For HW#2

Theorem 1. (Method of Transformation) *Let X be a continuous random variable with density function f_X and the set of possible values A. For the invertible function $h : A \to \mathbb{R}$, let $Y = h(X)$ be a random variable with the set of possible values $B = h(A)$. Suppose that the inverse of $y = h(x)$ is the function $x = h^{-1}(y)$, which is differentiable for all values of $y \in B$. Then f_Y, the density function of Y is given by*

$$f_Y(y) = f_X(h^{-1}(y))|(h^{-1})'(y)|$$

Let Θ be a uniform random variable over $[-\pi, \pi]$. Then we have

$$f_\Theta(\theta) = \begin{cases} \dfrac{1}{2\pi} & \text{if } \theta \in [-\pi, \pi] \\ 0 & otherwise. \end{cases}$$

Let $Y = f_m \cdot \cos(\Theta)$ be a random variable, where f_m is a constant. So $Y : [-f_m, f_m] \to [-\pi, \pi]$. Note that we seperate the domain of Θ into two parts, i.e., $[0, \pi]$ and $[-\pi, 0]$, so that we can find the inverse of $y = f_m \cdot \cos(\theta)$ for each part.

Calculate $\dfrac{d\theta}{dy}$ for each part and we obtain

$$\frac{d\theta}{dy} = \begin{cases} \dfrac{1}{f_m} \cdot \dfrac{1}{\sqrt{1 - (\frac{y}{f_m})^2}} & \text{for } \theta \in [-\pi, 0] \\ \dfrac{1}{f_m} \cdot \dfrac{-1}{\sqrt{1 - (\frac{y}{f_m})^2}} & \text{for } \theta \in [0, \pi] \end{cases}$$

For $y \in (-f_m, f_m)$, we have

$$\begin{aligned} f_Y(y) &= \sum_{y = f_m \cdot \cos(\theta)} f_\Theta(\theta = \cos^{-1}(\frac{y}{f_m})) \cdot |\frac{d\theta}{dy}| \\ &= \frac{1}{2\pi} \cdot \frac{2}{f_m} \cdot \frac{1}{\sqrt{1 - (\frac{y}{f_m})^2}} \\ &= \frac{1}{\pi f_m} \cdot \frac{1}{\sqrt{1 - (\frac{y}{f_m})^2}} \end{aligned}$$

and $f_Y(y) = 0$ otherwise.

To find $F_Y(y)$, the cumulative distribution function of Y, we just integral $f_Y(y)$. Note that for $y \in (-f_m, f_m)$,

$$\begin{aligned}
\int_{-f_m}^{y} f_Y(\tilde{y}) \cdot d\tilde{y} &= \int_{-f_m}^{y} \frac{1}{\pi f_m} \cdot \frac{1}{\sqrt{1-(\frac{\tilde{y}}{f_m})^2}} d\tilde{y} \\
&= \Big[\frac{1}{\pi f_m} \cdot f_m \cdot \sin^{-1}(\frac{\tilde{y}}{f_m})\Big]_{-f_m}^{y} \\
&= \frac{1}{\pi}\Big[{}^{-1}(\frac{y}{f_m}) + \frac{\pi}{2}\Big] \\
&= \frac{1}{2} + \frac{1}{\pi} \cdot \sin^{-1}(\frac{y}{f_m})
\end{aligned}$$

Therefore we have

$$F_Y(y) = \begin{cases} 0 & \text{for } y \in (-\infty, -f_m) \\ \frac{1}{2} + \frac{1}{\pi} \cdot \sin^{-1}\big(\frac{y}{f_m}\big) & \text{for } y \in [-f_m, f_m] \\ 1 & \text{for } y \in (f_m, \infty) \end{cases}$$