Convolutional Neural Networks

Kernel : A kernel is a small matrix of weights that is convolved with the input image in order to produce a set of feature maps

$$A = \begin{array}{|c|c|c|c|c|c|}\hline 3 & 2 & 7 & 1 & 6 & 3 \\ \hline 4 & 1 & 1 & 0 & 9 & 8 \\ \hline 8 & 2 & 0 & 1 & 2 & 4 \\ \hline 5 & 6 & 0 & 7 & 1 & 4 \\ \hline 0 & 5 & 4 & 9 & 6 & 0 \\ \hline 4 & 6 & 2 & 4 & 0 & 5 \\ \hline \end{array} = (A_{ij})_{6\times 6}$$

[Here, $n = 6$]

feat. extractor

$$K = \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 2 & 1 & 2 \\ \hline 2 & 2 & 1 \\ \hline \end{array} = (k_{ij})_{3\times 3}$$

[Here $f = 3$]

$H = (h_{ij})$

Let us consider a stride (s) = 1

$$h_{11} = 3\times1 + 2\times2 + 7\times2 + 2\times4 + 1\times1 + 2\times1 + 8\times2 + 2\times2 + 0\times1$$

$$= \underline{52}$$

$$h_{12} = 2\times1 + 7\times2 + 1\times2 + 1\times2 + 1\times1 + 0\cdot\times2 + 2\times2 + 0\times2 + 1\times1 = \underline{26}$$

$$A = \begin{array}{|c|c|c|c|c|c|}\hline 3 & 2 & 7 & 1 & 6 & 3 \\ \hline 4 & 1 & 1 & 0 & 9 & 8 \\ \hline 8 & 2 & 0 & 1 & 2 & 4 \\ \hline 5 & 6 & 0 & 7 & 1 & 4 \\ \hline 0 & 5 & 4 & 9 & 6 & 0 \\ \hline 4 & 6 & 2 & 4 & 0 & 5 \\ \hline \end{array}$$

[Operations Count :- Addition = 8
Multiplication = 9]

$$K = \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 2 & 1 & 2 \\ \hline 2 & 2 & 1 \\ \hline \end{array}$$

$$H = (h_{11} \quad h_{12} \quad h_{13} \quad h_{14})$$

$$h_{13} = 7\times1 + 1\times2 + 6\times2 + 1\times2 + 0\times1 + 9\times2 + 0\times2 + 1\times2 + 2\times1 = \underline{45}$$

$$A = \begin{array}{|c|c|c|c|c|c|}\hline 3 & 2 & 7 & 1 & 6 & 3 \\ \hline 4 & 1 & 1 & 0 & 9 & 8 \\ \hline 8 & 2 & 0 & 1 & 2 & 4 \\ \hline 5 & 6 & 0 & 7 & 1 & 4 \\ \hline 0 & 5 & 4 & 9 & 6 & 0 \\ \hline 4 & 6 & 2 & 4 & 0 & 5 \\ \hline \end{array}$$

$$h_{14} = 1\times1 + 6\times2 + 3\times2 + 0\times2 + 9\times1 + 8\times2 + 1\times2 + 2\times2 + 4\times1 = 54$$

$$K = \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 2 & 1 & 2 \\ \hline 2 & 2 & 1 \\ \hline \end{array}$$

Let us generalize h_{14} a bit

$$h_{14} = A_{14}K_{11} + A_{15}K_{12} + A_{16}K_{13} + A_{24}K_{21} + A_{25}K_{22} + A_{26}K_{23} + A_{34}K_{31} + A_{35}K_{32} + A_{36}K_{33}$$

$$A = \begin{array}{|c|c|c|c|c|c|}\hline 3 & 2 & 7 & 1 & 6 & 3 \\ \hline 4 & 1 & 1 & 0 & 9 & 8 \\ \hline 8 & 2 & 0 & 1 & 2 & 4 \\ \hline 5 & 6 & 0 & 7 & 1 & 4 \\ \hline 0 & 5 & 4 & 9 & 6 & 0 \\ \hline 4 & 6 & 2 & 4 & 0 & 5 \\ \hline \end{array}$$

$i \rightarrow$

$j \downarrow$

This can be written as.

$$h_{14} = \sum_{i=1}^{3}\sum_{j=1}^{3} A_{1+i-1,\,4+j-1}\, K_{ij}$$

$$K = \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 2 & 1 & 2 \\ \hline 2 & 2 & 1 \\ \hline \end{array}$$

This can be written as

$$h_{14} = \sum_{i=1}^{3} \sum_{j=1}^{3} A_{1+i-1,\, 4+j-1} \, K_{ij} =$$

$K =$

1	2	2
2	1	2
2	2	1

$$\boxed{h_{m,k} = \sum_{i,j=1}^{f} A_{m+i-1,\, k+j-1} \, K_{ij}}$$

$f \rightarrow$ Size of kernel

$\rightarrow$ Convolution operation.

Let us use this formula to obtain h_{21}

$A =$

3	2	7	1	6	3
4	1	1	0	9	8
8	2	0	1	2	4
5	6	0	7	1	4
0	5	4	9	6	0
4	6	2	4	0	5

$K =$

1	2	2
2	1	2
2	2	1

$$h_{21} = A_{21} K_{11} + A_{22} K_{12} + A_{23} K_{13} + A_{31} K_{21} + A_{32} K_{22} + A_{33} K_{23} + A_{41} K_{31} + A_{42} K_{32} + A_{43} K_{33}$$

$[m = 2, k = 1]$

$1 \le i, j \le 3$

$=$

$=$

$$= 4 \times 1 + 1 \times 2 + 1 \times 2 + 8 \times 2 + 2 \times 1 + 0 \times 2 + 5 \times 2 + 6 \times 2 + 0 \times 1 = 48$$

$$h_{22} = A_{22} K_{11} + A_{23} K_{12} + A_{24} K_{13} + A_{32} K_{21} + A_{33} K_{22} + A_{34} K_{23} + A_{42} K_{31} + A_{43} K_{32} + A_{44} K_{33}$$

$$= 1 \times 1 + 1 \times 2 + 0 \times 2 + 2 \times 2 + 0 \times 1 + 1 \times 2 + 6 \times 2 + 0 \times 2 + 7 \times 1 = 28$$

$$h_{23} = A_{23} K_{11} + A_{24} K_{12} + A_{25} K_{13} + A_{33} K_{21} + A_{34} K_{22} + A_{35} K_{23} + A_{43} K_{31} + A_{44} K_{32} + A_{45} K_{33}$$

$$= 1 \times 1 + 0 \times 2 + 9 \times 2 + 0 \times 2 + 1 \times 1 + 2 \times 2 + 0 \times 2 + 7 \times 2 + 1 \times 1 = 39$$

$$h_{24} = A_{24} K_{11} + A_{25} K_{12} + A_{26} K_{13} + A_{34} K_{21} + A_{35} K_{22} + A_{36} K_{23} + K_{44} K_{31} + A_{45} K_{32} + A_{46} K_{33}$$

$= 0 \times 1 + 9 \times 2 + 8 \times 2 + 1 \times 2 + 2 \times$ [illegible] $+ 4 \times 2 + 7 \times 2 + 1 \times 2 + 4 \times 1 = 66$

$$h_{31} = 8 \times 1 + 2 \times 2 + 0 \times 2 + 5 \times 2 + 6 \times 1 + 0 \times 2 + 0 \times 2 + 5 \times 2 + 4 \times 1 = 42$$

$$h_{32} = 2 \times 1 + 0 \times 2 + 1 \times 2 + 6 \times 2 + 0 \times 1 + 7 \times 2 + 5 \times 2 + 4 \times 2 + 9 \times 1 = 57$$

$$h_{33} = 0 \times 1 + 1 \times 2 + 2 \times 2 + 0 \times 2 + 7 \times 1 + 1 \times 2 + 4 \times 2 + 9 \times 2 + 6 \times 1 = 47$$

$$h_{34} = 1 \times 1 + 2 \times 2 + 4 \times 2 + 7 \times 2 + 1 \times 1 + 4 \times 2 + 9 \times 2 + 6 \times 2 + 0 \times 1 = 66$$

$$h_{41} = 5 \times 1 + 6 \times 2 + 0 \times 2 + 0 \times 2 + 5 \times 1 + 4 \times 2 + 4 \times 2 + 6 \times 2 + 2 \times 1 = 52$$

$$h_{31} = 8\times1 + 2\times2 + 0\times2 + 5\times2 + 6\times1 + 0\times2 + 0\times2 + 5\times2 + 4\times1 = 42$$

$$h_{32} = 2\times1 + 0\times2 + 1\times2 + 6\times2 + 0\times1 + 7\times2 + 5\times2 + 4\times2 + 9\times1 = 57$$

$$h_{33} = 0\times1 + 1\times2 + 2\times2 + 0\times2 + 7\times1 + 1\times2 + 4\times2 + 9\times2 + 6\times1 = 47$$

$$h_{34} = 1\times1 + 2\times2 + 4\times2 + 7\times2 + 1\times1 + 4\times2 + 9\times2 + 6\times2 + 0\times1 = 66$$

$$h_{41} = 5\times1 + 6\times2 + 0\times2 + 0\times2 + 5\times1 + 4\times2 + 4\times2 + 6\times2 + 2\times1 = 52$$

$$h_{42} = 6\times1 + 0\times2 + 7\times2 + 5\times2 + 4\times1 + 9\times2 + 6\times2 + 2\times2 + 4\times1 = 72$$

$$h_{43} = 0\times1 + 7\times2 + 1\times2 + 4\times2 + 9\times1 + 6\times2 + 2\times2 + 4\times2 + 0\times1 = 57$$

$$h_{44} = 7\times1 + 1\times2 + 4\times2 + 9\times2 + 6\times1 + 0\times2 + 4\times2 + 0\times2 + 5\times1 = 54$$

The output matrix after performing convolutions is

$$Y = \begin{pmatrix} 52 & 26 & 45 & 54 \\ 48 & 28 & 39 & 66 \\ 42 & 57 & 47 & 66 \\ 52 & 72 & 57 & 54 \end{pmatrix}_{4\times4}$$

$16\times9 = 144$

$= 128$

We observed previously that for one value of Y, we do 9 multiplications & 8 additions.

Total multiplications done $= 4 \times 4 \times 9 = 144$
Total additions done $= 4 \times 4 \times 8 = 128$

If we consider any $n \times n$ input matrix
$f \times f$ Kernel size & stride $= s$

The output matrix after applying convolution will be of size $\underbrace{\left\lfloor \frac{n-f}{s} + 1 \right\rfloor \times \left\lfloor \frac{n-f}{s} + 1 \right\rfloor}$

$n = 6,\ f = 3$
$\left\lfloor \frac{6-3}{1} + 1 \right\rfloor = 4$

Hence; number of multiplications
$$= \underbrace{\left\lfloor \frac{n-f}{s} + 1 \right\rfloor} \times \underbrace{\left\lfloor \frac{n-f}{s} + 1 \right\rfloor} \times \underbrace{f \times f}$$

Number of additions
$$= \left\lfloor \frac{n-f}{s} + 1 \right\rfloor \times \left\lfloor \frac{n-f}{s} + 1 \right\rfloor \times \underbrace{(f \times f - 1)}$$

We used formula for convolution as
$$h_{m,k} = \sum_{i,j=1}^{f} A_{m+i-1,\, k+j-1}\, K_{ij}$$

$200 \times 200 \times 3$ $\quad n = 200$
3×3 $\quad f = 3$

$$\left\lfloor \frac{n-f}{s} + 1 \right\rfloor = \left\lfloor \frac{200-3}{1} + 1 \right\rfloor = 198$$

$198 \times 198 \times 3$

$n = 200$
$f = 5$

$$\left\lfloor \frac{n-f}{s} + 1 \right\rfloor = \frac{195}{1} + 1 = 196$$

$196 \times 196 \times 3$