# Mixed Strategy Nash Equilibrium

# Introduction

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We now introduce a strategic game where players can randomize their action choice instead of having a deterministic action. There're two ways that we can model our game:

1. **Population distribution**: each player chooses an action deterministically, but we randomly choose a player from the population
2. **Probabilistic action distribution**: the same player plays every round, but their action choice follows an unchanging probabilistic distribution
   These representations are equivalent, but we'll choose the second approach

## Stochastic Steady State

- **steady state**: a fixed action set in which no players can change their action to benefit themselves
- **stochastic steady state**: a situation where no players can change their **probabilistic distribution** to benefit themselves. Note that outcomes can vary, unlike steady states

## Bernoulli Payoff Function

To represent our new notion, we'll use some more tools:

- **lottery**: an outcome distribution, i.e. the list of how likely an outcome will happen
- **vNM preferences (von Neumann and Morgenstern)**: an axiomatic way for us to rank two different lotteries (previously, we can only compare two deterministic actions). Such a payoff for a lottery $a$ containing $n$ outcomes is given by $u(a) = \sum_{i=1}^{n} p_i u(a_i)$ where $\sum_{i=1}^{n} p_i = 1$
- **Bernoulli payoff function**: any payoff function that can represent vNM preferences by letting you calculate the expected value $\mathrm{E}[u(a)] = \sum_{i=1}^{n} p_i u(a_i)$

Combine together, we'll obtain a distribution of each outcome that looks something like this (assuming each player's action choice is independent of one another)

| | $L\ (q)$ | $R\ (1-q)$ |
|---|---|---|
| $T\ (p)$ | $pq$ | $p(1-q)$ |
| $B\ (1-p)$ | $(1-p)q$ | $(1-p)(1-q)$ |

**Figure 107.1** The probabilities of the four outcomes in a two-player two-action strategic game when player 1's mixed strategy is $(p, 1-p)$ and player 2's mixed strategy is $(q, 1-q)$.

# Definition

> ⓘ Strategic Game (with vNM Preferences)
>
> - a set of **players**
> - for each player, a set of **actions**
> - for each player, **preferences** regarding lotteries using a Bernoulli payoff function

Two games being equivalent are similar to our original strategic games, but checking that two players have the same Bernoulli payoff function is more complex, as we need to check for all possible lotteries of an action set as well

# Equivalent Bernoulli Payoff Function

> ⓘ Equivalence of Bernoulli payoff functions (Linearity)
>
> Two Bernoulli payoff functions for the same player (which has $n$ actions) $u$ and $v$ are equivalent **if and only if** there exists $a > 0$ and $b$ such that $u(p) = av(p) + b$ for all lotteries $p$

**Proof sketch**:
1. **"If" clause**: this is straightforward due to linearity
2. **"Only if" clause**:
Since there're a finite number of pure strategy outcomes, let $x_m$ and $x_M$ be the pure strategy outcomes such that $u(x_m) = \min_x u(x)$ and $u(x_M) = \max_x u(x)$. Assume that $u(x_m) < u(x_M)$, otherwise it's trivial
Let $a = \frac{u(x_M)-u(x_m)}{v(x_M)-v(x_m)} > 0$ and $b = u(x_m) - av(x_m)$. Let us show $u = av + b$
- **Observation 1**: By the way we set up $a$ and $b$, we can see $u = av + b$ for $x_m$ and $x_M$
- **Observation 2:** Each lottery $x$ can be rewritten as a lottery of $x_m$ and $x_M$. Indeed, there exists a unique $p \in [0, 1]$ such that $u(x) = pu(x_m) + (1-p)u(x_M)$, since we chose $x_m$ and $x_M$ to be on the two ends. Due to linearity, this is equivalent to $u(px_m + (1-p)x_M)$, so $x$ is equivalent to the unique lottery containing 2 pure strategies $x_m$ and $x_M$ with probability $p$ and $1-p$
Let $x$ be a given lottery. Let us show $u(x) = av(x) + b$. Indeed, there exists a unique $p$ and $q$ such that $u(x) = u(px_m + (1-p)x_M)$ and $v(x) = v(qx_m + (1-q)x_M)$.
Since $x$ is equivalent to the lottery $px_m + (1-p)x_M$, $v(x) = v(px_m + (1-p)x_M)$
Equivalently: $v(px_m + (1-p)x_M) = v(qx_m + (1-q)x_M)$
Due to linearity, $p = q$, otherwise one would be greater than the other
Therefore, $u(x) = pu(x_m) + (1-p)u(x_M)$ and $v(x) = pv(x_m) + (1-p)v(x_M)$. Since we know $u = av + b$ holds true for $x_m$ and $x_M$, and these are linear functions, it also follows that $u = av + b$ for any $x$, which conclude our proof ■

# Mixed Strategy Nash Equilibrium

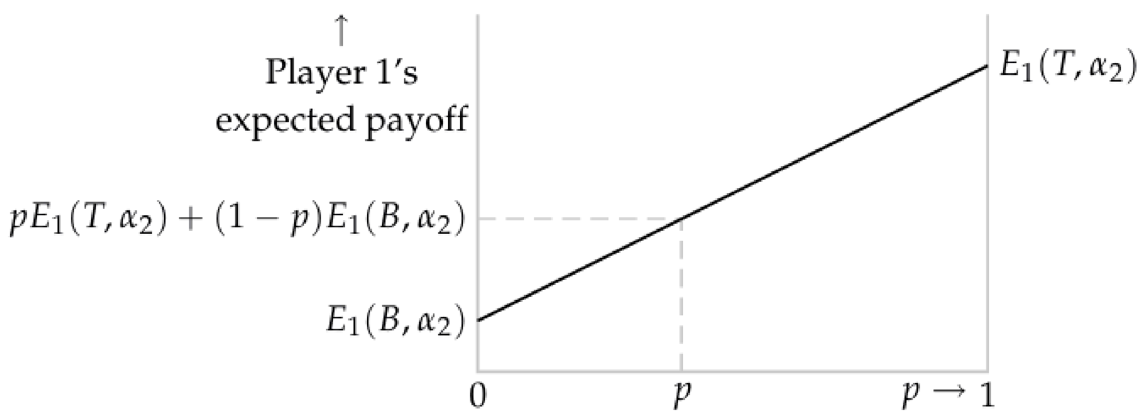
**Notations**:

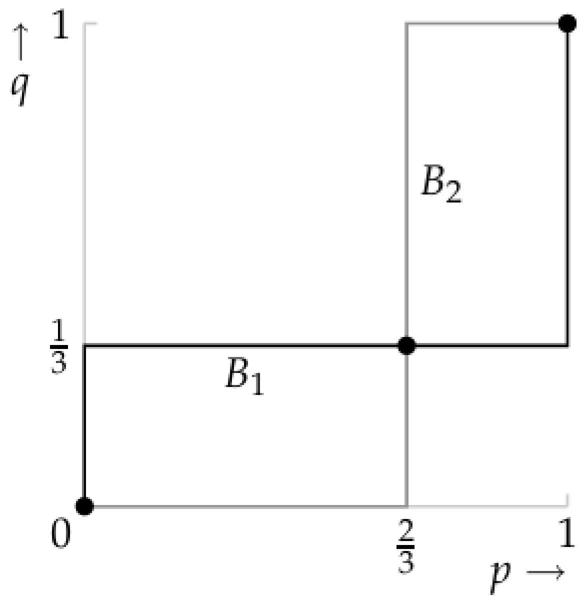
- $\alpha_i$: player i's action lottery
- $\alpha_i(a_k)$: the probability that player i assign to his action $a_k$ in the lottery/**mixed strategy** $\alpha$
  If a player chooses a **pure strategy**, we still denote it as $a_i$

> ⓘ Mixed Strategy Nash Equilibrium
>
> A mixed strategy profile $\alpha^*$ is an equilibrium if no player can change their action profile/lottery and be better off. In other words, for each player $i$:
>
> $$U_i(\alpha^*) \geq U_i(\alpha_i, \alpha^*_{-i})$$
>
> for every mixed strategy $\alpha_i$ of player $i$

## Best Response Functions

All the notions of a best response function still hold. Let $B_i(\alpha_{-i})$ the set of player $i$'s best mixed strategies given other players' mixed strategies $\alpha_{-i}$, then
$\alpha^*$ is an equilibrium if and only if $\alpha^*_i \in B_i(\alpha^*_{-i})$ for each player $i$
🛑 **NOTE:** For 2x2 games the best response set is always either a pure strategy or every possible lotteries of the 2 pure strategies (if the 2 pure strategies have the same utility)






*An example of overlaying the 2 best response functions for a 2x2 game, where $p$ and $q$ are how much each player leans completely towards their first action*

## An Useful Characterization

Normally, we'd iterate the expected utility of a mixed strategy profile into components containing pure strategies only. However, we can write this more compactly through the perspective of any player $i$:

$$U_i(\alpha) = \sum_{a_k \in A_k} p_k U_i(a_k, \alpha_{-i})$$

In other words, we're breaking down the utility of a mixed strategy profile into expected value of the player's pure action $a_k$ the remaining mixed strategy $\alpha_{-i}$, summing over all actions in $A_k$ with weight $p_k$, how much player $i$ weighs the action $a_k$ in his own mixed action. We can use this to check if a mixed profile is an equilibrium or not.

> ⓘ PROPOSITION
>
> A mixed action profile $\alpha^*$ is an equilibrium **if and only if** for each player $i$
>
> 1. The expected payoff, given $\alpha^*_{-i}$, to every action to which $\alpha^*_i$ assigns positive probability is the same (and equals $U_i(\alpha^*)$
> 2. The expected payoff to every action to which was assigned zero probability is at most the expected payoff to any action that was assigned a positive probability, i.e.
>
> $$U_i(a_m, \alpha_{-i}) \leq U_i(a_n, \alpha_{-i})$$
>
> for all $a_m, a_n$ in which $p_m = 0$ and $p_n > 0$

**Quick intuition**:

- If (1) is not true, there's an action in the mixed action that has lesser expected value so it's not optimal, so we should replace it to get better payoff.
- If (2) is not true, there's an action not assigned in the mixed action that has better expected value so it's not optimal, so we should add this action into our mixed action to get better payoff.
  **Application**: We'd check the utility for each action of player $i$ with the rest of the profile $\alpha_{-i}$ and make sure they equal (if present in the mixed profile) or no more than anything that's present (if not present in the mixed profile)

  > PROPOSITION
  >
  > Every strategic game with vNM preferences in which each player has a finite number of actions has a mixed strategy Nash equilibrium

(I'll have to think about this more. Nothing comes up right now)
Since we have a finite number of actions, we have a finite number of pure strategies, so we can choose a max and a min, i.e. the utility range is bound. We can also create any lottery on this range, so each player's expected payoff function is bounded and continuous on that range, so it'll intersect somewhere somehow idk

# Dominated Actions

We can extend the notion of a dominating action to a mixed strategy as well. We say that for player $i$, his mixed strategy $\alpha_i$:

- **strictly dominates** his action $a_i'$ if $u_i(\alpha_i, a_{-i}) > u_i(a_i', a_{-i})$ for every list $a_{-i}$ of the other players' actions.
- **weakly dominates** his action $a_i'$ if $u_i(\alpha_i, a_{-i}) \geq u_i(a_i', a_{-i})$ for every list $a_{-i}$ of the other players' actions AND $u_i(\alpha_i, a_{-i}) > u_i(a_i', a_{-i})$ for some list $a_{-i}$ of the other players' actions.
  🛑 **NOTE: This notion extends for pure strategy list $a_{-i}$ only**

| | $L$ | $R$ |
|---|---|---|
| $T$ | 1 | 1 |
| $M$ | 4 | 0 |
| $B$ | 0 | 3 |

**Figure 117.1** Player 1's payoffs in a strategic game with vNM preferences. The action $T$ of player 1 is strictly dominated by the mixed strategy that assigns probability $\frac{1}{2}$ to $M$ and probability $\frac{1}{2}$ to $B$.

A similar result regarding equilibrium follows: **a strictly dominated action is not used with a positive probability in any mixed strategy equilibrium**

> ⓘ PROPOSITION
>
> Every strategic game with vNM preferences in which each player has finitely many actions has a mixed strategy Nash equilibrium in which no player's strategy is weakly dominated.

(No clue on this one either)

# Pure Equilibria With Randomization

If an equilibrium contains pure strategies only, we can go back on forth between games where there's no randomization and there's randomization

> ⓘ PROPOSITION
>
> Equilibria when the players are not allowed to randomize remain equilibria when they are allowed to randomize, and any pure equilibria that exist when they are allowed to randomize are equilibria when they are not allowed to randomize.

We can easily check the two conditions of a mixed strategy equilibrium to show that this is true, and it makes sense intuitively.

# Equilibrium in a Single Population

We can also extend the notion of symmetry to strategic game with vNM preferences
A two-player strategic game with vNM preferences and every player has the same action set is **symmetric** if $u_1(a_1, a_2) = u_2(a_2, a_1)$ for every action pair $(a_1, a_2)$. If so, a mixed strategy profile $\alpha^*$ is a **symmetric mixed strategy Nash equilibrium** if it's a mixed strategy equilibrium and $\alpha^*$ is the same for each player $i$.

> ⓘ PROPOSITION
>
> Every strategic game with vNM preferences in which each player has the same finite set of actions has a symmetric mixed strategy Nash equilibrium.

# Extension

# Finding All Mixed Strategy Nash Equilibria

The following systematic method is suggested: (brute-force)

- For each player $i$, choose an action subset $S_i \in A_i$
- Check if there's a mixed strategy profile $\alpha$ such that for each player $i$, the set of actions that are assigned positive probabilities is exactly $S_i$, and it satisfies the two criteria for $\alpha$ to be an equilibrium
- Repeat for every collection of subsets of the players' sets of actions
  (possibly reverse engineer this to create a game with a desired number of equilibria or equilibrium satisfying some conditions)