Parity Bit Problem solving using RNN

Question 5. [1+4+5 = 10 marks]

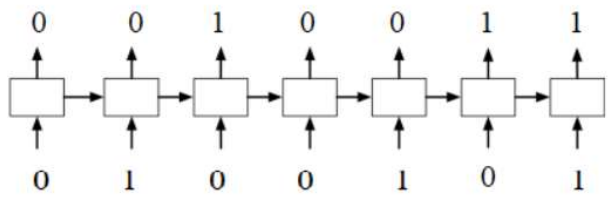
Design a recurrent neural network that outputs a parity bit for binary sequences of arbitrary length. The inputs are given as binary sequences from right to left and output of 1 is generated when number of '0's in the string seen so far is even. For instance, the input string 1010010 would generate an output as follows:

x_t 0 1 0 0 1 0 1

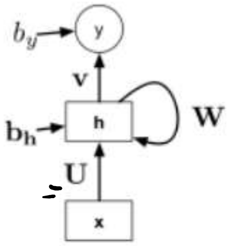
y_t 0 0 1 0 0 1 1

- Input: 0,1,0,0,1,0,1 x_t
- Correct output: 0,0,1,0,0,1,1 y_t

There is one input unit corresponding to the input bit, and one output unit. Therefore, the pattern of inputs and outputs for this example would be:



The RNN has one input unit **x**, two hidden units **h**, and one output unit y. All of the units use the hard threshold activation function, i.e., output is 1 if total weighted input is >=bias, else 0.



$W = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$

$\begin{pmatrix} -1 \\ -1 \end{pmatrix}$

$V = (1, -1)$

$\boxed{\text{Note, at time t, } \mathbf{h}_t = \text{step}(\mathbf{W}\mathbf{h}_{t-1} + \mathbf{U}\,\mathbf{x} - \mathbf{b}_h) \text{ and } y_t = \text{step}(\mathbf{v}\mathbf{h}_t - b_y)}$

We have input & output as below -

	1	2	3	4	5	6	7
Input	0	1	0	0	1	0	1
Output	0	0	1	0	0	1	1

Let $x_t :=$ input at time step t

$y_t :=$ output at time step t.

We have $y_t = \begin{cases} +1 & \text{if } y_{t-1} = 1 \text{ \& } x_t = 1,\ \underbrace{y_{t-1} = 0}\text{, }\underbrace{x_t = 0} \end{cases}$

We have $y_t = \begin{cases} +1 & \text{if } y_{t-1} = 1 \ \& \ x_t = 1, \ y_{t-1} \ldots \\ 0 & \text{else.} \end{cases}$

y_{t-1}	x_t	y_t
1	0	0
1	1	1
0	0	1
0	1	0

[At $t-1$ we have even zeroes, At t we get $x_t = 0$ & we have odd zeroes. $\Rightarrow y_t = 0$]

[At $t-1$ we have odd zeroes. $x_t = 0 \Rightarrow$ at time t we have even zeroes

$\Rightarrow y_t = 1$]

At $t = 0$ we can have $y_0 = 0$ or $y_0 = 1$ [initial starting point]

If $y_0 = 1 \Rightarrow$ initially we start with even number of zeroes

If $y_0 = 0 \Rightarrow$ initially we start with odd number of zeroes

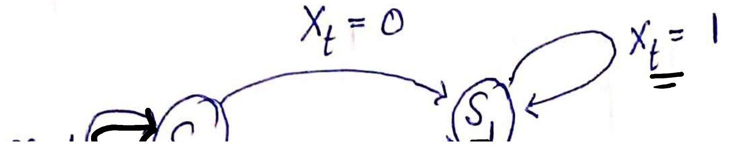
Let us define 3 states S_0, S_1, S_2. We start with $y_0 = 1$

(ie, state $0 \to S_0$ is even)

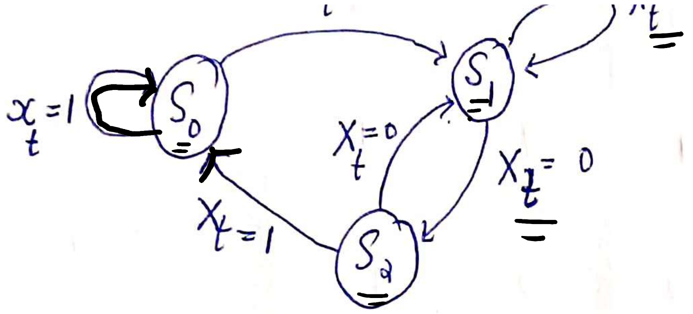
Two transitions are possible depending on x_t value (or x_1 value)

If $x_1 = 0$ it will go to odd state (say S_1)

If $x_1 = 1$ it will remain in S_0.

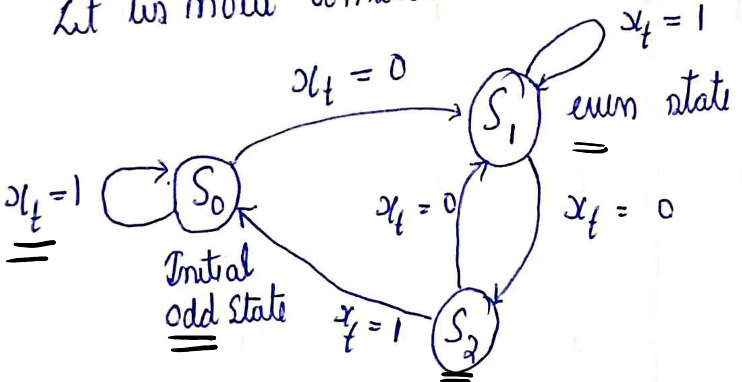


[Here;
S_0 := initial even state]



Here;
$S_0 :=$ initial even state
$S_1 :=$ odd state
$S_2 :=$ even state

Let us now consider a case where $y_0 = 0$ (S_0 is odd)



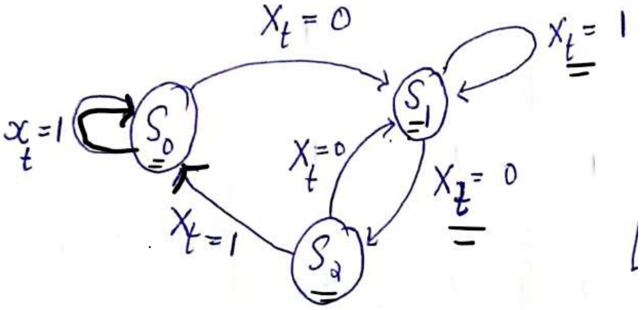
Here;
$S_0 :=$ initial odd state
$S_1 :=$ even state
$S_2 :=$ odd state

We actually get same transition diagram with the definition of S_0, S_1 & S_2 changed.

We have transition between 3 states & we can solve it with 2 hidden states - (h_1, h_2)

$$S_0\ S_1\ S_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$M, F \rightarrow (1), (0)$



We assume S_0 as $[0, 0]$
S_1 as $[0, 1]$
S_2 as $[1, 1]$

We have the transition shown below

$h_1(t-1)$	$h_2(t-1)$	x_t	y_t	$h_1(t)$	$h_2(t)$	
						$S_0 \rightarrow S_1$
(S_0) 0	0	0	$\boxed{0}$	0	1	$S_1 \rightarrow S_1$

(S_0) 0	0	0	0	1		$S_1 \to S_1$
(S_1) 0	1	1	0	0	1	
0	1	0	1	1	1	$S_1 \to S_2$
						$S_2 \to S_1$
1	1	0	0	0	1	
0	1	1	0	0	1	$S_1 \to S_1$
0	1	0	1	1	1	$S_1 \to S_2$
1	1	1	1	0	0	$S_2 \to S_0$

We will now represent y_t as a recurrence in form of h_t & h_{t-1}.

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We get the following distinct combination for h_1, h_2, y_t

At t,

$$\begin{pmatrix} h_1 & h_2 \\ 0 & 1 \\ 1 & 1 \\ 0 & 0 \end{pmatrix} \quad \& \quad \begin{pmatrix} y \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

$h_1(t)$	$h_2(t)$	$h_1(t) - h_2(t) + 1 + b$	y
0	1	b	0
1	1	$1+b$	1
0	0	$1+b$	1

We can say $y_t = h_1(t) + h_2(t)^c$

$h_1(t) \cdot [1 - h_2(t)]$

Here target of 0 is attained when $b < 0$

$b + \ldots = -0.5$

We can say $y_t = h_1(t) + h_2(t)$
$= h_1(t) + [1 - h_2(t)]$

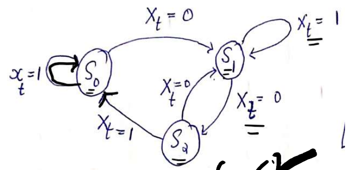
$$\boxed{y_t = \underbrace{h_1(t) - h_2(t) + 1}}$$

Here target y
when $b < 0$
We select $b = -0.5$
we get $h_1(t) - h_2(t) + 0.5$
as decision boundary.

We can now model the transition between states for h_1 & h_2

Let us consider first for $h_1(t)$.

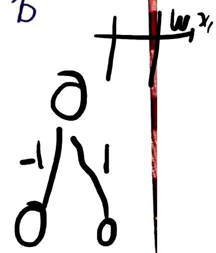
		S_0 $[0,0]$	S_1 $[0,1]$	S_2 $[1,1]$
x_t	0	0	1	0
	1	0	0	0



$S_0[0,0]$ $S_1(0,1)$ $S_2(1,1)$

We can say $h_1(t) = [1 - h_1(t)] + h_2(t-1) + [1 - x_t] + b$

$$\boxed{h_1(t) = -h_1(t-1) + h_2(t-1) - x_t + 2 - b}$$



Here b is a constant which we have to determine.

Let us observe $h_1(t)$ based on $h_1(t-1)$, $h_2(t-1)$ & x_t & then estimate b.

$h_1(t-1)$	$h_2(t-1)$	x_t	$h_1(t)$	$-h_1(t-1) + h_2(t-1) - x_t + 2 - b$
0	0	0	0	$2-b$
0	0	1	0	$1-b$
0	1	0	1	$3+(-b)$
0	1	1	0	$2-b$
1	1	0	0	$2-b$
[illegible]	[illegible]	1	0	$1-b$

-1	1	0	$-$	$\ldots b$
-1	1	1	0	$1-b$

Here we see $h_1(t) = 1$ if $3-b \geq 0$
$\Rightarrow b \leq 3$

We select $b = 2.5$

On doing so we get

$$-h_1(t-1) + h_2(t-1) - x_t + 2 - 2.5$$

$$= -h_1(t-1) + h_2(t-1) - x_t - 0.5$$

We can say $h_1(t) = \begin{cases} 1 & \text{if } -\underbrace{h_1(t-1) + h_2(t-1) - x_t}\geq 0.5 \\ 0 & \text{else.} \end{cases}$

Similarly we can get $h_2(t)$ as below -

$$\begin{array}{cc|ccc} & & S_0 & S_1 & S_2 \\ & & [0,0] & [0,1] & [1,1] \\ x_t & 0 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 0 \end{array}$$

We will use the step function $-h_1(t-1) + h_2(t-1) - x_t + 2 - b$

$h_1(t-1)$	$h_2(t-1)$	x_t	$h_2(t)$	$-h_1(t-1) + \overline{h_2}(t-1) - x_t + 2 - b$
0	0	0	1	$2-b$
				$1-b$

0	0	0		
0	0	1	0	1 - b
0	1	0	1	3 - b
0	1	1	1	2 - b
1	1	1	0	1 - b
1	1	0	1	2 - b

We say $h_2(t)$ is -ve (zero class) when $1 - b < 0$
$\Rightarrow \boxed{b > 1}$

We select $b = 1.5$

We have following recurrence relations.

$$y_t = \text{step}\left(\underbrace{h_1(t) - h_2(t)}+\underbrace{0.5}\right)$$

i.e. $y_t = \begin{cases} 1 & \text{if } \underbrace{h_1(t) - h_2(t) + 0.5} > 0 \\ 0 & \text{else} \end{cases}$

$$\Rightarrow y_t = \text{step}\left(\underbrace{(1 \quad -1)}\overbrace{\begin{pmatrix} h_1(t) \\ h_2(t) \end{pmatrix}} + 0.5\right)$$

$$\Rightarrow \boxed{y_t = \text{step}\left(V h_t - (b_y) \right)}$$

where $V = (1, -1)$

$h_t = \begin{pmatrix} h_1(t) \\ h_2(t) \end{pmatrix}$

$p \times m$

(h_1, h_2)

$V s_t = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$

$y_t = f(V s_t + c)$

$s_t = g(U x_t + W s_{t-1} + b)$

$$h_t = \begin{pmatrix} h_1(t) \\ h_2(t) \end{pmatrix}$$

$$b_y = -0.5$$

Also, $\underline{\underline{h_1}}(t) = \begin{cases} 1 & \text{if } \underline{-h_1(t-1) + h_2(t-1)} - x_t - 0.5 \geq 0 \\ 0 & \text{else} \end{cases}$ $\quad U = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$

$\underline{b}$

$\underline{\underline{h_2}}(t) = \begin{cases} 1 & \text{if } -\underline{h_1(t-1) + h_2}(t-1) - x_t + 0.5 \geq 0 \\ 0 & \text{else} \end{cases}$ $\quad W = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$

$$\Rightarrow \quad h(t) = \begin{pmatrix} h_1(t) \\ h_2(t) \end{pmatrix} = \text{step}\left(\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} h(t-1) + \begin{pmatrix} -1 \\ -1 \end{pmatrix} x_t + \begin{pmatrix} -0.5 \\ 0.5 \end{pmatrix}\right)$$

$$\boxed{h(t) = \text{step}(Ux_t + Wh_{t-1} - b_h)}$$

where $U = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$, $W = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$

$$b_h = \begin{pmatrix} 0.5 \\ -0.5 \end{pmatrix}$$