Chapter 4

——Material covered in class——

1. Show the trace of Making_Change(3.65).

2. Show the trace of the greedy algorithm which solves the fractional knapsack problem with the following inputs.

(a)

i	1	2	3	4	5	6
v_i	6	2	1	8	3	5
w_i	6	10	3	5	1	3

Take $W = 20$ as the maximum weight allowed.

(b)

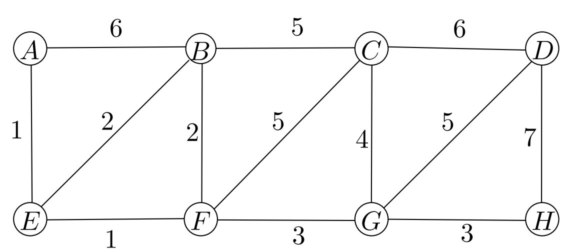
i	1	2	3	4	5	6
v_i	20	24	9	15	25	36
w_i	2	4	1	5	5	6

Take $W = 18$ as the maximum weight allowed.

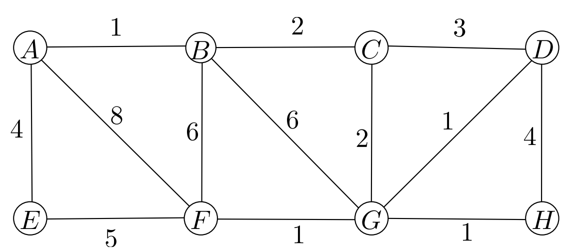
3. Find an input to the fractional knapsack problem for which there is more than one optimal solution.

4. Show the trace of Kruskal's algorithm on the following two inputs. Include the details of the Union-Find data structure in your trace.

(a)



(b)



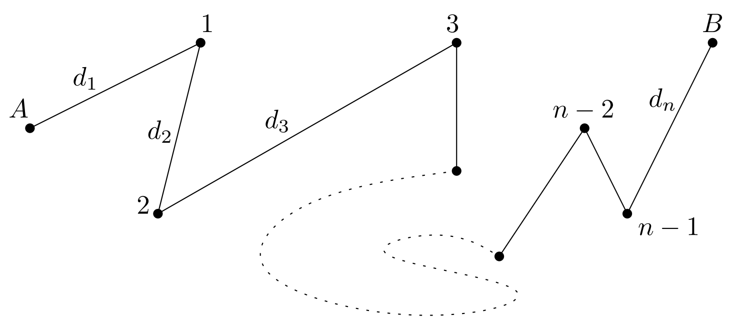
5. Show the trace of Prim's algorithm on the two inputs of Question 4, starting at vertex A.

6. If we *reverse* Kruskal algorithm, do we get a correct algorithm to compute MSTs? More precisely, what do you think of the following algorithm?

   - Start with graph $G = (V, E)$
   - Go through E in decreasing order of edge weights.
   - For each edge, check if deleting the edge will further disconnect the graph.
   - Perform any deletion that does not lead to additional disconnection.

   If you think that the algorithm is correct, prove it. If you think it is not, find a counter-example. https://en.wikipedia.org/wiki/Reverse-delete_algorithm

7. Donald Trump wants to drive his car from point A to point B. When the tank of the car is full, he can drive up to distance D (at the beginning of the trip, the tank is full). During his trip, he will cross n gaz stations where he can fill the tank of his car (the n-th gaz station being at B). Let d_i $(1 \leq i \leq n)$ be the distance between the $(i-1)$-th and the i-th gaz station. We denote by d_1 the distance between A and the first gaz station, and by d_n the distance between the $(n-1)$-th gaz station and the n-th gaz station at B.



   We suppose that $d_i \leq D$ for all $1 \leq i \leq n$. Donald wants to minimize the number of stops he needs to make. He knows the location of all gaz stations before leaving from A.

   (a) What greedy criterion should you use to solve this problem?

   (b) Prove that your greedy criterion does lead to an optimal solution.

   Hint: Start with an optimal solution S. Show that if S is not the greedy solution S^*, then you can convert S into S^* without increasing the number of stops, then conclude.

   (c) Explain why your greedy criterion leads to an algorithm that takes $O(n)$ time.

(d) Find an input to this problem for which there is an optimal solution that does not satisfy your greedy criterion. Does this contradict anything?

8. Can Prim's algorithm be used to find a minimum spanning tree in a graph with some negative weights? Justify your answer.

9. Can Kruskal's algorithm be used to find a minimum spanning tree in a graph with some negative weights? Justify your answer.

10. Consider the fractional knapsack problem. In class, we studied a version where a solution $X = (x_1, x_2, ..., x_n)$ must satisfy $0 \leq x_i \leq 1$ for all $1 \leq i \leq n$. In other words, for each $1 \leq i \leq n$ we can bring at most one object i in the knapsack.

    (a) What if a solution $X = (x_1, x_2, ..., x_n)$ must satisfy $0 \leq x_i \leq 2$ for all $1 \leq i \leq n$?

    (b) What if a solution $X = (x_1, x_2, ..., x_n)$ must satisfy $0 \leq x_i \leq \frac{3}{2}$ for all $1 \leq i \leq n$?

—–Proofs—–

11. Let $G = (V, E)$ be an undirected and connected graph. Suppose that each edge $\{u, v\} \in E$ has a weight $\mathrm{wt}(u, v) > 0$.

    In the section about minimum spanning trees, we are looking for a subgraph G' (subgraph of G) such that

    - the vertex set of G' is V,
    - G' is connected
    - and weight(G') is minimum.

    Prove that G' is a tree (i.e. G' is acyclic and connected).

12. Assume that the set of coins available in a given Country is $\{5, 10\}$ and prove that the algorithm Making_Change(x) leads to an optimal solution.

13. Consider an undirected graph $G = (V, E)$ with ***distinct*** non-negative edge weights. That is, for each $\{u, v\} \in E$, $\mathrm{wt}(u, v) \geq 0$. Suppose that you have computed a minimum spanning tree of G using Kruskal, and that you have also computed shortest paths to all nodes from a source node $s \in V$. Now suppose that each edge weight is increased by 1: for each $\{u, v\} \in E$, the new weight of $\{u, v\}$ is $\mathrm{wt}'(u, v) = \mathrm{wt}(u, v) + 1$.

    (a) Prove or disprove: the minimum spanning tree changes.

    (b) Prove or disprove: the shortest paths do not change.

14. Consider an undirected graph $G = (V, E)$ with non-negative edge weights. Prove or disprove: the shortest path between two nodes is necessarily part of some MST of G.

15. Consider an undirected graph $G = (V, E)$ with ***distinct*** non-negative edge weights. Assume that G has a cycle containing the heaviest edge e_M. Prove or disprove: e_M cannot be part of any MST of G.

16. Let $G = (V, E)$ be an undirected graph. Prove that if all its edge weights are distinct, then it has a unique minimum spanning tree.