

Supplementary materials for the paper

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1 DERIVATION OF THE FLUX

The expression of the numerical coagulation flux $F_{\text{coag}}^{\text{nc}}[\tilde{g}]$ is

$$\left\{ \begin{aligned} F_{\text{coag}}^{\text{nc}}[\tilde{g}](x, t) &= \sum_{l'=1}^N \sum_{i'=0}^k \sum_{l=1}^N \sum_{i=0}^k g_{l'}^{i'}(t) g_l^i(t) T(x, x_{\min}, x_{\max}, i', i, l', l), \\ T(x, x_{\min}, x_{\max}, i', i, l', l) &= \\ &\int_{x_{\min}}^x \mathcal{K}_1(u) \phi_{i'}(\xi_{l'}(u)) [\theta(u - x_{l'-1/2}) - \theta(u - x_{l'+1/2})] \int_{x-u+x_{\min}}^{x_{\max}} \frac{\mathcal{K}_2(v)}{v} \phi_i(\xi_l(v)) [\theta(v - x_{l-1/2}) - \theta(v - x_{l+1/2})] dv du. \end{aligned} \right. \quad (1)$$

With the change of variable $\xi_l = \frac{2}{h_l}(x - x_l)$, the inner integral writes

$$T_{\text{inner}}(x, x_{\min}, x_{\max}, i, l) = \int_{x-u+x_{\min}}^{x_{\max}} f_2(\xi_l) \left[\theta\left(\xi_l - \frac{2}{h_l}(x_{l-1/2} - x_l)\right) - \theta\left(\xi_l - \frac{2}{h_l}(x_{l+1/2} - x_l)\right) \right] d\xi_l, \quad (2)$$

with

$$f_2(\xi_l) \equiv \mathcal{K}_2\left(\frac{h_l}{2}\xi_l + x_l\right) \frac{\phi_i(\xi_l)}{\frac{h_l}{2}\xi_l + x_l}. \quad (3)$$

In the following, we will take advantage several times of the identity

$$\left\{ \begin{aligned} a < b, c < b, d \leq b, c < d \\ \int_a^b f(x) [\theta(x - c) - \theta(x - d)] dx &= \int_c^b f(x) dx + \theta(a - c) \int_a^c f(x) dx - \theta(b - d) \int_d^b f(x) dx - \theta(b - d) \theta(a - d) \int_a^d f(x) dx. \end{aligned} \right. \quad (4)$$

Eq. 4 gives for the term T_{inner}

$$\begin{aligned} T_{\text{inner}}(x, x_{\min}, x_{\max}, i, l) &= \\ &\int_{\frac{2}{h_l}(x_{l-1/2}-x_l)}^{\frac{2}{h_l}(x_{\max}-x_l)} f_2(\xi_l) d\xi_l + \theta\left(x - \frac{h_{l'}}{2}\xi_{l'} - x_{l'} + x_{\min} - x_{l-1/2}\right) \int_{\frac{2}{h_l}(x-\frac{h_{l'}}{2}\xi_{l'}-x_{l'}+x_{\min}-x_l)}^{\frac{2}{h_l}(x_{l-1/2}-x_l)} f_2(\xi_l) d\xi_l \\ &- \theta(x_{\max} - x_{l+1/2}) \left\{ \int_{\frac{2}{h_l}(x_{l+1/2}-x_l)}^{\frac{2}{h_l}(x_{\max}-x_l)} f_2(\xi_l) d\xi_l + \theta\left(x - \frac{h_{l'}}{2}\xi_{l'} - x_{l'} + x_{\min} - x_{l+1/2}\right) \int_{\frac{2}{h_l}(x-\frac{h_{l'}}{2}\xi_{l'}-x_{l'}+x_{\min}-x_l)}^{\frac{2}{h_l}(x_{l+1/2}-x_l)} f_2(\xi_l) d\xi_l \right\}. \end{aligned} \quad (5)$$

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Therefore, T writes

$$\begin{aligned}
 T(x, x_{\min}, x_{\max}, i', i, l', l) &= \frac{h_l}{2} \frac{h_{l'}}{2} \\
 &\left[\frac{\frac{2}{h_l}(x_{l+1/2}-x_l)}{\frac{2}{h_l}(x_{l-1/2}-x_l)} \int_{\frac{2}{h_l}(x_{l-1/2}-x_l)}^{\frac{2}{h_l}(x_{l+1/2}-x_l)} f_2(\xi_l) d\xi_l \left\{ \frac{\frac{2}{h_{l'}}(x-x_{l'})}{\frac{2}{h_{l'}}(x_{\min}-x_{l'})} \int_{\frac{2}{h_{l'}}(x_{\min}-x_{l'})}^{\frac{2}{h_{l'}}(x-x_{l'})} f_1(\xi_{l'}) \left[\theta\left(\xi_{l'} - \frac{2}{h_{l'}}(x_{l'-1/2}-x_{l'})\right) - \theta\left(\xi_{l'} - \frac{2}{h_{l'}}(x_{l'+1/2}-x_{l'})\right) \right] d\xi_{l'} \right\} \right. \\
 &+ \int_{\frac{2}{h_{l'}}(x_{\min}-x_{l'})}^{\frac{2}{h_{l'}}(x-x_{l'})} \int_{\frac{2}{h_{l'}}(x_{l-1/2}-x_l)}^{\frac{2}{h_{l'}}(x_{l+1/2}-x_l)} f_1(\xi_{l'}) f_2(\xi_l) \\
 &\left. \left[\theta\left(\xi_{l'} - \frac{2}{h_{l'}}(x_{l'-1/2}-x_{l'})\right) - \theta\left(\xi_{l'} - \frac{2}{h_{l'}}(x_{l'+1/2}-x_{l'})\right) \right] \theta\left(\frac{2}{h_{l'}}(x-x_{l-1/2}+x_{\min}-x_{l'})-\xi_{l'}\right) d\xi_l d\xi_{l'} \right. \\
 &- \int_{\frac{2}{h_{l'}}(x_{\min}-x_{l'})}^{\frac{2}{h_{l'}}(x-x_{l'})} \int_{\frac{2}{h_{l'}}(x_{l+1/2}-x_l)}^{\frac{2}{h_{l'}}(x_{l+1/2}-x_l)} f_1(\xi_{l'}) f_2(\xi_l) \\
 &\left. \left[\theta\left(\xi_{l'} - \frac{2}{h_{l'}}(x_{l'-1/2}-x_{l'})\right) - \theta\left(\xi_{l'} - \frac{2}{h_{l'}}(x_{l'+1/2}-x_{l'})\right) \right] \theta\left(\frac{2}{h_{l'}}(x-x_{l+1/2}+x_{\min}-x_{l'})-\xi_{l'}\right) d\xi_l d\xi_{l'} \right. \\
 &\left. \right],
 \end{aligned} \tag{6}$$

with $f_1(\xi_{l'}) \equiv \mathcal{K}_1(\xi_{l'})\phi_{i'}(\xi_{l'})$ and $f_2(\xi_l) \equiv \mathcal{K}_2\left(\frac{h_l}{2}\xi_l + x_l\right)\frac{\phi_i(\xi_l)}{\frac{h_l}{2}\xi_l + x_l}$.

1.1 Simple integral

In Eq. 6, the rule for the simple integral with the Heaviside function is

$$\begin{cases} a < b, a \leq c, a < d, c < d \\ \int_a^b f(x) [\theta(x-c) - \theta(x-d)] dx = \theta(b-c) \left[\int_c^b f(x) dx + \theta(a-c) \int_a^c f(x) dx \right] - \theta(b-d) \int_a^b f(x) dx. \end{cases} \tag{7}$$

The simple integral writes

$$\begin{aligned}
 &\bullet \int_{\frac{2}{h_{l'}}(x_{\min}-x_{l'})}^{\frac{2}{h_{l'}}(x-x_{l'})} f_1(\xi_{l'}) \left[\theta\left(\xi_{l'} - \frac{2}{h_{l'}}(x_{l'-1/2}-x_{l'})\right) - \theta\left(\xi_{l'} - \frac{2}{h_{l'}}(x_{l'+1/2}-x_{l'})\right) \right] d\xi_{l'} \\
 &= \theta(x-x_{l'-1/2}) \left[\int_{\frac{2}{h_{l'}}(x_{l'-1/2}-x_{l'})}^{\frac{2}{h_{l'}}(x-x_{l'})} f_1(\xi_{l'}) d\xi_{l'} + \underbrace{\theta(x_{\min}-x_{l'-1/2}) \int_{\frac{2}{h_{l'}}(x_{\min}-x_{l'})}^{\frac{2}{h_{l'}}(x_{l'-1/2}-x_{l'})} f_1(\xi_{l'}) d\xi_{l'}}_{0 \text{ since } x_{l-1/2} \geq x_{\min}} \right] \\
 &- \theta(x-x_{l'+1/2}) \int_{\frac{2}{h_{l'}}(x_{l'+1/2}-x_{l'})}^{\frac{2}{h_{l'}}(x-x_{l'})} f_1(\xi_{l'}) d\xi_{l'} \\
 &= \theta(x-x_{l'-1/2}) \int_{\frac{2}{h_{l'}}(x_{l'-1/2}-x_{l'})}^{\frac{2}{h_{l'}}(x-x_{l'})} f_1(\xi_{l'}) d\xi_{l'} - \theta(x-x_{l'+1/2}) \int_{\frac{2}{h_{l'}}(x_{l'+1/2}-x_{l'})}^{\frac{2}{h_{l'}}(x-x_{l'})} f_1(\xi_{l'}) d\xi_{l'}.
 \end{aligned} \tag{8}$$

1.2 First double integral

In Eq. 6, the rule for the first double integral with the Heaviside function is

$$\left\{ \begin{array}{l} a < b, a \leq c, a < d, c < d, e \leq b \\ \int_a^b f(x) [\theta(x-c) - \theta(x-d)] \theta(e-x) dx \\ = \theta(b-c) \left[-\theta(a-c)\theta(e-a) \left\{ \int_e^a f(x) dx + \theta(e-b) \int_b^e f(x) dx \right\} \right. \\ \quad \left. + (1 - \theta(a-c)) \left[\theta(b-c)\theta(e-c) \left\{ \int_c^e f(x) dx + \theta(e-b) \int_e^b f(x) dx \right\} + \theta(c-b)\theta(e-b) \left\{ \int_e^b f(x) dx + \theta(e-c) \int_c^e f(x) dx \right\} \right] \right. \\ \quad \left. - \theta(b-d) \left[\theta(b-d)\theta(e-d) \left\{ \int_d^e f(x) dx + \theta(e-b) \int_e^b f(x) dx \right\} + \theta(d-b)\theta(e-b) \left\{ \int_e^b f(x) dx + \theta(e-d) \int_d^e f(x) dx \right\} \right] \right] \end{array} \right\}. \quad (9)$$

Let denote

$$F_{2,1}(\xi_{l'}) \equiv \int_{\frac{2}{h_l} \left(x - \frac{h_{l'}}{2} \xi_{l'} - x_{l'} + x_{\min} - x_l \right)}^{\frac{2}{h_l} (x_{l-1/2} - x_l)} f_2(\xi_l) d\xi_l, \quad F_{2,2}(\xi_{l'}) \equiv \int_{\frac{2}{h_l} \left(x - \frac{h_{l'}}{2} \xi_{l'} - x_{l'} + x_{\min} - x_l \right)}^{\frac{2}{h_l} (x_{l+1/2} - x_l)} f_2(\xi_l) d\xi_l. \quad (10)$$

The double integral writes

$$(1) \equiv \int_{\frac{2}{h_{l'}} (x_{\min} - x_{l'})}^{\frac{2}{h_{l'}} (x - x_{l'})} f_1(\xi_{l'}) F_{2,1}(\xi_{l'}) \left[\theta \left(\xi_{l'} - \frac{2}{h_{l'}} (x_{l'-1/2} - x_{l'}) \right) - \theta \left(\xi_{l'} - \frac{2}{h_{l'}} (x_{l'+1/2} - x_{l'}) \right) \right] \\ \theta \left(\frac{2}{h_{l'}} (x - x_{l-1/2} + x_{\min} - x_{l'}) - \xi_{l'} \right) d\xi_l d\xi_{l'} \quad (11)$$

$$\begin{aligned}
(1) &= \theta(x - x_{l'-1/2}) \\
&\left[-\theta(x_{\min} - x_{l'-1/2})\theta(x - x_{l-1/2}) \right. \\
&\quad \left. \left\{ \underbrace{\int_{\frac{2}{h_{l'}}(x-x_{l-1/2}+x_{\min}-x_{l'})}^{\frac{2}{h_{l'}}(x_{\min}-x_{l'})} f_1(\xi_{l'})F_{2,1}(\xi_{l'})d\xi_{l'} + \theta(x_{\min} - x_{l-1/2}) \int_{\frac{2}{h_{l'}}(x-x_{l'})}^{\frac{2}{h_{l'}}(x-x_{l-1/2}+x_{\min}-x_{l'})} f_1(\xi_{l'})F_{2,1}(\xi_{l'})d\xi_{l'}}_{0 \text{ since } x_{l-1/2} \geq x_{\min}} \right\} \right. \\
&\quad + (1 - \theta(x_{\min} - x_{l'-1/2})) \\
&\quad \left[\theta(x - x_{l'-1/2})\theta(x - x_{l-1/2} + x_{\min} - x_{l'-1/2}) \right. \\
&\quad \left. \left\{ \underbrace{\int_{\frac{2}{h_{l'}}(x_{l'-1/2}-x_{l'})}^{\frac{2}{h_{l'}}(x-x_{l-1/2}+x_{\min}-x_{l'})} f_1(\xi_{l'})F_{2,1}(\xi_{l'})d\xi_{l'} + \theta(x_{\min} - x_{l-1/2}) \int_{\frac{2}{h_{l'}}(x-x_{l-1/2}+x_{\min}-x_{l'})}^{\frac{2}{h_{l'}}(x-x_{l'})} f_1(\xi_{l'})F_{2,1}(\xi_{l'})d\xi_{l'}}_{0 \text{ since } x_{l-1/2} \geq x_{\min}} \right\} \right. \\
&\quad + \theta(x_{l'-1/2} - x)\theta(x_{\min} - x_{l-1/2}) \\
&\quad \left. \left\{ \underbrace{\int_{\frac{2}{h_{l'}}(x-x_{l-1/2}+x_{\min}-x_{l'})}^{\frac{2}{h_{l'}}(x-x_{l'})} f_1(\xi_{l'})F_{2,1}(\xi_{l'})d\xi_{l'}}_{0 \text{ since } x_{l-1/2} \geq x_{\min}} \right. \right. \\
&\quad \left. \left. + \theta(x - x_{l-1/2} + x_{\min} - x_{l'-1/2}) \int_{\frac{2}{h_{l'}}(x_{l'-1/2}-x_{l'})}^{\frac{2}{h_{l'}}(x-x_{l-1/2}+x_{\min}-x_{l'})} f_1(\xi_{l'})F_{2,1}(\xi_{l'})d\xi_{l'} \right\} \right] \\
&- \theta(x - x_{l'+1/2}) \\
&\left[\theta(x - x_{l'+1/2})\theta(x - x_{l-1/2} + x_{\min} - x_{l'+1/2}) \right. \\
&\quad \left. \left\{ \underbrace{\int_{\frac{2}{h_{l'}}(x_{l'+1/2}-x_{l'})}^{\frac{2}{h_{l'}}(x-x_{l-1/2}+x_{\min}-x_{l'})} f_1(\xi_{l'})F_{2,1}(\xi_{l'})d\xi_{l'} + \theta(x_{\min} - x_{l-1/2}) \int_{\frac{2}{h_{l'}}(x-x_{l-1/2}+x_{\min}-x_{l'})}^{\frac{2}{h_{l'}}(x-x_{l'})} f_1(\xi_{l'})F_{2,1}(\xi_{l'})d\xi_{l'}}_{0 \text{ since } x_{l-1/2} \geq x_{\min}} \right\} \right. \\
&\quad + \theta(x_{l'+1/2} - x)\theta(x_{\min} - x_{l-1/2}) \\
&\quad \left. \left\{ \int_{\frac{2}{h_{l'}}(x-x_{l-1/2}+x_{\min}-x_{l'})}^{\frac{2}{h_{l'}}(x-x_{l'})} f_1(\xi_{l'})F_{2,1}(\xi_{l'})d\xi_{l'} \right. \right. \\
&\quad \left. \left. + \theta(x - x_{l-1/2} + x_{\min} - x_{l'+1/2}) \int_{\frac{2}{h_{l'}}(x_{l'+1/2}-x_{l'})}^{\frac{2}{h_{l'}}(x-x_{l-1/2}+x_{\min}-x_{l'})} f_1(\xi_{l'})F_{2,1}(\xi_{l'})d\xi_{l'} \right\} \right].
\end{aligned} \tag{12}$$

$$\begin{aligned}
 (1) &= \theta(x - x_{l'-1/2}) \\
 &\left[\begin{aligned} & -\theta(x_{\min} - x_{l'-1/2})\theta(x - x_{l-1/2}) \int_{\frac{2}{h_{l'}}(x - x_{l-1/2} + x_{\min} - x_{l'})}^{\frac{2}{h_{l'}}(x_{\min} - x_{l'})} f_1(\xi_{l'})F_{2,1}(\xi_{l'})d\xi_{l'} \\ & + (1 - \theta(x_{\min} - x_{l'-1/2})) \int_{\frac{2}{h_{l'}}(x_{l'-1/2} - x_{l'})}^{\frac{2}{h_{l'}}(x - x_{l-1/2} + x_{\min} - x_{l'})} f_1(\xi_{l'})F_{2,1}(\xi_{l'})d\xi_{l'} \\ & + \underbrace{\theta(x_{l'-1/2} - x)\theta(x_{\min} - x_{l-1/2})\theta(x - x_{l-1/2} + x_{\min} - x_{l'-1/2}) \int_{\frac{2}{h_{l'}}(x_{l'-1/2} - x_{l'})}^{\frac{2}{h_{l'}}(x - x_{l-1/2} + x_{\min} - x_{l'})} f_1(\xi_{l'})F_{2,1}(\xi_{l'})d\xi_{l'}}_{0 \text{ since } x_{l-1/2} \geq x_{\min}} \end{aligned} \right] \\
 &- \theta(x - x_{l'+1/2}) \\
 &\left[\begin{aligned} & \theta(x - x_{l'+1/2})\theta(x - x_{l-1/2} + x_{\min} - x_{l'+1/2}) \int_{\frac{2}{h_{l'}}(x_{l'+1/2} - x_{l'})}^{\frac{2}{h_{l'}}(x - x_{l-1/2} + x_{\min} - x_{l'})} f_1(\xi_{l'})F_{2,1}(\xi_{l'})d\xi_{l'} \\ & + \underbrace{\theta(x_{l'+1/2} - x)\theta(x_{\min} - x_{l-1/2})\theta(x - x_{l-1/2} + x_{\min} - x_{l'+1/2}) \int_{\frac{2}{h_{l'}}(x_{l'+1/2} - x_{l'})}^{\frac{2}{h_{l'}}(x - x_{l-1/2} + x_{\min} - x_{l'})} f_1(\xi_{l'})F_{2,1}(\xi_{l'})d\xi_{l'}}_{0 \text{ since } x_{l-1/2} \geq x_{\min}} \end{aligned} \right] \\
 (1) &= \theta(x - x_{l'-1/2})\theta(x - x_{l-1/2} + x_{\min} - x_{l'-1/2}) \int_{\frac{2}{h_{l'}}(x_{l'-1/2} - x_{l'})}^{\frac{2}{h_{l'}}(x - x_{l-1/2} + x_{\min} - x_{l'})} f_1(\xi_{l'})F_{2,1}(\xi_{l'})d\xi_{l'} \\
 &+ \theta(x - x_{l'+1/2})\theta(x - x_{l-1/2} + x_{\min} - x_{l'+1/2}) \int_{\frac{2}{h_{l'}}(x_{l'+1/2} - x_{l'})}^{\frac{2}{h_{l'}}(x - x_{l-1/2} + x_{\min} - x_{l'})} f_1(\xi_{l'})F_{2,1}(\xi_{l'})d\xi_{l'}.
 \end{aligned} \tag{13}$$

1.3 Second double integral

In Eq. 6, the rule for the second double integral with the Heaviside function is

$$\begin{cases} a < b, a \leq c, a < d, c < d, e < b \\ \int_a^b f(x) [\theta(x - c) - \theta(x - d)] \theta(e - x) dx \\ = \theta(b - d)\theta(e - d) \int_e^d f(x) dx + \theta(b - c) \left[(\theta(a - c) - 1)\theta(b - c)\theta(e - c) \int_e^c f(x) dx + \theta(a - c)\theta(e - a) \int_a^e f(x) dx \right], \end{cases} \tag{15}$$

The double integral writes

$$\begin{aligned}
 (2) &\equiv \int_{\frac{2}{h_{l'}}(x_{\min} - x_{l'})}^{\frac{2}{h_{l'}}(x - x_{l'})} f_1(\xi_{l'})F_{2,2}(\xi_{l'}) \left[\theta\left(\xi_{l'} - \frac{2}{h_{l'}}(x_{l'-1/2} - x_{l'})\right) - \theta\left(\xi_{l'} - \frac{2}{h_{l'}}(x_{l'+1/2} - x_{l'})\right) \right] \\
 &\quad \theta\left(\frac{2}{h_{l'}}(x - x_{l+1/2} + x_{\min} - x_{l'}) - \xi_{l'}\right) d\xi_{l'} d\xi_{l'}
 \end{aligned} \tag{16}$$

$$\begin{aligned}
(2) = & \theta(x - x_{l'+1/2})\theta(x - x_{l+1/2} + x_{\min} - x_{l'+1/2}) \int_{\frac{2}{h_{l'}}(x - x_{l+1/2} + x_{\min} - x_{l'})}^{\frac{2}{h_{l'}}(x_{l'+1/2} - x_{l'})} f_1(\xi_{l'}) F_{2,2}(\xi_{l'}) d\xi_{l'} \\
& + \theta(x - x_{l'-1/2}) \\
& \left[(\theta(x_{\min} - x_{l'-1/2}) - 1)\theta(x - x_{l'-1/2})\theta(x - x_{l+1/2} + x_{\min} - x_{l'-1/2}) \int_{\frac{2}{h_{l'}}(x - x_{l+1/2} + x_{\min} - x_{l'})}^{\frac{2}{h_{l'}}(x_{l'-1/2} - x_{l'})} f_1(\xi_{l'}) F_{2,2}(\xi_{l'}) d\xi_{l'} \right. \\
& \left. + \theta(x_{\min} - x_{l'-1/2})\theta(x - x_{l+1/2} + x_{\min} - x_{\min}) \int_{\frac{2}{h_{l'}}(x_{\min} - x_{l'})}^{\frac{2}{h_{l'}}(x - x_{l+1/2} + x_{\min} - x_{l'})} f_1(\xi_{l'}) F_{2,2}(\xi_{l'}) d\xi_{l'} \right], \tag{17}
\end{aligned}$$

$$\begin{aligned}
(2) = & \theta(x - x_{l'-1/2})\theta(x_{\min} - x_{l'-1/2})\theta(x - x_{l+1/2} + x_{\min} - x_{l'-1/2}) \int_{\frac{2}{h_{l'}}(x - x_{l+1/2} + x_{\min} - x_{l'})}^{\frac{2}{h_{l'}}(x_{l'-1/2} - x_{l'})} f_1(\xi_{l'}) F_{2,2}(\xi_{l'}) d\xi_{l'} \\
& + \theta(x - x_{l'-1/2})\theta(x_{\min} - x_{l'-1/2})\theta(x - x_{l+1/2}) \int_{\frac{2}{h_{l'}}(x_{\min} - x_{l'})}^{\frac{2}{h_{l'}}(x - x_{l+1/2} + x_{\min} - x_{l'})} f_1(\xi_{l'}) F_{2,2}(\xi_{l'}) d\xi_{l'} \\
& \underbrace{\hspace{15em}}_{0 \text{ since } x_{l'-1/2} \geq x_{\min}} \\
& + \theta(x - x_{l'+1/2})\theta(x - x_{l+1/2} + x_{\min} - x_{l'+1/2}) \int_{\frac{2}{h_{l'}}(x - x_{l+1/2} + x_{\min} - x_{l'})}^{\frac{2}{h_{l'}}(x_{l'+1/2} - x_{l'})} f_1(\xi_{l'}) F_{2,2}(\xi_{l'}) d\xi_{l'} \\
& - \theta(x - x_{l'-1/2})\theta(x - x_{l+1/2} + x_{\min} - x_{l'-1/2}) \int_{\frac{2}{h_{l'}}(x - x_{l+1/2} + x_{\min} - x_{l'})}^{\frac{2}{h_{l'}}(x_{l'-1/2} - x_{l'})} f_1(\xi_{l'}) F_{2,2}(\xi_{l'}) d\xi_{l'}. \tag{18}
\end{aligned}$$

$$\begin{aligned}
(2) = & \theta(x - x_{l'+1/2})\theta(x - x_{l+1/2} + x_{\min} - x_{l'+1/2}) \int_{\frac{2}{h_{l'}}(x - x_{l+1/2} + x_{\min} - x_{l'})}^{\frac{2}{h_{l'}}(x_{l'+1/2} - x_{l'})} f_1(\xi_{l'}) F_{2,2}(\xi_{l'}) d\xi_{l'} \\
& + \theta(x - x_{l'-1/2})\theta(x - x_{l+1/2} + x_{\min} - x_{l'-1/2}) \int_{\frac{2}{h_{l'}}(x_{l'-1/2} - x_{l'})}^{\frac{2}{h_{l'}}(x - x_{l+1/2} + x_{\min} - x_{l'})} f_1(\xi_{l'}) F_{2,2}(\xi_{l'}) d\xi_{l'}. \tag{19}
\end{aligned}$$

Then, T writes

$$\begin{aligned}
 T(x, x_{\min}, x_{\max}, i', i, l', l) &= \frac{h_l}{2} \frac{h_{l'}}{2} \\
 &\left[\int_{\frac{2}{h_l}(x_{l-1/2}-x_l)}^{\frac{2}{h_l}(x_{l+1/2}-x_l)} f_2(\xi_l) d\xi_l \left(\theta(x - x_{l'-1/2}) \int_{\frac{2}{h_{l'}}(x_{l'-1/2}-x_{l'})}^{\frac{2}{h_{l'}}(x-x_{l'})} f_1(\xi_{l'}) d\xi_{l'} - \theta(x - x_{l'+1/2}) \int_{\frac{2}{h_{l'}}(x_{l'+1/2}-x_{l'})}^{\frac{2}{h_{l'}}(x-x_{l'})} f_1(\xi_{l'}) d\xi_{l'} \right) \right. \\
 &\quad + \theta(x - x_{l'-1/2}) \\
 &\quad \left[\theta(x - x_{l-1/2} + x_{\min} - x_{l'-1/2}) \int_{\frac{2}{h_{l'}}(x_{l'-1/2}-x_{l'})}^{\frac{2}{h_{l'}}(x-x_{l-1/2}+x_{\min}-x_{l'})} \int_{\frac{2}{h_l}(x_{l-1/2}-x_l)}^{\frac{2}{h_l}(x_{l+1/2}-x_l)} f_1(\xi_{l'}) f_2(\xi_l) d\xi_l d\xi_{l'} \right. \\
 &\quad \left. - \theta(x_{\max} - x_{l+1/2}) \theta(x - x_{l+1/2} + x_{\min} - x_{l'-1/2}) \int_{\frac{2}{h_{l'}}(x_{l'-1/2}-x_{l'})}^{\frac{2}{h_{l'}}(x-x_{l+1/2}+x_{\min}-x_{l'})} \int_{\frac{2}{h_l}(x_{l+1/2}-x_l)}^{\frac{2}{h_l}(x_{l+1/2}-x_l)} f_1(\xi_{l'}) f_2(\xi_l) d\xi_l d\xi_{l'} \right] \\
 &\quad + \theta(x - x_{l'+1/2}) \\
 &\quad \left[\theta(x - x_{l-1/2} + x_{\min} - x_{l'+1/2}) \int_{\frac{2}{h_{l'}}(x_{l'-1/2}-x_{l'})}^{\frac{2}{h_{l'}}(x_{l'+1/2}-x_{l'})} \int_{\frac{2}{h_l}(x_{l-1/2}-x_l)}^{\frac{2}{h_l}(x_{l+1/2}-x_l)} f_1(\xi_{l'}) f_2(\xi_l) d\xi_l d\xi_{l'} \right. \\
 &\quad \left. - \theta(x_{\max} - x_{l+1/2}) \theta(x - x_{l+1/2} + x_{\min} - x_{l'+1/2}) \int_{\frac{2}{h_{l'}}(x_{l'-1/2}-x_{l'})}^{\frac{2}{h_{l'}}(x_{l'+1/2}-x_{l'})} \int_{\frac{2}{h_l}(x_{l+1/2}-x_l)}^{\frac{2}{h_l}(x_{l+1/2}-x_l)} f_1(\xi_{l'}) f_2(\xi_l) d\xi_l d\xi_{l'} \right] \Bigg]. \quad (20)
 \end{aligned}$$

To reduce as much as possible the number of integrals to evaluate, we define the following terms for the scheme

$$\begin{aligned}
 T_{\phi_i} &\equiv \int_{\frac{2}{h_l}(x_{l-1/2}-x_l)}^{\frac{2}{h_l}(x_{l+1/2}-x_l)} f_2(\xi_l) d\xi_l, T_{\phi_{i'}, \text{mix}} \equiv \int_{\frac{2}{h_{l'}}(x_{l'-1/2}-x_{l'})}^{\frac{2}{h_{l'}}(x_{l'+1/2}-x_{l'})} f_1(\xi_{l'}) d\xi_{l'}, T_{\phi_{i'}, \text{term1}} \equiv \int_{\frac{2}{h_{l'}}(x_{l'-1/2}-x_{l'})}^{\frac{2}{h_{l'}}(x-x_{l'})} f_1(\xi_{l'}) d\xi_{l'}, \\
 T_{\phi_{i'}, \phi_i, \text{allmix}} &\equiv \int_{\frac{2}{h_{l'}}(x_{l'-1/2}-x_{l'})}^{\frac{2}{h_{l'}}(x_{l'+1/2}-x_{l'})} \int_{\frac{2}{h_l}(x_{l+1/2}-x_l)}^{\frac{2}{h_l}(x_{l-1/2}-x_l)} f_1(\xi_{l'}) f_2(\xi_l) d\xi_l d\xi_{l'}, \\
 T_{\phi_{i'}, \phi_i, \text{mix.P1term1.P2term1}} &\equiv \int_{\frac{2}{h_{l'}}(x_{l'-1/2}-x_{l'})}^{\frac{2}{h_{l'}}(x_{l'+1/2}-x_{l'})} \int_{\frac{2}{h_l}(x_{l-1/2}-x_l)}^{\frac{2}{h_l}(x_{l+1/2}-x_l)} f_1(\xi_{l'}) f_2(\xi_l) d\xi_l d\xi_{l'}, \\
 T_{\phi_{i'}, \phi_i, \text{P1term1}} &\equiv \int_{\frac{2}{h_{l'}}(x_{l'-1/2}-x_{l'})}^{\frac{2}{h_{l'}}(x-x_{l-1/2}+x_{\min}-x_{l'})} \int_{\frac{2}{h_l}(x_{l-1/2}-x_l)}^{\frac{2}{h_l}(x_{l+1/2}-x_l)} f_1(\xi_{l'}) f_2(\xi_l) d\xi_l d\xi_{l'}, \\
 T_{\phi_{i'}, \phi_i, \text{P1term2}} &\equiv \int_{\frac{2}{h_{l'}}(x_{l'-1/2}-x_{l'})}^{\frac{2}{h_{l'}}(x-x_{l+1/2}+x_{\min}-x_{l'})} \int_{\frac{2}{h_l}(x_{l+1/2}-x_l)}^{\frac{2}{h_l}(x_{l+1/2}-x_l)} f_1(\xi_{l'}) f_2(\xi_l) d\xi_l d\xi_{l'}. \quad (21)
 \end{aligned}$$

1.4 Fortran scheme for flux term

The Fortran scheme to evaluate $T(x, x_{\min}, x_{\max}, i', i, l', l)$ writes with terms in Eq. 21

```

1  res1=0
2  res2=0
3  if ( $x > x_{l'+1/2}$ ) then
4      res1 =  $T_{\phi_i} \times T_{\phi_{i'}, \text{mix}}$ 
5      if ( $x > x_{l+1/2} + x_{l'+1/2} - x_{\min}$ ) then
6          res2 =  $T_{\phi_{i'}, \phi_i, \text{allmix}}$ 
7      else if ( $x \leq x_{l+1/2} + x_{l'+1/2} - x_{\min}$  and  $x > x_{l+1/2} + x_{l'-1/2} - x_{\min}$  and  $x > x_{l-1/2} + x_{l'+1/2} - x_{\min}$ ) then
8          res2 =  $T_{\phi_{i'}, \phi_i, \text{mix.P1term1.P2term1}} - T_{\phi_{i'}, \phi_i, \text{P1term2}}$ 
9      else if ( $x \leq x_{l+1/2} + x_{l'+1/2} - x_{\min}$  and  $x \leq x_{l+1/2} + x_{l'-1/2} - x_{\min}$  and  $x > x_{l-1/2} + x_{l'+1/2} - x_{\min}$ ) then
10         res2 =  $T_{\phi_{i'}, \phi_i, \text{mix.P1term1.P2term1}}$ 
11     else if ( $x \leq x_{l+1/2} + x_{l'+1/2} - x_{\min}$  and  $x > x_{l+1/2} + x_{l'-1/2} - x_{\min}$  and  $x \leq x_{l-1/2} + x_{l'+1/2} - x_{\min}$ ) then
12         res2 =  $T_{\phi_{i'}, \phi_i, \text{P1term1}} - T_{\phi_{i'}, \phi_i, \text{P1term2}}$ 
13     else if ( $x \leq x_{l+1/2} + x_{l'+1/2} - x_{\min}$  and  $x \leq x_{l+1/2} + x_{l'-1/2} - x_{\min}$  and  $x \leq x_{l-1/2} + x_{l'+1/2} - x_{\min}$  and
14          $x > x_{l-1/2} + x_{l'-1/2} - x_{\min}$ ) then
15         res2 =  $T_{\phi_{i'}, \phi_i, \text{P1term1}}$ 
16     else
17         res2 = 0
18     endif
19 else if ( $x \leq x_{l'+1/2}$  and  $x > x_{l'-1/2}$ ) then
20     res1 =  $T_{\phi_i} \times T_{\phi_{i'}, \text{term1}}$ 
21     if ( $x > x_{l+1/2} + x_{l'-1/2} - x_{\min}$ ) then
22         res2 =  $T_{\phi_{i'}, \phi_i, \text{P1term1}} - T_{\phi_{i'}, \phi_i, \text{P1term2}}$ 
23     else if ( $x \leq x_{l+1/2} + x_{l'-1/2} - x_{\min}$  and  $x > x_{l-1/2} + x_{l'-1/2} - x_{\min}$ ) then
24         res2 =  $T_{\phi_{i'}, \phi_i, \text{P1term1}}$ 
25     else
26         res2 = 0
27 else
28     res1=0
29     res2=0
30 endif
31
32 T = (res1+res2)*hl*hl'/4

```


2 DERIVATION OF INTEGRAL OF THE FLUX

The term with the integral of the numerical flux, that we note $\mathcal{F}_{\text{coag}}^{\text{nc}}$, writes

$$\left\{ \begin{aligned} \mathcal{F}_{\text{coag}}^{\text{nc}}[\tilde{g}, j, k](t) &= \sum_{l'=1}^N \sum_{i'=0}^k \sum_{l=1}^N \sum_{i=0}^k g_{i'}^{i'}(t) g_i^i(t) \mathcal{T}(x_{\min}, x_{\max}, j, k, i', i, l', l) \\ \mathcal{T}(x_{\min}, x_{\max}, j, k, i', i, l', l) &\equiv \int_{I_j} \int_{x_{\min}}^x \int_{x-u+x_{\min}}^{x_{\max}} \frac{\mathcal{K}(u, v)}{v} \partial_x \phi_k(\xi_j(x)) \phi_{i'}(\xi_{l'}(u)) [\theta(u - x_{l'-1/2}) - \theta(u - x_{l'+1/2})] \\ &\quad \phi_i(\xi_l(v)) [\theta(v - x_{l-1/2}) - \theta(v - x_{l+1/2})] dv du dx, \\ \mathcal{T}(x_{\min}, x_{\max}, j, k, i', i, l', l) &\equiv \int_{I_j} T(x, x_{\min}, x_{\max}, i', i, l', l) \partial_x \phi_k(\xi_j(x)) dx \end{aligned} \right. \quad (22)$$

Multiplying $T(x, x_{\min}, x_{\max}, i', i, l', l)$ in Eq. 20 by $\partial_x \phi_k(\xi_j(x))$, \mathcal{T} writes, with $\xi_j = \frac{2}{h_j}(x - x_j)$,

$$\begin{aligned} \mathcal{T}(x_{\min}, x_{\max}, j, k, i', i, l', l) &= \frac{h_l}{2} \frac{h_{l'}}{2} \\ &\left[\int_{\frac{2}{h_l}(x_{l-1/2}-x_l)}^{\frac{2}{h_l}(x_{l+1/2}-x_l)} f_2(\xi_l) d\xi_l \times \right. \\ &\left(\int_{\frac{2}{h_j}(x_{j-1/2}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)} \int_{\frac{2}{h_{l'}}(x_{l'-1/2}-x_{l'})}^{\frac{2}{h_{l'}}(\frac{h_j}{2}\xi_j+x_j-x_{l'})} f_1(\xi_{l'}) \partial_{\xi_j} \phi_k(\xi_j) \theta\left(\xi_j - \frac{2}{h_j}(x_{l'-1/2}-x_j)\right) d\xi_j \right. \\ &\quad \left. - \int_{\frac{2}{h_j}(x_{j-1/2}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)} \int_{\frac{2}{h_{l'}}(x_{l'+1/2}-x_{l'})}^{\frac{2}{h_{l'}}(\frac{h_j}{2}\xi_j+x_j-x_{l'})} f_1(\xi_{l'}) \partial_{\xi_j} \phi_k(\xi_j) \theta\left(\xi_j - \frac{2}{h_j}(x_{l'+1/2}-x_j)\right) d\xi_{l'} d\xi_j \right) \\ &+ \int_{\frac{2}{h_j}(x_{j-1/2}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)} \int_{\frac{2}{h_{l'}}(x_{l'-1/2}-x_{l'})}^{\frac{2}{h_{l'}}(\frac{h_j}{2}\xi_j+x_j-x_{l-1/2}+x_{\min}-x_{l'})} \int_{\frac{2}{h_l}(x_{l-1/2}-x_l)}^{\frac{2}{h_l}(x_{l+1/2}-x_l)} f_1(\xi_{l'}) f_2(\xi_l) \partial_{\xi_j} \phi_k(\xi_j) \\ &\quad \times \theta\left(\xi_j - \frac{2}{h_j}(x_{l'-1/2}-x_j)\right) \theta\left(\xi_j - \frac{2}{h_j}(x_{l-1/2}+x_{l'-1/2}-x_{\min}-x_j)\right) d\xi_l d\xi_{l'} d\xi_j \\ &- \int_{\frac{2}{h_j}(x_{j-1/2}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)} \int_{\frac{2}{h_{l'}}(x_{l'-1/2}-x_{l'})}^{\frac{2}{h_{l'}}(\frac{h_j}{2}\xi_j+x_j-x_{l+1/2}+x_{\min}-x_{l'})} \int_{\frac{2}{h_l}(x_{l+1/2}-x_l)}^{\frac{2}{h_l}(x_{l+1/2}-x_l)} f_1(\xi_{l'}) f_2(\xi_l) \partial_{\xi_j} \phi_k(\xi_j) \\ &\quad \times \theta\left(\xi_j - \frac{2}{h_j}(x_{l'-1/2}-x_j)\right) \theta\left(\xi_j - \frac{2}{h_j}(x_{l+1/2}+x_{l'-1/2}-x_{\min}-x_j)\right) d\xi_l d\xi_{l'} d\xi_j \\ &+ \int_{\frac{2}{h_j}(x_{j-1/2}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)} \int_{\frac{2}{h_{l'}}(x_{l'+1/2}-x_{l'})}^{\frac{2}{h_{l'}}(\frac{h_j}{2}\xi_j+x_j-x_{l+1/2}+x_{\min}-x_{l'})} \int_{\frac{2}{h_l}(x_{l-1/2}-x_l)}^{\frac{2}{h_l}(x_{l+1/2}-x_l)} f_1(\xi_{l'}) f_2(\xi_l) \partial_{\xi_j} \phi_k(\xi_j) \\ &\quad \times \theta\left(\xi_j - \frac{2}{h_j}(x_{l'+1/2}-x_j)\right) \theta\left(\xi_j - \frac{2}{h_j}(x_{l-1/2}+x_{l'+1/2}-x_{\min}-x_j)\right) d\xi_l d\xi_{l'} d\xi_j \\ &- \int_{\frac{2}{h_j}(x_{j-1/2}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)} \int_{\frac{2}{h_{l'}}(x_{l'+1/2}-x_{l'})}^{\frac{2}{h_{l'}}(\frac{h_j}{2}\xi_j+x_j-x_{l+1/2}+x_{\min}-x_{l'})} \int_{\frac{2}{h_l}(x_{l+1/2}-x_l)}^{\frac{2}{h_l}(x_{l+1/2}-x_l)} f_1(\xi_{l'}) f_2(\xi_l) \partial_{\xi_j} \phi_k(\xi_j) \\ &\quad \times \theta\left(\xi_j - \frac{2}{h_j}(x_{l'+1/2}-x_j)\right) \theta\left(\xi_j - \frac{2}{h_j}(x_{l+1/2}+x_{l'+1/2}-x_{\min}-x_j)\right) d\xi_l d\xi_{l'} d\xi_j \left. \right]. \end{aligned} \quad (23)$$

2.1 Derivation of double integrals

The rule for the double integrals is

$$\begin{cases} a < b \\ \int_a^b f(x)\theta(x-c)dx = \theta(b-c) \left(\int_c^b f(x)dx + \theta(a-c) \int_a^c f(x)dx \right). \end{cases} \quad (24)$$

Then we obtain

$$\begin{aligned} & \bullet \int_{\frac{2}{h_j}(x_{j-1/2}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)} \int_{\frac{2}{h_{l'}}(x_{l'-1/2}-x_{l'})}^{\frac{2}{h_{l'}}(x-x_{l'})} f_1(\xi_{l'}) \partial_{\xi_j} \phi_k(\xi_j) \theta \left(\xi_j - \frac{2}{h_j}(x_{l'-1/2}-x_j) \right) d\xi_j \\ &= \theta(x_{j+1/2} - x_{l'-1/2}) \\ & \left(\int_{\frac{2}{h_j}(x_{l'-1/2}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)} \int_{\frac{2}{h_{l'}}(x_{l'-1/2}-x_{l'})}^{\frac{2}{h_{l'}}(x-x_{l'})} f_1(\xi_{l'}) \partial_{\xi_j} \phi_k(\xi_j) d\xi_j + \theta(x_{j-1/2} - x_{l'-1/2}) \int_{\frac{2}{h_j}(x_{j-1/2}-x_j)}^{\frac{2}{h_j}(x_{l'-1/2}-x_j)} \int_{\frac{2}{h_{l'}}(x_{l'-1/2}-x_{l'})}^{\frac{2}{h_{l'}}(x-x_{l'})} f_1(\xi_{l'}) \partial_{\xi_j} \phi_k(\xi_j) d\xi_j \right), \\ & \bullet \int_{\frac{2}{h_j}(x_{j-1/2}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)} \int_{\frac{2}{h_{l'}}(x_{l'+1/2}-x_{l'})}^{\frac{2}{h_{l'}}(x-x_{l'})} f_1(\xi_{l'}) \partial_{\xi_j} \phi_k(\xi_j) \theta \left(\xi_j - \frac{2}{h_j}(x_{l'+1/2}-x_j) \right) d\xi_j \\ &= \theta(x_{j+1/2} - x_{l'+1/2}) \\ & \left(\int_{\frac{2}{h_j}(x_{l'+1/2}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)} \int_{\frac{2}{h_{l'}}(x_{l'+1/2}-x_{l'})}^{\frac{2}{h_{l'}}(x-x_{l'})} f_1(\xi_{l'}) \partial_{\xi_j} \phi_k(\xi_j) d\xi_j + \theta(x_{j-1/2} - x_{l'+1/2}) \int_{\frac{2}{h_j}(x_{j-1/2}-x_j)}^{\frac{2}{h_j}(x_{l'+1/2}-x_j)} \int_{\frac{2}{h_{l'}}(x_{l'+1/2}-x_{l'})}^{\frac{2}{h_{l'}}(x-x_{l'})} f_1(\xi_{l'}) \partial_{\xi_j} \phi_k(\xi_j) d\xi_j \right). \end{aligned} \quad (25)$$

2.2 Derivation of the triple integrals

The rule for the first and third triple integrals is

$$\begin{cases} a < b, c \leq d \\ \int_a^b f(x)\theta(x-c)\theta(x-d)dx = \theta(b-c) \left[\theta(a-c)\theta(b-d) \left\{ \int_d^b f(x)dx + \theta(a-d) \int_a^d f(x)dx \right\} \right. \\ \left. + (1-\theta(a-c)) \left[\theta(c-b)\theta(c-d) \left\{ \int_c^d f(x)dx + \theta(b-d) \int_d^b f(x)dx \right\} \right. \right. \\ \left. \left. + \theta(b-c)\theta(b-d) \left\{ \int_d^b f(x)dx + \theta(c-d) \int_c^d f(x)dx \right\} \right] \right] \end{cases} \quad (26)$$

Let denote

$$\begin{aligned} F_1(\xi_j) &\equiv \int_{\frac{2}{h_{l'}}(x_{l'-1/2}-x_{l'})}^{\frac{2}{h_{l'}}(x_{l'+1/2}-x_{l'})} \int_{\frac{2}{h_l}(x_{l-1/2}-x_l)}^{\frac{2}{h_l}(x_{l+1/2}-x_l)} f_1(\xi_{l'}) f_2(\xi_l) \partial_{\xi_j} \phi_k(\xi_j) d\xi_l d\xi_{l'}, \\ F_3(\xi_j) &\equiv \int_{\frac{2}{h_{l'}}(x_{l'+1/2}-x_{l'})}^{\frac{2}{h_{l'}}(x_{l'+1/2}-x_{l'})} \int_{\frac{2}{h_l}(x_{l-1/2}-x_l)}^{\frac{2}{h_l}(x_{l+1/2}-x_l)} f_1(\xi_{l'}) f_2(\xi_l) \partial_{\xi_j} \phi_k(\xi_j) d\xi_l d\xi_{l'}. \end{aligned} \quad (27)$$

The first triple integral writes

$$(1) \equiv \int_{\frac{2}{h_j}(x_{j-1/2}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)} F_1(\xi_j) \theta \left(\xi_j - \frac{2}{h_j}(x_{l'-1/2}-x_j) \right) \theta \left(\xi_j - \frac{2}{h_j}(x_{l-1/2}+x_{l'-1/2}-x_{\min}-x_j) \right) d\xi_j \quad (28)$$

$$\begin{aligned} (1) &= \theta(x_{j+1/2}-x_{l'-1/2}) \\ &\left[\theta(x_{j-1/2}-x_{l'-1/2}) \theta(x_{j+1/2}+x_{\min}-x_{l-1/2}-x_{l'-1/2}) \right. \\ &\quad \left. \left\{ \int_{\frac{2}{h_j}(x_{l-1/2}+x_{l'-1/2}-x_{\min}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)} F_1(\xi_j) d\xi_j + \theta(x_{j-1/2}+x_{\min}-x_{l-1/2}-x_{l'-1/2}) \int_{\frac{2}{h_j}(x_{j-1/2}-x_j)}^{\frac{2}{h_j}(x_{l-1/2}+x_{l'-1/2}-x_{\min}-x_j)} F_1(\xi_j) d\xi_j \right\} \right. \\ &\quad + (1 - \theta(x_{j-1/2}-x_{l'-1/2})) \\ &\quad \left[\theta(x_{l'-1/2}-x_{j+1/2}) \theta(x_{l'-1/2}+x_{\min}-x_{l-1/2}-x_{l'-1/2}) \right. \\ &\quad \left. \left\{ \int_{\frac{2}{h_j}(x_{l'-1/2}-x_j)}^{\frac{2}{h_j}(x_{l-1/2}+x_{l'-1/2}-x_{\min}-x_j)} F_1(\xi_j) d\xi_j + \theta(x_{j+1/2}+x_{\min}-x_{l-1/2}-x_{l'-1/2}) \int_{\frac{2}{h_j}(x_{l-1/2}+x_{l'-1/2}-x_{\min}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)} F_1(\xi_j) d\xi_j \right\} \right. \\ &\quad + \theta(x_{j+1/2}-x_{l'-1/2}) \theta(x_{j+1/2}+x_{\min}-x_{l-1/2}-x_{l'-1/2}) \\ &\quad \left. \left. \left. \left\{ \int_{\frac{2}{h_j}(x_{l-1/2}+x_{l'-1/2}-x_{\min}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)} F_1(\xi_j) d\xi_j + \theta(x_{l'-1/2}+x_{\min}-x_{l-1/2}-x_{l'-1/2}) \int_{\frac{2}{h_j}(x_{l'-1/2}-x_j)}^{\frac{2}{h_j}(x_{l-1/2}+x_{l'-1/2}-x_{\min}-x_j)} F_1(\xi_j) d\xi_j \right\} \right. \right. \right. \\ &\quad \left. \left. \left. \underbrace{\hspace{10cm}}_{0 \text{ since } x_{l-1/2} \geq x_{\min}} \right. \right. \right] \quad (29) \end{aligned}$$

$$\begin{aligned} (1) &= \theta(x_{j+1/2}-x_{l'-1/2}) \\ &\left[\theta(x_{j-1/2}-x_{l'-1/2}) \theta(x_{j+1/2}+x_{\min}-x_{l-1/2}-x_{l'-1/2}) \right. \\ &\quad \left\{ \int_{\frac{2}{h_j}(x_{l-1/2}+x_{l'-1/2}-x_{\min}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)} F_1(\xi_j) d\xi_j + \theta(x_{j-1/2}+x_{\min}-x_{l-1/2}-x_{l'-1/2}) \int_{\frac{2}{h_j}(x_{j-1/2}-x_j)}^{\frac{2}{h_j}(x_{l-1/2}+x_{l'-1/2}-x_{\min}-x_j)} F_1(\xi_j) d\xi_j \right\} \\ &\quad + (1 - \theta(x_{j-1/2}-x_{l'-1/2})) \\ &\quad \left[\theta(x_{l'-1/2}-x_{j+1/2}) \theta(x_{\min}-x_{l-1/2}) \int_{\frac{2}{h_j}(x_{l'-1/2}-x_j)}^{\frac{2}{h_j}(x_{l-1/2}+x_{l'-1/2}-x_{\min}-x_j)} F_1(\xi_j) d\xi_j \right. \\ &\quad + \theta(x_{l'-1/2}-x_{j+1/2}) \theta(x_{\min}-x_{l-1/2}) \theta(x_{j+1/2}+x_{\min}-x_{l-1/2}-x_{l'-1/2}) \int_{\frac{2}{h_j}(x_{l-1/2}+x_{l'-1/2}-x_{\min}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)} F_1(\xi_j) d\xi_j \\ &\quad \left. \left. \underbrace{\hspace{10cm}}_{0 \text{ since } x_{l-1/2} \geq x_{\min}} \right. \right. \\ &\quad \left. \left. \left. \left. \theta(x_{j+1/2}-x_{l'-1/2}) \theta(x_{j+1/2}+x_{\min}-x_{l-1/2}-x_{l'-1/2}) \int_{\frac{2}{h_j}(x_{l-1/2}+x_{l'-1/2}-x_{\min}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)} F_1(\xi_j) d\xi_j \right. \right. \right. \right] \quad (30) \end{aligned}$$

$$\begin{aligned}
(1) = & \theta(x_{j+1/2} - x_{l'-1/2})\theta(x_{j-1/2} - x_{l'-1/2})\theta(x_{j+1/2} + x_{\min} - x_{l-1/2} - x_{l'-1/2}) \int_{\frac{2}{h_j}(x_{l-1/2} + x_{l'-1/2} - x_{\min} - x_j)}^{\frac{2}{h_j}(x_{j+1/2} - x_j)} F_1(\xi_j) d\xi_j \\
& + \theta(x_{j+1/2} - x_{l'-1/2})\theta(x_{j-1/2} - x_{l'-1/2})\theta(x_{j+1/2} + x_{\min} - x_{l-1/2} - x_{l'-1/2}) \\
& \theta(x_{j-1/2} + x_{\min} - x_{l-1/2} - x_{l'-1/2}) \int_{\frac{2}{h_j}(x_{j-1/2} - x_j)}^{\frac{2}{h_j}(x_{l-1/2} + x_{l'-1/2} - x_{\min} - x_j)} F_1(\xi_j) d\xi_j \\
& - \theta(x_{j-1/2} - x_{l'-1/2})\theta(x_{j+1/2} - x_{l'-1/2})^2 \theta(x_{j+1/2} + x_{\min} - x_{l-1/2} - x_{l'-1/2}) \int_{\frac{2}{h_j}(x_{l-1/2} + x_{l'-1/2} - x_{\min} - x_j)}^{\frac{2}{h_j}(x_{j+1/2} - x_j)} F_1(\xi_j) d\xi_j \\
& + \theta(x_{j+1/2} - x_{l'-1/2})^2 \theta(x_{j+1/2} + x_{\min} - x_{l-1/2} - x_{l'-1/2}) \int_{\frac{2}{h_j}(x_{l-1/2} + x_{l'-1/2} - x_{\min} - x_j)}^{\frac{2}{h_j}(x_{j+1/2} - x_j)} F_1(\xi_j) d\xi_j
\end{aligned} \tag{31}$$

$$\begin{aligned}
(1) = & \theta(x_{j+1/2} - x_{l'-1/2})\theta(x_{j+1/2} + x_{\min} - x_{l-1/2} - x_{l'-1/2}) \\
& \left[\int_{\frac{2}{h_j}(x_{l-1/2} + x_{l'-1/2} - x_{\min} - x_j)}^{\frac{2}{h_j}(x_{j+1/2} - x_j)} F_1(\xi_j) d\xi_j \right. \\
& \left. + \theta(x_{j-1/2} - x_{l'-1/2})\theta(x_{j-1/2} + x_{\min} - x_{l-1/2} - x_{l'-1/2}) \int_{\frac{2}{h_j}(x_{j-1/2} - x_j)}^{\frac{2}{h_j}(x_{l-1/2} + x_{l'-1/2} - x_{\min} - x_j)} F_1(\xi_j) d\xi_j \right].
\end{aligned} \tag{32}$$

The third triple integral writes, with the similar simplifications,

$$(3) \equiv \int_{\frac{2}{h_j}(x_{j-1/2} - x_j)}^{\frac{2}{h_j}(x_{j+1/2} - x_j)} F_3(\xi_j) \theta\left(\xi_j - \frac{2}{h_j}(x_{l'+1/2} - x_j)\right) \theta\left(\xi_j - \frac{2}{h_j}(x_{l-1/2} + x_{l'+1/2} - x_{\min} - x_j)\right) d\xi_j \tag{33}$$

$$\begin{aligned}
(3) = & \theta(x_{j+1/2} - x_{l'+1/2})\theta(x_{j+1/2} + x_{\min} - x_{l-1/2} - x_{l'+1/2}) \\
& \left[\int_{\frac{2}{h_j}(x_{l-1/2} + x_{l'+1/2} - x_{\min} - x_j)}^{\frac{2}{h_j}(x_{j+1/2} - x_j)} F_3(\xi_j) d\xi_j \right. \\
& \left. + \theta(x_{j-1/2} - x_{l'+1/2})\theta(x_{j-1/2} + x_{\min} - x_{l-1/2} - x_{l'+1/2}) \int_{\frac{2}{h_j}(x_{j-1/2} - x_j)}^{\frac{2}{h_j}(x_{l-1/2} + x_{l'+1/2} - x_{\min} - x_j)} F_3(\xi_j) d\xi_j \right].
\end{aligned} \tag{34}$$

The rule for the second and fourth triple integrals is

$$\left\{ \begin{aligned} & a < b, c < d \\ & \int_a^b f(x)\theta(x-c)\theta(x-d)dx = \\ & \theta(b-c)\theta(b-d) \left[(1-\theta(a-c))\theta(b-c) \int_d^b f(x)dx + \theta(a-c) \left(\int_d^b f(x)dx + \theta(a-d) \int_a^d f(x)dx \right) \right] \\ & = \theta(b-d) \left[\int_d^b f(x)dx - \theta(a-c)\theta^2(b-c) \int_d^b f(x)dx + \theta(b-c)\theta(a-c) \int_d^b f(x)dx + \theta(b-c)\theta(a-c)\theta(a-d) \int_a^d f(x)dx \right] \\ & = \theta(b-c)\theta(b-d) \left[\int_d^b f(x)dx + \theta(a-c)\theta(a-d) \int_a^d f(x)dx \right]. \end{aligned} \right. \quad (35)$$

Let denote

$$\begin{aligned} F_2(\xi_j) &\equiv \int_{\frac{2}{h_{l'}} \left(\frac{h_j}{2} \xi_j + x_j - x_{l+1/2} + x_{\min} - x_{l'} \right)}^{\frac{2}{h_{l'}} (x_{l+1/2} - x_{l'})} \int_{\frac{2}{h_l} \left(\frac{h_j}{2} \xi_j + x_j - \frac{h_{l'}}{2} \xi_{l'} - x_{l'} + x_{\min} - x_l \right)}^{\frac{2}{h_l} (x_{l+1/2} - x_l)} f_1(\xi_{l'}) f_2(\xi_l) \partial_{\xi_j} \phi_k(\xi_j) d\xi_l d\xi_{l'}, \\ F_4(\xi_j) &\equiv \int_{\frac{2}{h_{l'}} \left(\frac{h_j}{2} \xi_j + x_j - x_{l+1/2} + x_{\min} - x_{l'} \right)}^{\frac{2}{h_{l'}} (x_{l'+1/2} - x_{l'})} \int_{\frac{2}{h_l} \left(\frac{h_j}{2} \xi_j + x_j - \frac{h_{l'}}{2} \xi_{l'} - x_{l'} + x_{\min} - x_l \right)}^{\frac{2}{h_l} (x_{l+1/2} - x_l)} f_1(\xi_{l'}) f_2(\xi_l) \partial_{\xi_j} \phi_k(\xi_j) d\xi_l d\xi_{l'}. \end{aligned} \quad (36)$$

The second triple integral writes

$$(2) \equiv \int_{\frac{2}{h_j} (x_{j-1/2} - x_j)}^{\frac{2}{h_j} (x_{j+1/2} - x_j)} F_2(\xi_j) \theta \left(\xi_j - \frac{2}{h_j} (x_{l'-1/2} - x_j) \right) \theta \left(\xi_j - \frac{2}{h_j} (x_{l+1/2} + x_{l'-1/2} - x_{\min} - x_j) \right) d\xi_l d\xi_{l'} d\xi_j. \quad (37)$$

$$\begin{aligned} (2) &= \theta(x_{j+1/2} - x_{l'-1/2}) \theta(x_{j+1/2} - x_{l+1/2} - x_{l'-1/2} + x_{\min}) \\ &\left[\int_{\frac{2}{h_j} (x_{l+1/2} + x_{l'-1/2} - x_{\min} - x_j)}^{\frac{2}{h_j} (x_{j+1/2} - x_j)} F_2(\xi_j) d\xi_j \right. \\ &\quad \left. + \theta(x_{j-1/2} - x_{l'-1/2}) \theta(x_{j-1/2} - x_{l+1/2} - x_{l'-1/2} + x_{\min}) \int_{\frac{2}{h_j} (x_{j-1/2} - x_j)}^{\frac{2}{h_j} (x_{l+1/2} + x_{l'-1/2} - x_{\min} - x_j)} F_2(\xi_j) d\xi_j \right]. \end{aligned} \quad (38)$$

The fourth triple integral writes

$$(4) \equiv \int_{\frac{2}{h_j} (x_{j-1/2} - x_j)}^{\frac{2}{h_j} (x_{j+1/2} - x_j)} F_4(\xi_j) \theta \left(\xi_j - \frac{2}{h_j} (x_{l'+1/2} - x_j) \right) \theta \left(\xi_j - \frac{2}{h_j} (x_{l+1/2} + x_{l'+1/2} - x_{\min} - x_j) \right) d\xi_l d\xi_{l'} d\xi_j. \quad (39)$$

$$\begin{aligned} (4) &= \theta(x_{j+1/2} - x_{l'+1/2}) \theta(x_{j+1/2} - x_{l+1/2} - x_{l'+1/2} + x_{\min}) \\ &\left[\int_{\frac{2}{h_j} (x_{l+1/2} + x_{l'+1/2} - x_{\min} - x_j)}^{\frac{2}{h_j} (x_{j+1/2} - x_j)} F_4(\xi_j) d\xi_j \right. \\ &\quad \left. + \theta(x_{j-1/2} - x_{l'+1/2}) \theta(x_{j-1/2} - x_{l+1/2} - x_{l'+1/2} + x_{\min}) \int_{\frac{2}{h_j} (x_{j-1/2} - x_j)}^{\frac{2}{h_j} (x_{l+1/2} + x_{l'+1/2} - x_{\min} - x_j)} F_4(\xi_j) d\xi_j \right]. \end{aligned} \quad (40)$$

Then \mathcal{T} writes

$$\begin{aligned}
\mathcal{T}(x_{\min}, x_{\max}, j, k, i', i, l', l) &= \frac{h_l}{2} \frac{h_{l'}}{2} \times \\
&\left[\frac{\frac{2}{h_l}(x_{l+1/2} - x_l)}{\frac{2}{h_l}(x_{l-1/2} - x_l)} \int f_2(\xi_l) d\xi_l \times \right. \\
&\left. \left\{ \theta(x_{j+1/2} - x_{l'-1/2}) \right. \right. \\
&\left. \left(\int_{\frac{\frac{2}{h_j}(x_{j+1/2} - x_j)}{\frac{2}{h_j}(x_{l'-1/2} - x_j)} \int_{\frac{\frac{2}{h_{l'}}(\frac{h_j}{2}\xi_j + x_j - x_{l'})}{\frac{2}{h_{l'}}(x_{l'-1/2} - x_{l'})}} f_1(\xi_{l'}) \partial_{\xi_j} \phi_k(\xi_j) d\xi_j + \theta(x_{j-1/2} - x_{l'-1/2}) \int_{\frac{\frac{2}{h_j}(x_{j-1/2} - x_j)}{\frac{2}{h_j}(x_{l'-1/2} - x_j)} \int_{\frac{\frac{2}{h_{l'}}(\frac{h_j}{2}\xi_j + x_j - x_{l'})}{\frac{2}{h_{l'}}(x_{l'-1/2} - x_{l'})}} f_1(\xi_{l'}) \partial_{\xi_j} \phi_k(\xi_j) d\xi_j \right. \right. \\
&\left. \left. - \theta(x_{j+1/2} - x_{l'+1/2}) \left(\int_{\frac{\frac{2}{h_j}(x_{j+1/2} - x_j)}{\frac{2}{h_j}(x_{l'+1/2} - x_j)} \int_{\frac{\frac{2}{h_{l'}}(\frac{h_j}{2}\xi_j + x_j - x_{l'})}{\frac{2}{h_{l'}}(x_{l'+1/2} - x_{l'})}} f_1(\xi_{l'}) \partial_{\xi_j} \phi_k(\xi_j) d\xi_j + \theta(x_{j-1/2} - x_{l'+1/2}) \int_{\frac{\frac{2}{h_j}(x_{j-1/2} - x_j)}{\frac{2}{h_j}(x_{l'+1/2} - x_j)} \int_{\frac{\frac{2}{h_{l'}}(\frac{h_j}{2}\xi_j + x_j - x_{l'})}{\frac{2}{h_{l'}}(x_{l'+1/2} - x_{l'})}} f_1(\xi_{l'}) \partial_{\xi_j} \phi_k(\xi_j) d\xi_j \right) \right\} \\
&+ \theta(x_{j+1/2} - x_{l'-1/2}) \theta(x_{j+1/2} + x_{\min} - x_{l-1/2} - x_{l'-1/2}) \\
&\left[\int_{\frac{\frac{2}{h_j}(x_{j+1/2} - x_j)}{\frac{2}{h_j}(x_{l-1/2} + x_{l'-1/2} - x_{\min} - x_j)}} F_1(\xi_j) d\xi_j \right. \\
&\left. + \theta(x_{j-1/2} - x_{l'-1/2}) \theta(x_{j-1/2} + x_{\min} - x_{l-1/2} - x_{l'-1/2}) \int_{\frac{\frac{2}{h_j}(x_{j-1/2} - x_j)}{\frac{2}{h_j}(x_{l-1/2} + x_{l'-1/2} - x_{\min} - x_j)}} F_1(\xi_j) d\xi_j \right] \\
&+ \theta(x_{j+1/2} - x_{l'+1/2}) \theta(x_{j+1/2} + x_{\min} - x_{l-1/2} - x_{l'+1/2}) \\
&\left[\int_{\frac{\frac{2}{h_j}(x_{j+1/2} - x_j)}{\frac{2}{h_j}(x_{l-1/2} + x_{l'+1/2} - x_{\min} - x_j)}} F_3(\xi_j) d\xi_j \right. \\
&\left. + \theta(x_{j-1/2} - x_{l'+1/2}) \theta(x_{j-1/2} + x_{\min} - x_{l-1/2} - x_{l'+1/2}) \int_{\frac{\frac{2}{h_j}(x_{j-1/2} - x_j)}{\frac{2}{h_j}(x_{l-1/2} + x_{l'+1/2} - x_{\min} - x_j)}} F_3(\xi_j) d\xi_j \right]
\end{aligned} \tag{41}$$

$$\begin{aligned}
& -\theta(x_{j+1/2} - x_{l'-1/2})\theta(x_{j+1/2} - x_{l+1/2} - x_{l'-1/2} + x_{\min}) \\
& \left[\int_{\frac{2}{h_j}(x_{l+1/2}+x_{l'-1/2}-x_{\min}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)} F_2(\xi_j) d\xi_j \right. \\
& \left. +\theta(x_{j-1/2} - x_{l'-1/2})\theta(x_{j-1/2} - x_{l+1/2} - x_{l'-1/2} + x_{\min}) \int_{\frac{2}{h_j}(x_{j-1/2}-x_j)}^{\frac{2}{h_j}(x_{l+1/2}+x_{l'-1/2}-x_{\min}-x_j)} F_2(\xi_j) d\xi_j \right] \\
& -\theta(x_{j+1/2} - x_{l'+1/2})\theta(x_{j+1/2} - x_{l+1/2} - x_{l'+1/2} + x_{\min}) \\
& \left[\int_{\frac{2}{h_j}(x_{l+1/2}+x_{l'+1/2}-x_{\min}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)} F_4(\xi_j) d\xi_j \right. \\
& \left. +\theta(x_{j-1/2} - x_{l'+1/2})\theta(x_{j-1/2} - x_{l+1/2} - x_{l'+1/2} + x_{\min}) \int_{\frac{2}{h_j}(x_{j-1/2}-x_j)}^{\frac{2}{h_j}(x_{l+1/2}+x_{l'+1/2}-x_{\min}-x_j)} F_4(\xi_j) d\xi_j \right] \Bigg].
\end{aligned}$$

To reduce as much as possible the number of integrals to evaluate, we define the following term for the scheme

$$\begin{aligned}
\mathcal{T}_{\phi_i} &\equiv \int_{\frac{2}{h_l}(x_{l-1/2}-x_l)}^{\frac{2}{h_j}(x_{l+1/2}-x_l)} f_2(\xi_l) d\xi_l, \mathcal{T}_{\phi_{i'}, \text{allmix}} \equiv \int_{\frac{2}{h_j}(x_{j-1/2}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)} \int_{\frac{2}{h_{l'}}(x_{l'-1/2}-x_{l'})}^{\frac{2}{h_{l'}}(x_{l'+1/2}-x_{l'})} f_1(\xi_{l'}) \partial_{\xi_j} \phi_k(\xi_j) d\xi_{l'} d\xi_j, \\
\mathcal{T}_{\phi_{i'}, \text{mixP1}} &\equiv \int_{\frac{2}{h_j}(x_{j-1/2}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)} \int_{\frac{2}{h_{l'}}(x_{l'-1/2}-x_{l'})}^{\frac{2}{h_{l'}}(\frac{h_j}{2}\xi_j+x_j-x_{l'})} f_1(\xi_{l'}) \partial_{\xi_j} \phi_k(\xi_j) d\xi_{l'} d\xi_j, \\
\mathcal{T}_{\phi_{i'}, \text{P1term1}} &\equiv \int_{\frac{2}{h_j}(x_{l'-1/2}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)} \int_{\frac{2}{h_{l'}}(x_{l'-1/2}-x_{l'})}^{\frac{2}{h_{l'}}(\frac{h_j}{2}\xi_j+x_j-x_{l'})} f_1(\xi_{l'}) \partial_{\xi_j} \phi_k(\xi_j) d\xi_{l'} d\xi_j, \\
\mathcal{T}_{\phi_{i'}, \text{P2term1}} &\equiv \int_{\frac{2}{h_j}(x_{l'+1/2}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)} \int_{\frac{2}{h_{l'}}(x_{l'+1/2}-x_{l'})}^{\frac{2}{h_{l'}}(\frac{h_j}{2}\xi_j+x_j-x_{l'})} f_1(\xi_{l'}) \partial_{\xi_j} \phi_k(\xi_j) d\xi_{l'} d\xi_j, \\
\mathcal{T}_{\phi_{i'}, \phi_i, \text{allmix}} &\equiv \int_{\frac{2}{h_j}(x_{j-1/2}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)} \int_{\frac{2}{h_{l'}}(x_{l'-1/2}-x_{l'})}^{\frac{2}{h_{l'}}(x_{l'+1/2}-x_{l'})} \int_{\frac{2}{h_l}(x_{l-1/2}-x_l)}^{\frac{2}{h_l}(x_{l+1/2}-x_l)} f_1(\xi_{l'}) f_2(\xi_l) \partial_{\xi_j} \phi_k(\xi_j) d\xi_l d\xi_{l'} d\xi_j \\
\mathcal{T}_{\phi_{i'}, \phi_i, \text{mixterm1term3}} &\equiv \int_{\frac{2}{h_j}(x_{j-1/2}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)} \int_{\frac{2}{h_{l'}}(x_{l'-1/2}-x_{l'})}^{\frac{2}{h_{l'}}(x_{l'+1/2}-x_{l'})} \int_{\frac{2}{h_l}(x_{l-1/2}-x_l)}^{\frac{2}{h_l}(x_{l+1/2}-x_l)} f_1(\xi_{l'}) f_2(\xi_l) \partial_{\xi_j} \phi_k(\xi_j) d\xi_l d\xi_{l'} d\xi_j \\
\mathcal{T}_{\phi_{i'}, \phi_i, \text{mixterm1}} &\equiv \int_{\frac{2}{h_j}(x_{j-1/2}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)} \int_{\frac{2}{h_{l'}}(x_{l'-1/2}-x_{l'})}^{\frac{2}{h_{l'}}(\frac{h_j}{2}\xi_j+x_j-x_{l-1/2}+x_{\min}-x_{l'})} \int_{\frac{2}{h_l}(x_{l-1/2}-x_l)}^{\frac{2}{h_l}(x_{l+1/2}-x_l)} f_1(\xi_{l'}) f_2(\xi_l) \partial_{\xi_j} \phi_k(\xi_j) d\xi_l d\xi_{l'} d\xi_j
\end{aligned} \tag{42}$$

$$\begin{aligned}
\mathcal{T}_{\phi_{i'}, \phi_i, \text{mixterm2}} &\equiv \int_{\frac{2}{h_j}(x_{j-1/2}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)} \int_{\frac{2}{h_{l'}}(x_{l'-1/2}-x_{l'})}^{\frac{2}{h_{l'}}(\frac{h_j}{2}\xi_j+x_j-x_{l+1/2}+x_{\min}-x_{l'})} \int_{\frac{2}{h_l}(x_{l+1/2}-x_l)}^{\frac{2}{h_l}(\frac{h_j}{2}\xi_j+x_j-\frac{h_{l'}}{2}\xi_{l'}-x_{l'}+x_{\min}-x_l)} f_1(\xi_{l'})f_2(\xi_l)\partial_{\xi_j}\phi_k(\xi_j)d\xi_l d\xi_{l'} d\xi_j \\
\mathcal{T}_{\phi_{i'}, \phi_i, \text{term1}} &\equiv \int_{\frac{2}{h_j}(x_{l-1/2}+x_{l'-1/2}-x_{\min}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)} \int_{\frac{2}{h_{l'}}(x_{l'-1/2}-x_{l'})}^{\frac{2}{h_{l'}}(\frac{h_j}{2}\xi_j+x_j-x_{l-1/2}+x_{\min}-x_{l'})} \int_{\frac{2}{h_l}(x_{l-1/2}-x_l)}^{\frac{2}{h_l}(\frac{h_j}{2}\xi_j+x_j-\frac{h_{l'}}{2}\xi_{l'}-x_{l'}+x_{\min}-x_l)} f_1(\xi_{l'})f_2(\xi_l)\partial_{\xi_j}\phi_k(\xi_j)d\xi_l d\xi_{l'} d\xi_j \\
\mathcal{T}_{\phi_{i'}, \phi_i, \text{term2}} &\equiv \int_{\frac{2}{h_j}(x_{l+1/2}+x_{l'-1/2}-x_{\min}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)} \int_{\frac{2}{h_{l'}}(x_{l'-1/2}-x_{l'})}^{\frac{2}{h_{l'}}(\frac{h_j}{2}\xi_j+x_j-x_{l+1/2}+x_{\min}-x_{l'})} \int_{\frac{2}{h_l}(x_{l+1/2}-x_l)}^{\frac{2}{h_l}(\frac{h_j}{2}\xi_j+x_j-\frac{h_{l'}}{2}\xi_{l'}-x_{l'}+x_{\min}-x_l)} f_1(\xi_{l'})f_2(\xi_l)\partial_{\xi_j}\phi_k(\xi_j)d\xi_l d\xi_{l'} d\xi_j \\
\mathcal{T}_{\phi_{i'}, \phi_i, \text{term3}} &\equiv \int_{\frac{2}{h_j}(x_{l-1/2}+x_{l'+1/2}-x_{\min}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)} \int_{\frac{2}{h_{l'}}(x_{l'+1/2}-x_{l'})}^{\frac{2}{h_{l'}}(\frac{h_j}{2}\xi_j+x_j-x_{l-1/2}+x_{\min}-x_{l'})} \int_{\frac{2}{h_l}(x_{l-1/2}-x_l)}^{\frac{2}{h_l}(\frac{h_j}{2}\xi_j+x_j-\frac{h_{l'}}{2}\xi_{l'}-x_{l'}+x_{\min}-x_l)} f_1(\xi_{l'})f_2(\xi_l)\partial_{\xi_j}\phi_k(\xi_j)d\xi_l d\xi_{l'} d\xi_j \\
\mathcal{T}_{\phi_{i'}, \phi_i, \text{term4}} &\equiv \int_{\frac{2}{h_j}(x_{l+1/2}+x_{l'+1/2}-x_{\min}-x_j)}^{\frac{2}{h_j}(x_{j+1/2}-x_j)} \int_{\frac{2}{h_{l'}}(x_{l'+1/2}-x_{l'})}^{\frac{2}{h_{l'}}(\frac{h_j}{2}\xi_j+x_j-x_{l+1/2}+x_{\min}-x_{l'})} \int_{\frac{2}{h_l}(x_{l+1/2}-x_l)}^{\frac{2}{h_l}(\frac{h_j}{2}\xi_j+x_j-\frac{h_{l'}}{2}\xi_{l'}-x_{l'}+x_{\min}-x_l)} f_1(\xi_{l'})f_2(\xi_l)\partial_{\xi_j}\phi_k(\xi_j)d\xi_l d\xi_{l'} d\xi_j.
\end{aligned}$$

2.3 Fortran scheme for integral of the flux term

Finally, the Fortran scheme to evaluate $\mathcal{T}(x, x_{\min}, x_{\max}, j, k, i', i, l', l)$ writes

```

1  res1 = 0
2  if ( $x_{j+1/2} > x_{l'+1/2}$ ) then
3    if ( $x_{j-1/2} > x_{l'+1/2}$ ) then
4      res1 =  $\mathcal{T}_{\phi_i} \times \mathcal{T}_{\phi_{i'}, \text{allmix}}$ 
5    else if ( $x_{j-1/2} \leq x_{l'+1/2}$  and  $x_{j-1/2} > x_{l'-1/2}$ ) then
6      res1 =  $\mathcal{T}_{\phi_i} \times (\mathcal{T}_{\phi_{i'}, \text{mixP1}} - \mathcal{T}_{\phi_{i'}, \text{P2term1}})$ 
7    else
8      res1 =  $\mathcal{T}_{\phi_i} \times (\mathcal{T}_{\phi_{i'}, \text{P1term1}} - \mathcal{T}_{\phi_{i'}, \text{P2term1}})$ 
9    endif
10
11 else if ( $x_{j+1/2} \leq x_{l'+1/2}$  and  $x_{j+1/2} > x_{l'-1/2}$ ) then
12   if ( $x_{j-1/2} > x_{l'-1/2}$ ) then
13     res1 =  $\mathcal{T}_{\phi_i} \times \mathcal{T}_{\phi_{i'}, \text{mixP1}}$ 
14   else
15     res1 =  $\mathcal{T}_{\phi_i} \times \mathcal{T}_{\phi_{i'}, \text{P1term1}}$ 
16   endif
17 else
18   res1 = 0
19 endif
20
21
22 res2 = 0
23 if ( $x_{j+1/2} > x_{l'+1/2}$ ) then
24   if ( $x_{j+1/2} > x_{l+1/2} + x_{l'+1/2} - x_{\min}$ ) then
25     if ( $x_{j-1/2} > x_{l'+1/2}$ ) then
26       if ( $x_{j-1/2} > x_{l+1/2} + x_{l'+1/2} - x_{\min}$ ) then
27         res2 =  $\mathcal{T}_{\phi_{i'}, \phi_i, \text{allmix}}$ 
28       else if ( $x_{j-1/2} \leq x_{l+1/2} + x_{l'+1/2} - x_{\min}$  and  $x_{j-1/2} > x_{l-1/2} + x_{l'+1/2} - x_{\min}$  and
29          $x_{j-1/2} > x_{l+1/2} + x_{l'-1/2} - x_{\min}$ ) then
30         res2 =  $\mathcal{T}_{\phi_{i'}, \phi_i, \text{mixterm1term3}} - \mathcal{T}_{\phi_{i'}, \phi_i, \text{mixterm2}} - \mathcal{T}_{\phi_{i'}, \phi_i, \text{term4}}$ 
31       else if ( $x_{j-1/2} \leq x_{l+1/2} + x_{l'+1/2} - x_{\min}$  and  $x_{j-1/2} > x_{l-1/2} + x_{l'+1/2} - x_{\min}$  and
32          $x_{j-1/2} \leq x_{l+1/2} + x_{l'-1/2} - x_{\min}$ ) then
33         res2 =  $\mathcal{T}_{\phi_{i'}, \phi_i, \text{mixterm1term3}} - \mathcal{T}_{\phi_{i'}, \phi_i, \text{term2}} - \mathcal{T}_{\phi_{i'}, \phi_i, \text{term4}}$ 
34       else if ( $x_{j-1/2} \leq x_{l+1/2} + x_{l'+1/2} - x_{\min}$  and  $x_{j-1/2} \leq x_{l-1/2} + x_{l'+1/2} - x_{\min}$  and
35          $x_{j-1/2} > x_{l+1/2} + x_{l'-1/2} - x_{\min}$ ) then
36         res2 =  $\mathcal{T}_{\phi_{i'}, \phi_i, \text{mixterm1}} - \mathcal{T}_{\phi_{i'}, \phi_i, \text{mixterm2}} + \mathcal{T}_{\phi_{i'}, \phi_i, \text{term3}} - \mathcal{T}_{\text{term4}}$ 
37       else if ( $x_{j-1/2} \leq x_{l+1/2} + x_{l'+1/2} - x_{\min}$  and  $x_{j-1/2} \leq x_{l-1/2} + x_{l'+1/2} - x_{\min}$  and
38          $x_{j-1/2} \leq x_{l+1/2} + x_{l'-1/2} - x_{\min}$  and  $x_{j-1/2} > x_{l-1/2} + x_{l'-1/2} - x_{\min}$ ) then
39         res2 =  $\mathcal{T}_{\phi_{i'}, \phi_i, \text{mixterm1}} - \mathcal{T}_{\phi_{i'}, \phi_i, \text{term2}} + \mathcal{T}_{\phi_{i'}, \phi_i, \text{term3}} - \mathcal{T}_{\phi_{i'}, \phi_i, \text{term4}}$ 
40       else
41         res2 =  $\mathcal{T}_{\phi_{i'}, \phi_i, \text{term1}} - \mathcal{T}_{\phi_{i'}, \phi_i, \text{term2}} + \mathcal{T}_{\phi_{i'}, \phi_i, \text{term3}} - \mathcal{T}_{\phi_{i'}, \phi_i, \text{term4}}$ 
42       endif
43     else if ( $x_{j-1/2} \leq x_{l'+1/2}$  and  $x_{j-1/2} > x_{l'-1/2}$ ) then
44       if ( $x_{j-1/2} > x_{l+1/2} + x_{l'-1/2} - x_{\min}$ ) then
45         res2 =  $\mathcal{T}_{\phi_{i'}, \phi_i, \text{mixterm1}} - \mathcal{T}_{\phi_{i'}, \phi_i, \text{mixterm2}} + \mathcal{T}_{\phi_{i'}, \phi_i, \text{term3}} - \mathcal{T}_{\phi_{i'}, \phi_i, \text{term4}}$ 
46       else if ( $x_{j-1/2} \leq x_{l+1/2} + x_{l'-1/2} - x_{\min}$ 
47         and  $x_{j-1/2} > x_{l-1/2} + x_{l'-1/2} - x_{\min}$ ) then
48         res2 =  $\mathcal{T}_{\phi_{i'}, \phi_i, \text{mixterm1}} - \mathcal{T}_{\phi_{i'}, \phi_i, \text{term2}} + \mathcal{T}_{\phi_{i'}, \phi_i, \text{term3}} - \mathcal{T}_{\phi_{i'}, \phi_i, \text{term4}}$ 
49       else
50         res2 =  $\mathcal{T}_{\phi_{i'}, \phi_i, \text{term1}} - \mathcal{T}_{\phi_{i'}, \phi_i, \text{term2}} + \mathcal{T}_{\phi_{i'}, \phi_i, \text{term3}} - \mathcal{T}_{\phi_{i'}, \phi_i, \text{term4}}$ 
51       endif
52     else
53       res2 =  $\mathcal{T}_{\phi_{i'}, \phi_i, \text{term1}} - \mathcal{T}_{\phi_{i'}, \phi_i, \text{term2}} + \mathcal{T}_{\phi_{i'}, \phi_i, \text{term3}} - \mathcal{T}_{\phi_{i'}, \phi_i, \text{term4}}$ 
54     endif

```

```

55     else if ( $x_{j+1/2} \leq x_{l+1/2} + x_{l'+1/2} - x_{\min}$  and  $x_{j+1/2} > x_{l-1/2} + x_{l'+1/2} - x_{\min}$  and
56          $x_{j+1/2} > x_{l+1/2} + x_{l'-1/2} - x_{\min}$ ) then
57         if ( $x_{j-1/2} > x_{l'+1/2}$ ) then
58             if ( $x_{j-1/2} > x_{l-1/2} + x_{l'+1/2} - x_{\min}$  and  $x_{j-1/2} > x_{l+1/2} + x_{l'-1/2} - x_{\min}$ ) then
59                  $\text{res2} = \mathcal{T}_{\phi_{i'}, \phi_i, \text{mixterm1term3}} - \mathcal{T}_{\phi_{i'}, \phi_i, \text{mixterm2}}$ 
60             else if ( $x_{j-1/2} > x_{l-1/2} + x_{l'+1/2} - x_{\min}$  and  $x_{j-1/2} \leq x_{l+1/2} + x_{l'-1/2} - x_{\min}$ ) then
61                  $\text{res2} = \mathcal{T}_{\phi_{i'}, \phi_i, \text{mixterm1term3}} - \mathcal{T}_{\phi_{i'}, \phi_i, \text{term2}}$ 
62             else if ( $x_{j-1/2} \leq x_{l-1/2} + x_{l'+1/2} - x_{\min}$  and  $x_{j-1/2} > x_{l+1/2} + x_{l'-1/2} - x_{\min}$ ) then
63                  $\text{res2} = \mathcal{T}_{\phi_{i'}, \phi_i, \text{mixterm1}} - \mathcal{T}_{\phi_{i'}, \phi_i, \text{mixterm2}} + \mathcal{T}_{\phi_{i'}, \phi_i, \text{term3}}$ 
64             else if ( $x_{j-1/2} \leq x_{l-1/2} + x_{l'+1/2} - x_{\min}$  and  $x_{j-1/2} \leq x_{l+1/2} + x_{l'-1/2} - x_{\min}$  and
65                  $x_{j-1/2} > x_{l-1/2} + x_{l'-1/2} - x_{\min}$ ) then
66                  $\text{res2} = \mathcal{T}_{\phi_{i'}, \phi_i, \text{mixterm1}} - \mathcal{T}_{\phi_{i'}, \phi_i, \text{term2}} + \mathcal{T}_{\phi_{i'}, \phi_i, \text{term3}}$ 
67             else
68                  $\text{res2} = \mathcal{T}_{\phi_{i'}, \phi_i, \text{term1}} - \mathcal{T}_{\phi_{i'}, \phi_i, \text{term2}} + \mathcal{T}_{\phi_{i'}, \phi_i, \text{term3}}$ 
69             endif
70         else if ( $x_{j-1/2} \leq x_{l'+1/2}$  and  $x_{j-1/2} > x_{l'-1/2}$ ) then
71             if ( $x_{j-1/2} > x_{l+1/2} + x_{l'-1/2} - x_{\min}$ ) then
72                  $\text{res2} = \mathcal{T}_{\phi_{i'}, \phi_i, \text{mixterm1}} - \mathcal{T}_{\phi_{i'}, \phi_i, \text{mixterm2}} + \mathcal{T}_{\phi_{i'}, \phi_i, \text{term3}}$ 
73             else if ( $x_{j-1/2} \leq x_{l+1/2} + x_{l'-1/2} - x_{\min}$  and  $x_{j-1/2} > x_{l-1/2} + x_{l'-1/2} - x_{\min}$ ) then
74                  $\text{res2} = \mathcal{T}_{\phi_{i'}, \phi_i, \text{mixterm1}} - \mathcal{T}_{\phi_{i'}, \phi_i, \text{term2}} + \mathcal{T}_{\phi_{i'}, \phi_i, \text{term3}}$ 
75             else
76                  $\text{res2} = \mathcal{T}_{\phi_{i'}, \phi_i, \text{term1}} - \mathcal{T}_{\phi_{i'}, \phi_i, \text{term2}} + \mathcal{T}_{\phi_{i'}, \phi_i, \text{term3}}$ 
77             endif
78         else
79              $\text{res2} = \mathcal{T}_{\phi_{i'}, \phi_i, \text{term1}} - \mathcal{T}_{\phi_{i'}, \phi_i, \text{term2}} + \mathcal{T}_{\phi_{i'}, \phi_i, \text{term3}}$ 
80         endif
81     else if ( $x_{j+1/2} \leq x_{l+1/2} + x_{l'+1/2} - x_{\min}$  and  $x_{j+1/2} > x_{l-1/2} + x_{l'+1/2} - x_{\min}$  and
82          $x_{j+1/2} \leq x_{l+1/2} + x_{l'-1/2} - x_{\min}$ ) then
83         if ( $x_{j-1/2} > x_{l'+1/2}$ ) then
84             if ( $x_{j-1/2} > x_{l-1/2} + x_{l'+1/2} - x_{\min}$ ) then
85                  $\text{res2} = \mathcal{T}_{\phi_{i'}, \phi_i, \text{mixterm1term3}}$ 
86             else if ( $x_{j-1/2} \leq x_{l-1/2} + x_{l'+1/2} - x_{\min}$  and  $x_{j-1/2} > x_{l-1/2} + x_{l'-1/2} - x_{\min}$ ) then
87                  $\text{res2} = \mathcal{T}_{\phi_{i'}, \phi_i, \text{mixterm1}} + \mathcal{T}_{\phi_{i'}, \phi_i, \text{term3}}$ 
88             else
89                  $\text{res2} = \mathcal{T}_{\phi_{i'}, \phi_i, \text{term1}} + \mathcal{T}_{\phi_{i'}, \phi_i, \text{term3}}$ 
90             endif
91         else if ( $x_{j-1/2} \leq x_{l'+1/2}$  and  $x_{j-1/2} > x_{l'-1/2}$ ) then
92             if ( $x_{j-1/2} > x_{l-1/2} + x_{l'-1/2} - x_{\min}$ ) then
93                  $\text{res2} = \mathcal{T}_{\phi_{i'}, \phi_i, \text{mixterm1}} + \mathcal{T}_{\phi_{i'}, \phi_i, \text{term3}}$ 
94             else
95                  $\text{res2} = \mathcal{T}_{\phi_{i'}, \phi_i, \text{term1}} + \mathcal{T}_{\phi_{i'}, \phi_i, \text{term3}}$ 
96             endif
97         else
98              $\text{res2} = \mathcal{T}_{\phi_{i'}, \phi_i, \text{term1}} + \mathcal{T}_{\phi_{i'}, \phi_i, \text{term3}}$ 
99         endif
100     else if ( $x_{j+1/2} \leq x_{l+1/2} + x_{l'+1/2} - x_{\min}$  and  $x_{j+1/2} \leq x_{l-1/2} + x_{l'+1/2} - x_{\min}$  and
101          $x_{j+1/2} > x_{l+1/2} + x_{l'-1/2} - x_{\min}$ ) then
102         if ( $x_{j-1/2} > x_{l'-1/2}$ ) then
103             if ( $x_{j-1/2} > x_{l+1/2} + x_{l'-1/2} - x_{\min}$ ) then
104                  $\text{res2} = \mathcal{T}_{\phi_{i'}, \phi_i, \text{mixterm1}} - \mathcal{T}_{\phi_{i'}, \phi_i, \text{mixterm2}}$ 
105             else if ( $x_{j-1/2} \leq x_{l+1/2} + x_{l'-1/2} - x_{\min}$  and  $x_{j-1/2} > x_{l-1/2} + x_{l'-1/2} - x_{\min}$ ) then
106                  $\text{res2} = \mathcal{T}_{\phi_{i'}, \phi_i, \text{mixterm1}} - \mathcal{T}_{\phi_{i'}, \phi_i, \text{term2}}$ 
107             else
108                  $\text{res2} = \mathcal{T}_{\phi_{i'}, \phi_i, \text{term1}} - \mathcal{T}_{\phi_{i'}, \phi_i, \text{term2}}$ 
109             endif
110         else
111              $\text{res2} = \mathcal{T}_{\phi_{i'}, \phi_i, \text{term1}} - \mathcal{T}_{\phi_{i'}, \phi_i, \text{term2}}$ 

```

```

112     endif
113 else if (xj+1/2 ≤ xl+1/2 + xl'+1/2 - xmin and xj+1/2 ≤ xl-1/2 + xl'+1/2 - xmin and
114         xj+1/2 ≤ xl+1/2 + xl'-1/2 - xmin and xj+1/2 > xl-1/2 + xl'-1/2 - xmin) then
115     if (xj-1/2 > xl'-1/2) then
116         if (xj-1/2 > xl-1/2 + xl'-1/2 - xmin) then
117             res2 =  $\mathcal{T}_{\phi_{i'}, \phi_i, \text{mixterm1}}$ 
118         else
119             res2 =  $\mathcal{T}_{\phi_{i'}, \phi_i, \text{term1}}$ 
120         endif
121     else
122         res2 =  $\mathcal{T}_{\phi_{i'}, \phi_i, \text{term1}}$ 
123     endif
124 else
125     res2 = 0
126 endif
127 else if (xj+1/2 ≤ xl'+1/2 and xj+1/2 > xl'-1/2) then
128     if (xj+1/2 > xl+1/2 + xl'-1/2 - xmin) then
129         if (xj-1/2 > xl'-1/2) then
130             if (xj-1/2 > xl+1/2 + xl'-1/2 - xmin) then
131                 res2 =  $\mathcal{T}_{\phi_{i'}, \phi_i, \text{mixterm1}}$  -  $\mathcal{T}_{\phi_{i'}, \phi_i, \text{mixterm2}}$ 
132             else if (xj-1/2 ≤ xl+1/2 + xl'-1/2 - xmin and xj-1/2 > xl-1/2 + xl'-1/2 - xmin) then
133                 res2 =  $\mathcal{T}_{\phi_{i'}, \phi_i, \text{mixterm1}}$  -  $\mathcal{T}_{\phi_{i'}, \phi_i, \text{term2}}$ 
134             else
135                 res2 =  $\mathcal{T}_{\phi_{i'}, \phi_i, \text{term1}}$  -  $\mathcal{T}_{\phi_{i'}, \phi_i, \text{term2}}$ 
136             endif
137         else
138             res2 =  $\mathcal{T}_{\phi_{i'}, \phi_i, \text{term1}}$  -  $\mathcal{T}_{\phi_{i'}, \phi_i, \text{term2}}$ 
139         endif
140     else if (xj+1/2 ≤ xl+1/2 + xl'-1/2 - xmin and xj+1/2 > xl-1/2 + xl'-1/2 - xmin) then
141         if (xj-1/2 > xl'-1/2) then
142             if (xj-1/2 > xl-1/2 + xl'-1/2 - xmin) then
143                 res2 =  $\mathcal{T}_{\phi_{i'}, \phi_i, \text{mixterm1}}$ 
144             else
145                 res2 =  $\mathcal{T}_{\phi_{i'}, \phi_i, \text{term1}}$ 
146             endif
147         else
148             res2 =  $\mathcal{T}_{\phi_{i'}, \phi_i, \text{term1}}$ 
149         endif
150     else
151         res2 = 0
152     endif
153 else
154     res2 = 0
155 endif

```