Optimal Control for Nuclear Reactors

Calculus of Variations Final Presentation

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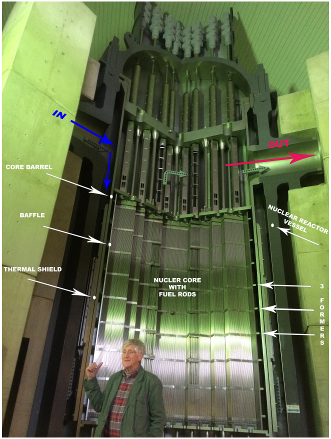
Basics of Nuclear Power

1. Just boils water to generate steam
2. Energy released via fission chain reactions in fissile Uranium 235 (in fuel rods)
3. Self-sustaining chain reaction that's stable due to negative feedback



Motivation

We would like to understand how to best control the chain reaction to acheive a desired end state, e.g.

1. Power uprate
2. Reactor startup
3. Minimize power peaking in the core

Nominally operate at *steady state*, but these state changes are necessarily *dynamic*. Safety and performance relevant. Optimal control used historically to investigate these problems.

Control Mechanisms

In light water reactors (LWRs), there are two main control mechanisms

1. Control rods: controlling *rapid* changes in the chain reaction
    1. Relevant for reactor dynamics
2. Chemical shim: controlling *slow* changes in in the chain reaction

Relevant Quantities in Reactor Physics

Interested only in neutrons and their interations (fission, absorption, etc).

Quantities

- ϕ (the scalar flux): number of neutrons impinging on a cross sectional area per second.
- Σ (the macroscopic cross section): characterizes the probabilitiy for a given interaction (measured in units of 1/L)

The Reaction Rate

With the scalar flux and the cross section, we can calculate reaction rates (e.g. of fission)

$$R_x = \Sigma_x \phi$$

units of $1/\text{cm}^3$ (a reaction rate density), and subsequently the *power*

$$P = \text{“}\int_V \Sigma_x \phi dV\text{”}$$

Fission Chain Reactions and Controllability

Controlling the Chain Reaction

The fission process is incredibly fast, and in a nuclear reactor, the time between a neutron's birth via fission and its next fission reaction is on the order of μs.

1. This would make controlling a reactor nearly impossible
2. The response to small changes in the neutron multiplication factor result in *extremely* rapid changes to power

Prompt and Delayed Neutrons

During a fission event, some neutrons are released (nearly) immediately from the fissioning nucleus: **prompt neutrons**, and some are emitted by subsequent decaay of fission products **delayed neutrons**.

β: The fraction of fission neutrons that are delayed ($\approx 0.7\%$)

Governing Equations

Simplification of the general Boltzman equation from statistical mechanics

The Linear Boltzman (Neutron Transport) Equation

State space $\boldsymbol{r}$, $\boldsymbol{\Omega}$, E, t (or $\boldsymbol{r}$, $\boldsymbol{v}$, t) $\implies \psi(\boldsymbol{r}, \boldsymbol{\Omega}, E, t)$, the angular flux

$$\begin{aligned} \frac{1}{v}\frac{\partial \psi}{\partial t} &= \hat{T}(t)\psi + \hat{F}_p(t)\int_{4\pi} \psi d\boldsymbol{\Omega} + \sum_{i=1}^{6} \varepsilon_i(\boldsymbol{r}, E, t) - \hat{L}(t)\psi \\ \frac{\partial \varepsilon_i}{\partial t} &= -\lambda_i \varepsilon_i + \hat{F}_{d,i}\int_{4\pi} \psi d\boldsymbol{\Omega} \qquad i = 1, \cdots, 6 \end{aligned} \tag{1}$$

The Operators

All linear operators

1. $\hat{T}$: Scattering
2. $\hat{F}_p$: Prompt fission
3. $\hat{F}_d$: Delayed fission
4. $\hat{L}$: Leakage

Note that $\hat{F}_p + \hat{F}_d = \hat{F}_{tot}$

The Factorization

To derive a simplified model of the transport equation that's suitable for practical calculations, we need to *factorize*.

Motivation

The main idea is to separate the variation of the flux into a fast and slow component (due to delayed neutrons). Factorization of the form

$$\psi(\boldsymbol{r}, \boldsymbol{\Omega}, E, t) = A(t)\Psi(\boldsymbol{r}, \boldsymbol{\Omega}, E; t)$$

$A(t)$ called the *amplitude* function and $\Psi(\boldsymbol{r}, \boldsymbol{\Omega}, E; t)$ the shape function.

This factorization is *always* possible (not excluding any variables) and is not (yet) unique.

The Shape Equations

Substituting the factorization into the transport equation

$$
\begin{aligned}
&A(t)\frac{1}{v}\frac{\partial \Psi}{\partial t} + \Psi \frac{1}{v}\frac{dA}{dt} \\
&= A(t)\hat{T}(t)\Psi + A(t)\hat{F}_p(t)\int_{4\pi} \Psi \, d\boldsymbol{\Omega} + \sum_{i=1}^{6} \varepsilon_i(\boldsymbol{r}, E, t) - A(t)\hat{L}(t)\Psi
\end{aligned}
$$

$$
\frac{\partial \varepsilon_i}{\partial t} = -\lambda_i \varepsilon_i + A(t)\hat{F}_{d,i}\int_{4\pi} \Psi \, d\boldsymbol{\Omega} \qquad i = 1, \cdots, 6
$$

Called the shape equations, however, $A(t)$ is an unknown so as it stands these are not well-posed.

The Adjoint Equation

The Reference Reactor

First define a reference reactor in steady state. The steady state problem is

$$\hat{L}_0(t)\psi_0 = \hat{T}_0(t)\psi_0 + \frac{1}{k}\hat{F}_{tot,0}\int_{4\pi}\psi_0 d\mathbf{\Omega}$$

k is a constant necessary for ensuring a solution exists. Physically it represents the neutron multiplication factor. For a critical reactor, $k = 1$. Introduce the "transport operator" $\hat{H}_0$

$$(-\hat{L}_0 - \hat{T}_0 + \hat{F}_{tot,0})\psi_0 \equiv \hat{H}_0\psi_0 = 0$$

The Adjoint Equation

Define the inner product as the integral over the entire phase space (excluding t), then can define adjoint equation

$$\hat{H}_0^\dagger \psi_0^\dagger = 0$$

Projecting Onto the Adjoint

$\psi_0^\dagger$ can be interpreted as an importance function.

$$A(t)\left\langle \psi_0^\dagger \middle| \frac{1}{v}\frac{\partial \Psi}{\partial t}\right\rangle + \left\langle \psi_0^\dagger \middle| \frac{1}{v}\Psi\right\rangle \frac{dA}{dt}$$

$$= A(t)\left\langle \psi_0^\dagger \middle| \hat{T}(t)\Psi\right\rangle + A(t)\left\langle \psi_0^\dagger \middle| \hat{F}_p(t)\int_{4\pi}\Psi d\boldsymbol{\Omega}\right\rangle + \cdots$$

$$+ \sum_{i=1}^{6}\left\langle \psi_0^\dagger \middle| \varepsilon_i(\boldsymbol{r}, E, t)\right\rangle - A(t)\left\langle \psi_0^\dagger \middle| \hat{L}(t)\Psi\right\rangle$$

$$\left\langle \psi_0^\dagger \middle| \frac{\partial \varepsilon_i}{\partial t}\right\rangle = -\lambda_i\left\langle \psi_0^\dagger \middle| \varepsilon_i\right\rangle + A(t)\left\langle \psi_0^\dagger \middle| \hat{F}_{d,i}\int_{4\pi}\Psi d\boldsymbol{\Omega}\right\rangle \qquad i = 1, \cdots, 6$$

Projecting Onto the Adjoint (Continued)

The adjoint ($\psi_0^\dagger$) is assumed constant in time, and so inner product and time derivatives can be interchanged.

$$
\begin{aligned}
&A(t)\frac{\partial}{\partial t}\left\langle \psi_0^\dagger \middle| \frac{1}{v}\Psi \right\rangle + \left\langle \psi_0^\dagger \middle| \frac{1}{v}\Psi \right\rangle \frac{dA}{dt} \\
&= A(t)\left\langle \psi_0^\dagger \middle| \hat{T}(t)\Psi \right\rangle + A(t)\left\langle \psi_0^\dagger \middle| \hat{F}_p(t)\int_{4\pi}\Psi\, d\mathbf{\Omega} \right\rangle + \cdots \\
&+ \sum_{i=1}^{6}\left\langle \psi_0^\dagger \middle| \varepsilon_i(\boldsymbol{r}, E, t) \right\rangle - A(t)\left\langle \psi_0^\dagger \middle| \hat{L}(t)\Psi \right\rangle
\end{aligned}
$$

$$
\frac{\partial}{\partial t}\left\langle \psi_0^\dagger \middle| \varepsilon_i \right\rangle = -\lambda_i \left\langle \psi_0^\dagger \middle| \varepsilon_i \right\rangle + A(t)\left\langle \psi_0^\dagger \middle| \hat{F}_{d,i}\int_{4\pi}\Psi\, d\mathbf{\Omega} \right\rangle \qquad i = 1, \cdots, 6
$$

Making the Factorization Unique

Since the factorization is not yet uniquely defined, we may make it uniquely defined by choosing the convenient condition

$$\frac{\partial}{\partial t}\left\langle \psi_0^\dagger \middle| \frac{1}{v}\Psi \right\rangle = 0$$

which allows us to eliminate the first term in the projected equations. If we divide each term by the quantity $\left\langle \psi_0^\dagger \middle| \hat{F}_{tot,0}\Psi \right\rangle$ (the importance of the fission neutrons distributed according to the shape Ψ).

$$\frac{\left\langle \psi_0^\dagger \middle| \frac{1}{v}\Psi \right\rangle}{\left\langle \psi_0^\dagger \middle| \hat{F}_{tot,0}\Psi \right\rangle}\frac{dA}{dt} = A(t)\frac{\left\langle \psi_0^\dagger \middle| (\hat{T}(t) - \hat{L}(t) + \hat{F}_p(t))\Psi \right\rangle}{\left\langle \psi_0^\dagger \middle| \hat{F}_{tot,0}\Psi \right\rangle} + \sum_{i=1}^{6}\frac{\left\langle \psi_0^\dagger \middle| \varepsilon_i(\boldsymbol{r}, E, t) \right\rangle}{\left\langle \psi_0^\dagger \middle| \hat{F}_{tot,0}\Psi \right\rangle}$$

$$\frac{\partial}{\partial t}\frac{\left\langle \psi_0^\dagger \middle| \varepsilon_i \right\rangle}{\left\langle \psi_0^\dagger \middle| \hat{F}_{tot,0}\Psi \right\rangle} = -\lambda_i\frac{\left\langle \psi_0^\dagger \middle| \varepsilon_i \right\rangle}{\left\langle \psi_0^\dagger \middle| \hat{F}_{tot,0}\Psi \right\rangle} + A(t)\frac{\left\langle \psi_0^\dagger \middle| \hat{F}_{d,i}\int_{4\pi}\Psi d\boldsymbol{\Omega} \right\rangle}{\left\langle \psi_0^\dagger \middle| \hat{F}_{tot,0}\Psi \right\rangle} \qquad i = 1, \cdots ,$$

The Physical Interpretations

- The effective generation time of prompt neutrons

$$\Lambda(t) = \frac{\left\langle \psi_0^\dagger \middle| \frac{1}{v}\Psi \right\rangle}{\left\langle \psi_0^\dagger \middle| \hat{F}_{tot,0}\Psi \right\rangle}$$

- The Reactivity (related to the multiplication factor)

$$\rho(t) = \frac{\left\langle \psi_0^\dagger \middle| \hat{H}(t)\Psi \right\rangle}{\left\langle \psi_0^\dagger \middle| \hat{F}_{tot,0}\Psi \right\rangle}$$

- The effective delayed neutron fraction for the ith group

$$\tilde{\beta}_i(t) = \frac{\left\langle \psi_0^\dagger \middle| \hat{F}_{d,i}\Psi \right\rangle}{\left\langle \psi_0^\dagger \middle| \hat{F}_{tot,0}\Psi \right\rangle}$$

The Physical Interpretations (Continued)

- The total effective delayed neutron fraction

$$\tilde{\beta}(t) = \sum_{i=1}^{6} \tilde{\beta}_i(t)$$

- The total effective delayed neutron emission for the ith group

$$\tilde{C}_i(t) = \frac{\left\langle \psi_0^\dagger \middle| \varepsilon_i(t) \right\rangle}{\left\langle \psi_0^\dagger \middle| \hat{F}_{tot,0}\Psi \right\rangle}$$

Using these parameters, we can rewrite the projected equations

$$\frac{dA(t)}{dt} = \frac{\rho(t) - \tilde{\beta}(t)}{\Lambda(t)} A(t) + \sum_{i=1}^{6} \lambda_i \tilde{C}_i(t)$$

$$\frac{\partial}{\partial t} \tilde{C}_i(t) = -\lambda_i \tilde{C}_i(t) + \frac{\tilde{\beta}_i(t)}{\Lambda(t)} A(t) \qquad i = 1, \cdots, 6$$

The Point Kinetic Model

- The above system of equation has a nonlinear coupling through the shape function
- Very hard to solve
- A simplified model can be obtained by neglecting spatial dependence

Assuming that

$$\Psi(\boldsymbol{r},\boldsymbol{\Omega},E;t) \approx \Psi(\boldsymbol{r},\boldsymbol{\Omega},E;t=0) = \psi_0(\boldsymbol{r},\boldsymbol{\Omega},E)$$
$$A(t=0)=1$$

The coefficients in the shape equation become constants. Usually rescale the equations so that $A(t) \to P(t)$ i.e. the total power.

The Point Kinetic Equations

$$\frac{dP(t)}{dt} = \frac{\rho - \tilde{\beta}}{\Lambda} P(t) + \sum_{i=1}^{6} \lambda_i \tilde{C}_i(t)$$
$$\frac{\partial}{\partial t}\tilde{C}_i(t) = -\lambda_i \tilde{C}_i(t) + \frac{\tilde{\beta}_i}{\Lambda} P(t) \qquad i = 1, \cdots, 6$$

Feedback

- The parameters Λ, λ_i, $\tilde{\beta}$, $\tilde{\beta}_i$ determined from nuclear data
- Cannot be solved without ρ.
  - Typically treated as a function of time $\rho(t) = \rho_{ex}(t) + \rho_f(t)$
  - Feedback from fuel/moderator temperature on reactivity via doppler broadening, etc.
- Practically, $\rho_{ex}(t)$ given, and $\rho_{ex}(t)$ calculated from a feeback model.

A Lumped Feedback Model

Temperature influenced by $P(t)$ via

$$\frac{d}{dt}\begin{pmatrix} T_f \\ T_m \end{pmatrix} + \begin{pmatrix} -b & b \\ c & -c-d \end{pmatrix}\begin{pmatrix} T_f \\ T_m \end{pmatrix} = \begin{pmatrix} aP(t) \\ dT_{in} \end{pmatrix}$$

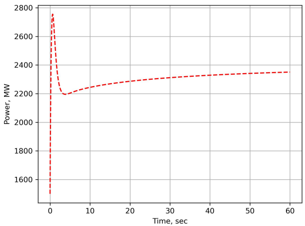
Reactivity influenced by temperature via

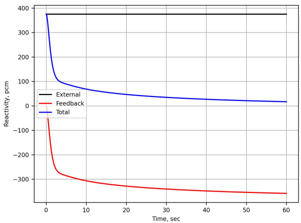
$$\rho_f(t) = \alpha_T^f(T_f - T_{f,0}) + \alpha_T^m(T_m - T_{m,0})$$

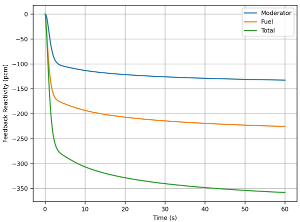
Nonlinear feedback

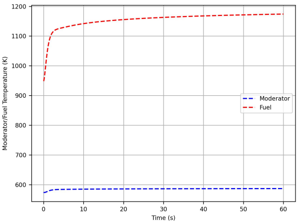
An Example

Response of a reactor to a step reactivity insertion (e.g. by ejection of a control rod)









Formulating Control Problems

- Our control parameter is the external reactivity $\rho_{ex}(t)$ (from the control rods)
- Want to acheive some change of state
- E.g. a power uprate $P_0 \to P_1$

A example problem might be performing a power uprate in *minimal* time. The optimal control problem is finding the trajectory $\rho_{ex}(t)$ that *minimizes* the uprate time T.

- Not simply a matter of inserting the largest possible reactivity then instantly withdrawing it to acheive the desired power
    - This delayed neutrons will lead to further reactivity changes that must be accounted for before the final state can be reached
- Operational limits on reactivity insertion, and power overshoot
    - Positive reactivity insertion less than β (sub-prompt criticality)
    - Power overshoot must be less than αP_1, where $\alpha = 1.5$ (for example).

The State Space

The set of state variables can be written in vector form

$$\boldsymbol{X} \equiv \begin{pmatrix} P \\ C_1 \\ \vdots \\ C_6 \\ T_f \\ T_m \end{pmatrix}$$

where $\boldsymbol{X}$ satisfies the following veector differential equation

$$\frac{d}{dt}\boldsymbol{X} = \begin{pmatrix} \frac{\rho(t)-\beta}{\Lambda} & \lambda_1 & \cdots & \lambda_6 & 0 & 0 \\ \frac{\beta_1}{\Lambda} & -\lambda_1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{\beta_6}{\Lambda} & 0 & \cdots & -\lambda_g & 0 & 0 \\ a & 0 & \cdots & 0 & -b & b \\ 0 & 0 & \cdots & 0 & c & -c-d \end{pmatrix} \boldsymbol{X} + \begin{pmatrix} 0 \\ \vdots \\ dT_{in} \end{pmatrix}$$

The State Space (Continued)

Caveats

- *Nonelinear* first order problem, because of the product $\rho(t)P(t)$ (since $\rho(t)$ depends on T_f, T_m).
- Not really fair to write as a matrix equation.
- If we were to write as $\boldsymbol{X}' = \boldsymbol{f}(t, \boldsymbol{X}(t), \rho_{ext}(t))$, we see that $\boldsymbol{f}$ depends on the control $\rho_{ext}(t)$.
  - This makes solving the costate equation *very* difficult

Operational Limits on the State Space

1. $0 \leq P(t) \leq \alpha P_1$
2. $T_f, T_m > 0$
3. $C_i > 0$
4. Might want to impose some temperature constraints, but similar to power constraint

Formulating the Optimal Control Problem

Assuming the conditions on $\boldsymbol{f}$ hold, we may apply the Pontryagin minimum principle to express the optimal control $\overline{\rho}_{ex}$ that minimizes the uprate time T as that for which

$$\boldsymbol{\lambda}'(t) = \left(\partial_{\boldsymbol{X}}\boldsymbol{f}\right)^* \boldsymbol{\lambda}(t)$$
$$\overline{\rho}_{ext}(t) = \operatorname{argmin}_{\rho_{ext}\in U} \left\langle \boldsymbol{\lambda}(t), \boldsymbol{f}(t, \overline{\boldsymbol{X}}, \rho_{ext(t)}) \right\rangle$$

Where U is the set of all piecewise continuous functions with maximum less than β. Let $d(t) = P(t) - P_1$, then

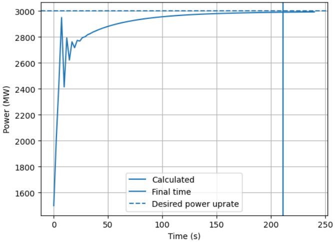
$$\boldsymbol{\nabla} d = \begin{pmatrix} \frac{\partial d}{\partial P} \\ \frac{\partial d}{\partial C_1} \\ \vdots \\ \frac{\partial d}{\partial T_m} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

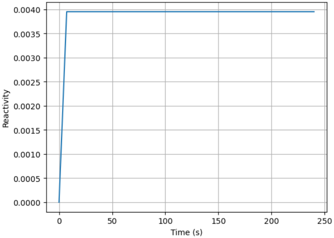
so $C = \{\boldsymbol{x} \in \mathbb{R}^9 | x_1 < P_1\}$, we see that $d = 0$ on ∂C and $\boldsymbol{\nabla} d \neq 0$ on ∂C, so we have the following terminal condition on the costate

$$\boldsymbol{\lambda}(T) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Numerical Alternatives

- Due to the nonlinear term in $\boldsymbol{f}$, and the dependence on ρ_{ext}, the costate equation cannot be solved separately, then used to compute the optimal control by minimizing.
- An alternative is to use a numerical constrained optimization algorithm to minimize the uprate time T by varying a number of points in a discretized representation of $\rho_{ext}(t)$
- Didn't have time to fully work out the details of this
- An example (non-optimal) trajectory





References

[1] David L Hetrick. Dynamics of nuclear reactors. 1971.