Compilation of Proofs for the Master's Comprehensive Exam in Complex Variables

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Contents

1 Cauchy's Inequalities

Statement. *Let $f(z)$ be analytic in the disk where $|z - z_0| \leq r$, and let $M = max\ |f(z)|$ in the disk. Let a_n be the n^{th} coefficient of the Taylor Series of f about z_0. Then $|a_n| \leq M/\ r^n$.*

Proof. By Cauchy's Formula for derivatives, we have

$$a_n = \frac{f^{(n)}(z_0)}{n!}$$

$$a_n = \frac{1}{2\pi i} \oint_{|w-z_0|=r} \frac{f(w)}{(w - z_0)^{n+1}}\ dw$$

Estimating the integral, we get

$$|a_n| \leq \frac{1}{2\pi} \cdot \frac{M}{r^{n+1}} \cdot 2\pi r$$

$$|a_n| \leq \frac{M}{r^n}$$

□

2 Liouville's Theorem

Statement. *The only bounded entire functions are constant.*

Proof. Let $f(z)$ be a bounded, entire function and let z be a fixed point of $\mathbb{C}$.

Let $C = \{w \in \mathbb{C} : |w| = R\}$ where $R \gg 1$ and $R \gg |z|$.

By Cauchy's Integral Formula,

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(w)}{w - z}\ dw$$

and

$$f(0) = \frac{1}{2\pi i} \oint_C \frac{f(w)}{w}\ dw$$

Therefore

$$|f(z) - f(0)| = \left| \frac{1}{2\pi i} \oint_C \left(\frac{f(w)}{w - z} - \frac{f(w)}{w} \right)\ dw \right|$$

$$|f(z) - f(0)| \leq \frac{1}{2\pi} \oint_C \frac{|f(w)| \cdot |z|}{|w(w - z)|}\ |dw|$$

Since f is bounded, $|f(z)| \leq M < \infty\ \forall\ z \in \mathbb{C}$, therefore

$$|f(z) - f(0)| \leq \frac{M \cdot |z|}{2\pi} \oint_C \frac{1}{R(R - |z|)}\ |dw|$$

Since $2\pi R$ is the length of C and $R(R-|z|)$ is constant with respect to w,

$$|f(z)-f(0)| \leq \frac{M\cdot|z|}{2\pi}\cdot\frac{2\pi R}{R(R-|z|)} = \frac{M\cdot|z|}{R-|z|} \to 0 \text{ as } R\to\infty$$

Therefore, $f(z)=f(0)$. Since z was any point of $\mathbb{C}$, f is constant. □

3 Riemann's Theorem on Removable Singularities

Statement. *If f is analytic and bounded in some punctured neighborhood of z_0, then $\lim_{z\to z_0} f(z)$ exists. If $f(z_0)$ is defined as this limit, then f becomes analytic at z_0.*

Proof. Define $g(z) = \begin{cases} (z-z_0)^2 f(z) & \text{if } z\neq z_0 \\ 0 & \text{if } z = z_0 \end{cases}$

We show g is analytic in a neighborhood of z_0. g is differentiable at z_0 because

$$g'(z_0) = \lim_{z\to z_0}\frac{g(z)-g(z_0)}{z-z_0} = \lim_{z\to z_0}\frac{(z-z_0)^2 f(z)-0}{z-z_0} = \lim_{z\to z_0}(z-z_0)f(z) = 0$$

The last step is valid since $|f(z)|$ is bounded. If $z\neq z_0$,

$$g'(z) = 2(z-z_0)f(z) + (z-z_0)^2 f'(z)$$

So g is analytic in a neighborhood of z_0. Therefore, it coincides with its Taylor Series centered at z_0, that is,

$$g(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + a_3(z-z_0)^3 + \cdots$$

But, $a_0 = g(z_0) = 0$ and $a_1 = g'(z_0) = 0$, therefore

$$g(z) = a_2(z-z_0)^2 + a_3(z-z_0)^3 + \cdots$$

$$f(z) = a_2 + a_3(z-z_0) + \cdots$$

Set $f(z_0) = a_2$. Since the power series is valid in a neighborhood of z_0 and $f(z)$ is bounded there, f is analytic at z_0.

□

4 Rouché's Theorem

Statement. *Let f and g be analytic inside and on a simple, closed curve C and suppose that $|g(z)| < |f(z)|$ when z is on C. Then $f(z)$ and $f(z)+g(z)$ have the same number of zeroes inside C.*

Proof. Let N be the number of zeroes of $f(z) + g(z)$ inside C.
By the Argument Principle,

$$N = \frac{1}{2\pi i} \oint_C \frac{f' + g'}{f + g}\ dz$$

$$N = \frac{1}{2\pi i} \oint_C \frac{f'}{f}\ dz + \frac{1}{2\pi i} \oint_C \left(\frac{f' + g'}{f + g} - \frac{f'}{f} \right)\ dz$$

Let $I = \oint_C \left(\frac{f'+g'}{f+g} - \frac{f'}{f} \right)\ dz$
If we can show that $I = 0$, then we are done. Find a common denominator and

$$I = \oint_C \frac{g'f - gf'}{f(f+g)}\ dz$$

$$I = \oint_C \frac{\frac{g'f - gf'}{f^2}}{1 + \frac{g}{f}}\ dz$$

Let $\varphi = \frac{g}{f}$, then

$$I = \oint_C \frac{\varphi'}{1 + \varphi}\ dz$$

Since $|\varphi| < 1$ on C,

$$I = \oint_C \varphi'(1 - \varphi + \varphi^2 - \varphi^3 + \cdots)\ dz$$

$$I = \oint_C \varphi'\ dz - \oint_C \varphi'\varphi\ dz + \oint_C \varphi'\varphi^2\ dz - \oint_C \varphi'\varphi^3\ dz + \cdots$$

$$I = \oint_C (\varphi)'\ dz - \frac{1}{2} \oint_C (\varphi^2)'\ dz + \frac{1}{3} \oint_C (\varphi^3)'\ dz - \frac{1}{4} \oint_C (\varphi^4)'\ dz + \cdots$$

We can see each integral is of a derivative and therefore zero, thus $I = 0$. $\square$

5 Open Mapping Theorem

Statement. *If a nonconstant function f is analytic on a connected, open set S, then the image of S under $w = f(z)$ is an open set.*

Proof. Let $f(z)$ be a nonconstant analytic function on a connected, open set S.
Let $z_0 \in S$ and set $w_0 = f(z_0)$.
Define $g(z) = f(z) - w_0$, a translation. $g(z)$ is nonconstant, analytic, and $g(z_0) = 0$, which must be an isolated zero of g.
Therefore, there is an $r > 0$ such that the disk $\gamma = \{|z - z_0| \leq r\}$ contains no other zeroes of g.
Let $\delta = \min\limits_{|z-z_0|=r} |g(z)|$. Let Δ be the open disk $\{|w - w_0| < \delta\}$.
Then for any z on the circle $C = \{z : |z - z_0| = r\}$ we have $|w - w_0| < |g(z)|$ for some fixed

$w \in \Delta$.
By Rouché's Theorem, $g(z)$ and $g(z) - (w - w_0)$ have the same number of zeroes inside the disk C.
Since g has at least one zero inside C, $g(z) - w + w_0 = f(z) - w_0 - w + w_0 = f(z) - w$ has at least one zero inside C.
So $w = f(z)$ for at least one z inside C. In other words, w is in the image under f of the disk bounded by γ. Therefore, every point w_0 in the image of f has a neighborhood contained in the image of f. $\square$

6 Schwarz' Lemma

Statement. *If f is analytic in a closed disk Δ of radius 1 centered at z_0, $f(z_0) = 0$, and $|f(z)| \leq M$ on the boundary of Δ; then $|f(z)| \leq M|z - z_0|$ for z inside Δ. Equality holds for some interior point of Δ (other than z_0) if and only if $f(z) \equiv Me^{i\theta}(z - z_0)$ for some real θ.*

Proof. Consider $g(z) = f(z)/(z - z_0)$ in Δ. Then $|g(z)| \leq M$ on the boundary of Δ and g has a removable singularity at z_0, which we may suppose has been removed. The maximum principle for g says that $|g(z)| \leq M$ inside Δ, whence $|f(z)| \leq M|z - z_0|$. If equality holds at an interior point, then g is a constant and $|g(z)| = M$. $\square$

7 Pringsheim's Lemma

Statement. *If $f(z) = \sum_{k=0}^{\infty} a_k z^k$, where $a_k \geq 0$ and the series has radius of convergence R, then we cannot make a direct analytic continuation of f to a neighborhood of the point R.*

Proof. Assume that $R = 1$ (ie., consider $f(Rz)$ instead.) and that $f(z)$ has an analytic continuation past 1.
The extended function is analytic in some disk D cenetered at 1 and the union of D with the unit disk $\{|z| < 1\}$ contains a disk centered at 1/2 with radius just greater than 1/2. Since

$$f^{(n)}\left(\frac{1}{2}\right) = \sum_{k=n}^{\infty} \frac{k!}{(k-n)!} a_k \left(\frac{1}{2}\right)^{k-n}$$

The Taylor Series of f about 1/2 for $x \in \mathbb{R}$ slightly larger than 1 is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}\left(\frac{1}{2}\right)}{n!} \left(x - \frac{1}{2}\right)^n$$

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(x - \frac{1}{2}\right)^n \sum_{k=n}^{\infty} \frac{k!}{(k-n)!} a_k \left(\frac{1}{2}\right)^{k-n}$$

Everything here is positive, so we can change the order of summation

$$f(x) = \sum_{k=0}^{\infty} a_k \sum_{n=0}^{k} \frac{1}{n!}\left(x - \frac{1}{2}\right)^n \frac{k!}{(k-n)!}\left(\frac{1}{2}\right)^{k-n}$$

The inner sum is the binomial expanson of

$$\left[\left(x - \frac{1}{2}\right) + \frac{1}{2}\right]^k = x^k$$

So we have

$$f(x) = \sum_{k=0}^{\infty} a_k x^k \text{ for some } x > 1$$

This series is the original power series for f, but now with $x > 1$. Convergence of this series for some $x > 1$ implies convergence for all z with $|z| < x$, so the Maclaurin series of f has radius of convergence greater than 1, a contradiction to our hypothesis. This shows that f cannot be continued directly past the point 1. □

8 Abel's Theorem

Statement. *If* $\sum_{n=0}^{\infty} a_n z^n$ *converges to* $f(z)$ *when* $|z| < R$ *and* $\sum_{n=0}^{\infty} a_n R^n$ *converges to* A*, then* $\lim_{x \to R^-} f(x) = A$.

Proof. Assume that $R = 1$ and $A = 0$. (ie., consider $g(z) = f(z/R) - A$ instead.)
We have $\sum_n a_n = 0$ and want to prove $\lim_{x \to 1^-} f(x) = 0$.

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$s_n = a_0 + a_1 + \cdots + a_n$$

$$f(x) = s_0 + (s_1 - s_0)x + (s_2 - s_1)x^2 + \cdots + (s_n - s_{n-1})x^n + \cdots$$

We can rearrange the terms of $f(x)$ because it is absolutely convergent.

$$f(x) = s_0(1-x) + s_1 x(1-x) + s_2 x^2(1-x) + \cdots + s_n x^n(1-x) + \cdots$$

Let $\epsilon > 0$ and since $f(x) = 1$,

$$\exists\ N(\epsilon) : |s_n| < \epsilon\ \forall\ n \geq N(\epsilon)$$

$$|f(x)| \leq (1-x)\left|s_0 + s_1 x + \cdots + s_{N-1}x^{N-1}\right| + (1-x)(|s_N|x^N + |s_{N+1}|n^{N+1} + \cdots)$$

$$|f(x)| \leq (1-x)(|s_0| + |s_1| + \cdots + |s_{N-1}|) + (1-x)\epsilon x^N(1 + x + x^2 + \cdots)$$

$$|f(x)| \leq (1-x)(|s_0| + |s_1| + \cdots + |s_{N-1}|) + \frac{(1-x)\epsilon x^N}{(1-x)}$$

$$\limsup_{x\to 1^-} |f(x)| \le \limsup_{x\to 1^-} \left[(1-x)(|s_0|+|s_1|+\cdots+|s_{N-1}|)+\epsilon x^N)\right]$$

$$\limsup_{x\to 1^-} |f(x)| \le \epsilon$$

Since ϵ can be made arbitrarily small,

$$\limsup_{x\to 1^-} f(x) = 0$$

$\square$

References

[1] Ralph P. Boas, *Invitation to Complex Analysis*, 2e, 2010, ISBN 978-0-88385-764-9

[2] Stephen D. Fisher, *Complex Variables*, 2e, 1999, ISBN 0-486-40679-2