Approximation and Linear Programming

CSCI 570

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DISCUSSION 14

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Outline

- Problems with Solutions

Problem 1

- The *Bin Packing* problem is as follows. You have an infinite supply of bins, each of which can hold M maximum weight. You also have n objects, each of which has a (possibly distinct) weight w_i (any given w_i is at most M). Our goal is to partition the objects into bins, such that no bin holds more than M total weight, and that we use as few bins as possible. This problem in general is **NP**-hard.
- Give a 2-approximation to the *Bin Packing* problem. That is, give an algorithm that will compute a valid partitioning of objects into bins, such that no bin holds more than M weight, and our algorithm uses at most twice as many bins as the optimal solution. Prove that the approximation ratio of your algorithm is two.
- Use the following as an example

  w = {10, 20, 15, 5, 25, 35, 30, 40}

  M = 45

Solution 1

- Let's try using a greedy algorithm for this problem
- First we need to sort the set w

  $w = \{10, 20, 15, 5, 25, 35, 30, 40\}$

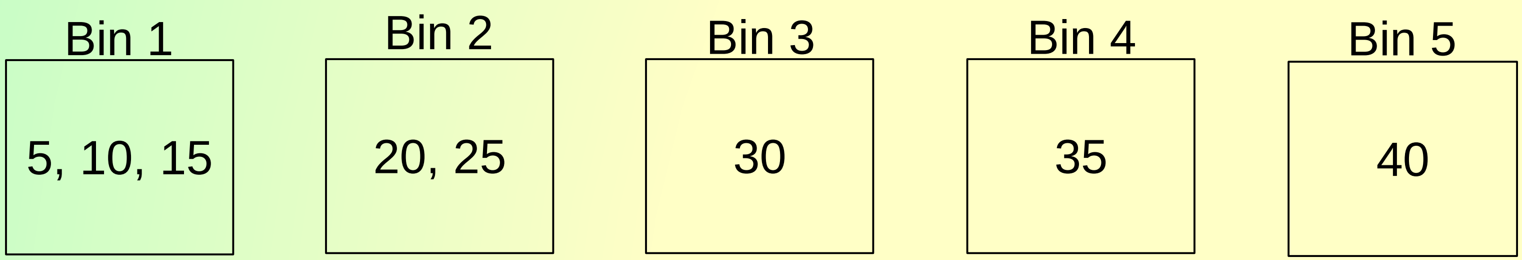
  $= \{5, 10, 15, 20, 25, 30, 35, 40\}$

  $M = 45$
- Then we place objects in the first bin until placing another object would exceed M
  - Place objects in subsequent bins in a similar fashion

Bin 1	Bin 2	Bin 3	Bin 4	Bin 5
5, 10, 15	20, 25	30	35	40

Solution 1

- Now consider the sum of any two bins, bin i and bin i+1
- The sum of bin i and bin i+1 will never be less than M
  - If the sum was less than M, all of the items would be in bin i
- So even though this may not be an optimal solution, we know that two bins can never be combined into one
  - That shows that our solution is no more than two times the value of the optimal solution



Problem 2

- Suppose you are given a set of positive integers A: $a_1\ a_2 \ldots a_n$ and a positive integer B. A subset $S \subseteq A$ is called *feasible* if the sum of the numbers in S does not exceed B.
- The sum of the numbers in S will be called the *total sum* of S. You would like to select a feasible subset S of A whose total sum is as large as possible.
- Give a linear-time algorithm for this problem that finds a feasible set $S \subseteq A$ whose total sum is at least half as large as the maximum total sum of any feasible set $S' \subseteq A$. Prove that your algorithm achieves this guarantee. You may assume that each $a_i \leq B$.

- **Example**: If A = {14, 2, 3, 4, 20} and B = 21 then the optimal solution is the subset S = {20}

Solution 2

- Let's try using a greedy algorithm for this problem
- First we need to sort the set A

  $A = \{14, 2, 3, 4, 20\}$

  $= \{2, 3, 4, 14, 20\}$

  $B = 21$

- Then choose values from small to large such that the sum does not exceed B

  $X = 2 + 3 + 4 = 9$

- Compare X to the value of the next element in the list (let's call it Y), which is 14
  - We know that $X + Y > B$ or we would have added Y into X
- That means that the larger of X and Y must be at least $B / 2$
  - Therefore no feasible set could be more than twice as big

Solution 2

- As another example, let's take the following

  $A = \{4, 10, 12\}$

  $B = 21$

- Then choose values from small to large such that the sum does not exceed B

  $X = 4 + 10 = 14$

- Compare X to the value of the next element in the list (let's call it Y), which is 12
  - We know that $X + Y > B$ or we would have added Y into X
- That means that the larger of X and Y must be at least $B / 2$

Problem 3

- A cargo plane can carry a maximum weight of 100 tons and a maximum volume of 60 cubic meters. There are three materials to be transported, and the cargo company may choose to carry any amount of each, up to the maximum available limits given below.
  - Material 1 has density 2 tons/cubic meter, maximum available amount 40 cubic meters, and revenue $1,000 per cubic meter.
  - Material 2 has density 1 ton/cubic meter, maximum available amount 30 cubic meters, and revenue $1,200 per cubic meter.
  - Material 3 has density 3 tons/cubic meter, maximum available amount 20 cubic meters, and revenue $12,000 per cubic meter.
- Write a linear program that optimizes revenue within the constraints. You do not need to solve the linear program.

Solution 3

A cargo plane can carry a maximum weight of 100 tons and a maximum volume of 60 cubic meters.

Material 1 has density 2 tons/cubic meter, maximum available amount 40 cubic meters, and revenue $1,000 per cubic meter.

Material 2 has density 1 ton/cubic meter, maximum available amount 30 cubic meters, and revenue $1,200 per cubic meter.

Material 3 has density 3 tons/cubic meter, maximum available amount 20 cubic meters, and revenue $12,000 per cubic meter.

- Let M1, M2, and M3 denote the cubic meters of the three materials we're going to transport.
- We want to maximize the profit, specifically maximize

  T = 1000 * M1 + 2000 * M2 + 12000 * M3

- Subject to these constraints:

  M1 + M2 + M3 <= 60 // 60 cubic meters space available

  2 * M1 + M2 + 3 * M3 <= 100 // 100 tons weight capacity

  M1 <= 40 // maximum amount of material 1

  M2 <= 30 // maximum amount of material 2

  M3 <= 20 // maximum amount of material 3

Solution 3

- We want to maximize the profit, specifically maximize

  T = 1000 * M1 + 2000 * M2 + 12000 * M3

- Subject to these constraints:

M1 + M2 + M3 <= 60	// 60 cubic meters space available
2 * M1 + M2 + 3 * M3 <= 100	// 100 tons weight capacity
M1 <= 40	// maximum amount of material 1
M2 <= 30	// maximum amount of material 2
M3 <= 20	// maximum amount of material 3

M3 = 20

M2 = 30

M1 = 5

T = 1000 * 5 + 2000 * 30 + 12000 * 20

T = 5,000 + 60,000 + 240,000 = 305,000