EE 131A
Probability
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Discussion Set 1
Monday, October 05, 2020
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Reading: Chapter 2 of *Probability, Statistics, and Random Processes* by A. Leon-Garcia

1. (Problem 2.13 of the book)

   Solution: Using the Venn diagram in Figure 1, the event "exactly one of the events A and B occurs" can be expressed as $(A \cap B^c) \cup (A^c \cap B)$

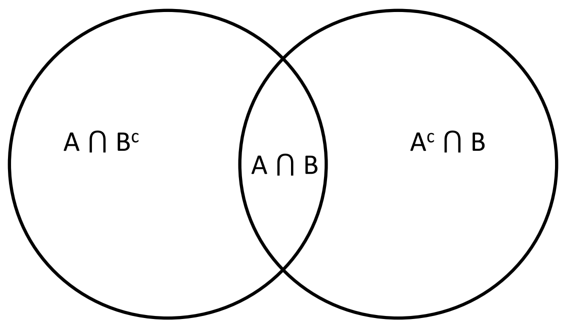

Figure 1: Decomposition of $A \cup B$ into three disjoint sets.

2. There is a box containing 20 lamps. Eight of these lamps are defective and do not work correctly, although they look exactly as the normal ones. We test the lamps one by one until we find all defective lamps. What is the probability of the event that the last defective lamp is found on the 12th test.

   Solution: Let the outcome of each test be marked as F (Failure) or S (Success). There are 20 experiments (each one corresponds to testing one lamp) which are performed one by one. The outcomes can be one of $\binom{20}{8}$ different arrangements,. In $\binom{11}{7}$ of them, the last defective lamp is found on the 12th test (The other 7 defective lamp are found during the first 11 tests). Consequently, the probability is:

$$P(\text{the last defective lamp is found on the 12th test}) = \frac{\binom{11}{7}}{\binom{20}{8}}.$$

3. *Birthday paradox.* We have a class with N students. How large must N be for the probability of two students sharing the same birthday to be at least 1/2?
   Solution: We can compute instead the probability that no students share a birthday. The first student has 365 choices of birthdays. The 2nd student has 364 choices (since he can't share with the first student). The 3rd student has 363 choices. The Nth

student has $365 - N + 1$ choices. Overall, the probability is:

$$\frac{365}{365} \times \frac{364}{365} \times \frac{363}{365} \times \ldots \times \frac{365 - N + 1}{365} = \frac{365!}{365^N \times (365 - N)!}$$
$$= \frac{\binom{365}{N} N!}{365^N}.$$

So our probability is $1 - \frac{\binom{365}{N} N!}{365^N}$. Plugging this into a calculator and trying some small values, we see that $N = 23$ gives 0.5073, which is greater than 1/2, while $N = 22$ gives 0.4757. So the answer is just $N = 23$ (a surprisingly small number)!

If we wanted the minimum N to *guarantee* a shared birthday (i.e., probability 1), we would get $N = 366$, since everyone could have birthdays on different days of the year. But if we want probability at least 0.999, $N = 70$, not even a fifth of 366, is enough! Try it yourself.

4. Suppose n_1 indistinguishable white balls and n_2 indistinguishable black balls are randomly put into m distinguishable boxes. What is the probability of the event "No box is empty"?

   Solution: If we put n indistinguishable balls into k distinguishable boxes, the number of possible outcomes is $\binom{n+k-1}{k-1}$.

   Let A_i be the event "box i is empty", $1 \le i \le m$. Suppose x_i and y_i, $1 \le i \le m$, represent the number of white balls and black balls in box i, respectively. Then for any $I \subset \{1, 2, \cdots, m\}$,

$$\begin{aligned} P(\bigcap_{i \in I} A_i) &= P(x_i = y_i = 0, \forall i \in I) \\ &= P(\sum_{i \notin I} x_i = n_1, x_i \ge 0; \sum_{i \notin I} y_i = n_2, y_i \ge 0) \\ &= \frac{\binom{n_1+m-|I|-1}{n_1}\binom{n_2+m-|I|-1}{n_2}}{\binom{n_1+m-1}{n_1}\binom{n_2+m-1}{n_2}}. \end{aligned} \tag{1}$$

   Therefore the probability (let it be P) we are gonna compute is:

$$\begin{aligned} P &= 1 - P(\bigcup_{i=1}^{m} A_i) \\ &= 1 + \sum_{k=1}^{m} (-1)^k \sum_{I \subset \{1, \cdots, m\}, |I| = k} P(\bigcap_{i \in I} A_i) \\ &= 1 + \sum_{k=1}^{m} (-1)^k \binom{m}{k} \frac{\binom{n_1+m-k-1}{n_1}\binom{n_2+m-k-1}{n_2}}{\binom{n_1+m-1}{n_1}\binom{n_2+m-1}{n_2}}. \end{aligned} \tag{2}$$

5. (Harder) Suppose there are n indistinguishable white balls and n indistinguishable black balls in a container. There are n students, each of them randomly picks up two

balls from the container together. What is the probability of the event "each student picks up exactly one white ball and one black ball"?

Solution: Since the white balls are indistinguishable and black balls are also indistinguishable, and each student picks up 2 balls, the outcome depends only on the number of white balls that each student has. Suppose the i-th student has x_i white balls, then $0 \leq x_i \leq 2$, and $x_1 + x_2 + \cdots + x_n = n$. The totally number of the outcomes equals to the total number of solutions to the equation.

Suppose $I_i = \{j : x_j = i, 1 \leq j \leq n\}$, $0 \leq i \leq 2$, and $n_i = |I_i|$. Then $n_1 + 2n_2 = n$ and $n_0, n_1 + n_2 = n$, which has solutions $(n_0, n_1, n_2) = (m, n - 2m, m)$, $0 \leq m \leq \lfloor \frac{n}{2} \rfloor$. And for each solution specified by m, there are $\binom{n}{m,m,n-2m}$ ways to specify the solution to the original problem. Therefore the total number of outcomes is:

$$\sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{m, m, n-2m}. \tag{3}$$

Thus the probability is: original problem. Therefore the total number of outcomes is:

$$\frac{1}{\sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{m,m,n-2m}}. \tag{4}$$