CSI - 3105 Design & Analysis of Algorithms
Course 7

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Fall 2019

Cyclic Directed Graphs

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Step 1 : Run DFS (including pre/post-numbers)

Step 2 : For each non-tree edge (v, u), test if

$$pre(u) < pre(v) < post(v) < post(u).$$

- If "yes" for at least one non-tree edge, return "cyclic".
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Running time: $O(|V| + |E|)$.

Back to Topological Ordering

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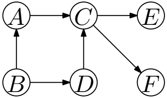

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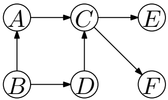

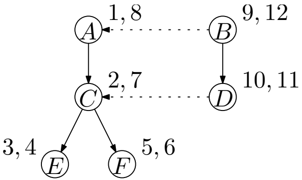

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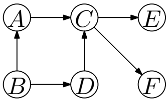

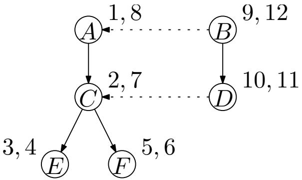

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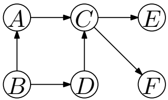

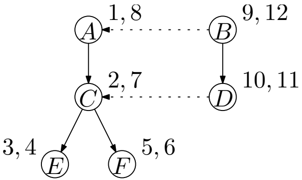

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$B \quad D \quad A \quad C \quad F \quad E$

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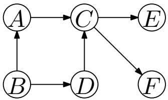

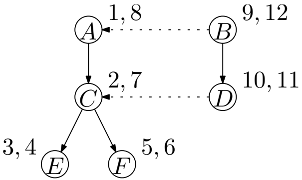

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Topologial ordering:

	B	D	A	C	F	E
#	1	2	3	4	5	6

Correctness: Let (v, u) be any edge in G.

To show: in topological sorting:
number of v < number of u

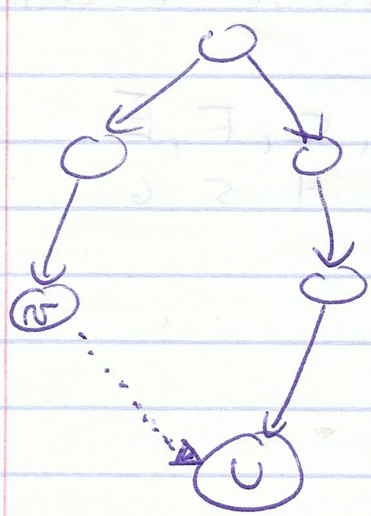
$$\text{post}(v) > \text{post}(u)$$

By contradiction. Assume $\text{post}(v) < \text{post}(u)$

Therefore, (v, u) is not a tree edge and not a forward edge (see page 73).

Since G is acyclic, (v, u) is not a back edge.

Hence, (v, u) is a cross edge.



But since $\text{post}(v) < \text{post}(u)$, (v, u) is not a cross edge, which is a contradiction.

$\square$

Shortest Paths

Input :

- A directed graph $G = (V, E)$, where each edge $(u, v) \in E$ has a weight $wt(u, v) > 0$.
- A vertex $s \in V$ (called the *source*).

Output :

- For each vertex $v \in V$,

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General Approach

For each vertex $v \in V$, maintain variable

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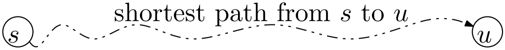
Loop: Pick a vertex u for which $d(u) = \delta(s, u)$. For each edge u, v,

$$d(v) = \min\{d(v), d(u) + wt(u, v)\}$$



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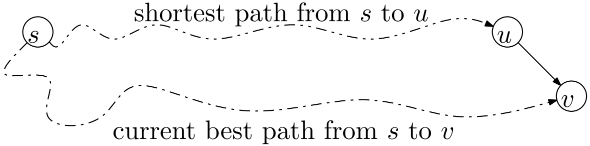

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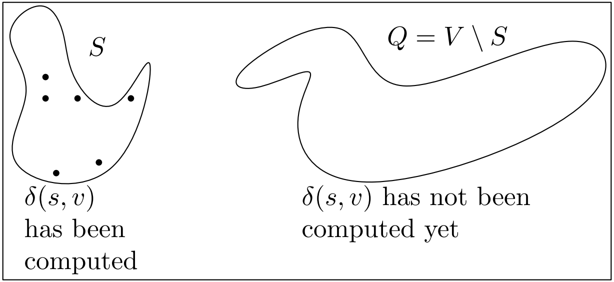
Maintain $S \subset V$ such that for all $v \in S$:

$$d(v) = \delta(s, v), \qquad \text{i.e., we know } \delta(s, v)$$



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Then for each edge (u, v),

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Algorithm *Dijkstra*(G, s)

for each vertex $v \in V$ **do**
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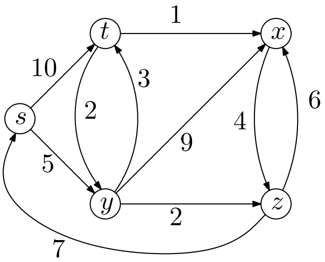
1
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3
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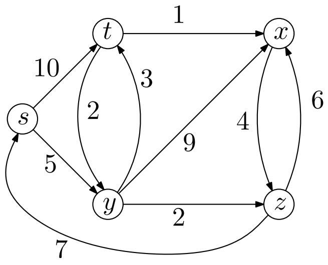
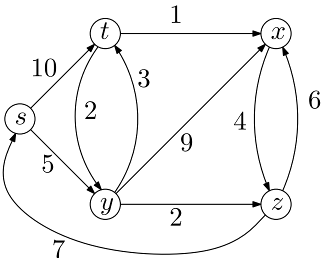
7



Q	s	t	x	y	z
d	0	∞	∞	∞	∞



Q	s	t	x	y	z
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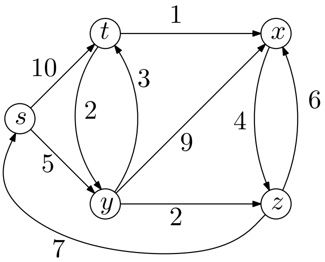
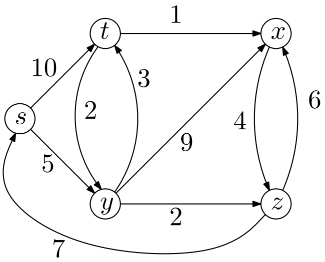
- $u = s$
- $\delta(s, s) = d(s) = 0$
- delete s from Q
- update $d(t)$ and $d(y)$



Q	t	x	y	z
d	10	∞	5	∞



Q	t	x	y	z
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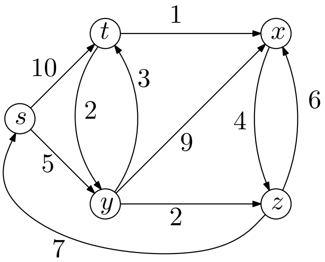
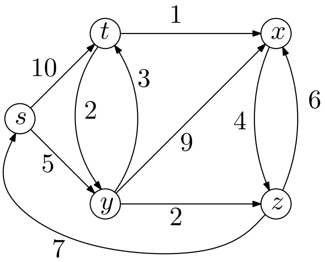
- $u = y$
- $\delta(s, y) = d(y) = 5$
- delete y from Q
- update $d(t)$, $d(x)$ and $d(z)$



Q	t	x	z
d	8	14	7



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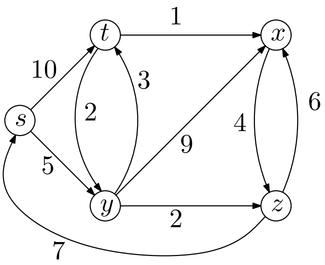
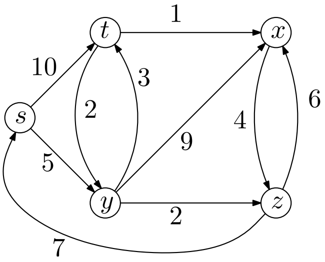
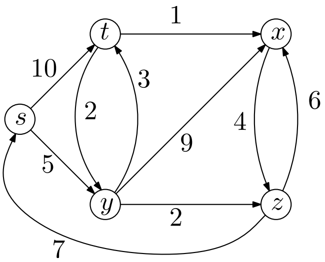


Q	t	x
d	8	13



Q	t	x
d	8	13

- $u = t$
- $\delta(s, t) = d(t) = 8$
- delete t from Q
- update $d(x)$ and $d(y)$



Q	x
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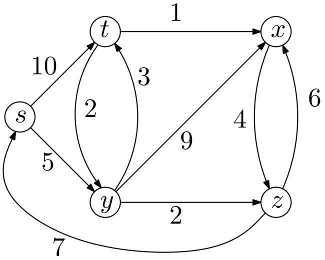

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- $u = x$
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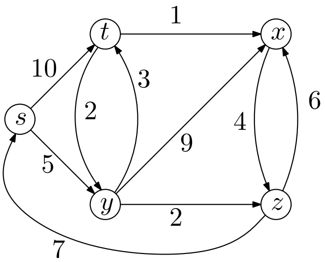
- delete x from Q
- update $d(z)$



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$Q = \{\,\}$: done!

What is the running time of Dijkstra?

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Total time for one iteration:

$$O(\log(n)) + O(outdegree(u) \cdot \log(n))$$

Total running time:

$$O(n) + O\left(\sum_{u \in V} (\log(n) + outdegree(u) \cdot \log(n))\right)$$

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\end{aligned}
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Note: Using a data structure called Fibonacci Heap to store Q, we can do $O(n\log(n) + m)$ time.