1.

if $j=0$

$$\text{cov}(Y_n, Y_n) = \mathbb{E}[(Y_n)(Y_n)] - \mathbb{E}[Y_n]\,\mathbb{E}[Y_n]$$

$$= \mathbb{E}[(X_n + X_{n+1} + X_{n+2})^2] - \left[\mathbb{E}[X_n] + \mathbb{E}[X_{n+1}] + \mathbb{E}[n+2]\right]^2$$

$$= \mathbb{E}[X_n^2 + X_{n+1}^2 + X_{n+2}^2 + 2X_nX_{n+1} + 2X_nX_{n+2} + 2X_{n+1}X_{n+2}] - \left[\mathbb{E}[X_n]^2 + \mathbb{E}[X_{n+1}]^2 + \mathbb{E}[X_{n+2}]^2 + 2\mathbb{E}[X_n]\mathbb{E}[X_{n+1}] + 2\mathbb{E}[X_{n+1}]\mathbb{E}[X_{n+2}] + 2\mathbb{E}[X_n]\mathbb{E}[X_{n+2}]\right]$$

$$= \mathbb{E}[X_n^2] - \mathbb{E}[X_n]^2 + \mathbb{E}[X_{n+1}^2] - \mathbb{E}[X_{n+1}]^2 + \mathbb{E}[X_{n+2}^2] - \mathbb{E}[X_{n+2}]^2$$

$$= 3\sigma^2$$

if $j \neq (1 \text{ or } 2)$

$$\text{cov}(Y_n, Y_{n+j}) = \mathbb{E}[Y_n Y_{n+j}] - \mathbb{E}[Y_n]\,\mathbb{E}[Y_{n+j}]$$

$$= \mathbb{E}[(X_n + X_{n+1} + X_{n+2})(X_{n+j} + X_{n+j+1} + X_{n+j+2})] - \mathbb{E}[(X_n + X_{n+1} + X_{n+2})]\,\mathbb{E}[X_{n+j} + X_{n+j+1} + X_{n+j+2}]$$

$$= \mathbb{E}[X_nX_{n+j} + X_nX_{n+j+1} + X_nX_{n+j+2} + X_{n+1}X_{n+j} + X_{n+1}X_{n+j+1} + X_{n+j+2}X_{n+1} + X_{n+2}X_{n+j} + X_{n+2}X_{n+j+1} + X_{n+2}X_{n+j+2}] - [9\pi^2]$$

$$= 9\pi^2 - 9\pi^2 = \boxed{0}$$

$$
\begin{aligned}
&\mathbb{E}[X_n X_{n+j}] && - \mathbb{E}[X_n]\,\mathbb{E}[X_{n+j}] \\
+ &\mathbb{E}[X_n X_{n+j+1}] && - \mathbb{E}[X_n]\,\mathbb{E}[X_{n+j+1}] \\
+ &\mathbb{E}[X_n X_{n+j+2}] && - \mathbb{E}[X_n]\,\mathbb{E}[X_{n+j+2}] \\
+ &\mathbb{E}[X_{n+1} X_{n+j}] && - \mathbb{E}[X_{n+1}]\,\mathbb{E}[X_{n+j}] \\
+ &\mathbb{E}[X_{n+1} X_{n+j+1}] && - \mathbb{E}[X_{n+1}]\,\mathbb{E}[X_{n+j+1}] \\
+ &\mathbb{E}[X_{n+1} X_{n+j+2}] && - \mathbb{E}[X_{n+1}]\,\mathbb{E}[X_{n+j+2}] \\
+ &\mathbb{E}[X_{n+2} X_{n+j}] && - \mathbb{E}[X_{n+2}]\,\mathbb{E}[X_{n+j}] \\
+ &\mathbb{E}[X_{n+2} X_{n+j+1}] && - \mathbb{E}[X_{n+2}]\,\mathbb{E}[X_{n+j+1}] \\
+ &\mathbb{E}[X_{n+2} X_{n+j+2}] && - \mathbb{E}[X_{n+2}]\,\mathbb{E}[X_{n+j+1}]
\end{aligned}
$$

if $j=1$ then $\mathrm{var}(X_{n+1}) + \mathrm{var}(X_{n+2})$

$= 2\sigma^2$

if $j=2$ then $\mathrm{var}(X_{n+2})$

$= \sigma^2$

$$
\boxed{\mathrm{cov}(Y_n, Y_{n+j}) = \begin{cases} 3\sigma^2 & j=0 \\ 2\sigma^2 & j=1 \\ \sigma^2 & j=2 \\ 0 & j>2 \end{cases}}
$$

(2)

$$\boxed{\mathbb{E}[S_n] = n\mu}$$

$$\mathrm{Var}[S_n] = \sum_{i=1}^{n} \mathrm{var}(x_i) + \sum_{j=1}\sum_{\substack{i=1 \\ i\neq j}} \mathrm{cov}(x_i, x_j)$$

$$= n\sigma^2 + \sum_{j=1}\sum_{\substack{i=1 \\ i\neq j}} \sigma^2 p^{|i-j|}$$

$$n\sigma^2 + \sigma^2 \sum_{j=1}\sum_{\substack{i=1 \\ i\neq j}} p^{|i-j|}$$

$$n\sigma^2 + \sigma^2 \left[2p \sum_{i=1}^{n-1} \frac{(1-p^i)}{1-p} \right.$$

$$n\sigma^2 + \frac{2\sigma^2 p}{1-p} \sum_{i=1}^{n-1} (1-p^i)$$

$$\mathrm{var}[S_n] = n\sigma^2 + \frac{2\sigma^2 p}{1-p}\left[(n-1) - p\frac{(1-p^{n-1})}{1-p} \right]$$

j \ i	1	2	3	4	...	n
1		2	3	4	...	n
2	1		3	4	...	n
3	1	2		4	...	n
4	1	2	3		...	n
⋮	1	2	3	4	...	n
n	1	2	3	4	5 ...	

j \ i	1	2	3	4	...	n
1		1	2	3	...	n-1
2	1		1	2	...	n-2
3	2	1		1	...	n-3
4	3	2	1			n-4
⋮	⋮	⋮	⋮			
n	n-1	n-2	n-3			

n-4

n-(n-1)

j \ i	1	2	3	4	5
1		1	2	3	4
2	1		1	2	3
3	2	1		1	2
4	3	2	1		1
5	4	3	2	1	

$$\frac{p\,2}{1-p}\left[4 - \frac{(1-p^4)}{1-p}\right]$$

$$2\sum_{i=1}^{4} \frac{p(1-p^i)}{1-p}$$

③

a. $N(50000, 10000^2) = x_i$ $\quad x_i \sim$ iid

$$Z = 74000 + 11 \cdot 47000 - \left(\sum_{i=1}^{11} x_i\right)$$

$$\mathbb{E}[Z] = 74000 + 11 \cdot 47000 - 11 \cdot 50000$$

$$Var[Z] = 11 \cdot 10000^2$$

$$\Phi\left(\frac{20000 - \mathbb{E}[Z]}{\sqrt{Var[Z]}}\right) = \boxed{.2633 = P[\text{supply of gas will be below } 20000 \text{ gallons}]}$$

ⓑ.

$$\mathbb{E}[Z] = 74000 + 11 \cdot X - 11 \cdot 50000$$
$$Var[Z] = 11 \cdot 10000^2$$

$$\Phi\left(\frac{20000 - \mathbb{E}[Z]}{\sqrt{Var[Z]}}\right) = .005$$

$$\frac{20000 - \mathbb{E}[Z]}{\sqrt{Var[Z]}} = -2.5758$$

$$20000 + \sqrt{Var[Z]} \cdot 2.5758 = 74000 + 11X - 11 \cdot 50000$$

$$\frac{20000 + \sqrt{Var[Z]} \cdot 2.5758 - 74000 + 11 \cdot 50000}{11} = X$$

$X = 52857$ gallons

$\boxed{52857 \text{ gallons per week}}$

(4)

$\mathbb{E}[S_n] = 100 \cdot 0.5 = 50$
$Var[S_n] = 100 \cdot 0.25 = 25$

$P[40 < S_n < 60]$

$$P\left[\frac{40-\mathbb{E}[S_n]}{\sqrt{Var[S_n]}} < \frac{S_n - \mathbb{E}[S_n]}{\sqrt{Var[S_n]}} < \frac{60-\mathbb{E}[S_n]}{\sqrt{Var[S_n]}}\right]$$

$$= \Phi\left(\frac{60-50}{\sqrt{25}}\right) - \Phi\left(\frac{40-50}{\sqrt{25}}\right) = \boxed{0.9545 = P[40 < S_n \leq 60]}$$

$=$

$P[50 < S_n < 53]$

$$= \Phi\left(\frac{53-50}{\sqrt{25}}\right) - \Phi\left(\frac{50-50}{\sqrt{25}}\right) = \boxed{0.2257 = P[50 < S_n \leq 53]}$$

(5)

X_i is iid w/ $X_i \sim e^{-x}$

$\mathbb{E}[X_i] = 1$
$Var[X_i] = 1$

$$\left(1 - \Phi\left(\frac{15-n}{\sqrt{n}}\right)\right) \geq .99$$

$$.01 \geq \Phi\left(\frac{15-n}{\sqrt{n}}\right)$$

$$-2.3263 \geq \frac{15-n}{\sqrt{n}}$$

$$-2.3263\sqrt{n} + n \geq 15$$

$\boxed{\text{28 pens}}$