On the Shortest Path between two Vertices in Permutation Polytope

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Definitions

Definition 1

Extreme points. Let $A \subset \mathbb{R}^n$ be a set. A point $a \in A$ is called an *extreme point* of A provided for any two points $b, c \in A$ such that $\frac{b+c}{2} = a$ one must have $b = c = a$. The set of all extreme points of A is denoted $\mathrm{ex}(A)$.

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Extreme points. Let $A \subset \mathbb{R}^n$ be a set. A point $a \in A$ is called an *extreme point* of A provided for any two points $b, c \in A$ such that $\frac{b+c}{2} = a$ one must have $b = c = a$. The set of all extreme points of A is denoted $\mathrm{ex}(A)$.

Definition 2

Polytope. The convex hull of a finite set of points in $\mathbb{R}^n$ is called a *polytope*.

Definition 3

Polyhedron. Let $c_1, ..., c_m$ be vectors from $\mathbb{R}^n$ and let $\beta_1, ..., \beta_m$ be real numbers and additionally $\langle \cdot, \cdot \rangle$ be the standard inner product. The set:

$$P = \{x \in \mathbb{R}^n : \langle c_i, x \rangle \leq \beta_i, i = 1, .., m\}$$

is called **polyhedron.**

Extreme Points Theorem

Notation

An extreme point of a polyhedron is called a *vertex*.

Theorem 4

Let $P \subset \mathbb{R}^n$ be a polyhedron:

$$P = \{x \in \mathbb{R}^n : \langle c_i, x \rangle \leq \beta_i, i = 1, .., m\}$$

where $c_i \in \mathbb{R}^n$ and $\beta_i \in \mathbb{R}$ for $i = 1, .., m$. For $u \in P$ let

$$I(u) = \{i : \langle c_i, u \rangle = \beta_i\}$$

be a set of the inequalities that are active on u. Then u is a vertex of P if and only if the set of vectors $\{c_i : i \in I(u)\}$ linearly spans the vector space $\mathbb{R}^n$. In particular if u is a vertex of P the set $I(u)$ contains at least n indices: $|I(u)| \geq n$

Permutation Polytope

Definition 5

Let us fix a point $x = (\xi_1, ..., \xi_n) \in \mathbb{R}^n$. For a permutation σ of the set $\{1, ..., n\}$ let $\sigma(x)$ be the vector $y = (\eta_1, ..., \eta_n)$ where $\eta_i = \xi_{\sigma^{-1}(i)}$ Let S_n be the symmetric group of all permutations of the set $\{1, ..., n\}$. Let us define the *permutation polytope* $P(x)$ by

$$P(x) = conv(\sigma(x) : \sigma \in S_n)$$

in words: we permute the coordinates of a given vector x in all possible ways and take the convex hull of resulting vectors.

Permutohedron

Theorem 6

Permutation Polytope $P(x)$ where $x = (1, 2, ..., n) \in \mathbb{R}^n$ lies in a $n-1$ dimensional affine subspace of euclidan $\mathbb{R}^n$ space.

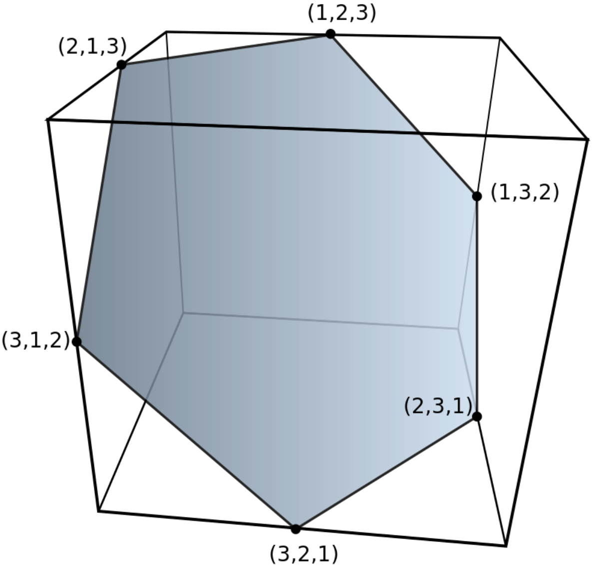
Definition 7

Permutation Polytope $P(x)$ where $x = (1, ..., n) \in \mathbb{R}^n$ is called *Permutohedron*.

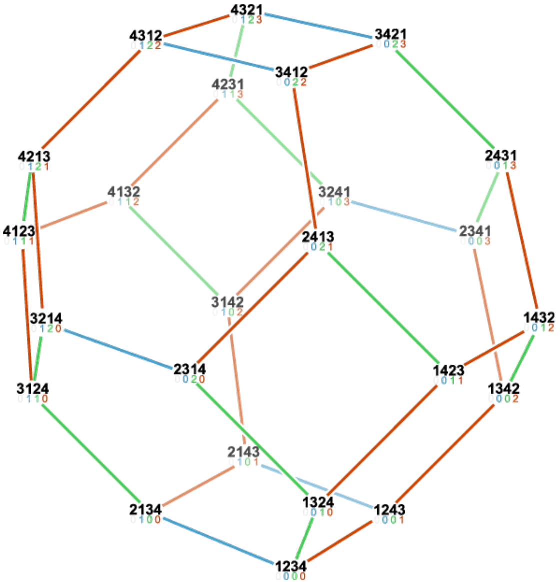
Theorem 8

Every permutation is a vertex of Permutohedron.

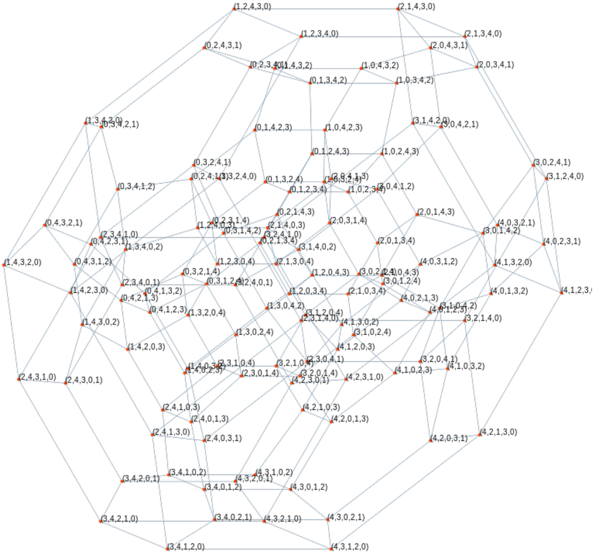
Permutohedron of point $x = (1, 2, 3)$



Permutohedron of point $x = (1, 2, 3, 4)$



Permutohedron of point $x = (1, 2, 3, 4, 5)$



Facets and Faces of Polyhedron

Let P be a polyhedron defined as earlier. For $u \in P$ we define a set inequalities that are active on u as following:

$$I(u) = \{i : \langle c_i, u \rangle = \beta_i\}$$

Definition 9

Face. Let $P \subset \mathbb{R}^n$ be a polyhedron. Let $F \subset P$ be a set of all points $u \in P$ such that some $i_1, ..., i_k \in I(u)$ for all u. Set F is called a Face. by dimension of a Face we understand $d = n - dim(span(c_{i_1}, .., c_{i_k}))$

Facets and Faces of Polyhedron

Theorem 10

Every face of Polyhedron is a Polyhedron.

Proposition/Definition Face of $n-1$ dimension is called a *Facet.*
Face of 1 dimension is called an *Edge.*
Face of 0 dimension is a *Vertex*.
Every Edge contains exactly two vertices.
If there exists and edge between two vertices then the vertices are called *adjacent.*

Edges of Permutohedron

Theorem 11

Let $x_1, x_2 \in \mathbb{R}^n$ be two vertices of Permutohedron associated with permutations σ_1, σ_2. There is an edge between x_1 and x_2, which means they are adjacent if and only if there exist transposition (permutation) $\sigma_i = (i \; i+1)$ such that $\sigma_1 = \sigma_2 \circ \sigma_i$.

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Definition 12

Let $[n] = \{1, ..., n\}$ be a set of n first natural numbers. Set of points $x \in \mathbb{R}^n$ fulfilling $2^n - 2$ inequalities and one equality:

$$\sum_{i \in J} x_i \geq \frac{|J|(|J|+1)}{2}, J \subset [n], J \neq \emptyset, J \neq [n]$$

$$\sum_{i=1}^{n} x_i = \frac{n(n+1)}{2}$$

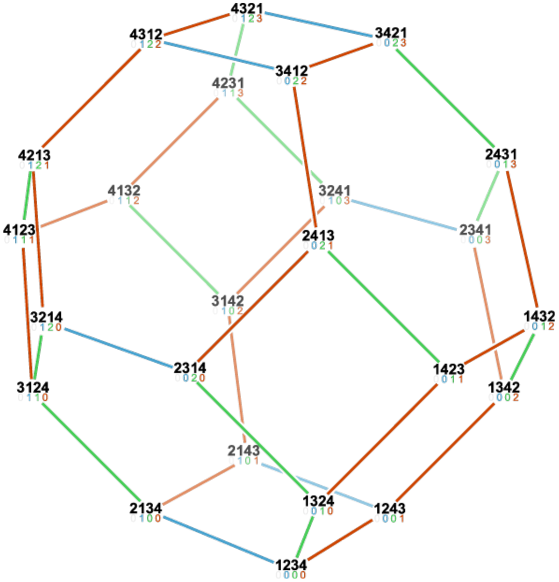
is a Permutohedron.

Example

Example 13

$x = (3, 4, 1, 2) \in \mathbb{R}^4$
adjacent vertices to the vertex x are:
$x_1 = x \circ (1\ 2) = (3, 4, 2, 1)$
$x_2 = x \circ (2\ 3) = (2, 4, 1, 3)$
$x_3 = x \circ (3\ 4) = (4, 3, 1, 2)$

Permutohedron of point $x = (1, 2, 3, 4)$



Adjacent Vertices as Matrices

$$\sigma = (2,4,3,1),\ X^\sigma = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

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$$(2,4,3,1) \circ (1\ 2) = (1,4,3,2)$$

$$\begin{bmatrix} \mathbf{0} & 0 & 0 & \mathbf{1} \\ \mathbf{1} & 0 & 0 & \mathbf{0} \\ \mathbf{0} & 0 & 1 & \mathbf{0} \\ \mathbf{0} & 1 & 0 & \mathbf{0} \end{bmatrix} \xrightarrow{(1\ 2)} \begin{bmatrix} \mathbf{1} & 0 & 0 & \mathbf{0} \\ \mathbf{0} & 0 & 0 & \mathbf{1} \\ \mathbf{0} & 0 & 1 & \mathbf{0} \\ \mathbf{0} & 1 & 0 & \mathbf{0} \end{bmatrix}$$

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Adjacent vertices to vertex $(2, 4, 3, 1)$

$(2, 4, 3, 1) \circ (1\ 2) = (1, 4, 3, 2)$

$$\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \xrightarrow{(1\ 2)} \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$(2, 4, 3, 1) \circ (2\ 3) = (3, 4, 2, 1)$

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ 0 & 1 & 0 & 0 \end{bmatrix} \xrightarrow{(2\ 3)} \begin{bmatrix} 0 & 0 & 0 & 1 \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$(2, 4, 3, 1) \circ (3\ 4) = (2, 3, 4, 1)$

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \end{bmatrix} \xrightarrow{(3\ 4)} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \end{bmatrix}$$

The Shortest Path between Two Vertices of Permutohedron

$\sigma_1 = (2, 4, 3, 1)$

$$X^{\sigma_1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$\sigma_2 = (2, 3, 1, 4)$

$$X^{\sigma_2} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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$\sigma_1 = (2, 4, 3, 1)$

$$X^{\sigma_1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$\sigma_2 = (2, 3, 1, 4)$

$$X^{\sigma_2} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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$\sigma_1 = (2,4,3,1)$ $\sigma_2 = (2,3,1,4)$

$$X^{\sigma_1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad X^{\sigma_2} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ 0 & 1 & 0 & 0 \end{bmatrix} \xrightarrow{(2\ 3)} \begin{bmatrix} 0 & 0 & 0 & 1 \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \xrightarrow{(1\ 2)} \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

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$$\sigma_1 \circ (23) \circ (12) \circ (23) \circ (34) = \sigma_2$$

The Shortest Path between Two Vertices of Permutohedron

$\sigma_1 = (2, 4, 3, 1)$

$$\begin{bmatrix} \mathbf{4}: \\ \mathbf{2}: \\ \mathbf{1}: \\ \mathbf{3}: \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$\sigma_2 = (2, 3, 1, 4)$

$$\begin{bmatrix} \mathbf{1}: \\ \mathbf{2}: \\ \mathbf{3}: \\ \mathbf{4}: \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The Shortest Path between Two Vertices of Permutohedron

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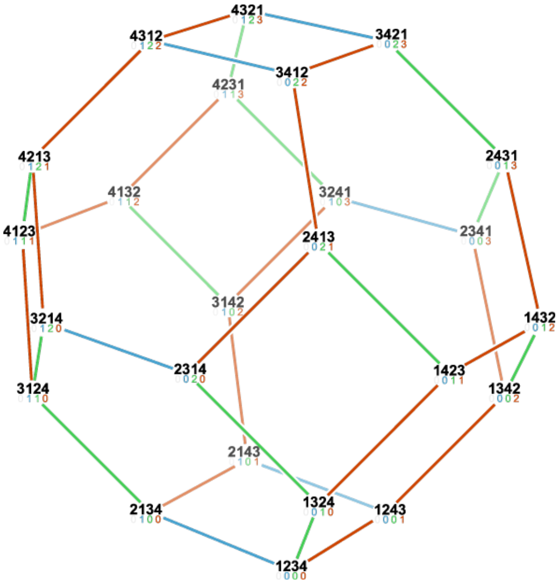
Problem of finding the shortest path between two vertices is the same as problem of sorting an array swapping only adjacent elements.

Permutohedron vertices coloring



Further Questions and Ideas.

1. Finite Groups found in Permutohedron.
2. Derangements in Permuatohedron.
3. Inverse Permutations in Permutohedron.
4. Charachteristics of all Faces of Permutohedron.

Literature

A. Barvinok *A Course in Convexity* American Mathematical Society, 2002.

G. Lancia and P. Serafini *Compact Extended Linear Programming Models* http://ndl.ethernet.edu.et/bitstream/123456789/71466/1/77.pdf

M. Goemans *Smallest Compact Formulation for the Permutahedron* https://math.mit.edu/~goemans/PAPERS/permutahedron.pdf