Chapter 1

An Appendix

This appendix it's aimed to set up the conventions / notation that I will use in the rest of the thesis and to refresh the reader(and myself) some topics. The metric is the mostly plus $\eta = (- + ++)$.The notation will be the same as the book Wess & Bagger [referecia]. If the reader is not familiar with this concepts keep going that in the end I will make a connection with the usual Dirac stuff.

Let $\mathbf{M}$ be a two-by-two matrix with $det\mathbf{M} = 1$ or $\mathbf{M} \in SL(2, C)$ this are matrices with complex values and unit determinant. One thing to note, is that the number of generators of this group. We have 4 complex entries (8 real) and the constrain from the unit determinant, give two equations (real part = 1 and imaginary = 0). Thus we have $8 - 2 = 6$ generators, the same as our old friend The Lorentz Group $SO(3, 1)$ with 3 boosts + 3 rotations . Now we introduce the the dotted and undotted indices. The spinor with dotted indices transform under the $(0, 1/2)$ representation of Lorentz group and spinor with undotted indices transform under $(1/2, 0)$ conjugate representation.The spinor indices take values $\alpha = 1, 2 \quad \dot{\alpha} = \dot{1}, \dot{2}$.

$$\Psi'_\alpha = M_\alpha{}^\beta \Psi_\beta \quad ; \quad \Psi'^\alpha = (M^{-1})_\beta{}^\alpha \Psi^\beta \tag{1.1}$$

$$\bar{\Psi}'_{\dot{\alpha}} = (M^*)_{\dot{\alpha}}{}^{\dot{\beta}} \bar{\Psi}_{\dot{\beta}} \quad ; \quad \bar{\Psi}'^{\dot{\alpha}} = (M^*)^{-1}{}_{\dot{\beta}}{}^{\dot{\alpha}} \bar{\Psi}^{\dot{\beta}} \tag{1.2}$$

We recall that any $2X2$ matrix can be written as linear combination of the Pauli matrices plus the identity. Let me call this basis as $\sigma^m = (-I, \vec{\sigma})$, where $m = 0, ..., 3$.

$$\mathbf{P} = P_m \sigma^{\mathbf{m}} = -IP_0 + \vec{P} \cdot \vec{\sigma} = \begin{pmatrix} -P_0 + P_3 & P_1 - iP_2 \\ P_1 + iP_2 & -P_0 - P_3 \end{pmatrix} \tag{1.3}$$

We can see that $\mathbf{P}$ is hermitian ($\mathbf{P} = \mathbf{P}^\dagger$). A nice property of the matrix P is that $\det\mathbf{P} = P_0^2 - \vec{P}\cdot\vec{P} = -\eta^{mn}P_mP_n$. Using the fact that $\mathbf{P}$ is hermitian we can write another matrix $\mathbf{P'}$ as:

$$\mathbf{P'} = \mathbf{MPM}^\dagger \tag{1.4}$$

$$\mathbf{P'}^\dagger = (\mathbf{MPM}^\dagger)^\dagger = \mathbf{MPM}^\dagger = \mathbf{P'} \tag{1.5}$$

Both $\mathbf{P'}$ and $\mathbf{P}$ can be written as linear combination of σ^m. The determinant of $\mathbf{P'}$ (because the determinant of $\mathbf{M}$ is one and $det[ABC] = det[A]det[B].det[C]$) is equal to the determinant of $\mathbf{P'}$.

$$\det\mathbf{P'} = -\eta^{mn}P'_mP'_n = -\eta^{mn}P_mP_n \tag{1.6}$$

Now we start to see the connection between the Lorentz group and this matrices.This transformation correspond to a Lorentz transformation, that's cool. Before we continue let's appreciate what we have done. We stared defining a 2×2 matrix $\mathbf{M}$ that had determinant one (you could say unimodular), and we noted that any 2×2 hermitian matrix $\mathbf{P}$ could be expanded as a linear combination of σ^m and the determinant of this was the inner product of a Lorentz four vector, i.e, $\eta^{mn}P_mP_n$. Finally we found a transformation that is the same as the Lorentz Transformation.

Lets take a look on the index structure of $\mathbf{P}$. From 1.1 that $\mathbf{M}^\dagger \equiv (\mathbf{M^T})^* = ((M_\alpha{}^\beta)^T)^* = (M^\beta{}_\alpha)^* = M^{\dot\beta}{}_{\dot\alpha}$. Thus we can rewrite 1.4 as:

$$P_{\alpha\dot\alpha} = M_\alpha{}^\beta P_{\beta\dot\beta}M^{\dot\beta}{}_{\dot\alpha} \tag{1.7}$$

And the index structure of the Pauli matrices : $\sigma^m = \sigma^m_{\alpha\dot\alpha}$.

In the Weyl basis the gamma matrix is:

$$\gamma^m = \begin{pmatrix} 0 & \sigma^m \\ \bar\sigma^m & 0 \end{pmatrix} \tag{1.8}$$

where the $\sigma^m = (-I, \vec\sigma)$ and $\bar\sigma^m = (-I, -\vec\sigma)$, I and $\vec\sigma$ are the identity and Pauli matrices. The gamma matrix act on a 4 components spinor

$$\Psi = \begin{pmatrix} \psi_\alpha \\ \bar\chi^{\dot\alpha} \end{pmatrix}$$

so the index structure of the σ^m and $\bar{\sigma}^m$ are

$$(\bar{\sigma}^m)^{\dot{\alpha}\alpha} \quad \texttt{and} \quad (\sigma^m)_{\alpha\dot{\alpha}} \tag{1.9}$$