansion constant strictly greater than some fixed positive constant c.

It should be mentioned that the use of the notion of 'expander' is not consistent in the literature. What is called 'an expander' here is sometimes (cf West [12]) called 'enlarger' (while 'expander' has a different meaning). It also varies which of the above properties is used for definition. However, when considering d-regular graphs for fixed d, it is of no essential difference whether we take the isoperimetric or the expansion constant in order to define expanders: each of them can be bounded in terms of the other and d.

Much more noticeable is that some authors consider bipartite regular graphs on $2n$ vertices instead of graphs as in the above definition. For instance the probabilistic proof of existence of expanders, as can be found in West [12] (as guided exercise) or Lubotzky [7], seems to be easier to carry out for bipartite graphs. A d-regular bipartite graph with partition classes I and O of order n is said to be an (n, d, c)-expander if for all subsets $A \subset I$ of order at most $\frac{n}{2}$ the neighbourhood satisfies $N(A) \geq (1 + c)|A|$.

Surprisingly, Lubotzky [7] claims that this is no restriction after all: one can pass from one notion to the other. He does not give all details, in particular not for the interesting direction from bipartite regular graphs to regular graphs on half the number of vertices.

Given a d-regular graph G on n vertices, we obtain a bipartite d-regular graph on n vertices by adjoining a copy G' of G, and replacing corresponding copies $uv \in E(G)$ and $u'v' \in E(G')$ of edges by uv' and uv'. Taking into account that the expansion in the bipartite graph is of the form $(1 + c)$, edges uu' connecting the two copies of each original vertex are added as well. Thus in the new $(d + 1)$-regular graph the neighbours of a vertex $v \in V(G)$ are exactly the copies in G' of its original neighbours and its own copy v'. $V(G)$ and $V(G')$ are the partition classes. This construction obviously respects the expansion properties: the neighbourhood of a subset $A \subset V(G)$ in the new graph may be written as $A' \cup N(A)'$.

For the other direction: given a bipartite d-regular graph G on $2n$ vertices, we seek a regular graph on n vertices. The idea is to try to 'reverse' the previous construction by identifying pairs of vertices containing one from each class using Hall's Theorem. Unfortunately, [7] does not describe how this should work. As Hall's condition is satisfied by regular bipartite graphs we find a perfect matching in G or $\tilde{G}$, the bipartite graph whose partition classes are those of G with edges exactly the non-edges of G between the classes. Identifying matching pairs in G leaves us with a $(2d - 1)$- or $2d$-regular multigraph.

How do we obtain a *regular and simple* graph? A possible answer might be to consider matchings in G and restrict the attention to expanders with girth at least 5.

Likewise some other graph invariants, the isoperimetric constant as defined above is rather hard to determine; in order to determine the infimum one would need to go over half of all possible subsets of V. We would like to find expressions easier to work with than this infimum. There are various ways to estimate the expansion constant. One result—see Alon and Spencer [2, Thm 9.2.1] for instance—in this direction is

Theorem 1. *Let $G = (V, E)$ be a d-regular graph of order n and let λ_1 be the second largest eigenvalue of the adjacency matrix of G. Then for every $W \subset V$*

$$\frac{|e(W, V \smallsetminus W)|}{|W|} = \frac{|\partial W|}{|W|} \geq \frac{d - \lambda_1}{n} |V \smallsetminus W| .$$

This theorem implies that $h(G) \geq \frac{d-\lambda_1}{2}$. Thus it does give a lower bound which can be computed efficiently given the adjacency matrix of a graph. It does not tell how good this bound is, nor how much $h(G)$ and $d - \lambda_1$ correlate, however, it suggests some questions:

- How in general does the second largest eigenvalue of the adjacency matrix relate to common parameters of graphs?
- If there are such consequences, what are the structural consequences to graphs forced by a large *spectral gap*, $d - \lambda_1$?
- How can we estimate λ_1?

The remainder of this essay is divided into two sections. The first one is mainly devoted to estimates of λ_1—which inevitably involve aspects of the first two questions. Estimates chosen are an inequality due to Chung relating the essential spectral radius $\lambda(G)$—that is, the maximum absolute value of non-trivial eigenvalues—and the diameter of G, Nilli's lower bound on $\lambda_1(G)$ yielding the extremal property of Ramanujan graphs and Cioabă's elementary proof of Serre's unpublished result improving Nilli's theorem.

The second section makes a first step towards explicit constructions of Ramanujan graphs—one of the most prevailing aims in this area. We will follow Lubotzky's presentation given in [7] of a construction due to Alon and Milman.

1 General bounds on λ_1 and $\lambda(X)$

Some ideas or methods used for eigenvalue estimates are quite common. We will introduce the methods of the first two theorems by making little observations first.

1.1 Chung's bound

1.1.1 A method to obtain a bound on the diameter

The graphs we are looking for should not have unnecessarily large diameter. Let k denote the diameter of a graph G with adjacency matrix A. A well known way—which may be found in Biggs [3]—to obtain a bound on k is this one. As $[A^\ell]_{ij}$ counts the number of $i - j$-walks of length ℓ in G, this entry is 0 unless the distance $d(i, j)$ is at most ℓ. In G we find vertices v_0 and v_k of maximum distance $k = \operatorname{diam}(G)$. Let $v_0 v_1 \dots v_k$ be a $v_0 - v_k$-path of minimum length. Considering the entries $[A^j]_{v_0, v_r}$, $r \leq k$, we find that $I, A, \dots, A^k$ are linearly independent, i.e. the *adjacency algebra* has dimension strictly greater than the diameter of G.

Since A is symmetric, the minimum polynomial associated to A has only simple roots and (by the Cayley-Hamilton theorem) divides the characteristic polynomial of A. Thus, the dimension of the adjacency algebra equals the number of different eigenvalues of the adjacency matrix, and in particular we obtain: the number of different eigenvalues is a strict upper bound on the diameter of G.

1.1.2 Chung's bound

Since G is d-regular its largest eigenvalue equals d and corresponds to constant eigenfunctions (cf Biggs [3, Prop. 3.1]). If G is not bipartite, all other eigenvalues λ_i satisfy $|\lambda_i| < d$. Using this fact, there is a nice inequality due to F.R.K. Chung, see [9], relating for a d-regular non-bipartite graph G

$$\lambda(G) := \max\{|\lambda_i| \mid |\lambda_i| < d\}$$

to the diameter k, which in particular implies that graphs with large spectral gap, i.e. small $\lambda(G)$, have small diameter.

Picking up the ideas from the previous argument, a natural number r is an upper bound on the diameter k of a graph corresponding to A if A^r has no entry "zero".

In order to pass on to eigenvalues, let $u_0, u_1, \ldots, u_{n-1}$ be an orthonormal basis of eigenvectors of A and $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$ the corresponding eigenvalues. We may assume that $d = \lambda_0 > \lambda_1 \geq \cdots \geq \lambda_{n-1} > -d$ and $u_0 = \frac{1}{\sqrt{n}}(1, \ldots 1)^t$. Thus

$$A^r = \sum_{i=0}^{n-1} \lambda_i^r u_i u_i^t \ .$$

The idea is now to use the inequalities $[A^k]_{x,y} > 0$ for all entries (x, y), and the fact that all but one eigenvalue, namely d, lie within the $\lambda(G)$-ball around 0, to obtain a bound on k.

By choice of u_0, the triangle inequality, Cauchy-Schwarz, and normality of the basis

$$\begin{aligned}
[A^r]_{x,y} &\geq \frac{d^r}{n} - \left| \sum_{i>0}^{n-1} \lambda_i^r (u_i)_x (u_i)_y^t \right| \\
&\geq \frac{d^r}{n} - \lambda^r(G) \left| \sum_{i>0}^{n-1} (u_i)_x (u_i)_y^t \right| \\
&\geq \frac{d^r}{n} - \lambda^r(G) \left(\sum_{i>0}^{n-1} (u_i)_x^2 \right)^{1/2} \left(\sum_{i>0}^{n-1} (u_i)_y^2 \right)^{1/2} \\
&= \frac{d^r}{n} - \lambda^r(G) \left(1 - (u_0)_x^2\right)^{1/2} \left(1 - (u_0)_y^2\right)^{1/2} \\
&= \frac{d^r}{n} - \lambda^r(G) \left(1 - \frac{1}{n}\right) \ .
\end{aligned}$$

Thus, if r satisfies

$$\begin{aligned} & \frac{d^r}{n} - \lambda^r(G)\left(1 - \frac{1}{n}\right) > 0 \\ \Leftrightarrow \quad & d^r > \lambda^r(G)\,(n-1) \\ \Leftrightarrow \quad & r > \frac{\log(n-1)}{\log(\frac{d}{\lambda(G)})} \end{aligned}$$

then r is at least as big as the diameter.

We proved that

$$r = \frac{\log(n-1)}{\log(\frac{d}{\lambda(G)})} + 1$$

is an upper bound for the diameter. Having $\lambda(G)$ small implies a small upper *bound* on the diameter.

1.2 Nilli's theorem

1.2.1 The Rayleigh method and a proof of Theorem 1

In order to obtain further connections and bounds on eigenvalues, some insight into the actual meaning of an eigenvalue is needed. This is provided by linear algebra in form of the Rayleigh quotient.

Let $A \in \mathrm{M}(n, \mathbb{R})$ be a real symmetric matrix. Then

$$\lambda_i = \min_{v \perp e_{n-1}, \ldots, e_{i+1}} \frac{\langle Av, v\rangle}{\langle v, v\rangle} \ ,$$

where $\lambda_{n-1} \leq \lambda_{n-2} \leq \cdots \leq \lambda_0$ are the eigenvalues of A and $e_{n-1}, \ldots e_0$ are the corresponding eigenvectors.

This means, to obtain an upper bound on λ_i all we have to do is to evaluate the quotient $\frac{\langle Av,v\rangle}{\langle v,v\rangle}$ at some $v \in \{u \mid u \perp e_{n-1}, \ldots, e_{i+1}\}$.

Since our graphs are all d-regular we may consider the Laplacian $L = dI - A$ instead of the adjacency matrix without loosing information on the eigenvalues of A, as L has eigenvalues $0 = d - \lambda_0 < d - \lambda_1 \leq \cdots \leq d - \lambda_{n-1}$. The eigenfunctions corresponding to 0 are the constants. Therefore, any specific non-zero v, such that $\langle v, \mathbf{1}\rangle = 0$, where $\mathbf{1} = (1, \ldots, 1)^t$, gives an upper bound

$$\frac{\langle Lv, v\rangle}{\langle v, v\rangle} \geq d - \lambda_1 \ .$$

As L contains all information about our graph it should be possible to use this inequality to relate λ_1 to more obvious properties of the graph. There is one frequently employed choice of v which will be developed now. Firstly, we would like to simplify $\langle Lv, v\rangle$ and express it in more familiar terms:

Taking the entries of L and v to be indexed by the vertices of our graph, we can write (cf [2, p.138])

$$\begin{aligned}
\langle Lv, v\rangle &= \sum_{i,j\in V(G)} v_i v_j L_{i,j} = \sum_{i\in V(G)} \{dv_i^2 - \sum_{\substack{j\in V(G)\\ ij\in E(G)}} v_i v_j\} \\
&= \sum_{i\in V(G)} dv_i^2 - 2\sum_{ij\in E(G)} v_i v_j \\
&= \sum_{ij\in E(G)} (v_i - v_j)^2 \ . \qquad (1)
\end{aligned}$$

Note that there is nothing special about the regular case here. This equally holds in general when d denotes the average degree.

The last expression shows that edges within a class of vertices on which v is constant make no contribution towards $\langle Lv, v\rangle$. In view of (1), $\langle Lv, v\rangle$ can be used to count edges between classes on which v is constant. This gives us the possibility to include the expansion rate: recalling its definition, what we need is a function which returns $|\partial A|$, i.e. counts edges between two classes $A \subset V$ and $V \setminus A$. It has to be constant on each class with constants chosen such that the arithmetic mean is 0. As we do not need to worry about scalar factors this naturally leads to

$$v_i = \begin{cases} n-a & \text{if} \quad i\in A \ , \\ -a & \text{if} \quad i \in V\setminus A \ , \end{cases}$$

where $a = |A|$.

Putting things together, we obtain

$$\begin{aligned}
d - \lambda_1 &\le \frac{\langle Lv, v\rangle}{\langle v, v\rangle} \\
&= \frac{\sum_{ij\in E(G)} (v_i - v_j)^2}{a(n-a)^2 + (n-a)a^2} \\
&= \frac{n^2\,|e(A, V\setminus A)|}{a(n-a)n} \\
&= \frac{n\,|\partial A|}{a(n-a)} \ ,
\end{aligned}$$

completing a proof for the Theorem 1 stated in the introduction. In particular, if we pick A such that $a \le \frac{1}{2}n$, we obtain

$$\frac{d-\lambda_1}{2} \le \inf_{A:a\le\frac{1}{2}n} \left\{(d-\lambda_1)\frac{n-a}{n}\right\} \le \inf_{A:a\le\frac{1}{2}n} \left\{\frac{|\partial A|}{a}\right\} = h(G) \ .$$

1.2.2 Nilli's theorem

In [10] A. Nilli gives an amazing refinement of this technique. His approach basically uses the same idea as above but with a much subtler choice of function. The plan is to define a function on two disjoint sets A, B of vertices, again. If we do not insist on the function being constant on each class, (1) suggests that things become easier when choosing A and B such that none of them has neighbours in the other. Let us see why this is so:

If we wish to obtain an upper bound for $d - \lambda_1$ by the Rayleigh method, we need to make sure that our test function f (which was called v before) has mean 0. This can be obtained by choosing non-zero weights a and b for $f|_A$ and $f|_B$ (the restrictions to each of the disjoint classes). However, by this 'abstract' choice we have no control over what happens, thus seek an estimate of the eigenvalue to which none of the weights contributes. If A and B, however, satisfy the above disjointness criterion, both $\langle Lf, f\rangle$ and $\langle f, f\rangle$ naturally split into sums according to the two vertex sets. By bilinearity, these are of the form $a^2\tilde{f}_1 + b^2\tilde{f}_2$ and $a^2 f_1 + b^2 f_2$ and we may employ the following inequality on sums in fractions:

$$\text{(2)} \qquad \max\left\{\frac{\tilde{f}_1}{f_1}, \frac{\tilde{f}_2}{f_2}\right\} = \max\left\{\frac{a^2\tilde{f}_1}{a^2 f_1}, \frac{b^2\tilde{f}_2}{b^2 f_2}\right\} \geq \frac{a^2\tilde{f}_1 + b^2\tilde{f}_2}{a^2 f_1 + b^2 f_2} .$$

This bound, indead, is independent of a and b.

In order to estimate $\langle Lf, f\rangle$ later, we will need a suitable way to describe how edges in A and B lie with respect to which values f takes on their endvertices. A convenient way to describe the set of edges is to split the graphs $G_{[A\cup\Gamma(A)]}$ and $G_{[B\cup\Gamma(B)]}$ into classes V_A^i (resp. V_B^i) of common distance i from a fixed edge[2] in A (resp. B). Edges either lie within one class or connect subsequent ones. This gives us a much better way to get hold of where edges lie than just considering the adjacency matrix. In addition, we are going to ignore those edges within classes by making f constant there.

Also, we get an estimate on the size of classes for free. A vertex has at most $d-1$ neighbours in its successor class. Therefore, $\left|V_A^i\right| \leq (d-1)\left|V_A^{i-1}\right|$ holds for all $0 < i$ (similarly for B).

By inequality (2) and symmetry it suffices to consider the quotient $\frac{f_1}{\tilde{f}_1}$. Put

$$f(x) = (d-1)^{-i/2} \quad \text{if} \quad x \in V_A^i .$$

Then

$$\langle f|_A, f|_A\rangle = \sum_{i=0}^{k} \frac{\left|V_A^i\right|}{(d-1)^{2(i/2)}} ,$$

where k is maximum such that $V_A^i \neq \emptyset$, and

$$\text{(3)} \qquad \langle Lf|_A, f|_A\rangle \leq \sum_{i=0}^{k-1} \left|V_A^i\right|(d-1)\left((d-1)^{-i/2} - (d-1)^{-(i+1)/2}\right)^2 + \left|V_A^k\right|\frac{d-1}{(d-1)^k},$$

since $E(V_A^i, V_A^{i+1}) \leq \left|V_A^i\right|(d-1)$.

The RHS of (3) equals

$$\sum_{i=0}^{k-1} \frac{\left|V_A^i\right|}{(d-1)^i}\left((d-1)^{1/2} - 1\right)^2 + \left|V_A^k\right|\frac{d-1}{(d-1)^k}$$

$$= \sum_{i=0}^{k-1} \frac{\left|V_A^i\right|}{(d-1)^i}(d - 2\sqrt{d-1}) + \left|V_A^k\right|\frac{d-1}{(d-1)^k} .$$

[2] It might be more natural to split the graph into equal-distance sets with respect to a central *vertex*. Choosing an edge instead, bears the advantage of not producing a special case for the vertex set at distance 1 when considering upper bounds on the order of these sets later.

Completing the sum that looks like a scalar multiple of $\langle f|_A, f|_A \rangle$, gives:

$$\begin{aligned} & \sum_{i=0}^{k} \frac{\left|V_A^i\right|}{(d-1)^i}(d-2\sqrt{d-1}) + \left|V_A^k\right| \frac{2\sqrt{d-1}-1}{(d-1)^k} \\ (4) \qquad & = \langle f|_A, f|_A \rangle (d-2\sqrt{d-1}) + \left|V_A^k\right| \frac{2\sqrt{d-1}-1}{(d-1)^k} \ . \end{aligned}$$

What to do with the last term of (4)? By the above mentioned inequality $\left|V_A^i\right| \leq (d-1)\left|V_A^{i-1}\right|$ we have

$$\frac{|V_A^i|}{(d-1)^i} \leq \frac{|V_A^j|}{(d-1)^j} \ ,$$

whenever $i < j$. Thus $\frac{|V_A^k|}{(d-1)^k}$ is at most the arithmetic mean of all these fractions, i.e. $\frac{\langle f|_A, f|_A \rangle}{k+1}$.

Putting things together, we obtain

$$\begin{aligned} \frac{\tilde{f}_1}{f_1} & = \frac{\langle Lf|_A, f|_A \rangle}{\langle f|_A, f|_A \rangle} \\ & \leq \frac{1}{\langle f|_A, f|_A \rangle}\left(\langle f|_A, f|_A \rangle (d-2\sqrt{d-1}) + (2\sqrt{d-1}-1)\frac{\langle f|_A, f|_A \rangle}{k+1} \right) \\ (5) \qquad & = (d-2\sqrt{d-1}) + \frac{2\sqrt{d-1}-1}{k+1} \ . \end{aligned}$$

Thus, the actual choice of function f was such that

- $(f_i - f_{i-1})^2$ is only dependent on i,
- the $\left|V_A^i\right|$ cancel out in $\frac{\langle Lf|_A, f|_A \rangle}{\langle f|_A, f|_A \rangle}$,
- the extra term $\left|V_A^k\right| f_k^2$ can be bounded in terms of $\langle f|_A, f|_A \rangle$.

The estimate (5) still contains an undefined value: k. In the face of inequality (2) above, we are interested in the maximum value of (5) and its analogous expression for $f|_B$. Thus, we will obtain the best upper bound on $d - \lambda_1$ when the minimum of values of k for A and B is maximized. This will be obtained by choosing both A and B of maximal diameter: the two central edges with respect to which the rings of equal distance are defined about have to be two edges at the maximum distance[3] $2k+2$ or $2k+3$ in G. Choosing A to be the set of vertices at distance at most k from one of them and B the set of vertices at distance at most k from the other, gives us sets that satisfy the correct disjointness property and shows that $k+1$ is at most half the "edge-diameter".

We proved the following theorem.

Theorem 2 (A. Nilli)**.** *Let G be a d-regular graph in which there are two edges of distance at least $2k+2$. Then*

$$d - \lambda_1 \leq d - 2\sqrt{d-1} + \frac{2\sqrt{d-1}-1}{k+1} \ .$$

[3] The distance between two edges is defined to be the distance of the corresponding sets of endvertices.

The importance of this theorem lies in a genuine consequence about Ramanujan graphs.

Definition. *A d-regular graph is called* Ramanujan *if its second largest eigenvalue λ_1 is at most $2\sqrt{d-1}$.*

Nilli's theorem implies a theorem first proven by Alon and Boppona—see Nilli [10] for reference—which states that asymptotically the second largest eigenvalue of a d-regular graph is at least $2\sqrt{d-1}$.

Considering the fact that a restricted maximum degree (e.g. requiring the graph to be d-regular) implies that there is a fixed upper bound on the number of vertices at distance at most i from a fixed vertex v_0, we find that the diameter of a family $(G_j)_{j\in\mathbb{N}}$ of d-regular graphs satisfying $|G_j| \to \infty$ as $j \to \infty$ tends to infinity as well.

Together with Theorem 2 this implies

$$\liminf \lambda_1(G_j) \geq 2\sqrt{d-1} \ .$$

This means that families of Ramanujan graphs, i.e. families of d-regular Ramanujan graphs with order tending to infinity, are the solution to an extremal problem: they are extremal with respect to the asymptotic spectral gap.

When thinking about extremal properties one might expect that they imply some sort of structural restriction (which is true here to some extend for expansion—cf Theorem 1). In this view a somewhat astonishing fact is that a large spectral gap, the defining property of Ramanujan graphs, implies best possible estimate of the following pseudorandom property (see Alon and Spencer [2, Cor. 9.2.5]):

Theorem 3. *Let G be a d-regular graph of order n and let λ denote the largest (in absolute value) non-trivial eigenvalue of G. For every two sets of vertices B and C with $|B| = bn$, $|C| = cn$*

$$|E(B,C) - cbdn| \leq \lambda\sqrt{bc}n \ .$$

If we had picked a random d-regular graph on n vertices, each of the bn vertices in B would be expected to have about cd vertices in C. The bound is best for smallest λ. However, this theorem does *not* tell how good the bound is. Since the assertion is closely related to the expansion constant, the proof is very short, but does not seem to use any unexpected ideas. It can be found in [2].

1.3 Cioabă's proof of Serre's theorem

In connection to the question of how strong the Ramanujan property is, an unpublished theorem by Serre gives a partial answer:

Theorem 4. *Let $\varepsilon > 0$ and $d \in \mathbb{N}$, then there exists a constant $C > 0$, which depends on ε and d only, such that for any d-regular graph G at least $C\,|G|$ of its $|G|$ eigenvalues lie in $[(2-\epsilon)\sqrt{d-1}, d]$.*

This theorem says that, asymptotically, Ramanujan graphs will have a positive proportion of their eigenvalues situated within a very small interval

$$[(2-\epsilon)\sqrt{d-1}, 2\sqrt{d-1}] \ ,$$

since the contribution of λ_0 vanishes. This seems to be a rather strong property.

How do we think about the proof? The theorem basically makes a statement on the distribution of eigenvalues of regular graphs. It is not surprising that the proof given in Davidoff, Sarnak and Valette [5] following Serre's (original?) approach falls back on measures. There is a new proof due to Cioabă [4] for this theorem which is both elementary and very short. It starts off with the same ideas as Serre's approach: instead of directly estimating eigenvalues one might consider the trace of some (even) power of A. This will help since on the one hand we can, again, use the comprised graph theoretical meaning, i.e. numbers of walks of fixed length, to estimate the trace of some power of A. On the other hand we know that all eigenvalues of A are contained in $[-d, d]$. So we can shift the eigenvalues by considering the matrix $dI + A$ towards $\mathbb{R}_0^+$ such that when taking even powers, their ordering will be preserved. $\operatorname{tr}(dI + A)^{2s}$ will thus give information on the second largest eigenvalue, while binomial expansion of $(dI + A)^{2s}$ will enable us to apply the estimates of $\operatorname{tr}(A^{2j})$.

While Serre's proof uses Chebychev-polynomials, recurrence relations and generating functions to estimate $\operatorname{tr}(A^{2j})$, Cioabă employs a connection to Catalan numbers. We shall see how this works.

$[A^{2s}]_{j,j}$ equals the number of closed walks in G starting at vertex j. The trace of $(dI+A)^{2s}$ will be bounded from both directions later. For that purpose we first seek a lower bound on $\operatorname{tr}(A^{2s})$:

Instead of all closed walks at j we will only count such walks that lift to closed walks in the universal cover of G: the infinite d-regular tree T_d. This will lead to the lower bound on $\operatorname{tr}(A^{2s})$.

A closed walk of length $2s$ in T_d can be viewed as a sequence of digits "1" and "-1" of length $2s$ which sums to 0 and such that every initial segment has a non-negative sum. Such a ±1-sequence shall be referred to as a *valid* sequence. Let $v_0 \in p^{-1}(j)$ be the root of T_d, where $p : T_d \to G$ denotes the covering projection. Then a "1" at k-th position in the sequence corresponds to the walk increasing the distance to v_0 by one at the i-th step. Similarly, a "-1" corresponds to decreasing the distance by one. For the total sum to vanish we need to have exactly s entries "1" in a sequence. Corresponding to every valid sequence we find $d(d-1)^{s-1}$ possible walks, as there are several choices each time we increase the distance to the root by one. Now, the problem of how many valid ±1-sequences exist is just the "10- and 20- notes change problem" as can be found in Aigner [1, p.174, p.300]: its answer is $\frac{1}{s+1}\binom{2s}{s}$, the s-th Catalan number.

There are $\binom{2s}{s}$ sequences with exactly s entries "1" in total. Every sequence that ignores the second requirement has a shortest negative initial segment, containing one more entry "-1" then there are entries "1". Swapping labels "1" and "-1" within this initial segment gives a sequence containing $s+1$ entries "1" and $s-1$ entries "-1". Conversely, given any such sequence containing exactly $s+1$ entries "1", there is a shortest initial segment containing more entries "1" than entries "-1". Swapping labels "1" and "-1" within this initial segment leads to a ±1-sequence with exactly s entries "1" that fails the initial-segment-condition. As the described relation is bijective, there are

$$\binom{2s}{s} - \binom{2s}{s+1} = \frac{s+1}{s+1}\binom{2s}{s} - \frac{s}{s+1}\binom{2s}{s} = \frac{1}{s+1}\binom{2s}{s}$$

valid sequences.

Hence, we obtain the estimate

$$\operatorname{tr}(A^{2s}) \geq |G|\, d(d-1)^{s-1}\frac{1}{s+1}\binom{2s}{s}\ .$$

To apply this in order to obtain a lower bound for $\operatorname{tr}(dI+A)^{2s}$ we will need to expand the product first:

$$\operatorname{tr}(dI+A)^{2s} = \sum_{j=0}^{2s}\binom{2s}{j} d^j \operatorname{tr}(A^{2s-j})\ .$$

The estimate above only applies to even exponents. It turns out that the odd ones can just be omitted. Note that the entries on the diagonal of any power of A are non-negative. Thus, neglecting these exponents, gives us

$$\begin{aligned}
\operatorname{tr}(dI+A)^{2s} &\geq \sum_{j=0}^{s}\binom{2s}{2j} d^{2j}\operatorname{tr}(A^{2s-2j}) \\
&\geq \sum_{j=0}^{s}\binom{2s}{2j} d^{2j} n d(d-1)^{s-j-1}\frac{1}{s-j+1}\binom{2s-2j}{s-j} \\
&> \sum_{j=0}^{s}\binom{2s}{2j} d^{2j} n\sqrt{d-1}^{2s-2j}\frac{1}{s-j+1}\binom{2s-2j}{s-j}\ .
\end{aligned}$$

This almost looks like part of the binomial expansion of the term $(d+2\sqrt{d-1})^{2s}$. The factor $\frac{1}{s-j+1}\binom{2s-2j}{s-j}$ does not belong to it, and a factor 2^{2s-2j} is missing. However, $\binom{2k}{k} > \frac{4^k}{k+1}$, for $\binom{2}{1} = 2 \geq \frac{2^2}{1+1}$ and inductively

$$\begin{aligned}
\binom{2(s+1)}{s+1} &= \binom{2s}{s}\frac{(2s+2)(2s+1)}{(s+1)^2} \geq \frac{4^s}{s+1}\frac{(2s+2)(2s+1)}{(s+1)^2}\frac{s+2}{s+2} \\
&= \frac{2\cdot 4^s}{s+2}\frac{(2s+1)(s+2)}{(s+1)^2} = \frac{2\cdot 4^s}{s+2}\frac{2s^2+(4+1)s+2}{(s+1)^2} \\
&> \frac{4^{s+1}}{s+2}\ .
\end{aligned}$$

Thus, we may continue this estimate by

$$\begin{aligned}
\operatorname{tr}(dI+A)^{2s} &> \sum_{j=0}^{s}\binom{2s}{2j} d^{2j} n\sqrt{d-1}^{2s-2j}\frac{1}{s-j+1}\binom{2s-2j}{s-j} \\
&> \sum_{j=0}^{s}\binom{2s}{2j} d^{2j} n(2\sqrt{d-1})^{2s-2j}\frac{1}{(s-j+1)^2} \\
&> n\frac{1}{(s+1)^2}\sum_{j=0}^{s}\binom{2s}{2j} d^{2j}(2\sqrt{d-1})^{2s-2j} \\
&= \frac{n}{2(s+1)^2}\left((d+2\sqrt{d-1})^{2s} + (d-2\sqrt{d-1})^{2s}\right) \\
&> \frac{n}{2(s+1)^2}(d+2\sqrt{d-1})^{2s}\ .
\end{aligned}$$

The reason for dropping terms in the last step will become apparent later: they simply will not be needed to bound m/n away from zero. It is, however, easier not to carry them along until then.

To bound $\mathrm{tr}(dI+A)^{2s}$ above, a very common strategy is used: the value we are interested in, i.e. the number of m of eigenvalues of A in $[(2-\varepsilon)\sqrt{d-1}, d]$, already provides us with bounds: there are $|G|-m = n-m$ eigenvalues of value less than $(2-\varepsilon)\sqrt{d-1}$. All the other m eigenvalues are at most d. Surprisingly, this simple estimate is all we need for the upper bound.

$$\begin{aligned}
\mathrm{tr}(dI+A)^{2s} &= \sum_{i=0}^{n-1}(d+\lambda_i)^{2s} \\
&< (n-m)\left(d+(2-\varepsilon)\sqrt{d-1}\right)^{2s} + m(2d)^{2s} \\
&= n\left(d+(2-\varepsilon)\sqrt{d-1}\right)^{2s} \\
&\quad + \ m\left((2d)^{2s} - (d+(2-\varepsilon)\sqrt{d-1})^{2s}\right)
\end{aligned}$$

What we aim for is to show that the proportion $\frac{m}{n}$ is bounded away from 0 by some fixed $c(\varepsilon,k)$. By the two above bounds we find the following inequality on $\frac{m}{n}$:

$$\frac{m}{n} > \frac{\frac{1}{2(s+1)^2}(d+2\sqrt{d-1})^{2s} - (d+(2-\varepsilon)\sqrt{d-1})^{2s}}{(2d)^{2s} - (d+(2-\varepsilon)\sqrt{d-1})^{2s}} \ .$$

What remains is to show that for some $s = s_0$ the RHS is strictly positive. This would give us the desired $c(\varepsilon,k)$. The denominator is always positive (as $s \geq 1$) and can be ignored. Considering the numerator, it is positive iff

$$\begin{aligned}
\frac{1}{2(s+1)^2} &> \frac{(d+(2-\varepsilon)\sqrt{d-1})^{2s}}{(d+2\sqrt{d-1})^{2s}} \quad \Leftrightarrow \\
2^{-1/2s}(s+1)^{-1/s} &> \frac{d+(2-\varepsilon)\sqrt{d-1}}{d+2\sqrt{d-1}}
\end{aligned}$$

which holds for *every* large enough s, for

$$\lim_{s\to\infty} 2^{-1/2s}(s+1)^{-1/s} = \lim_{s\to\infty}(s+1)^{-1/s} = (\lim_{s\to\infty}(s+1)^{1/s})^{-1} = 1 \ .$$

Note that the last step is proven in very much the same way as $\lim_{n\to\infty}\sqrt[n]{n} = 1$. This completes the proof of Serre's theorem.

2 Theoretical basics of an explicit construction

A whole branch of research done in connection with Ramanujan graphs seems to be concerned with explicit constructions of families of such graphs. According to [5, Thm 0.10] explicit constructions of d-regular Ramanujan graphs are known in cases where $d-1$ is prime or a prime power while the general case remains open. Lubotzky raised the question of whether every locally finite tree which covers a finite graph also covers a Ramanujan graph or maybe even infinitely many. In answering this question with 'No', Lubotzky and Nagnibeda [8] suggest that it might not be unlikely that the conjectured existence of Ramanujan graphs for general d does not hold after all.

According to [5, 9, 7] the first explicit constructions were found independently by Margulis, and Lubotzky, Phillips and Sarnak. These and further constructions (not including the elementary approach in [5]) seem to make heavy use of rather advanced tools from algebra and other areas. At least for the early attempts representation theory seems to play an important role. As such tools do not appear in the classical parts of discrete mathematics, it is interesting to note that there is an elementary relation between eigenvalues and characters. It is possible (and not hard to deduce) to express the eigenvalues of a Cayley graph $X(G, S)$ by character sums $\lambda_\chi = \sum_{s \in S} \chi(s)$ for the irreducible characters χ of G (cf Murty [9, Thm 6]). Still we do not know how the proofs of such explicit constructions look like nor what they involve.

In order to get an idea of what kind of theory leads to an explicit construction, we will describe Alon and Milman's way to explicitly construct families of expanders in the remaining part. We will restrict ourselves to expanders as the original papers for construction of Ramanujan graphs (esp. estimate of expansion constant) seem to be too involved. And instead of following the original paper, we will take hold on Lubotzky's presentation in [7] since it provides some of the necessary prerequisites. Lubotzky's way of presenting things is if not terse then still limited to the very essential; often short arguments are simply omitted. Here we attempt to fill in some of the 'gaps' within the explanation of individual steps and provide definitions of concepts that might not be part of undergraduate level.

2.1 Definitions

To begin with we need some definitions from topological group theory and measure theory (as can be found in Halmos [6]): A *topological group* itself is a topological space G endowed with a group structure on G such that multiplication and taking inverses are continuous—or, equivalently, such that the map $G \times G \to G$, $(x, y) \mapsto x^{-1}y$ is continuous. A *locally compact* group is a topological group such that the underlying space is locally compact, i.e. every point has a neighbourhood with compact closure.

The advantage locally compact groups provide is the existence of a corresponding *Haar measure* (Halmos [6, § 58]) which makes it possible to transfer most of the theorems about linear representations of finite groups to unitary representations of these groups (see Serre [11, Ch.4] for reference).

The elements of the σ-algebra generated by the set of compact subsets of G are called the *Borel sets* of G. A *Haar measure* μ on a locally compact group is a Borel measure (i.e., it is finite on compact sets) with the additional properties of being non-zero on non-empty Borel-open sets and translation invariant in the sense of $\mu(xE) = \mu(E)$ for all $x \in G$ and every Borel set E.

2.2 Amenable groups

Equipped with these basic definitions we can proceed to those relevant for constructions (cf Lubotzky [7]). The bridge between groups and graphs used are

Cayley graphs[4] $X(G, K)$. The following notion of an amenable group becomes interesting in the sense that it tells us (from view of topological groups) about the expansion properties of Cayley graphs of locally compact groups with respect to compact subsets. Amenable groups will not be needed for the Alon-Milman construction (though there are connections between amenability and the Kazhdan property to be introduced later). They are nevertheless included here as they provide an example of how the expanding property may be translated to topological groups.

Let G be a locally compact group and λ a Haar measure. G is *amenable* if it satisfies Følner's property

(F) For every $\varepsilon > 0$ and every compact set $F \subset G$, there is a Borel set $U \subseteq G$ such that
$$\lambda(xU \triangle U) < \varepsilon\lambda(U),$$
for all $x \in K$ and the symmetric difference $xU \triangle U$.

For discrete (and thus necessarily locally compact) groups the Haar measure is given by $\lambda(U) = |U|$ (up to a constant factor maybe). Indeed, singleton sets are compact, and hence we obtain $\lambda(x) = \lambda(xe) = \lambda(e)$ for all $x \in G$, where e denotes the identity of G.

The inclusion

$$KU \triangle U \subseteq \bigcup_{x \in K} (xU \triangle U) \qquad \text{yields} \qquad |KU \triangle U| \leq \sum_{x \in K} |xU \triangle U|$$

for finite K. Hereby, and since compact the sets are exactly those which are finite, property (F) can be replaced by

$$|KU \triangle U| < \varepsilon\,|U| \quad \text{for all finite } K$$

in the discrete case. It is this inequality that provides the key to the mentioned connection (which was unfortunately not worked out completely in [7]): Given any Cayley graph $X(G, K)$ on a discrete group G with respect to finite K, the set ∂U of neighbours of a subset U of vertices is given by $KU \setminus U \subset KU \triangle U$. Thus, if G is amenable we find, given any $\varepsilon > 0$, a subset of vertices U with smaller expansion $\frac{|\partial U|}{|U|} \leq \frac{|KU \triangle U|}{|U|} < \varepsilon$. $X(G, K)$ is not an expander for any ε. Conversely, if $X(G, K)$ is not an expander, we find for every $\varepsilon > 0$ subsets U of vertices such that $\frac{|\partial U|}{|U|} < \varepsilon$. As $|KU| \geq |U|$ their symmetric difference satisfies $|KU \triangle U| \leq 2 \cdot |KU \setminus U| = 2 \cdot |\partial U|$. Thus G is amenable.

2.3 Kazhdan property and preliminary propositions

Returning to our actual aim, this section contains all prerequisites still missing for the final construction.

For a locally compact group a *unitary representation* on a Hilbert space H is a continuous group homomorphism $\rho : G \to U(H)$ into the group of unitary operators on H.

[4]Let G be a group and $S \subset G$ as subset. Then the Cayley graph $X(G, S)$ has vertex set G and xy is an edge if there is some $s \in S$ such that $x = sy$. Usually S is required to be symmetric, i.e. $S = S^{-1}$ such that the resulting graph is simple.

Definition (Fell topology)**.** *Let G be a locally compact group and let $\hat{G}$ denote the set of equivalence classes of irreducible unitary representations of G.*

The Fell topology *is the topology on $\hat{G}$ whose basic open neighbourhoods $W(K, \varepsilon, v)$ of (ρ, H), where $K \subset G$ is a compact subset, $\varepsilon > 0$ and $v \in H$ has norm 1, are given by*

$$W(K, \varepsilon, v) = \left\{ (\sigma, H') \in \hat{G} \;\middle|\; \begin{array}{c} \exists v' \in H' \text{ such that} \\ |\langle v, \rho(g)v\rangle - \langle v', \sigma(g)v'\rangle| < \varepsilon \quad \forall g \in K \end{array} \right\} .$$

(ρ, H) is said to be weakly contained *in (σ, H') if given any compact set $K \subset G$ and $\varepsilon > 0$, there is a sequence of vectors (v'_i) in H' such that $(g \mapsto \langle v, \rho(g)v\rangle)$ is the uniform limit of $(g \mapsto \langle v'_i, \sigma(g)v'_i\rangle)$ on K.*

The (only) case important for us is that of the one-dimensional trivial representation ρ_0 of G. In this case, we can even reformulate the property of being weakly contained in such a way that it has a somewhat more obvious geometric interpretation. Note that if v has norm 1, $\langle \rho_0(g)v, v\rangle = 1$.

Since

$$\begin{aligned} \|\sigma(g)v' - v'\| &= \langle \sigma(g)v' - v', \sigma(g)v' - v'\rangle^{1/2} \\ &= \sqrt{2 - 2\langle \sigma(g)v', v'\rangle} \end{aligned}$$

we have

$$|1 - \langle \sigma(g)v', v'\rangle| < \varepsilon \quad \Leftrightarrow \quad \|\sigma(g)v' - v'\| < \sqrt{2\varepsilon} \, .$$

The second inequality says that for small ε the image $\sigma(g)$ of g leaves v' almost invariant. (σ, H') is said to *have almost invariant vectors*, if given any compact set K and any $\varepsilon > 0$, we find a unit vector $v' \in H'$ such that $\|\sigma(g)v' - v'\| < \varepsilon$ for all $g \in K$.

Thus, ρ_0 is weakly contained in σ if and only σ has almost invariant vectors. ρ_0 is said to be *contained* in σ if σ has a subrepresentation that is isomorphic to ρ_0, which is equivalent to saying that σ has an invariant one-dimensional subspace, i.e. an invariant vector.

Definition (Kazhdan group)**.** *A locally compact group is said to be* Kazhdan *(or to have property (T)) if ρ_0 is isolated in the Fell topology, i.e., there are K, ε, v such that the neighbourhood $W(K, \varepsilon, v)$ of ρ_0 only contains ρ_0.*

In other words, a group is Kazhdan iff there is a compact subset $K \subset G$ and an $\varepsilon > 0$ such that for every non trivial irreducible representation (σ, H) and every $v \in H$ there is some $g \in K$ such that

$$\|\sigma(g)v - v\| > \varepsilon \, \|v\|$$

holds.

We will need the following equivalent assertion later. Note that the equivalence does require proof[5].

Proposition 1. *A locally compact group G is Kazhdan iff there are $\varepsilon > 0$ as well as a compact subset $K \subset G$, such that for any (σ, H') without invariant vectors and for every $v \in H'$ we find $g \in K$ with $\|\sigma(g)v - v\| \geq \varepsilon$.*

[5]Regrettably, non of the references given in [7] that are available to me gives a self-contained proof. References therein were not available at all.

If Γ is a discrete group the compact subsets are exactly the finite subsets. See [7, 3.2.5] for the statement of the following proposition.

Proposition 2. *Let Γ be a finitely generated discrete Kazhdan group and let S be a finite generating set. Then there exists $\varepsilon > 0$ such that for every unitary representation (σ, H) without invariant vectors and every $v \in H$ there is some $s \in S$ such that*

$$\|\sigma(s)v - v\| > \varepsilon \|v\| \ .$$

Proof: Since Γ is Kazhdan there is (by Proposition 1) a finite K and $\tilde{\varepsilon} > 0$ such that for every unitary representation (σ, H) without invariant vectors and every $v \in H$ we find $g \in K$ such that

$$\|\sigma(g)v - v\| > \tilde{\varepsilon} \|v\| \ .$$

Every $g \in K$ can be expressed as a word in S. Choose a fixed S-word for every $g \in K$ and let ℓ be the maximal length of chosen words. Suppose that there is no ε as claimed. In particular let (ρ, H') be a witness for $\varepsilon = \frac{\tilde{\varepsilon}}{\ell}$, i.e. a non-trivial irreducible representation for which we find $v \in H'$ such that for all $s \in S$

$$\|\rho(s)v - v\| \leq \frac{\tilde{\varepsilon}}{\ell} \|v\| \ .$$

Let g be any element of K and let $g = s_1 s_2 \dots s_j$, $j < \ell$, be its chosen S-word. We have

$$\begin{aligned}
\|\sigma(g)v - v\| &= \|\sigma(s_1 s_2 \dots s_j)v - v\| \\
&= \| \ \sigma(s_1 s_2 \dots s_{j-1})(\sigma(s_j)v - v) + \\
&\quad\ \sigma(s_1 s_2 \dots s_{j-2})(\sigma(s_{j-1})v - v) + \\
&\quad\ \dots + (\sigma(s_1)v - v) \ \| \\
&\leq \|\sigma(s_1 s_2 \dots s_{j-1})(\sigma(s_j)v - v)\| + \\
&\quad \|\sigma(s_1 s_2 \dots s_{j-2})(\sigma(s_{j-1})v - v)\| + \\
&\quad \dots + \|\sigma(s_1)v - v\| \\
&< \ell \frac{\tilde{\varepsilon}}{\ell} \|v\| \ ,
\end{aligned}$$

where we used the fact that the length is invariant by unitary operators. Clearly, this contradicts the choice of K. □

2.4 Alon and Milman's construction

Now we are in the position of actually applying all these tools. The plan is as follows. To obtain graphs from groups we will use Cayley graphs again. The Kazhdan property shall be used to bound the expansion away from zero. As we are seeking a family of expanding graphs, we somehow need to do this for all graphs together. If we are given an infinite Kazhdan group with an infinite number of normal subgroups such that the corresponding quotients are finite, we may on the one hand use these finite groups to obtain Cayley graphs. On the other hand by considering compositions with quotient maps, we will obtain representations of the original group into unitary groups of group algebras of the finite quotients. This will enable us to use information provided by the Kazhdan property within the individual quotients.

How is it possible to relate the Kazhdan property and expansion? We can choose any suitable representation to begin with and hope to find a subrepresentation without invariant vectors to apply the Kazhdan property. Recalling the proof of theorem 1 (cf p.6) we already know functions on G (i.e. $\in \mathbb{C}G$) that can be used to count partition edges. These will provide the missing link as they turn out to be vectors contained in such subrepresentations without invariant vectors.

The easiest unitary representations on $\mathbb{C}G$ to be taken into account are the special cases of the permutation representation: the regular representations.

Note that, when choosing our infinite group to be discrete, there is no need to worry about continuity restriction. A reason of why it is desirable to work with finitely generated discrete groups lies in the previous proposition 2, as will become apparent in the proof.

Theorem 5. *Let Γ be a finitely generated discrete Kazhdan group. Let $\mathcal{N}$ be a family of finite index normal subgroups of Γ and $S \cup S^{-1}$ a finite generating set. (We may and shall assume that $S = S \cup S^{-1}$.) Considering S as generating set for each of the finite groups Γ/N, where $N \in \mathcal{N}$, the family $X(\Gamma/N, S)$ of Cayley graphs is a family of (n,k,c)-expanders for $n = |\Gamma/N|$, $k = |S|$, and some $c > 0$.*

Proof: Let $N \in \mathcal{N}$ be a fixed member and $H = L^2(\Gamma/N)$ the Hilbert space of square summable complex functions on the finite group $V := \Gamma/N$. The corresponding norm is $\|f\|^2 = \sum_{x \in V} |f(x)|^2$. If we ignore the norm of H, we can regard its elements as elements in $\mathbb{C}V$ and consider the right regular representation $\rho_\Gamma : \Gamma \to U(H)$ obtained by the action

$$\rho_\Gamma(\gamma)(f)(x) = f(x\gamma)$$

of Γ on H. Let H_0 denote the subspace of H of functions with vanishing mean: $H_0 = \{f \in H \mid \sum_{x \in V} f(x) = 0\}$. This is an obvious candidate for invariant subspaces under permutation representations. H splits into the direct sum $H = H_0 \oplus \mathbb{C}_{\chi_V}$, where $\mathbb{C}_{\chi_V}$ denotes the space of constants on $V = \Gamma/N$. Since the permutation action of Γ on Γ/N by multiplication from the right is transitive, there are no invariant functions under ρ_Γ but those in $\mathbb{C}_{\chi_V}$, i.e. the constants. As Γ is Kazhdan, proposition 2 applies and there is some $\varepsilon > 0$ such that for every representation (ρ, H) without almost invariant vectors and every $f \in H$ there is some $s \in S$ such that $\|\rho(s)f - f\| > \varepsilon \|f\|$. Thus, given any particular $f \in H_0$, we find $\gamma \in S$ such that for every

$$\|\rho_\Gamma(\gamma)f - f\| > \varepsilon \|f\| \ .$$

Now we are in the position to apply edge-counting functions. If we choose f exactly as in the proof of theorem 1 (cf p.6), that is, given $A \subset V = \Gamma/N$ we put

$$f(x) = \begin{cases} n - a & \text{if} \quad x \in A \ , \\ -a & \text{if} \quad x \in V \setminus A \ , \end{cases}$$

where $a = |A|$, then f belongs to H_0 and $\|\rho_\Gamma(\gamma)f - f\|$ counts those edges in $\partial A = e(A, V \setminus A)$ which are generated by γ. Here lies the importance of the previous proposition: we need the group element in the Kazhdan property to belong to S, for otherwise $\|\rho_\Gamma(\gamma)f - f\|$ cannot be interpreted in terms of edges in the Cayley graph as will be done now.

We shall denote the subset of edges within $e(A, V \setminus A)$ which are generated by γ by $E_\gamma(A, V \setminus A)$, including those edges xy twice for which $x = y\gamma$ and $y = x\gamma$. Thus $E_\gamma(A, V \setminus A)$ is a set of *disjoint* edges, some of which may be contained twice. Then

$$\begin{aligned} \|\rho_\Gamma(\gamma)f - f\| &= \sum_{x\in V} |f(x\gamma) - f(x)|^2 \\ &= n^2 E_\gamma(A, V \setminus A) \end{aligned}$$

and thus

$$\begin{aligned} N(A) &\geq \frac{1}{2} E_\gamma(A, V \setminus A) \\ &= \frac{1}{2n^2} \|\rho_\Gamma(\gamma)f - f\| \\ &\geq \frac{\varepsilon}{2n^2} \|f\|^2 \\ &= \frac{\varepsilon}{2n^2} a(n-a)n \ . \end{aligned}$$

Therefore

$$\frac{N(A)}{a} \geq \frac{\varepsilon}{2} \cdot \frac{n}{n-a} \geq \frac{\varepsilon}{2} \ ,$$

which shows that the graphs $X(\Gamma/N, S)$ are a family of $(n, |S|, \frac{\varepsilon}{2})$-expanders.

It is not hard to guess that there are connections between the notions of amenable and Kazhdan groups, which will, however, not be further discussed here. What remains—and will by far be the more complicated part—would be to prove that explicit infinite groups (after having identified them) enjoy the Kazhdan property. Examples, though their presentation is not self-contained, are given in Lubotzky [7].

Concluding Remarks

In this essay the focus was restricted to some elementary bounds on eigenvalues and a very first insight into explicit constructions of expanders.

There is a wide range of ongoing research in this area. Murty gives a short overview over some open (at state of 2003) problems in [9]. There are generalizations in different directions of the notion of a Ramanujan graph: making use of the spectral radius of the universal cover, the definition can be extended to non-regular graphs; see Lubotzky and Nagnibeda [8]. Other generalizations go into higher dimensions.

A fascinating aspects of the topic is the wide range of mathematical disciplines that are involved. At very small scale this was seen in Section 2 of this essay. An overview over some achievements in connection with expanding graphs is given on `http://www.ipam.ucla.edu/programs/agg2004/`. Most astonishingly, applications of expanders seem to reach as far as to three-dimensional geometry. The above mentioned overview, for example, points out that a recent manuscript by Lackenby[6] was showing how a conjecture of Lubotzky and Sarnak might be used within a proof of the virtually Haken Conjecture for hyperbolic three-manifolds.

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