# Supplementary materials for the paper

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## 1 DERIVATION OF THE FLUX

The expression of the numerical coagulation flux $F^{\mathrm{nc}}_{\mathrm{coag}}[\tilde{g}]$ is

$$\begin{cases} F^{\mathrm{nc}}_{\mathrm{coag}}[\tilde{g}](x,t) = \sum\limits_{l'=1}^{N}\sum\limits_{i'=0}^{k}\sum\limits_{l=1}^{N}\sum\limits_{i=0}^{k} g_{l'}^{i'}(t) g_l^i(t) T(x, x_{\min}, x_{\max}, i', i, l', l), \\ T(x, x_{\min}, x_{\max}, i', i, l', l) = \\ \int\limits_{x_{\min}}^{x} \mathcal{K}_1(u)\phi_{i'}(\xi_{l'}(u))[\theta(u - x_{l'-1/2}) - \theta(u - x_{l'+1/2})] \int\limits_{x-u+x_{\min}}^{x_{\max}} \frac{\mathcal{K}_2(v)}{v}\phi_i(\xi_l(v))[\theta(v - x_{l-1/2}) - \theta(v - x_{l+1/2})]\mathrm{d}v\mathrm{d}u. \end{cases} \tag{1}$$

With the change of variable $\xi_l = \frac{2}{h_l}(x - x_l)$, the inner integral writes

$$T_{\mathrm{inner}}(x, x_{\min}, x_{\max}, i, l) = \int\limits_{x-u+x_{\min}}^{x_{\max}} f_2(\xi_l) \left[\theta\left(\xi_l - \frac{2}{h_l}(x_{l-1/2} - x_l)\right) - \theta\left(\xi_l - \frac{2}{h_l}(x_{l+1/2} - x_l)\right)\right] \mathrm{d}\xi_l, \tag{2}$$

with

$$f_2(\xi_l) \equiv \mathcal{K}_2\left(\frac{h_l}{2}\xi_l + x_l\right) \frac{\phi_i(\xi_l)}{\frac{h_l}{2}\xi_l + x_l}. \tag{3}$$

In the following, we will take advantage several times of the identity

$$\begin{cases} a < b, c < b, d \leq b, c < d \\ \int_a^b f(x)[\theta(x-c) - \theta(x-d)]\mathrm{d}x = \int_c^b f(x)\mathrm{d}x + \theta(a-c)\int_a^c f(x)\mathrm{d}x - \theta(b-d)\int_d^b f(x)\mathrm{d}x - \theta(b-d)\theta(a-d)\int_a^d f(x)\mathrm{d}x. \end{cases} \tag{4}$$

Eq. 4 gives for the term $T_{\mathrm{inner}}$

$$\begin{aligned} &T_{\mathrm{inner}}(x, x_{\min}, x_{\max}, i, l) = \\ &\int\limits_{\frac{2}{h_l}(x_{l-1/2}-x_l)}^{\frac{2}{h_l}(x_{\max}-x_l)} f_2(\xi_l)\mathrm{d}\xi_l + \theta\left(x - \frac{h_{l'}}{2}\xi_{l'} - x_{l'} + x_{\min} - x_{l-1/2}\right) \int\limits_{\frac{2}{h_l}(x - \frac{h_{l'}}{2}\xi_{l'} - x_{l'} + x_{\min} - x_l)}^{\frac{2}{h_l}(x_{l-1/2}-x_l)} f_2(\xi_l)\mathrm{d}\xi_l \\ &- \theta\left(x_{\max} - x_{l+1/2}\right) \left\{ \int\limits_{\frac{2}{h_l}(x_{l+1/2}-x_l)}^{\frac{2}{h_l}(x_{\max}-x_l)} f_2(\xi_l)\mathrm{d}\xi_l + \theta\left(x - \frac{h_{l'}}{2}\xi_{l'} - x_{l'} + x_{\min} - x_{l+1/2}\right) \int\limits_{\frac{2}{h_l}(x - \frac{h_{l'}}{2}\xi_{l'} - x_{l'} + x_{\min} - x_l)}^{\frac{2}{h_l}(x_{l+1/2}-x_l)} f_2(\xi_l)\mathrm{d}\xi_l \right\}. \end{aligned} \tag{5}$$

Therefore, $T$ writes

$$
\begin{aligned}
&T(x,x_{\min},x_{\max},i',i,l',l) = \frac{h_l}{2}\frac{h_{l'}}{2} \\
&\left[\int\limits_{\frac{2}{h_l}\left(x_{l-1/2}-x_l\right)}^{\frac{2}{h_l}\left(x_{l+1/2}-x_l\right)} f_2(\xi_l)\mathrm{d}\xi_l \left\{ \int\limits_{\frac{2}{h_{l'}}\left(x_{\min}-x_{l'}\right)}^{\frac{2}{h_{l'}}\left(x-x_{l'}\right)} f_1(\xi_{l'}) \left[\theta\left(\xi_{l'}-\frac{2}{h_{l'}}\left(x_{l'-1/2}-x_{l'}\right)\right)-\theta\left(\xi_{l'}-\frac{2}{h_{l'}}\left(x_{l'+1/2}-x_{l'}\right)\right)\right]\mathrm{d}\xi_{l'} \right\}\right. \\
&\quad + \int\limits_{\frac{2}{h_{l'}}\left(x_{\min}-x_{l'}\right)}^{\frac{2}{h_{l'}}\left(x-x_{l'}\right)} \int\limits_{\frac{2}{h_l}\left(x-\frac{h_{l'}}{2}\xi_{l'}-x_{l'}+x_{\min}-x_l\right)}^{\frac{2}{h_{l'}}\left(x_{l-1/2}-x_l\right)} f_1(\xi_{l'})f_2(\xi_l) \\
&\qquad \left[\theta\left(\xi_{l'}-\frac{2}{h_{l'}}\left(x_{l'-1/2}-x_{l'}\right)\right)-\theta\left(\xi_{l'}-\frac{2}{h_{l'}}\left(x_{l'+1/2}-x_{l'}\right)\right)\right]\theta\left(\frac{2}{h_{l'}}\left(x-x_{l-1/2}+x_{\min}-x_{l'}\right)-\xi_{l'}\right)\mathrm{d}\xi_l\mathrm{d}\xi_{l'} \\
&\quad - \int\limits_{\frac{2}{h_{l'}}\left(x_{\min}-x_{l'}\right)}^{\frac{2}{h_{l'}}\left(x-x_{l'}\right)} \int\limits_{\frac{2}{h_l}\left(x-\frac{h_{l'}}{2}\xi_{l'}-x_{l'}+x_{\min}-x_l\right)}^{\frac{2}{h_{l'}}\left(x_{l+1/2}-x_l\right)} f_1(\xi_{l'})f_2(\xi_l) \\
&\qquad \left[\theta\left(\xi_{l'}-\frac{2}{h_{l'}}\left(x_{l'-1/2}-x_{l'}\right)\right)-\theta\left(\xi_{l'}-\frac{2}{h_{l'}}\left(x_{l'+1/2}-x_{l'}\right)\right)\right]\theta\left(\frac{2}{h_{l'}}\left(x-x_{l+1/2}+x_{\min}-x_{l'}\right)-\xi_{l'}\right)\mathrm{d}\xi_l\mathrm{d}\xi_{l'} \\
&\left.\vphantom{\int}\right],
\end{aligned} \tag{6}
$$

with $f_1(\xi_{l'}) \equiv \mathcal{K}_1(\xi_{l'})\phi_{i'}(\xi_{l'})$ and $f_2(\xi_l) \equiv \mathcal{K}_2\left(\frac{h_l}{2}\xi_l + x_l\right)\frac{\phi_i(\xi_l)}{\frac{h_l}{2}\xi_l + x_l}$.

### 1.1 Simple integral

In Eq. 6, the rule for the simple integral with the Heaviside function is

$$
\begin{cases}
a < b, a \leq c, a < d, c < d \\
\displaystyle\int_a^b f(x)\left[\theta(x-c)-\theta(x-d)\right]\mathrm{d}x = \theta(b-c)\left[\int_c^b f(x)\mathrm{d}x + \theta(a-c)\int_a^c f(x)\mathrm{d}x\right] - \theta(b-d)\int_d^b f(x)\mathrm{d}x.
\end{cases} \tag{7}
$$

The simple integral writes

- $$
\begin{aligned}
&\int\limits_{\frac{2}{h_{l'}}\left(x_{\min}-x_{l'}\right)}^{\frac{2}{h_{l'}}\left(x-x_{l'}\right)} f_1(\xi_{l'})\left[\theta\left(\xi_{l'}-\frac{2}{h_{l'}}(x_{l'-1/2}-x_{l'})\right)-\theta\left(\xi_{l'}-\frac{2}{h_{l'}}(x_{l'+1/2}-x_{l'})\right)\right]\mathrm{d}\xi_{l'} \\
&=\theta(x-x_{l'-1/2})\left[\int\limits_{\frac{2}{h_{l'}}\left(x_{l'-1/2}-x_{l'}\right)}^{\frac{2}{h_{l'}}\left(x-x_{l'}\right)} f_1(\xi_{l'})\mathrm{d}\xi_{l'} + \underbrace{\theta(x_{\min}-x_{l'-1/2})\int\limits_{\frac{2}{h_{l'}}\left(x_{\min}-x_{l'}\right)}^{\frac{2}{h_{l'}}\left(x_{l'-1/2}-x_{l'}\right)} f_1(\xi_{l'})\mathrm{d}\xi_{l'}}_{0\text{ since }x_{l-1/2}\geq x_{\min}}\right] \\
&\quad -\theta(x-x_{l'+1/2})\int\limits_{\frac{2}{h_{l'}}\left(x_{l'+1/2}-x_{l'}\right)}^{\frac{2}{h_{l'}}\left(x-x_{l'}\right)} f_1(\xi_{l'})\mathrm{d}\xi_{l'} \\
&= \theta(x-x_{l'-1/2})\int\limits_{\frac{2}{h_{l'}}\left(x_{l'-1/2}-x_{l'}\right)}^{\frac{2}{h_{l'}}\left(x-x_{l'}\right)} f_1(\xi_{l'})\mathrm{d}\xi_{l'} - \theta(x-x_{l'+1/2})\int\limits_{\frac{2}{h_{l'}}\left(x_{l'+1/2}-x_{l'}\right)}^{\frac{2}{h_{l'}}\left(x-x_{l'}\right)} f_1(\xi_{l'})\mathrm{d}\xi_{l'}.
\end{aligned} \tag{8}
$$

### 1.2 First double integral

In Eq. 6, the rule for the first double integral with the Heaviside function is

$$
\begin{cases}
a<b, a\le c, a<d, c<d, e\le b\\
\int_a^b f(x)\left[\theta(x-c)-\theta(x-d)\right]\theta(e-x)\mathrm{d}x\\
=\theta(b-c)\left[-\theta(a-c)\theta(e-a)\left\{\int\limits_e^a f(x)\mathrm{d}x+\theta(e-b)\int\limits_b^e f(x)\mathrm{d}x\right\}\right.\\
\qquad +\left(1-\theta(a-c)\right)\\
\qquad\quad \left.\left[\theta(b-c)\theta(e-c)\left\{\int\limits_c^e f(x)\mathrm{d}x+\theta(e-b)\int\limits_e^b f(x)\mathrm{d}x\right\}+\theta(c-b)\theta(e-b)\left\{\int\limits_e^b f(x)\mathrm{d}x+\theta(e-c)\int\limits_c^e f(x)\mathrm{d}x\right\}\right]\right]\\
\quad -\theta(b-d)\left[\theta(b-d)\theta(e-d)\left\{\int\limits_d^e f(x)\mathrm{d}x+\theta(e-b)\int\limits_e^b f(x)\mathrm{d}x\right\}+\theta(d-b)\theta(e-b)\left\{\int\limits_e^b f(x)\mathrm{d}x+\theta(e-d)\int\limits_d^e f(x)\mathrm{d}x\right\}\right].
\end{cases}
\tag{9}
$$

Let denote

$$
F_{2,1}(\xi_{l'})\equiv\int\limits_{\frac{2}{h_l}\left(x-\frac{h_{l'}}{2}\xi_{l'}-x_{l'}+x_{\min}-x_l\right)}^{\frac{2}{h_l}\left(x_{l-1/2}-x_l\right)} f_2(\xi_l)\mathrm{d}\xi_l,\ \ F_{2,2}(\xi_{l'})\equiv\int\limits_{\frac{2}{h_l}\left(x-\frac{h_{l'}}{2}\xi_{l'}-x_{l'}+x_{\min}-x_l\right)}^{\frac{2}{h_l}\left(x_{l+1/2}-x_l\right)} f_2(\xi_l)\mathrm{d}\xi_l. \tag{10}
$$

The double integral writes

$$
\begin{aligned}
(1)\equiv\int\limits_{\frac{2}{h_{l'}}(x_{\min}-x_{l'})}^{\frac{2}{h_{l'}}(x-x_{l'})} & f_1(\xi_{l'})F_{2,1}(\xi_{l'})\left[\theta\left(\xi_{l'}-\frac{2}{h_{l'}}(x_{l'-1/2}-x_{l'})\right)-\theta\left(\xi_{l'}-\frac{2}{h_{l'}}(x_{l'+1/2}-x_{l'})\right)\right]\\
&\theta\left(\frac{2}{h_{l'}}(x-x_{l-1/2}+x_{\min}-x_{l'})-\xi_{l'}\right)\mathrm{d}\xi_l\mathrm{d}\xi_{l'}
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
(1) = {} & \theta(x - x_{l'-1/2}) \\
& \Bigg[ - \theta(x_{\min} - x_{l'-1/2})\theta(x - x_{l-1/2}) \\
& \left\{ \int_{\frac{2}{h_{l'}}(x - x_{l-1/2} + x_{\min} - x_{l'})}^{\frac{2}{h_{l'}}(x_{\min} - x_{l'})} f_1(\xi_{l'})F_{2,1}(\xi_{l'})\mathrm{d}\xi_{l'} + \underbrace{\theta(x_{\min} - x_{l-1/2}) \int_{\frac{2}{h_{l'}}(x - x_{l'})}^{\frac{2}{h_{l'}}(x - x_{l-1/2} + x_{\min} - x_{l'})} f_1(\xi_{l'})F_{2,1}(\xi_{l'})\mathrm{d}\xi_{l'}}_{0 \text{ since } x_{l-1/2} \geq x_{\min}} \right\} \\
& + (1 - \theta(x_{\min} - x_{l'-1/2})) \\
& \Bigg[ \theta(x - x_{l'-1/2})\theta(x - x_{l-1/2} + x_{min} - x_{l'-1/2}) \\
& \left\{ \int_{\frac{2}{h_{l'}}(x_{l'-1/2} - x_{l'})}^{\frac{2}{h_{l'}}(x - x_{l-1/2} + x_{\min} - x_{l'})} f_1(\xi_{l'})F_{2,1}(\xi_{l'})\mathrm{d}\xi_{l'} + \underbrace{\theta(x_{\min} - x_{l-1/2}) \int_{\frac{2}{h_{l'}}(x - x_{l-1/2} + x_{\min} - x_{l'})}^{\frac{2}{h_{l'}}(x - x_{l'})} f_1(\xi_{l'})F_{2,1}(\xi_{l'})\mathrm{d}\xi_{l'}}_{0 \text{ since } x_{l-1/2} \geq x_{\min}} \right\} \\
& + \theta(x_{l'-1/2} - x)\theta(x_{\min} - x_{l-1/2}) \\
& \left\{ \underbrace{\int_{\frac{2}{h_{l'}}(x - x_{l-1/2} + x_{\min} - x_{l'})}^{\frac{2}{h_{l'}}(x - x_{l'})} f_1(\xi_{l'})F_{2,1}(\xi_{l'})\mathrm{d}\xi_{l'}}_{0 \text{ since } x_{l-1/2} \geq x_{\min}} \right. \\
& \left. + \theta(x - x_{l-1/2} + x_{\min} - x_{l'-1/2}) \int_{\frac{2}{h_{l'}}(x_{l'-1/2} - x_{l'})}^{\frac{2}{h_{l'}}(x - x_{l-1/2} + x_{\min} - x_{l'})} f_1(\xi_{l'})F_{2,1}(\xi_{l'})\mathrm{d}\xi_{l'} \right\} \Bigg] \Bigg] \\
& - \theta(x - x_{l'+1/2}) \\
& \Bigg[ \theta(x - x_{l'+1/2})\theta(x - x_{l-1/2} + x_{\min} - x_{l'+1/2}) \\
& \left\{ \int_{\frac{2}{h_{l'}}(x_{l'+1/2} - x_{l'})}^{\frac{2}{h_{l'}}(x - x_{l-1/2} + x_{\min} - x_{l'})} f_1(\xi_{l'})F_{2,1}(\xi_{l'})\mathrm{d}\xi_{l'} + \underbrace{\theta(x_{\min} - x_{l-1/2}) \int_{\frac{2}{h_{l'}}(x - x_{l-1/2} + x_{\min} - x_{l'})}^{\frac{2}{h_{l'}}(x - x_{l'})} f_1(\xi_{l'})F_{2,1}(\xi_{l'})\mathrm{d}\xi_{l'}}_{0 \text{ since } x_{l-1/2} \geq x_{\min}} \right\} \\
& + \theta(x_{l'+1/2} - x)\theta(x_{\min} - x_{l-1/2}) \\
& \left\{ \int_{\frac{2}{h_{l'}}(x - x_{l-1/2} + x_{\min} - x_{l'})}^{\frac{2}{h_{l'}}(x - x_{l'})} f_1(\xi_{l'})F_{2,1}(\xi_{l'})\mathrm{d}\xi_{l'} \right. \\
& \left. + \theta(x - x_{l-1/2} + x_{\min} - x_{l'+1/2}) \int_{\frac{2}{h_{l'}}(x_{l'+1/2} - x_{l'})}^{\frac{2}{h_{l'}}(x - x_{l-1/2} + x_{\min} - x_{l'})} f_1(\xi_{l'})F_{2,1}(\xi_{l'})\mathrm{d}\xi_{l'} \right\} \Bigg] .
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
(1) = {} & \theta(x - x_{l'-1/2}) \\
& \Bigg[ -\theta(x_{\min} - x_{l'-1/2})\theta(x - x_{l-1/2}) \int\limits_{\frac{2}{h_{l'}}(x - x_{l-1/2} + x_{\min} - x_{l'})}^{\frac{2}{h_{l'}}(x_{\min} - x_{l'})} f_1(\xi_{l'}) F_{2,1}(\xi_{l'}) \mathrm{d}\xi_{l'} \\
& + (1 - \theta(x_{min} - x_{l'-1/2})) \\
& \Bigg[ \theta(x - x_{l'-1/2})\theta(x - x_{l-1/2} + x_{min} - x_{l'-1/2}) \int\limits_{\frac{2}{h_{l'}}(x_{l'-1/2} - x_{l'})}^{\frac{2}{h_{l'}}(x - x_{l-1/2} + x_{\min} - x_{l'})} f_1(\xi_{l'}) F_{2,1}(\xi_{l'}) \mathrm{d}\xi_{l'} \\
& + \theta(x_{l'-1/2} - x)\theta(x_{\min} - x_{l-1/2}) \underbrace{\theta(x - x_{l-1/2} + x_{\min} - x_{l'-1/2}) \int\limits_{\frac{2}{h_{l'}}(x_{l'-1/2} - x_{l'})}^{\frac{2}{h_{l'}}(x - x_{l-1/2} + x_{\min} - x_{l'})} f_1(\xi_{l'}) F_{2,1}(\xi_{l'}) \mathrm{d}\xi_{l'}}_{0 \text{ since } x_{l-1/2} \geq x_{\min}} \Bigg] \Bigg] \\
& - \theta(x - x_{l'+1/2}) \\
& \Bigg[ \theta(x - x_{l'+1/2})\theta(x - x_{l-1/2} + x_{\min} - x_{l'+1/2}) \int\limits_{\frac{2}{h_{l'}}(x_{l'+1/2} - x_{l'})}^{\frac{2}{h_{l'}}(x - x_{l-1/2} + x_{\min} - x_{l'})} f_1(\xi_{l'}) F_{2,1}(\xi_{l'}) \mathrm{d}\xi_{l'} \\
& + \theta(x_{l'+1/2} - x)\theta(x_{\min} - x_{l-1/2}) \underbrace{\theta(x - x_{l-1/2} + x_{\min} - x_{l'+1/2}) \int\limits_{\frac{2}{h_{l'}}(x_{l'+1/2} - x_{l'})}^{\frac{2}{h_{l'}}(x - x_{l-1/2} + x_{\min} - x_{l'})} f_1(\xi_{l'}) F_{2,1}(\xi_{l'}) \mathrm{d}\xi_{l'}}_{0 \text{ since } x_{l-1/2} \geq x_{\min}} \Bigg]
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
(1) = {} & \theta(x - x_{l'-1/2})\theta(x - x_{l-1/2} + x_{min} - x_{l'-1/2}) \int\limits_{\frac{2}{h_{l'}}(x_{l'-1/2} - x_{l'})}^{\frac{2}{h_{l'}}(x - x_{l-1/2} + x_{\min} - x_{l'})} f_1(\xi_{l'}) F_{2,1}(\xi_{l'}) \mathrm{d}\xi_{l'} \\
& + \theta(x - x_{l'+1/2})\theta(x - x_{l-1/2} + x_{\min} - x_{l'+1/2}) \int\limits_{\frac{2}{h_{l'}}(x - x_{l-1/2} + x_{\min} - x_{l'})}^{\frac{2}{h_{l'}}(x_{l'+1/2} - x_{l'})} f_1(\xi_{l'}) F_{2,1}(\xi_{l'}) \mathrm{d}\xi_{l'}.
\end{aligned}
\tag{14}
$$

### 1.3 Second double integral

In Eq. 6, the rule for the second double integral with the Heaviside function is

$$
\begin{cases}
a < b, a \leq c, a < d, c < d, e < b \\
\displaystyle\int_a^b f(x) \left[\theta(x - c) - \theta(x - d)\right] \theta(e - x) \mathrm{d}x \\
\displaystyle = \theta(b - d)\theta(e - d) \int_e^d f(x) \mathrm{d}x + \theta(b - c) \left[ (\theta(a - c) - 1)\theta(b - c)\theta(e - c) \int_e^c f(x) \mathrm{d}x + \theta(a - c)\theta(e - a) \int_a^e f(x) \mathrm{d}x \right],
\end{cases}
\tag{15}
$$

The double integral writes

$$
\begin{aligned}
(2) \equiv & \int\limits_{\frac{2}{h_{l'}}(x_{\min} - x_{l'})}^{\frac{2}{h_{l'}}(x - x_{l'})} f_1(\xi_{l'}) F_{2,2}(\xi_{l'}) \left[ \theta\left( \xi_{l'} - \frac{2}{h_{l'}}(x_{l'-1/2} - x_{l'}) \right) - \theta\left( \xi_{l'} - \frac{2}{h_{l'}}(x_{l'+1/2} - x_{l'}) \right) \right] \\
& \qquad \theta\left( \frac{2}{h_{l'}}(x - x_{l+1/2} + x_{\min} - x_{l'}) - \xi_{l'} \right) \mathrm{d}\xi_l \mathrm{d}\xi_{l'}
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
(2) = {} & \theta(x - x_{l'+1/2})\theta(x - x_{l+1/2} + x_{\min} - x_{l'+1/2}) \int\limits_{\frac{2}{h_{l'}}(x - x_{l+1/2} + x_{\min} - x_{l'})}^{\frac{2}{h_{l'}}(x_{l'+1/2} - x_{l'})} f_1(\xi_{l'}) F_{2,2}(\xi_{l'}) \mathrm{d}\xi_{l'} \\
& + \theta(x - x_{l'-1/2}) \\
& \Bigg[ (\theta(x_{\min} - x_{l'-1/2}) - 1)\theta(x - x_{l'-1/2})\theta(x - x_{l+1/2} + x_{\min} - x_{l'-1/2}) \int\limits_{\frac{2}{h_{l'}}(x - x_{l+1/2} + x_{\min} - x_{l'})}^{\frac{2}{h_{l'}}(x_{l'-1/2} - x_{l'})} f_1(\xi_{l'}) F_{2,2}(\xi_{l'}) \mathrm{d}\xi_{l'} \\
& \quad + \theta(x_{\min} - x_{l'-1/2})\theta(x - x_{l+1/2} + x_{\min} - x_{\min}) \int\limits_{\frac{2}{h_{l'}}(x_{\min} - x_{l'})}^{\frac{2}{h_{l'}}(x - x_{l+1/2} + x_{\min} - x_{l'})} f_1(\xi_{l'}) F_{2,2}(\xi_{l'}) \mathrm{d}\xi_{l'} \Bigg],
\end{aligned}
\tag{17}
$$

$$
\begin{aligned}
(2) = {} & \theta(x - x_{l'-1/2})\theta(x_{\min} - x_{l'-1/2})\theta(x - x_{l+1/2} + x_{\min} - x_{l'-1/2}) \int\limits_{\frac{2}{h_{l'}}(x - x_{l+1/2} + x_{\min} - x_{l'})}^{\frac{2}{h_{l'}}(x_{l'-1/2} - x_{l'})} f_1(\xi_{l'}) F_{2,2}(\xi_{l'}) \mathrm{d}\xi_{l'} \\
& \underbrace{+ \theta(x - x_{l'-1/2})\theta(x_{\min} - x_{l'-1/2})\theta(x - x_{l+1/2}) \int\limits_{\frac{2}{h_{l'}}(x_{\min} - x_{l'})}^{\frac{2}{h_{l'}}(x - x_{l+1/2} + x_{\min} - x_{l'})} f_1(\xi_{l'}) F_{2,2}(\xi_{l'}) \mathrm{d}\xi_{l'}}_{0 \text{ since } x_{l'-1/2} \geq x_{\min}} \\
& + \theta(x - x_{l'+1/2})\theta(x - x_{l+1/2} + x_{\min} - x_{l'+1/2}) \int\limits_{\frac{2}{h_{l'}}(x - x_{l+1/2} + x_{\min} - x_{l'})}^{\frac{2}{h_{l'}}(x_{l'+1/2} - x_{l'})} f_1(\xi_{l'}) F_{2,2}(\xi_{l'}) \mathrm{d}\xi_{l'} \\
& - \theta(x - x_{l'-1/2})\theta(x - x_{l+1/2} + x_{\min} - x_{l'-1/2}) \int\limits_{\frac{2}{h_{l'}}(x - x_{l+1/2} + x_{\min} - x_{l'})}^{\frac{2}{h_{l'}}(x_{l'-1/2} - x_{l'})} f_1(\xi_{l'}) F_{2,2}(\xi_{l'}) \mathrm{d}\xi_{l'}.
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
(2) = {} & \theta(x - x_{l'+1/2})\theta(x - x_{l+1/2} + x_{\min} - x_{l'+1/2}) \int\limits_{\frac{2}{h_{l'}}(x - x_{l+1/2} + x_{\min} - x_{l'})}^{\frac{2}{h_{l'}}(x_{l'+1/2} - x_{l'})} f_1(\xi_{l'}) F_{2,2}(\xi_{l'}) \mathrm{d}\xi_{l'} \\
& + \theta(x - x_{l'-1/2})\theta(x - x_{l+1/2} + x_{\min} - x_{l'-1/2}) \int\limits_{\frac{2}{h_{l'}}(x_{l'-1/2} - x_{l'})}^{\frac{2}{h_{l'}}(x - x_{l+1/2} + x_{\min} - x_{l'})} f_1(\xi_{l'}) F_{2,2}(\xi_{l'}) \mathrm{d}\xi_{l'}.
\end{aligned}
\tag{19}
$$

Then, $T$ writes

$$
\begin{aligned}
&T(x,x_{\min},x_{\max},i',i,l',l)=\frac{h_l}{2}\frac{h_{l'}}{2}\\
&\left[\int\limits_{\frac{2}{h_l}(x_{l-1/2}-x_l)}^{\frac{2}{h_l}(x_{l+1/2}-x_l)} f_2(\xi_l)\mathrm{d}\xi_l\left(\theta(x-x_{l'-1/2})\int\limits_{\frac{2}{h_{l'}}(x_{l'-1/2}-x_{l'})}^{\frac{2}{h_{l'}}(x-x_{l'})} f_1(\xi_{l'})\mathrm{d}\xi_{l'}-\theta(x-x_{l'+1/2})\int\limits_{\frac{2}{h_{l'}}(x_{l'+1/2}-x_{l'})}^{\frac{2}{h_{l'}}(x-x_{l'})} f_1(\xi_{l'})\mathrm{d}\xi_{l'}\right)\right.\\
&\quad+\theta(x-x_{l'-1/2})\\
&\quad\left[\theta(x-x_{l-1/2}+x_{\min}-x_{l'-1/2})\int\limits_{\frac{2}{h_{l'}}(x_{l'-1/2}-x_{l'})}^{\frac{2}{h_{l'}}(x-x_{l-1/2}+x_{\min}-x_{l'})}\int\limits_{\frac{2}{h_l}(x-\frac{h_{l'}}{2}\xi_{l'}-x_{l'}+x_{\min}-x_l)}^{\frac{2}{h_l}(x_{l-1/2}-x_l)} f_1(\xi_{l'})f_2(\xi_l)\mathrm{d}\xi_l\mathrm{d}\xi_{l'}\right.\\
&\quad\left.-\theta(x_{\max}-x_{l+1/2})\theta(x-x_{l+1/2}+x_{\min}-x_{l'-1/2})\int\limits_{\frac{2}{h_{l'}}(x_{l'-1/2}-x_{l'})}^{\frac{2}{h_{l'}}(x-x_{l+1/2}+x_{\min}-x_{l'})}\int\limits_{\frac{2}{h_l}(x-\frac{h_{l'}}{2}\xi_{l'}-x_{l'}+x_{\min}-x_l)}^{\frac{2}{h_l}(x_{l+1/2}-x_l)} f_1(\xi_{l'})f_2(\xi_l)\mathrm{d}\xi_l\mathrm{d}\xi_{l'}\right]\\
&\quad+\theta(x-x_{l'+1/2})\\
&\quad\left[\theta(x-x_{l-1/2}+x_{\min}-x_{l'+1/2})\int\limits_{\frac{2}{h_{l'}}(x-x_{l-1/2}+x_{\min}-x_{l'})}^{\frac{2}{h_{l'}}(x_{l'+1/2}-x_{l'})}\int\limits_{\frac{2}{h_l}(x-\frac{h_{l'}}{2}\xi_{l'}-x_{l'}+x_{\min}-x_l)}^{\frac{2}{h_l}(x_{l-1/2}-x_l)} f_1(\xi_{l'})f_2(\xi_l)\mathrm{d}\xi_l\mathrm{d}\xi_{l'}\right.\\
&\quad\left.\left.-\theta(x_{\max}-x_{l+1/2})\theta(x-x_{l+1/2}+x_{\min}-x_{l'+1/2})\int\limits_{\frac{2}{h_{l'}}(x-x_{l+1/2}+x_{\min}-x_{l'})}^{\frac{2}{h_{l'}}(x_{l'+1/2}-x_{l'})}\int\limits_{\frac{2}{h_l}(x-\frac{h_{l'}}{2}\xi_{l'}-x_{l'}+x_{\min}-x_l)}^{\frac{2}{h_l}(x_{l+1/2}-x_l)} f_1(\xi_{l'})f_2(\xi_l)\mathrm{d}\xi_l\mathrm{d}\xi_{l'}\right]\right].
\end{aligned}
\tag{20}
$$

To reduce as much as possible the number of integrals to evaluate, we define the following terms for the scheme

$$
\begin{aligned}
&T_{\phi_i}\equiv\int\limits_{\frac{2}{h_l}(x_{l-1/2}-x_l)}^{\frac{2}{h_l}(x_{l+1/2}-x_l)} f_2(\xi_l)\mathrm{d}\xi_l,\,T_{\phi_{i'},\mathrm{mix}}\equiv\int\limits_{\frac{2}{h_{l'}}(x_{l'-1/2}-x_{l'})}^{\frac{2}{h_{l'}}(x_{l'+1/2}-x_{l'})} f_1(\xi_{l'})\mathrm{d}\xi_{l'},\,T_{\phi_{i'},\mathrm{term1}}\equiv\int\limits_{\frac{2}{h_{l'}}(x_{l'-1/2}-x_{l'})}^{\frac{2}{h_{l'}}(x-x_{l'})} f_1(\xi_{l'})\mathrm{d}\xi_{l'},\\
&T_{\phi_{i'},\phi_i,\mathrm{allmix}}\equiv\int\limits_{\frac{2}{h_{l'}}(x_{l'-1/2}-x_{l'})}^{\frac{2}{h_{l'}}(x_{l'+1/2}-x_{l'})}\int\limits_{\frac{2}{h_l}(x_{l+1/2}-x_l)}^{\frac{2}{h_l}(x_{l-1/2}-x_l)} f_1(\xi_{l'})f_2(\xi_l)\mathrm{d}\xi_l\mathrm{d}\xi_{l'},\\
&T_{\phi_{i'},\phi_i,\mathrm{mix\_P1term1\_P2term1}}\equiv\int\limits_{\frac{2}{h_{l'}}(x_{l'-1/2}-x_{l'})}^{\frac{2}{h_{l'}}(x_{l'+1/2}-x_{l'})}\int\limits_{\frac{2}{h_l}(x-\frac{h_{l'}}{2}\xi_{l'}-x_{l'}+x_{\min}-x_l)}^{\frac{2}{h_l}(x_{l-1/2}-x_l)} f_1(\xi_{l'})f_2(\xi_l)\mathrm{d}\xi_l\mathrm{d}\xi_{l'}\\
&T_{\phi_{i'},\phi_i,\mathrm{P1term1}}\equiv\int\limits_{\frac{2}{h_{l'}}(x_{l'-1/2}-x_{l'})}^{\frac{2}{h_{l'}}(x-x_{l-1/2}+x_{\min}-x_{l'})}\int\limits_{\frac{2}{h_l}(x-\frac{h_{l'}}{2}\xi_{l'}-x_{l'}+x_{\min}-x_l)}^{\frac{2}{h_l}(x_{l-1/2}-x_l)} f_1(\xi_{l'})f_2(\xi_l)\mathrm{d}\xi_l\mathrm{d}\xi_{l'}\\
&T_{\phi_{i'},\phi_i,\mathrm{P1term2}}\equiv\int\limits_{\frac{2}{h_{l'}}(x_{l'-1/2}-x_{l'})}^{\frac{2}{h_{l'}}(x-x_{l+1/2}+x_{\min}-x_{l'})}\int\limits_{\frac{2}{h_l}(x-\frac{h_{l'}}{2}\xi_{l'}-x_{l'}+x_{\min}-x_l)}^{\frac{2}{h_l}(x_{l+1/2}-x_l)} f_1(\xi_{l'})f_2(\xi_l)\mathrm{d}\xi_l\mathrm{d}\xi_{l'}.
\end{aligned}
\tag{21}
$$

### 1.4 Fortran scheme for flux term

The `Fortran` scheme to evaluate $T(x, x_{\min}, x_{\max}, i', i, l', l)$ writes with terms in Eq. 21

```
res1=0
res2=0
if ($x > x_{l'+1/2}$) then
   res1 = $T_{\phi_i} \times T_{\phi_{i'},\mathrm{mix}}$
   if ($x > x_{l+1/2} + x_{l'+1/2} - x_{\min}$) then
      res2 = $T_{\phi_{i'},\phi_i,\mathrm{allmix}}$
   else if ($x \leq x_{l+1/2} + x_{l'+1/2} - x_{\min}$ and $x > x_{l+1/2} + x_{l'-1/2} - x_{\min}$ and $x > x_{l-1/2} + x_{l'+1/2} - x_{\min}$) then
      res2 = $T_{\phi_{i'},\phi_i,\mathrm{mix\_P1term1\_P2term1}} - T_{\phi_{i'},\phi_i,\mathrm{P1term2}}$
   else if ($x \leq x_{l+1/2} + x_{l'+1/2} - x_{\min}$ and $x \leq x_{l+1/2} + x_{l'-1/2} - x_{\min}$ and $x > x_{l-1/2} + x_{l'+1/2} - x_{\min}$) then
      res2 = $T_{\phi_{i'},\phi_i,\mathrm{mix\_P1term1\_P2term1}}$
   else if ($x \leq x_{l+1/2} + x_{l'+1/2} - x_{\min}$ and $x > x_{l+1/2} + x_{l'-1/2} - x_{\min}$ and $x \leq x_{l-1/2} + x_{l'+1/2} - x_{\min}$) then
      res2 = $T_{\phi_{i'},\phi_i,\mathrm{P1term1}} - T_{\phi_{i'},\phi_i,\mathrm{P1term2}}$
   else if ($x \leq x_{l+1/2} + x_{l'+1/2} - x_{\min}$ and $x \leq x_{l+1/2} + x_{l'-1/2} - x_{\min}$ and $x \leq x_{l-1/2} + x_{l'+1/2} - x_{\min}$ and
            $x > x_{l-1/2} + x_{l'-1/2} - x_{\min}$) then
      res2 = $T_{\phi_{i'},\phi_i,\mathrm{P1term1}}$
   else
      res2 = 0
   endif
else if ($x \leq x_{l'+1/2}$ and $x > x_{l'-1/2}$) then
   res1 = $T_{\phi_i} \times T_{\phi_{i'},\mathrm{term1}}$
   if ($x > x_{l+1/2} + x_{l'-1/2} - x_{\min}$) then
      res2 = $T_{\phi_{i'},\phi_i,\mathrm{P1term1}} - T_{\phi_{i'},\phi_i,\mathrm{P1term2}}$
   else if ($x \leq x_{l+1/2} + x_{l'-1/2} - x_{\min}$ and $x > x_{l-1/2} + x_{l'-1/2} - x_{\min}$) then
      res2 = $T_{\phi_{i'},\phi_i,\mathrm{P1term1}}$
   else
      res2 = 0
else
   res1=0
   res2=0
endif

T = (res1+res2)*hl*hl'/4
```

## 2 DERIVATION OF INTEGRAL OF THE FLUX

The term with the integral of the numerical flux, that we note $\mathcal{F}^{\rm nc}_{\rm coag}$, writes

$$
\begin{cases}
\mathcal{F}^{\rm nc}_{\rm coag}[\tilde{g},j,k](t) = \displaystyle\sum_{l'=1}^{N}\sum_{i'=0}^{k}\sum_{l=1}^{N}\sum_{i=0}^{k} g_{l'}^{i'}(t)\, g_l^{i}(t)\mathcal{T}\left(x_{\min},x_{\max},j,k,i',i,l',l\right)\\
\mathcal{T}\left(x_{\min},x_{\max},j,k,i',i,l',l\right) \equiv \displaystyle\int\limits_{I_j}\int\limits_{x_{\min}}^{x}\int\limits_{x-u+x_{\min}}^{x_{\max}} \frac{\mathcal{K}(u,v)}{v}\,\partial_x\phi_k(\xi_j(x))\phi_{i'}(\xi_{l'}(u))[\theta(u-x_{l'-1/2})-\theta(u-x_{l'+1/2})]\\
\qquad\qquad \phi_i(\xi_l(v))[\theta(v-x_{l-1/2})-\theta(v-x_{l+1/2})]\mathrm{d}v\,\mathrm{d}u\,\mathrm{d}x,\\
\mathcal{T}\left(x_{\min},x_{\max},j,k,i',i,l',l\right) \equiv \displaystyle\int\limits_{I_j} T(x,x_{\min},x_{\max},i',i,l',l)\partial_x\phi_k(\xi_j(x))\mathrm{d}x
\end{cases}
\tag{22}
$$

Multiplying $T(x,x_{\min},x_{\max},i',i,l',l)$ in Eq. 20 by $\partial_x\phi_k\left(\xi_j(x)\right)$, $\mathcal{T}$ writes, with $\xi_j = \frac{2}{h_j}\left(x-x_j\right)$,

$$
\begin{aligned}
&\mathcal{T}\left(x_{\min},x_{\max},j,k,i',i,l',l\right) = \frac{h_l}{2}\frac{h_{l'}}{2}\\
&\Bigg[\int\limits_{\frac{2}{h_l}\left(x_{l-1/2}-x_l\right)}^{\frac{2}{h_l}\left(x_{l+1/2}-x_l\right)} f_2(\xi_l)\mathrm{d}\xi_l \times\\
&\Bigg(\int\limits_{\frac{2}{h_j}\left(x_{j-1/2}-x_j\right)}^{\frac{2}{h_j}\left(x_{j+1/2}-x_j\right)}\int\limits_{\frac{2}{h_{l'}}\left(x_{l'-1/2}-x_{l'}\right)}^{\frac{2}{h_{l'}}\left(\frac{h_j}{2}\xi_j+x_j-x_{l'}\right)} f_1(\xi_{l'})\partial_{\xi_j}\phi_k(\xi_j)\theta\left(\xi_j-\frac{2}{h_j}(x_{l'-1/2}-x_j)\right)\mathrm{d}\xi_j\\
&\quad-\int\limits_{\frac{2}{h_j}\left(x_{j-1/2}-x_j\right)}^{\frac{2}{h_j}\left(x_{j+1/2}-x_j\right)}\int\limits_{\frac{2}{h_{l'}}\left(x_{l'+1/2}-x_{l'}\right)}^{\frac{2}{h_{l'}}\left(\frac{h_j}{2}\xi_j+x_j-x_{l'}\right)} f_1(\xi_{l'})\partial_{\xi_j}\phi_k(\xi_j)\theta\left(\xi_j-\frac{2}{h_j}(x_{l'+1/2}-x_j)\right)\mathrm{d}\xi_{l'}\mathrm{d}\xi_j\Bigg)\\
&\quad+\int\limits_{\frac{2}{h_j}\left(x_{j-1/2}-x_j\right)}^{\frac{2}{h_j}\left(x_{j+1/2}-x_j\right)}\int\limits_{\frac{2}{h_{l'}}\left(x_{l'-1/2}-x_{l'}\right)}^{\frac{2}{h_{l'}}\left(\frac{h_j}{2}\xi_j+x_j-x_{l-1/2}+x_{\min}-x_{l'}\right)}\int\limits_{\frac{2}{h_l}\left(\frac{h_j}{2}\xi_j+x_j-\frac{h_{l'}}{2}\xi_{l'}-x_{l'}+x_{\min}-x_l\right)}^{\frac{2}{h_l}\left(x_{l-1/2}-x_l\right)} f_1(\xi_{l'})f_2(\xi_l)\partial_{\xi_j}\phi_k(\xi_j)\\
&\quad\times\theta\left(\xi_j-\frac{2}{h_j}(x_{l'-1/2}-x_j)\right)\theta\left(\xi_j-\frac{2}{h_j}(x_{l-1/2}+x_{l'-1/2}-x_{\min}-x_j)\right)\mathrm{d}\xi_l\mathrm{d}\xi_{l'}\mathrm{d}\xi_j\\
&\quad-\int\limits_{\frac{2}{h_j}\left(x_{j-1/2}-x_j\right)}^{\frac{2}{h_j}\left(x_{j+1/2}-x_j\right)}\int\limits_{\frac{2}{h_{l'}}\left(x_{l'-1/2}-x_{l'}\right)}^{\frac{2}{h_{l'}}\left(\frac{h_j}{2}\xi_j+x_j-x_{l+1/2}+x_{\min}-x_{l'}\right)}\int\limits_{\frac{2}{h_l}\left(\frac{h_j}{2}\xi_j+x_j-\frac{h_{l'}}{2}\xi_{l'}-x_{l'}+x_{\min}-x_l\right)}^{\frac{2}{h_l}\left(x_{l+1/2}-x_l\right)} f_1(\xi_{l'})f_2(\xi_l)\partial_{\xi_j}\phi_k(\xi_j)\\
&\quad\times\theta\left(\xi_j-\frac{2}{h_j}(x_{l'-1/2}-x_j)\right)\theta\left(\xi_j-\frac{2}{h_j}(x_{l+1/2}+x_{l'-1/2}-x_{\min}-x_j)\right)\mathrm{d}\xi_l\mathrm{d}\xi_{l'}\mathrm{d}\xi_j\\
&\quad+\int\limits_{\frac{2}{h_j}\left(x_{j-1/2}-x_j\right)}^{\frac{2}{h_j}\left(x_{j+1/2}-x_j\right)}\int\limits_{\frac{2}{h_{l'}}\left(\frac{h_j}{2}\xi_j+x_j-x_{l-1/2}+x_{\min}-x_{l'}\right)}^{\frac{2}{h_{l'}}\left(x_{l'+1/2}-x_{l'}\right)}\int\limits_{\frac{2}{h_l}\left(\frac{h_j}{2}\xi_j+x_j-\frac{h_{l'}}{2}\xi_{l'}-x_{l'}+x_{\min}-x_l\right)}^{\frac{2}{h_l}\left(x_{l-1/2}-x_l\right)} f_1(\xi_{l'})f_2(\xi_l)\partial_{\xi_j}\phi_k(\xi_j)\\
&\quad\times\theta\left(\xi_j-\frac{2}{h_j}(x_{l'+1/2}-x_j)\right)\theta\left(\xi_j-\frac{2}{h_j}(x_{l-1/2}+x_{l'+1/2}-x_{\min}-x_j)\right)\mathrm{d}\xi_l\mathrm{d}\xi_{l'}\mathrm{d}\xi_j\\
&\quad-\int\limits_{\frac{2}{h_j}\left(x_{j-1/2}-x_j\right)}^{\frac{2}{h_j}\left(x_{j+1/2}-x_j\right)}\int\limits_{\frac{2}{h_{l'}}\left(\frac{h_j}{2}\xi_j+x_j-x_{l+1/2}+x_{\min}-x_{l'}\right)}^{\frac{2}{h_{l'}}\left(x_{l'+1/2}-x_{l'}\right)}\int\limits_{\frac{2}{h_l}\left(\frac{h_j}{2}\xi_j+x_j-\frac{h_{l'}}{2}\xi_{l'}-x_{l'}+x_{\min}-x_l\right)}^{\frac{2}{h_l}\left(x_{l+1/2}-x_l\right)} f_1(\xi_{l'})f_2(\xi_l)\partial_{\xi_j}\phi_k(\xi_j)\\
&\quad\times\theta\left(\xi_j-\frac{2}{h_j}(x_{l'+1/2}-x_j)\right)\theta\left(\xi_j-\frac{2}{h_j}\left(x_{l+1/2}+x_{l'+1/2}-x_{\min}-x_j\right)\right)\mathrm{d}\xi_l\mathrm{d}\xi_{l'}\mathrm{d}\xi_j\Bigg].
\end{aligned}
\tag{23}
$$

### 2.1 Derivation of double integrals

The rule for the double integrals is

$$\begin{cases} a < b \\ \int\limits_a^b f(x)\theta(x-c)\mathrm{d}x = \theta(b-c)\left(\int\limits_c^b f(x)\mathrm{d}x + \theta(a-c)\int\limits_a^c f(x)\mathrm{d}x\right). \end{cases} \tag{24}$$

Then we obtain

$$\begin{aligned}
&\bullet \int\limits_{\frac{2}{h_j}(x_{j-1/2}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)} \int\limits_{\frac{2}{h_{l'}}(x_{l'-1/2}-x_{l'})}^{\frac{2}{h_{l'}}(x-x_{l'})} f_1(\xi_{l'})\partial_{\xi_j}\phi_k(\xi_j)\theta\left(\xi_j - \frac{2}{h_j}(x_{l'-1/2}-x_j)\right)\mathrm{d}\xi_j \\
&= \theta(x_{j+1/2}-x_{l'-1/2}) \\
&\left( \int\limits_{\frac{2}{h_j}(x_{l'-1/2}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)} \int\limits_{\frac{2}{h_{l'}}(x_{l'-1/2}-x_{l'})}^{\frac{2}{h_{l'}}(x-x_{l'})} f_1(\xi_{l'})\partial_{\xi_j}\phi_k(\xi_j)\mathrm{d}\xi_j + \theta(x_{j-1/2}-x_{l'-1/2}) \int\limits_{\frac{2}{h_j}(x_{j-1/2}-x_j)}^{\frac{2}{h_j}(x_{l'-1/2}-x_j)} \int\limits_{\frac{2}{h_{l'}}(x_{l'-1/2}-x_{l'})}^{\frac{2}{h_{l'}}(x-x_{l'})} f_1(\xi_{l'})\partial_{\xi_j}\phi_k(\xi_j)\mathrm{d}\xi_j \right), \\
&\bullet \int\limits_{\frac{2}{h_j}(x_{j-1/2}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)} \int\limits_{\frac{2}{h_{l'}}(x_{l'+1/2}-x_{l'})}^{\frac{2}{h_{l'}}(x-x_{l'})} f_1(\xi_{l'})\partial_{\xi_j}\phi_k(\xi_j)\theta\left(\xi_j - \frac{2}{h_j}(x_{l'+1/2}-x_j)\right)\mathrm{d}\xi_j \\
&= \theta(x_{j+1/2}-x_{l'+1/2}) \\
&\left( \int\limits_{\frac{2}{h_j}(x_{l'+1/2}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)} \int\limits_{\frac{2}{h_{l'}}(x_{l'+1/2}-x_{l'})}^{\frac{2}{h_{l'}}(x-x_{l'})} f_1(\xi_{l'})\partial_{\xi_j}\phi_k(\xi_j)\mathrm{d}\xi_j + \theta(x_{j-1/2}-x_{l'+1/2}) \int\limits_{\frac{2}{h_j}(x_{j-1/2}-x_j)}^{\frac{2}{h_j}(x_{l'+1/2}-x_j)} \int\limits_{\frac{2}{h_{l'}}(x_{l'+1/2}-x_{l'})}^{\frac{2}{h_{l'}}(x-x_{l'})} f_1(\xi_{l'})\partial_{\xi_j}\phi_k(\xi_j)\mathrm{d}\xi_j \right).
\end{aligned} \tag{25}$$

### 2.2 Derivation of the triple integrals

The rule for the first and third triple integrals is

$$\begin{cases} a < b, c \leq d \\ \int_a^b f(x)\theta(x-c)\theta(x-d)\mathrm{d}x = \theta(b-c)\left[\theta(a-c)\theta(b-d)\left\{\int\limits_d^b f(x)\mathrm{d}x + \theta(a-d)\int\limits_a^d f(x)\mathrm{d}x\right\}\right. \\ + (1-\theta(a-c))\left[\theta(c-b)\theta(c-d)\left\{\int\limits_c^d f(x)\mathrm{d}x + \theta(b-d)\int\limits_d^b f(x)\mathrm{d}x\right\}\right. \\ \qquad \left.\left. +\theta(b-c)\theta(b-d)\left\{\int\limits_d^b f(x)\mathrm{d}x + \theta(c-d)\int\limits_c^d f(x)\mathrm{d}x\right\}\right]\right] \end{cases} \tag{26}$$

Let denote

$$\begin{aligned}
F_1(\xi_j) &\equiv \int\limits_{\frac{2}{h_{l'}}\left(x_{l'-1/2}-x_{l'}\right)}^{\frac{2}{h_{l'}}\left(\frac{h_j}{2}\xi_j + x_j - x_{l-1/2} + x_{\min} - x_{l'}\right)} \int\limits_{\frac{2}{h_l}\left(\frac{h_j}{2}\xi_j + x_j - \frac{h_{l'}}{2}\xi_{l'} - x_{l'} + x_{\min} - x_l\right)}^{\frac{2}{h_l}\left(x_{l-1/2}-x_l\right)} f_1(\xi_{l'})f_2(\xi_l)\partial_{\xi_j}\phi_k(\xi_j)\mathrm{d}\xi_l\mathrm{d}\xi_{l'}, \\
F_3(\xi_j) &\equiv \int\limits_{\frac{2}{h_{l'}}\left(\frac{h_j}{2}\xi_j + x_j - x_{l-1/2} + x_{\min} - x_{l'}\right)}^{\frac{2}{h_{l'}}\left(x_{l'+1/2}-x_{l'}\right)} \int\limits_{\frac{2}{h_l}\left(\frac{h_j}{2}\xi_j + x_j - \frac{h_{l'}}{2}\xi_{l'} - x_{l'} + x_{\min} - x_l\right)}^{\frac{2}{h_l}\left(x_{l-1/2}-x_l\right)} f_1(\xi_{l'})f_2(\xi_l)\partial_{\xi_j}\phi_k(\xi_j)\mathrm{d}\xi_l\mathrm{d}\xi_{l'}.
\end{aligned} \tag{27}$$

The first triple integral writes

$$(1) \equiv \int\limits_{\frac{2}{h_j}(x_{j-1/2}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)} F_1(\xi_j)\theta\left(\xi_j - \frac{2}{h_j}(x_{l'-1/2}-x_j)\right)\theta\left(\xi_j - \frac{2}{h_j}(x_{l-1/2}+x_{l'-1/2}-x_{\min}-x_j)\right)\mathrm{d}\xi_j \tag{28}$$

$$\begin{aligned}
(1) = {} & \theta(x_{j+1/2}-x_{l'-1/2}) \\
& \Bigg[\theta(x_{j-1/2}-x_{l'-1/2})\theta(x_{j+1/2}+x_{\min}-x_{l-1/2}-x_{l'-1/2}) \\
& \Bigg\{\int\limits_{\frac{2}{h_j}(x_{l-1/2}+x_{l'-1/2}-x_{\min}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)} F_1(\xi_j)\mathrm{d}\xi_j + \theta(x_{j-1/2}+x_{\min}-x_{l-1/2}-x_{l'-1/2}) \int\limits_{\frac{2}{h_j}(x_{j-1/2}-x_j)}^{\frac{2}{h_j}(x_{l-1/2}+x_{l'-1/2}-x_{\min}-x_j)} F_1(\xi_j)\mathrm{d}\xi_j\Bigg\} \\
& + (1-\theta(x_{j-1/2}-x_{l'-1/2})) \\
& \Bigg[\theta(x_{l'-1/2}-x_{j+1/2})\theta(x_{l'-1/2}+x_{\min}-x_{l-1/2}-x_{l'-1/2}) \\
& \Bigg\{\int\limits_{\frac{2}{h_j}(x_{l'-1/2}-x_j)}^{\frac{2}{h_j}(x_{l-1/2}+x_{l'-1/2}-x_{\min}-x_j)} F_1(\xi_j)\mathrm{d}\xi_j + \theta(x_{j+1/2}+x_{\min}-x_{l-1/2}-x_{l'-1/2}) \int\limits_{\frac{2}{h_j}(x_{l-1/2}+x_{l'-1/2}-x_{\min}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)} F_1(\xi_j)\mathrm{d}\xi_j\Bigg\} \\
& + \theta(x_{j+1/2}-x_{l'-1/2})\theta(x_{j+1/2}+x_{\min}-x_{l-1/2}-x_{l'-1/2}) \\
& \Bigg\{\int\limits_{\frac{2}{h_j}(x_{l-1/2}+x_{l'-1/2}-x_{\min}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)} F_1(\xi_j)\mathrm{d}\xi_j + \underbrace{\theta(x_{l'-1/2}+x_{\min}-x_{l-1/2}-x_{l'-1/2}) \int\limits_{\frac{2}{h_j}(x_{l'-1/2}-x_j)}^{\frac{2}{h_j}(x_{l-1/2}+x_{l'-1/2}-x_{\min}-x_j)} F_1(\xi_j)\mathrm{d}\xi_j}_{\text{0 since } x_{l-1/2}\geq x_{\min}}\Bigg\}\Bigg]\Bigg]
\end{aligned} \tag{29}$$

$$\begin{aligned}
(1) = {} & \theta(x_{j+1/2}-x_{l'-1/2}) \\
& \Bigg[\theta(x_{j-1/2}-x_{l'-1/2})\theta(x_{j+1/2}+x_{\min}-x_{l-1/2}-x_{l'-1/2}) \\
& \Bigg\{\int\limits_{\frac{2}{h_j}(x_{l-1/2}+x_{l'-1/2}-x_{\min}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)} F_1(\xi_j)\mathrm{d}\xi_j + \theta(x_{j-1/2}+x_{\min}-x_{l-1/2}-x_{l'-1/2}) \int\limits_{\frac{2}{h_j}(x_{j-1/2}-x_j)}^{\frac{2}{h_j}(x_{l-1/2}+x_{l'-1/2}-x_{\min}-x_j)} F_1(\xi_j)\mathrm{d}\xi_j\Bigg\} \\
& + (1-\theta(x_{j-1/2}-x_{l'-1/2})) \\
& \Bigg[\theta(x_{l'-1/2}-x_{j+1/2})\theta(x_{\min}-x_{l-1/2}) \int\limits_{\frac{2}{h_j}(x_{l'-1/2}-x_j)}^{\frac{2}{h_j}(x_{l-1/2}+x_{l'-1/2}-x_{\min}-x_j)} F_1(\xi_j)\mathrm{d}\xi_j \\
& + \underbrace{\theta(x_{l'-1/2}-x_{j+1/2})\theta(x_{\min}-x_{l-1/2})\,\theta(x_{j+1/2}+x_{\min}-x_{l-1/2}-x_{l'-1/2}) \int\limits_{\frac{2}{h_j}(x_{l-1/2}+x_{l'-1/2}-x_{\min}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)} F_1(\xi_j)\mathrm{d}\xi_j}_{\text{0 since } x_{l-1/2}\geq x_{\min}} \\
& + \theta(x_{j+1/2}-x_{l'-1/2})\theta(x_{j+1/2}+x_{\min}-x_{l-1/2}-x_{l'-1/2}) \int\limits_{\frac{2}{h_j}(x_{l-1/2}+x_{l'-1/2}-x_{\min}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)} F_1(\xi_j)\mathrm{d}\xi_j\Bigg]\Bigg]
\end{aligned} \tag{30}$$

$$
\begin{aligned}
(1) =& \cancel{\theta(x_{j+1/2} - x_{l'-1/2})\theta(x_{j-1/2} - x_{l'-1/2})\theta(x_{j+1/2} + x_{\min} - x_{l-1/2} - x_{l'-1/2}) \int\limits_{\frac{2}{h_j}(x_{l-1/2}+x_{l'-1/2}-x_{\min}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)} F_1(\xi_j)\mathrm{d}\xi_j} \\
&+ \theta(x_{j+1/2} - x_{l'-1/2})\theta(x_{j-1/2} - x_{l'-1/2})\theta(x_{j+1/2} + x_{\min} - x_{l-1/2} - x_{l'-1/2}) \\
&\quad \theta(x_{j-1/2} + x_{\min} - x_{l-1/2} - x_{l'-1/2}) \int\limits_{\frac{2}{h_j}(x_{j-1/2}-x_j)}^{\frac{2}{h_j}(x_{l-1/2}+x_{l'-1/2}-x_{\min}-x_j)} F_1(\xi_j)\mathrm{d}\xi_j \\
&- \cancel{\theta(x_{j-1/2} - x_{l'-1/2})\theta(x_{j+1/2} - x_{l'-1/2})^2\theta(x_{j+1/2} + x_{\min} - x_{l-1/2} - x_{l'-1/2}) \int\limits_{\frac{2}{h_j}(x_{l-1/2}+x_{l'-1/2}-x_{\min}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)} F_1(\xi_j)\mathrm{d}\xi_j} \\
&+ \theta(x_{j+1/2} - x_{l'-1/2})^2\theta(x_{j+1/2} + x_{\min} - x_{l-1/2} - x_{l'-1/2}) \int\limits_{\frac{2}{h_j}(x_{l-1/2}+x_{l'-1/2}-x_{\min}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)} F_1(\xi_j)\mathrm{d}\xi_j
\end{aligned}
\tag{31}
$$

$$
\begin{aligned}
(1) =& \ \theta(x_{j+1/2} - x_{l'-1/2})\theta(x_{j+1/2} + x_{\min} - x_{l-1/2} - x_{l'-1/2}) \\
&\left[ \int\limits_{\frac{2}{h_j}(x_{l-1/2}+x_{l'-1/2}-x_{\min}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)} F_1(\xi_j)\mathrm{d}\xi_j \right. \\
&\left. +\theta(x_{j-1/2} - x_{l'-1/2})\theta(x_{j-1/2} + x_{\min} - x_{l-1/2} - x_{l'-1/2}) \int\limits_{\frac{2}{h_j}(x_{j-1/2}-x_j)}^{\frac{2}{h_j}(x_{l-1/2}+x_{l'-1/2}-x_{\min}-x_j)} F_1(\xi_j)\mathrm{d}\xi_j \right].
\end{aligned}
\tag{32}
$$

The third triple integral writes, with the similar simplifications,

$$
(3) \equiv \int\limits_{\frac{2}{h_j}(x_{j-1/2}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)} F_3(\xi_j)\theta\left(\xi_j - \frac{2}{h_j}(x_{l'+1/2} - x_j)\right)\theta\left(\xi_j - \frac{2}{h_j}(x_{l-1/2} + x_{l'+1/2} - x_{\min} - x_j)\right)\mathrm{d}\xi_j
\tag{33}
$$

$$
\begin{aligned}
(3) =& \ \theta(x_{j+1/2} - x_{l'+1/2})\theta(x_{j+1/2} + x_{\min} - x_{l-1/2} - x_{l'+1/2}) \\
&\left[ \int\limits_{\frac{2}{h_j}(x_{l-1/2}+x_{l'+1/2}-x_{\min}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)} F_3(\xi_j)\mathrm{d}\xi_j \right. \\
&\left. +\theta(x_{j-1/2} - x_{l'+1/2})\theta(x_{j-1/2} + x_{\min} - x_{l-1/2} - x_{l'+1/2}) \int\limits_{\frac{2}{h_j}(x_{j-1/2}-x_j)}^{\frac{2}{h_j}(x_{l-1/2}+x_{l'+1/2}-x_{\min}-x_j)} F_3(\xi_j)\mathrm{d}\xi_j \right].
\end{aligned}
\tag{34}
$$

The rule for the second and fourth triple integrals is

$$
\begin{cases}
a < b, c < d \\
\int\limits_a^b f(x)\theta(x-c)\theta(x-d)\mathrm{d}x = \\
\theta(b-c)\theta(b-d)\left[(1-\theta(a-c))\theta(b-c)\int\limits_d^b f(x)\mathrm{d}x + \theta(a-c)\left(\int\limits_d^b f(x)\mathrm{d}x + \theta(a-d)\int\limits_a^d f(x)\mathrm{d}x\right)\right] \\
= \theta(b-d)\left[\int\limits_d^b f(x)\mathrm{d}x \cancel{-\theta(a-c)\theta^2(b-c)\int\limits_d^b f(x)\mathrm{d}x + \theta(b-c)\theta(a-c)\int\limits_d^b f(x)\mathrm{d}x} + \theta(b-c)\theta(a-c)\theta(a-d)\int\limits_a^d f(x)\mathrm{d}x\right] \\
= \theta(b-c)\theta(b-d)\left[\int\limits_d^b f(x)\mathrm{d}x + \theta(a-c)\theta(a-d)\int\limits_a^d f(x)\mathrm{d}x\right].
\end{cases}
\tag{35}
$$

Let denote

$$
\begin{aligned}
F_2(\xi_j) &\equiv \int\limits_{\frac{2}{h_{l'}}\left(x_{l'-1/2}-x_{l'}\right)}^{\frac{2}{h_{l'}}\left(\frac{h_j}{2}\xi_j+x_j-x_{l+1/2}+x_{\min}-x_{l'}\right)} \int\limits_{\frac{2}{h_l}\left(\frac{h_j}{2}\xi_j+x_j-\frac{h_{l'}}{2}\xi_{l'}-x_{l'}+x_{\min}-x_l\right)}^{\frac{2}{h_l}\left(x_{l+1/2}-x_l\right)} f_1(\xi_{l'})f_2(\xi_l)\partial_{\xi_j}\phi_k(\xi_j)\mathrm{d}\xi_l\mathrm{d}\xi_{l'}, \\
F_4(\xi_j) &\equiv \int\limits_{\frac{2}{h_{l'}}\left(\frac{h_j}{2}\xi_j+x_j-x_{l+1/2}+x_{\min}-x_{l'}\right)}^{\frac{2}{h_{l'}}\left(x_{l'+1/2}-x_{l'}\right)} \int\limits_{\frac{2}{h_l}\left(\frac{h_j}{2}\xi_j+x_j-\frac{h_{l'}}{2}\xi_{l'}-x_{l'}+x_{\min}-x_l\right)}^{\frac{2}{h_l}\left(x_{l+1/2}-x_l\right)} f_1(\xi_{l'})f_2(\xi_l)\partial_{\xi_j}\phi_k(\xi_j)\mathrm{d}\xi_l\mathrm{d}\xi_{l'}.
\end{aligned}
\tag{36}
$$

The second triple integral writes

$$
(2) \equiv \int\limits_{\frac{2}{h_j}(x_{j-1/2}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)} F_2(\xi_j)\,\theta\left(\xi_j-\frac{2}{h_j}(x_{l'-1/2}-x_j)\right)\theta\left(\xi_j-\frac{2}{h_j}(x_{l+1/2}+x_{l'-1/2}-x_{\min}-x_j)\right)\mathrm{d}\xi_l\mathrm{d}\xi_{l'}\mathrm{d}\xi_j.
\tag{37}
$$

$$
\begin{aligned}
(2) = {}& \theta(x_{j+1/2}-x_{l'-1/2})\theta(x_{j+1/2}-x_{l+1/2}-x_{l'-1/2}+x_{\min}) \\
& \left[\int\limits_{\frac{2}{h_j}(x_{l+1/2}+x_{l'-1/2}-x_{\min}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)} F_2(\xi_j)\,\mathrm{d}\xi_j \right. \\
& \left. +\theta(x_{j-1/2}-x_{l'-1/2})\theta(x_{j-1/2}-x_{l+1/2}-x_{l'-1/2}+x_{\min}) \int\limits_{\frac{2}{h_j}(x_{j-1/2}-x_j)}^{\frac{2}{h_j}(x_{l+1/2}+x_{l'-1/2}-x_{\min}-x_j)} F_2(\xi_j)\,\mathrm{d}\xi_j \right].
\end{aligned}
\tag{38}
$$

The fourth triple integral writes

$$
(4) \equiv \int\limits_{\frac{2}{h_j}(x_{j-1/2}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)} F_4(\xi_j)\,\theta\left(\xi_j-\frac{2}{h_j}(x_{l'+1/2}-x_j)\right)\theta\left(\xi_j-\frac{2}{h_j}(x_{l+1/2}+x_{l'+1/2}-x_{\min}-x_j)\right)\mathrm{d}\xi_l\mathrm{d}\xi_{l'}\mathrm{d}\xi_j.
\tag{39}
$$

$$
\begin{aligned}
(4) = {}& \theta(x_{j+1/2}-x_{l'+1/2})\theta(x_{j+1/2}-x_{l+1/2}-x_{l'+1/2}+x_{\min}) \\
& \left[\int\limits_{\frac{2}{h_j}(x_{l+1/2}+x_{l'+1/2}-x_{\min}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)} F_4(\xi_j)\,\mathrm{d}\xi_j \right. \\
& \left. +\theta(x_{j-1/2}-x_{l'+1/2})\theta(x_{j-1/2}-x_{l+1/2}-x_{l'+1/2}+x_{\min}) \int\limits_{\frac{2}{h_j}(x_{j-1/2}-x_j)}^{\frac{2}{h_j}(x_{l+1/2}+x_{l'+1/2}-x_{\min}-x_j)} F_4(\xi_j)\,\mathrm{d}\xi_j \right].
\end{aligned}
\tag{40}
$$

Then $\mathcal{T}$ writes

$$
\begin{aligned}
&\mathcal{T}\left(x_{\min}, x_{\max}, j, k, i', i, l', l\right) = \frac{h_l}{2}\frac{h_{l'}}{2}\times \\
&\left[\int\limits_{\frac{2}{h_l}\left(x_{l-1/2}-x_l\right)}^{\frac{2}{h_l}\left(x_{l+1/2}-x_l\right)} f_2(\xi_l)\mathrm{d}\xi_l \times\right. \\
&\quad\left\{\theta(x_{j+1/2}-x_{l'-1/2})\right. \\
&\quad\left(\int\limits_{\frac{2}{h_j}(x_{l'-1/2}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)} \int\limits_{\frac{2}{h_{l'}}(x_{l'-1/2}-x_{l'})}^{\frac{2}{h_{l'}}(\frac{h_j}{2}\xi_j+x_j-x_{l'})} f_1(\xi_{l'})\partial_{\xi_j}\phi_k(\xi_j)\mathrm{d}\xi_j + \theta(x_{j-1/2}-x_{l'-1/2}) \int\limits_{\frac{2}{h_j}(x_{j-1/2}-x_j)}^{\frac{2}{h_j}(x_{l'-1/2}-x_j)} \int\limits_{\frac{2}{h_{l'}}(x_{l'-1/2}-x_{l'})}^{\frac{2}{h_{l'}}(\frac{h_j}{2}\xi_j+x_j-x_{l'})} f_1(\xi_{l'})\partial_{\xi_j}\phi_k(\xi_j)\mathrm{d}\xi_j\right) \\
&\quad - \theta(x_{j+1/2}-x_{l'+1/2}) \\
&\quad\left.\left(\int\limits_{\frac{2}{h_j}(x_{l'+1/2}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)} \int\limits_{\frac{2}{h_{l'}}(x_{l'+1/2}-x_{l'})}^{\frac{2}{h_{l'}}(\frac{h_j}{2}\xi_j+x_j-x_{l'})} f_1(\xi_{l'})\partial_{\xi_j}\phi_k(\xi_j)\mathrm{d}\xi_j + \theta(x_{j-1/2}-x_{l'+1/2}) \int\limits_{\frac{2}{h_j}(x_{j-1/2}-x_j)}^{\frac{2}{h_j}(x_{l'+1/2}-x_j)} \int\limits_{\frac{2}{h_{l'}}(x_{l'+1/2}-x_{l'})}^{\frac{2}{h_{l'}}(\frac{h_j}{2}\xi_j+x_j-x_{l'})} f_1(\xi_{l'})\partial_{\xi_j}\phi_k(\xi_j)\mathrm{d}\xi_j\right)\right\} \\
&\quad + \theta(x_{j+1/2}-x_{l'-1/2})\theta(x_{j+1/2}+x_{\min}-x_{l-1/2}-x_{l'-1/2}) \\
&\quad\left[\int\limits_{\frac{2}{h_j}(x_{l-1/2}+x_{l'-1/2}-x_{\min}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)} F_1(\xi_j)\mathrm{d}\xi_j\right. \\
&\quad\quad\left. +\theta(x_{j-1/2}-x_{l'-1/2})\theta(x_{j-1/2}+x_{\min}-x_{l-1/2}-x_{l'-1/2}) \int\limits_{\frac{2}{h_j}(x_{j-1/2}-x_j)}^{\frac{2}{h_j}(x_{l-1/2}+x_{l'-1/2}-x_{\min}-x_j)} F_1(\xi_j)\mathrm{d}\xi_j\right] \\
&\quad + \theta(x_{j+1/2}-x_{l'+1/2})\theta(x_{j+1/2}+x_{\min}-x_{l-1/2}-x_{l'+1/2}) \\
&\quad\left[\int\limits_{\frac{2}{h_j}(x_{l-1/2}+x_{l'+1/2}-x_{\min}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)} F_3(\xi_j)\mathrm{d}\xi_j\right. \\
&\quad\quad\left. +\theta(x_{j-1/2}-x_{l'+1/2})\theta(x_{j-1/2}+x_{\min}-x_{l-1/2}-x_{l'+1/2}) \int\limits_{\frac{2}{h_j}(x_{j-1/2}-x_j)}^{\frac{2}{h_j}(x_{l-1/2}+x_{l'+1/2}-x_{\min}-x_j)} F_3(\xi_j)\mathrm{d}\xi_j\right]
\end{aligned}
\tag{41}
$$

$$
\begin{aligned}
&-\theta(x_{j+1/2}-x_{l'-1/2})\theta(x_{j+1/2}-x_{l+1/2}-x_{l'-1/2}+x_{\min})\\
&\left[\int\limits_{\frac{2}{h_j}(x_{l+1/2}+x_{l'-1/2}-x_{\min}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)} F_2\left(\xi_j\right)\mathrm{d}\xi_j\right.\\
&\left.+\theta(x_{j-1/2}-x_{l'-1/2})\theta(x_{j-1/2}-x_{l+1/2}-x_{l'-1/2}+x_{\min})\int\limits_{\frac{2}{h_j}(x_{j-1/2}-x_j)}^{\frac{2}{h_j}(x_{l+1/2}+x_{l'-1/2}-x_{\min}-x_j)} F_2\left(\xi_j\right)\mathrm{d}\xi_j\right]\\
&-\theta(x_{j+1/2}-x_{l'+1/2})\theta(x_{j+1/2}-x_{l+1/2}-x_{l'+1/2}+x_{\min})\\
&\left[\int\limits_{\frac{2}{h_j}(x_{l+1/2}+x_{l'+1/2}-x_{\min}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)} F_4\left(\xi_j\right)\mathrm{d}\xi_j\right.\\
&\left.\left.+\theta(x_{j-1/2}-x_{l'+1/2})\theta(x_{j-1/2}-x_{l+1/2}-x_{l'+1/2}+x_{\min})\int\limits_{\frac{2}{h_j}(x_{j-1/2}-x_j)}^{\frac{2}{h_j}(x_{l+1/2}+x_{l'+1/2}-x_{\min}-x_j)} F_4\left(\xi_j\right)\mathrm{d}\xi_j\right]\right].
\end{aligned}
$$

To reduce as much as possible the number of integrals to evaluate, we define the following term for the scheme

$$
\begin{aligned}
&\mathcal{T}_{\phi_i}\equiv\int\limits_{\frac{2}{h_l}(x_{l-1/2}-x_l)}^{\frac{2}{h_l}(x_{l+1/2}-x_l)} f_2(\xi_l)\mathrm{d}\xi_l,\ \mathcal{T}_{\phi_{i'},\mathrm{allmix}}\equiv\int\limits_{\frac{2}{h_j}(x_{j-1/2}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)}\int\limits_{\frac{2}{h_{l'}}(x_{l'-1/2}-x_{l'})}^{\frac{2}{h_{l'}}(x_{l'+1/2}-x_{l'})} f_1(\xi_{l'})\partial_{\xi_j}\phi_k(\xi_j)\mathrm{d}\xi_{l'}\mathrm{d}\xi_j,\\
&\mathcal{T}_{\phi_{i'},\mathrm{mixP1}}\equiv\int\limits_{\frac{2}{h_j}(x_{j-1/2}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)}\int\limits_{\frac{2}{h_{l'}}(x_{l'-1/2}-x_{l'})}^{\frac{2}{h_{l'}}(\frac{h_j}{2}\xi_j+x_j-x_{l'})} f_1(\xi_{l'})\partial_{\xi_j}\phi_k(\xi_j)\mathrm{d}\xi_{l'}\mathrm{d}\xi_j,\\
&\mathcal{T}_{\phi_{i'},\mathrm{P1term1}}\equiv\int\limits_{\frac{2}{h_j}(x_{l'-1/2}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)}\int\limits_{\frac{2}{h_{l'}}(x_{l'-1/2}-x_{l'})}^{\frac{2}{h_{l'}}(\frac{h_j}{2}\xi_j+x_j-x_{l'})} f_1(\xi_{l'})\partial_{\xi_j}\phi_k(\xi_j)\mathrm{d}\xi_{l'}\mathrm{d}\xi_j,\\
&\mathcal{T}_{\phi_{i'},\mathrm{P2term1}}\equiv\int\limits_{\frac{2}{h_j}(x_{l'+1/2}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)}\int\limits_{\frac{2}{h_{l'}}(x_{l'+1/2}-x_{l'})}^{\frac{2}{h_{l'}}(\frac{h_j}{2}\xi_j+x_j-x_{l'})} f_1(\xi_{l'})\partial_{\xi_j}\phi_k(\xi_j)\mathrm{d}\xi_{l'}\mathrm{d}\xi_j,\\
&\mathcal{T}_{\phi_{i'},\phi_i,\mathrm{allmix}}\equiv\int\limits_{\frac{2}{h_j}(x_{j-1/2}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)}\int\limits_{\frac{2}{h_{l'}}(x_{l'-1/2}-x_{l'})}^{\frac{2}{h_{l'}}(x_{l'+1/2}-x_{l'})}\int\limits_{\frac{2}{h_l}(x_{l+1/2}-x_l)}^{\frac{2}{h_l}(x_{l-1/2}-x_l)} f_1(\xi_{l'})f_2(\xi_l)\partial_{\xi_j}\phi_k(\xi_j)\mathrm{d}\xi_l\mathrm{d}\xi_{l'}\mathrm{d}\xi_j\\
&\mathcal{T}_{\phi_{i'},\phi_i,\mathrm{mixterm1term3}}\equiv\int\limits_{\frac{2}{h_j}(x_{j-1/2}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)}\int\limits_{\frac{2}{h_{l'}}(x_{l'-1/2}-x_{l'})}^{\frac{2}{h_{l'}}(x_{l'+1/2}-x_{l'})}\int\limits_{\frac{2}{h_l}(\frac{h_j}{2}\xi_j+x_j-\frac{h_{l'}}{2}\xi_{l'}-x_{l'}+x_{\min}-x_l)}^{\frac{2}{h_l}(x_{l-1/2}-x_l)} f_1(\xi_{l'})f_2(\xi_l)\partial_{\xi_j}\phi_k(\xi_j)\mathrm{d}\xi_l\mathrm{d}\xi_{l'}\mathrm{d}\xi_j\\
&\mathcal{T}_{\phi_{i'},\phi_i,\mathrm{mixterm1}}\equiv\int\limits_{\frac{2}{h_j}(x_{j-1/2}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)}\int\limits_{\frac{2}{h_{l'}}(x_{l'-1/2}-x_{l'})}^{\frac{2}{h_{l'}}(\frac{h_j}{2}\xi_j+x_j-x_{l-1/2}+x_{\min}-x_{l'})}\int\limits_{\frac{2}{h_l}(\frac{h_j}{2}\xi_j+x_j-\frac{h_{l'}}{2}\xi_{l'}-x_{l'}+x_{\min}-x_l)}^{\frac{2}{h_l}(x_{l-1/2}-x_l)} f_1(\xi_{l'})f_2(\xi_l)\partial_{\xi_j}\phi_k(\xi_j)\mathrm{d}\xi_l\mathrm{d}\xi_{l'}\mathrm{d}\xi_j
\end{aligned}
\tag{42}
$$

$$
\begin{aligned}
\mathcal{T}_{\phi_{i'},\phi_i,\mathrm{mixterm2}} \equiv & \int\limits_{\frac{2}{h_j}(x_{j-1/2}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)} \int\limits_{\frac{2}{h_{l'}}(x_{l'-1/2}-x_{l'})}^{\frac{2}{h_{l'}}(\frac{h_j}{2}\xi_j+x_j-x_{l+1/2}+x_{\min}-x_{l'})} \int\limits_{\frac{2}{h_l}(\frac{h_j}{2}\xi_j+x_j-\frac{h_{l'}}{2}\xi_{l'}-x_{l'}+x_{\min}-x_l)}^{\frac{2}{h_l}(x_{l+1/2}-x_l)} f_1(\xi_{l'}) f_2(\xi_l) \partial_{\xi_j}\phi_k(\xi_j) \mathrm{d}\xi_l \mathrm{d}\xi_{l'} \mathrm{d}\xi_j \\
\mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term1}} \equiv & \int\limits_{\frac{2}{h_j}(x_{l-1/2}+x_{l'-1/2}-x_{\min}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)} \int\limits_{\frac{2}{h_{l'}}(x_{l'-1/2}-x_{l'})}^{\frac{2}{h_{l'}}(\frac{h_j}{2}\xi_j+x_j-x_{l-1/2}+x_{\min}-x_{l'})} \int\limits_{\frac{2}{h_l}(\frac{h_j}{2}\xi_j+x_j-\frac{h_{l'}}{2}\xi_{l'}-x_{l'}+x_{\min}-x_l)}^{\frac{2}{h_l}(x_{l-1/2}-x_l)} f_1(\xi_{l'}) f_2(\xi_l) \partial_{\xi_j}\phi_k(\xi_j) \mathrm{d}\xi_l \mathrm{d}\xi_{l'} \mathrm{d}\xi_j \\
\mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term2}} \equiv & \int\limits_{\frac{2}{h_j}(x_{l+1/2}+x_{l'-1/2}-x_{\min}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)} \int\limits_{\frac{2}{h_{l'}}(x_{l'-1/2}-x_{l'})}^{\frac{2}{h_{l'}}(\frac{h_j}{2}\xi_j+x_j-x_{l+1/2}+x_{\min}-x_{l'})} \int\limits_{\frac{2}{h_l}(\frac{h_j}{2}\xi_j+x_j-\frac{h_{l'}}{2}\xi_{l'}-x_{l'}+x_{\min}-x_l)}^{\frac{2}{h_l}(x_{l+1/2}-x_l)} f_1(\xi_{l'}) f_2(\xi_l) \partial_{\xi_j}\phi_k(\xi_j) \mathrm{d}\xi_l \mathrm{d}\xi_{l'} \mathrm{d}\xi_j \\
\mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term3}} \equiv & \int\limits_{\frac{2}{h_j}(x_{l-1/2}+x_{l'+1/2}-x_{\min}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)} \int\limits_{\frac{2}{h_{l'}}(\frac{h_j}{2}\xi_j+x_j-x_{l-1/2}+x_{\min}-x_{l'})}^{\frac{2}{h_{l'}}(x_{l'+1/2}-x_{l'})} \int\limits_{\frac{2}{h_l}(\frac{h_j}{2}\xi_j+x_j-\frac{h_{l'}}{2}\xi_{l'}-x_{l'}+x_{\min}-x_l)}^{\frac{2}{h_l}(x_{l-1/2}-x_l)} f_1(\xi_{l'}) f_2(\xi_l) \partial_{\xi_j}\phi_k(\xi_j) \mathrm{d}\xi_l \mathrm{d}\xi_{l'} \mathrm{d}\xi_j \\
\mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term4}} \equiv & \int\limits_{\frac{2}{h_j}(x_{l+1/2}+x_{l'+1/2}-x_{\min}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)} \int\limits_{\frac{2}{h_{l'}}(\frac{h_j}{2}\xi_j+x_j-x_{l+1/2}+x_{\min}-x_{l'})}^{\frac{2}{h_{l'}}(x_{l'+1/2}-x_{l'})} \int\limits_{\frac{2}{h_l}(\frac{h_j}{2}\xi_j+x_j-\frac{h_{l'}}{2}\xi_{l'}-x_{l'}+x_{\min}-x_l)}^{\frac{2}{h_l}(x_{l+1/2}-x_l)} f_1(\xi_{l'}) f_2(\xi_l) \partial_{\xi_j}\phi_k(\xi_j) \mathrm{d}\xi_l \mathrm{d}\xi_{l'} \mathrm{d}\xi_j .
\end{aligned}
$$

### 2.3 Fortran scheme for integral of the flux term

Finally, the `Fortran` scheme to evaluate $\mathcal{T}(x, x_{\min}, x_{\max}, j, k, i', i, l', l)$ writes

```
res1 = 0
if ($x_{j+1/2} > x_{l'+1/2}$) then
   if ($x_{j-1/2} > x_{l'+1/2}$) then
      res1 = $\mathcal{T}_{\phi_i} \times \mathcal{T}_{\phi_{i'},\mathrm{allmix}}$
   else if ($x_{j-1/2} \leq x_{l'+1/2}$ and $x_{j-1/2} > x_{l'-1/2}$) then
      res1 = $\mathcal{T}_{\phi_i} \times (\mathcal{T}_{\phi_{i'},\mathrm{mixP1}} - \mathcal{T}_{\phi_{i'},\mathrm{P2term1}})$
   else
      res1 = $\mathcal{T}_{\phi_i} \times (\mathcal{T}_{\phi_{i'},\mathrm{P1term1}} - \mathcal{T}_{\phi_{i'},\mathrm{P2term1}})$
   endif

else if ($x_{j+1/2} \leq x_{l'+1/2}$ and $x_{j+1/2} > x_{l'-1/2}$) then
   if ($x_{j-1/2} > x_{l'-1/2}$) then
      res1 = $\mathcal{T}_{\phi_i} \times \mathcal{T}_{\phi_{i'},\mathrm{mixP1}}$
   else
      res1 = $\mathcal{T}_{\phi_i} \times \mathcal{T}_{\phi_{i'},\mathrm{P1term1}}$
   endif
else
   res1 = 0
endif


res2 = 0
if ($x_{j+1/2} > x_{l'+1/2}$) then
   if ($x_{j+1/2} > x_{l+1/2} + x_{l'+1/2} - x_{\min}$) then
      if ($x_{j-1/2} > x_{l'+1/2}$) then
         if ($x_{j-1/2} > x_{l+1/2} + x_{l'+1/2} - x_{\min}$) then
            res2 = $\mathcal{T}_{\phi_{i'},\phi_i,\mathrm{allmix}}$
         else if ($x_{j-1/2} \leq x_{l+1/2} + x_{l'+1/2} - x_{\min}$ and $x_{j-1/2} > x_{l-1/2} + x_{l'+1/2} - x_{\min}$ and
                  $x_{j-1/2} > x_{l+1/2} + x_{l'-1/2} - x_{\min}$) then
            res2 = $\mathcal{T}_{\phi_{i'},\phi_i,\mathrm{mixterm1term3}} - \mathcal{T}_{\phi_{i'},\phi_i,\mathrm{mixterm2}} - \mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term4}}$
         else if ($x_{j-1/2} \leq x_{l+1/2} + x_{l'+1/2} - x_{\min}$ and $x_{j-1/2} > x_{l-1/2} + x_{l'+1/2} - x_{\min}$ and
                  $x_{j-1/2} \leq x_{l+1/2} + x_{l'-1/2} - x_{\min}$) then
            res2 = $\mathcal{T}_{\phi_{i'},\phi_i,\mathrm{mixterm1term3}} - \mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term2}} - \mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term4}}$
         else if ($x_{j-1/2} \leq x_{l+1/2} + x_{l'+1/2} - x_{\min}$ and $x_{j-1/2} \leq x_{l-1/2} + x_{l'+1/2} - x_{\min}$ and
                  $x_{j-1/2} > x_{l+1/2} + x_{l'-1/2} - x_{\min}$) then
            res2 = $\mathcal{T}_{\phi_{i'},\phi_i,\mathrm{mixterm1}} - \mathcal{T}_{\phi_{i'},\phi_i,\mathrm{mixterm2}} + \mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term3}} - \mathcal{T}_{\mathrm{term4}}$
         else if ($x_{j-1/2} \leq x_{l+1/2} + x_{l'+1/2} - x_{\min}$ and $x_{j-1/2} \leq x_{l-1/2} + x_{l'+1/2} - x_{\min}$ and
                  $x_{j-1/2} \leq x_{l+1/2} + x_{l'-1/2} - x_{\min}$ and $x_{j-1/2} > x_{l-1/2} + x_{l'-1/2} - x_{\min}$) then
            res2 = $\mathcal{T}_{\phi_{i'},\phi_i,\mathrm{mixterm1}} - \mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term2}} + \mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term3}} - \mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term4}}$
         else
            res2 = $\mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term1}} - \mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term2}} + \mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term3}} - \mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term4}}$
         endif
      else if ($x_{j-1/2} \leq x_{l'+1/2}$ and $x_{j-1/2} > x_{l'-1/2}$) then
         if ($x_{j-1/2} > x_{l+1/2} + x_{l'-1/2} - x_{\min}$) then
            res2 = $\mathcal{T}_{\phi_{i'},\phi_i,\mathrm{mixterm1}} - \mathcal{T}_{\phi_{i'},\phi_i,\mathrm{mixterm2}} + \mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term3}} - \mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term4}}$
         else if ($x_{j-1/2} \leq x_{l+1/2} + x_{l'-1/2} - x_{\min}$
                  and $x_{j-1/2} > x_{l-1/2} + x_{l'-1/2} - x_{\min}$) then
            res2 = $\mathcal{T}_{\phi_{i'},\phi_i,\mathrm{mixterm1}} - \mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term2}} + \mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term3}} - \mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term4}}$
         else
            res2 = $\mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term1}} - \mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term2}} + \mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term3}} - \mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term4}}$
         endif
      else
         res2 = $\mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term1}} - \mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term2}} + \mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term3}} - \mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term4}}$
      endif
```

```
    else if ($x_{j+1/2} \leq x_{l+1/2} + x_{l'+1/2} - x_{\min}$ and $x_{j+1/2} > x_{l-1/2} + x_{l'+1/2} - x_{\min}$ and
            $x_{j+1/2} > x_{l+1/2} + x_{l'-1/2} - x_{\min}$) then
        if ($x_{j-1/2} > x_{l'+1/2}$) then
            if ($x_{j-1/2} > x_{l-1/2} + x_{l'+1/2} - x_{\min}$ and $x_{j-1/2} > x_{l+1/2} + x_{l'-1/2} - x_{\min}$) then
                res2 = $\mathcal{T}_{\phi_{i'},\phi_i,\mathrm{mixterm1term3}} - \mathcal{T}_{\phi_{i'},\phi_i,\mathrm{mixterm2}}$
            else if ($x_{j-1/2} > x_{l-1/2} + x_{l'+1/2} - x_{\min}$ and $x_{j-1/2} \leq x_{l+1/2} + x_{l'-1/2} - x_{\min}$) then
                res2 = $\mathcal{T}_{\phi_{i'},\phi_i,\mathrm{mixterm1term3}} - \mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term2}}$
            else if ($x_{j-1/2} \leq x_{l-1/2} + x_{l'+1/2} - x_{\min}$ and $x_{j-1/2} > x_{l+1/2} + x_{l'-1/2} - x_{\min}$) then
                res2 = $\mathcal{T}_{\phi_{i'},\phi_i,\mathrm{mixterm1}} - \mathcal{T}_{\phi_{i'},\phi_i,\mathrm{mixterm2}} + \mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term3}}$
            else if ($x_{j-1/2} \leq x_{l-1/2} + x_{l'+1/2} - x_{\min}$ and $x_{j-1/2} \leq x_{l+1/2} + x_{l'-1/2} - x_{\min}$ and
                    $x_{j-1/2} > x_{l-1/2} + x_{l'-1/2} - x_{\min}$) then
                res2 = $\mathcal{T}_{\phi_{i'},\phi_i,\mathrm{mixterm1}} - \mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term2}} + \mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term3}}$
            else
                res2 = $\mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term1}} - \mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term2}} + \mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term3}}$
            endif
        else if ($x_{j-1/2} \leq x_{l'+1/2}$ and $x_{j-1/2} > x_{l'-1/2}$) then
            if ($x_{j-1/2} > x_{l+1/2} + x_{l'-1/2} - x_{\min}$) then
                res2 = $\mathcal{T}_{\phi_{i'},\phi_i,\mathrm{mixterm1}} - \mathcal{T}_{\phi_{i'},\phi_i,\mathrm{mixterm2}} + \mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term3}}$
            else if ($x_{j-1/2} \leq x_{l+1/2} + x_{l'-1/2} - x_{\min}$ and $x_{j-1/2} > x_{l-1/2} + x_{l'-1/2} - x_{\min}$) then
                res2 = $\mathcal{T}_{\phi_{i'},\phi_i,\mathrm{mixterm1}} - \mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term2}} + \mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term3}}$
            else
                res2 = $\mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term1}} - \mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term2}} + \mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term3}}$
            endif
        else
            res2 = $\mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term1}} - \mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term2}} + \mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term3}}$
        endif
    else if ($x_{j+1/2} \leq x_{l+1/2} + x_{l'+1/2} - x_{\min}$ and $x_{j+1/2} > x_{l-1/2} + x_{l'+1/2} - x_{\min}$ and
            $x_{j+1/2} \leq x_{l+1/2} + x_{l'-1/2} - x_{\min}$) then
        if ($x_{j-1/2} > x_{l'+1/2}$) then
            if ($x_{j-1/2} > x_{l-1/2} + x_{l'+1/2} - x_{\min}$) then
                res2 = $\mathcal{T}_{\phi_{i'},\phi_i,\mathrm{mixterm1term3}}$
            else if ($x_{j-1/2} \leq x_{l-1/2} + x_{l'+1/2} - x_{\min}$ and $x_{j-1/2} > x_{l-1/2} + x_{l'-1/2} - x_{\min}$) then
                res2 = $\mathcal{T}_{\phi_{i'},\phi_i,\mathrm{mixterm1}} + \mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term3}}$
            else
                res2 = $\mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term1}} + \mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term3}}$
            endif
        else if ($x_{j-1/2} \leq x_{l'+1/2}$ and $x_{j-1/2} > x_{l'-1/2}$) then
            if ($x_{j-1/2} > x_{l-1/2} + x_{l'-1/2} - x_{\min}$) then
                res2 = $\mathcal{T}_{\phi_{i'},\phi_i,\mathrm{mixterm1}} + \mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term3}}$
            else
                res2 = $\mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term1}} + \mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term3}}$
            endif
        else
            res2 = $\mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term1}} + \mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term3}}$
        endif
    else if ($x_{j+1/2} \leq x_{l+1/2} + x_{l'+1/2} - x_{\min}$ and $x_{j+1/2} \leq x_{l-1/2} + x_{l'+1/2} - x_{\min}$ and
            $x_{j+1/2} > x_{l+1/2} + x_{l'-1/2} - x_{\min}$) then
        if ($x_{j-1/2} > x_{l'-1/2}$) then
            if ($x_{j-1/2} > x_{l+1/2} + x_{l'-1/2} - x_{\min}$) then
                res2 = $\mathcal{T}_{\phi_{i'},\phi_i,\mathrm{mixterm1}} - \mathcal{T}_{\phi_{i'},\phi_i,\mathrm{mixterm2}}$
            else if ($x_{j-1/2} \leq x_{l+1/2} + x_{l'-1/2} - x_{\min}$ and $x_{j-1/2} > x_{l-1/2} + x_{l'-1/2} - x_{\min}$) then
                res2 = $\mathcal{T}_{\phi_{i'},\phi_i,\mathrm{mixterm1}} - \mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term2}}$
            else
                res2 = $\mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term1}} - \mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term2}}$
            endif
        else
            res2 = $\mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term1}} - \mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term2}}$
```

```
    endif
  else if ($x_{j+1/2} \leq x_{l+1/2} + x_{l'+1/2} - x_{\min}$ and $x_{j+1/2} \leq x_{l-1/2} + x_{l'+1/2} - x_{\min}$ and
          $x_{j+1/2} \leq x_{l+1/2} + x_{l'-1/2} - x_{\min}$ and $x_{j+1/2} > x_{l-1/2} + x_{l'-1/2} - x_{\min}$) then
    if ($x_{j-1/2} > x_{l'-1/2}$) then
      if ($x_{j-1/2} > x_{l-1/2} + x_{l'-1/2} - x_{\min}$) then
        res2 = $\mathcal{T}_{\phi_{i'},\phi_i,\mathrm{mixterm1}}$
      else
        res2 = $\mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term1}}$
      endif
    else
      res2 = $\mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term1}}$
    endif
  else
    res2 = 0
  endif
else if ($x_{j+1/2} \leq x_{l'+1/2}$ and $x_{j+1/2} > x_{l'-1/2}$) then
  if ($x_{j+1/2} > x_{l+1/2} + x_{l'-1/2} - x_{\min}$) then
    if ($x_{j-1/2} > x_{l'-1/2}$) then
      if ($x_{j-1/2} > x_{l+1/2} + x_{l'-1/2} - x_{\min}$) then
        res2 = $\mathcal{T}_{\phi_{i'},\phi_i,\mathrm{mixterm1}} - \mathcal{T}_{\phi_{i'},\phi_i,\mathrm{mixterm2}}$
      else if ($x_{j-1/2} \leq x_{l+1/2} + x_{l'-1/2} - x_{\min}$ and $x_{j-1/2} > x_{l-1/2} + x_{l'-1/2} - x_{\min}$) then
        res2 = $\mathcal{T}_{\phi_{i'},\phi_i,\mathrm{mixterm1}} - \mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term2}}$
      else
        res2 = $\mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term1}} - \mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term2}}$
      endif
    else
      res2 = $\mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term1}} - \mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term2}}$
    endif
  else if ($x_{j+1/2} \leq x_{l+1/2} + x_{l'-1/2} - x_{\min}$ and $x_{j+1/2} > x_{l-1/2} + x_{l'-1/2} - x_{\min}$) then
    if ($x_{j-1/2} > x_{l'-1/2}$) then
      if ($x_{j-1/2} > x_{l-1/2} + x_{l'-1/2} - x_{\min}$) then
        res2 = $\mathcal{T}_{\phi_{i'},\phi_i,\mathrm{mixterm1}}$
      else
        res2 = $\mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term1}}$
      endif
    else
      res2 = $\mathcal{T}_{\phi_{i'},\phi_i,\mathrm{term1}}$
    endif
  else
    res2 = 0
  endif
else
  res2 = 0
endif
```