CSI - 3105 Design & Analysis of Algorithms
Course 11

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Minimum Spanning Tree

We are given a graph $G = (V, E)$ that is undirected and connected. Each edge $\{u, v\} \in E$ has a weight $wt(u, v)$.

We want to compute a subgraph G' of G such that

- The vertex set of G' is V,
- G' is connected,
- and $weight(G')$ is minimum, where

$$weight(G') = \text{sum of weights of edges in } G'.$$

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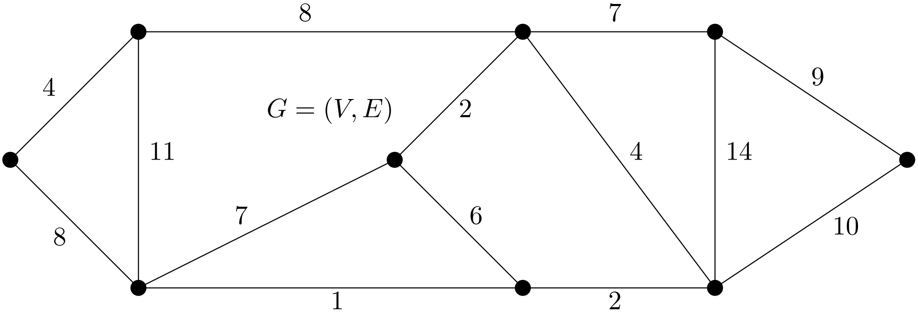
We can prove that G' must be a tree (connected and no cycles). Do you see why?

G' is called a *Minimum Spanning Tree of G* (MST of G).

Example:



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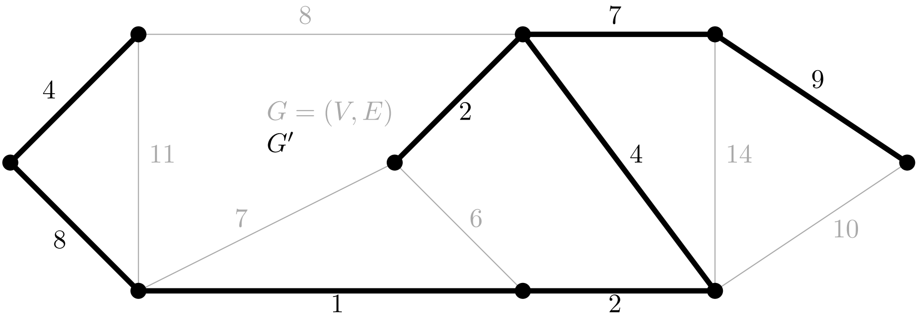

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- G' is connected,
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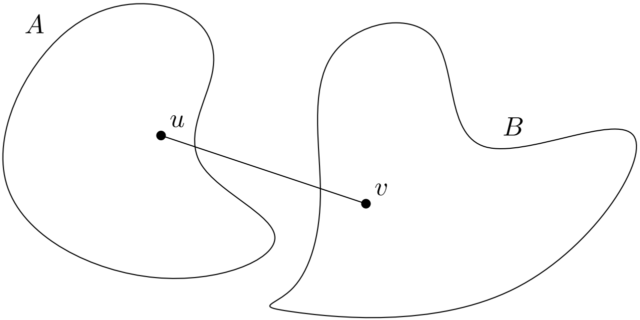
Fundamental Lemma

Lemma

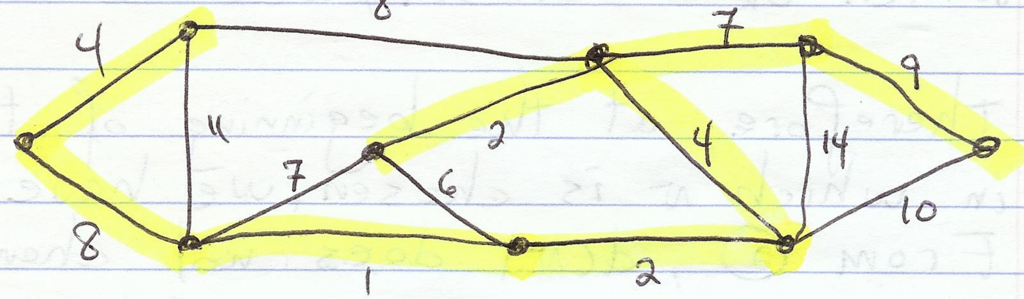
Let $G = (V, E)$ be an undirected and connected graph, where each edge $\{u, v\} \in E$ has a weight $wt(u, v)$.

Split V into A and B. Let $\{u, v\} \in E$ be a shortest edge connecting A and B. Then there is an MST of G that contains $\{u, v\}$.

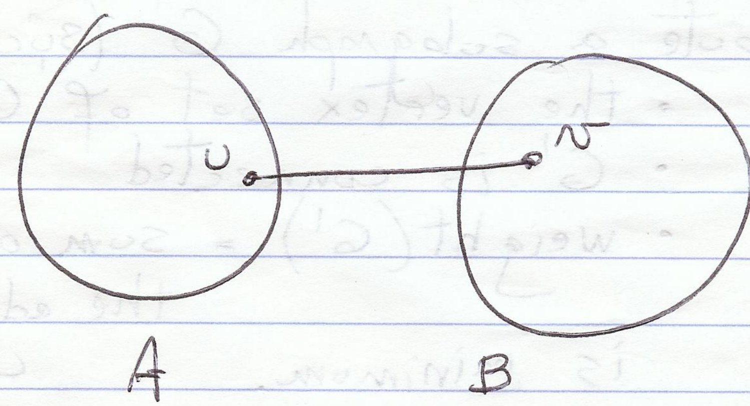


PROOF:

Example :

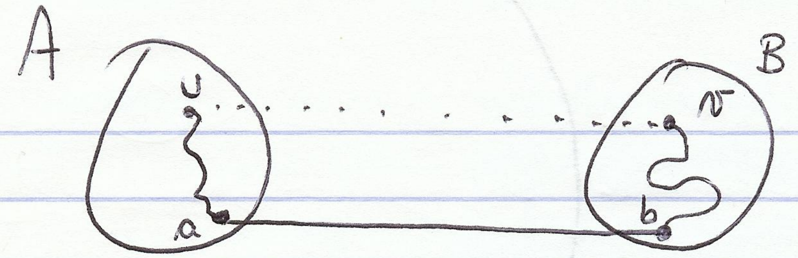


Lemma : Split V into A and B. Let $\{u, v\}$ be a shortest edge connecting A and B. Then there is an MST of G that contains the edge $\{u, v\}$.



proof : Let T be an MST of G. If $\{u, v\}$ is an edge of T, we are done! Assume $\{u, v\}$ is not an edge of T.

Since T is connected, there is a path in T between u and v.



This path contains an edge $\{a,b\}$ with $a \in A$ and $b \in B$. Define

$$T' = T \text{ minus } \{a,b\} \text{ plus } \{u,v\}$$

$$\text{weight}(T) \leq \text{weight}(T') \qquad \text{since } T \text{ is an MST}$$

$$= \text{weight}(T) - wt(a,b) + \underbrace{wt(u,v)}_{\leq\, wt(a,b) \text{ by the assumption}}$$

$$\leq \text{weight}(T)$$

Thus, $\text{weight}(T') = \text{weight}(T)$.

Therefore, T' is an MST that contains the edge $\{u,v\}$. □

From the previous lemma, any algorithm that follows this greedy scheme is guaranteed to work:

$X = \{\}$ (edges picked so far)
repeat until $|X| = |V| - 1$
- pick a set $S \subset V$ for which X has no edges between S and $V \setminus S$.
- let $e \in E$ be the minimum-weight edge betwen S and $V \setminus S$.
- $X = X \cup \{e\}$

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About the Union-Find Data Structure

$$\Big($$

Before presenting a first algorithm to compute an MST, we first open a parenthesis and study a data structure called *Union-find*.

Given n sets, each of size one,

$$A_1 = \{1\}, \quad A_2 = \{2\}, \qquad \cdots \qquad A_n = \{n\},$$

process a sequence of operations, where each operation is one of

Union(A, B, C):

Set $C = A \cup B$

$A = \{\ \}$

$B = \{\ \}$

Find(x):

Return the name of the set that contains x.

The sequence consists of

$n-1$ **Union** operations

m **Find** operations

which can be done in any arbitrary order.

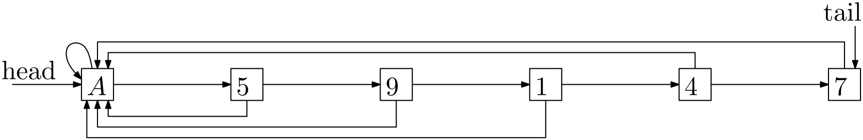
We are interested in the total time to process any such sequence.

Store each set in a list:

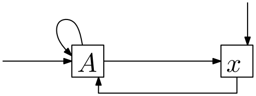
- the list has a pointer to the head and a pointer to the tail
- the first node stores the name of the set
- each other node stores one element of the set
- each node u stores two pointers:

  next(u) the next node in the list
  back(u) first node in the list

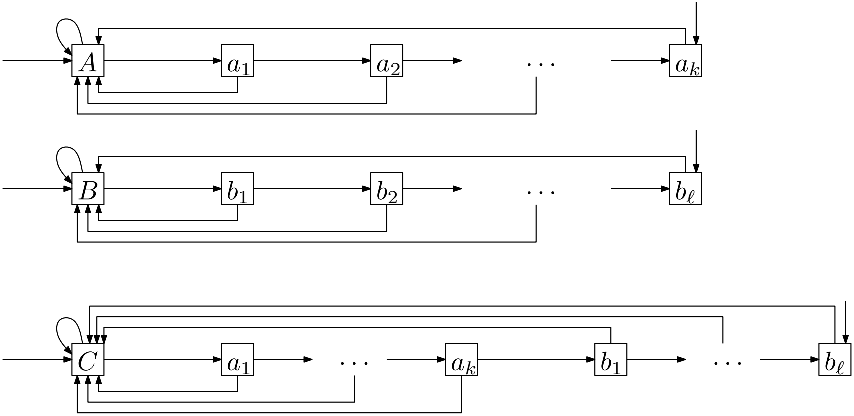
$A = \{1, 4, 5, 7, 9\}$



Start: for each set $A = \{x\}$:

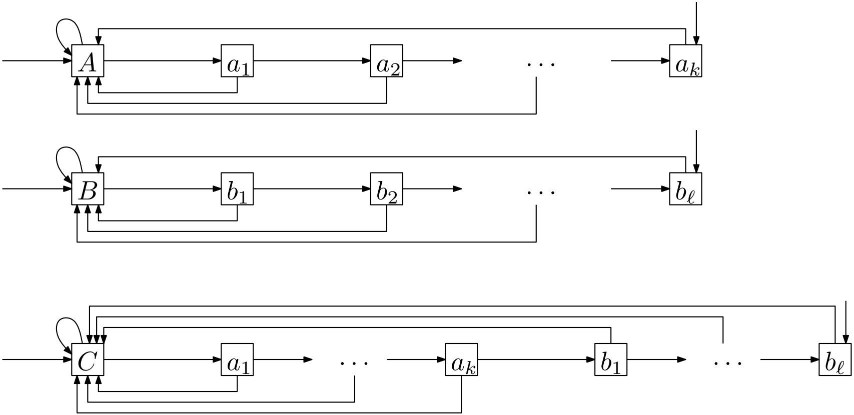


Union(A, B, C):



Append the list B at the end of the list A, do some pointer arithmetic, change the name in the head of the new list from A to C.

Union(A, B, C):



Time $= O(\ell) = O(\text{size of } B)$

Find(x): follow the back pointer from the node storing x to the head of the list and return the name stored at the head.

Time $= O(1)$

Example:

Union	Time
$\{2\}, \{1\}$	1
$\{3\}, \{2, 1\}$	2
$\{4\}, \{3, 2, 1\}$	3
$\vdots$	$\vdots$
$\{n\}, \{n-1, n-2, \ldots, 2, 1\}$	$n-1$

Example:

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$\{2\}, \{1\}$	1
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$\vdots$	$\vdots$
$\{n\}, \{n-1, n-2, ..., 2, 1\}$	$n-1$

Total time $= 1 + 2 + 3 + ... + n - 1 = O(n^2)$.

Better solution:

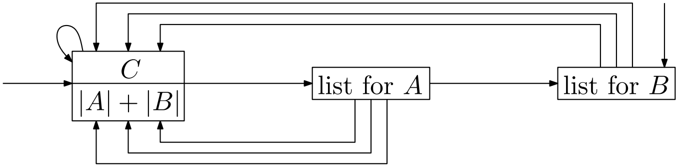
for each list, the head stores

- name of the set
- size of the set

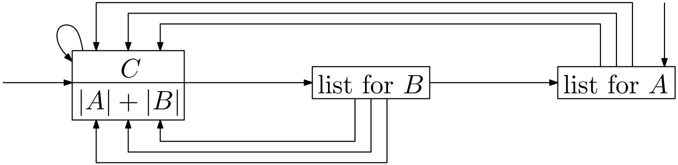
Find(x) takes $O(1)$ time, as before.

Union(A, B, C):

If $|A| \geq |B|$:



If $|A| < |B|$:



Time $= O(\min\{|A|, |B|\}) = O(\text{number of back-pointers that are changed})$

What is the total time for a sequence of $n-1$ **Union** operations:

$$\begin{aligned}\text{Total time} &= \text{total number of back-pointer changes} \\ &= \sum_{x=1}^{n} \text{total number of times that back}(x) \text{ is changed}\end{aligned}$$

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Consider an element x. How many times do we change back(x)?

Start: x is in a set of size 1.

First time that back(x) is changed:

the set containing x is merged with a set of size ≥ 1.

Hence, the new set containing x has size ≥ 2.

Second time that back(x) is changed:

the set containing x is merged with a set of size ≥ 2.

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Since there are n elements, back(x) is changed $\leq \log_2(n)$ times.

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Conclusion: Any sequence of $n-1$ **Union** and m **Find** operations takes $O(m + n \log(n))$ time.

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