1.

a) $f_X(x) = \int_1^5 \frac{x}{5} + \frac{y}{20}\, dy$

$= \left[\frac{x}{3}y + \frac{y^2}{40}\right]_1^5$

$= \left[x + \frac{25}{40}\right] - \left[\frac{x}{5} + \frac{1}{40}\right]$

$= \frac{4x}{5} + \frac{24}{40}$

$= \frac{32x+24}{40} = \frac{4x+3}{5}$

$f_Y(y) = \int_0^1 \frac{x}{5} + \frac{y}{20}\, dx$

$= \left[\frac{x^2}{10} + \frac{yx}{20}\right]_0^1$

$= \left[\frac{1}{10} + \frac{y}{20}\right] -$

$= \frac{2+y}{20}$

$f_X(x)\, f_Y(y) = \frac{(32x+24)(2+y)}{40 \cdot 20}$

$\frac{64x + 32xy + 48 + 24y}{80}$

$= \frac{64x}{80} + \frac{32xy}{80} + \frac{48}{80} + \frac{24y}{80}$

$\boxed{X \text{ and } Y \text{ are not independent}}$

$x=1,\ y=1$

$f_{XY}(1,1) = \frac{1}{5} + \frac{1}{20} = \frac{5}{20} = \frac{1}{4}$

$f_X(1) f_Y(1) = \frac{7}{5} \cdot \frac{3}{20} = \frac{21}{100} \neq \frac{1}{4} = f_{X,Y}(1,1)$

(b) $\int_0^1 \int_1^5 xy\left(\frac{x}{5} + \frac{y}{20}\right) dy\, dx$

$\int_0^1 \int_1^5 \frac{x^2 y}{5} + \frac{xy^2}{20}\, dy\, dx$

$\int_0^1 \left[\frac{x^2y^2}{10} + \frac{xy^3}{60}\right]_1^5 dx$

$\int_0^1 \left[\frac{25x^2}{10} + \frac{125}{60}x\right] - \left[\frac{x^2}{10} + \frac{x}{60}\right] dx$

$\int_0^1 \frac{24x^2}{10} + \frac{124}{60}x\, dx$

$\left[\frac{24x^3}{30} + \frac{124x^2}{120}\right]_0^1$

$\frac{24}{30} + \frac{124}{120}$

$\mathbb{E}[XY] = \frac{96+124}{120}$

$= \frac{220}{120} = \frac{22}{12} = \frac{11}{6}$

$\mathbb{E}[X] = \int_0^1 x\left(\frac{32x+24}{40}\right)$

$\frac{1}{40}\int_0^1 32x^2 + 24x\, dx$

$\frac{1}{40}\left[\frac{32x^3}{3} + 12x^2\right]_0^1 = \frac{1}{40}\left[\frac{32}{3} + \frac{36}{3}\right] = \frac{68}{120} = \frac{34}{60} = \frac{17}{30}$

$$\mathbb{E}[y] = \int_1^5 y\left[\frac{2+y}{20}\right]$$

$$\int_1^5 \left[\frac{2y+y^2}{20}\right]$$

$$\left[\frac{y^2}{20} + \frac{y^3}{40}\right]_1^5$$

$$\left[\frac{25}{20} + \frac{125}{40}\right] - \left[\frac{1}{20} + \frac{1}{60}\right]$$

$$\frac{75+125}{60} - \frac{4}{40}$$

$$= \frac{196}{60} = \frac{98}{30} = \frac{49}{15}$$

$$\text{cov}(x,y) = \frac{11}{6} - \left(\frac{17}{30}\right)\left(\frac{49}{15}\right) = \boxed{\frac{-4}{225} = \text{cov}(x,y)}$$

(2) $\Phi_Y(w) = \mathbb{E}[e^{jwY}]$

(a). $\Phi_{X+N}(w) = \mathbb{E}[e^{jw[X+N]}]$ X and N independent so

$= \mathbb{E}[e^{jwX} e^{jwN}]$

$= \mathbb{E}[e^{jwX}]\,\mathbb{E}[e^{jwN}]$ $\Phi(w)$ for Gaussian is $= e^{jwm - \frac{\sigma^2 w^2}{2}}$

$= \Phi_X(w)\,\Phi_N(w)$

$= e^{-\frac{\sigma_X^2 w^2}{2}} e^{-\frac{\sigma_N^2 w^2}{2}}$

$= e^{\frac{-\sigma_X^2 w^2 - \sigma_N^2 w^2}{2}}$

$= e^{-\frac{w^2(\sigma_X^2 + \sigma_N^2)}{2}}$

char function of Gaussian so

$Y \sim N(0, \sigma_X^2 + \sigma_N^2)$

$Var(Y) = \sigma_X^2 + \sigma_N^2$

$Var(X) = \sigma_X^2$

$Cov(X,Y) = \mathbb{E}[XY] - m_X m_Y$ (with $m_X = 0$, $m_Y = 0$)

$= \mathbb{E}[XY]$

$= \mathbb{E}[X(X+N)]$

$= E[X^2] + E[XN]$

$= E[X^2] + E[X]E[N]$

$= E[X^2]$

$E[X^2] + E^2[X] = \sigma_X^2$

$\boxed{Cov(X,Y) = \sigma_X^2}$

$$\boxed{\rho_{X,Y} = \frac{\sigma_X^2}{\sqrt{\sigma_X^2}\sqrt{\sigma_X^2+\sigma_N^2}} = \frac{\sigma_X}{\sqrt{\sigma_X^2+\sigma_N^2}}}$$

(b)

$$\frac{d}{da}\mathbb{E}[(X-aY)^2] = \mathbb{E}\left[\frac{d}{da}(X-aY)^2\right]$$

$$= \mathbb{E}[2(X-aY)(-Y)]$$

$$= -2\,\mathbb{E}[(X-aY)(Y)]$$

$$= -2\,\mathbb{E}[XY-aY^2]$$

$$= -2\left[\mathbb{E}[XY] - a\,\mathbb{E}[Y^2]\right]$$

$$= -2\left[\sigma_X^2 - a\,\mathbb{E}[(X+N)^2]\right]$$

$$= -2\left[\sigma_X^2 - a\,\mathbb{E}[X^2+XN+N^2]\right]$$

$$-2\left[\sigma_X^2 - a[\sigma_X^2+\sigma_N^2]\right] = 0$$

$$\sigma_X^2 - a[\sigma_X^2+\sigma_N^2] = 0$$

$$\boxed{\frac{\sigma_X^2}{\sigma_X^2+\sigma_N^2} = a}$$

(c).

$$\mathbb{E}[(X-aY)^2]$$

$$\mathbb{E}[X^2-2aXY+a^2Y^2]$$

$$\mathbb{E}[X^2]-2a\mathbb{E}[XY]+a^2\,\mathbb{E}[Y^2]$$

$$\sigma_X^2 - 2a\,\sigma_X^2 + a^2\,\mathbb{E}[(X+N)^2]$$

$$\sigma_X^2 - \frac{2\sigma_X^4}{\sigma_X^2+\sigma_N^2} + a^2\,\mathbb{E}[X^2+2XN+N^2]$$

$$\sigma_X^2 - \frac{2\sigma_X^4}{\sigma_X^2+\sigma_N^2} + a^2(\sigma_X^2+\sigma_N^2)$$

$$\frac{\sigma_X^2(\sigma_X^2+\sigma_N^2) - 2\sigma_X^4 + \sigma_X^4}{\sigma_X^2+\sigma_N^2}$$

$$\frac{\sigma_X^4+\sigma_X^2\sigma_N^2-\cancel{2\sigma_X^4}+\cancel{\sigma_X^4}}{\sigma_X^2+\sigma_N^2}$$

$$\frac{\sigma_X^2\sigma_N^2}{\sigma_X^2+\sigma_N^2} = \text{MSE}$$

(3)

$Z = \frac{X}{X+Y}$

hm?

$Z(x+y) = x$

$Zx + Zy = x$

$Zy = x - xZ$

$y = \frac{x(1-z)}{z}$

$y = x\frac{(1-z)}{z}$

$X, Y \quad 0 < X < \infty$

$0 \leq Y \leq \infty$

$Z = g(X,Y) = \frac{X}{X+Y}$

$f_{X,Y}(x,y) = f_X(x) f_Y(y) \quad X \perp\!\!\!\perp Y \quad A_z = \{(x,y) : \frac{x}{x+y} \leq z\}$

A_z

$$F_Z(z) = \int_0^\infty \int_{\frac{x(1-z)}{z}}^\infty \lambda^2 e^{-\lambda x} e^{-\lambda y} \, dy \, dx$$

$x \leq xz + yz$

$x - xz \leq yz$

$\frac{x(1-z)}{z} \leq y$

$$\int_0^\infty \lambda e^{-\lambda x} \left[-e^{-\lambda y} \right]_{\frac{x(1-z)}{z}}^\infty dx$$

$$\int_0^\infty \lambda e^{-\lambda x} e^{\lambda x - \frac{\lambda x}{z}} dx$$

$$\int_0^\infty \lambda e^{-\lambda x + \lambda x - \frac{\lambda x}{z}} dx$$

$$\int_0^\infty \lambda e^{-\frac{\lambda x}{z}} dx$$

$$\left[-z e^{-\frac{\lambda x}{z}} \right]_0^\infty dx$$

$\frac{x}{x+y} = z$

if $x=0 \rightarrow \frac{0}{y} \rightarrow 0 = z$

if $y=0 \rightarrow \frac{x}{x} \rightarrow 1 = z$

if $x=\infty \rightarrow \frac{\infty}{\infty+y} \rightarrow 1 = z$

if $y=\infty \rightarrow \frac{x}{x+\infty} \rightarrow 0 = z$

$0 \leq z \leq 1$

$F_Z(z) \overset{wp}{=} z$ for $0 \leq z \leq 1$, $\overset{wp}{}$ 1 for $z \geq 1$, $\overset{wp}{}$ 0 for $z < 0$

$f_Z(z) = 1 \quad 0 \leq z \leq 1 \quad 0$ else

(4)

(a) $f_X(x) = \int_{-\infty}^{\infty} x\, f_{xy}(x,y)\, dx$

$$\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-p^2}} \int_{-\infty}^{\infty} x \exp\left[\frac{-1}{2(1-p^2)}\left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} - \frac{2pxy}{\sigma_x\sigma_y}\right)\right]$$

$$\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-p^2}} \exp\left[\frac{-1}{2(1-p^2)}\left(\frac{y^2}{\sigma_y^2}\right)\right] \int_{-\infty}^{\infty} x\, e^{\frac{-1}{2(1-p^2)}\left(\frac{x^2}{\sigma_x^2} - \frac{2pxy}{\sigma_x\sigma_y}\right)} dx$$

$$c \int_{-\infty}^{\infty} e^{\frac{-1}{2(1-p^2)}\left(\frac{x^2}{\sigma_x^2} - \frac{2pxy}{\sigma_x\sigma_y}\right)}$$

$$c \int_{-\infty}^{\infty} e^{\frac{-1}{2(1-p^2)\sigma_x^2}\left(x^2 - \frac{2pxy\sigma_x}{\sigma_y}\right)}$$

$$c \int_{-\infty}^{\infty} e^{\frac{-1}{2(1-p^2)\sigma_x^2}\left(x^2 - 2pxy\frac{\sigma_x}{\sigma_y}\right)}$$

$$c \int_{-\infty}^{\infty} e^{\frac{-1}{2(1-p^2)\sigma_x^2}\left(x^2 - 2x\left(py\frac{\sigma_x}{\sigma_y}\right) + \left(py\frac{\sigma_x}{\sigma_y}\right)^2 - \left(py\frac{\sigma_x}{\sigma_y}\right)^2\right)}$$

$$c \int_{-\infty}^{\infty} e^{\left[\frac{-1}{2(1-p^2)\sigma_x^2}\left(x - py\frac{\sigma_x}{\sigma_y}\right)^2 + \frac{\left(py\frac{\sigma_x}{\sigma_y}\right)^2}{2(1-p^2)\sigma_x^2}\right]}$$

$$c\, e^{\left[\frac{\left(py\frac{\sigma_x}{\sigma_y}\right)^2}{2(1-p^2)\sigma_x^2}\right]} \int_{-\infty}^{\infty} \underbrace{e^{\frac{-1}{2(1-p^2)\sigma_x^2}\left(x - py\frac{\sigma_x}{\sigma_y}\right)^2}}_{\text{gaussian w/ std} = \sigma_x\sqrt{1-p^2}}$$

mean $= py\frac{\sigma_x}{\sigma_y}$

$$e^{\left[\frac{\left(py\frac{\sigma_x}{\sigma_y}\right)^2}{2(1-p^2)\sigma_x^2}\right]} \cdot \sqrt{2\pi}\,\sigma_x(1-p^2)^{\frac{1}{2}} \cdot c \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma_x(1-p^2)^{1/2}} e^{\frac{-1}{2(1-p^2)\sigma_x^2}\left(x - py\frac{\sigma_x}{\sigma_y}\right)^2}}_{\text{Gaussian so integral } = 1!}$$

$$e^{\frac{\left(\rho y \frac{\sigma_x}{\sigma_y}\right)^2}{2(1-\rho^2)\sigma_x^2}} \cdot \frac{1}{2\pi \cancel{\sigma_x}\sigma_y \cancel{(1-\rho^2)^{1/2}}} \cdot \sqrt{2\pi}\, \cancel{\sigma_x} \cancel{(1-\rho^2)^{1/2}} \cdot \exp\left(\frac{-1\, y^2}{2(1-\rho^2)\sigma_y^2}\right)$$

$$e^{\frac{\rho^2 y^2 \frac{\cancel{\sigma_x^2}}{\sigma_y^2}}{2(1-\rho^2)\cancel{\sigma_x^2}}} \cdot \frac{1}{2\pi\sigma_y}\sqrt{2\pi} \cdot \exp\left(\frac{-y^2}{2(1-\rho^2)\sigma_y^2}\right)$$

$$\frac{1}{(2\pi)^{1/2}\sigma_y} \cdot e^{\frac{\rho^2 y^2 - y^2}{2(1-\rho^2)\sigma_y^2}}$$

$$\frac{1}{\sqrt{2\pi}\,\sigma_y} \cdot e^{\frac{y^2(\rho^2-1)}{2(1-\rho^2)\sigma_y^2}}$$

$$\frac{1}{\sqrt{2\pi}\,\sigma_y} \cdot e^{\frac{-y^2}{2\sigma_y^2}} = \text{Gaussian w/ mean} = 0$$

$f_Y(y) = N(0, \sigma_y^2)$ Variance $= \sigma_y^2$

(b) $$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left[\frac{-1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} - \frac{2\rho xy}{\sigma_x\sigma_y}\right)\right]}{\frac{1}{\sqrt{2\pi}\,\sigma_y} \exp\left[\frac{-y^2}{2\sigma_y^2}\right]}$$

$$= \text{Constant} \cdot \exp\left[\frac{-1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} - \frac{2\rho xy}{\sigma_x\sigma_y}\right)\right.$$

constant $\cdot$ $\exp\left[\frac{-1}{2(1-p^2)}\left(\frac{x^2}{\sigma_x^2} - \frac{2pxy}{\sigma_x\sigma_y}\right)\right]$

constant $\cdot$ $\exp\left[\frac{-1}{2(1-p^2)\sigma_x^2}\left(x^2 - 2x\left(\frac{py}{\sigma_y}\sigma_x\right) + p\left(\frac{y\sigma_x}{\sigma_y}\right)^2 - \left(\frac{py\sigma_x}{\sigma_y}\right)^2\right)\right]$

constant $\cdot$

$$\exp\left[\frac{-1}{2(1-p^2)\sigma_x^2}\left(x - py\frac{\sigma_x}{\sigma_y}\right)^2 + \frac{\left(py\frac{\sigma_x}{\sigma_y}\right)^2}{2(1-p^2)\sigma_x^2}\right]$$

constant $\cdot$

$$\underbrace{\exp\left[\frac{-1}{2(1-p^2)\sigma_x^2}\left(x - py\frac{\sigma_x}{\sigma_y}\right)^2\right]}_{\text{Gaussian unnormalized...}}$$

$$f_{X|Y}(x|y) = \frac{1}{\sqrt{2\pi}\,(1-p^2)^{1/2}\sigma_x}\exp\left[\frac{-1}{2(1-p^2)\sigma_x^2}\left(x - py\frac{\sigma_x}{\sigma_y}\right)^2\right]$$

$$\boxed{f_{X|Y}(x|y) = N\left(\underbrace{py\frac{\sigma_x}{\sigma_y}}_{\text{mean}}, \underbrace{(1-p^2)\sigma_x^2}_{\text{variance}}\right)}$$

$$\boxed{\begin{aligned} \mathbb{E}_{X|Y}(x|y) &= \frac{py\sigma_x}{\sigma_y} \\ \sigma^2_{X|Y} &= (1-p^2)\sigma_x^2 \end{aligned}}$$