CSI - 3105 Design & Analysis of Algorithms
Course 19

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Fall 2019

Example of a Reduction

$VERTEX - COVER = \{(G, K) \mid$ graph $G = (V, E)$ contains a vertex cover with K vertices.$\}$

Vertex cover: subset V' of V such that for each edge $\{u, v\}$ of G, at least one of u and v is in V'.

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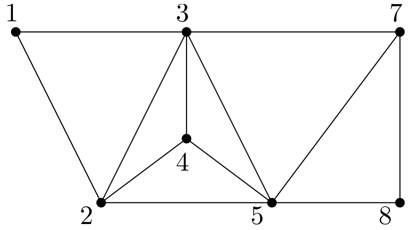

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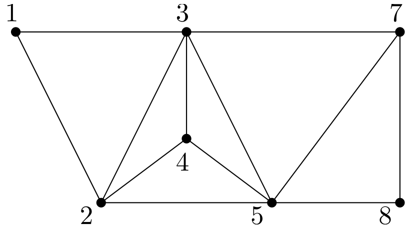

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$$CLIQUE \leq_P VERTEX - COVER.$$

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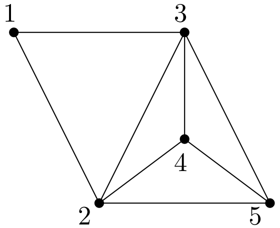

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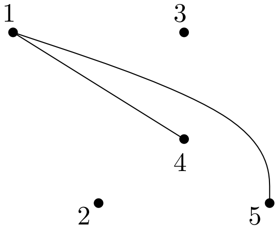

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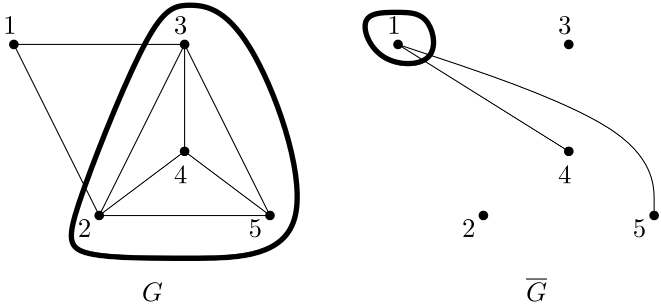

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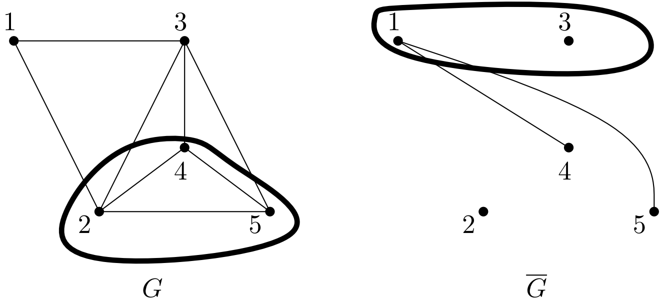

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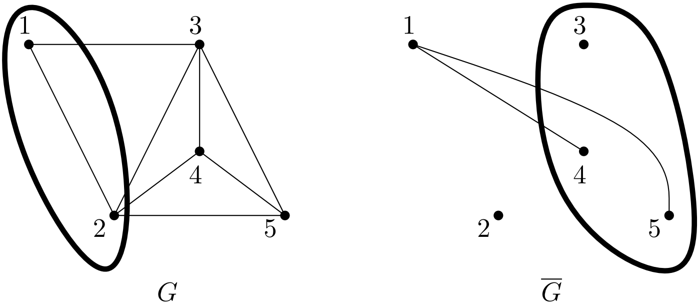

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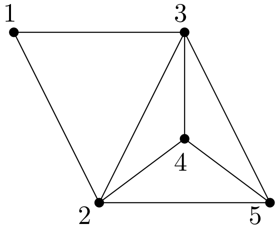

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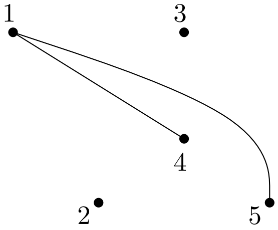

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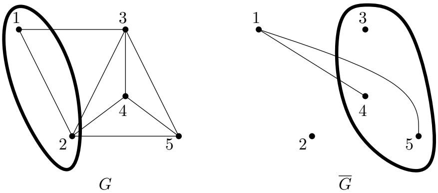
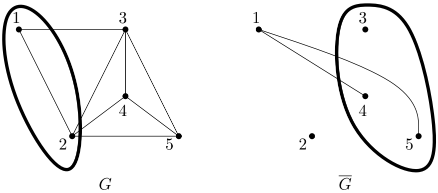
Is this a coincidence?



To prove the polynomial-time reduction $CLIQUE \leq_P VERTEX - COVER$, we need a function f which satisfies the *famous* 3 *properties*.



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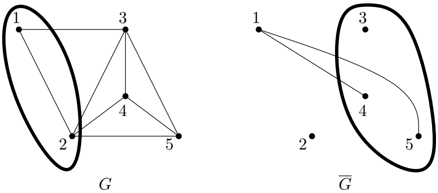
$$f(G, K) = \left(\overline{G}, n - K\right)$$



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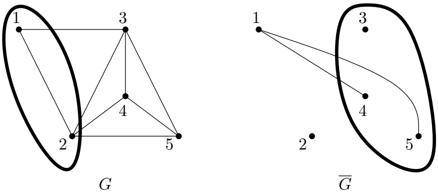

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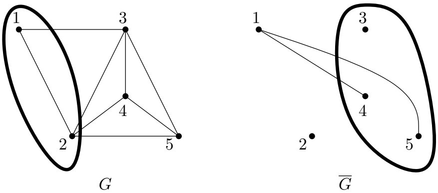

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1. f maps inputs for CLIQUE to inputs for VERTEX-COVER.
2. Is it true that G has a clique of size K if and only if $\overline{G}$ has a vertex cover of size $n - K$?
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And we have

$$|V \setminus V'| = |V| - |V'| = n - K.$$

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And we have

$$|V \setminus V^*| = |V| - |V^*| = n - (n - K) = K.$$

$\square$

About 3SAT

$$\phi = (x_1 \vee \neg x_1 \vee \neg x_2) \wedge (x_2 \vee x_3 \vee x_4) \wedge (x_1 \vee x_3 \vee \neg x_4) \wedge (\neg x_1 \vee x_3 \vee \neg x_4)$$

About 3SAT

Consider a Boolean formula ϕ with variables $x_1, x_2, \dots, x_n$ of the form

$$\phi = C_1 \wedge C_2 \wedge \dots \wedge C_m,$$

where each C_i is of the form

$$C_i = \ell_1^i \vee \ell_2^i \vee \ell_3^i.$$

Each ℓ_j^i is a variable or the negation of a variable. C_i is called a *clause* and ℓ_j^i is called a *literal*.

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We say that ϕ is *satisfiable* is there exists a truth value for each of $x_1, x_2, \ldots, x_n$ such that ϕ is true.

For the example, if $x_1 = 0$, $x_2 = 1$, $x_3 = 0$ and $x_4 = 0$, then $\phi = 1$, hence ϕ is satisfiable.

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Somehow, we have to "encode" a satisfiable formula ϕ as an independent set of size K in a graph.

Let ϕ be an input for 3SAT

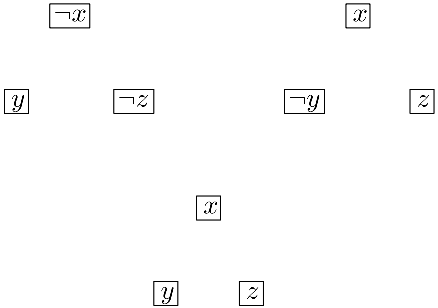
$$\phi = C_1 \wedge C_2 \wedge \ldots \wedge C_m,$$

each clause C_i is the disjunction ($\vee$) of 3 literals.

(G, K) is obtained as follows:

- $K = m$ (number of clauses)
- G has $3m$ vertices, one vertex for each literal.
    - For each clause, the literals in this clause form a triangle in G.
    - Additionally, there is an edge between any pair of opposite literals.

$$\phi = (\neg x \vee y \vee \neg z) \wedge (x \vee \neg y \vee z) \wedge (x \vee y \vee z)$$



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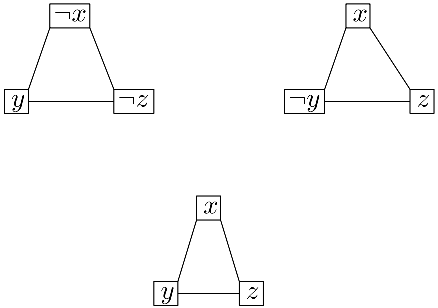

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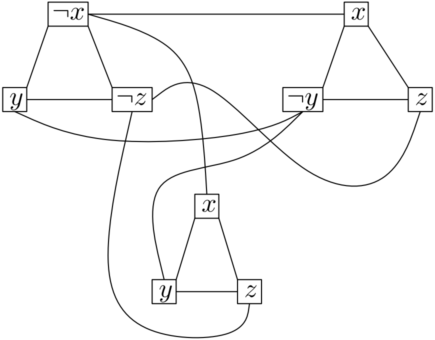

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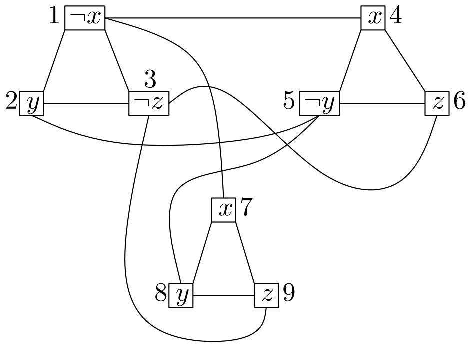

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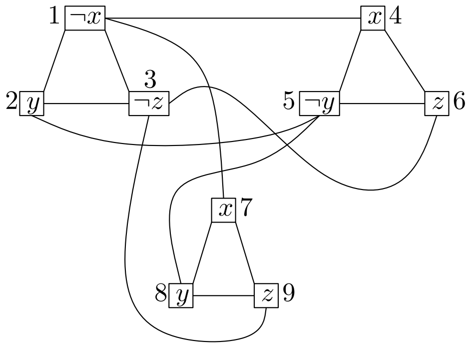

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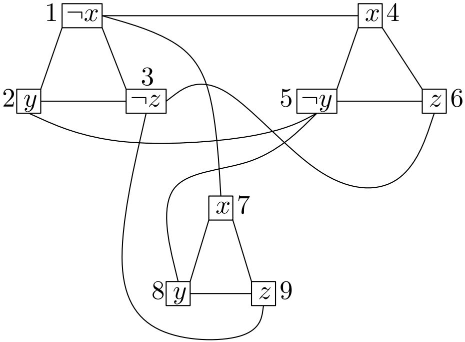

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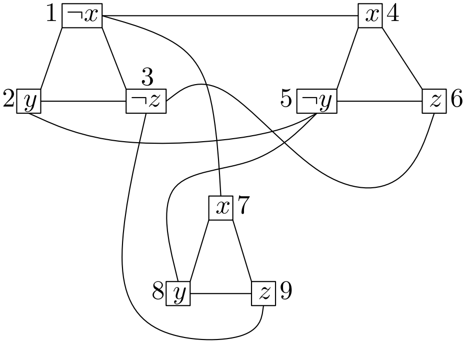

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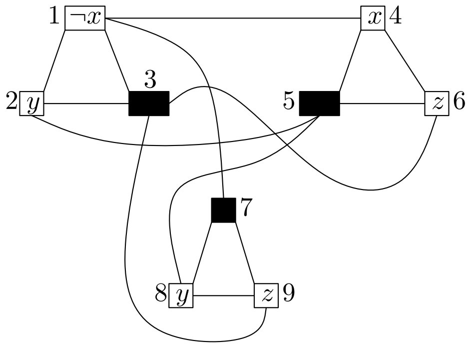

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- second clause: $\neg y = \text{TRUE}$, choose vertex 5
- third: $x = \text{TRUE}$, choose vertex 7

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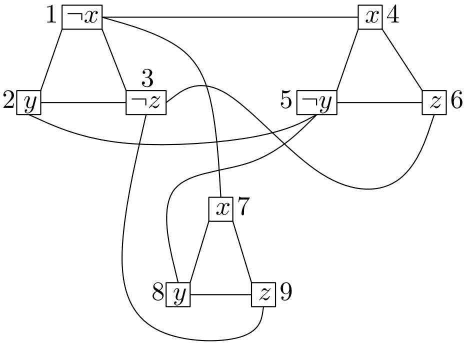
Thus,

$$\phi \in 3SAT \Longrightarrow (G, K) \in INDEP - SET.$$



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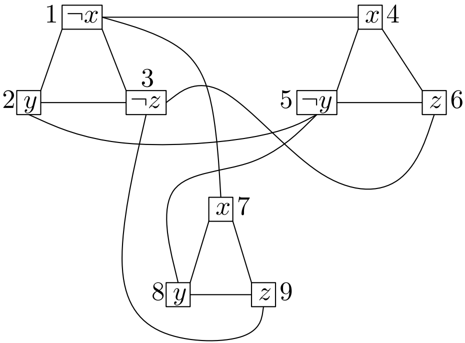

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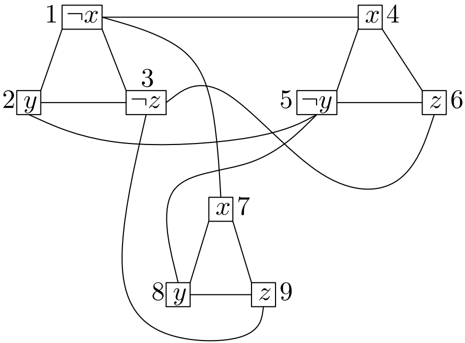

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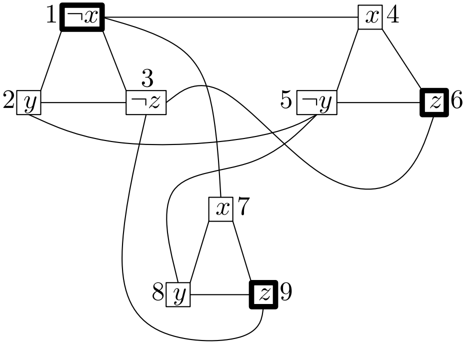

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For each triangle, let v be the vertex that is in the independent set.

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Thus,

$$(G, K) \in INDEP - SET \Longrightarrow \phi \in 3SAT.$$

What is the time to compute $(G, K) = f(\phi)$? Using brute-force to compute the edges of G, we can do it in

$$O\left((3m)^2\right) = O\left(m^2\right) = O\left((\#\text{of clauses in } \phi)^2\right) = \text{ polynomial time.}$$

$\square$