CSCI567 Machine Learning (Spring 2018)

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Lecture on March 5, 2018

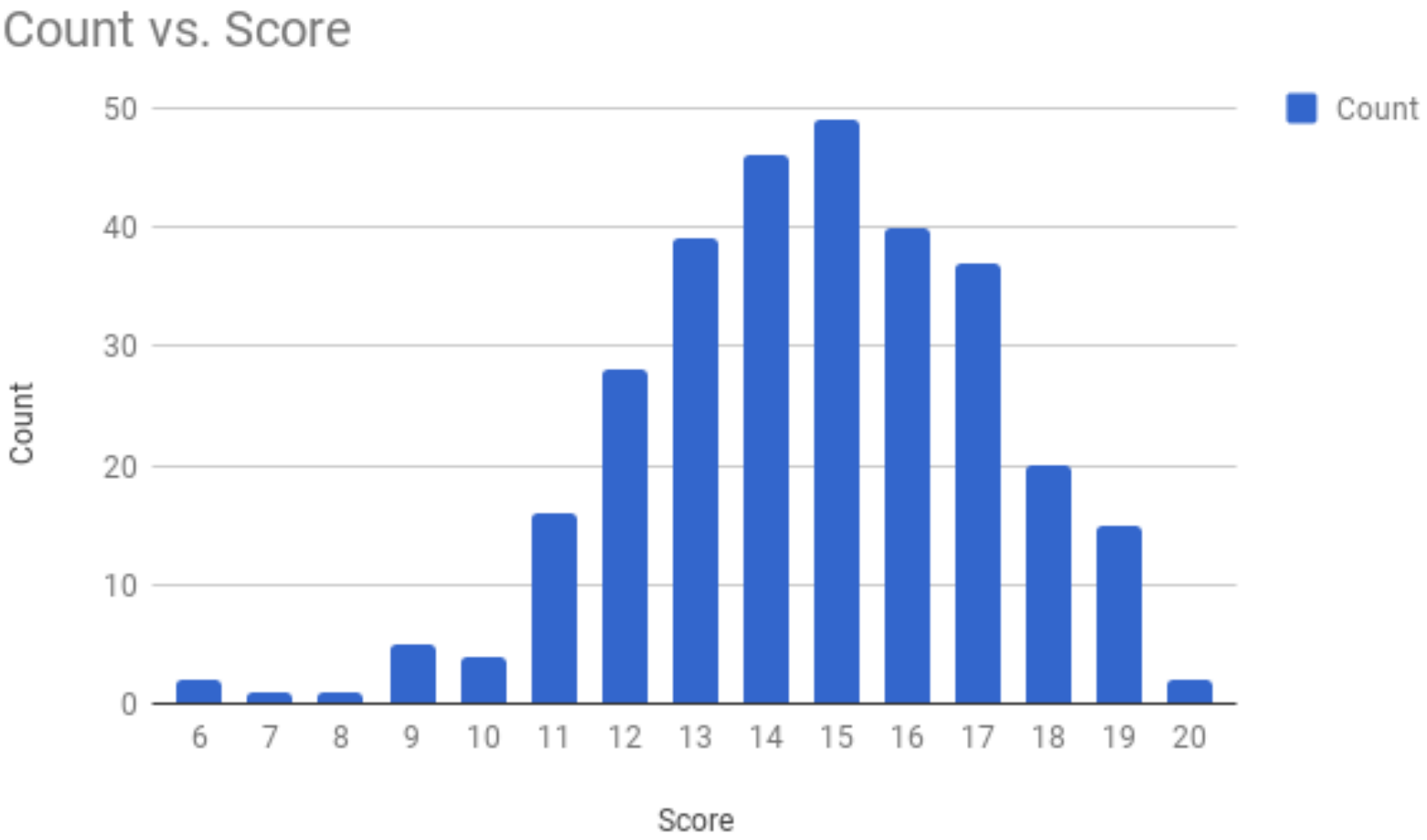
Outline

Outline

Administrative stuff on Quiz 1

- Grades are up
- Friday's regrades are under review
  - Reminder about spurious regrade requests
- PA3/PS3 due dates
  - Reminder about grace days

Breakdown of MC scores



Notes about quiz meanings

- At present time, I do not plan
  - to discuss curve
  - to disregard Quiz 1
  - to discount Quiz 1
- You *can* do well without free response
- But you should strive for more.

Outline

A tree model for deciding where to eat

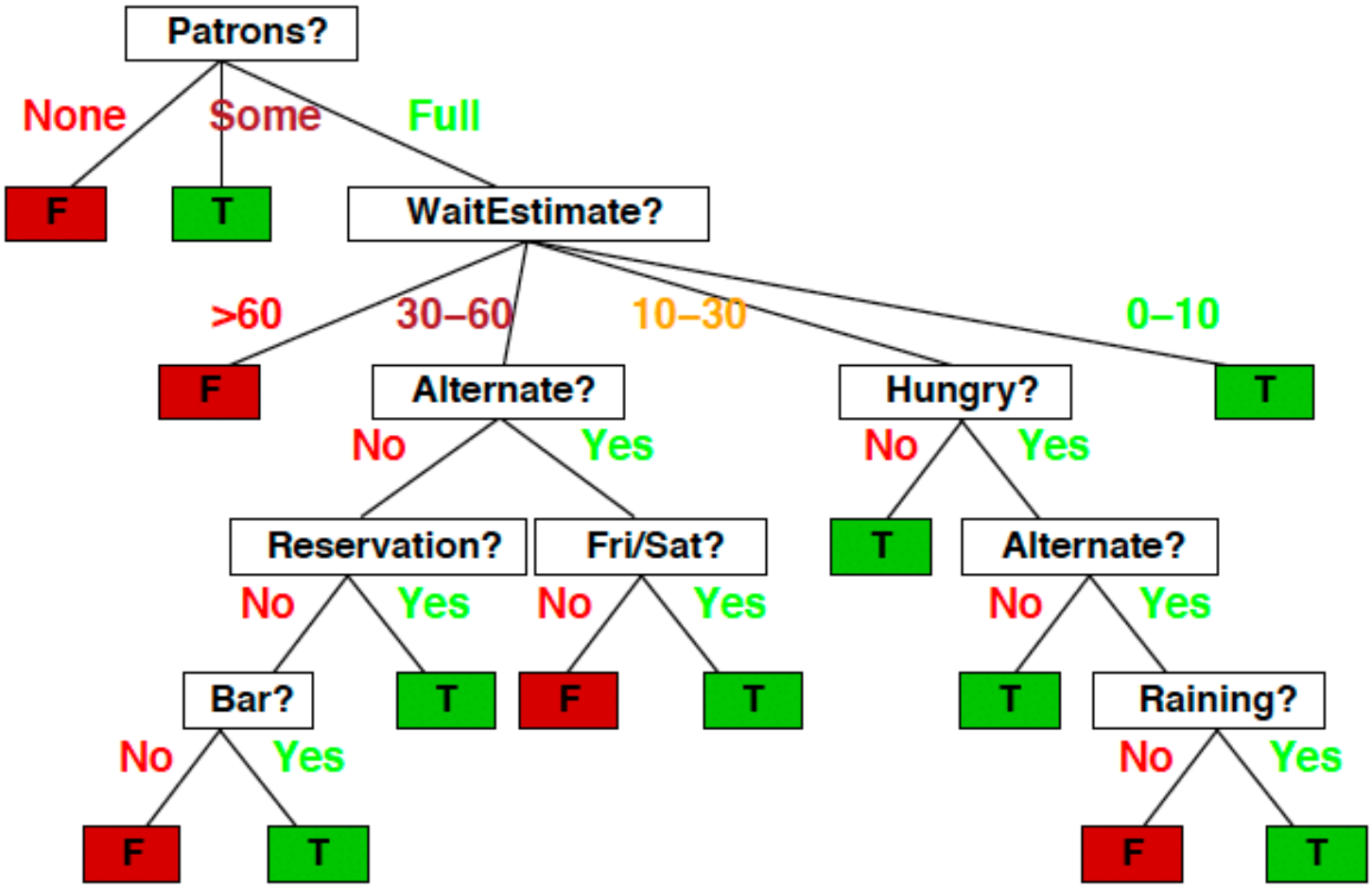
Choosing a restaurant

(Example from Russell & Norvig, AIMA)

Example	Attributes										Target
	Alt	*Bar*	*Fri*	*Hun*	*Pat*	*Price*	*Rain*	*Res*	*Type*	*Est*	*WillWait*
X_1	*T*	*F*	*F*	*T*	*Some*	*$$$*	*F*	*T*	*French*	*0–10*	*T*
X_2	*T*	*F*	*F*	*T*	*Full*	*$*	*F*	*F*	*Thai*	*30–60*	*F*
X_3	*F*	*T*	*F*	*F*	*Some*	*$*	*F*	*F*	*Burger*	*0–10*	*T*
X_4	*T*	*F*	*T*	*T*	*Full*	*$*	*F*	*F*	*Thai*	*10–30*	*T*
X_5	*T*	*F*	*T*	*F*	*Full*	*$$$*	*F*	*T*	*French*	*>60*	*F*
X_6	*F*	*T*	*F*	*T*	*Some*	*$$*	*T*	*T*	*Italian*	*0–10*	*T*
X_7	*F*	*T*	*F*	*F*	*None*	*$*	*T*	*F*	*Burger*	*0–10*	*F*
X_8	*F*	*F*	*F*	*T*	*Some*	*$$*	*T*	*T*	*Thai*	*0–10*	*T*
X_9	*F*	*T*	*T*	*F*	*Full*	*$*	*T*	*F*	*Burger*	*>60*	*F*
X_{10}	*T*	*T*	*T*	*T*	*Full*	*$$$*	*F*	*T*	*Italian*	*10–30*	*F*
X_{11}	*F*	*F*	*F*	*F*	*None*	*$*	*F*	*F*	*Thai*	*0–10*	*F*
X_{12}	*T*	*T*	*T*	*T*	*Full*	*$*	*F*	*F*	*Burger*	*30–60*	*T*

Classification of examples is positive (T) or negative (F)

Greedily we build the tree and get this



Outline

Boosting

High-level idea: combine a lot of classifiers

- Sequentially construct those classifiers one at a time
- Use *weak* classifiers to arrive at complex decision boundaries

Our plan

- Describe AdaBoost algorithm
- Derive the algorithm

How Boosting algorithm works?

- Given: N samples $\{\boldsymbol{x}_n, y_n\}$, where $y_n \in \{+1, -1\}$, and some ways of constructing weak (or base) classifiers
- Initialize weights $w_1(n) = \frac{1}{N}$ for every training sample.
- For t= 1 to T
  1. Train a weak classifier $h_t(\boldsymbol{x})$ based on the current weight $w_t(n)$, by minimizing the *weighted* classification error

$$\epsilon_t = \sum_n w_t(n) \mathbb{I}[y_n \neq h_t(\boldsymbol{x}_n)]$$

  2. Calculate weights for combining classifiers $\beta_t = \frac{1}{2} \log \frac{1-\epsilon_t}{\epsilon_t}$
  3. Update weights

$$w_{t+1}(n) \propto w_t(n) e^{-\beta_t y_n h_t(\boldsymbol{x}_n)}$$

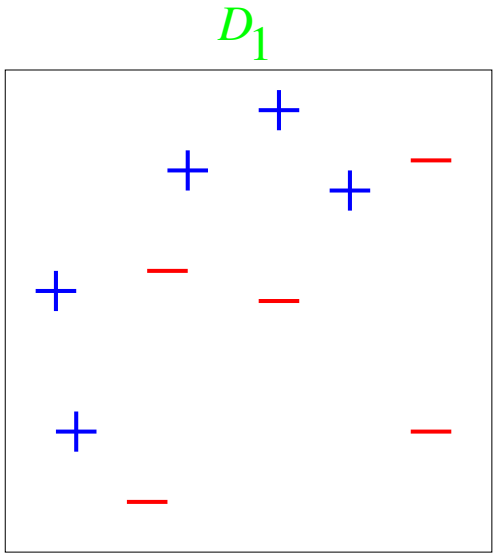
     and normalize them such that $\sum_n w_{t+1}(n) = 1$.
- Output the final classifier

$$h[\boldsymbol{x}] = \text{sign}\left[\sum_{t=1}^{T} \beta_t h_t(\boldsymbol{x})\right]$$

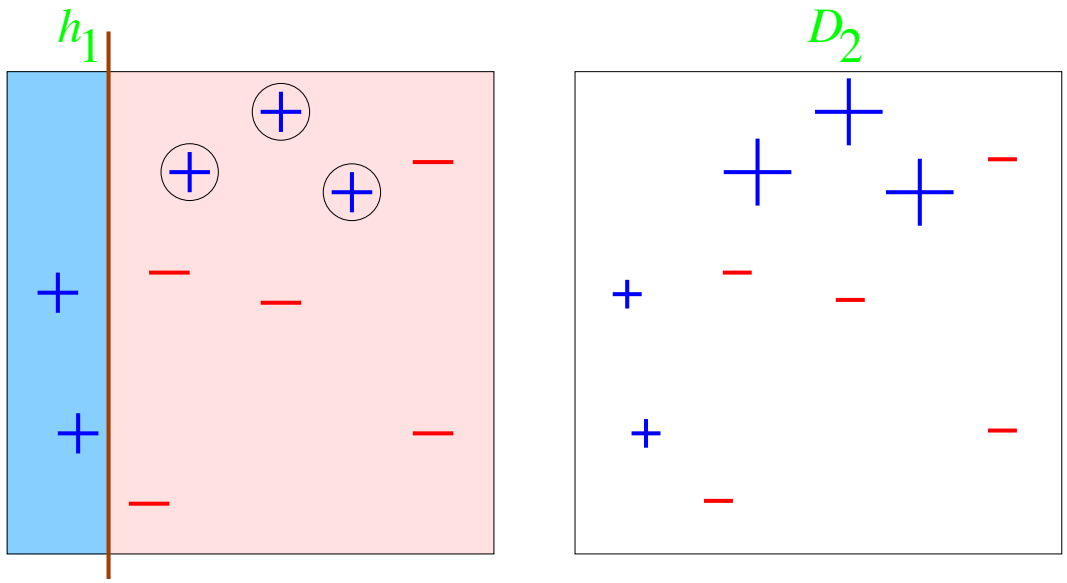
Example

10 data points

- Base classifier $h(\cdot)$: either horizontal or vertical lines (these are called *decision stumps*, classifying data based on a single attribute)
- The data points are clearly not linear separable.
- In the beginning, all data points have equal weights (the size of the data markers "+" or "-")

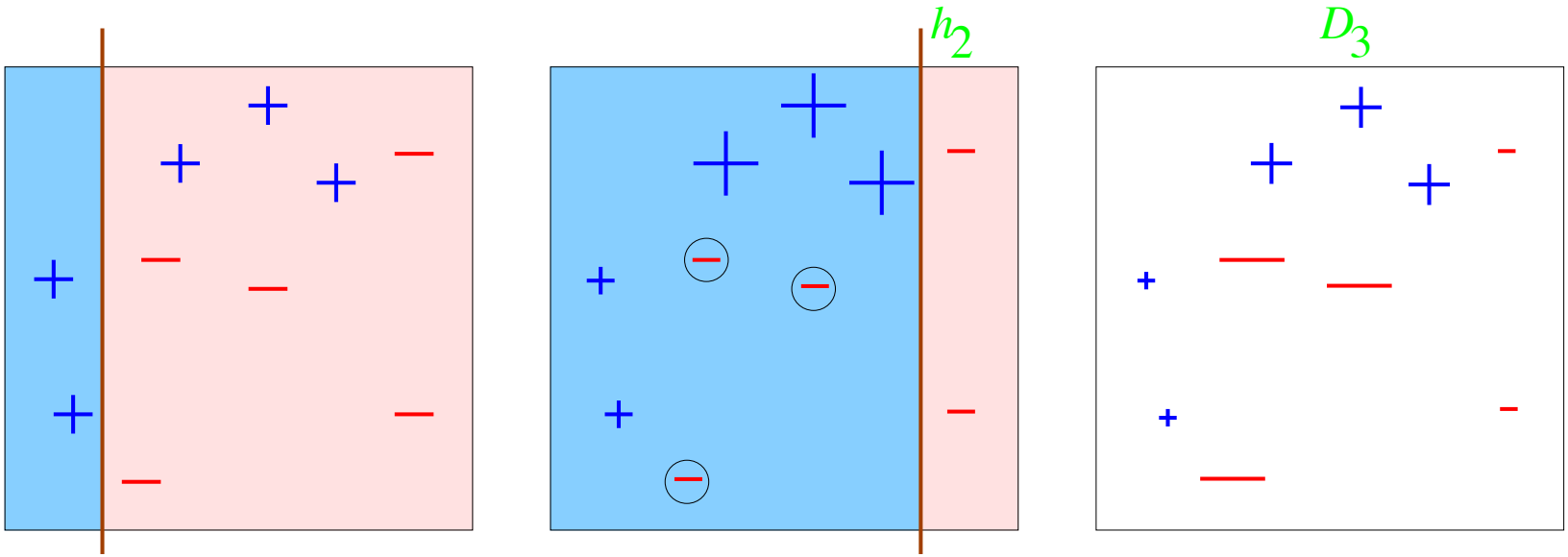


Round 1: $t = 1$



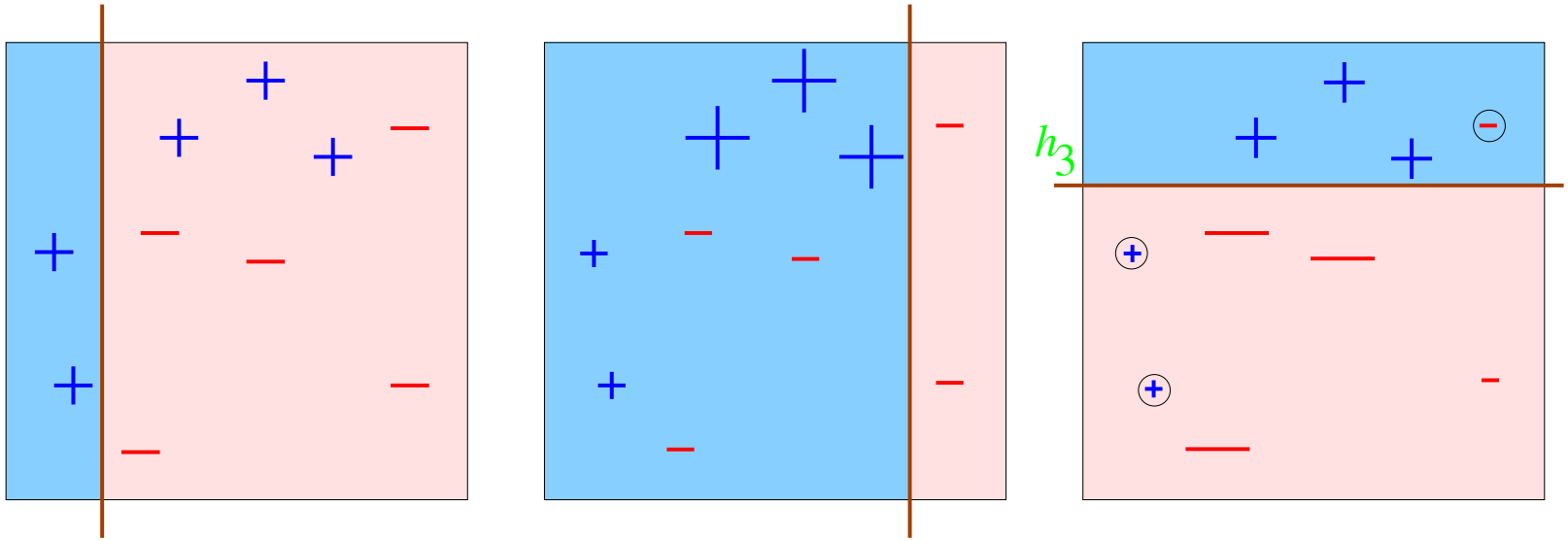
- 3 misclassified (with circles): $\epsilon_1 = 0.3 \rightarrow \beta_1 = 0.42$.
- Weights recomputed; the 3 misclassified data points receive larger weights

Round 2: $t = 2$



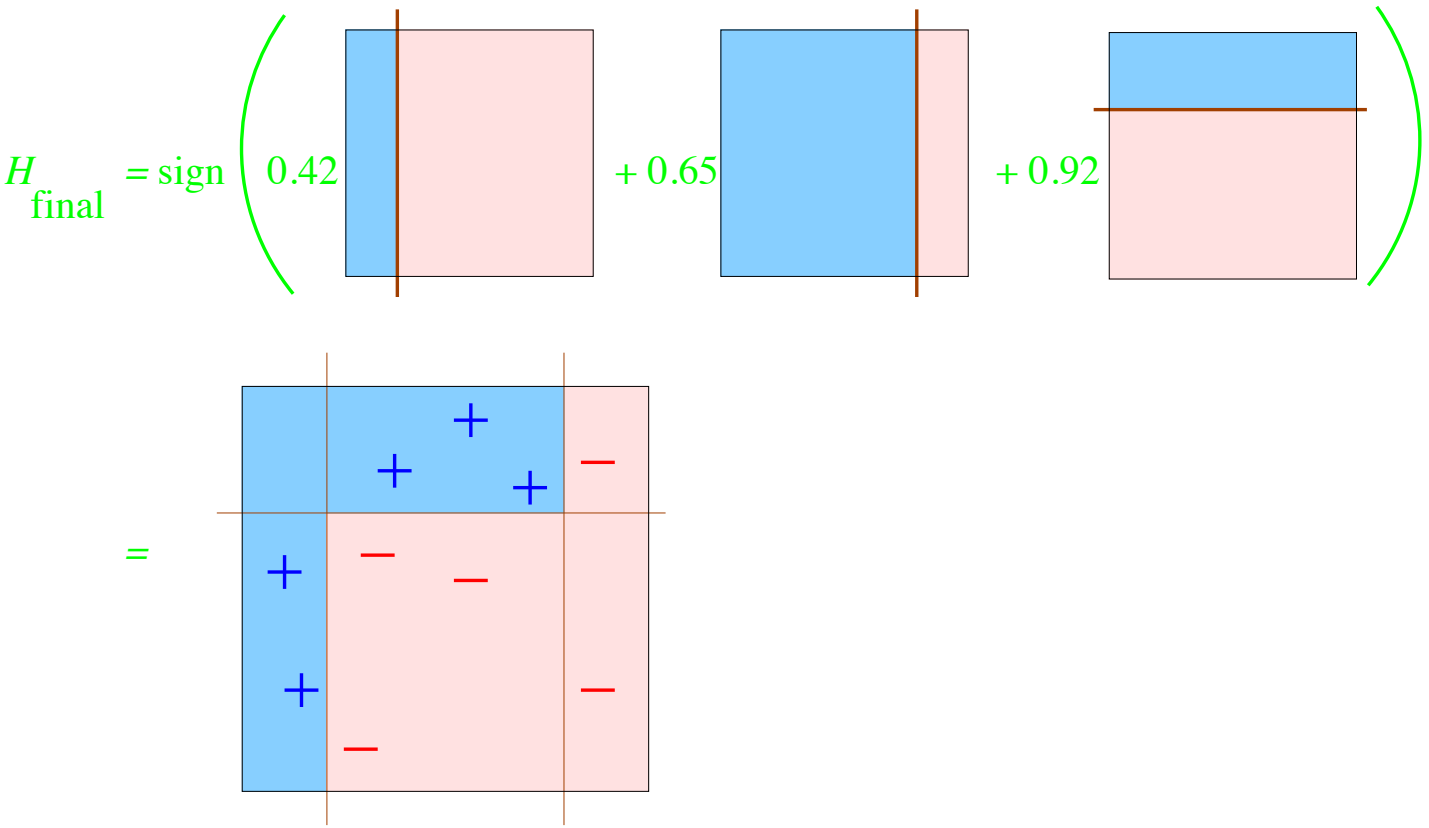
- 3 misclassified (with circles): $\epsilon_2 = 0.21 \rightarrow \beta_2 = 0.65$.
  Note that $\epsilon_2 \neq 0.3$ as those 3 data points have weights less than $1/10$
- Weights recomputed; the 3 misclassified data points receive larger weights. Note that the data points classified correctly on round $t = 1$ receive much smaller weights as they have been consistently classified correctly

Round 3: $t = 3$



- 3 misclassified (with circles): $\epsilon_3 = 0.14 \rightarrow \beta_3 = 0.92$.
- Note that those previously correctly classified data points are now misclassified — however, we might be lucky on this as if they have been consistently classified correctly, then this round's mistake is probably not a big deal.

Final classifier: combining 3 classifiers



- all data points are now classified correctly!

Why AdaBoost works?

We will show next that it minimizes a loss function related to classification error.

Classification loss

- Suppose we want to have a classifier

$$h(\boldsymbol{x}) = \text{sign}[f(\boldsymbol{x})] = \begin{cases} 1 & \text{if } f(\boldsymbol{x}) > 0 \\ -1 & \text{if } f(\boldsymbol{x}) < 0 \end{cases}$$

- our loss function is thus

$$\ell(h(\boldsymbol{x}), y) = \begin{cases} 0 & \text{if } yf(\boldsymbol{x}) > 0 \\ 1 & \text{if } yf(\boldsymbol{x}) < 0 \end{cases}$$

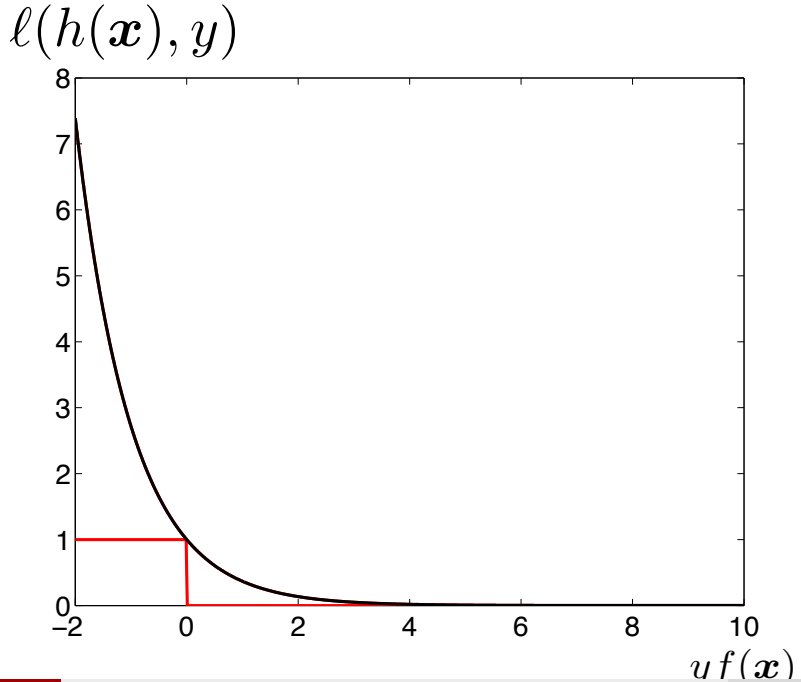
Namely, the function $f(\boldsymbol{x})$ and the target label y should have the same sign to avoid a loss of 1.

Exponential loss

The previous loss function $\ell(h(\boldsymbol{x}), y)$ is difficult to optimize. Instead, we will use the following loss function

$$\ell^{\text{EXP}}(h(\boldsymbol{x}), y) = e^{-yf(\boldsymbol{x})}$$

This loss function will function as a surrogate to the true loss function $\ell(h(\boldsymbol{x}, y)$. However, $\ell^{\text{EXP}}(h(\boldsymbol{x}), y)$ is easier to handle numerically as it is differentiable, see below the contrast between the red and black curves



Choosing the t-th classifier

Suppose we have built a classifier $f_{t-1}(\boldsymbol{x})$, and we want to improve it by adding a new classifier $h_t(\boldsymbol{x})$ to construct a new classifier

$$f(\boldsymbol{x}) = f_{t-1}(\boldsymbol{x}) + \beta_t h_t(\boldsymbol{x})$$

how can we choose optimally the new classifier $h_t(\boldsymbol{x})$ and the combination coefficient β_t? The strategy we will use is to greedily *minimize the exponential loss function*.

$$\begin{aligned}(h_t^*(\boldsymbol{x}), \beta_t^*) &= \arg\min_{(h_t(\boldsymbol{x}),\beta_t)} \sum_n e^{-y_n f(\boldsymbol{x}_n)} \\ &= \arg\min_{(h_t(\boldsymbol{x}),\beta_t)} \sum_n e^{-y_n [f_{t-1}(\boldsymbol{x}_n) + \beta_t h_t(\boldsymbol{x}_n)]}\end{aligned}$$

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$$\begin{aligned}
(h_t^*(\boldsymbol{x}), \beta_t^*) &= \arg\min_{(h_t(\boldsymbol{x}),\beta_t)} \sum_n e^{-y_n f(\boldsymbol{x}_n)} \\
&= \arg\min_{(h_t(\boldsymbol{x}),\beta_t)} \sum_n e^{-y_n [f_{t-1}(\boldsymbol{x}_n) + \beta_t h_t(\boldsymbol{x}_n)]} \\
&= \arg\min_{(h_t(\boldsymbol{x}),\beta_t)} \sum_n w_t(n) e^{-y_n \beta_t h_t(\boldsymbol{x}_n)}
\end{aligned}$$

where we have used $w_t(n)$ as a shorthand for $e^{-y_n f_{t-1}(\boldsymbol{x}_n)}$

The new classifier

We decompose the *weighted* loss function (by $w_t(n)$) into two parts

$$\sum_n w_t(n) e^{-y_n \beta_t h_t(\boldsymbol{x}_n)}$$

$$= \sum_n w_t(n) e^{\beta_t} \mathbb{I}[y_n \neq h_t(\boldsymbol{x}_n)] + \sum_n w_t(n) e^{-\beta_t} \mathbb{I}[y_n = h_t(\boldsymbol{x}_n)]$$

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&= \sum_n w_t(n) e^{\beta_t} \mathbb{I}[y_n \neq h_t(\boldsymbol{x}_n)] + \sum_n w_t(n) e^{-\beta_t} (1 - \mathbb{I}[y_n \neq h_t(\boldsymbol{x}_n)])
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&= (e^{\beta_t} - e^{-\beta_t}) \sum_n w_t(n) \mathbb{I}[y_n \neq h_t(\boldsymbol{x}_n)] + e^{-\beta_t} \sum_n w_t(n)
\end{aligned}$$

We have used the following properties to derive the above

- $y_n h_t(\boldsymbol{x}_n)$ is either 1 or -1 as $h_t(\boldsymbol{x}_n)$ is the output of a binary classifier.
- The indicator function $\mathbb{I}[y_n = h_t(\boldsymbol{x}_n)]$ is binary, either 0 or 1. Thus, it equals to $1 - \mathbb{I}[y_n \neq h_t(\boldsymbol{x}_n)]$.

Minimizing the weighted classification error

Thus, we would want to choose $h_t(\boldsymbol{x}_n)$ such that

$$h_t^*(\boldsymbol{x}) = \arg\min_{h_t(\boldsymbol{x})} \epsilon_t = \sum_n w_t(n) \mathbb{I}[y_n \neq h_t(\boldsymbol{x}_n)]$$

Namely, the weighted classification error is minimized — precisely *train a weak classifier based on the current weight* $w_t(n)$ *on the slide* **How Boosting algorithm works?**.

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Namely, the weighted classification error is minimized — precisely *train a weak classifier based on the current weight* $w_t(n)$ *on the slide* **How Boosting algorithm works?**.

Remarks We can safely assume that $w_t(\boldsymbol{x}_n)$ is normalized so that $\sum_n w_t(\boldsymbol{x}_n) = 1$. This normalization requirement can be easily maintained by changing the weights to

$$w_t(\boldsymbol{x}_n) \leftarrow \frac{w_t(\boldsymbol{x}_n)}{\sum_{n'} w_t(\boldsymbol{x}_{n'})}$$

This change *does not* affect how to choose $h_t^*(\boldsymbol{x})$, as the term $\sum_{n'} w_t(\boldsymbol{x}_{n'})$ is a constant with respect to n.

How to choose β_t?

We will select β_t to minimize

$$(e^{\beta_t} - e^{-\beta_t}) \sum_n w_t(n) \mathbb{I}[y_n \neq h_t(\boldsymbol{x}_n)] + e^{-\beta_t} \sum_n w_t(n)$$

We assume $\sum_n w_t(n)$ is now 1 (cf. the previous slide's Remarks). We take derivative with respect to β_t and set to zero, and derive the optimal β_t as

$$\beta_t^* = \frac{1}{2} \log \frac{1 - \epsilon_t}{\epsilon_t}$$

which is precisely what is on the slide **How Boosting algorithm works?**

Take-home exercise. Verify the solution

Updating the weights

Now that we have improved our classifier into

$$f(\boldsymbol{x}) = f_{t-1}(\boldsymbol{x}) + \beta_t^* h_t^*(\boldsymbol{x})$$

At the t-th iteration, we will need to compute the weights for the above classifier, which is,

$$w_{t+1}(n) = e^{-y_n f(\boldsymbol{x}_n)} = e^{-y_n [f_{t-1}(\boldsymbol{x}) + \beta_t^* h_t^*(\boldsymbol{x}_n)]}$$

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$$= w_t(n) e^{-y_n \beta_t^* h_t^*(\boldsymbol{x}_n)}$$

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Remarks The key point is that the misclassified data point will get its weight increased, while the correctly data point will get its weight decreased.

Remarks

Note that the AdaBoost algorithm itself never specifies how we would get $h_t^*(\boldsymbol{x})$ as long as it minimizes the weighted classification error

$$\epsilon_t = \sum_n w_t(n)\mathbb{I}[y_n \neq h_t^*(\boldsymbol{x}_n)]$$

In this aspect, the AdaBoost algorithm is a meta-algorithm and can be used with any classifier where we can do the above.

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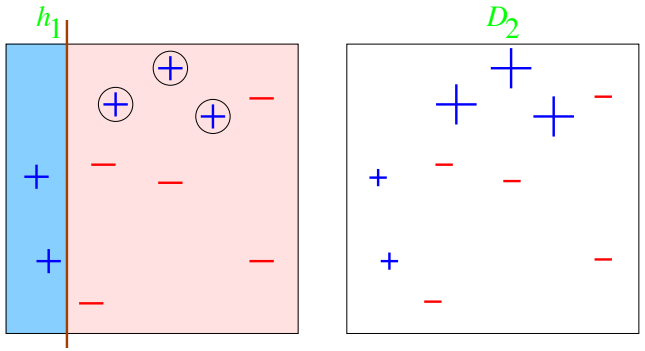
In this aspect, the AdaBoost algorithm is a meta-algorithm and can be used with any classifier where we can do the above.

Ex. How do we choose the decision stump classifier given the weights at the second round of the following distribution?



We can simply enumerate all possible ways of putting vertical and horizontal lines to separate the data points into two classes and find the one with the smallest weighted classification error!

Nonlinear basis learned by boosting

Two-stage process

- Get $\text{SIGN}[f_1(\boldsymbol{x})]$, $\text{SIGN}[f_2(\boldsymbol{x})]$,$\cdots$,
- Combine into a linear classification model

$$y = \text{SIGN}\left\{\sum_t \beta_t \text{SIGN}[f_t(\boldsymbol{x})]\right\}$$

Equivalently, each stage learns a nonlinear basis $\phi_t(\boldsymbol{x}) = \text{SIGN}[f_t(\boldsymbol{x})]$.

One thought is then, why not learning the basis functions and the classifier at the same time?