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Scattering Amplitudes in Twistor String Theory

by

Matheus Loss Lize

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"We did it, we bashed them, wee Potters the one,
and Voldys gone moldy, so now lets have fun!"

- Peeves

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Abstract

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Moonstone (also known as the wishing stone[1]) is found in a variety of colors. Its supposed magical effects include helping a person gain emotional balance. Since Harry spent much of book five emotionally unbalanced, it is perhaps fitting that he was forced to write an essay on the stone's use in Potions-making. It is a gemstone of medium value. Moonstones are a milky colour and shine very brightly, almost as though they are a source of their own light. They are a useful potion ingredient; powdered moonstones are used as an ingredient for the Draught of Peace and in several Love Potions. Powdered Moonstone is also an ingredient in in Potion No. 86 which is likely an experimental potion. Moonstones were also known to be among the gems set into Muriel's tiara.

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Advice given by Helga Hufflepuff has been a great help in...

To my beloved Ernie Macmillan for all the...

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Speed of Light $c = 2.997\ 924\ 58 \times 10^{8}\ \mathrm{ms}^{-\mathrm{S}}$ (exact)

For/Dedicated to/To my…

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2.1 Null momenta

We start by remembering (appendix A) that the Lorentz Group $SO(3,1)$ is isomorthic to $SL(2,\mathbb{C})$. A Lorentz vector p_m ($m = 0,1,2,3$) can be constructed by the product of two spinors, one from $(1/2,0)$ representation and one from $(0,1/2)$. The spinor λ_α, $\alpha = 1,2$, that transforms under $(1/2,0)$ are called left handed Chiral spinors and $\bar{\lambda}_{\dot{\alpha}}$, $\dot{\alpha} = \dot{1},\dot{2}$ that transforms under$(0,1/2)$ are called the right handed Chiral spinors.

The map that take the Lorentz vector index m to two spinors index $(\alpha\dot{\alpha})$ is done by a linear combination of Pauli Matrices plus identity $[\sigma^m = (-I,\vec{\sigma})]$:

$$p_{\alpha\dot{\alpha}} = p_m \sigma^m{}_{\alpha\dot{\alpha}} = -p_0 I + \vec{\sigma}\cdot\vec{p} \tag{2.1}$$

As a consequence of (**??**) we get:

$$\det(p_{\alpha\dot{\alpha}}) = -p^m p_m \tag{2.2}$$

the minus sign is due to the metric $\eta_{mn} = (-+++)$. Note that we are using p to denote different objects, but you will be able to spot the difference by the context. Because we are interested in massless particles, $p^2 = 0$ implies $\det(p_{\alpha\dot{\alpha}}) = 0$. So p has one eigenvalue equal to zero, the rank of the 2×2 matrix goes down $2 \to 1$. Then we can write the matrix p as a product of two component spinors:

$$p_{\alpha\dot{\alpha}} = \lambda_\alpha \bar{\lambda}_{\dot{\alpha}} \tag{2.3}$$

these spinors are usually called helicity spinors (now you get the chapter name). Looking at (**??**) you could make the observation that $\lambda, \bar{\lambda}$ are not sufficient to determinate p, there is a scale freedom:

$$(\lambda, \bar{\lambda}) \to (t\lambda, t^{-1}\bar{\lambda}) \quad t \in \mathbb{C}^* \tag{2.4}$$

The momentum p_m is real, so $p_{\alpha\dot{\alpha}} = p^*_{\alpha\dot{\alpha}} \Rightarrow \bar{\lambda}^*_{\dot{\alpha}} = \lambda_\alpha$, and the scale parameter t become just a phase $e^{i\theta}$. The spinors $(\lambda, \bar{\lambda})$ are not independent. This is the real world, but we are theoretical physicists and we can do almost everything, for example consider the case where the momentum is complex, then we get rid off the constrain $\bar{\lambda}^*_{\dot{\alpha}} = \lambda_\alpha$. Your mathematician friend would say that you are complexing the Lorentz group and tell you that it is locally isomorphic to $SO(3,1,\mathbb{C}) \cong SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$. The reasons to consider a complex momentum, will be clear when we start calculating scattering amplitudes, so hold up. Of course in the end of the day, we have to return to real world and set p to be real.

Another way to make the spinors $(\lambda, \bar{\lambda})$ be independent, is to consider the Lorentz group with a different signature $\eta = (-+++) \to (--++)$, or we if you want to be dramatic, change a space dimension to a time one. We can use the fact that $SO(2,2) \cong SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$, and write $(\lambda, \bar{\lambda})$ as two real independent spinors.(Appendeix A).

A quick exercise is to count the degrees of freedom for each case. In complex momentum we have $4 \times 2 - 2 = 6$ real parameters, the two is due to $\det p = 0$ gives two constraints (real and complex). Or for complex $(\lambda, \bar{\lambda})$ minus the scale $2 \times 2 - 1 = 3$ complex or 6 reals. In the real momentum case we have $4 - 1 = 3$ real parameters.

Given two spinors λ, μ the Lorentz invariant object is:

$$\langle \lambda, \mu \rangle \equiv \varepsilon^{\alpha\beta} \lambda_\alpha \mu_\beta = \lambda_\alpha \mu^\alpha \tag{2.5}$$

where we use the antisymmetric tensor $\varepsilon^{\alpha\beta}$ defined as $\varepsilon_{12} = \varepsilon^{21} = -1$ and $\varepsilon^{\alpha\beta}\varepsilon_{\beta\rho} = \delta^\alpha{}_\rho$, to lower and raise the indices. From the definition (**??**) and the tensor $\varepsilon^{\alpha\beta}$ the product $\langle \lambda, \mu \rangle = -\langle \mu, \lambda \rangle$ is antisymmetric. In particular, if $\langle \lambda, \mu \rangle = 0$ implies $\mu \sim \lambda$. This is true because λ, μ are commuting variables, note that if they were anti commuting $\langle \lambda, \mu \rangle = \langle \mu, \lambda \rangle$ as in the appendix.

The same thing is valid for two dotted spinors:

$$[\lambda, \mu] \equiv \varepsilon^{\dot{\alpha}\dot{\beta}} \bar{\lambda}_{\dot{\alpha}} \bar{\mu}_{\dot{\beta}} = \bar{\lambda}_{\dot{\alpha}} \bar{\mu}^{\dot{\alpha}} \tag{2.6}$$

Let's see how others objects looks in the spinor formalism. Given two null momentum $p^{\dot{\alpha}\alpha} = \lambda^{\alpha}\bar{\lambda}^{\dot{\alpha}}, \quad q_{\alpha\dot{\alpha}} = \mu_{\alpha}\bar{\mu}_{\dot{\alpha}}$:

$$(p+q)^2 = 2p \cdot q = 2\lambda^{\alpha}\bar{\lambda}^{\dot{\alpha}}\mu_{\alpha}\bar{\mu}_{\dot{\alpha}} = 2\langle\lambda, \mu\rangle[\lambda, \mu] \tag{2.7}$$

For instance if we label different momentum by a number $p^{(1)}, p^{(2)}, p^{(3)}, p^{(4)}$... we can simplify even more the notation. Using the numbers to identity the momentum then write $(p^{(1)} + p^{(2)})^2 = 2\langle 1, 2\rangle[1, 2]$. This notation is common in the literature, and very elegant.

Recall that the scattering amplitudes for gluons are described by momentum p_m and polarization vectors ϵ^m. We already have p in spinors, now we need the polarization vector. From Yang Mills Equation of motion in momentum space:

$$p \cdot \epsilon = 0 \tag{2.8}$$

this represents the fact that the polarization vector ϵ^m does not have longitudinal components and it has a gauge symmetry $\epsilon^m \to \epsilon^m + bp^m$. To construct ϵ^m as a bi-spinor we can guess the result using its symmetries. Also there is not so many objects to use. Let me define $\epsilon^{+}_{\alpha\dot{\alpha}} = d^{-1}\mu_{\alpha}\bar{\lambda}_{\dot{\alpha}}$ where d is a Lorentz invariant object, and μ_{α} is an arbitrary spinor. The polarization has to be invariant under a scale $\mu \to a\mu$ because μ is arbitrary. Thus $d \sim \mu_{\alpha}$, but d is Lorentz invariant, and the only object that we have to contract is λ^{α}. The result is:

$$\epsilon^{+}_{\alpha\dot{\alpha}} = \frac{\mu_{\alpha}\bar{\lambda}_{\dot{\alpha}}}{\langle\mu, \lambda\rangle} \tag{2.9}$$

Just by looking at (**??**) we see that $\epsilon^{+}_{\alpha\dot{\alpha}}p^{\dot{\alpha}\alpha} = 0$ due $[\lambda, \lambda] = 0$. The gauge transformation($\epsilon^m \to \epsilon^m + bp^m$) is now translated to a spinor shift ($\mu_{\alpha} \to a\mu_{\alpha} + b\lambda_{\alpha}$). Then we get:

$$\epsilon^{+}_{\alpha\dot{\alpha}} \to \epsilon^{+}_{\alpha\dot{\alpha}} + b\frac{\lambda_{\alpha}\lambda_{\dot{\alpha}}}{\langle\mu, \lambda\rangle} \tag{2.10}$$

that is what we expect. We know that the polarization vector (the photon) has two degrees of freedom. So we have another polarization. Doing the same process we find:

$$\epsilon^{-}_{\alpha\dot{\alpha}} = \frac{\bar{\mu}_{\dot{\alpha}}\lambda_{\alpha}}{[\mu, \lambda]} \tag{2.11}$$

Note that this vectors are normalized $\epsilon^- \cdot \epsilon^+ = 1$.

Now we return to the scaling and make the connection to helicity. Helicity is the quantum number that tell us how the spinor change under a rotation. It is the spin for a massless particle. The helicity is a Lorentz conserved quantity. Can be define as the projection of the spin operator $\vec{S}$ in to the 3-momentum $\vec{p}$

$$h \equiv \frac{\vec{S} \cdot \vec{p}}{|\vec{p}|} \tag{2.12}$$

In the massless case the particle is moving at the speed of light. So we can not do a boost that invert the direction of the rotation.

Let's write the massless Dirac equation in terms of Weyl basis:

$$(\bar{\sigma}^m)^{\dot{\alpha}\alpha} \partial_m \psi_\alpha = 0 \tag{2.13}$$

$$(\sigma^m)_{\alpha\dot{\alpha}} \partial_m \bar{\psi}^{\dot{\alpha}} = 0 \tag{2.14}$$

we see that if we multiply by $\sigma^n_{\dot{\alpha}\alpha} \partial_n$ we get the massless Klein-Gordon equation (using $\mathrm{Tr}\sigma^n \bar{\sigma}^m = -2\eta^{mn}$):

$$\partial^m \partial_m \psi_\alpha = 0 \tag{2.15}$$

So it has a plane wave solution $\psi_\alpha = L_\alpha e^{ip\cdot x}$, for a constant L_α Where from (**??**) L_α must satisfy $L_\alpha p^{\dot{\alpha}\alpha} = \langle L, \lambda \rangle \bar{\lambda}^{\dot{\alpha}} = 0$ that implies $L_\alpha = c\lambda_\alpha$. We get the wave function

$$\psi_\alpha = c\lambda_\alpha e^{ip^{\dot{\alpha}\alpha} x_{\alpha\dot{\alpha}}} \tag{2.16}$$

The spinor transforms as a rotation by angle θ around the $\vec{n}$ direction as(appendix),

$$\psi_\alpha = e^{i\frac{\vec{\theta}\cdot\vec{n}}{2}} \psi_\alpha \tag{2.17}$$

this implies that λ carries half units of angular momentum. Now if we define the scalar parameter $t \equiv e^{i\frac{\vec{\theta}\cdot\vec{n}}{2}}$ we see that the wave function scales as t^{-2h} if $h = -1/2$. So we say that λ has negative helicity $h = -1/2$.

We can go on and find the wave function for $\bar{\psi}^{\dot{\alpha}}$. This will define a wave function for helicity $h = +1/2$. Because $\bar{\psi}_{\dot{\alpha}}$ transforms as the complex conjugate of ψ_α. Thus the i on the rotation give a minus sign, and flips the h.

$$\bar{\psi}_{\dot{\alpha}} = c\bar{\lambda}_{\dot{\alpha}} e^{ip^{\dot{\alpha}\alpha} x_{\alpha\dot{\alpha}}} \tag{2.18}$$

Now we can understand why the polarization vector were label by ϵ^+, ϵ^-. Under the scaling $(\lambda, \bar{\lambda}) \to (t\lambda, t^{-1}\bar{\lambda})$ the polarization vectors scale as

$$(\epsilon^+,\ \epsilon^-) \to (t^{-2}\epsilon^+,\ t^{+2}\epsilon^-) = (t^{-2h}\epsilon^+,\ t^{-2h}\epsilon^-) \tag{2.19}$$

then ϵ^+ has helicity $+1$ and ϵ^- has helicity $h = -1$.

show that indeed these polarization are $+, -$ helicity

Consider a function that transforms as $f(e^{i\theta}x) = e^{i\theta h} f(x)$. We can thing the function as $f(x) = x^h$ and this satisfy $x\partial_x f(x) = hf(x)$. A wave function $\psi(\lambda, \bar{\lambda})$ will satisfy a similar equation

$$\left(\lambda^\alpha \frac{\partial}{\partial \lambda^\alpha} - \bar{\lambda}^{\dot{\alpha}} \frac{\partial}{\partial \bar{\lambda}^{\dot{\alpha}}}\right) \psi(\lambda, \bar{\lambda}) = -2h\psi(\lambda, \bar{\lambda}) \tag{2.20}$$

sometimes this constrain is called the auxiliary condition.

2.2 Scattering Amplitudes

The scattering amplitude A for n gluons is a functions of the external momentum (the asymptotic state limit) $p_1, \ldots, p_n$ and the polarization vectors $\epsilon_1, \ldots, \epsilon_n$. By Lorentz Invariance of A, we only have combination of Lorentz Invariance objects ($p_i \cdot p_j$, $\epsilon_i \cdot p_j$, $\epsilon_i \cdot \epsilon_j$). As we saw we can specify a particle with spin by its momentum $p^{(i)}_{\alpha\dot{\alpha}} = \lambda^{(i)}_\alpha \bar{\lambda}^{(i)}_{\dot{\alpha}}$ and helicity $h^{(i)}$. Thus the amplitude is a function of Lorentz invariant object expressed in term of $(\lambda_\alpha, \bar{\lambda}_{\dot{\alpha}}, h)$

$$A = A(\lambda^{(1)}_\alpha, \bar{\lambda}^{(1)}_{\dot{\alpha}}, h^{(1)}; \ldots; \lambda^{(n)}_\alpha, \bar{\lambda}^{(n)}_{\dot{\alpha}}, h^{(n)}) \tag{2.21}$$

Recall that under crossing we have $p \to -p$ and $\epsilon \to \epsilon^*$ or $\epsilon^+ \to \epsilon^-$. Then we can treat as all particle as outgoing and then use crossing to find the other helicity amplitudes.

Now in the same spirit the amplitude is a function of the spinors. Then it also satisfy a auxiliary condition for each particle $\lambda^{(i)}, \bar{\lambda}^{(i)}$:

$$\left(\lambda^{\alpha(i)}\frac{\partial}{\partial\lambda^{\alpha(i)}} - \bar{\lambda}^{\dot{\alpha}(i)}\frac{\partial}{\partial\bar{\lambda}^{\dot{\alpha}(i)}}\right) A(\lambda^{(i)}, \bar{\lambda}^{(i)}, h^{(i)}) = -2h^{(i)} A(\lambda^{(i)}, \bar{\lambda}^{(i)}, h^{(i)}) \tag{2.22}$$

Add a picture with the amplitude and labels

We can write the auxiliary condition in another form. It is easy to see how the amplitude change under the scaling

$$A(t\lambda^{(i)}, t^{-1}\bar{\lambda}^{(i)}, h^{(i)}) = t^{-2h^{(i)}} A(\lambda^{(i)}, \bar{\lambda}^{(i)}, h^{(i)}) \tag{2.23}$$

That is to say that the amplitude transforms homogeneously with weight $-2h^{(i)}$, where $h^{(i)}$ is the helicity of the particle i.

2.3 MHV Amplitudes

To appreciate the power of the spinor helicity formalism let us consider tree level scattering amplitudes for the Yang-Mills Theory. The Yang Mills action is

$$S_{YM} = -\frac{1}{2}\int dx^4 Tr(F^{mn}F_{mn}) \tag{2.24}$$

$$\textbf{with} \tag{2.25}$$

$$F_{mn} \equiv F^A_{mn}T^A = \partial_{[m}A^A_{n]}T^A - igA^B_m A^C_n[T^B, T^C] \tag{2.26}$$

and T^A are generators of the gauge group $SU(N)$ with $A = 1, ...N^2 - 1$, counting the number of generators. The generators satisfy the algebra with the standard normalization:

$$[T^A, T^B] = if^{ABC}T^C \quad ; \quad Tr(T^AT^B) = \frac{1}{2}\delta^{AB} \tag{2.27}$$

where f^{ABC} is the structure constant and it is antisymmetric in all indices. From (**??**) $f^{ABC} = -2iTr([T^A, T^B]T^C)$.

Amplitudes for the gluons will have products of Color(generatos) Traces. But in the end we have a simplification (only single traces). The full amplitude can be written as a sum over the cyclic permutation of the Color Traces. To convince you that this really happen recall that for the gauge group $SU(N)$ we have this identity (M, N are combination of $T's$):

$$Tr(MT^E)Tr(NT^E) = \frac{1}{2}Tr(MN) \tag{2.28}$$

where we used the fact that $Tr(T^A) = 0$ and $E's$ are summed. From $Tr(F^2)$ we get two types of self interaction term $A^A A^B \partial_m A^C \sim gp_m Tr([T^A, T^B]T^C)$ and $A^A A^B A^C A^D \sim Tr([T^A, T^B][T^C, T^D]$. The propagator clue together two vertex by a δ^{AB}. Then color indices are summed over and this kind of term reduce to a single trace.

In the end, for n particles we have cyclic permutation of $Tr(T^{A_1}T^{A_2}...T^{A_n})$ and the amplitude can be factorize as:

$$A_n = g^{n-2}(2\pi)^4\delta(\sum_{i=1}^{n} p_i)\mathcal{A}(p_1, h_1, ..., p_n, h_n)Tr(T^{A_1}T^{A_2}...T^{A_n}) + permutations \tag{2.29}$$

We can concentrate in the color free Amplitude $\mathcal{A}(1, 2, \ldots, n)$. Dimension analysis is a very power full tool. Let's use it to have a feeling how the Tree-Amplitude behaves. From (**??**) we have $[dx^4] = M^{-4}$; $[F] = M^2$; $[\partial] = M^1$; $[A] = M^1$; $[g] = M^0$, where M is the mass dimension.

We have two vertices, the cubic vertex $V_3 \sim gf^{ABC}p$, from $AA\partial A$ that has mass dimension 1 ($[V_3] = M^1$). And the quartic vertex $V_4 \sim g^2 f^2$, from A^4 that has mass dimension zero ($[V_4] = M^0$). From the picture is quit easy to see that the number of V_3 is always greater by one the number of propagators P (please do not get confuse with another P, I know that my imagination is not good). Also for n particle scattering, the number of V_3 is $n - 2$. The mass dimension of the propagator is -2 ($[P] = M^{-2}$).

Add the picture with the diagrams

In conclusion the mass dimension of n particle scattering amplitude is

$$[\mathcal{A}_n] = \frac{M^{n-2}}{(M^2)^{n-3}} = M^{4-n} \tag{2.30}$$

A important note is that, in the numerator, the powers of momenta (M) can not be greater than $n - 2$.

A general tree amplitude is a product of Lorentz scalars

$$[\mathcal{A}_n] = \sum_{diagrams} \frac{\sum \prod(\epsilon_i \cdot \epsilon_j) \prod(\epsilon_i \cdot p_j) \prod(p_i \cdot p_j)}{\prod P^2} \tag{2.31}$$

in term of spinors the polarization (??)-(??) products take the form

$$\epsilon_i^+ \cdot \epsilon_j^+ \propto \langle \mu_i \mu_j \rangle \quad ; \quad \epsilon_i^- \cdot \epsilon_j^- \propto [\mu_i \mu_j] \quad ; \quad \epsilon_i^- \cdot \epsilon_j^+ \propto \langle \lambda_i \mu_j \rangle [\mu_i \lambda_j] \tag{2.32}$$

where μ_i represents the gauge freedom that we have. Finally we can attack the amplitude with all the tools and information.

Let us start with all plus polarization amplitude $\mathcal{A}_n(1^+2^+ \dots n^+)$. For n particles we have n polarization ϵ^+. From (??) we see that if we choose $\mu_1 = \mu_2 = \cdots = \mu$ the product $\prod(\epsilon_i^+ \cdot \epsilon_j^+) = 0$ due $\langle \mu\mu \rangle = 0$. Then the only way for the amplitude not to be zero is to have $\prod(\epsilon_i \cdot p_j)$. But as we saw the amplitude in the numerator have n polarizations and need n momenta to create a Lorentz scalar. But it can have only $n-2$ powers of momenta $p's$, thus it is zero. You see that? In few lines we were able to evaluate ALL the plus helicity amplitudes. That's powerfull.

We can continue, and see what else we learn about the amplitude with only one negative helicity, $\mathcal{A}_n(1^+ \dots k^- \dots n^+)$. Again we use the freedom that we have and choose $\mu_1 = \mu_2 = \cdots = \lambda_k$ then all the terms $\epsilon_i^+ \cdot \epsilon_j^+$ vanish where $i,j \neq k$. And from $\epsilon_k^- \cdot \epsilon_j^+ \propto \langle \lambda_k \mu_j \rangle [\mu_k \lambda_j]$ we also see that it is zero. Thus we get the same problem as before. We need n powers of momenta and we can only have $n-2$. We conclude that $\mathcal{A}_n(1^+ \dots k^- \dots n^+) = 0$.

Now that we got this far let us do more, and calculate $\mathcal{A}_n(1^-, 2^-, 3^+ \dots\dots n^+)$. Choosing $\mu_i = \lambda_1$ for $i \geq 3$, then $\epsilon_i^+ \cdot \epsilon_j^+ = 0$ for $i,j \geq 3$. Set $\mu_1 = \mu_2 = \lambda_k$. Then the only non zero polarizations are $\epsilon_2^- \cdot \epsilon_i^+ \propto \langle \lambda_2 \mu_i \rangle [\mu_2 \lambda_i]$ for $i \geq 3 \quad \& \quad i \neq k$. Now we see that we can have a term with two polarization, and we can fill in with $n-2$ momentum contraction $\epsilon_i \cdot p_j$. Thus we don't have a vanishing amplitude.

This Amplitude $\mathcal{A}_n(1^-, 2^-, 3^+ \dots n^+)$ is called the Maximally Helicity Violating (MHV). Before we continue let understand this name. Because we choose all the particle to be outgoing, to see the physical ($2 \to n-2$) scattering, we have to use crossing. Recall that crossing symmetry interchange the helicity between the incoming particle and the outgoing particle. Then the first amplitude $\mathcal{A}_n(1^+, 2^+ \dots n^+)$ describes $1^-2^- \to 3^+ \dots n^+$ scattering, that's two particle incoming and $n-2$ particles outgoing. This 'violate' the helicity conservation. We can do the same thing for the other amplitudes.

Then $\mathcal{A}_n(1^+, 2^- \dots n^+)$ describes $1^-2^+ \to 3^+ \dots n^+$ also violet but it is zero. Finally $\mathcal{A}_n(1^+, 2^+, 3^-, 4^- \dots n^+)$ describes $1^-2^- \to 3^-, 4^-, 5^+ \dots n^+$ is the processes that can be maximal violated that is not zero.

Now that we understand the name, let us jump to a final result. The MHV amplitude has a nice closed formula that has postulated by Parke & Taylor and proved by Berends and Giele. The prove was using off-shell recursion method. This formula can be proven by the BCFW recursion Relations that use on-shell methods and complex momenta. But this kind of Technique is beyond the scope of this thesis. Then the tree-level amplitude for two negative helicity particles ($\{p^r, h^r = -\}, \{p^s, h^s = -\}$) and $n-2$ positive helicity particles is (without the color factor and momentum conservation)

$$\mathcal{A}(r^-, s^-) = \frac{\langle \lambda_r, \lambda_s \rangle^4}{\prod_{i=1}^n \langle \lambda_i, \lambda_{i+1} \rangle} \tag{2.33}$$

This is a very elegant and simple formula. Unfortunately we can not prove this formula here, but we can cheek if the symmetries that we found hold. The Lorentz is trivial because (**??**) is made of Lorentz Invariant objects $\langle \lambda_i, \lambda_j, \rangle$. The scaling is also simple. We note that the denominator has two spinor for each particle (i), *i.e.*, $\prod \langle \lambda_i, \lambda_{i+1}, \rangle \sim \lambda^{(i)}\lambda^{(i)}$. Thus under the scaling $\lambda^{(i)} \to t\lambda^{(i)}$ the amplitude get a factor of t^{-2}. And in the numerator a factor of t^4, but note that the numerator gets this factor only for negative helicity r, s.

Then for a particle with positive helicity $h = +1$ $(i) \neq r, s$ the amplitude scales as

$$\mathcal{A}(r^-, s^-, ti^+) = t^{-2}\mathcal{A}(r^-, s^-, i^+) = t^{-2h^{(i)}}\mathcal{A}(r^-, s^-, i^+) \tag{2.34}$$

and for a particle with negative helicity $h^s = -1$

$$\mathcal{A}(r^-, ts^-) = t^{+2}\mathcal{A}(r^-, s^-) = t^{-2h^{(s)}}\mathcal{A}(r^-, s^-) \tag{2.35}$$

As we expected from the auxiliary condition (**??**). Another symmetry that Yang-Mills Theory have is conformal symmetry. Hold up to the next chapter that we will explain what conformal transformations are and prove that the Yang-Mills and MHV amplitude are conformal invariant. We are going to see that the reason for the amplitudes with one or zero flipped helicity are zero, is a hidden symmetry a super symmetry in the Super Yang-Mills theory (that is a lot of super- words).

2.4 $\mathcal{N}$ =4 Super Yang-Mills Theory

We are interesting in the supersymmetric version of Yang Mills, but to get there I will make a review of supersymmetry, but a very brief one. So we are in the same page. Supersymmetry is a very interesting subject and we hope to find it in the LHC soon. The supersymmetry algebra is the only graded Lie algebra of symmetries of the S-matrix in a relativist quantum field theory, this result was proven by Haag, Sohnius and Lopuszanki. In a way, we are enlarging the Poicaré algebra by adding fermionic generators $(Q, \bar{Q})$. These fermionic generators (Grassmann) propose a symmetry between fermionic and bosonic fields. The supersymmetry algebra is given by

$$\{Q^A_\alpha, \bar{Q}_{\dot{\alpha}B}\} = 2\sigma^m_{\alpha\dot{\alpha}} P_m \delta^A{}_B \qquad \{Q^A_\alpha, Q^B_\beta\} = \varepsilon_{\alpha\beta} Z^{AB} \tag{2.36}$$

where m and $\alpha\dot{\alpha}$ are the usual Lorentz and spinor index respectively. The index A labels the number of supersymmetries, runs from $1 \ldots \mathcal{N}$. Thus the total number of supercharges is $l \times \mathcal{N}$, l is the dimension of spinor representation. The $\bar{Q}_{\dot{\alpha}} \equiv (Q_\alpha)^\dagger$ is the adjoint of operator of Q_α. The $Z^{AB} = -Z^{BA}$ are called the central charges and only exist for $\mathcal{N} > 1$, because they are antisymmetric. But they are zero in the massless case[1].

Note that the algebra is preserved under a $U(\mathcal{N})$ transformation called R-Symmetry:

$$Q'^A_\alpha = R^A{}_B Q^B_\alpha \quad \bar{Q}'_{\dot{\alpha}A} = R^{-1B}_{\ A} \bar{Q}_{\dot{\alpha}B} \qquad R^A{}_B \in U(\mathcal{N}) \tag{2.37}$$

Let us analyze the massless spectrum in four dimensions. In the frame $P_m = (-E, 0, 0, E)$ we have the algebra (**??**)

$$\{Q^A_\alpha, \bar{Q}_{\dot{\alpha}B}\} = \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} \delta^A{}_B \tag{2.38}$$

In the upper side we can define the creation and annihilation operators $a^\dagger_A, a^A$ by

$$a^A = \frac{Q^A_1}{2\sqrt{E}} \quad ; \quad a^\dagger_A = (a^A)^\dagger = \frac{\bar{Q}_{\dot{1}A}}{2\sqrt{E}} \tag{2.39}$$

with this normalization we get $\{a^A, a^\dagger_B\} = \delta^A{}_B$. We can use $(a^\dagger_A, a^A)$ to raise and lower the helicity of a state by $\frac{1}{2}$. So we know how to create the hilbert space. We start with

[1] $\{Q^A_2, \bar{Q}^B_2\} = 0 \Rightarrow \langle\psi|\bar{Q}^B_2 Q^A_2|\psi\rangle = 0$ which implies that $Q^A_2|\psi\rangle = 0$. Then Z^{AB} is zero since $Q^A_2 = 0$

the highest helicity h_{max} state $a^{\dagger}_{A}|h\rangle = 0$ and use the annihilation operator to decrease the helicity.

$$|h\rangle \quad , a^{A}|h\rangle \quad , a^{A_1} \cdots a^{A_{\mathcal{N}}}|h\rangle \tag{2.40}$$

The highest helicity is $h_{max} = h_{min} + \frac{\mathcal{N}}{2}$, thus we have $2^{\mathcal{N}}$ states. We can see that by looking the string of $a^{A_1}...a^{A_{\mathcal{N}}}$ and note that in each space we can have or not a a^{A_i}. Each state are $\binom{\mathcal{N}}{n}$ degenerate, where n is the number of a^{A_i} in the state.

If we want to keep helicity $|h| < 1$ than the $\mathcal{N} = 4$ is the maximal amount of supersymmetry that we can have in four dimension. And that is the Theory that we want to study. One interesting fact is that CPT theories usually double the number of states, because it needs the opposite helicity. But our theory is self-dual CPT, *i.e.* it has $h = \pm 1;\ \pm 1/2$. The spectrum can be summarize in the table **??**.

states $\|h\rangle$	Number of states	Name
$\|1\rangle$	1	gluon
$a^{A}\|h\rangle = \|1/2\rangle$	4	gluion
$a^{A}a^{B}\|h\rangle = \|0\rangle$	6	scalar
$a^{A}a^{B}a^{C}\|h\rangle = \|-1/2\rangle$	4	anti-gluion
$a^{1}a^{2}a^{3}a^{4}\|h\rangle = \|-1\rangle$	1	gluon

Table 2.1: Super Yang Mills $\mathcal{N} = 4$ Spectrum

The action that has this spectrum can be written compactly as a $\mathcal{N} = 1$ in ten dimension

$$S_{YM} = -\frac{1}{4g^2_{YM}} \int dx^{10} Tr\left(F_{MN}F^{MN} - 2i\bar{\psi}\Gamma^{M}D_{M}\psi\right) \tag{2.41}$$

M, N runs from $(0, \ldots, 9)$ and ψ is a $10d$ Majorana-Weyl spinor that has the minimal representation 16 real degrees of freedom. The Γ^{M} are the gamma matrices in $10d$. By dimension reduction $(M) \to (m, i)$ we get the $\mathcal{N} = 4 \quad d = 4$ Super Yang-Mills theory. Where m run from 0 to 3 is the $4d$ Lorentz index and i is the 6 compacted dimensions. To see that is true, note that (**??**) has $\mathcal{N} \times l = 1 \times 16$ super charges (l is the spinor dimension), and no fields are higher that spin 1. The only theory in 4 dimension that has the same properties is $\mathcal{N} = 4$ SYM.

After the dimension reduction, the action for $\mathcal{N} = 4$ SYM in $d = 4$ become

$$S_{YM} = -\frac{1}{2g^2_{YM}} \int dx^4 \left(\frac{1}{2}(F^{mn})^2 + (D_m X^i)^2 - [X^i, X^j]^2 + i\bar{\psi}^I \gamma^m D_m \psi^I + \bar{\psi}^I \Gamma^i_{IJ}[X^i, \psi^J]\right) \tag{2.42}$$

Here $(field)^2$ means the proper contraction of indices. The ψ^I are four $4d$ Weyl spinors $(I = 1\dots 4)$, γ^m the $4d$ gamma matrices, and Γ^i_{IJ} the gamma matrices for $SO(6)$. Also the X^i are six real scalars, which i labels the $SO(6)$ global R-symmetry. This scalars come from the gauge field under the dimension reduction $A^M \to A^m + X^i$. All the fields (A^m, X^i, ψ^I) are in the adjoint of the gauge group (color group) which we choose to be $SU(N)$, *i.e.* $A^m \equiv A^{m(a)}T^{(a)}$ where $a = 1, \dots, N^2 - 1$. Please do not get confuse with N- number of the gauge group & $\mathcal{N}$- number of supersymmetries.

It is convenient to rewrite the X^i six real scalars as six complex field $\phi^{IJ} = -\phi^{JI}$ with $I, J = 1, \dots 4$ transforming in the fully anti-symmetric 2-index representation of $SU(4)$. It also has to satisfy the condition $\bar{\phi}_{IJ} = \frac{1}{2}\epsilon_{IJKL}\phi^{KL}$.

Remember that the R-Symmetry of the algebra was $U(4) = SU(4) \times U(1)$, for $\mathcal{N} = 4$............

2.4.1 On-shell Superspace and Superamplitudes

The on-shell degrees of freedom of $\mathcal{N} = 4$ SYM can be group together by an on-shell chiral superfield Φ. If we introduce the Grassmann odd variables η_A labeled by the $SU(4)$ index $A = 1\dots 4$, the table **??** can be grouped in

$$\Phi(p,\eta) = g^+(p) + \eta_A\psi^A(p) - \frac{1}{2!}\eta_A\eta_B S^{AB}(p) - \frac{1}{3!}\eta_A\eta_B\eta_C\psi^{ABC}(p) + \frac{1}{4!}\eta_A\eta_B\eta_C\eta_D\epsilon^{ABCD}g^-(p) \tag{2.43}$$

Fields	Helicity	Name(Type)	$SU(4)_R$ representation
g^+	1	gluon (B)	singlet
ψ^A	1/2	gluion (F)	fudamental (**4**)
S^{AB}	0	scalar (B)	anti-symmetric (**6**)
$\psi^{ABC} \sim \bar{\psi}_A$	$-1/2$	anti-gluion (F)	anti-fund ($\bar{\mathbf{4}}$)
g^-	-1	gluon (B)	singlet

Table 2.2: Super Yang Mills spectrum and its global R-symmetry transformation

The super amplitude is parametrized by the external super legs that depends on the usual helicity spinors plus the new grassmann parameter $\{\lambda_\alpha, \bar{\lambda}_{\dot{\alpha}}, \eta_A\}$. We can think of $\Phi_i \equiv \Phi(p_{(i)}, \eta_{(i)})$ as the super-wavefunction for the $i'th$ external particle of the super amplitude $\mathbb{A}_n(\Phi_1, \Phi_2, \dots, \Phi_n)$. In the same spirit of super space formalism, the super amplitude is to be understood as the power series expansion in η. The $SU(4)_R$-symmetry requires that the external states form a $SU(4)$ singlet. For example, the amplitudes from the previous section can be extract as

$$\mathbb{A}_n\Big|_{\eta^0} = A_n(1^+, 2^+, \ldots, n^+) \tag{2.44}$$

$$\mathbb{A}_n\Big|_{\eta_1^4} = A_n(1^-, 2^+, \ldots, n^+) \qquad\qquad \eta_i^4 = \eta_{i1}\eta_{i2}\eta_{i3}\eta_{i4} \tag{2.45}$$

$$\mathbb{A}_n\Big|_{\eta_1^4\eta_2^4} = A_n(1^-, 2^-, 3^+ \ldots, n^+) \tag{2.46}$$

We could also extract using Grassmann integrals. If we define the helicity operator

$$h_i = 1 - \frac{1}{2}\eta_A^i \frac{\partial}{\partial \eta_A^i} \tag{2.47}$$

when h_i acts on each component of the superfield Φ_i it gives the helicity of each particle. For example:

$$h_i g^+ = +1 g^+(p) \quad ; \quad h_i \eta_A \psi^A(p) = +\frac{1}{2}\eta_A \psi^A(p) \tag{2.48}$$

Using the same logic, when h_i acts on the Superamplitude gives the helicity of the particle. One can see this because $h_i \mathbb{A}_n(\Phi_i) = (1 - \frac{K}{2})\mathbb{A}_n(\Phi_i)$ where K counts the degree in η_i. When $K = 0$ it represents the positive gluon g^+ and for $K = 4$ it represents the negative gluon g^-.

Let us see how the Supersymmetry invariance of the Superamplitude imposes that the two amplitudes (**??**)-(**??**) are zero. The Supersymmetry generators in the on-shell Superspace are

$$Q_\alpha^A = \sqrt{2}\sum_{i=1}^{n} \lambda_\alpha^{(i)} \eta_{(i)}^A \quad ; \quad \bar{Q}_{\dot\alpha B} = \sqrt{2}\sum_{i=1}^{n} \bar{\lambda}_{\dot\alpha}^{(i)} \frac{\partial}{\partial \eta_{(i)}^B} \tag{2.49}$$

here we are summing over all the external particle ($i = 1, \ldots n$). These generators satisfy the algebra

$$\{Q_\alpha^A, \bar{Q}_{\dot\alpha B}\} = 2P_{\alpha\dot\alpha}\ \delta^A{}_B \quad ; \quad P_{\alpha\dot\alpha} = \sum_{i=1}^{n} \lambda_\alpha^{(i)} \bar{\lambda}_{\dot\alpha}^{(i)} \tag{2.50}$$

where $P_{\alpha\dot\alpha}$ is the translation generator for the external particles. Recall that the Amplitudes have the delta of momentum conservation $\delta^4(\sum_{i=1}^{n} \lambda_\alpha^{(i)} \bar{\lambda}_{\dot\alpha}^{(i)})$ on it. Thus

$$\{Q_\alpha^A, \bar{Q}_{\dot\alpha B}\}\mathbb{A}_n = 2P_{\alpha\dot\alpha}\delta^A{}_B \mathbb{A}_n \quad \Rightarrow \quad Q_\alpha^A \mathbb{A}_n = \bar{Q}_{\dot\alpha B}\mathbb{A}_n = 0 \tag{2.51}$$

We have that the Supersymmetry generators annihilate the super amplitude. Note that the generator Q^A_α act multiplicatively, and we can write it as a Grassmann delta function, from the property [2] $\delta(\eta - \eta_0) = \eta - \eta_0$

Then we use the statement that $Q^A_\alpha \mathbb{A}_n = 0$ to write $\mathbb{A}_n \propto \delta^8(Q^A_\alpha)$ such that $Q^A_\alpha \delta^8(Q^A_\alpha) = 0$. Then the Superamplitude can be decomposed as

$$\mathbb{A}_n(\lambda_i, \bar{\lambda}_i, \eta_i) = \delta^8(Q^A_\alpha)\delta^4(P)\mathbb{A}^{(K)}_n(\lambda_i, \bar{\lambda}_i, \eta_i) \tag{2.52}$$

where K is the degree in η, where $\mathbb{A}^{(K)}_n \sim \mathcal{O}(\eta^K)$. Note that the $SU(4)_R$ symmetry impose that $\mathbb{A}^{(K)}_n$ have an η expansion of the form

$$\mathbb{A}^{(K)}_n = \mathbb{A}^{(0)}_n + \mathbb{A}^{(4)}_n + \mathbb{A}^{(8)}_n + \cdots + \mathbb{A}^{(4n-16)}_n \tag{2.53}$$

The explicit expression for the Grassmann delta is

$$\delta^8(Q^A_\alpha) = \prod_{\alpha=1}^{2}\prod_{A=1}^{4}\left(\sum_{i=1}^{n}\lambda^{(i)}_\alpha \eta^A_{(i)}\right) = \frac{1}{2}\prod_{A=1}^{4}\sum_{i,j=1}^{n}\langle ij\rangle \eta^A_{(i)}\eta^A_{(j)}\,{}^{[3]} \tag{2.54}$$

we used $\langle\lambda_i, \lambda_j\rangle = \langle ij\rangle$. Finally we can see that the expansion on the super-amplitude starts at order η^8 from the Grassmann delta. Then the amplitudes (**??**)-(**??**) are zero because they are order zero and forth in η. As I promised these two amplitudes are zero due to a hidden supersymmetry.

The MHV amplitude with particles r, s with negative helicity, can be extracted from the superamplidute as

$$\begin{aligned} A_n(r^-, s^-) &= \int d^4\eta_s d^4\eta_r \mathbb{A}^K_n \\ &= \delta^4(P)\int d^4\eta_s d^4\eta_r \prod_{A=1}^{4}\sum_{i,j=1}^{n}\langle ij\rangle \eta^A_{(i)}\eta^A_{(j)}\mathbb{A}^K_n \\ &= \delta^4(P)\langle sr\rangle^4 \mathbb{A}^0_n \end{aligned} \tag{2.55}$$

we can read from (**??**) that $\mathbb{A}^0_n$ (zero order expansion in η) is given by

[2] To see that use a test function $F(\eta) = F_0 + F_1\eta$ and the integration rules $\int d\eta\eta = 1 \quad \int d\eta 1 = 0$ and prove $\int d\eta\delta(\eta - \eta_0)F(\eta) = F(\eta_0)$.

[3] for Grassmann variables we have $\theta_1\theta_2 = \frac{1}{2}\theta^\alpha\theta_\alpha$, then $Q^A_1 Q^A_2 = \frac{1}{2}Q^{\alpha A}Q^A_\alpha$

$$\mathbb{A}_n^0 = \frac{1}{\langle 12\rangle\langle 23\rangle \ldots \langle n1\rangle} \tag{2.56}$$

We can also see that the power in $\langle sr\rangle^4$ is due to $\mathcal{N} = 4$ supercharges.

$$h = \frac{1}{2}\left(-\lambda^{\alpha}\frac{\partial}{\partial\lambda^{\alpha}} + \bar{\lambda}^{\dot{\alpha}}\frac{\partial}{\partial\bar{\lambda}^{\dot{\alpha}}} + \eta_A\frac{\partial}{\partial\eta_A}\right) \tag{2.57}$$

Chapter 3

Twistors

As a motivation to study Twistors, I will start with the conformal transformations and then see that the generators in twistor variables (something that I have not told you what they are) are easier to deal with than the spinors variables $(\lambda, \bar{\lambda})$.

3.1 Conformal Invariance

Lets first give a reason why conformal transformation are interesting. I will do this by study the symmetries of Maxwell theory. Conformal transformations on an space are those that preserve locally angles between two lines, mathematically this definition is:

$$g_{lk}(x)\frac{\partial x^l}{\partial x'^m}\frac{\partial x^k}{\partial x'^n} = \Lambda(x^{'})g_{mn}(x^{'}) \tag{3.1}$$

In our case will be sufficient to study the flat metric $\eta_{mn} = (-, + + ...+)$, the dots are because it's useful to consider d dimensions. Then the equation that we want to solve is:

$$\eta_{lk}\frac{\partial x^l}{\partial x'^m}\frac{\partial x^k}{\partial x'^n} = \Lambda(x^{'})\eta_{mn} \tag{3.2}$$

We could also write this equation in infinitesimal form using $x^{'m} = x^m + \epsilon\xi^m(x) + \mathcal{O}(\epsilon^2)$, where ϵ is small. So if we just do a Taylor expansion in $\Lambda(x^{'}) = 1 - \epsilon K(x) + \mathcal{O}(\epsilon^2)$ and $\frac{\partial x^k}{\partial x'^n} = \delta^k_{\ n} - \epsilon\partial_n\xi^k + \mathcal{O}(\epsilon^2)$ we see that in the infinitesimal formal is given by

$$\partial_m\xi_n + \partial_n\xi_m = K(x)\eta_{mn} \tag{3.3}$$

we can find an even simpler equation if we take the trace of this equation to isolate $K(x)$ and we get:

$$\partial_m \xi_n + \partial_n \xi_m = \frac{2}{d}(\partial \cdot \xi)\eta_{mn} \tag{3.4}$$

where the d appears from $\eta_{mn}\eta^{mn} = d$ is the dimension of the spacetime. Maybe here is a good place to stop and give you the motivation that I promised.

The Maxwell theory in d-flat dimensions is given by the action

$$S_{MW} = -\frac{1}{4g}\int dx^d F^{mn} F_{mn} \tag{3.5}$$

where $F_{mn} = \partial_m A_n - \partial_n A_m$ is the usual field strength, and g is the coupling constant. So lets see how this action transform under a conformal transformation. The potential transform as $A^{'}_m(x^{'}) = \frac{\partial x^l}{\partial x^{'m}} A_l(x)$ then under $x^{\prime}_n = x_n - \epsilon\xi_n$:

$$A^{'}_m(x^{'}) = A^{'}_m(x) - \epsilon\xi^n \partial_n A^{'}_m(x) = A_m(x) + \partial_m(\epsilon\xi^l)A_l(x)$$

Then the infinitesimal transformation is given by

$$\delta_\xi A_m \equiv A^{'}_m(x) - A_m(x) = \epsilon\xi^n\partial_n A_m(x) + \partial_m(\epsilon\xi^l)A_l(x) \tag{3.6}$$

this infinitesimal transformation is the Lie derivative $\mathcal{L}_\xi A_m$. Now if we do the same thing for the tensor F_{mn} we get

$$\delta_\xi F_{mn} = \epsilon\xi^r\partial_r F_{mn}(x) + \partial_m(\epsilon\xi^l)F_{ln}(x) + \partial_n(\epsilon\xi^l)F_{ml}(x) \tag{3.7}$$

Now we have all the ingredients we can calculate the variation of the action

$$\delta_\xi S_{MW} = -\frac{1}{2g}\int dx^d F^{mn}\{\xi^r\partial_r F_{mn} + \partial_m(\xi^l)F_{ln} + \partial_n(\xi^l)F_{ml}\} \tag{3.8}$$

$$= -\frac{1}{2g}\int dx^d\{\frac{1}{2}\partial_r(\xi^r F^2) - \frac{1}{2}\partial\cdot\xi(F^2) + F^{mr}F^n{}_r(\partial_m\xi_n + \partial_n\xi_m)\} \tag{3.9}$$

$$= -\frac{1}{4g}\int dx^d\{\partial_r(\xi^r F^2) + (\frac{4}{d} - 1)F^2\} \tag{3.10}$$

here $F^2 \equiv F^{mn}F_{mn}$ and to go from the second line to third I used the equation (**??**). The Maxwell theory is conformal invariant only in $d = 4$ it is a total derivative. That's our motivation to study conformal transformation, the Maxwell theory has more symmetry than just Lorentz transformation.

3.1.1 Generators of conformal algebra

Now that we are motivated, let's study more deeply the conformal transformations, but not so deep, even because we just want the generators, thus if you want a more elaborate and complete solution you could go here and here[T. Zee, franscesco] . To do so, we have to find the vector that solves the equation (**??**). ....rewrite..... One could solve by brute force, but I choose to guess the answer and see if ti works.

First let's manipulate (**??**) by acting with ∂^m and then ∂^n we get:

$$d\partial^2\xi_n = (2-d)\partial_n(\partial\cdot\xi) \quad ; \quad (d-1)\partial^2(\partial\cdot\xi) = 0 \tag{3.11}$$

From one see that $d = 2$ things become simpler. If you write $\partial_1 = \partial_z + i\partial_{\bar{z}}$ and $\partial_2 = \partial_z - i\partial_{\bar{z}}$ in complex variables $z, \bar{z}$, you find (in $\eta = (1,1)$) the Laplacian $\partial_z\partial_{\bar{z}}\xi(z,\bar{z}) = 0$. This implies that $\xi(z,\bar{z})$ has infinity solutions, for example, $\xi(z,\bar{z}) = f(z)$ or $g(\bar{z})$. This is what we have in string theory the world-sheet has two dimension conformal symmetry.

From the second equation in (**??**) when $d \neq 1$ the vector is $\xi \sim x^2$. From equation (**??**) if we set $\Lambda(x) = 1$ then $K(x) = 0$ we get $\partial_m\xi_n = -\partial_n\xi_m$ and $\xi_m = a_m + b_m{}^n x_n$ with $b_{mn} = -b_{nm}$. This is Translation and Lorentz transformation, i.e the Poincar acento transformations. Now if we make rescaling on $x' = \lambda x$ we see that $\Lambda = \lambda^2$ so it's a valid transformation, these are called dilations. We are almost done, we still can have order quadratic in x. There are not some many ways to write a second order term. A good start could be $\xi_m = a_m x^2 + (b\cdot x)x_m$. If one plug this guess in the equations and do some manipulations, you get the right answer $\xi_m = d^n(\eta_{mn}x^2 - 2x_n x_m)$ this are called the Special Conformal Transformations(SCTs). Just to summarize the conformal vector is given by:

$$\xi_m = \underbrace{a_m}_{Translation} + \underbrace{b_m{}^n x_n}_{Lorentz} + \underbrace{c x_m}_{Dilation} + \underbrace{d^n(\eta_{mn}x^2 - 2x_n x_m)}_{SCT} \tag{3.12}$$

One thing to recall is that all this analysis was done for the Minkowski metric. Now we can read off the generator G for each transformation from (**??**), if we use the differential operator representation, from the definition $\delta_\xi x_m = \xi \cdot G x_m$, we get:

$$P_m = -i\partial_m \quad \text{and} \quad M_{mn} = i(x_m\partial_n - x_n\partial_m) \tag{3.13a}$$

$$D = -ix^m\partial_m \quad \text{and} \quad K^m = -i(\eta^{mn}x^2 - 2x^m x^n)\partial_n \tag{3.13b}$$

with these generators we can go and calculate the algebra they form. With $[P_m, X_n] = i\eta_{mn}$ and $[A, BC] = [A, B]C + B[A, C]$ you can calculate all the commutators.

$$[D, P^m] = iP^m \quad ; \quad [D, K^m] = -iK^m \quad ; \quad [D, M^{mn}] = 0 \tag{3.14}$$

$$[M^{mn}, P^l] = -\eta^{ml}P^n + \eta^{nl}P^m \quad ; \quad [M^{mn}, K^l] = -\eta^{ml}K^n + \eta^{nl}K^m \tag{3.15}$$

$$[M^{mn}, M^{lr}] = i(-\eta^{ml}M^{nr} - \eta^{nr}M^{ml} + \eta^{nl}M^{mr} + \eta^{mr}M^{nl}) \tag{3.16}$$

$$[K^m, P^n] = 2i(-M^{mn} + \eta^{mn}D) \tag{3.17}$$

We can just look the generator and count how many they are. In d-dimension $M : \frac{d(d-1)}{2}$, $P : \ d$, $K : \ d$ and $D : \ 1$ a total of $\frac{(d+1)(d+2)}{2}$. What is this group? If we look for the number of generators in the Lorentz group(M) and compare with the conformal group is just a shit on $d \to d+2$, i.e, $\frac{d(d-1)}{2} \to \frac{(d+1)(d+2)}{2}$, thus on good guest the Conformal group would be $SO(q, p)$ with $q + p = d + 2$. The conformal group in $d = 4$ is $SO(4, 2)$. (maybe a appendix about that or even here)

3.2 Conformal invariance of the MHV amplitudes

As I promised in the chapter 2 (make a ref link) when I introduced the MHV tree amplitude formula (**??**) to check the symmetries. As we just saw the Yang-Mills theory has a bigger group that the Poicare symmetry, it is also invariant under conformal transformations. Thus the amplitude should also be invariant under these transformations. To be able to calculate this invariance we have to find the generators (**??**)-(**??**) in the spinor $(\lambda, \bar{\lambda})$ representation.

The momentum generator are just the multiplication operator

$$P_{\alpha\dot{\alpha}} = \lambda_\alpha \bar{\lambda}_{\dot{\alpha}} \tag{3.18}$$

The Lorentz Generators can be found by looking how a spinor transform(appendix)

$$\delta\lambda_\alpha = \frac{i}{2} \underbrace{\omega_{mn}(\sigma^{mn})_\alpha{}^\beta}_{\Omega_\alpha{}^\beta} \lambda_\beta \tag{3.19}$$

the variation if define as

$$\delta\lambda_\rho = \Omega^{\alpha\beta} m_{\alpha\beta} \lambda_\rho \tag{3.20}$$

where $\Omega^{\alpha\beta}$ are the coefficients of the transformation and $m_{\alpha\beta}$ is the generator of the transformation. Thus the Lorentz generator is a first order differential operator given by

$$m_{\alpha\beta} = \frac{i}{2}\left(\lambda_\alpha \frac{\partial}{\partial\lambda^\beta} + \lambda_\beta \frac{\partial}{\partial\lambda^\alpha}\right) \tag{3.21}$$

The same thing for the dotted indices:

$$\bar{m}_{\dot{\alpha}\dot{\beta}} = \frac{i}{2}\left(\bar{\lambda}_{\dot{\alpha}} \frac{\partial}{\partial\bar{\lambda}^{\dot{\beta}}} + \bar{\lambda}_{\dot{\beta}} \frac{\partial}{\partial\bar{\lambda}^{\dot{\alpha}}}\right) \tag{3.22}$$

They are symmetric in (α, β) and $(\dot{\alpha}, \dot{\beta})$ this gives $4-1=3$ generators in a total of 6 for both $m, \bar{m}$. They can be thought as the projection $m_{\alpha\beta} = M^{mn}\sigma_{\alpha\beta}$ and $\bar{m}_{\dot{\alpha}\dot{\beta}} = M^{mn}\bar{\sigma}_{\dot{\alpha}\dot{\beta}}$ where $\sigma_{\alpha\beta}$ & $\bar{\sigma}_{\dot{\alpha}\dot{\beta}}$ are the Lorentz generators (appendix). (Note we use the $\varepsilon_{\dot{\alpha}\dot{\beta}}$ and $\varepsilon_{\alpha\beta}$ to raise and lower the indices).

We have to find the generators of dilation and special conformal transformation. From the action of dilation of momentum $[D, P^m] = iP^m$ we can associate a dilation weight $+1$ and a dilation weight -1 for K^m. A natural guess for the dilation is

$$d = \frac{i}{2}\left(\lambda^\alpha \frac{\partial}{\partial\lambda^\alpha} + \bar{\lambda}^{\dot{\alpha}} \frac{\partial}{\partial\bar{\lambda}^{\dot{\alpha}}} + c\right) \tag{3.23}$$

we see that $[d, \lambda_\alpha] = \frac{i}{2}\lambda_\alpha$ and $[d, \bar{\lambda}_{\dot{\alpha}}] = \frac{i}{2}\bar{\lambda}_{\dot{\alpha}}$, for any constant c. Thus is natural to associate the dilation weight $1/2$ for the $(\lambda, \bar{\lambda})$ spinors.

To construct the special conformal transformation operator is not so trivial to guess

$$K_{\alpha\dot{\alpha}} = \frac{\partial^2}{\partial\lambda^\alpha \partial\bar{\lambda}^{\dot{\alpha}}} \tag{3.24}$$

but is the simplest operator that has the right dilation weight -1 and the commutator with d

$$[d, K_{\alpha\dot{\alpha}}] = -iK_{\alpha\dot{\alpha}} \tag{3.25}$$

If works it is fine, but keep in mind the second order differential operator are not so nice, and this may seen as a motivation to introduce Twistors. One way to fix the constant c in the dilation operator is using the commutation relation

$$[K_{\alpha\dot{\alpha}}, P^{\dot{\beta}\beta}] = -i(\delta_\alpha{}^\beta \bar{m}_{\dot{\alpha}}{}^{\dot{\beta}} + \delta_{\dot{\alpha}}{}^{\dot{\beta}} m_\alpha{}^\beta + \delta_\alpha{}^\beta \delta_{\dot{\alpha}}{}^{\dot{\beta}} d) \tag{3.26}$$

doing the calculation on the left side by just plugging the definition of K and P we find

$$[K_{\alpha\dot{\alpha}}, P^{\dot{\beta}\beta}] = \delta_\alpha{}^\beta \delta_{\dot{\alpha}}{}^{\dot{\beta}} + \delta_\alpha{}^\beta \bar{\lambda}^{\dot{\beta}} \frac{\partial}{\partial\bar{\lambda}^{\dot{\alpha}}} + \delta_{\dot{\alpha}}{}^{\dot{\beta}} \lambda^\beta \frac{\partial}{\partial\lambda^\alpha} \tag{3.27}$$

using the antisymmetric property of two spinors $(A_\alpha B_\beta - A_\beta B_\alpha) = \varepsilon_{\alpha\beta} A^\rho B_\rho$. (To see this just plug the numbers). Thus we can split $\lambda^\beta \frac{\partial}{\partial\lambda^\alpha}$ in symmetric plus antisymmetric part

$$\lambda^\beta \frac{\partial}{\partial\lambda^\alpha} = -im_\alpha{}^\beta + \frac{1}{2}\delta_\alpha{}^\beta \lambda^\rho \frac{\partial}{\partial\lambda^\rho} \tag{3.28}$$

the same is true for the dotted index

$$\bar{\lambda}^{\dot{\beta}} \frac{\partial}{\partial\bar{\lambda}^{\dot{\alpha}}} = -i\bar{m}_{\dot{\alpha}}{}^{\dot{\beta}} + \frac{1}{2}\delta_{\dot{\alpha}}{}^{\dot{\beta}} \lambda^{\dot{\rho}} \frac{\partial}{\partial\bar{\lambda}^{\dot{\rho}}} \tag{3.29}$$

we used the fact that the symmetric part is proportional to the Lorentz generator.

Finally we can identify the left side of (**??**) and find the dilation operator is

$$d = \frac{i}{2}\left(\lambda^\alpha \frac{\partial}{\partial\lambda^\alpha} + \bar{\lambda}^{\dot{\alpha}} \frac{\partial}{\partial\bar{\lambda}^{\dot{\alpha}}} + 2\right) \tag{3.30}$$

Now we are ready to prove that the MHV amplitude is invariant under conformal transformations. First the generators that we found were for one particle. For n particle is just the sum of the individual particle.

Let me remind you the form of the MHV-Amplitude (without the color trace):

$$A_n(r^-, s^-) = g^{n-2}(2\pi)^4\delta^4\left(\sum_i^n \lambda_\alpha^{(i)}\bar{\lambda}_{\dot{\alpha}}^{(i)}\right)\frac{\langle\lambda_r, \lambda_s, \rangle^4}{\prod_{i=1}^n\langle\lambda_i, \lambda_{i+1}\rangle} \tag{3.31}$$

– Translation operator:

$$P_{\alpha\dot{\alpha}} = \sum_{i=1}^n \lambda_\alpha^{(i)}\bar{\lambda}_{\dot{\alpha}}^{(i)} \tag{3.32}$$

then

$$P_{\alpha\dot{\alpha}}A_n = 0 \tag{3.33}$$

the delta function gives zero.

– Lorentz operator:

$$m_{\alpha\beta} = \frac{i}{2}\sum_{i=1}^n\left(\lambda_\alpha^{(i)}\frac{\partial}{\partial\lambda^{\beta(i)}} + \lambda_\beta^{(i)}\frac{\partial}{\partial\lambda^{\alpha(i)}}\right) \tag{3.34}$$

the Lorentz is manifest in the MHV amplitude because it only depends on Lorentz Invariant objects $\langle\lambda^{(i)}, \lambda^{(j)}\rangle$.

——-maybe add the calculation——-

– Dilation operator:

$$d = \frac{i}{2}\sum_{i=1}^n\left(\lambda^{\alpha(i)}\frac{\partial}{\partial\lambda^{\alpha(i)}} + \bar{\lambda}^{\dot{\alpha}(i)}\frac{\partial}{\partial\bar{\lambda}^{\dot{\alpha}(i)}} + 2\right) \tag{3.35}$$

As we saw the the dilation operator measures the weight in mass units. Then the operator will give the mass weight plus n when act on the amplitude

$$dA_n = ([A_n] + n)A_n = \left([\langle\lambda^{(r)}, \lambda^{(s)}\rangle^4] + [\delta^4(p)] + [\frac{1}{\prod_{i=1}^n\langle\lambda_i, \lambda_{i+1}\rangle}] + n\right)A_n \tag{3.36}$$

Thus we have $[\lambda] = 1/2$; $[\langle\lambda^{(i)}, \lambda^{(j)}\rangle^4] = 4$; and $[\delta^4(p)] = -4$. Also$[\frac{1}{\langle\lambda_i,\lambda_{i+1}\rangle}] = -1$

We have

$$dA_n = (4 - 4 + n(-1) + n)A_n = 0 \tag{3.37}$$

–Special Conformal operator

$$K_{\alpha\dot{\alpha}} = \sum_{i=1}^{n} \frac{\partial^2}{\partial\lambda^{\alpha(i)}\partial\bar{\lambda}^{\dot{\alpha}(i)}} \tag{3.38}$$

To prove that the invariance of the MHV amplitude we note that $\bar{\lambda}$ is on the delta function only so

$$K_{\alpha\dot{\alpha}}A_n = \sum_{i=1}^{n} \left[\frac{\partial}{\partial\lambda^{\alpha(i)}} \left(\frac{\partial}{\partial\bar{\lambda}^{\dot{\alpha}(i)}} \delta^4(p) \right) \mathcal{A}_n + \left(\frac{\partial}{\partial\bar{\lambda}^{\dot{\alpha}(i)}} \delta^4(p) \right) \frac{\partial}{\partial\lambda^{\alpha(i)}} \mathcal{A}_n \right] \tag{3.39}$$

using the chain rule $\frac{\partial}{\partial\bar{\lambda}^{\dot{\alpha}(i)}} = \frac{\partial P^{\dot{\beta}\beta}}{\partial\bar{\lambda}^{\dot{\alpha}(i)}} \frac{\partial}{\partial P^{\dot{\beta}\beta}} = \lambda^{\beta(i)} \frac{\partial}{\partial P^{\dot{\alpha}\beta}}$ we get

$$K_{\alpha\dot{\alpha}}A_n = \left(n\frac{\partial}{\partial P^{\dot{\alpha}\alpha}}\delta^4(p) + P^{\dot{\beta}\beta} \frac{\partial}{\partial P^{\dot{\alpha}\beta}} \frac{\partial}{\partial P^{\alpha\dot{\beta}}} \delta^4(p) \right) \mathcal{A}_n + \sum_{i=1}^{n} \left(\frac{\partial}{\partial P^{\dot{\alpha}\beta}} \delta^4(p) \right) \lambda^{\beta(i)} \frac{\partial}{\partial\lambda^{\alpha(i)}} \mathcal{A}_n \tag{3.40}$$

The last term we use the decomposition symmetry plus antisymmetry (**??**). Keeping in mind that $m_\alpha{}^\beta \mathcal{A}_n = 0$. And that the antisymmetric piece is the λ part of the dilation operator, thus

$$\lambda^\beta \frac{\partial}{\partial\lambda^\alpha} \mathcal{A}_n = [\mathcal{A}_n] = (4 - n)\mathcal{A}_n \tag{3.41}$$

The last piece of information is to note that

$$P^{\dot{\beta}\beta} \frac{\partial}{\partial P^{\dot{\alpha}\beta}} \frac{\partial}{\partial P^{\alpha\dot{\beta}}} \delta^4(p) = -4\frac{\partial}{\partial P^{\dot{\alpha}\alpha}} \delta^4(p) \tag{3.42}$$

this can be seen as a property of delta function $\delta^{'}(x)f(x) = -f^{'}(x)\delta(x)$ and $x\delta^{''}(x)f(x) = -2\delta^{'}(x)f(x)$. The extra 2 in the formula is due to $\frac{\partial P^{\beta\dot{\beta}}}{\partial P^{\beta\dot{\alpha}}} = 2\delta^{\dot{\beta}}{}_{\dot{\alpha}}$ compare to the simple example.

Plugging (**??**)-(**??**) in to (**??**), we see that the MHV amplitude is indeed invariant under special conformal transformations.

3.3 Twistor Space as a conformal representation

Let us consider a space that the conformal generators are simpler. This space is called Twistor space.

Consider the wave function $\psi(\lambda, \bar{\lambda})$ and let us make a Fourier transformation in the i particle in two ways

$$\tilde{\psi}(Z_i) = \int d^2\bar{\lambda}_i \psi(\lambda_i, \bar{\lambda}_i) \exp(i\bar{\lambda}^{\dot{\alpha}}_{(i)}\mu_{\dot{\alpha}(i)}) \tag{3.43}$$

where $Z_i \equiv (\lambda_i, \mu_i)$ denote a 4 component vector. If we consider the metric $SO(2,2)$ the spinors $(\lambda, \bar{\lambda})$ are real and independent. Thus (**??**) is the usual Fourier transformation, and the variable μ can be interpreted as the conjugate of $\bar{\lambda}$, in the same sense p is the conjugate of x.

The conformal group in the signature $SO(2,2)$ is given by $SL(4,R)$. Note that the Z in this metric is a 4 real component vector that transforms naturally under $SL(4,R)$. The Z is called twistor lives in R^4 the twistor space.

As you probably note the scaling $(\lambda, \bar{\lambda}) \to (t\lambda, t^{-1}\bar{\lambda})$ plays a important role, by connecting the helicity of the particle and how the amplitude change under this scaling. Just to remind you

$$\mathcal{A}(t\lambda_i, t^{-1}\bar{\lambda}_i) = t^{-2h_i}\mathcal{A}(\lambda_i, \bar{\lambda}_i) \tag{3.44}$$

If we transform the amplitude to the Twistor space

$$\tilde{\mathcal{A}}(Z_i) = \int d^2\bar{\lambda}_i \mathcal{A}(\lambda_i, \bar{\lambda}_i) \exp(i\bar{\lambda}^{\dot{\alpha}}_{(i)}\mu_{\dot{\alpha}(i)}) \tag{3.45}$$

Under $Z_i(\lambda_i, \mu_i) \to tZ_i(\lambda_i, \mu_i)$ the amplitude change as

$$\tilde{\mathcal{A}}(tZ_i) = \int d^2\bar{\lambda}_i \mathcal{A}(t\lambda_i, \bar{\lambda}_i) \exp(i\bar{\lambda}^{\dot{\alpha}}_{(i)}t\mu_{\dot{\alpha}(i)}) = t^{-2(h_i+1)}\tilde{\mathcal{A}}(Z_i) \tag{3.46}$$

where the t^{-2} came from redefining the $d^2\bar{\lambda}_i \to t^{-2}d^2\bar{\lambda}_i$. From (**??**) the $\tilde{\mathcal{A}}(Z_i)$ transforms homogeneously with weight $-2(h_i+1)$ in the Z_i variable. Thus we can define our helicity operator in twistor variables as

$$h = -\frac{1}{2}\left(Z_i^I \frac{\partial}{\partial Z_i^I} + 2\right) \tag{3.47}$$

To find the generators $G(\lambda, \bar{\lambda})$ in the twistor space $\tilde{G}(\lambda, \mu)$ we use the Fourier transformation

$$\tilde{G}(\lambda, \mu)\tilde{\psi}(\lambda, \mu) = \int d^2\bar{\lambda}_i G(\lambda, \bar{\lambda})\psi(\lambda, \bar{\lambda}) \exp(i\bar{\lambda}^{\dot{\alpha}}\mu_{\dot{\alpha}}) \tag{3.48}$$

Doing carefully you find that

$$\bar{\lambda}_{\dot{\alpha}} \frac{\partial}{\partial \bar{\lambda}^{\dot{\beta}}} \to -\varepsilon_{\dot{\alpha}\dot{\beta}} + \mu_{\dot{\beta}} \frac{\partial}{\partial \mu^{\dot{\alpha}}} \tag{3.49}$$

the rest of the transformation are trivial so we find the generators in twistor variables as

$$m_{\alpha\beta} = \frac{i}{2}\sum_{i=1}^{n}\left(\lambda_{\alpha}^{(i)} \frac{\partial}{\partial \lambda^{\beta(i)}} + \lambda_{\beta}^{(i)} \frac{\partial}{\partial \lambda^{\alpha(i)}}\right) \tag{3.50}$$

$$\bar{m}_{\dot{\alpha}\dot{\beta}} = \frac{i}{2}\sum_{i=1}^{n}\left(\bar{\mu}_{\dot{\alpha}}^{(i)} \frac{\partial}{\partial \mu^{\dot{\beta}(i)}} + \mu_{\dot{\beta}}^{(i)} \frac{\partial}{\partial \mu^{\dot{\alpha}(i)}}\right) \tag{3.51}$$

– Dilation operator:

$$d = \frac{i}{2}\sum_{i=1}^{n}\left(\lambda^{\alpha(i)} \frac{\partial}{\partial \lambda^{\alpha(i)}} - \mu^{\dot{\alpha}(i)} \frac{\partial}{\partial \mu^{\dot{\alpha}(i)}}\right) \tag{3.52}$$

–Special Conformal operator

$$K_{\alpha\dot{\alpha}} = i\sum_{i=1}^{n} \mu_{\dot{\alpha}} \frac{\partial}{\partial \lambda^{\alpha(i)}} \tag{3.53}$$

– Momentum operator

$$P_{\alpha\dot{\alpha}} = i\sum_{i=1}^{n} \lambda_{\alpha}^{(i)} \frac{\partial}{\partial \mu^{\dot{\alpha}(i)}} \tag{3.54}$$

3.3.1 amplitudes and Twistor (re-think the name)

Now that we know how the conformla generatos look like in the twistor matheus

To appreciate even more Twistors we will take a look on the

3.3.2 The ambitwistor

In (**??**) we choose to Fourier transform $\bar{\lambda}$ but we could have done the opposite

$$\tilde{\mathcal{A}}(W_i) = \int d^2\lambda_i \mathcal{A}(\lambda_i, \bar{\lambda}_i) \exp(i\bar{\mu}^{\alpha}_{(i)}\lambda_{\alpha(i)}) \tag{3.55}$$

here we define $W = (\bar{\mu}, \bar{\lambda})$ called the ambitwistor variable. Than using the power of scaling again we get

$$\tilde{\mathcal{A}}(tW_i) = \int d^2\lambda_i \mathcal{A}(\lambda_i, t\bar{\lambda}_i) \exp(it\bar{\mu}^{\alpha}_{(i)}\lambda_{\alpha(i)}) = t^{2(h-1)}\tilde{\mathcal{A}}(W_i) \tag{3.56}$$

using $\mathcal{A}(t^{-}\lambda_i, t\bar{\lambda}_i) = t^{2h_i}\mathcal{A}(\lambda_i, \bar{\lambda}_i)$. Note that ig the particle i has positive helicity then $\mathcal{A}(tW_i) = \mathcal{A}(W_i)$ and the same is true for a particle with negative helicity $\mathcal{A}(tZ_i) = \mathcal{A}(Z_i)$. The W can be interpreted as the conjugate of Z. In the same way we did for the μ and $\bar{\lambda}$. We can construct a invariant Lorentz object $Z \cdot W = \lambda\bar{\mu} + \mu\bar{\lambda}$. With these constrains we can calculate for example the 4 gluon scattering $\mathcal{A}(1^+, 2^+, 3^-, 4^-)$ using the fact that

$$\begin{aligned}\mathcal{A}(W_1^+, W_2^+, Z_3^-, Z_4^-) =& \mathcal{A}(tW_1^+, W_2^+, Z_3^-, Z_4^-) = \mathcal{A}(W_1^+, tW_2^+, Z_3^-, Z_4^-) = \\ =& \mathcal{A}(W_1^+, W_2^+, tZ_3^-, Z_4^-) = \mathcal{A}(W_1^+, W_2^+, Z_3^-, tZ_4^-)\end{aligned} \tag{3.57}$$

so it seems that it does not depend on the $W's$ and $Z's$. It seems that in twistor variable the amplitude is just a constant. Actually the amplitude will depend the sign of each Lorentz product.

$$\mathcal{A}(W_1^+, W_2^+, Z_3^-, Z_4^-) = sign(Z_3 \cdot W_1) sign(Z_3 \cdot W_2) sign(Z_4 \cdot W_1) sign(Z_4 \cdot W_2) \tag{3.58}$$

That is amazing! we were able to find the 4 gluon scattering in very few lines using just scaling and the right variables.

3.4 Super conformal Transformation

Chapter 4

First order Lagrangian

4.1 bc Conformal Field Theory

4.1.1 Properties

4.2 Linear Dilaton and bc are the same?

4.2.1 Bosonization

Chapter 5

Berkovits's Twistor String Action

5.1 Bosonic action

5.1.1 conformal symmetry and generators

5.1.2 particle action (maybe)

5.2 Adding Super Things

5.2.1 Stress Energy Tensor; vertex Operator; Q-brst ; Current Algebra

5.2.2 Instanton number (d) and helicity (h)

5.3 The MHV amplitude from Twistor String

Chapter 6

Twistors in Ten Dimensions

6.1 Motivation

6.2 The goal and the difficulty

6.3 Mason Ambtwistor and Scattering amplitudes

Appendix A

Dotted or Undotted

This appendix it's aimed to set up the conventions / notation that I will use in the rest of the thesis and to refresh the reader (and myself) some topics. The metric is the mostly plus $\eta = (- + ++)$. The notation will be the same as the book Wess & Bagger [referecia]. If the reader is not familiar with these concepts keep going that in the end I will make a connection with the usual Dirac stuff.

Let $\mathbf{M}$ be a two-by-two matrix with $\det \mathbf{M} = 1$, i.e. $\mathbf{M} \in SL(2, C)$, this are matrices with complex values and unit determinant. One thing to note, is the number of generators of this group. We have 4 complex entries (8 real) and the constrain from the unit determinant give two equations (real part = 1 and imaginary = 0). Thus we have $8 - 2 = 6$ generators, the same as our old friend The Lorentz Group $SO(3, 1)$ with 3 boosts + 3 rotations. Now we introduce the the dotted and undotted indices. The spinor with dotted indices transform under the $(0, 1/2)$ representation of Lorentz group and spinor with undotted indices transform under $(1/2, 0)$ conjugate representation.The spinor indices take values $\alpha = 1, 2 \quad \dot{\alpha} = \dot{1}, \dot{2}$.

$$\psi'_\alpha = M_\alpha{}^\beta \psi_\beta \qquad ; \qquad \psi'^\alpha = (M^{-1})_\beta{}^\alpha \psi^\beta \tag{A.1a}$$

$$\bar{\psi}'_{\dot{\alpha}} = (M^*)_{\dot{\alpha}}{}^{\dot{\beta}} \bar{\psi}_{\dot{\beta}} \qquad ; \qquad \bar{\psi}'^{\dot{\alpha}} = (M^*)^{-1}{}_{\dot{\beta}}{}^{\dot{\alpha}} \bar{\psi}^{\dot{\beta}} \tag{A.1b}$$

We have two indices (dotted and undotted) the two representation are inequivalent, i.e. we can not find a matrix $\mathbf{C}$ such that $\mathbf{M} = \mathbf{C}\mathbf{M}^*\mathbf{C}^{-1}$. But the two representation in (**??**) are equivalent, thus exist a matrix ε such that $\mathbf{M} = \varepsilon \mathbf{M}^{-1T} \varepsilon^{-1}$. Hang on that we will see what this matrix is. The same is valid for the two transformation with dotted indices.

We recall that any 2×2 matrix can be written as linear combination of the Pauli matrices plus the identity. Let me call this basis as $\sigma^m = (-I, \vec{\sigma})$, where $m = 0, ..., 3$.

$$\mathbf{P} = P_m \sigma^{\mathbf{m}} = -IP_0 + \vec{P}\cdot\vec{\sigma} = \begin{pmatrix} -P_0 + P_3 & P_1 - iP_2 \\ P_1 + iP_2 & -P_0 - P_3 \end{pmatrix} \tag{A.2}$$

We can see that $\mathbf{P}$ is hermitian ($\mathbf{P} = \mathbf{P}^\dagger$). A nice property of the matrix P is that $\det \mathbf{P} = P_0^2 - \vec{P}\cdot\vec{P} = -\eta^{mn}P_m P_n$. Using the fact that $\mathbf{P}$ is hermitian we can write another matrix $\mathbf{P'}$ as:

$$\mathbf{P'} = \mathbf{MPM}^\dagger \tag{A.3}$$

$$\mathbf{P'}^\dagger = (\mathbf{MPM}^\dagger)^\dagger = \mathbf{MPM}^\dagger = \mathbf{P'} \tag{A.4}$$

Both $\mathbf{P'}$ and $\mathbf{P}$ can be written as linear combination of σ^m. The determinant of $\mathbf{P'}$ (because the determinant of $\mathbf{M}$ is one and $\det[ABC] = \det[A]\det[B]\det[C]$) is equal to the determinant of $\mathbf{P'}$.

$$\det \mathbf{P'} = -\eta^{mn}P'_m P'_n = -\eta^{mn}P_m P_n \tag{A.5}$$

Now we start to see the connection between the Lorentz group and this matrices.This transformation correspond to a Lorentz transformation, that's cool. Before we continue let's appreciate what we have done. We stared defining a 2×2 matrix $\mathbf{M}$ that had determinant one (you could say unimodular), and we noted that any 2×2 hermitian matrix $\mathbf{P}$ could be expanded as a linear combination of σ^m and the determinant of this was the inner product of a Lorentz four vector, i.e, $\eta^{mn}P_m P_n$. Finally we found a transformation that is the same as the Lorentz Transformation.

Lets take a look on the index structure of $\mathbf{P}$. From (**??**) that $\mathbf{M}^\dagger \equiv (\mathbf{M^T})^* = ((M_\alpha{}^\beta)^T)^* = (M^\beta{}_\alpha)^* = M^{\dot\beta}{}_{\dot\alpha}$. Thus we can rewrite (**??**) as:

$$P'_{\alpha\dot\alpha} = M_\alpha{}^\beta P_{\beta\dot\beta} M^{\dot\beta}{}_{\dot\alpha} \tag{A.6}$$

And the index structure of the Pauli matrices : $\sigma^m = \sigma^m_{\alpha\dot\alpha}$. Note that we use Latin indices for vectors and tensors and Greek indices for spinors. Now we return to the matrix ε that relate the two equivalent representation (**??**).

Let

$$\varepsilon = (\varepsilon_{\alpha\beta}) = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{A.7}$$

$$\varepsilon^{-1} = (\varepsilon^{\alpha\beta}) = \begin{pmatrix} \varepsilon^{11} & \varepsilon^{12} \\ \varepsilon^{21} & \varepsilon^{22} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{A.8}$$

with this matrix one can just plug in $\mathbf{M}^{-1T} = \varepsilon \mathbf{M} \varepsilon^{-1}$ and check that works. The $\varepsilon_{\alpha\beta}$ and $\varepsilon^{\alpha\beta}$ are antisymmetric tensors, and satisfy $\det \varepsilon = 1$ and $\varepsilon_{\alpha\beta}\varepsilon^{\beta\gamma} = \delta_{\alpha}{}^{\gamma}$.

we can write $\mathbf{M}^{-1T} = \varepsilon \mathbf{M} \varepsilon^{-1}$ with indices :

$$\varepsilon^{\alpha\beta} M_{\beta}{}^{\gamma} \varepsilon_{\gamma\rho} = (M^{-1T})_{\rho}{}^{\alpha} = (M^{-1})^{\alpha}{}_{\rho} \tag{A.9}$$

$$\psi'^{\alpha} = (M^{-1})^{\alpha}{}_{\rho} \psi^{\rho} = \varepsilon^{\alpha\beta} M_{\beta}{}^{\gamma} (\varepsilon_{\gamma\rho}\psi^{\rho}) \tag{A.10}$$

$$(\varepsilon_{\beta\alpha}\psi'^{\alpha}) = M_{\beta}{}^{\gamma} (\varepsilon_{\gamma\rho}\psi^{\rho}) \tag{A.11}$$

Thus $\varepsilon_{\gamma\rho}\psi^{\rho}$ transform as ψ_{γ} and the ε tensor can be used to lower and raise indices:

$$\psi_{\gamma} = \varepsilon_{\gamma\rho}\psi^{\rho} \quad ; \quad \psi^{\gamma} = \varepsilon^{\gamma\rho}\psi_{\rho} \tag{A.12}$$

Every thing that we have done for undotted indice can be done similar for dotted.

We are almost ready to see the connection to Dirac usual spinor. Before to do we take a look on some identities of the Pauli matrix. If we define another Pauli basis:

$$\bar{\sigma}^{m} = (-I, -\vec{\sigma}) \tag{A.13a}$$

with indices

$$(\bar{\sigma}^{m})^{\dot{\alpha}\alpha} = \varepsilon^{\dot{\alpha}\dot{\beta}} \varepsilon^{\alpha\beta} (\sigma^{m})_{\beta\dot{\beta}} \tag{A.13b}$$

we have some important identities:

$$\sigma^{(m}\bar{\sigma}^{n)} = (\sigma^{m}\bar{\sigma}^{n} + \sigma^{n}\bar{\sigma}^{m})_{\alpha}{}^{\beta} = -2\eta^{mn}\delta_{\alpha}{}^{\beta} \tag{A.14a}$$

$$\bar{\sigma}^{(m}\sigma^{n)} = (\bar{\sigma}^{m}\sigma^{n} + \bar{\sigma}^{n}\sigma^{m})_{\dot{\alpha}}{}^{\dot{\beta}} = -2\eta^{mn}\delta_{\dot{\alpha}}{}^{\dot{\beta}} \tag{A.14b}$$

and

$$\text{Tr}\ \sigma^m \bar{\sigma}^n = -2\eta^{mn} \tag{A.15a}$$

$$(\sigma^m)_{\alpha\dot{\alpha}} (\bar{\sigma}_m)^{\dot{\beta}\beta} = -2\delta_{\dot{\alpha}}{}^{\dot{\beta}} \delta_\alpha{}^\beta \tag{A.15b}$$

Now we can easy go back and forth between Lorentz indices and bispinor indices ($m \leftrightarrow \alpha\dot{\alpha}$):

$$p_{\alpha\dot{\alpha}} = p_m (\sigma^m)_{\alpha\dot{\alpha}} \quad ; \quad p_m = -\frac{1}{2} (\bar{\sigma}_m)^{\dot{\beta}\beta} p_{\beta\dot{\beta}} \tag{A.16}$$

As I promise, let's see the connection with the usual Dirac matrices and spinors. The Clifford algebra is (in the $(-,+++)$ metric):

$$\{\Gamma^m, \Gamma^n\} = -2I\eta^{mn} \tag{A.17}$$

In the Weyl basis the gamma matrix is :

$$\Gamma^m = \begin{pmatrix} 0 & \sigma^m \\ \bar{\sigma}^m & 0 \end{pmatrix} \tag{A.18}$$

One can easily see that this gamma matrix satisfy the Clifford algebra (**??**). The gamma matrix act on a 4 components spinor with index structure:

$$\Psi = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}$$

One can write the Dirac equation:

$$(\Gamma^m \partial_m + m)\Psi = 0 \tag{A.19a}$$

and in the weyl basis

$$((\bar{\sigma}^m)^{\dot{\alpha}\alpha} \partial_m + m)\psi_\alpha = 0 \tag{A.19b}$$

$$((\sigma^m)_{\alpha\dot{\alpha}} \partial_m + m)\bar{\chi}^{\dot{\alpha}} = 0 \tag{A.19c}$$

Remember also that the Lorentz generators were given by $S^{mn} = \frac{1}{4}[\Gamma^m, \Gamma^n]$, and the Dirac component spinor transform as $\Psi \to \exp(\frac{1}{2}\omega_{mn}S^{mn})\Psi$. Then the Lorentz group generator in the spinor representation become:

$$(\sigma^{mn})_\alpha{}^\beta = \frac{1}{4}(\sigma^m\bar{\sigma}^n - \sigma^n\bar{\sigma}^m)_\alpha{}^\beta \tag{A.20a}$$

$$(\bar{\sigma}^{mn})^{\dot{\alpha}}{}_{\dot{\beta}} = \frac{1}{4}(\bar{\sigma}^m\sigma^n - \bar{\sigma}^n\sigma^m)^{\dot{\alpha}}{}_{\dot{\beta}} \tag{A.20b}$$

This matrices seem strange when one looks for the first time, remember they are made of Pauli Matrices, in particular:

$$\sigma^{ij} = \bar{\sigma}^{ij} = -\frac{i}{2}\epsilon^{ijk}\sigma^k \quad \text{and} \quad \sigma^{0i} = -\bar{\sigma}^{0i} = \frac{1}{2}\sigma^i$$

these are rotations and boost respectively. The ϵ^{ijk} is the usual Levi-Civita symbol, and $i, j, k = 1, 2, 3$ with $\epsilon^{123} = 1$. Note that rotations act the same in booth spinors as opposed for boost. The Lorentz transformation acts as:

$$\psi'_\alpha = (e^{\frac{1}{2}\omega_{mn}\sigma^{mn}})_\alpha{}^\beta\psi_\beta \tag{A.21}$$

$$\bar{\psi}'^{\dot{\alpha}} = (e^{\frac{1}{2}\omega_{mn}\bar{\sigma}^{mn}})^{\dot{\alpha}}{}_{\dot{\beta}}\bar{\psi}^{\dot{\beta}} \tag{A.22}$$

Now let's make some checks and see that with these Lorentz transformation we get the right answers. We know what a rotation does on a vector P^m. Then if we use (**??**) and multiply by $\bar{\sigma}^m$ and take the trace using (**??**) we get:

$$P'^m = -\frac{1}{2}\,\mathrm{Tr}[\bar{\sigma}^m\mathbf{M}\sigma^n\mathbf{M}^\dagger]P_n \tag{A.23}$$

If we choose a rotation on z-axis:

$$\mathbf{M} = e^{\omega_{12}\sigma^{12}} = e^{\frac{i}{2}\theta\sigma^3} = I\cos(\theta/2) + i\sigma^3\sin(\theta/2)$$

where we used the fact $\omega_{12} = -\omega_{21}$ that kills the 1/2 in (**??**) and then we choose $\omega_{12} = -\theta$. Now that we have all the elements we can express (**??**) as:

$$P'^m = P^m\cos^2(\theta/2) - \frac{1}{2}P_n\sin^2(\theta/2)\mathrm{Tr}[\bar{\sigma}^m\sigma^3\sigma^n\sigma^3] - \frac{i}{2}\sin(\theta/2)\cos(\theta/2)P_n\mathrm{Tr}[\bar{\sigma}^m[\sigma^3, \sigma^n]] \tag{A.24}$$

with this expression and plus some sigma Trace identities and trigonometric one find:

$$P'^0 = P^0 \quad ; \quad \vec{P}' = \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \vec{P} \tag{A.25}$$

That's our rotation matrix, it worked! Let me say the trivial fact that if $\theta = 2\pi$ then $\vec{P}' = \vec{P}$. Now we do the same trivial statement for a spinor, under a full rotation $\psi' = e^{\frac{i}{2}2\pi\sigma^3}\psi = -\psi$, that's another way to see that it's a (1/2)

$$a \overset{(??)}{=} b \tag{A.26}$$

Bibliography