Competitive Exclusion Problem

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1 Problem Definition

Definition 1.1. *Let $G = (V, E)$ be an undirected finite graph. We will sometimes write $N = |V|$ and $M = |E|$ when necessary.*

1. *If $v, w \in V$, we define the distance between v, w, which we denote as $d(v, w)$, as the length of the shortest path between v and w.*
2. *The* **diameter** *of G, denoted $\operatorname{diam}(G)$, is defined as*

$$\operatorname{diam}(G) = \max_{v,w} d(v, w).$$

Exercise 1.2. *Show that $\operatorname{diam}(G) \leq N - 1$. Show an example where this bound is attained. What is the minimum possible value for $\operatorname{diam}(G)$?*

Solution 1.3. *Suppose $\operatorname{diam}(G) = M \geq N$, then there exists v, w so that $d(v, w) = M$. There is a sequence $\{u_n\}_{n=1}^{M+1}$ so that $u_1 = v, u_{M+1} = w$, and $(u_n, u_{n+1}) \in E$ for each $1 \leq n \leq M$. But since $M + 1 > N$ we have that by the "pigeon hole principle" there exists $1 \leq i < j \leq M$ so that $u_i = u_j$. But then the path $(u_1, u_2, ..., u_{i-1}, , u_j, u_{j+1}, ..., u_M, u_{M+1})$ is a path from v to w of length $M - (j - i)$. But $d(v, w) = M$, a contradiction. Thus $\operatorname{diam}(G) \leq N - 1$. QED*

Let us define the following construction. Choose a graph $G = (V, E)$, and define a function

$$f \colon \{0, 1, \ldots, diam(G)\} \to \mathbb{R},$$

and then define the following system of differential equations:

$$\frac{d}{dt} x_v = x_v(r_v - \sum_{w \in V} a_{vw} x_w), \tag{1}$$ eq:ODE

where

$$a_{vw} = f(d(v, w)), \quad r_v = \sum_{w \in V} a_{vw}.$$

Exercise 1.4. *The vector $\mathbf{1} = (1, 1, 1, \ldots, 1)$ is always a fixed point of (1).*

Solution 1.5.

$$\frac{d}{dt} x_v = 1(r_v - \sum_{w \in V} a_{vw} 1) = r_v - r_v = 0, \textit{ for each } v = 1, 2, ..., N$$

ex:Jacobian **Exercise 1.6.** *Compute the Jacobian of (1) at $x = \mathbf{1}$.*

Solution 1.7. *Let $g_i(\mathbf{x}) = x_i(r_i - \sum_{j=1}^N a_{ij}x_j)$, then*

$$\frac{\partial}{\partial x_j} g_i(\mathbf{x}) = \begin{cases} -a_{ij}x_i, & \text{for } i \neq j \\ r_i - a_{ii}x_i - \sum_{k=1}^N a_{ik}x_k, & \text{for } i = j \end{cases}$$

and so

$$\frac{\partial}{\partial x_j} g_i(\mathbf{1}) = -a_{ij}.$$

So the Jacobian at $\mathbf{1}$ is $-A$. In general the Jacobian at $\mathbf{x}$ is

$$\frac{\partial}{\partial x_j} g_i(\mathbf{x}) = (A(\mathbf{1} - \mathbf{x}))_i \delta_{ij} - a_{ij}x_i$$
$$J(\mathbf{x}) = \mathrm{diag}((A(\mathbf{1} - \mathbf{x}))) - A\mathrm{diag}(\mathbf{x}).$$

The convention we should always use is that $f(0) \geq f(1) \geq f(2) \geq \cdots \geq f(\mathrm{diam}(G)) \geq 0$. As long as all of the f's are nonnegative, then this is a competition model; as long as the function is decreasing then this means competition is strictest amongst closer neighbors. Of course we can technically choose $f \equiv 0$ but this is a degenerate (and trivial) case, so let us always assume that $f(0) > 0$.

Another way of writing (1) is as follows. Let us define the operator $\odot\colon \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ as follows:

$$(x \odot y)_i = x_i y_i.$$

In this sense, it is the "dot-times" operator of MATLAB. Then we can write

$$\frac{d}{dt}\mathbf{x} = \mathbf{x} \odot (r - A\mathbf{x}).$$

2 Case 1: The ring

Let us define the ring graph R_N as the graph with $V = [N]$ and $(i, j) \in E$ iff $i = j \pm 1 \pmod N$.

Exercise 2.1. *Show that* $\mathrm{diam}(R_N) = \lfloor N/2 \rfloor$.

Solution 2.2. *Case* $\mathrm{diam}(R_N) \leq \lfloor N/2 \rfloor$*:*
Assume $1 \leq i < j \leq N$, and let $k = j - i$. Then there are paths $(i, i+1, i+2, ..., i+k-1, i+k)$ and $(j, j+1 \pmod N), ..., j+N-k-1 \pmod N), j+N-k \pmod N))$ from i to j which are of lengths k and $N-k$ respectively. Thus $d(i,j) \leq \min\{k, N-k\} \leq \lfloor N/2 \rfloor$. Since i, j were arbitrary, $\mathrm{diam}(R_N) \leq \lfloor N/2 \rfloor$.
Case $\mathrm{diam}(R_N) \geq \lfloor N/2 \rfloor$*:*
Show $d(0, \lfloor N/2 \rfloor) = \lfloor N/2 \rfloor$.
Fix k. Suppose there exists an M and a path $(v_0 = 0, v_1, ..., v_M = k)$ which does not contain all integers between $\{0, 1, 2, ..., k\}$ or $\{0, N-1, N-2, ..., k\}$. Then there exists $0 < n < k < m < N$ so that $v_i \neq n, m$ for each i. Then $\exists i$ so that $v_i \in \{m+1, ..., N-1, 0, 1, ..., n-1\}$ and $v_{i+1} \in \{n+1, ..., k, ..., m-1\}$. But this is a contradiction as $v_i = v_{i+1} \pm 1 \pmod N$. Thus $d(0,k) \geq \min\{k, N-k\}$. So $d(0, \lfloor N/2 \rfloor) = \lfloor N/2 \rfloor$. Thus $\mathrm{diam}(R_N) = \lfloor N/2 \rfloor$.

Big question right now: for which functions f is $\mathbf{1}$ stable? Said another way, when does the Jacobian computed in Exercise 1.6 have all negative (or nonpositive) eigenvalues?

For convenience let us define $c_{N-k} := c_k$, for $k = 1, 2, ..., , \lfloor N/2 \rfloor$. We will call the diameter of the graph $D := \lfloor N/2 \rfloor$. We will look into how convexity of $\{c_0, c_1, ..., c_D\}$ affects the stability of the system 1.

Circulant matrices have eigenvalues of the form $\lambda_j = c_0 + c_{N-1}\omega_j + \cdots + c_1\omega_j^{N-1}$ and eigenvectors $v_j = (1, \omega_j, \omega_j^2, ..., \omega_j^{N-1})^T$ where $\omega_j = \exp(2\pi i j/N)$, and $i = \sqrt{-1}$. Note that $\omega_j^k + \omega_j^{N-k} = 2\cos(2\pi jk/N)$.

There is another choice of eigenvectors that is often more convenient:

$$\begin{cases} v_0 = \mathbf{1} \\ v_j = (1, \cos(2\pi j 1/N), \cos(2\pi j 2/N), ..., , \cos(2\pi j(N-1)/N))^T, \text{ for } 1 \leq j < N/2 \\ v_{N/2} = (1, -1, 1, -1, ..., 1, -1)^T \text{ if } 2|N \\ v_{N-j} = (0, \sin(2\pi j 1/N), \sin(2\pi j 2/N), ..., , \sin(2\pi j(N-1)/N))^T, \text{ for } 1 \leq j < N/2 \end{cases}$$

Then since A is circulant the eigenvalues of A are

$$\begin{aligned} \lambda_j =& c_0 + c_{N-1}\omega_j + \cdots + c_1\omega_j^{N-1} \\ =& c_0 + \sum_{k=1}^{N-1} \cos(\frac{2\pi j}{N}k)c_k, \end{aligned} \tag{2}$$

lin e-val

for any j. So $\lambda_{N-j} = c_0 + \sum_{k=1}^{N-1} \cos(\frac{2\pi(N-j)}{N}k)c_k = \lambda_j$.

Now using the fact that $1 + \sum_{k=1}^{N-1} \cos(\frac{2\pi j}{N}k) = 0$ (for $j \neq 0$), we have that lin e-val 2 becomes

$$\lambda_j = c_0 - c + \sum_{k=1}^{N-1} \cos(\frac{2\pi j}{N}k)(c_k - c) \text{ for any c} \in \mathbb{R}, \tag{3}$$

alt lin e-v

which is often more convenient.

lin dec f's **Lemma 2.3.** *. The series* $\sum_{k=1}^{M-1}(M-k)\cos(\omega k) = -\frac{M}{2} + \frac{1}{2}\frac{1-\cos(M\omega)}{1-\cos(\omega)}$.

Proof. Let $z = e^{i\omega}$. Then

$$\begin{aligned} \sum_{k=1}^{M-1}(M-k)\cos(\omega k) &= \Re\left\{\sum_{k=1}^{M-1}(M-k)e^{i\omega k}\right\} \\ &= \Re\left\{\sum_{k=1}^{M-1}(M-k)z^k\right\} \\ &= \Re\left\{\sum_{k=1}^{M-1}Mz^k - \sum_{k=1}^{M-1}kz^k\right\} \\ &= \Re\left\{\frac{M(z-z^M)}{1-z} - \frac{z(1-z^M - Mz^{M-1} + Mz^M)}{(1-z)^2}\right\} \\ &= \Re\left\{\frac{M(z-z^2-z^M+z^{M+1})}{(1-z)^2} - \frac{(z-z^{M+1}-Mz^M+Mz^{M+1})}{(1-z)^2}\right\} \\ &= \Re\left\{\frac{M(z-z^2)}{(1-z)^2} - \frac{(z-z^{M+1})}{(1-z)^2}\right\} \\ &= \Re\left\{\frac{Mz}{1-z} + \frac{z^M-1}{z^{-1}(1-z)^2}\right\} \\ &= \Re\left\{\frac{M+Mz^{-1}}{(1-z)(1+z^{-1})} + \frac{z^M-1}{2\cos(\omega)-2}\right\} \\ &= \frac{-M}{2} + \frac{1}{2}\frac{1-\cos(M\omega)}{1-\cos(\omega)} \end{aligned}$$

$\square$

lin dec f's **Theorem 2.4.** . *Suppose $\{c_0, c_1, ..., c_M\}$ are linear and decreasing for $M \le D$ and $c_M = c_{M+1} = \cdots = c_D = 0$. Then the eigenvalues of A are $\lambda_j = \frac{1}{M}\frac{1-\cos(\frac{2\pi j}{N}M)}{1-\cos(\frac{2\pi j}{N})}$, for $j = \{1, ..., N-1\}$.*

Proof. Consider an integer $M \le D$, and $c_0 - c_1 = \cdots = c_{M-1} - c_M = \frac{1}{M}, c_M = c_{M+1} = \cdots = c_{\lfloor \frac{N}{2} \rfloor} = 0$. Now we have that the eigenvalues are

$$\begin{aligned}\lambda_j &= c_0 + \sum_{k=1}^{N-1} \cos(\frac{2\pi j}{N}k)c_k, \\ &= 1 + \frac{2}{M}\sum_{k=1}^{M-1}(M-k)\cos(\frac{2\pi j}{N}k) \\ &= \frac{1}{M}\frac{1-\cos(\frac{2\pi j}{N}M)}{1-\cos(\frac{2\pi j}{N})}. \end{aligned} \tag{4}$$

See Lemma sum for lin dec f's 2.3 for the last equality. □

rmk:gcd **Remark 2.5.** *Now $\gcd(N, M) = 1$ if and only if we have a strict negative definite Jacobian. Indeed, if $\gcd(N, M) = 1$ then for $j \in \{1, 2, ..., N-1\}$, $\frac{Mj}{N} \notin \mathbb{N}$ and thus $\lambda_j = \frac{1}{M}\frac{1-\cos(\frac{2\pi j}{N}M)}{1-\cos(\frac{2\pi j}{N})} > 0$. Also, $\lambda_0 = \sum_{k=0}^{N-1} c_k = 1$. On the other hand if $\gcd(N, M) > 1$ we can let $j = \frac{N}{\gcd(N,M)} \in \{2, ..., N-1\}$. Then one can check that $\lambda_j, \lambda_{N-j} = 0$. Additionally, since $\omega_j^k + \omega_j^{N-k} = 2\cos(2\pi jk/N)$ we have that $v_j + v_{N-j} \in R^n$, thus $v_j + v_{N-j} \in \ker A$.*

Now $v_j + v_{N-j} \in \ker A$ implies that $A(\alpha(v_j + v_{N-j}) + \mathbf{1}) = A\mathbf{1}$. Thus we have a hyperplane of fixed points.

_convx_comb **Theorem 2.6.** *Consider the set $\{c_0, c_1, ..., c_D\}$. Define $\alpha_D = c_{D-1} - c_D$ and $\alpha_k = c_{k-1} - 2c_k + c_{k+1}$, for $k = 1, 2, ..., D-1$. Then the eigenvalues of A are $\lambda_j = \sum_{i=1}^{D} \alpha_i \frac{1-\cos(\frac{2\pi j}{N}i)}{1-\cos(\frac{2\pi j}{N})}$ for $j \in \{1, 2, ..., N-1\}$.*

Proof. By alt lin e-val 3 we have that

$$\begin{aligned}\lambda_j &= c_0 - c_D + \sum_{k=1}^{N-1}\cos(\frac{2\pi j}{N}k)(c_k - c_D), \\ &= c_0 - c_D + 2\sum_{k=1}^{D-1}\cos(\frac{2\pi j}{N}k)(c_k - c_D), \\ &= \sum_{i=1}^{D} i\alpha_i + 2\sum_{k=1}^{D-1}\cos(\frac{2\pi j}{N}k)\left[\sum_{i=k+1}^{D}(i-k)\alpha_i\right], \text{details left for the reader,} \\ &= \sum_{i=1}^{D} i\alpha_i + 2\sum_{1 \le k \le i-1 \le D-1}\cos(\frac{2\pi j}{N}k)(i-k)\alpha_i, \\ &= \alpha_1 + \sum_{i=2}^{D}\left[i\alpha_i + 2\alpha_i\sum_{k=1}^{i-1}(i-k)\cos(\frac{2\pi j}{N}k)\right], \\ &= \sum_{i=1}^{D}\alpha_i\frac{1-\cos(\frac{2\pi j}{N}i)}{1-\cos(\frac{2\pi j}{N})}, \text{by lemma sum for lin dec f's 2.3.} \end{aligned} \tag{5}$$

e-vals_ring

□

Corollary 2.7. *Suppose that $\{c_0, c_1, ..., c_D\}$ is convex and decreasing. That is $c_0 - c_1 \ge \cdots \ge c_{D-1} - c_D \ge 0$, $\alpha_D = c_{D-1} - c_D \ge 0$, and $\alpha_k = c_{k-1} - 2c_k + c_{k+1} \ge 0$ for $k = 1, 2, ..., D-1$. Then $\lambda_j = \sum_{i=1}^{M} \alpha_i \frac{1-\cos(\frac{2\pi j}{N}i)}{1-\cos(\frac{2\pi j}{N})} \ge 0$ for $j \in \{0, 1, 2, ..., N-1\}$. Also, $\gcd(N, \{i : \alpha_i \ne 0\}) = 1$ if and only if A is positive definite. In particular, strict convexity gives us a stable fixed point.*

Proof. Apply Theorem e-val_convx_comb 2.6 and see Remark rmk:gcd 2.5. $\square$

Corollary 2.8. *Suppose that $\{c_0, c_1, ..., c_D\}$ is concave and decreasing. That is $c_0 - c_1 \leq \cdots \leq c_{D-1} - c_D \leq 0$, $\alpha_D = c_{D-1} - c_D > 0$, and $\alpha_k = c_{k-1} - 2c_k + c_{k+1} \leq 0$ for $k = 1, 2, ..., D-1$. Then $\lambda_j = \sum_{i=1}^{M} \alpha_i \frac{1-\cos(\frac{2\pi j}{N} i)}{1-\cos(\frac{2\pi j}{N})}$ for $j \in \{0, 1, 2, ..., N-1\}$. Also if N is even and $\{c_0, c_1, ..., c_D\}$ is not linear, then $\lambda_2 < 0$.*

Proof. Apply Theorem e-val_convx_comb 2.6 and see Remark rmk:gcd 2.5. $\square$

Remark 2.9. *Note that we cannot guaranty instability in general here, but we can say that having concave, decreasing c_i's is equivalent to $\alpha_i \leq 0$ and $\alpha_M \geq \sum_{i=1}^{M} |\alpha_i|$, and so there always exists an $\omega \in [0, 2\pi)$ so that $\sum_{i=1}^{M} \alpha_i \frac{1-\cos(\omega i)}{1-\cos(\omega)} < 0$. Then we can ask whether or not a $j \in \mathbb{N}$ exists so that $\omega = \frac{2\pi j}{N}$.*

Suppose we want to more carefully analyze the stability in the concave case. Then we might want a polynomial form for the sum $\sum_{i=1}^{D} \alpha_i \frac{1-\cos(\frac{2\pi j}{N} i)}{1-\cos(\frac{2\pi j}{N})}$. In general

$$\cos(n\omega) = 2^{n-1}\cos^n(\omega) + n \sum_{k=1}^{D} \frac{(-1)^k}{k} \binom{n-k-1}{k-1} 2^{n-2k-1} \cos^{n-2k}(\omega),$$

so we have that

$$\alpha_n(1 - \cos(n\omega)) = \alpha_n - \alpha_n 2^{n-1}\cos^n(\omega) - \alpha_n n \sum_{k=1}^{D} \frac{(-1)^k}{k} \binom{n-k-1}{k-1} 2^{n-2k-1} \cos^{n-2k}(\omega).$$

Let $c_i := -\alpha_i 2^{i-1} - 2^{i-1} \sum_{k=1}^{\lfloor \frac{D-i}{2} \rfloor} \alpha_{i+2k} \frac{(-1)^k}{k} \binom{i+k-1}{k-1}$ and $c_0 := \sum_{k=1}^{D} \alpha_k - 2^{-1} \sum_{k=1}^{\lfloor \frac{D}{2} \rfloor} \alpha_{2k} \frac{(-1)^k}{k}$. Then we have that

$$\sum_{i=1}^{D} c_i \cos^i\left(\frac{2\pi j}{N}\right) = \sum_{i=1}^{D} \alpha_i \left(1 - \cos\left(\frac{2\pi j}{N} i\right)\right).$$

We originally asked does the stability improve when edges are added to the graph. Here is an example were stability is in fact lost. Consider the graph $G_1 = R_6$ and let G_2 have edges $(1,3), (2,4)$, and $(1,5)$ in addition to G_1. Let $c_0 = 1.51, c_1 = 1, c_2 = .5, c_3 = .3$. Then the corresponding matrices are

$$\begin{pmatrix} 1.51 & 1 & .5 & .3 & .5 & 1 \\ 1 & 1.51 & 1 & .5 & .3 & .5 \\ .5 & 1 & 1.51 & 1 & .5 & .3 \\ .3 & .5 & 1 & 1.51 & 1 & .5 \\ .5 & .3 & .5 & 1 & 1.51 & 1 \\ 1 & .5 & .3 & .5 & 1 & 1.51 \end{pmatrix} \text{ and } \begin{pmatrix} 1.51 & 1 & 1 & .5 & 1 & 1 \\ 1 & 1.51 & 1 & 1 & .5 & .5 \\ 1 & 1 & 1.51 & 1 & .5 & .5 \\ .5 & 1 & 1 & 1.51 & 1 & .5 \\ 1 & .5 & .5 & 1 & 1.51 & 1 \\ 1 & .5 & .5 & .5 & 1 & 1.51 \end{pmatrix}$$

respectively. The first matrix has the following eigenvalues $\{4.81, 1.71, 1.71, 0.31, 0.31, 0.21\}$ which are all positive as expected. The second matrix has eigenvalues $\{5.52674, 1.59281, 1.06746, 0.51, 0.439637, -0.0766491\}$ which contains a negative value and thus unstable. Thus the stability is lost after adding edges.

3 Case 2: Hamiltonian Cycles

In this section we will attempt to generalize from rings to Hamiltonian cycles. This is a natural step from the ring as any Hamiltonian cycle may be written as a ring with added edges. Also the class of connected graphs which are Hamiltonian cycles is quite large.

We will start by analyzing conditions of stability for graphs which come from adding an edge to a ring. In this section we will define $Q_A(\mathbf{x}) := \frac{\langle A\mathbf{x}, \mathbf{x} \rangle}{\langle \mathbf{x}, \mathbf{x} \rangle}$ for a matrix A.

Lemma 3.1. *Suppose that $\{v^1, ..., v^N\}$ are eigenvectors of a matrix A. Let $\lambda_1^A \leq ... \leq \lambda_N^A$ and $\lambda_1^B \leq ... \leq \lambda_N^B$ be the eigenvalues of the two symmetric $N \times N$ matrices A and B respectively.*

Then $\lambda_k^B \geq \lambda_k^A$ if $\min_{\mathbf{x} \in U} Q_B(\mathbf{x}) \geq \min_{\mathbf{x} \in U} Q_A(\mathbf{x})$ where $U := \operatorname{span}\{v^k, ..., v^N\}$.

Proof. Apply Min-Max theorem. □

Remark 3.2. *Next thing I will do is add an example of* **??** *and illustrations and shit... Also I will try to generalize* **??** *to all Hamiltonian cycles then I will prove that if f is not of the form $c_0 \geq c_1 = c_2 = ... = c_D$ then there exists a graph with unstable behavior.*

4 Determinant of the $\alpha - \lambda$ matrix

Theorem 2.6 gives us a matrix $M_N = \left(\frac{1-\cos(2\pi ij/N)}{1-\cos(2\pi i/N)}\right)_{i,j=1}^{D}$ so that

$$M_N \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_D \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_D \end{pmatrix}. \tag{6}$$

For simplicity consider (prove later)

$$\begin{aligned} L_N := & \left| \left(1 - \cos\left(\frac{2\pi ij}{N}\right)\right)_{i,j=1}^{D} \right|^2 \\ = & D \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}^* + 2 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}^* + \operatorname{diag} \begin{pmatrix} D/2 \\ D/2 \\ \vdots \\ D/2 \end{pmatrix} + \frac{D}{2} \mathbf{e}_N \mathbf{e}_N^*. \end{aligned}$$

Finish later...

5 Properties of the competitive Lotka-Volterra

Consider the directed set $(\wp([N]), \leq)$. For the duration of this section denote Q_S as the $|S| \times N$ matrix defined by $(Q_S)_{i,j} = \delta_{S_i,j}$, given $S \in \wp([N])$. Moreover, denote the associated projection $P_S := Q_S^* Q_S$. For example, if $S = \{1, 3, 4\}$ then

$$Q_{\{1,3,4\}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, P_{\{1,3,4\}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now $P_S = \operatorname{diag}(\sum_{i \in S} \mathbf{e}_i) \leq I$ is always true. Furthermore, we will use subscript notation to indicate the persistence of a population in our model:

$$\begin{aligned} \tilde{A}_S &:= Q_S A Q_S^* & A_S &:= P_S A P_S^* \\ \tilde{\mathbf{r}}_S &:= Q_S \mathbf{r} & \mathbf{r}_S &:= P_S \mathbf{r} \\ \tilde{\mathbf{p}}_S &:= \tilde{A}_S^{-1} \tilde{\mathbf{r}}_S & \mathbf{p}_S &:= A_S^\dagger \mathbf{r}_S \end{aligned} \tag{7}$$

proj_vars

Note that the pseudo inverse $A_S^\dagger$ is equal to $Q_S^*(Q_S A Q_S^*)^{-1} Q_S$, and that $\mathbf{p}_S = Q_S^* \tilde{\mathbf{p}}_S$. Typically $\tilde{A}_S$ is invertible and thus the equations 7 will suffice. If ever $\tilde{A}_S$ in not invertible defining $\tilde{\mathbf{p}}_S := \tilde{A}_S^\dagger \tilde{\mathbf{r}}_S$ will suffice.

Definition 5.1. *We say that $\mathbf{x} \in \mathbb{R}^N$ persists according to S if $P_S\mathbf{x} = \mathbf{x}$ and $Q_S\mathbf{x} \in \mathbb{R}^{|S|}_{++}$.*

Notice that for any $\mathbf{x} \in \mathbb{R}^N_+$, if $P_S\mathbf{x} = \mathbf{x}$ then the species corresponding to the set $[N] \setminus S$ are extinct. Moreover, if $Q_S\mathbf{x} \in \mathbb{R}^{|S|}_{++}$ then the species corresponding to the set S are not extinct. As an example consider $\mathbf{x} \in \mathbb{R}^4$,

$$Q_S\mathbf{x} = (x_1, x_3, x_4)^T \text{ and}$$
$$P_S\mathbf{x} = (x_1, 0, x_3, x_4)^T.$$

rm:sub_dyn **Remark 5.2.** *Suppose that S is given. If $\mathbf{x}$ persists according to S, then the dynamics of $\dot{\mathbf{x}} = \mathbf{x} \odot (\mathbf{r} - A\mathbf{x})$ is equivalent to $\dot{\mathbf{x}}_S = \mathbf{x}_S \odot (\tilde{\mathbf{r}}_S - \tilde{A}_S\mathbf{x}_S)$, where $\mathbf{x}_S \in \mathbb{R}^{|S|}_+$. Indeed,*

$$Q_S\dot{\mathbf{x}} = Q_S[\mathbf{x} \odot (\mathbf{r} - AP_S\mathbf{x})] = Q_S\mathbf{x} \odot (Q_S\mathbf{r} - Q_SAQ_S^*Q_S\mathbf{x}) = Q_S\mathbf{x} \odot (\tilde{\mathbf{r}}_S - \tilde{A}_SQ_S\mathbf{x}). \quad (8)$$ eq:sub_dyn

So if $\tilde{\mathbf{p}}_S$ fixes $\dot{\mathbf{x}}_S = \mathbf{x}_S \odot (\tilde{\mathbf{r}}_S - \tilde{A}_S\mathbf{x}_S)$ then $\mathbf{p}_S$ fixes $\dot{\mathbf{x}} = \mathbf{x} \odot (\mathbf{r} - A\mathbf{x})$. Be advised that $\mathbf{p}_S$ may not always in the domain of the dynamical system $\mathbb{R}^N_+$, this would then lead to instability.

:stable_sub **Theorem 5.3.** *Suppose we are given S and that $A = A^*$. If $\tilde{\mathbf{r}}_S \in \tilde{A}_S(\mathbb{R}^{|S|}_{++})$ and $\tilde{A}_S$ is positive definite, then $\tilde{A}_S^{-1}\tilde{\mathbf{r}}_S$ is a stable fixed point for the dynamical system $\dot{\mathbf{x}}_S = \mathbf{x}_S \odot (\tilde{\mathbf{r}}_S - \tilde{A}_S\mathbf{x}_S)$, where $\mathbf{x}_S \in \mathbb{R}^{|S|}_+$.*

Proof. By remark rm:sub_dyn 5.2, $\tilde{\mathbf{p}}_S = \tilde{A}_S^{-1}\tilde{\mathbf{r}}_S$ is a fixed point in $Q_S(\mathbb{R}^N_+)$, and $\tilde{A}_S$ which equals $Q_SAQ_S^*$ is self-adjoint. Consider the function

$$V(\mathbf{x}_S) := (\mathbf{x}_S - \mathbf{p}_S)^*\tilde{A}_S(\mathbf{x}_S - \mathbf{p}_S).$$

Then V is radially unbounded and positive definite (w.r.t. $\mathbf{x}_S$). Finally,

$$\begin{aligned}\frac{d}{dt}[V(\mathbf{x}_S(t))] &= \sum_{i\in S} V_{\mathbf{x}_S[i]}(\mathbf{x}_S)\frac{d\mathbf{x}_S[i]}{dt} \\ &= \sum_{i\in S} 2[\tilde{A}_S(\mathbf{x}_S - \mathbf{p}_S)][i]\mathbf{x}_S[i][\tilde{A}_S(\mathbf{p}_S - \mathbf{x}_S)][i] \\ &= -\sum_{i\in S} 2\mathbf{x}_S[i](\tilde{A}_S(\mathbf{x}_S - \mathbf{p}_S)[i])^2 \\ &< 0.\end{aligned}$$

Thus V is a Lyapunov function, and so $\tilde{\mathbf{p}}_S$ is a stable fixed point in $Q_S(\mathbb{R}^N_{++})$ for the dynamical system $\dot{\mathbf{x}}_S = \mathbf{x}_S \odot (\tilde{\mathbf{r}}_S - \tilde{A}_S\mathbf{x}_S)$. □

Note that $\mathbf{p}_S$ is not necessarily attracting in any open neighborhood of $\mathbb{R}^N_+$.

hm:incl_As **Theorem 5.4.** *Suppose $S_2 \subset S_1$ and $\tilde{A}_1 > 0$, then $\tilde{A}_2 > 0$.*

Proof. Notice that $Q_2P_1 = Q_2$, so $\tilde{A}_2 = Q_2AQ_2^* = Q_2P_1AP_1^*Q_2^* = Q_2Q_1^*\tilde{A}_1Q_1Q_2^*$. Thus $\tilde{A}_2$ is a $|S_2| \times |S_2|$ principal submatrix of $\tilde{A}_1$. Now applying the inclusion principle we have for the smallest eigenvalue of $\tilde{A}_2$, for $\tilde{A}_2(\lambda_0) \geq \tilde{A}_1(\lambda_0) > 0$. □

hm:exp_dec **Theorem 5.5.** *The population x_i does not decay faster than exponentially.*

Proof. By (bounded theorem hasn't been written yet, come back to this one) there is a vector $\mathbf{b}$ so that

$$\dot{x}_i = x_i\left(r_i - \sum_{j=1}^N A_{ij}x_j\right) \geq x_i\left(r_i - \sum_{j=1}^N A_{ij}b_j\right),$$

hence the conclusion. □

rgest_space **Theorem 5.6.** *Suppose $S_2 \subset S_1$, $\tilde{A}_1 > 0$, and $\mathbf{r}_1 \in A_1(\mathbb{R}^N_{++})$. Then $Q_2^* \tilde{A}_2^{-1} \tilde{\mathbf{r}}_2$ is not attracting in $\mathbb{R}^N_{++}$.*

Proof. Let $\mathbf{p}_2 = A_2^\dagger \mathbf{r}_2$ (that is $Q_2^* \tilde{A}_2^{-1} \tilde{\mathbf{r}}_2$), and $\mathbf{p}_1 = A_1^\dagger \mathbf{r}_1$. Now $\mathbf{p}_2$ fixes $\dot{\mathbf{x}} = \mathbf{x} \odot (\mathbf{r} - A\mathbf{x})$, so $Q_2 A(\mathbf{1} - \mathbf{p}_2) = 0$. Thus $\forall k \in S_2$ we have that $\sum_{i \in [N]} A[i,k](\mathbf{1} - \mathbf{p}_2[i]) = 0$.

Now $Q_1 A \mathbf{1} = Q_1 \mathbf{r}_1 \geq Q_1 \mathbf{r}_2 = Q_1 A \mathbf{p}_2$ which means that $Q_1 A(\mathbf{1} - \mathbf{p}_2) \geq 0$. On the other hand we have both $Q_1 A \mathbf{1} = Q_1 \mathbf{r} = \tilde{\mathbf{r}}_1 = \tilde{A}_1 \tilde{\mathbf{p}}_1$ and $Q_1 A \mathbf{p}_2 = \tilde{A}_1 Q_1 \mathbf{p}_2$, which implies that $Q_1 A(\mathbf{1} - \mathbf{p}_2) = \tilde{A}_1(\tilde{\mathbf{p}}_1 - Q_1 \mathbf{p}_2)$ which is not 0 since $\tilde{A}_1 > 0$. Thus $\exists k \in S_1 \setminus S_2$ so that $\sum_{i \in [N]} A[i,k](\mathbf{1} - \mathbf{p}_2[i]) > 0$.

Let $\epsilon = \sum_{i \in [N]} A[i,k](\mathbf{1} - \mathbf{p}_2[i])$. By continuity, for $\mathbf{x}$ in arbitrarily small neighborhoods of $\mathbf{p}_2$ we have $\dot{\mathbf{x}}[k] > \frac{\epsilon}{2} \mathbf{x}[k]$. Thus $\mathbf{p}_2$ is not attracting in $\mathbb{R}^N_{++}$. □