Nash Equilibrium

Introduction

Game Theory aims to explain situations in which decision makers interact. It consists of a collection of **models**, *a simplified yet sufficiently correct interpretation of the situation*.
A model derives power from its simplicity, and the assumptions it use should capture the essence of the situation, not irrelevant details.

Characteristics

Each game consists of:

- **Players**: who are the participants in this game
- **Actions**: a subset of all possible actions, determined by the **restrictions**, to form the current possible moves.
- **Preferences**: we rank the preferences using **payoff functions** u (or **utility function**), which only give the ranking of choices corresponding to a specific player. In general: $u(a) > u(b)$ if and only if action a is preferred to action b.

  > ⓘ Payoff Function Equivalence
  >
  > Two payoff functions are the same regarding the same action set A and player p if they give the **same ranking order** for every action.

- **Theory of rational choice**: *the action chosen by a player is at least as good as every other available action*. This allows for the possibility of indifference.
  🛑 **NOTE: Payoff functions cannot:**
- **How much an action is preferred over another**
- **How the difference between two choices a and b relate to two other choices c and d**

Strategic Games

A **strategic game** consists of **players**, **actions** and **preferences** over them for each player.
Players choose their action once and for all, and their action decision is not known to the others (one-move game).

Prisoner's Dilemma

A classic problem in which if players cooperate, they'll end up better than if each player acts in their best interest.
We need a payoff function that represents each player's preference ordering. One such can be given with the following utility scores:

		Suspect 2	
		Quiet	*Fink*
Suspect 1	*Quiet*	2, 2	0, 3
	Fink	3, 0	1, 1

ⓘ Game Equivalence

Two game are the same if we can assign each action from one game to another for each player and the payoff function still gives the same ordering. In other words, **there's an isomorphism between the two games**.

Some other games are exactly the same as Prisoner's Dilemma, for example: **The Arms Race** and **Duopoly**. Other common 2x2 games include:

- **Battle of the Sexes**: where there are 2 equilibria when both players choose the same action
- **Matching pennies**: a strictly competitive game where there's no cooperation or equilibrium
- **Stag Hunt**: a game where both player works together for shared result or both work alone. There are 2 equilibria, which players choose the same action, and working together is better. Any other profile contains at least 1 hunter choosing Hare and 1 choosing stag, so any Stag-choosing hunter would be better off switching to Hare, hence not an equilibrium.

Nash Equilibrium

Assumptions

1. Each player chooses their action according to the model of **rational choice**, given their belief about the other players' actions.
2. Every player's belief about the other players' actions **is correct**.

Beliefs are formed from past experience of playing the game. We also assume that each game is played in isolation, where opponents are randomly chosen from a pool of participants (so as to average out the playing experience).

Some notation: $a = (a_1, a_2, \cdots a_n)$ represents an action profile where the i-th player chooses action a_i. Suppose we have another action profile a' where it differs from a only in the i-th entry with a'_i. This is denoted as (a'_i, a_{-i}). (we can read this as *everyone keeps their original action in* a*, except for* i *who changes from* $a_i \to a'_i$)

A *Nash equilibrium* is an **action profile** a^* in which no player i can do better by choosing an action different from a_i^*, given that every other player j keeps their action a_J^*.

Definition - Nash equilibrium of strategic game with ordinal preferences

The action profile a^* is a **Nash equilibrium** if for every player i,

$$u_i(a^*) \geq u_i(a_i, a_{-i}^*) \text{ for every action } a_i \text{ of player } i$$

- A Nash equilibrium is **strict** if the outcome is worse off when any player changes their action. In other words, the $\geq$ inequality is replaced with >. If it's not strict, there're actions in which players are indifferent when choosing.

In plain English, we fix all but one player (assume it's i) action in a^*. Allow i to choose any action and verify (a_i, a_{-i}^*) is no better than a^*. Do this for every player.
🛑 **NOTE: This definition implies that a strategic game need not have a Nash equilibrium or have at most one.**

Testing Nash equilibrium

To test the theory of Nash equilibrium experimentally, we match each subject's preferences with the role of the player in the game we're examining. **Money is usually used to match the payoff functions**. If the outcome of the game doesn't match Nash equilibrium, this could mean our original assumptions are not true.

Questions to be answered

- How do we know when the outcome has converged? Is there a number of games that can determine this?
- How do we tell if data is close enough to the theory to support it?

Best Response Functions

To better generalize the idea of determining a Nash equilibrium to make the process of finding one more intuitive, we can use a **best response function**. We denote:

- A_i: the set of all possible actions for player i
- $B_i(a_{-i})$: the set of best actions player i can perform given all other players follow action profile a.
  Mathematically, the **best response function**
  $B_i(a_{-i}) = \{a_i \in A_i : u_i(a_i, a_{-i}) \geq u_i(a_i', a_{-i}) \; \forall a_i' \in A_i\}$

Proposition: Nash equilibrium equivalence

The action profile a^* is a Nash equilibrium if and only if every player's action is a best response to the other players' actions. In other words:

$$a_i^* \in B_i(a_{-i}^*) \; \forall \text{ players } i$$

Intuitively, we can think of such an action profile as the intersection of all best response functions. Moreover, if each response function only contains a *single* optimal action, then the Nash equilibrium is **unique**.

Dominated Actions

Sometimes despite whatever the other players do, there's an action that player i can choose that will guarantee the best payoff. For example: choosing *Snitch* in Prisoner's Dilemma. We say that for player i, his action a_i''

- **strictly dominates** his action a_i' if $u_i(a_i'', a_{-i}) > u_i(a_i', a_{-i})$ for every list a_{-i} of the other players' actions.
- **weakly dominates** his action a_i' if $u_i(a_i'', a_{-i}) \geq u_i(a_i', a_{-i})$ for every list a_{-i} of the other players' actions AND $u_i(a_i'', a_{-i}) > u_i(a_i', a_{-i})$ for some list a_{-i} of the other players' actions. (otherwise it'd be strictly dominating).
  Obviously, any action that is *dominated* is not used in any Nash equilibrium. To show domination this, fix an action profile a_{-i} and compare action a_i'' with a_i'.

Symmetric Game

A two-player game is **symmetric** if the players share the same action set and their preferences are represented by payoff function u_1 and u_2 such that $u_1(a_1, a_2) = u_2(a_2, a_1)$ for all action pairs (a_1, a_2).
For symmetrical game, an action profile a^* is a **symmetric Nash equilibrium** if it is a Nash equilibrium and a_i^* is the same for every player i. In other words, it's an equilibrium which lies on the diagonal.