Rings and Ideals

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2025

Definition

A **ring** is a set R equipped with two binary operations denoted $(+)$, $(\cdot)$ satisfying the following conditions:

1. The structure $(R, +)$ is an abelian group.
2. $\forall x, y, z \in R : (x \cdot y) \cdot z = x \cdot (y \cdot z)$ *(associativity)*
3. $\forall x, y, z \in R : x \cdot (y + z) = (x \cdot y) + (x \cdot z) \wedge$
   $(x + y) \cdot z = (x \cdot z) + (y \cdot z)$ *(distributivity)*

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Additionally:

1. *If there exists an identity in a monoid $(R, \cdot)$, we say that R is a **ring with identity***
2. *If a monoid $(R, \cdot)$ is commutative, we say that R is a **commutative ring**.*

Ring Examples

Remark

1. Later in this presentation, when we refer to a ring we mean commutative ring with identity.
2. If binary operations in a ring are implied or natural, we simply write ring R and we treat it like a set.

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Example

The following structures are Rings:

1. $(\mathbb{Z}, +, \cdot)$ - Ring of integer numbers.

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Example

The following structures are Rings:

1. $(\mathbb{Z}, +, \cdot)$ - Ring of integer numbers.
2. $\mathbb{R}[x_1, \ldots, x_n]$ - Ring of polynomials in n variables with real coefficients.

Definition

Let R be a ring. Let $I \subseteq R$ be any subset of R. We say, that I is an **ideal of ring R** if following conditions are satisfied:

1. $\forall a, b \in I : a + b \in I$
2. $\forall r \in R \land a \in I : r \cdot a \in I$

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Example

Let $R = \mathbb{Z}$ be a ring of integers. Then

$$I = \{a \in R \mid \exists k \in \mathbb{Z} : a = 3 \cdot k\} = \{\ldots, -6, -3, 0, 3, 6, \ldots\}$$

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Example

Let $R = \mathbb{R}[x, y]$ be a ring of real polynomials in two variables, $C = (x, y) \in \mathbb{R}^2 \mid x - y = 0$. Then

$$I = \{f \in R \mid \forall (x, y) \in C : f(x, y) = 0\}$$

is an ideal of R.

Properties of Ideals

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Theorem

Let R be a ring and let $I, J \subseteq R$ be ideals of R. Then $I \cap J$ is an ideal of R.

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Proof.

Let a, b be any elements of $I \cap J$. Then by definition $a \in I \wedge a \in J \wedge b \in I \wedge b \in J$. Since I is an ideal in R and $a, b \in I$ therefore $a + b \in I$. Same can be applied to get $a + b \in J$. Thus $a + b \in I \cap J$. Since I is an ideal of R then for all $r \in R$ $r \cdot a \in I$. Same can be applied to show that for all $r \in R$ it is true that $r \cdot a \in J$. Now we can state the fact that for all $r \in Rr \cdot a \in I \cap J$ which ends the proof. □

Let's focus on a case of polynomial ring $R = \mathbb{R}[x, y]$.

Definition

Let I be an ideal of R. We define $V(I)$ to be the intersection of set of roots of every polynomial in I:

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Now, we have defined nice connection between ideals of polynomial ring and sets on a real plane.

Example

Let $I = \{(x - 2y) \cdot f \mid f \in R\}$ be an ideal of R. Then

$$V(I) = \{(x, y) \in \mathbb{R}^2 \mid x - 2y = 0\} = \{(x, \frac{1}{2}x) \in \mathbb{R}^2 | x \in \mathbb{R}\}$$