CSI - 3105 Design & Analysis of Algorithms
Course 17

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For each of the 4 previous problems,

- Not known if it can be solved in polynomial time.

- If the answer to the question is YES, then
  - There is a "short" proof for this.

    Here, "short" means the length of the proof is "polynomial in the length of the input".

  - If someone gives us such a short proof, then we can "easily" verify this proof.

    Here, "easily" means "in polynomial time".

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A decision problem A is in NP if

- If for a given input I, the answer to the question $A(I)$ is YES, then there exists a proof/solution/certificate C such that
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The following problems are in NP:

HAM-CYCLE, TSP, SUBSET-SUM, CLIQUE

§6.2 A More Formal Approach Using Languages

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$$HAM-CYCLE = \{G \mid G \text{ is a graph that contains a Hamiltonian cycle}\}$$

$$\begin{aligned} TSP = \{(G,K) \mid\ & G \text{ is a complete directed graph } G=(V,E), \\ & \text{where each edge } (u,v) \in E \\ & \text{has a weight } wt(u,v) > 0, \\ & K \text{ is an integer} \\ & \text{and } G \text{ contains a Hamiltonian cycle} \\ & \text{with total weight at most } K.\} \end{aligned}$$

$$SUBSET-SUM = \{(S,t) \mid S \text{ is a set of integers, } t \text{ is an integer and } \exists S' \subseteq S \text{ such that } \sum_{x \in S'} x = t.\}$$

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$$CLIQUE = \{(G,K) \mid G \text{ is an undirected graph, } K \text{ is an integer and } G \text{ contains a clique of size } K.\}$$

Definition (Complexity Class P)

The language L (of a decision problem) is in P if the following is true. There exists an algorithm A and a constant $c \geq 1$ such that for any input x,

- If $x \in L$, then $A(x)$ returns YES.
- If $x \notin L$, then $A(x)$ returns NO.
- The running time of $A(x)$ is $O(n^c)$, where n is the length of x.

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The language L (of a decision problem) is in NP if the following is true. There exists an algorithm V and a constant $c \geq 1$ such that for any input x,

$x \in L \iff$ there exists a certificate y such that

- $|y| = O(|x|^c)$,
- $V(x, y)$ returns YES
- and the running time of $V(x, y)$ is polynomial in the length of x.

Observe that V is a verification algorithm. It has 2 input parameters.

We show that

$$\text{HAMCYCLE} = \{G : G \text{ is a graph that has a Hamiltonian cycle}\}$$

is in NP:

Verification algorithm V takes as input

- graph $G = (V, E)$, where $n = |V|$
- certificate $v_1, v_2, \ldots, v_k$

Step 1: check if $k = n$?

Step 2: check if $v_1, v_2, \ldots, v_k$ are all different?

Step 3: check if $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\}$ are edges in G.

Step 4: if Steps 1-3 were successful, return YES, otherwise, return NO.

G is in HAMCYCLE

$\Longleftrightarrow$

$\exists$ permutation $v_1, v_2, \ldots, v_n$ of G's vertex set such that
$\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\}$ are edges in G.

$\Longleftrightarrow$

$\exists$ certificate $(v_1, v_2, \ldots, v_K)$ with $K = n$ such that $V(G, (v_1, v_2, \ldots, v_K))$ returns YES

the length of the certificate

$= \#$ of vertices in G

$= O(\text{size of } G)$

Running time of $V = O((\text{size of } G)^2)$

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$V(x, y)$ does the following: it runs $A(x)$ and that's it! (It ignores y.)

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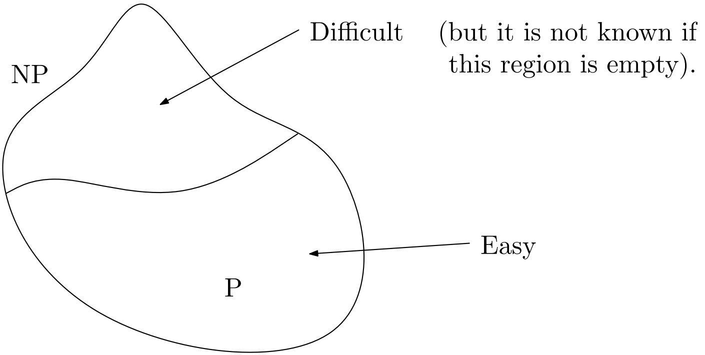
Therefore L is in NP. $\square$

Big Question

Is $P = NP$ or $P \neq NP$?

Most people believe that $P \neq NP$.



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Such an L must be "difficult".

So we should look at the "most difficult" problems.

But what does this mean?! How can we measure how difficult a problem is?!

§6.3 Reductions

Definition (Polynomial-Time Reduction)

Let L and L' be two languages. We say that L is *polynomial-time reducible to* L' if the following is true: There exists a function f which satisfies the following *famous* 3 *properties*:

1. f maps inputs for L to inputs for L'.
2. for every input x for L,

$$x \in L \Longleftrightarrow f(x) \in L'$$

3. for every input x for L, $f(x)$ can be computed in time that is polynomial in the length of x.

Notation: $L \leq_P L'$

What Does This Mean?

If we have a program A' that solves L', then we can use A' to solve L:

- Compute $x' = f(x)$
- Run A' on input x'.

Thus, we only have to write a program for the function f.

Example of a Reduction

$$CLIQUE = \{(G, K) \mid \text{graph } G \text{ has a clique with } K \text{ vertices.}\}$$
$$INDEP - SET = \{(G, K) \mid \text{graph } G \text{ has an independent set of } K \text{ vertices.}\}$$

Clique: each pair of vertices is connected by an edge.

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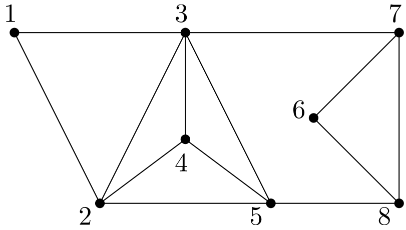

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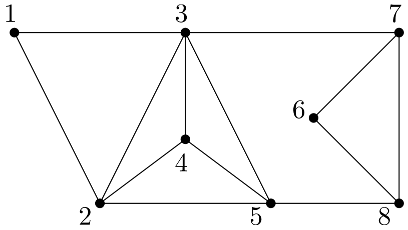

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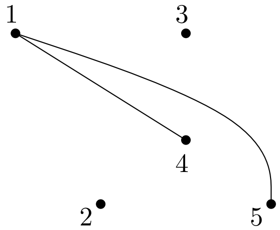
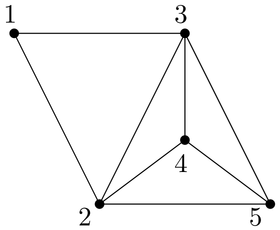
We want to show that

$$INDEP-SET \leq_P CLIQUE.$$

What is the connection between clique and independent set?



G



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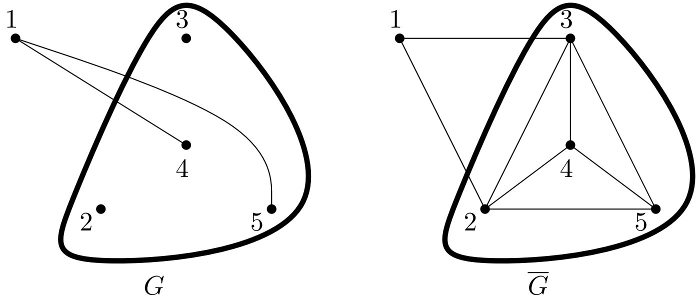

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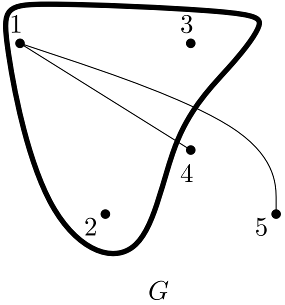

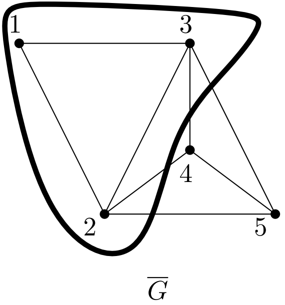

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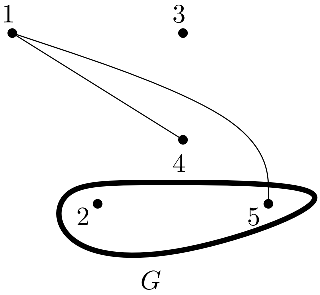

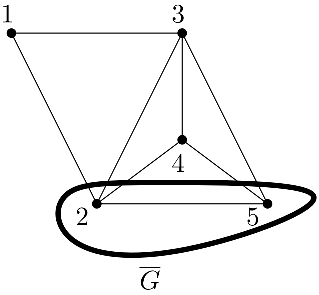

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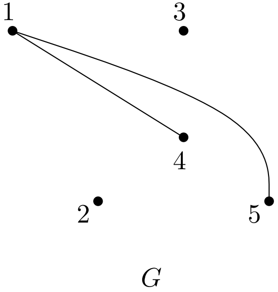
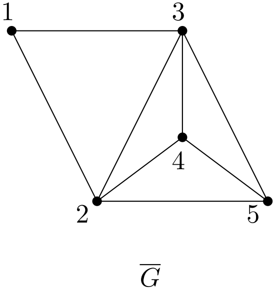
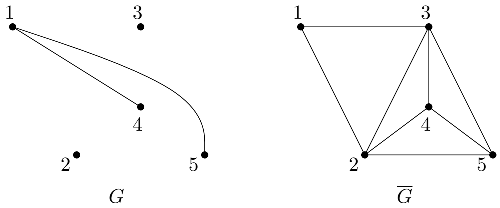




Is this a coincidence?



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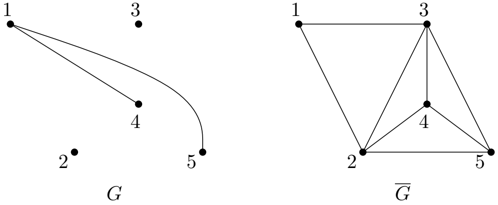

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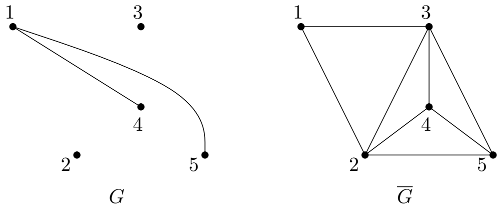

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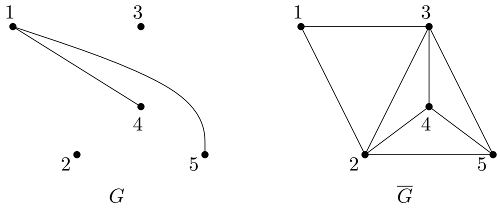

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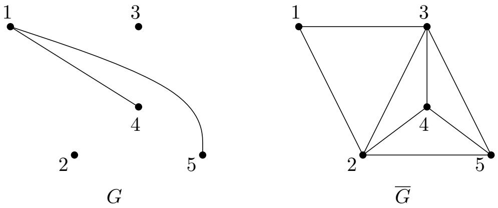

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Theorem

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Intuition:

- $L' \in P$ means "L' is easy".
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So L is easy. So $L \in P$.

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The running time of A is polynomial in the length of x. So $L \in P$. □