On the shortest Path between two Vertices in Permutation Polytope.

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Basics

Polytope. The convex hull of a finite set of points in $\mathbb{R}^d$ is called a *polytope*. Let $c_1, ..., c_m$ be vectors from $\mathbb{R}^d$ and let $\beta_1, ..., \beta_m$ be real numbers and additionally $\langle \cdot, \cdot \rangle$ be the standard inner product. The set:

$$P = \{x \in \mathbb{R}^d : \langle c_i, x \rangle \leq \beta_i, i = 1, .., m\}$$

is called *polyhedron*. If polyhedron is bounded it is also a Polytope. A *vertex* of polyhedron P is a point $a \in P$ provided that for any two points $b, c \in P$ such that $\frac{b+c}{2} = a$ one must have $b = c = a$.
Doubly stochastic Matrix. An $n \times n$ matrix $M = (\alpha_{ij})$ is called *doubly stochastic* if it is non-negative and the sum of elements in every row and every columns equals 1:

1. $\sum_{i=1}^{n} \alpha_{ij} = 1$ for $j = 1, ..., n$,
2. $\sum_{j=1}^{n} \alpha_{ij} = 1$ for $i = 1, ..., n$,
3. $\alpha_{ij} > 0$ for $i, j = 1, .., n$.

The polyhedron B_n of all $n \times n$ doubly stochastic matrices is called the *Birkhoff Polytope*, where we consider $n \times n$ matrix X as point in $\mathbb{R}^{n^2}$.

Birkhoff Polytope Properties

Birkhoff Polyope lies in $(n-1)^2$ dimensional affine subspace of $\mathbb{R}^{n^2}$.
What's more **Birkhoff - von Neumann Theorem** states that the vertices of B_n are exactly permutation matrices.
Now let consider a point $x \in \mathbb{R}^n$. We define a *Permutation Polytope* $P(x)$ to be a convex hull of all possible permutations of coordinates of point x. There is a simple connection between Birkhoff Polyope and Permutation Polyope. If we define $T : \mathbb{R}^{n^2} \to \mathbb{R}^n$ by $T(X) = Xa$ for every $X \in \mathbb{R}^{n^2}$ and some fixed vetor $a \in \mathbb{R}^n$, then it can be shown that $T(B_n) = P(a)$.
Since sum of coordinates for every point of Permutation Polyope is constant, therefore if $x \in \mathbb{R}^n$ then $P(x)$ lies in $n-1$ dimensional affine subspace of $\mathbb{R}^n$.

Equivalent definition

Let $[n] = \{1, ..., n\}$ be a set of n first natural numbers. Set of points $x \in \mathbb{R}^n$ fulfilling $2^n - 2$ inequalities and one equality:

$$\sum_{i \in J} x_i \geq \frac{|J|(|J|+1)}{2}, J \subset [n], J \neq \emptyset, J \neq [n]$$

$$\sum_{i=1}^{n} x_i = \frac{n(n+1)}{2}$$

is a Permutohedron.

Edges in Permutohedron

There exists edge between two vertices associated with two permutations σ_1, σ_2 if and only if there exists a transposition of two next elements $\sigma_3 = (i\ i+1)$, $i = 1, ..n-1$ such that $\sigma_2 = \sigma_3 \circ \sigma_1$.
It can be intuitively understood by the fact, that these are the closest neighbours in the standard Euclidian metric.

Question

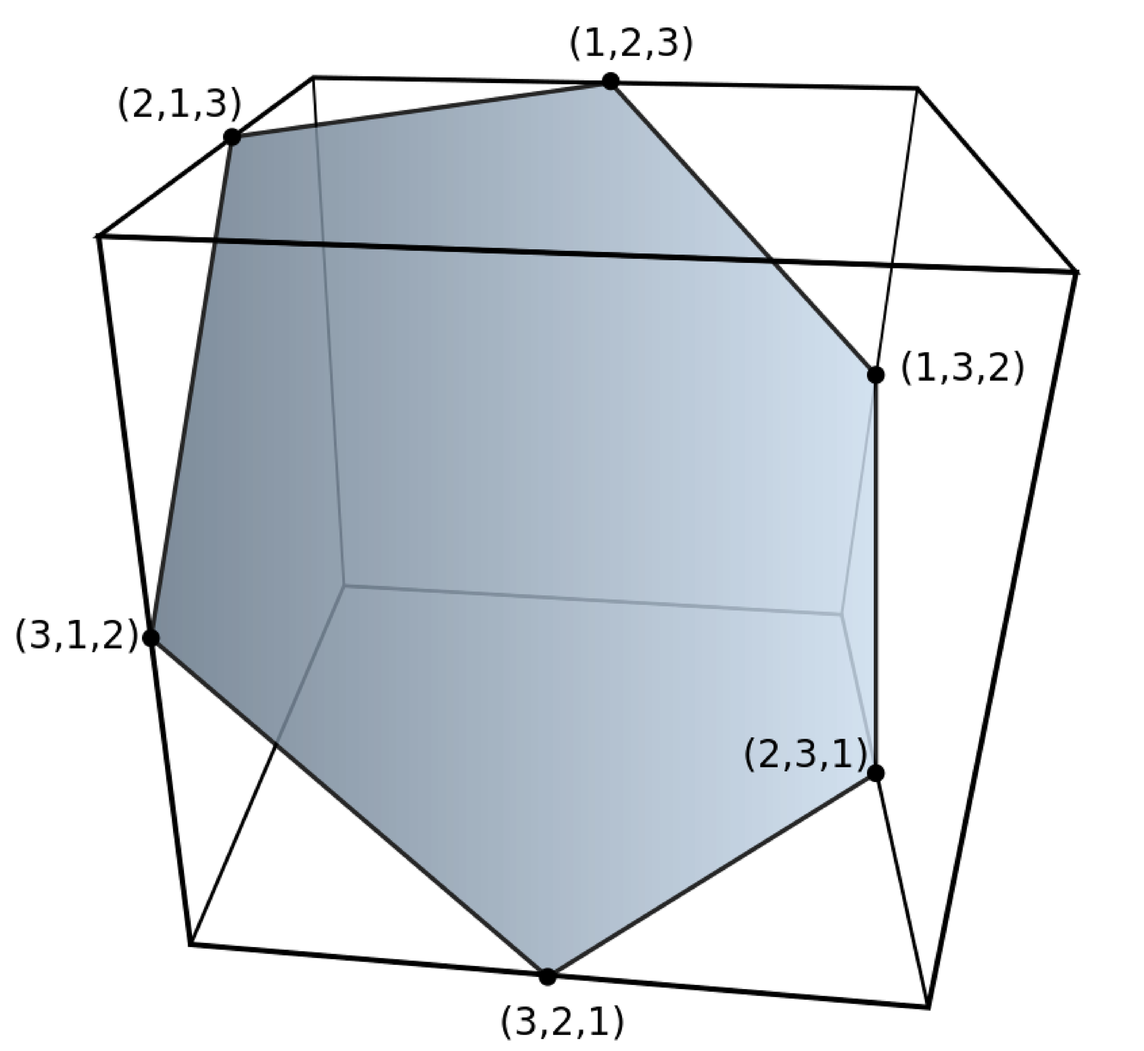
Since we can look at n-dimensional Permutohedron as graph, interesting question is, how to find the shortest path between two vertices.

Schur-Horn Theorem

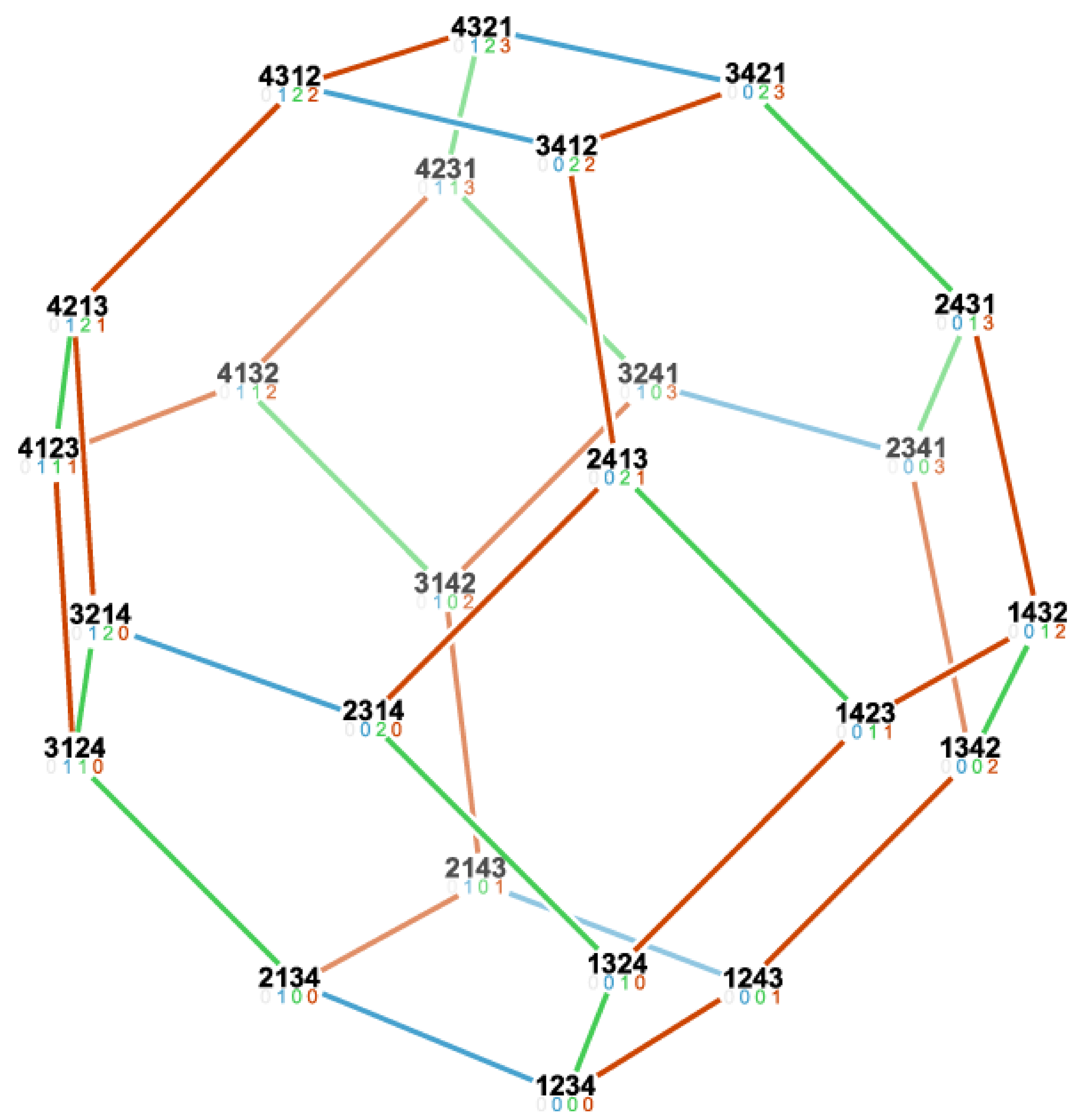
Schur-Horn Theorem. Let us fix a positive integer n and real numbers $\lambda_1, ..., \lambda n$. Let $l = (\lambda_1, ..., \lambda n) \in \mathbb{R}^n$ be a vector.

1. Let $A = (\alpha_{ij})$ be a $n \times n$ real symmetric matrix with the eigenvalues $\lambda_1, ..., \lambda_n$. Then the diagonal $a = (\alpha_{11}, ..., \alpha_{nn}$ lies in the permutation polytope $P(l) : a \in P(l)$ (Schur's Theorem).
2. Let $a \in P(l)$ be a point from the permutation polytope. Then there exists an $n \times n$ real symmetric matrix $A = (\alpha_{ij})$ with the eigenvalues $\lambda_1, ..., \lambda_n$ and the diagonal $a = (\alpha_{11}, ..., \alpha_{nn})$

Permutohedron of (1,2,3)



Permutation Polytope



$\sigma_1 = (2, 4, 3, 1)$ $\sigma_2 = (2, 3, 1, 4)$

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ 0 & 1 & 0 & 0 \end{bmatrix} \xrightarrow{(?)} \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \xrightarrow{(2\ 3)} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \xrightarrow{(1\ 2)}$$

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \xrightarrow{(2\ 3)} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \xrightarrow{(3\ 4)}$$

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Algorithm

$\sigma_1 = (2, 4, 3, 1)$

$$\begin{bmatrix} \mathbf{4}: \\ \mathbf{2}: \\ \mathbf{1}: \\ \mathbf{3}: \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$\sigma_2 = (2, 3, 1, 4)$

$$\begin{bmatrix} \mathbf{1}: \\ \mathbf{2}: \\ \mathbf{3}: \\ \mathbf{4}: \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Problem of finding the shortest path between two vertices is the same as problem of sorting an array swapping only adjacent elements.

Complexity of Algorithm

An array to be sorted using only consecutive swaps, needs in edge case $\frac{n(n-1)}{2}$ iterations - which is also the diameter of Permutohedron, therefore shown algorithm is the fastest possible if our goal is to find the shortest path.
If we only need a length of it, we can use approach similair to the merge sort algorithm, which can find an answer in $O(n \log(n))$ time but the exact path cannot be found that way.

References

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