CSI - 3105 Design & Analysis of Algorithms
Course 19

Jean-Lou De Carufel

Fall 2019

Theorem

The relation $\leq_P$ is transitive:

$$L \leq_P L' \quad \text{and} \quad L' \leq_P L'' \qquad \Longrightarrow \qquad L \leq_P L''$$

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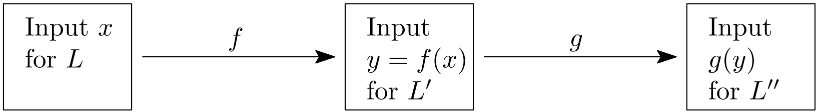

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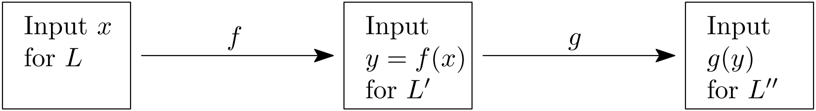

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Thus,

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The reduction from L to L'' is given by the function $g \circ f$. Given x, $(g \circ f)(x) = g(f(x))$ can be computed in time that is polynomial in the length of x (do you see why?)

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The language L is *NP-Complete* if

- $L \in NP$
- and for all $L' \in NP$, $L' \leq_P L$.

Intuitively, this means that L belongs to the most difficult problems in NP.

This is what we were looking for in §6.2.

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In order to apply this, we need *a first* NP-Complete problem.

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It is not even clear whether such a problem exists!!!

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We will show that CIRCUIT-SAT is NP-Complete.

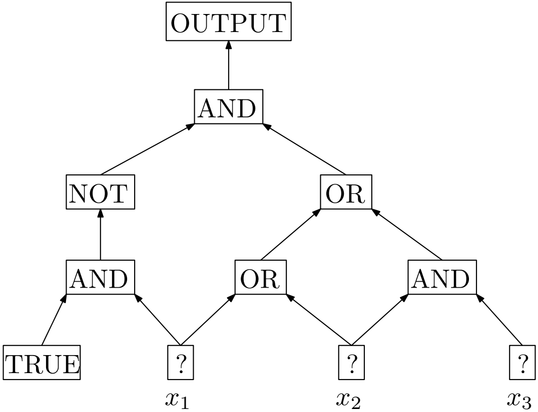
CIRCUIT-SAT

input: A Boolean circuit.

- Directed acyclic graph, where vertices are gates
- AND-gates and OR-gates have indegree 2
- Known input gates have indegree 0 and are labeled TRUE or FALSE.
- Unknown input gates have indegree 0 and are labeled "?".
- There is one output gate (whose outdegree is 0).

question: Is it possible to assign a truth-value to each unknown input gate, such that the output of the circuit is TRUE?

CIRCUIT-SAT



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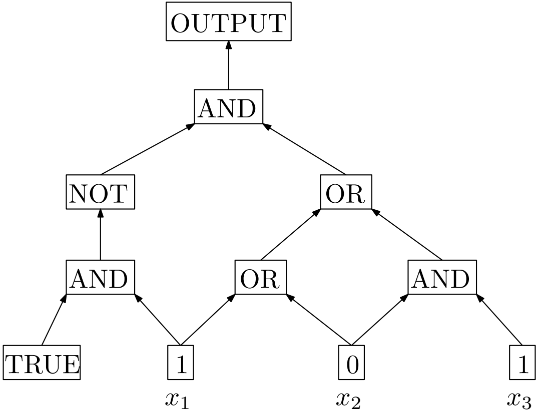

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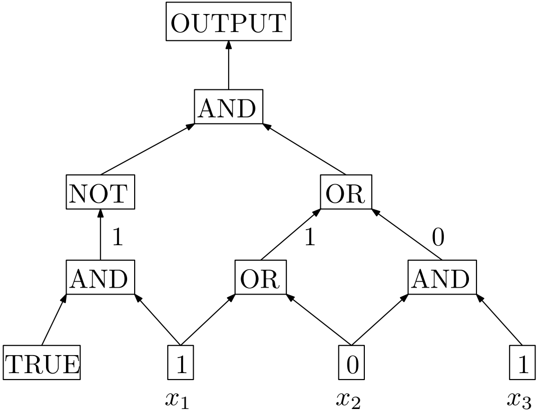

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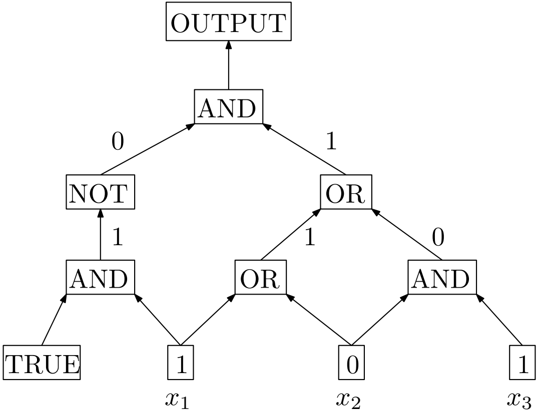

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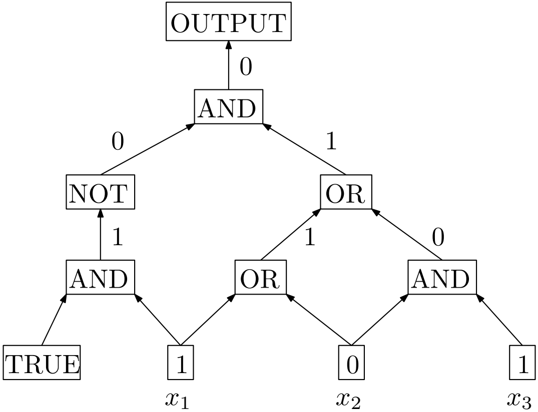

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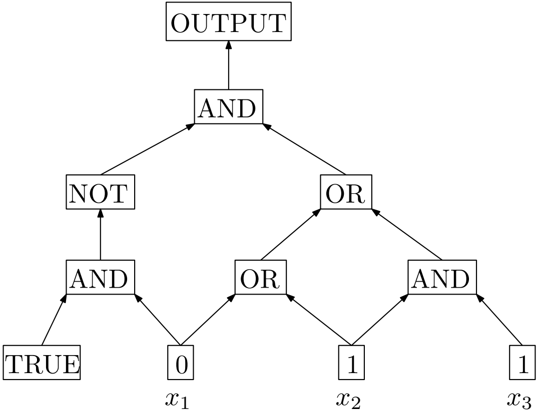

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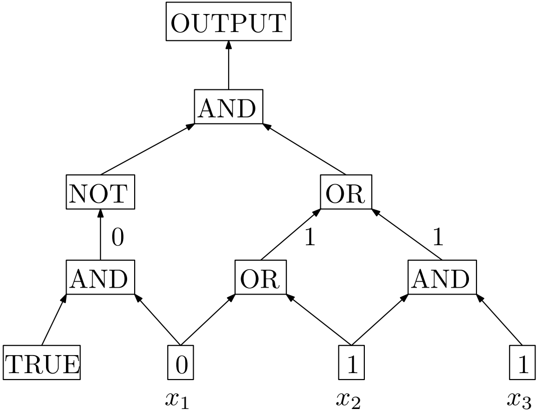

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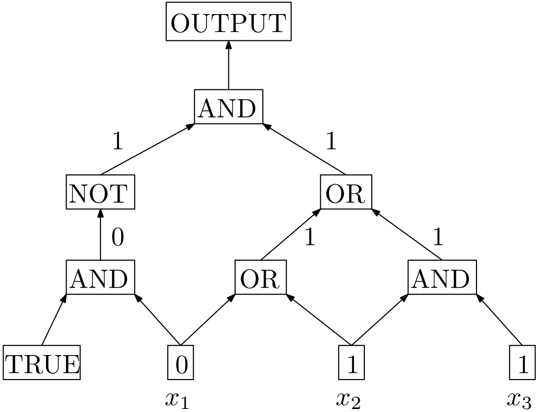

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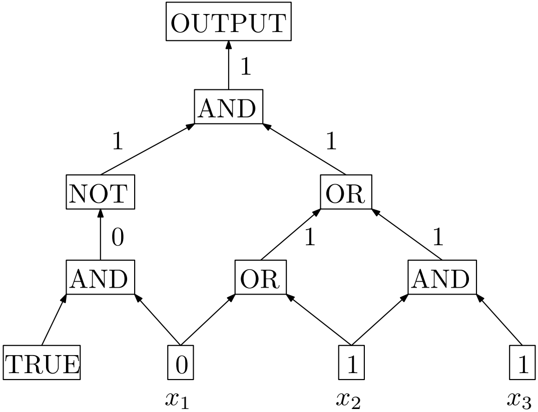

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Let $L \in NP$. We need a function f such that

1. f takes any input x for L and produces an input $B = f(x)$ for $CIRCUIT-SAT$.

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- The input to V is (x, y), where x is an input for L and y is a certificate.
- For every input x to L,

$$x \in L \iff \text{there exists a certificate } y \text{ such that}$$

$\cdot\ |y| \leq |x|^c$,

$\cdot\ V(x, y)$ returns YES

$\cdot$ and the running time of $V(x, y)$ is at most $|x|^{c'}$.

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- $V_x(y)$ runs $V(x, y)$
- If $V(x, y)$ terminates in at most $|x|^{c'}$ steps, then $V_x(y)$ terminates and returns the output of $V(x, y)$.
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Observe:

- Running time of Algorithm V_x is at most $|x|^{c'}$.
-

$$x \in L \iff \text{there exists an input } y \text{ for Algorithm } V_x \text{ such that } V_x(y) \text{ returns YES}$$

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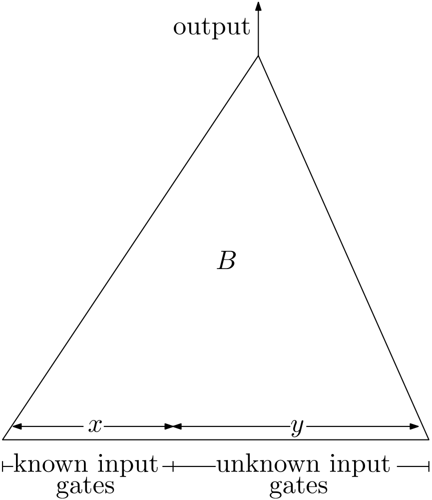

size of B:

$$O\left(|x| + |y| + |x|^{c'}\right)$$

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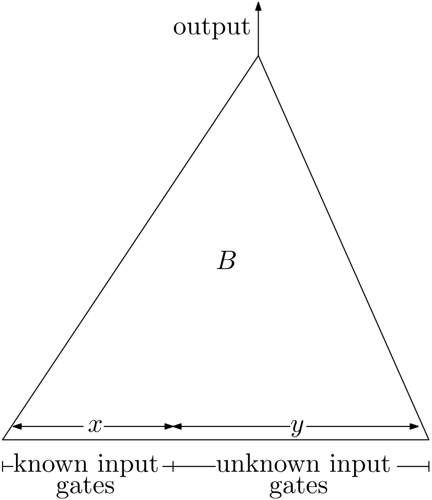

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The functions f maps x to B!

We have

$$x \in L \iff \text{There exists } y \text{ such that } V_x(y) \text{ returns TRUE}$$

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\begin{aligned}
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&\hspace{8cm} (\text{definition of } V_x) \\
&\iff \quad \text{There exists } y \text{ such that the output of } B \text{ is TRUE} \\
&\hspace{9cm} (\text{definition of } B) \\
&\iff \quad B \in CIRCUIT - SAT
\end{aligned}
$$

Conclusion: CIRCUIT-SAT is NP-COMPLETE!

Remember the following theorem (refer to Course 20).

Theorem

$$\left.\begin{array}{r} L \text{ is } NP\text{-Complete} \\ L \leq_P L' \\ L' \in NP \end{array}\right\} \implies L' \text{ is } NP\text{-Complete}$$

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The first item is easy: for a given truth-assignment of the variables, we can verify in polynomial time if the Boolean formula is true.

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1. f transforms any input (Boolean circuit) B for $CIRCUIT - SAT$ and produces an input $\phi = f(B)$ (Boolean formula) for 3SAT.
2. There exist truth-values for the unknown input gates such that B's output is true

   $$\iff$$

   There exist truth-values for the variables such that ϕ is true
3. $\phi = f(B)$ can be computed in time that is polynomial in the size of B.