We know that α^* is a Nash equilibrium if and only if $\alpha_i^* \in B_i(\alpha_{-i}^*)$, $\forall i$

Let B be the best-response functions for every player, defined as

$B(\alpha) = B_i(\alpha_{-i})$ where α is any mixed strategy profile

Then α^* is a Nash equilibrium if and only if $B(\alpha^*) \ni \alpha^*$

Let us show for any mixed strategy game, $\exists$ mixed strategy α^* such that $\alpha^* \in B(\alpha^*)$ using Kakutani's fixed-point theorem. Let X be the set of all mixed-strategies outcomes. We need to show:

(1) X is non-empty, closed, bounded, and convex set

(2) $B(x)$ is non-empty, $\forall x \in X$

(3) $B(x)$ is convex, $\forall x \in X$

(4) $\{(x,y) \mid y \in B(x)\}$ is closed

(1): • non-empty: Since any existing player has at least 1 action, $X \neq \emptyset$

• closed and bounded: I don't know how to prove this rigorously, since X is not a real subset, but intuitively, this is true since X contains the boundary, which are the pure strategies

• convex: Let $x, y \in X$ be given. Let us show $\alpha X + (1-\alpha) Y \in X$. Let $X = (x_1, x_2, \dots, x_n)$ and $Y = (y_1, y_2, \dots, y_n)$ where x_i, y_i are player i's mixed strategy

$\Rightarrow \alpha X + (1-\alpha) Y = (\alpha x_1 + (1-\alpha) y_1, \dots, (\alpha y_n + (1-\alpha) y_n)$

$\Rightarrow \alpha X + (1-\alpha) Y \in X$ since $\alpha x_i + (1-\alpha) y_i$ is another mixed strategy for player i

(2) $B(x)$ is non-empty, $\forall x \in X$

Let i be given $\Rightarrow B(x) = B_i(\alpha_{-i})$. Since player i has finite # actions,

Let a_j be action s.t. $u_i(a_j, \alpha_{-i}) = \max_k (a_k, \alpha_{-i})$ ~~then $u_i(a$~~

then $u_i(\alpha_i, \alpha_{-i}) \leq u_i(a_j, \alpha_{-i})$ $\forall$ mixed strategy of player i

$\Rightarrow a_j \in B(x) \Rightarrow B(x) \neq \emptyset$.

(3) $B(x)$ is convex

Let $x, y \in B(x)$ be given. Similar to (1), $\forall \alpha \in [0;1] \Rightarrow \alpha x + (1-\alpha) y \in X$

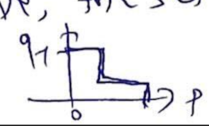
$\alpha x + (1-\alpha) y$ has the same utility score as x and y, which are the best, $\forall \alpha \in [0;1] \Rightarrow B(x)$ convex

$u_i(\alpha X + (1-\alpha) Y) = \alpha u_i(X) + (1-\alpha) u_i(Y)$

$= u_i(X) = u_i(Y)$ ~~= max~~, $\forall i$

(4) $\{(x;y) \mid y \in B(x)\}$ is closed

For this one, I don't know how to show this rigorously since again, X is not simply a real set. But intuitive, the set of best responses look something like this (for 2×2 games):



which is closed