Chapter 4

1 Show the trace of Making_Change(3.65).

Denomination	Remaining Amount of Money
-	3.65
2.00	1.65
1.00	0.65
0.25	0.40
0.25	0.15
0.10	0.05
0.05	0.00

So we need to give one 2.00, one 1.00, two 0.25, one 0.10 and one 0.05.

3 There are several different optimal solutions for the following input with $W = 4$.

i	1	2	3	4	5	6
v_i	20	24	9	15	25	36
w_i	2	4	1	5	5	6

Can you find them?

5 In the following tables, for each vertex v, we first write the value of minweight(v) and then the value of closest(v).

(a)

A	T	A		B		C		D		E		F		G		H	
$\{A\}$	$\{\}$			6	A	∞		∞		1	A	∞		∞		∞	
$\{A, E\}$	$\{EA\}$			2	E	∞		∞				1	E	∞		∞	
$\{A, E, F\}$	$\{EA, FE\}$			2	E	5	F	∞						3	F	∞	
$\{A, E, F, B\}$	$\{EA, FE, BE\}$					5	F	∞						3	F	∞	
$\{A, E, F, B, G\}$	$\{EA, FE, BE, GF\}$					4	G	5	G							3	G
$\{A, E, F, B, G, H\}$	$\{EA, FE, BE, GF, HG\}$					4	G	5	G								
$\{A, E, F, B, G, H, C\}$	$\{EA, FE, BE, GF, HG, CG\}$							5	G								
$\{A, E, F, B, G, H, C, D\}$	$\{EA, FE, BE, GF, HG, CG, DG\}$																

(b)

A	T	A		B		C		D		E		F		G		H	
$\{A\}$	$\{\}$			1	A	∞		∞		4	A	8	F	∞		∞	
$\{A, B\}$	$\{BA\}$					2	B	∞		4	A	6	B	6	B	∞	
$\{A, B, C\}$	$\{BA, CB\}$							3	C	4	A	6	B	2	C	∞	
$\{A, B, C, G\}$	$\{BA, CB, GC\}$							1	G	4	A	1	G			1	G
$\{A, B, C, G, D\}$	$\{BA, CB, GC, DG\}$									4	A	1	G			1	G
$\{A, B, C, G, D, F\}$	$\{BA, CB, GC, DG, FG\}$									4	A					1	G
$\{A, B, C, G, D, F, H\}$	$\{BA, CB, GC, DG, FG, HG\}$									4	A						
$\{A, B, C, G, D, F, H, E\}$	$\{BA, CB, GC, DG, FG, HG, EA\}$																

7 (a) You always go as far as you can. Formally: when you are at A or at any gaz station, you go to the farthest gaz station that is at distance at most D from your current position.

(b) Let S be an optimal solution and let S^* be the greedy solution. We denote the i-th element of S by $S[i]$ and the i-th element of S^* by $S^*[i]$. The number $S[i]$ corresponds to the distance between A and the i-th stop (when following strategy S). The number $S^*[i]$ corresponds to the distance between A and the i-th stop (when following strategy S^*). Notice that, since S is optimal, the number of stops in S is not larger than the number of stops in S^*.

If $S = S^*$, we are done. Therefore, suppose that $S \neq S^*$. We will prove by induction that S can be transformed into S^* without increasing the number of stops, from which we can conclude that S^* is optimal. If $S[1] = S^*[1]$, then the two solutions agree on the first stop. If $S[1] \neq S^*[1]$, then $S[1] < S^*[1]$ since $S^*[1]$ is the farthest we can travel from A. Moreover, since $S[2]$ is reachable from $S[1]$, we have $S[2]-S[1] \leq D$. Therefore, $S[2]-S^*[1] < S[2]-S[1] \leq D$. Consequently, if we assign the value $S^*[1]$ to $S[1]$, $S[2]$ is still reachable from the updated $S[1]$ and the two solutions agree on the first stop. Moreover, the number of stops in the updated S is still the same.

Suppose that the two solutions agree up to the i-th stop. If $S[i+1] = S^*[i+1]$, then the two solutions also agree on the $(i+1)$-th stop. If $S[i+1] \neq S^*[i+1]$, then $S[i+1] < S^*[i+1]$ since $S^*[i+1]$ is the farthest we can travel from $S[i] = S^*[i]$. Moreover, since $S[i+2]$ is reachable from $S[i+1]$, we have $S[i+2] - S[i+1] \leq D$. Therefore, $S[i+2] - S^*[i+1] < S[i+2] - S[i+1] \leq D$. Consequently, if we assign the value $S^*[i+1]$ to $S[i+1]$, $S[i+2]$ is still reachable from the updated $S[i+1]$ and the two solutions agree on the $i+1$-th stop. Moreover, the number of stops in the updated S is still the same.

Therefore, we can, one stop at a time, transform S into S^* without increasing the number of stops. So S^* is optimal.

(c) You need to scan once the path from A to B to build the sequence of stops using this greedy criterion, so it takes $O(n)$ time.

(d) Take $n = 3$, $d_1 = 4$, $d_2 = 1$, $d_3 = 4$ and $D = 5$. The following strategy is optimal, but not greedy. Stop at the first gaz station, then drive to B.

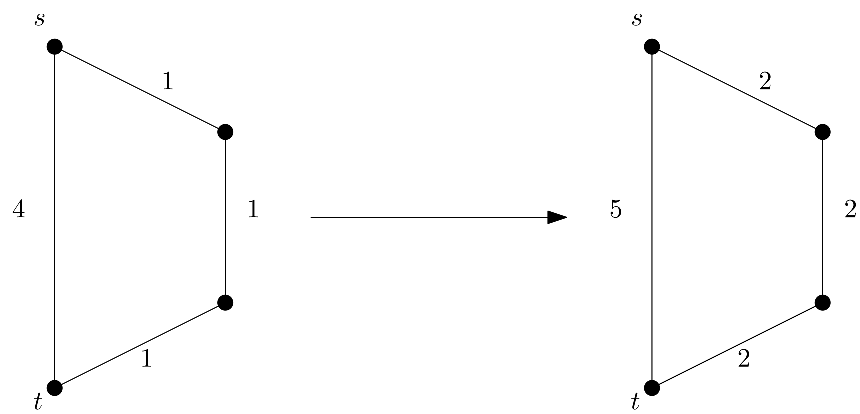
This does not contradict anything. We proved that the greedy solution is optimal, but we never proved that the greedy strategy was the only way to get an optimal strategy.

9 Yes it can. The correctness of Kruskal's algorithm relies on the lemma we proved in class about minimum spanning trees. If you assume that some weights are negative, you can re-write the exact same proof and everything works.

11 By the hypotheses, the graph G' must be connected. It remains to show that it is acyclic. We prove it by contradiction. Suppose it is cyclic. We can remove any edge on this cycle and G' stays connected. By removing an edge, the weight of G' decreases. This contradicts the fact that weight(G') is minimum.

13 (a) No, the minimum spanning tree does not change. To see this, let's run Kruskal to find the MST. Kruskal's algorithm will consider the edges in the same order, because we increased all of the edge weights by the same amount. We have the same graph structure, so Kruskal's algorithm will add the same edges to the MST. Since Kruskal's algorithm is correct, the MST cannot change.

(b) No, some shortest paths could change. The length of a path increases by the number of edges on that path so paths with a shorter number of edges may be shortest paths in the modified graph but not the original graph. Here is an example where the shortest path from s to t changes.



15 It is true: e_M cannot be part of any MST of G. We prove this by contradiction.

Let $e_M = \{a, b\}$, where $a, b \in V$. Assume that e_M is part of some MST T of G. The tree T can be decomposed into $T = e_M$ PLUS T_a PLUS T_b , where T_a is a tree containing a and T_b is a tree containing b. Observe that T_a and T_b are disjoint.

Since e_M is part of a cycle in G, and since T_a and T_b are disjoint, there is an edge $e' \neq e_M$ in E between a vertex of T_a and a vertex of T_b. Since e_M is the heaviest edge, then $wt(e') < wt(e_M)$. Consider the tree $T' = T$ MINUS e_M PLUS e'. We have

$$wt(T') = wt(T) - wt(e_M) + wt(e') < wt(T) - wt(e') + wt(e') = wt(T),$$

from which T does not have minimum weight. This is a contradiction since T was supposed to be an MST.