CSI - 3105 Design & Analysis of Algorithms
Course 2

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Fall 2019

The Fibonacci Numbers

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$$F_1 = 1$$
$$F_n = F_{n-1} + F_{n-2} \qquad (n \geq 2)$$

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Algorithm *fib*(n)

Input: An integer $n \geq 0$.

Output: F_n.

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if n ≤ 1 then
    return n
else
    return fib(n − 1) + fib(n − 2)
end if
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The Running Time of *fib*(n)

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If $n = 0$ or $n = 1$, we do

$$\left.\begin{array}{l}\text{comparison “}n \leq 1\text{”} \\ \text{return value}\end{array}\right\} \quad \text{2 steps}$$

The Running Time of $fib(n)$

If $n \geq 2$, we do

comparison "$n \leq 1$"	:	1 step
compute $n - 1$	:	1 step
call $fib(n - 1)$	:	?
compute $n - 2$	:	1 step
call $fib(n - 2)$	:	?
compute sum of two results	:	1 step
return output	:	1 step

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call $fib(n-1)$	:	$T(n-1)$ steps
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$$T(n) = \begin{cases} 2 & \text{if } n = 0, \\ 2 & \text{if } n = 1, \\ T(n-1) + T(n-2) + 5 & \text{if } n \geq 2. \end{cases}$$

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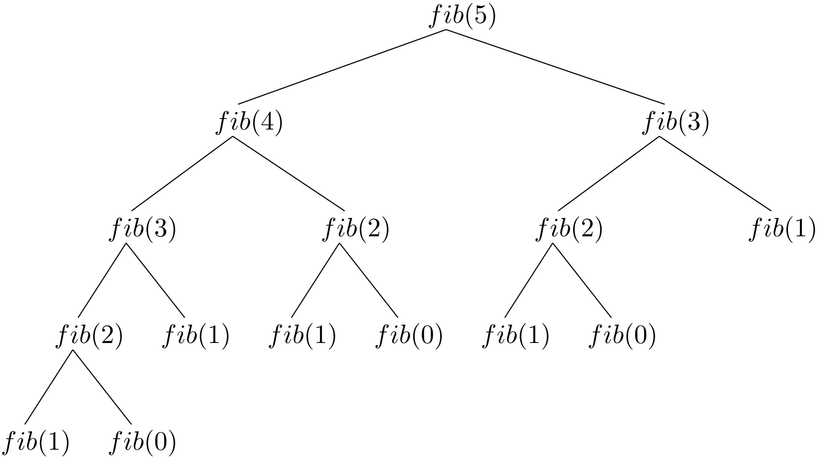
Why is *fib*(n) so slow?

The Running Time of *fib*(n)



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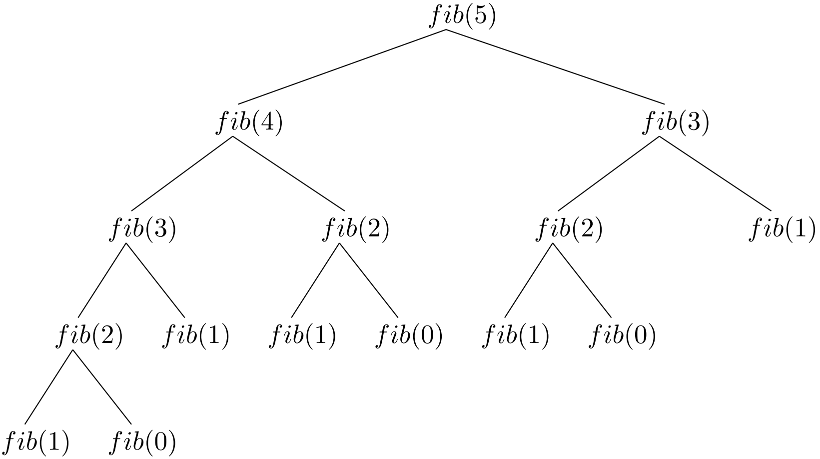

Too many things are called multiple times.

A Better Algorithm

Algorithm *fib2*(n)

Input: An integer $n \geq 0$.

Output: F_n.

```
if n ≤ 1 then
    return n
else
    initialize array f[0..n]
    f[0] = 0
    f[1] = 1
    for i = 2 to n do
        f[i] = f[i − 1] + f[i − 2]
    end for
    return f[n]
end if
```

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In our analysis, one step corresponds to

$$\left.\begin{array}{l}\text{comparison}\\ \text{addition}\\ \text{subtraction}\end{array}\right\}\quad \text{involving very large numbers}$$

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When running *fib*2(n), we do roughly n additions of numbers, each of these numbers is at most F_n, each of these numbers has roughly n bits.

Therefore, *fib*2(n) makes a *quadratic number* of bit-operations, i.e. $O(n^2)$ bit-operations.

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Therefore, the running time, in terms of bit-operations:

Running time of $fib(n)$:	exponential
Running time of $fib2(n)$:	quadratic

Chapter 2: Divide-and-Conquer Algorithms

To solve a problem of size n:

- **Divide** the problem into subproblems, each of size $< n$.
- **Conquer**: Solve each subproblem recursively (and independantly of the other subproblems).
- **Combine/Merge** the solutions to the subproblems into a solution to the original problem.

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For a given problem,

$\rightarrow$ How do we divide the problem, into how many subproblems?

$\rightarrow$ How to combine/merge?

Example: Merge Sort

To sort n numbers:

If $n \leq 1$: do nothing.

If $n \geq 2$: divide the n numbers arbitrarily into two sequences, both of size $n/2$, run Merge sort twice, once for each sequence.

Then merge the two sorted sequences into one sorted sequence.

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Hence, there is a constant $c > 0$ such that

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What do we do with this?

We solve by *unfolding*!

Solve this recurrence by <u>unfolding</u>:

Assume $n = 2^K$, $c = 1$

$$T(n) \le n + 2T(n/2)$$

$$\le n + 2\cdot\left(\frac{n}{2} + 2\cdot T\left(\frac{n}{4}\right)\right)$$

$$= 2n + 4T\left(\frac{n}{4}\right)$$

$$\le 2n + 4\left(\frac{n}{4} + 2\cdot T\left(\frac{n}{8}\right)\right)$$

$$= 3n + 8T\left(\frac{n}{8}\right)$$

$$\le 3n + 8\left(\frac{n}{8} + 2\cdot T\left(\frac{n}{16}\right)\right)$$

$$= 4n + 16T\left(\frac{n}{16}\right)$$

$\vdots$

$$\le Kn + 2^K T\left(\frac{n}{2^K}\right)$$

$$= Kn + n\cdot 1$$

$$= n\log_2(n) + n$$

since $\log_2(n) = \log_2(2^K) = K$

$$\le n\log_2(n) + n\cdot\log_2(n)$$

$$\le 2n\log_2(n)$$

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For a general n, we have

$$T(n) \leq \begin{cases} c & \text{if } n = 1, \\ T\left(\lfloor n/2 \rfloor\right) + T\left(\lceil n/2 \rceil\right) + c \cdot n & \text{if } n \geq 2, \end{cases}$$

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By induction, we can prove that $T(n) = O(n \log_2(n))$.