Time Series Management

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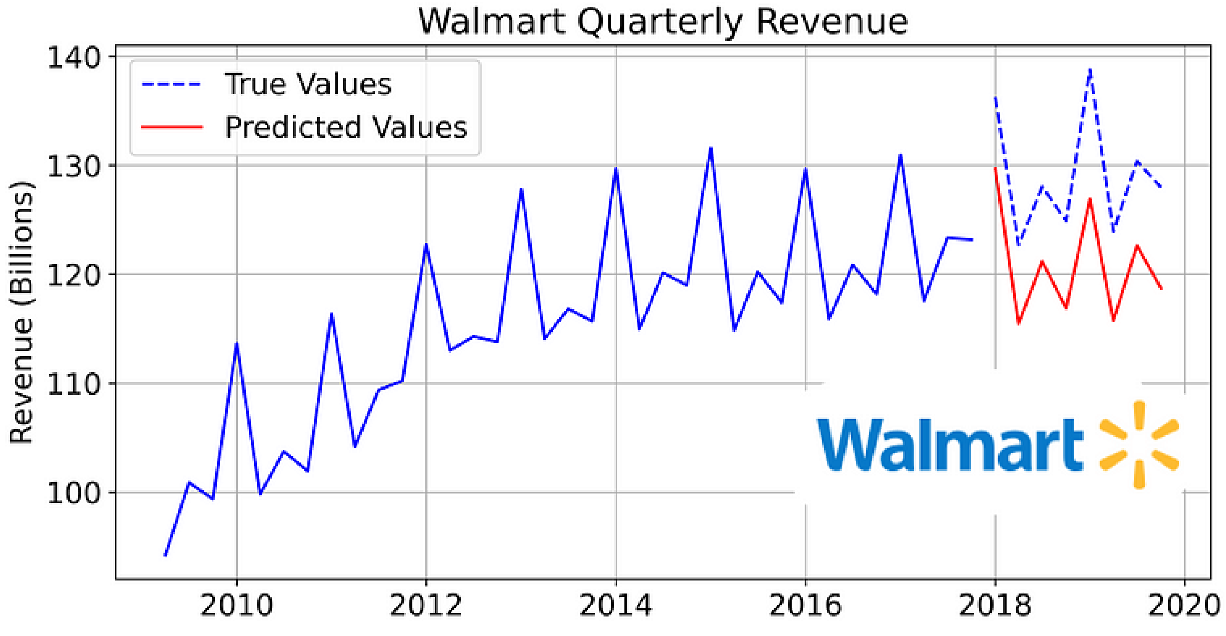
Syllabus

- ARIMA model
- Backshift Notation
- Model selection and forecasting
- MLE (Maximum likelihood estimation)
- AIC Akaike’s Information Criterion

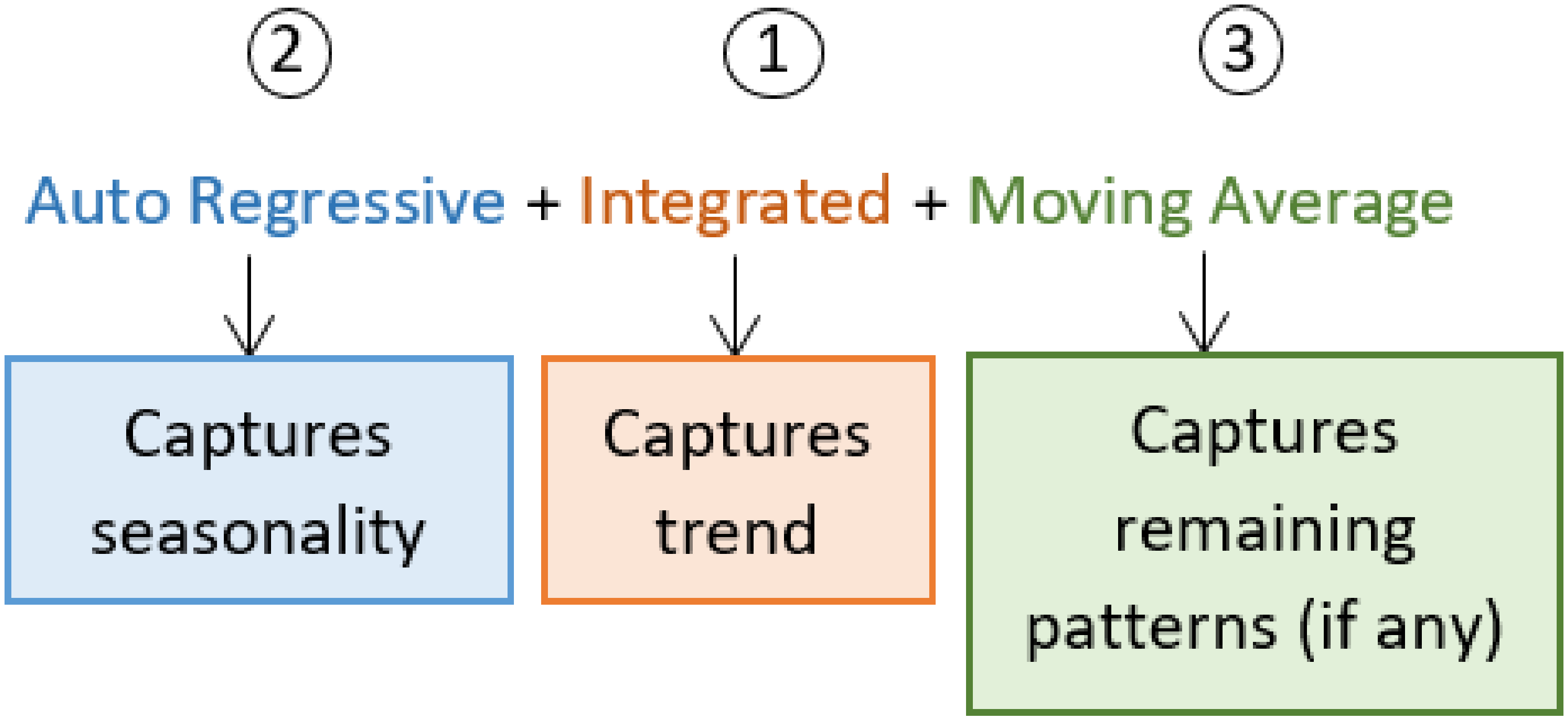
Time series data… quick recap

- A **univariate time series** is a sequence of measurements of the same variable collected over time. Most often, the measurements are made at regular time intervals.



https://www.kaggle.com/code/anushkaml/walmart-time-series-sales-forecasting

AR.I.MA



https://www.linkedin.com/pulse/time-series-part-2-introduction-arima-models-using-excel-agarwal

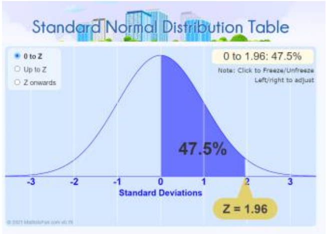
ACF and PACF

ACF (lag h) = $\rho_h = \dfrac{Covariance(X_{[t]}, X_{[t-h]})}{Std.Dev.(X_{[t]})Std.Dev.(X_{[t-h]})}$

Limites de signification ACF et PACF

$$S^+ = \frac{+Z}{N-h} \qquad S^- = \frac{-Z}{N-h}$$

N $\coloneqq$ $time\ series\ values$ $\qquad h \coloneqq lag$



Pick Z such that the area of N(0,1) between (– Z) and (Z) is equal to the desired amount of confidence (in general for 95% c.i. , Z = 1.96)

https://www.mathsisfun.com/data/standard-normal-distribution-table.html

Direct (partial) effect of $X_{[t-h]}$ on $X_{[t]}$, removing $X_{[t-h+1]}$, … , $X_{[t-1]}$

PACF (lag 3) = $\rho_{3,3} = \dfrac{Covariance(X_{[t]}, X_{[t-3]} | X_{[t-2]}, X_{[t-1]})}{\sqrt{Variance(X_{[t]} | X_{[t-1]}, X_{[t-2]}) Variance(X_{[t-3]} | X_{[t-1]}, X_{[t-2]})}}$

…

Levinson and Durbin recursive algorithm (for PACF computation)

$$\text{PACF (lag h)} = \rho_{h,h} = \frac{\rho_h - \sum_{j=1}^{h-1} \rho_{h-1,j}\, \rho_{h-j}}{1 - \sum_{j=1}^{h-1} \rho_{h-1,j}\, \rho_j}$$

where :

$$\rho_{h,j} = \rho_{h,j} - \rho_{h,h}\, \rho_{h-1,h-j}$$

for :

$$j = 1,2, \dots, h-1$$

Autoregression (AR) models

$$Y_t = c + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + w_t$$

Where c is a constant, ϕ_i are parameters to estimate and w_t the error term (iid ~N(0,σ_{ω_t}))

Constraints:

- For p = 1, -1 < ϕ_1 < 1.
- For p = 2, -1 < ϕ_2 < 1, ϕ_2 + ϕ_1 < 1 and ϕ_2 - ϕ_1< 1.
- For p = 3, more complicated conditions

Integrated model (d)

It is important to remove trends, or non-stationarity, from time series data prior to model building, since such autocorrelations dominate the ACF. One way of removing non-stationarity is through the method of differencing. :

$$Y'_t = Y_t - Y_{t-1}$$

Occasionally, such taking of first differences is insufficient to remove non-stationarity. In that case, second-order differences usually produce the desired effect:

$$Y''_t = Y'_t - Y'_{t-1}$$

d := order of the integration

Integrated model (seasonal)

A seasonal difference is the difference between an observation and the corresponding observation from the previous year, quarter or month as appropriate, where s is the number of time periods back. For example, with monthly data, s = 12 and the seasonal difference is obtained as:

$$Y'_t = Y_t - Y'_{t-12}$$

Moving Average (MA) models

$$Y_t = c + w_t - \theta_1 w_{t-1} - \theta_2 w_{t-2} + \cdots$$

Where c is a constant, θ_i are parameters to estimate and w_i the error terms (iid ~N(0,1))

Constraints:

- For q = 1, -1 < θ_1 < 1.
- For q = 2, -1 < θ_2< 1, θ_2 + θ_1 < 1.
- For q = 3, more complicated conditions

Backshift notation

The backshift notation is commonly used to represent ARIMA models. It uses the operator B, which shifts data back one period:

$$BY_t = Y_{t-1}$$

Two applications of B shift the data back two periods:

$$\mathrm{B}(BY_t) = B^2 Y_t = Y_{t-2}$$

In general, Bs represents “shift back s time periods”. Note that a first difference is represented by $1 - B$:

$$\mathrm{Y}_{\mathrm{t}}' = Y_t - Y_{t-1} = Y_t - BY_t = \text{(1-B)}\, Y_t$$

ARIMA(p,d,q) = AR(p) , I(d) , MA(q)

ARIMA(1,1,1) ➔ Y_t = c + $\phi_1 Y_{t-1} + \theta_1 w_{t-1} + w_t$

➔(1 – ϕ_1B)(1 – B)$Y_t = c + (1 + \theta_1 B)\, w_t$

The general expression of the ARIMA(p, d, q) in backshift notation :

ARIMA(p,d,q) ➔ (1 – ϕ_1B – ⋯ – $\phi_p B^p$)(1– B)$Y_t = c +$ (1 $+ \theta_1$B $+ \cdots + \theta_q B^q) w_t$

Model selection and forecasting

Phase 1 (Identification):

Preliminary analysis and data preparation:

- Difference data to obtain a stationary series.

Model selection:

- Analyse time plots, ACF, PACF to identify potential models.

Model selection and forecasting

Phase 2 (Estimation and testing (1/2)):

1. Estimate parameters in potential models
2. Select best model using suitable criterion…

3. If satisfying model is not found go back to Phase 1
4. Otherwise…

Model selection and forecasting

Phase 2 (Estimation and testing (1/2)):

1. **Estimate parameters in potential models**
2. Select best model using suitable criterion…

3. If satisfying model is not found go back to Phase 1
4. Otherwise…

Likelihood

Likelihood function := L(Model with parameters θ | observed data)

:= Prob(observed data | Model with parameters θ)

The likelihood of a fully-specified model with a set of parameters θ, given some observed data, is equal to the probability of observing these data, given the defined model with those specific parameter values.

Likelihood is a quantitative measure of model fit. Higher likelihoods correspond to a higher probability of the model producing the observed data (the data “fit” the model well).

Maximum Likelihood estimation (MLE)

$$\widehat{\Theta}_{MLE} = \underset{\Theta}{arg\,max}\, L(\Theta)$$

The goal of MLE is to find the parameter Θ that maximizes the likelihood function (or equivalently, the log-likelihood).

Likelihood of the Normal Distribution

Consider you have a sample of data $x_1, x_2, \ldots, x_n$ drawn from a normal distribution with unknown mean μ and variance σ^2.

$$f(x|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

The log-likelihood function is:

$$\log L(\mu,\sigma^2) = -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

Maximizing this with respect to μ and σ^2 gives the MLE estimates:

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\hat{\mu})^2$$

Sum of squares error (SSE):

$$\boldsymbol{SSE(\theta) = \sum_{i=1}^{T}(Y_i - f(Y_{i-1}, \dots; \theta))^2 = \sum_{i=1}^{T} r_i^2(\theta)}$$

Parameters

Model prediction

Find the values of the parameters which maximise the probability of obtaining the data that we have observed (minimize $\boldsymbol{SSE(\theta)}$)

AIC (Akaike's Information Criterion)

$$AIC = -2 \ln(L) + 2m$$

L:= likelyhood

$m = p + q$.

By varying the choices of p, q, we want to pick the lowest AIC.

AIC penalizes overfitting

BIC (Bayesian Information Criterion)

$$\text{BIC} = -2\ \ln(L) + k(ln(\text{n}))$$

L:= likelyhood

K := number of parameters to estimate

By varying the choices of p, q, we want to pick the lowest AIC.

The BIC introduces a penalty term for the number of parameters in the model. The penalty term is larger in BIC than in AIC.

Model quality - R^2

- The R^2 quantifies the degree of any linear correlation between Y_{obs} and $Y_{pred.}$

$$\boldsymbol{S(\theta) = \sum_{i=1}^{T} (Y_i - f(Y_{i-1}, \ldots; \theta))^2 = \sum_{i=1}^{T} r_i^2(\theta)}$$

$$\boldsymbol{S_{tot}(\theta) = \sum_{i=1}^{T} (Y_i - \bar{Y})^2}$$

$$R^2 = 1 - \frac{\boldsymbol{S(\theta)}}{\boldsymbol{S_{tot}(\theta)}}$$

Error

$$\boldsymbol{RMSE} = \sqrt{\frac{\sum_{i=1}^{T}(Y_i - \widehat{Y_i})^2}{N}}$$

Prediction

The RMSE (or RMSD) of a sample is the quadratic mean of the differences between the observed values and predicted ones. These deviations are called **residuals** when the calculations are performed over the data sample that was used for estimation and are called **errors** (or prediction errors) when computed out-of-sample.

Error

$$\boldsymbol{MAPE} = \sum_{i=1}^{N} \sqrt{\frac{(Y_i - \widehat{Y_i})^2}{Y_i}}$$

Prediction

The **mean absolute percentage error** (MAPE), also known as mean absolute percentage deviation (MAPD), is a measure of prediction accuracy of a forecasting method in statistics.

Model selection and forecasting

Phase 2 (Estimation and testing (2/2)):

Testing:

1. Check ACF/PACF of residuals (errors)
2. Are residuals white noise ?

Model selection and forecasting

Phase 3 (Application):

Use the model to forecast.

TS Analysis in Python

- SciPy
- NumPy;
- Matplotlib;
- Pandas;

statsmodels

https://www.statsmodels.org/stable/index.html
https://github.com/statsmodels/statsmodels/)

ACF and PACF

statsmodels

Import the data in a Pandas DataFrame

- from pandas import read_excel
- series = read_excel([filename], sheet_name='MAdata’,)

https://pandas.pydata.org/docs/reference/api/pandas.read_excel.html

ACF and PACF

statsmodels

Import functions to plot ACF and PACF

- from statsmodels.graphics.tsaplots import plot_acf, plot_pacf

- plot_acf([series], title='', lags=xx)
- plot_pacf([series], title='', lags=xx)

https://www.statsmodels.org/stable/generated/statsmodels.graphics.tsaplots.plot_acf.html#statsmodels.graphics.tsaplots.plot_acf

ARIMA MODEL statsmodels

Compute ARIMA MODEL

from statsmodels.tsa.arima.model import ARIMA.

model = ARIMA([series], order=(p, d, q)),

where p, d, and q represent the parameters of the model

https://www.statsmodels.org/stable/generated/statsmodels.tsa.arima.model.ARIMA.html

https://www.statsmodels.org/stable/dev/generated/statsmodels.base.model.GenericLikelihoodModelResults.aic.html#statsmodels.base.model.GenericLikelihoodModelResults.aic

TD - Consignes

- Givent the file **priceData.xlsx**

- Write a Python program to **compute the ACF and PACF** (test several lags) of the time series contained in **priceData.xlsx**

- Write a Python program to **estimate and compute the best ARIMA** model of the time series in **priceData.xlsx, based on AIC criterion.**
- **Which kind of model is choosen ?**

References

- Andrew V. Metcalfe, **Paul S.P. Cowpertwait, Introductory Time Series with R** (2009).

- Aileen Nielsen, **Practical Time Series Analysis: Prediction with Statistics and Machine Learning** (2019).

- Changquan Huang , Alla Petukhina, **Applied Time Series Analysis and Forecasting with Python** (2022)