CSI - 3105 Design & Analysis of Algorithms
Course 21

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Fall 2019

Remember the following theorem (refer to Course 20).

Theorem

$$\left.\begin{array}{r} L \text{ is } NP\text{-Complete} \\ L \leq_P L' \\ L' \in NP \end{array}\right\} \implies L' \text{ is } NP\text{-Complete}$$

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1. f transforms any input (a Boolean circuit) B for $CIRCUIT - SAT$ and produces an input $\phi = f(B)$ (a Boolean formula) for 3SAT.
2. There exist truth-values for the unknown input gates such that B's output is true

   $$\Longleftrightarrow$$

   There exist truth-values for the variables such that ϕ is true
3. $\phi = f(B)$ can be computed in time that is polynomial in the size of B.

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- one variable for each gate
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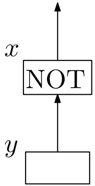
Unknown input gates:

$$x\ \boxed{?} \longleftrightarrow \text{clause: } x \vee \neg x$$

NOT-gates:



$\longleftrightarrow$

clauses: $(y \implies \neg x) \wedge (\neg y \implies x)$
same as

$$(\neg y \vee \neg x) \wedge (y \vee x)$$

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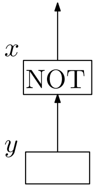

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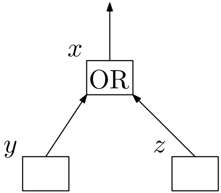
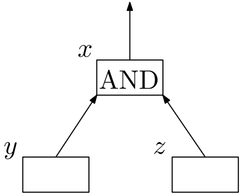


$\longleftrightarrow$

clauses: $(x \implies (y \vee z)) \wedge ((y \vee z) \implies x)$
same as
$$(\neg x \vee y \vee z) \wedge (\neg y \vee x) \wedge (\neg z \vee x)$$

AND-gates:



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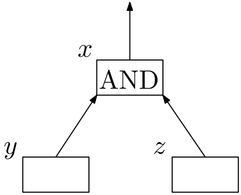

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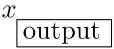
Output-gates:



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clause: x

Let ϕ be the conjunction of all these clauses.

By construction,

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- number of variables in ϕ: number of gates in B.
- each clause has at most 3 literals
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Conclusion: 3SAT is NP-Complete! $\square$

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- 3SAT is NP-Complete
- $3SAT \leq_P INDEP-SET$ (refer to Course 19)
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- $CLIQUE \leq_P VERTEX - COVER$ (refer to Course 18)

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How do we prove that $SUBSET - SUM \leq_P VERTEX - COVER$?

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We could find a function f which satisfies the famous 3 properties...

Exercise

(Exercise 20) Is there a problem in *NP* that is not *NP*-Complete?