(1)

(a) $P(G_2) = P(G_2 \cap R_1) + P(G_2 \cap G_1)$

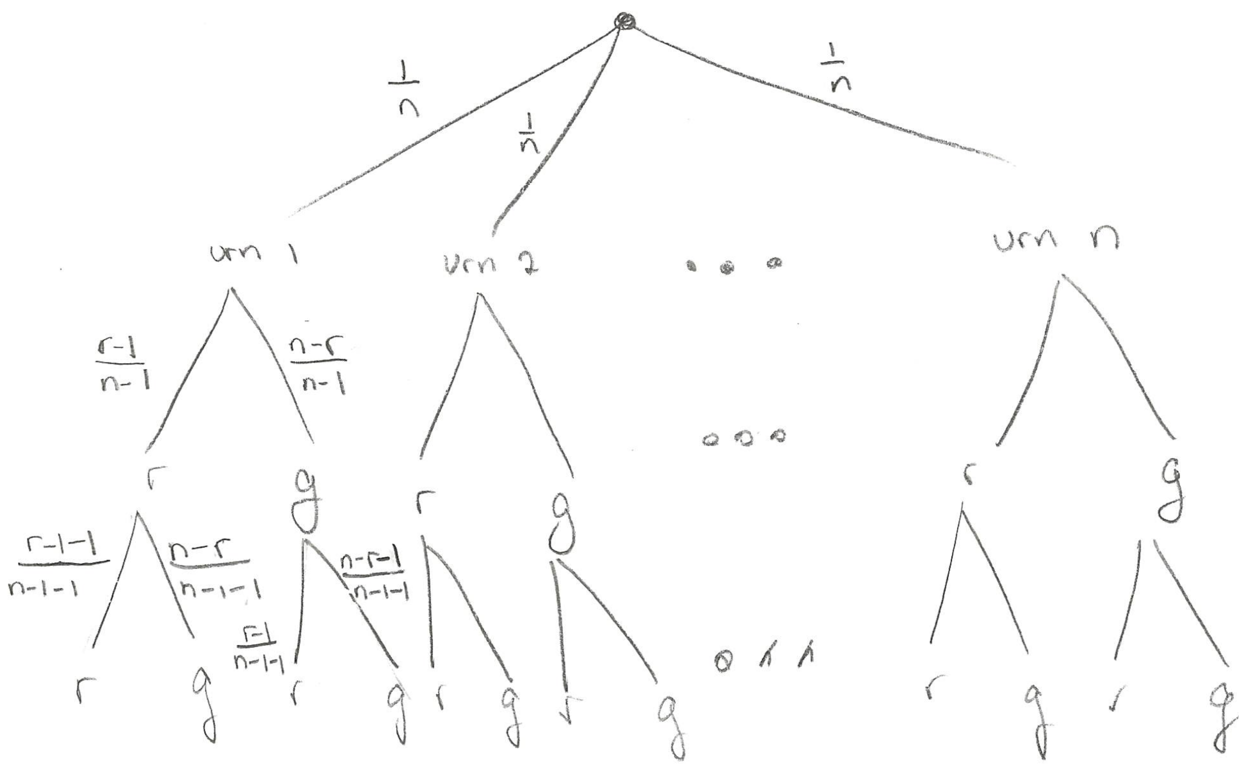
$= \sum_{k=1}^{n} P(G_2 \cap R_1 | U_k) P(U_k) + \sum_{k=1}^{n} P(G_2 \cap G_1 | U_k) P(U_k)$

$= \sum_{k=1}^{n} P(G_2 \cap R_1 \cap U_k) + \sum_{k=1}^{n} P(G_2 \cap G_1 \cap U_k)$

$= \sum_{k=1}^{n} P(G_2 | R_1, U_k) \cdot P(U_k, R_1) + \sum_{k=1}^{n} P(G_2 | G_1, U_k) \cdot P(U_k, G_1)$

$= \sum_{k=1}^{n} P(G_2 | R_1, U_k) \cdot P(R_1 | U_k) \cdot P(U_k) + \sum_{k=1}^{n} P(G_2 | G_1, U_k) \cdot P(G_1 | U_k) \cdot P(U_k)$

$= \sum_{k=1}^{n} \left[P(G_2 | R_1, U_k) P(R_1 | U_k) + P(G_2 | G_1, U_k) P(G_1 | U_k) \right] P(U_k)$



x) Cont.

$$\sum_{r=1}^{n} \frac{1}{n}\left[\frac{(n-r)\left[(r-1)+(n-r-1)\right]}{(n-1)(n-2)}\right]$$

$$\sum_{r=1}^{n} \frac{1}{n}\left[\frac{(n-r)\cancel{(n-2)}}{(n-1)\cancel{(n-2)}}\right]$$

$$\sum_{r=1}^{n} \frac{1}{n}\left[\frac{n-r}{n-1}\right]$$

$$\sum_{r=1}^{n} \frac{1}{n}\left(\left[\frac{n}{n-1}\right]-\left[\frac{r}{n-1}\right]\right)$$

$$\frac{n}{n}\left[\frac{n}{n-1}\right] - \frac{1}{n}\sum_{r=1}^{n}\frac{r}{n-1}$$

$$\frac{n}{n-1} - \frac{1}{(n)(n-1)}\sum_{r=1}^{n} r$$

$$\frac{n}{n-1} - \left(\frac{\cancel{n}(n+1)}{2}\right)\left(\frac{1}{(n-1)\cancel{n}}\right)$$

$$\frac{n}{n-1} - \frac{n+1}{2(n-1)}$$

$$\left(\frac{2n}{2(n-1)}\right) - \frac{n+1}{2(n-1)}$$

$$\frac{n-1}{2(n-1)} = \boxed{\frac{1}{2} = p(\text{second ball green})}$$

b)

$$P(g_2|g_1) = \frac{P(g_2 \cap g_1)}{P(g_1)}$$

$$= \frac{\sum_{r=1}^{n} P(g_2 \cap g_1 | U_r) P(U_r)}{\sum_{r=1}^{n} P(g_1 | U_r) P(U_r)}$$

$$\boxed{= \frac{\sum_{r=1}^{n} P(g_2|g_1, U_r) P(g_1|U_r) P(U_r)}{\sum_{r=1}^{n} P(g_1|U_r) P(U_r)}}$$

$$\boxed{\frac{\sum_{r=1}^{n} \frac{1}{n}\left(\frac{n-r}{n-1}\right)\left(\frac{n-r-1}{n-1-1}\right)}{\sum_{r=1}^{n} \frac{1}{n}\left(\frac{n-r}{n-1}\right)}}$$

$$\frac{P(g_2 \cap g_1 \cap U_r)}{P(U_r)} = \frac{P(g_2|g_1, U_r) P(g_1|U_r) P(U_r)}{P(U_r)} = P(g_2|g_1, U_r) P(g_1|U_r)$$

denominator

$$\sum_{r=1}^{n} \frac{n-r}{n-1}$$

$$\sum_{r=1}^{n} \frac{n-r}{n-1}$$

$$\sum_{r=1}^{n} \left[\frac{n}{n-1} - \frac{r}{n-1}\right]$$

$$\frac{n^2}{(n-1)} - \frac{n(n+1)}{2(n-1)}$$

$$\frac{2n^2 - n^2 - n}{2(n-1)} = \frac{n^2 - n}{2(n-1)} = \frac{n(n-1)}{2(n-1)} = \frac{n}{2}$$

numerator

$$\sum_{r=1}^{n} \left(\frac{(n-r)(n-r-1)}{(n-1)(n-2)} \right)$$

$$\sum_{r=1}^{n} \left(\frac{n^2 - nr - n - nr + r^2 + r}{(n-1)(n-2)} \right)$$

$$\sum_{r=1}^{n} \left[\frac{n^2}{(n-1)(n-2)} - \frac{2nr}{(n-1)(n-2)} - \frac{n}{(n-1)(n-2)} + \frac{r^2}{(n-1)(n-2)} + \frac{r}{(n-1)(n-2)} \right]$$

$$\frac{n^3}{(n-1)(n-2)} - \frac{2n^2(n+1)}{2(n-1)(n-2)} - \frac{n^2}{(n-1)(n-2)} + \frac{n(n+1)(2n+1)}{6(n-1)(n-2)} + \frac{n(n+1)}{2(n-1)(n-2)}$$

$$\frac{6n^3}{6(n-1)(n-2)} - \frac{(6n^3 + 6n^2)}{6(n-1)(n-2)} - \frac{6n^2}{6(n-1)(n-2)} + \frac{(n^2+n)(2n+1)}{6(n-1)(n-2)} + \frac{3n^2+3n}{6(n-1)(n-2)}$$

$$\frac{\cancel{6n^3} - \cancel{6n^3} - 6n^2 - 6n^2 + 2n^3 + \cancel{3n^2} + n + \cancel{3n^2} + 3n}{6(n-1)(n-2)}$$

$$\frac{-6n^2 + 2n^3 + 4n}{6(n-1)(n-2)}$$

$$\frac{2n(-3n + n^2 + 2)}{6(n-1)(n-2)}$$

$$\frac{1n}{3} \frac{\cancel{(n-2)}\cancel{(n-1)}}{\cancel{(n-1)}\cancel{(n-2)}}$$

$$= \frac{1n}{3}$$

$$\boxed{\frac{\text{num}}{\text{dem}} = \frac{\frac{n}{3}}{\frac{n}{2}} = \frac{2}{3} = P(G_2|G_1)}$$

(2)

(a) $S = \{ (1,1), (1,2), (1,3), (1,4), (1,5), (1,6),$
$(2,1), (2,2), (2,3), (2,4), (2,5), (2,6),$
$(3,1), (3,2), (3,3), (3,4), (3,5), (3,6),$
$(4,1), (4,2), (4,3), (4,4), (4,5), (4,6),$
$(5,1), (5,2), (5,3), (5,4), (5,5), (5,6),$
$(6,1), (6,2), (6,3), (6,4), (6,5), (6,6) \}$

(b) $\frac{6}{36} = \boxed{P(\text{same number both times}) = \frac{1}{6}}$

(c)

$|X_1 - X_2| = Y$

$\sum_{i=1}^{6} P(Y \geq X_1 = i \mid X_1 = i)$

or $\displaystyle \sum_{i=1}^{6}\sum_{j=1}^{6} \frac{\mathbb{1}(|i-j| \geq i)}{36}$

$\boxed{P(\text{abs}(X_2 - X_1) > X_1) = \frac{1}{4}}$

(3)

Base case $n=1$ $P(A_1) \geq P(A_1)$ ✓

induction method △

assume $n=i$ holds

$$P\left(\bigcap_{k=1}^{i} A_k\right) \geq \sum_{k=1}^{i} P(A_k) - (i-1)$$

prove holds for $n=i+1$

$B = \bigcap_{k=1}^{i} A_k \rightarrow P(B) \geq \sum_{k=1}^{i} P(A_k) - (i-1)$ add A_{i+1} to both sides

$$P(A_{i+1}) + P(B) \geq \sum_{k=1}^{i} P(A_k) - (i-1) + P(A_{i+1})$$

$$P(A_{i+1}) + P(B) \geq \sum_{k=1}^{i+1} P(A_k) - (i-1)$$

by definition $P(A_{i+1}) + P(B) = P(B \cap A_{i+1}) + P(B \cup A_{i+1})$

$$P(B \cap A_{i+1}) + P(B \cup A_{i+1}) \geq \sum_{k=1}^{i+1} P(A_k) - (i-1)$$

subtract 1 from both sides

$$P(B \cap A_{i+1}) + (P(B \cup A_{i+1}) - 1) \geq \sum_{k=1}^{i+1} P(A_k) - (i+1-1)$$

$$P\left(\bigcap_{k=1}^{i+1} A_k\right) \geq P\left(\bigcap_{k=1}^{i+1} A_k\right) + (\underbrace{P(B \cup A_{i+1}) - 1}_{\leq 0}) \geq \sum_{k=1}^{i+1} P(A_k) - (i+1-1)$$

So $\boxed{P\left(\bigcap_{k=1}^{i+1} A_k\right) \geq \sum_{k=1}^{i+1} P(A_k) - (i+1-1)}$

proved that $P\left(\bigcap_{k=1}^{i} A_k\right) \geq \sum_{k=1}^{i} P(A_k) - (i-1)$ by induction

induction method 2

case $n = i+1$

$$P\left(\bigcap_{k=1}^{i+1} A_k\right) \geq \sum_{k=1}^{i+1} A_k - (i+1-1)$$

$$P\left(\bigcap_{k=1}^{i+1} A_k\right) \geq \sum_{k=1}^{i} A_k - (i-1) + (P(A_{i+1}) - 1)$$

$$P\left(\bigcap_{k=1}^{i} A_k \cap A_{i+1}\right) \geq \sum_{k=1}^{i} A_k - (i-1) + (P(A_{i+1}) - 1)$$

by def $P(A \cap B) = P(A) + P(B) - P(A \cup C)$

$$P\left(\bigcap_{k=1}^{i} A_k\right) + P(A_{i+1}) - P\left(\bigcup_{k=1}^{i+1} A_k\right) \geq \sum_{k=1}^{i} A_k - (i-1) + (P(A_{i+1}) - 1)$$

$$P\left(\bigcap_{k=1}^{i} A_k\right) + P(A_{i+1}) - [P(A_{i+1}) - 1] - P\left(\bigcup_{k=1}^{i+1} A_k\right) \geq \sum_{k=1}^{i} A_k - (i-1)$$

$$P\left(\bigcap_{k=1}^{i} A_k\right) + \underbrace{\left[1 - P\left(\bigcup_{k=1}^{i+1} A_k\right)\right]}_{\geq 0} \geq \sum_{k=1}^{i} A_k - (i-1) \quad \star$$

so since we already assume $P\left(\bigcap_{k=1}^{i} A_k\right) \geq \sum_{k=1}^{i} A_k - (i-1)$ holds then $\star$ is true

Thus proved by induction $P\left(\bigcap_{k=1}^{i} A_k\right) \geq \sum_{k=1}^{i} A_k - (i-1)$

(4.)

$P(A) = \frac{3 \cdot 6}{36} = \frac{1}{2}$

$P(B) = \frac{3 \cdot 6}{36} = \frac{1}{2}$

$C = \{(3,6), (4,5), (6,3), (5,4)\}$ $\quad P(C) = \frac{1}{9}$

$P(\text{first die is 2 or 3}) = \frac{2 \cdot 6}{36} = \frac{1}{3} =$

$P(A \cap B) = \frac{1}{3} \neq P(A)P(B) = \frac{1}{4}$

$P(A \cap B) = P(A)P(B)$ ✗

$P(\text{first die is 1, 2, or 3 and sum is 9}) = \frac{1}{36} \neq P(A)P(C) = \left(\frac{1}{2}\right)\left(\frac{1}{9}\right) = \frac{1}{18}$

$P(A \cap C) = P(A)P(C)$ ✗

$P(\text{first die is 2, 3, or 6 and sum is 9}) = \frac{2}{36} = \frac{1}{18} = P(B)P(C) = \left(\frac{1}{2}\right)\left(\frac{1}{9}\right) = \frac{1}{18}$

$P(B \cap C) = P(B)P(C)$ ✓

$P(\text{first die is 2 or 3 and sum is 9}) = \frac{1}{36} = P(A)P(B)P(C) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{9}\right) = \frac{1}{36}$

$P(A \cap B \cap C) = P(A)P(B)P(C)$ ✓

No the events are not independent as $P(A \cap B) \neq P(A)P(B)$

$P(A \cap C) \neq P(A)P(C)$

(5)

HH
TH
HT
TT

A = first toss is heads

B = second toss is heads

C = both tosses are same

$P(A) = \frac{1}{2}$ $\quad P(A \cap B) = \frac{1}{4}$ $\quad P(A \cap B \cap C) = \frac{1}{4}$

$P(B) = \frac{1}{2}$ $\quad P(A \cap C) = \frac{1}{4}$

$P(C) = \frac{1}{2}$ $\quad P(B \cap C) = \frac{1}{4}$

$$\boxed{\begin{array}{l} P(A \cap B) = \frac{1}{4} = P(A)P(B) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4} \\ P(A \cap C) = \frac{1}{4} = P(A)P(C) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4} \\ P(B \cap C) = \frac{1}{4} = P(B)P(C) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4} \\ \\ P(A \cap B \cap C) = \frac{1}{4} \neq P(A)P(B)P(C) = \frac{1}{8} \end{array}}$$