(1) $\Phi_X(\omega) = e^{jm\omega - \frac{\sigma^2 \omega^2}{2}}$

$$N(m,\sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}}$$

$$\Phi_X(\omega) = E_X[e^{j\omega x}] = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}} \cdot e^{j\omega x} \, dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{j\omega x - \frac{(x-m)^2}{2\sigma^2}} \, dx$$

hint (1)

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-(x-K)^2 + 2mj\omega\sigma^2 - \omega^2\sigma^4}{2\sigma^2}} dx \quad \text{where } K = m + j\omega\sigma^2$$

$$= e^{\frac{2mj\omega\sigma^2 - \omega^2\sigma^4}{2\sigma^2}} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-K)^2} dx}_{\text{Gaussian w/ mean } m+j\omega\sigma^2 \text{, var } \sigma^2} \qquad K = m + j\omega\sigma^2$$

so $\int_{-\infty}^{\infty} N(m+j\omega\sigma^2, \sigma^2) = 1$

$$= e^{\frac{2mj\omega\sigma^2 - \omega^2\sigma^4}{2\sigma^2}}$$

$$= e^{\frac{2mj\omega\sigma^2}{2\sigma^2} - \frac{\omega^2\sigma^4}{2\sigma^2}}$$

$$\boxed{\Phi_X(\omega) = e^{\left(jm\omega - \frac{\omega^2\sigma^2}{2}\right)}}$$

2. $Y = aX + b$ $\qquad \frac{Y-b}{a} = X \qquad \frac{dx}{dy} = \frac{1}{a}$

$$f_Y(y) = f_X\left(\frac{Y-b}{a}\right) \cdot \left|\frac{dx}{dy}\right|$$

$$= f_X\left(\frac{Y-b}{a}\right) \cdot \left|\frac{1}{a}\right|$$

$$\Phi_Y(\omega) = \mathbb{E}[e^{j\omega Y}]$$

$$= \mathbb{E}[e^{j\omega(aX+b)}]$$

$$= \mathbb{E}[e^{j\omega aX + j\omega b}]$$

$$= e^{j\omega b}\,\mathbb{E}[e^{aj\omega X}]$$

$$= e^{j\omega b} \int_{-\infty}^{\infty} e^{aj\omega x} \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-m)^2}{2\sigma^2}}\, dx$$

$$= \frac{e^{j\omega b}}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{aj\omega x - \frac{(x-m)^2}{2\sigma^2}}\, dx$$

$$\frac{e^{j\omega b}}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{aj\omega x \frac{-x^2+2xm-m^2}{2\sigma^2}}\, dx$$

$$\frac{e^{j\omega b}}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{\frac{2\sigma^2 aj\omega x - x^2 + 2xm - m^2}{2\sigma^2}}\, dx$$

$$\frac{e^{j\omega b}}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{\frac{-x^2 + 2x(\sigma^2 aj\omega + m) - m^2}{2\sigma^2}}\, dx$$

$$\frac{e^{j\omega b}}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{\frac{-x^2 + 2x(\sigma^2 aj\omega + m) - (\sigma^2 aj\omega + m)^2 + (\sigma^2 aj\omega + m)^2 - m^2}{2\sigma^2}}\, dx$$

$$\frac{e^{j\omega b}}{\sqrt{2\pi}\sigma} \cdot e^{\frac{(\sigma^2 aj\omega + m)^2 - m^2}{2\sigma^2}} \int_{-\infty}^{\infty} e^{\frac{-x^2 + 2x(\sigma^2 aj\omega + m) - (\sigma^2 aj\omega + m)^2}{2\sigma^2}}\, dx$$

$$e^{j\omega b} \cdot e^{\frac{(\sigma^2 aj\omega + m)^2 - m^2}{2\sigma^2}} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x - [\sigma^2 aj\omega + m])^2}{2\sigma^2}}\, dx}_{\text{Gaussian}}$$

$$e^{j\omega b} \cdot e^{\frac{-\sigma^4 a^2 \omega^2 + 2\sigma^2 aj\omega m \cancel{+m^2} \cancel{-m^2}}{2\sigma^2}}$$

$$= e^{j\omega b + aj\omega m - \frac{\sigma^2 a^2 \omega^2}{2}} = \boxed{e^{\left(j\omega(b+m) - \frac{\sigma^2 a^2 \omega^2}{2}\right)} = \Phi_Y(\omega)}$$

You may conclude if $Y = aX + b$ and X is gaussian then

$$\boxed{m_Y = m_X + b \quad \text{and} \quad \begin{aligned} \sigma_Y &= \sigma_X a \\ \sigma_Y^2 &= \sigma_X^2 a^2 \end{aligned}}$$

③

a. $\mathbb{E}[x] = \int p(x)\, x \; dx$

$=$

$= \int [\, p(x \cap X \geq a) + p(x \cap X < a) \,]\, x \; dx$

$= \int [\, p(x \mid X \geq a)\, p(X \geq a) + p(x \mid X < a)\, p(X < a) \,]\, x \; dx$

$= \int p(x \mid X \geq a)\, x \; dx \; p(X \geq a) + \int p(x \mid X < a)\, x \; dx \; p(X < a)$

$\mathbb{E}[X] = \mathbb{E}[X \mid X \geq a]\, p(X \geq a) + \mathbb{E}[X \mid X < a]\, P[X < a]$

b. Markov inequality proof

$P(X \geq a) \leq \frac{\mathbb{E}[X]}{a}$

$\mathbb{E}[X] = \underbrace{\mathbb{E}[X \mid X < a]\, P[X < a]}_{\geq 0} + \mathbb{E}[X \mid X \geq a]\, P[X \geq a]$

$\frac{\mathbb{E}[X] - \mathbb{E}[X \mid X < a]\, P[X < a]}{\mathbb{E}[X \mid X \geq a]} = P[X \geq a]$

$\mathbb{E}[X \mid X < a]\, P[X < a] \geq 0$ by definition of problem $X > 0$ and $a > 0$ and $1 \geq P(\cdot) \geq 0$

$\frac{\mathbb{E}[X]}{\mathbb{E}[X \mid X \geq a]} \geq P[X \geq a]$

$\mathbb{E}[X \mid X \geq a] \geq a$

so

divide by less than before gives bigger number.

$\boxed{\frac{\mathbb{E}[X]}{a} \geq P[X \geq a]}$

(4)

$Y = \frac{X}{n}$ X is Binomial w/ n trials

$a=1$

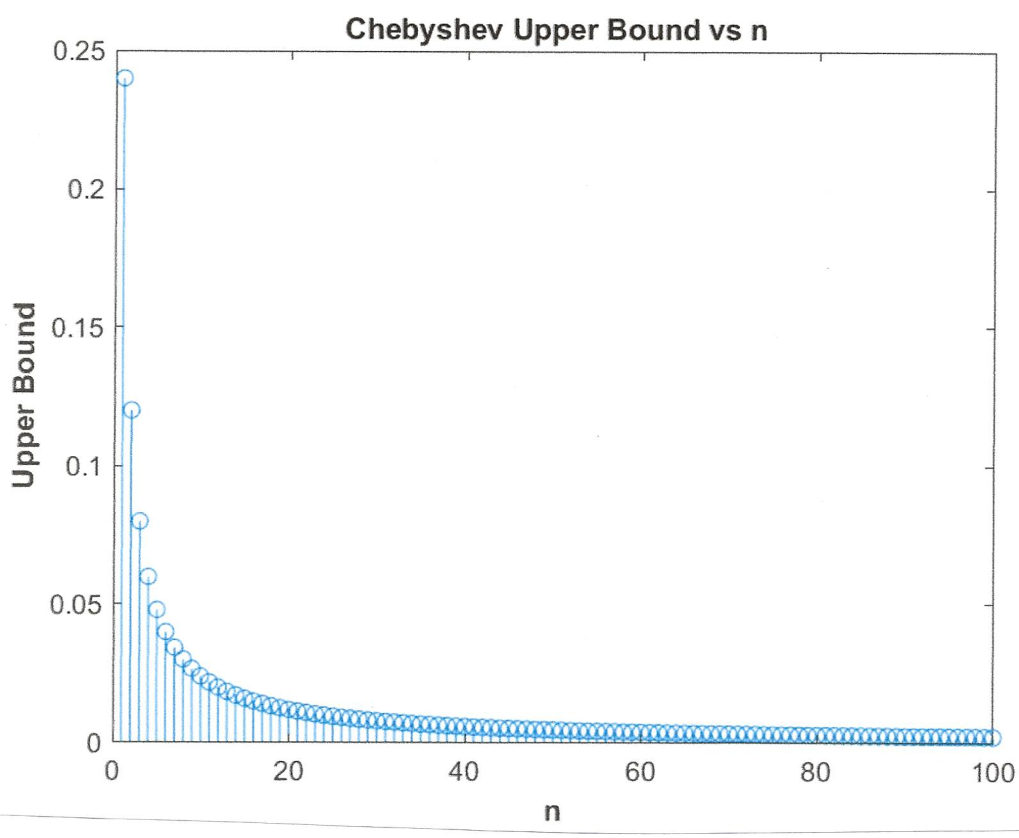
$E[Y] = \frac{np}{n} = p$

$Var[Y] = \frac{1}{n^2} \cdot np(1-p) = \frac{1}{n}(1-p)p$

Chebyshev bound

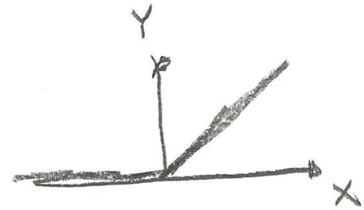
$$\boxed{P(|Y-p|>a) \leq \left(\frac{1}{n}(1-p)p\right)}$$

as $n \to \infty$ then the bound $\frac{1}{n}(1-p)p \to 0$



(3)

(a)



for $Y \geq 0$

$$f_Y(y) = \left|\frac{dx}{dy}\right| f_X(x)$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\,2} e^{\frac{-(y-2)^2}{8}} \qquad y > 0$$

for $Y < 0$

$$\Phi\left(\frac{0-2}{2}\right)$$

$$\boxed{f_Y(y) = \underbrace{u(y)}_{\text{step fn}} \cdot \frac{1}{2\sqrt{2\pi}} e^{\frac{-(y-2)^2}{8}} + \underbrace{\delta}_{\text{delta fn}}(\max(y,0))\,\Phi(-1)}$$

(b) Y is not a continuous random variable due to the delta function... It is a mixed random variable.