(1) $\mathbb{E}[e^{j\omega x}] = \Phi_x(\omega)$ $\qquad f_x(x) = \lambda e^{-\lambda x}$

$$\lambda \int_0^\infty e^{j\omega x} e^{-\lambda x} \, dx$$

$$\lambda \int_0^\infty e^{x(j\omega - \lambda)} \, dx$$

$$\frac{\lambda}{j\omega - \lambda} \left[e^{-x(\lambda - j\omega)} \right]_0^\infty$$

$$\frac{\lambda}{j\omega - \lambda} [0 - 1]$$

$$-\left[\frac{\lambda}{j\omega - \lambda}\right] = \frac{\lambda}{\lambda - j\omega}$$

$$\mathbb{E}[e^{sx}] = \frac{\lambda}{\lambda - s} \qquad s = j\omega$$

Chernoff $P[X \geq a] \leq e^{-as} \mathbb{E}[e^{st}]$

$$\leq e^{-as} \frac{\lambda}{\lambda - s}$$

$$H(s) = (e^{-as})\left(\frac{\lambda}{\lambda - s}\right)$$

$$\frac{dH(s)}{ds} = 0 = \frac{\lambda(as - a\lambda + 1) e^{-as}}{(s - \lambda)^2}$$

$$\lambda(as - a\lambda + 1) e^{-as} = 0$$

$$as - a\lambda + 1 = 0$$

$$as = a\lambda - 1$$

$$s = \frac{\lambda a - 1}{a}$$

$$\boxed{s = \lambda - \frac{1}{a}}$$

$[X \geq a] \geq e^{-a(\lambda - \frac{1}{a})} \dfrac{\lambda}{\lambda - (\lambda - 1/a)}$

$\geq e^{1-\lambda a} \dfrac{\lambda}{\frac{1}{a}}$

$\boxed{P[X>=a] \leq a\lambda e^{1-\lambda a} \quad \text{chernoff bound}}$

$1 - P[X \geq a] = 1 - e^{-\lambda a}$

$\boxed{P[X \geq a] = e^{-\lambda a} \quad \text{exact prob}}$

(2)

(a)

$X = \mathbb{1}(U > 1-p)$ where X is $\begin{cases} 0 & \text{if } U \leq p \\ 1 & \text{if } U > 1-p \end{cases}$

(b)

$X = \sum_{i=1}^{n} \mathbb{1}(U_i > 1-p)$

(c)

(1) generate iid Berouillis until the first success ... $Y_1, Y_2, \ldots, Y_L$

L be the index of the success.

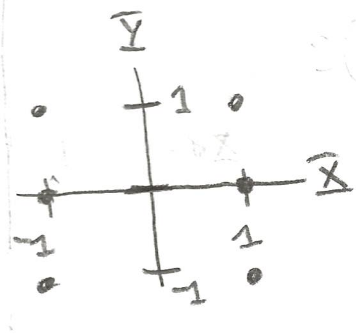
Success

when Y_L is 1 stop generating bernoullis, and the index L is the X value

3

a)

S_{XY}



$S = \{(-1,-1),(-1,0),(-1,1),(1,-1),(1,0),(1,1)\}$

$S = S_{X,Y}$

b)

$(-1,-1)\quad \left(\frac{1}{4}\right)(1-p-p_e)$

$(-1,0)\quad \left(\frac{1}{4}\right)(p_e)$

$(-1,1)\quad \left(\frac{1}{4}\right)(p)$

$(1,-1)\quad \left(\frac{3}{4}\right)(p)$

$(1,0)\quad \left(\frac{3}{4}\right)(p_e)$

$(1,1)\quad \left(\frac{3}{4}\right)(1-p-p_e)$

$\sum_S \sum = 1$

$\frac{1}{4} - \cancel{\frac{1}{4}p} - \cancel{\frac{1}{4}p_e} + \cancel{\frac{1}{4}p_e} + \cancel{\frac{1}{4}p} + \cancel{\frac{3}{4}p} + \cancel{\frac{3}{4}p_e} + \frac{3}{4} - \cancel{\frac{3}{4}p} - \cancel{\frac{3}{4}p_e}$

$\frac{1}{4} + \frac{3}{4} = 1$

c) $P[X \neq Y] = \frac{1}{4}p_e + \frac{1}{4}p + \frac{3}{4}p + \frac{3}{4}p_e = p + p_e$

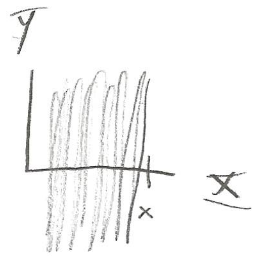
$P[Y=0] = \frac{1}{4}p_e + \frac{3}{4}p_e = p_e$

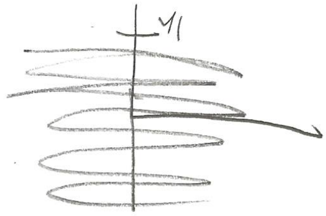
8

(a)

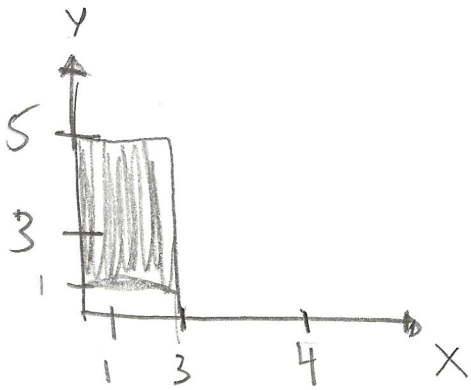
$F_X(x) = F_{X,Y}(x,\infty) = (1-\frac{1}{x^2})$ for $x > 1$

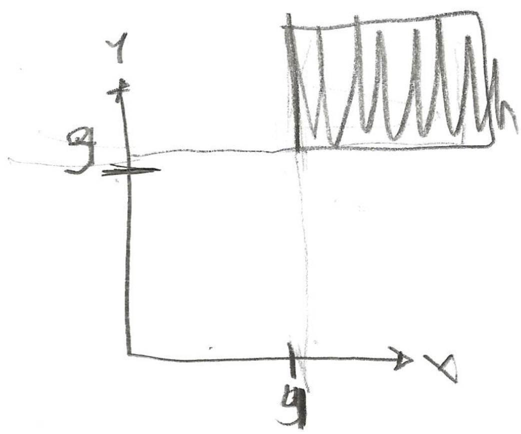
$F_Y(y) = F_{X,Y}(\infty,y) = (1-\frac{1}{y^2})$ for $y > 1$





(b)





$F_{XY}(3,5) = (1-\frac{1}{9})(1-\frac{1}{25}) = (\frac{8}{9})(\frac{24}{25}) = \boxed{0.8533}$

$\{X>4, Y>3\} = F_{X,Y}(\infty,\infty) - F_{X,Y}(4,\infty) - F_{X,Y}(\infty,3) + F_{X,Y}(4,3)$

$= 1 - \frac{15}{16} - \frac{8}{9} + (\frac{15}{16})(\frac{8}{9}) = \boxed{.0069}$

5.

a. $$\int_0^\infty \int_0^x c e^{-x} e^{-y}\, dx = \int_0^\infty c e^{-x}(1-e^{-x})\, dx = \frac{c}{2} = 1$$

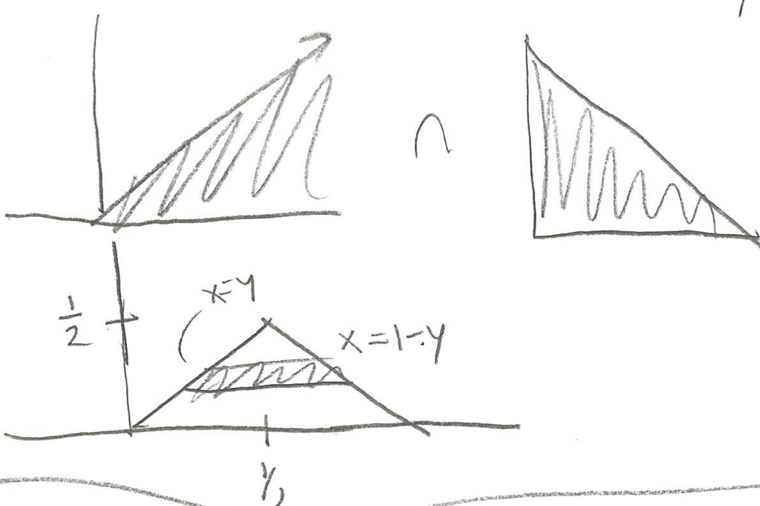
$c=2$

b. $$f_X(x) = 2\int_0^x e^{-x} e^{-y}\, dy = 2e^{-x}\left[-e^{-y}\right]_0^x = 2e^{-x}\left[1-e^{-x}\right] \quad 0 \le x \le \infty$$

$$f_Y(y) = 2\int_0^\infty e^{-x} e^{-y}\, dx = 2\int_y^\infty 2e^{-x} e^{-y}\, dx = 2e^{-2y} \quad 0 \le y \le \infty$$

c.

$x+y \le 1$



$$P[X+Y \le 1] = \int_0^{.5}\int_y^{1-y} 2e^{-x}e^{-y}\, dx\, dy = \int_0^{.5} 2e^{-y}\left[e^{-y} - e^{-(1-y)}\right] dy$$

$$= 1 - 2e^{-1}$$