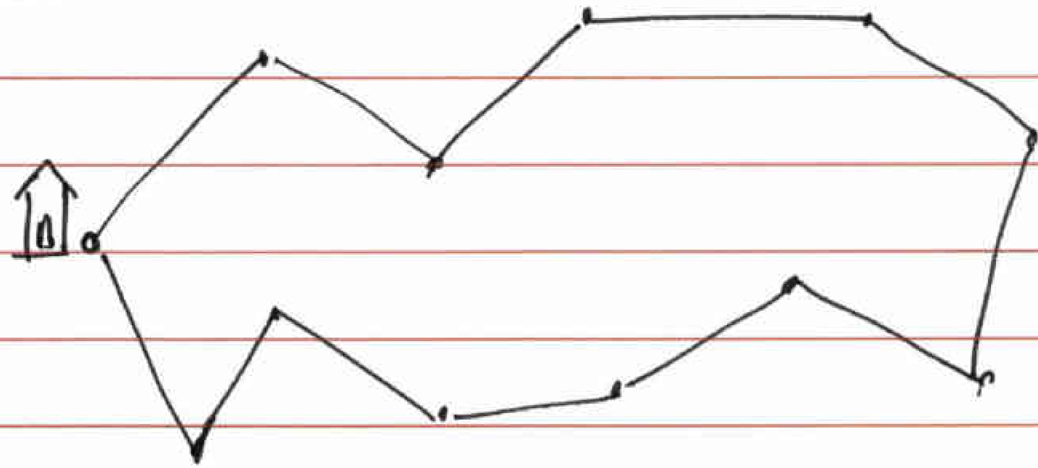
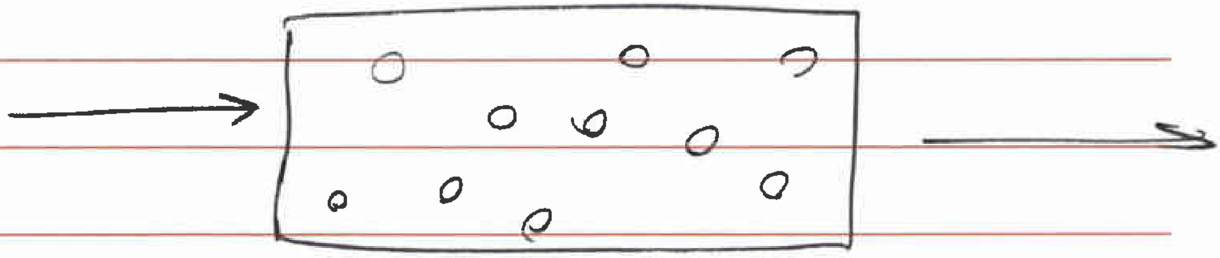
Traveling Salesman Problem (TSP)
&
Hamiltonian Cycle

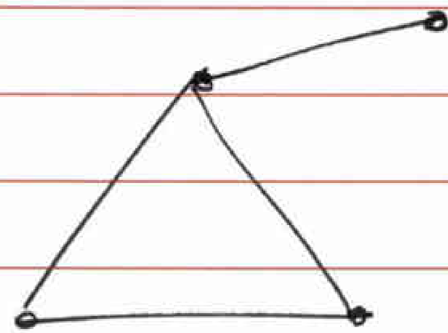




Given the set of distances, order $n$ cities in a tour $V_{i_1}, V_{i_2}, \ldots N_{i_n}$ with $i_1 = 1$ so t it minimizes

$$\sum d(V_{i_j}, V_{i_{j+1}}) + d(V_{i_n}, V_{i_1})$$

Decision version of TSP:
Given a set of distances on $n$ cities and a bound $D$, is there a tour of length at most $D$?



Def. A cycle $C$ in $G$ is a Hamiltonian Cycle if it visits each vertex exactly once

Show that the Ham. Cycle problem is NP-complete

1- Show that HC is ~~in~~ in NP.

- certificate : ordered list of the vertices on the HC

- Certifier : There is an edge between last & first nodes. ✓
  - There is an edge between each pair of adjacent nodes in the list. ✓
  - nodes do not repeat ✓
  - list ~~is~~ is of size $n$. ✓

$\Rightarrow$ HC problem is in NP

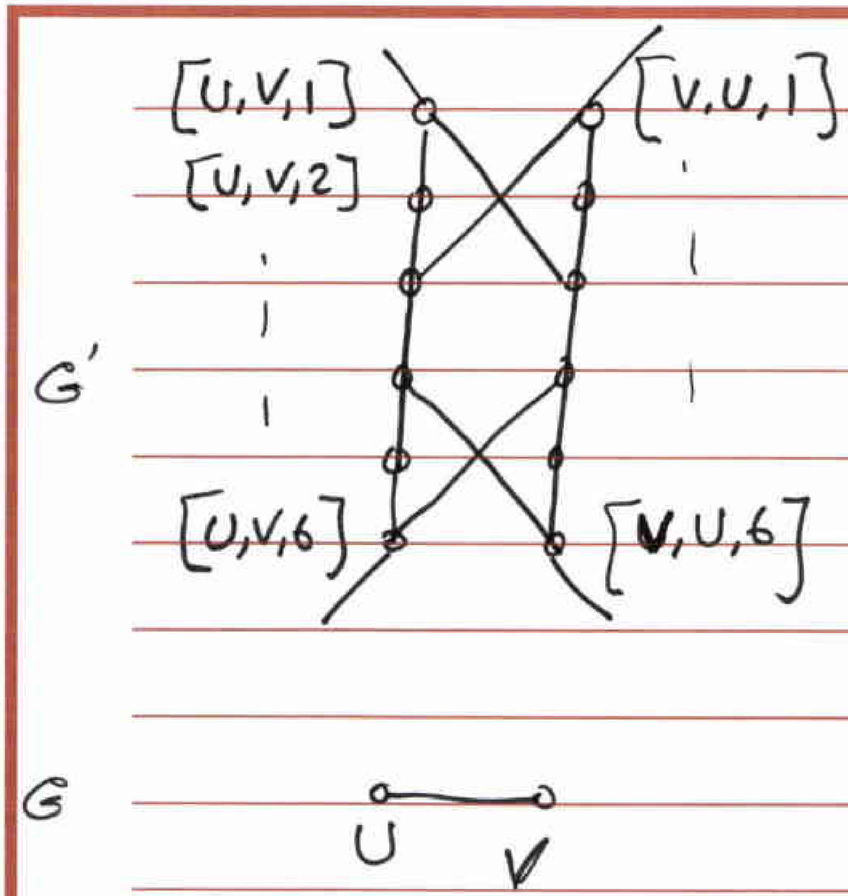
2- Choose vertex Cover

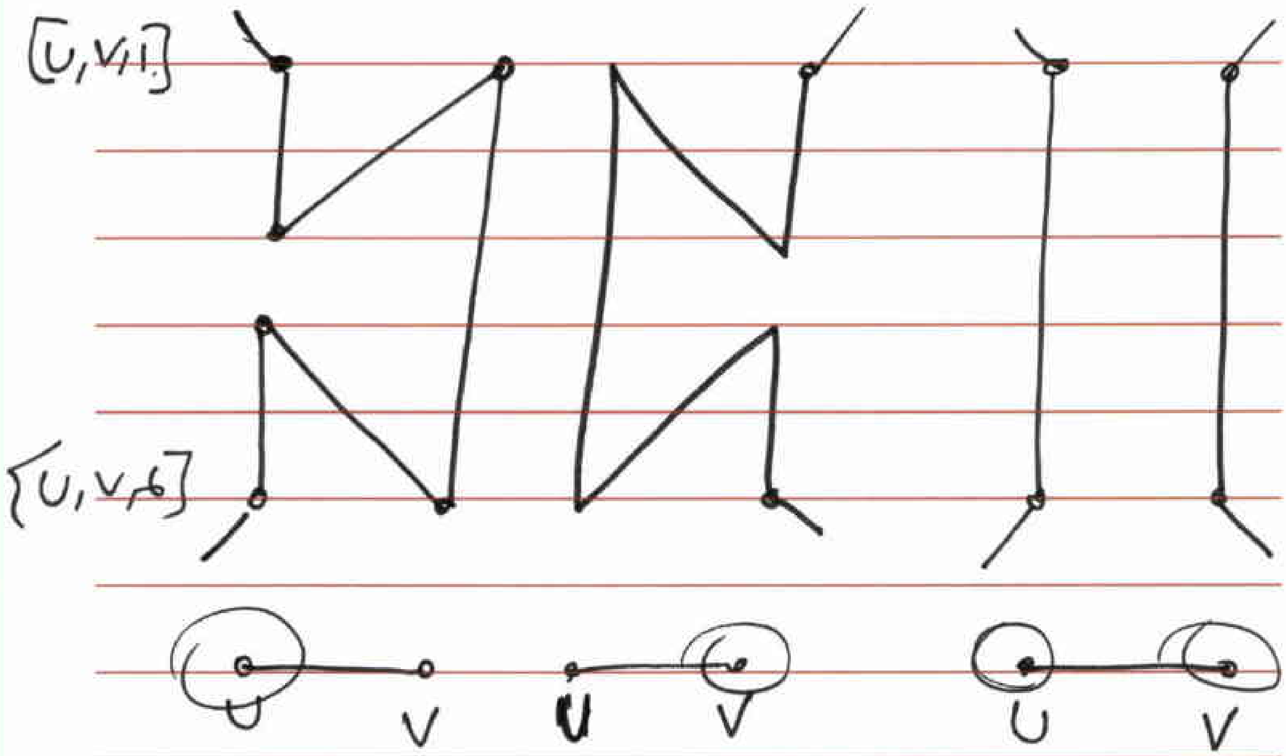
3- Show that vertex cover $\leq_p$ HC

Plan: Given an undirected graph $G=(V,E)$ and an integer $\underline{k}$, we construct $G'=(V',E')$ that has a Ham. Cycle iff $G$ has a vertex cover of size at most $\underline{k}$.

<u>Construction of $G'$</u>

For each edge in $G$, $G'$ will have one gadget $W_{UV}$ with the following node labeling



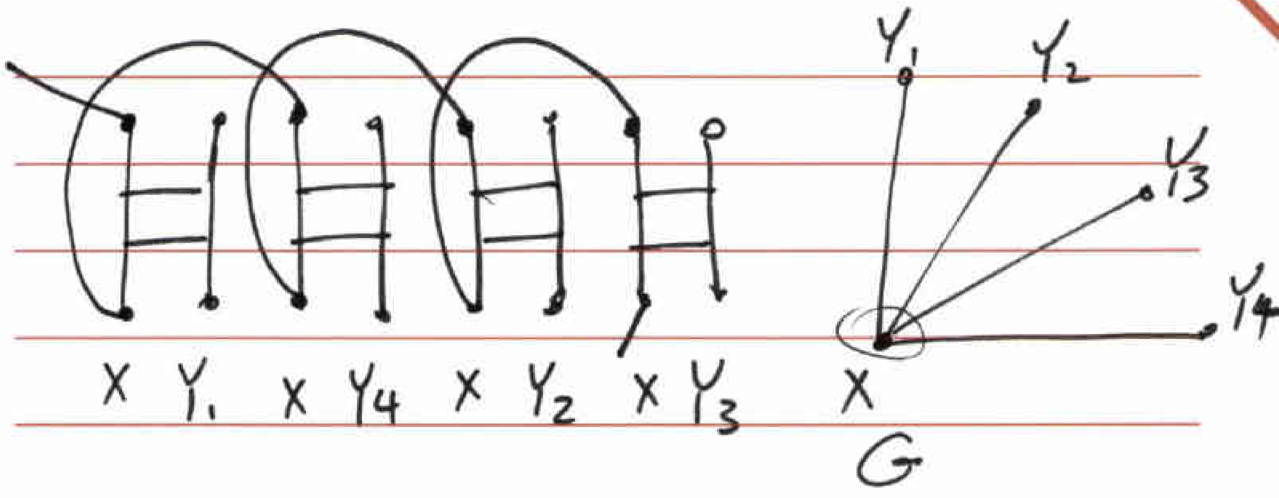



other vertices in $G'$

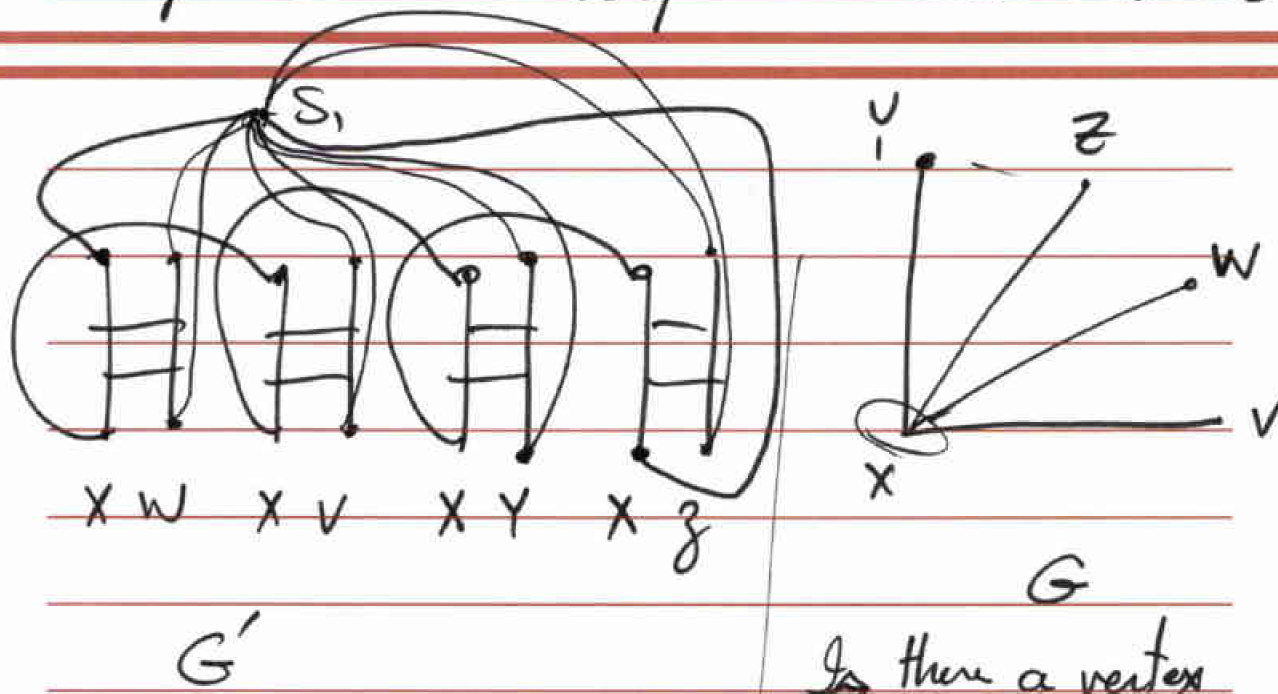
- selector vertices : there are $\underline{k}$ selector vertices in $G'$ : $s_1 \dots s_k$

- other edges in $G'$

1- For each vertex $u \in V$ we add edges to join pairs of gadgets in order to form a path going thru all the gadgets corresponding to edges incident on $u$ in $G$.

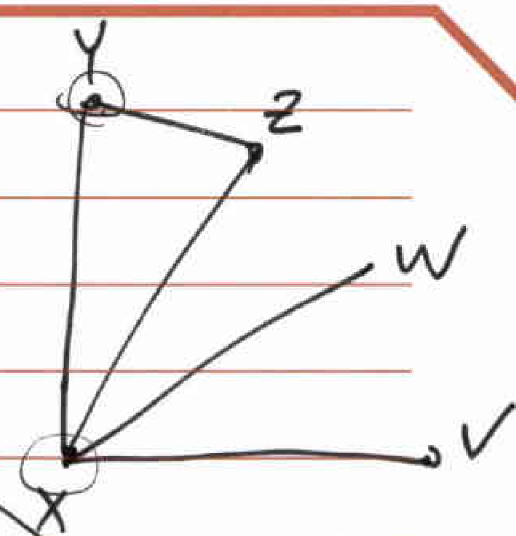


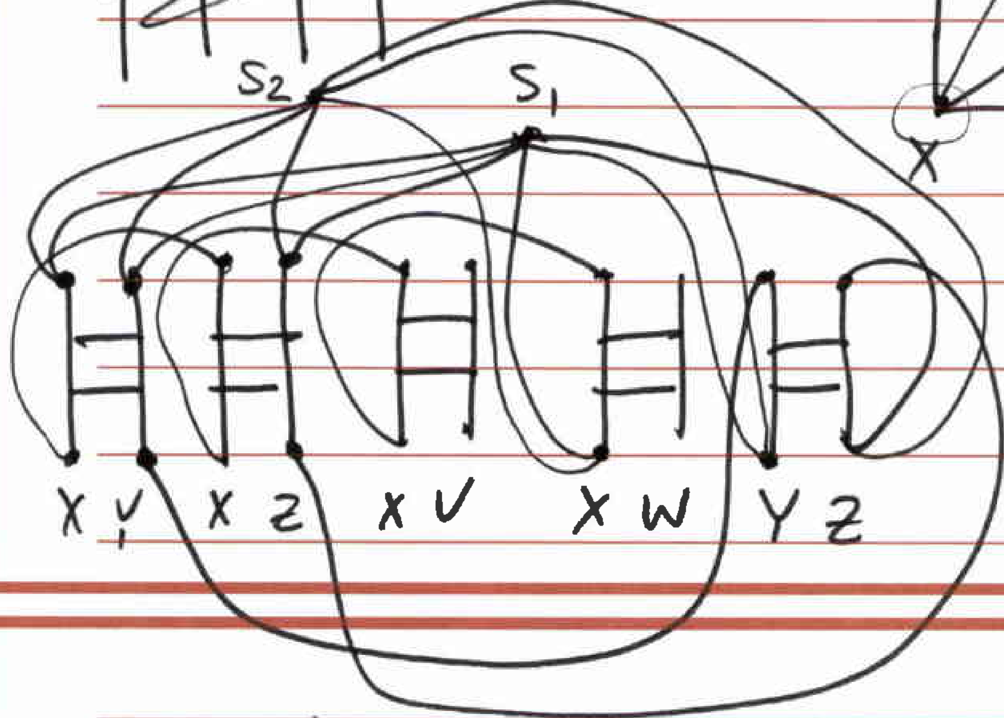

2- Final set of edges in G' join the first vertex $[x, y, 1]$ and last vertex $[x, y_{(\deg(x))}, 6]$ of each of these paths to each of the selector vertices.




Is there a vertex cover of size at most **1** in G?

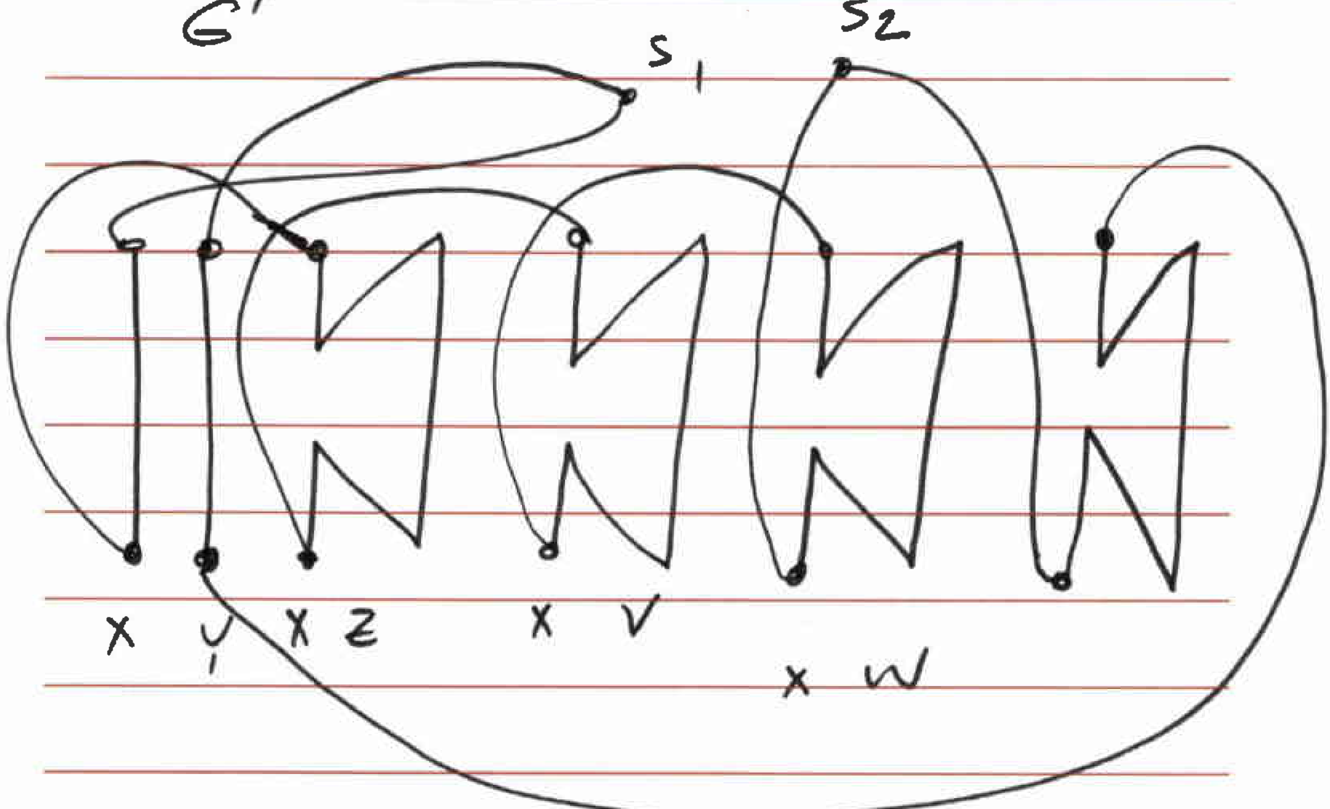
Is there a vertex cover
of size at most $\cancel{k}$ in G?
2



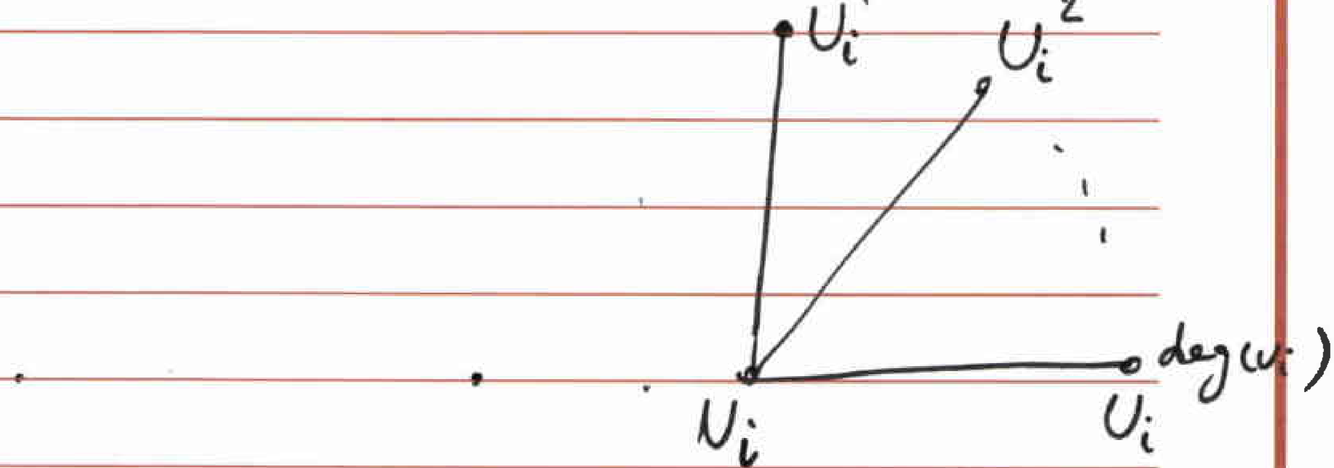

G'

G'



A) Suppose that $G = (V, E)$ has a vertex cover of size $k$. let vertex cover set

$$S = \{U_1, U_2, \ldots U_k\}$$




Form a HC in $G'$ by following these nodes in this order:
start at $s_1$ and go to

$[U_1, U_1^1, 1] \ldots [U_1, U_1^1, 6]$
$[U_1, U_1^2, 1] \ldots [U_1, U_1^2, 6]$
$\vdots$
$[U_1, U_1^{\deg(u_1)}, 6]$

then go to $s_2$
then to $[U_2, U_2^1, 1] \ldots [U_2, U_2^1, 6]$

$$\cdots \quad [u_2, u_2^{\deg(u_2)}, 6]$$

$\vdots$

$$[u_k, u_k^{\deg(u_k)}, 6]$$

then go back to $s_1$

B) Suppose $G'$ has a Ham. cycle $C$
Then the set

$$\underline{S} = \{ u_j \in V : (s_j, [u_j, u_j^1, 1]) \in C$$
$$\text{for some } 1 \le j \le k \}$$