CSI - 3105 Design & Analysis of Algorithms
Course 3

Jean-Lou De Carufel
http://cglab.ca/~jdecaruf/CSI3105.html

Fall 2019

Matrix Multiplication

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \qquad B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n,1} & b_{n2} & \cdots & b_{nn} \end{pmatrix}$$

$$\begin{aligned} C &= AB \\ c_{ij} &= \sum_{k=1}^{n} a_{ik} b_{kj} \end{aligned}$$

Matrix Multiplication

Example

$$\begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{-1} \\ \mathbf{0} & \mathbf{2} & \mathbf{3} \\ 1 & 5 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{4} & 0 & \mathbf{7} \\ \mathbf{-2} & 1 & \mathbf{0} \\ \mathbf{6} & 1 & \mathbf{-1} \end{pmatrix} = \begin{pmatrix} -2 & -1 & \mathbf{8} \\ \mathbf{14} & 5 & -3 \\ -6 & 5 & 7 \end{pmatrix}$$

Matrix Multiplication

Input: Two square matrices $A_{n \times n}$ and $B_{n \times n}$.
Output: AB.

Initialize matrix $C_{n \times n}$
for $i = 1$ to n **do**
    for $j = 1$ to n **do**
        $c_{ij} = 0$
        for $k = 1$ to n **do**
            $c_{ij} = c_{ij} + a_{ik} b_{kj}$
        end for
    end for
end for
return C

Matrix Multiplication

Input: Two square matrices $A_{n\times n}$ and $B_{n\times n}$.
Output: AB.

```
Initialize matrix C_{n×n}
for i = 1 to n do
    for j = 1 to n do
        c_{ij} = 0
        for k = 1 to n do
            c_{ij} = c_{ij} + a_{ik} b_{kj}
        end for
    end for
end for
return C
```

$$T(n) = \Theta\left(n^3\right)$$

How can we Divide and Conquer?

Assume $n = 2^k$.

$$\begin{pmatrix} 1 & 0 & -1 & 2 \\ 3 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 2 & 4 & 7 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 & 3 \\ -2 & 1 & -1 & 1 \\ 1 & 0 & 1 & 0 \\ 5 & 1 & -2 & -1 \end{pmatrix} = \begin{pmatrix} 9 & 3 & -3 & 1 \\ 4 & 5 & 4 & 9 \\ 5 & 1 & -2 & -1 \\ 4 & 7 & 5 & 9 \end{pmatrix}$$

How can we Divide and Conquer?

Assume $n = 2^k$.

$$\begin{pmatrix} 1 & 0 & -1 & 2 \\ 3 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 2 & 4 & 7 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 & 3 \\ -2 & 1 & -1 & 1 \\ 1 & 0 & 1 & 0 \\ 5 & 1 & -2 & -1 \end{pmatrix} = \begin{pmatrix} 9 & 3 & -3 & 1 \\ 4 & 5 & 4 & 9 \\ 5 & 1 & -2 & -1 \\ 4 & 7 & 5 & 9 \end{pmatrix}$$

$$\begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} & \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 2 & 4 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 7 & 1 \end{pmatrix} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ -2 & 1 \end{pmatrix} & \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ -2 & -1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 9 & 3 \\ 4 & 5 \end{pmatrix} & \begin{pmatrix} -3 & 1 \\ 4 & 9 \end{pmatrix} \\ \begin{pmatrix} 5 & 1 \\ 4 & 7 \end{pmatrix} & \begin{pmatrix} -2 & -1 \\ 5 & 9 \end{pmatrix} \end{pmatrix}$$

How can we Divide and Conquer?

Assume $n = 2^k$.

$$\begin{pmatrix} 1 & 0 & -1 & 2 \\ 3 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 2 & 4 & 7 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 & 3 \\ -2 & 1 & -1 & 1 \\ 1 & 0 & 1 & 0 \\ 5 & 1 & -2 & -1 \end{pmatrix} = \begin{pmatrix} 9 & 3 & -3 & 1 \\ 4 & 5 & 4 & 9 \\ 5 & 1 & -2 & -1 \\ 4 & 7 & 5 & 9 \end{pmatrix}$$

$$\begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} & \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 2 & 4 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 7 & 1 \end{pmatrix} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ -2 & 1 \end{pmatrix} & \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ -2 & -1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 9 & 3 \\ 4 & 5 \end{pmatrix} & \begin{pmatrix} -3 & 1 \\ 4 & 9 \end{pmatrix} \\ \begin{pmatrix} 5 & 1 \\ 4 & 7 \end{pmatrix} & \begin{pmatrix} -2 & -1 \\ 5 & 9 \end{pmatrix} \end{pmatrix}$$

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \qquad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

How can we Divide and Conquer?

Assume $n = 2^k$.

$$\begin{pmatrix} 1 & 0 & -1 & 2 \\ 3 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 2 & 4 & 7 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 & 3 \\ -2 & 1 & -1 & 1 \\ 1 & 0 & 1 & 0 \\ 5 & 1 & -2 & -1 \end{pmatrix} = \begin{pmatrix} 9 & 3 & -3 & 1 \\ 4 & 5 & 4 & 9 \\ 5 & 1 & -2 & -1 \\ 4 & 7 & 5 & 9 \end{pmatrix}$$

$$\begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} & \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 2 & 4 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 7 & 1 \end{pmatrix} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ -2 & 1 \end{pmatrix} & \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ -2 & -1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 9 & 3 \\ 4 & 5 \end{pmatrix} & \begin{pmatrix} -3 & 1 \\ 4 & 9 \end{pmatrix} \\ \begin{pmatrix} 5 & 1 \\ 4 & 7 \end{pmatrix} & \begin{pmatrix} -2 & -1 \\ 5 & 9 \end{pmatrix} \end{pmatrix}$$

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \qquad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

$$C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

$$\begin{aligned} C_{11} &= A_{11}B_{11} + A_{12}B_{21} \\ C_{12} &= A_{11}B_{12} + A_{12}B_{22} \\ C_{21} &= A_{21}B_{11} + A_{22}B_{21} \\ C_{22} &= A_{21}B_{12} + A_{22}B_{22} \end{aligned}$$

How can we Divide and Conquer?

Assume $n = 2^k$.

$$\begin{pmatrix} 1 & 0 & -1 & 2 \\ 3 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 2 & 4 & 7 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 & 3 \\ -2 & 1 & -1 & 1 \\ 1 & 0 & 1 & 0 \\ 5 & 1 & -2 & -1 \end{pmatrix} = \begin{pmatrix} 9 & 3 & -3 & 1 \\ 4 & 5 & 4 & 9 \\ 5 & 1 & -2 & -1 \\ 4 & 7 & 5 & 9 \end{pmatrix}$$

$$\begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} & \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 2 & 4 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 7 & 1 \end{pmatrix} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ -2 & 1 \end{pmatrix} & \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ -2 & -1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 9 & 3 \\ 4 & 5 \end{pmatrix} & \begin{pmatrix} -3 & 1 \\ 4 & 9 \end{pmatrix} \\ \begin{pmatrix} 5 & 1 \\ 4 & 7 \end{pmatrix} & \begin{pmatrix} -2 & -1 \\ 5 & 9 \end{pmatrix} \end{pmatrix}$$

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \qquad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

$$C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

$$\begin{aligned} C_{11} &= A_{11}B_{11} + A_{12}B_{21} \\ C_{12} &= A_{11}B_{12} + A_{12}B_{22} \\ C_{21} &= A_{21}B_{11} + A_{22}B_{21} \\ C_{22} &= A_{21}B_{12} + A_{22}B_{22} \\ T(n) = 8\,T(n/2) + 4\,(n/2)^2 &= 8\,T(n/2) + n^2 \end{aligned}$$

Strassen Algorithm (1969)

$$
\begin{array}{lllll}
S_1 = B_{12} - B_{22} & S_2 = A_{11} + A_{12} & S_3 = A_{21} + A_{22} & S_4 = B_{21} - B_{11} & S_5 = A_{11} + A_{22} \\
S_6 = B_{11} + B_{22} & S_7 = A_{12} - A_{22} & S_8 = B_{21} + B_{22} & S_9 = A_{11} - A_{21} & S_{10} = B_{11} + B_{12}
\end{array}
$$

Strassen Algorithm (1969)

$$\begin{array}{lllll} S_1 = B_{12} - B_{22} & S_2 = A_{11} + A_{12} & S_3 = A_{21} + A_{22} & S_4 = B_{21} - B_{11} & S_5 = A_{11} + A_{22} \\ S_6 = B_{11} + B_{22} & S_7 = A_{12} - A_{22} & S_8 = B_{21} + B_{22} & S_9 = A_{11} - A_{21} & S_{10} = B_{11} + B_{12} \end{array}$$

$$\begin{aligned} P_1 &= A_{11} S_1 = A_{11} B_{12} - A_{11} B_{22} \\ P_2 &= S_2 B_{22} = A_{11} B_{22} + A_{12} B_{22} \\ P_3 &= S_3 B_{11} = A_{21} B_{11} + A_{22} B_{11} \\ P_4 &= A_{22} S_4 = A_{22} B_{21} - A_{22} B_{11} \\ P_5 &= S_5 S_6 = A_{11} B_{11} + A_{11} B_{22} + A_{22} B_{11} + A_{22} B_{22} \\ P_6 &= S_7 S_8 = A_{12} B_{21} + A_{12} B_{22} - A_{22} B_{21} - A_{22} B_{22} \\ P_7 &= S_9 S_{10} = A_{11} B_{11} + A_{11} B_{12} - A_{21} B_{11} - A_{21} B_{12} \end{aligned}$$

Strassen Algorithm (1969)

$$\begin{array}{lllll}
S_1 = B_{12} - B_{22} & S_2 = A_{11} + A_{12} & S_3 = A_{21} + A_{22} & S_4 = B_{21} - B_{11} & S_5 = A_{11} + A_{22} \\
S_6 = B_{11} + B_{22} & S_7 = A_{12} - A_{22} & S_8 = B_{21} + B_{22} & S_9 = A_{11} - A_{21} & S_{10} = B_{11} + B_{12}
\end{array}$$

$$\begin{aligned}
P_1 &= A_{11} S_1 = A_{11} B_{12} - A_{11} B_{22} \\
P_2 &= S_2 B_{22} = A_{11} B_{22} + A_{12} B_{22} \\
P_3 &= S_3 B_{11} = A_{21} B_{11} + A_{22} B_{11} \\
P_4 &= A_{22} S_4 = A_{22} B_{21} - A_{22} B_{11} \\
P_5 &= S_5 S_6 = A_{11} B_{11} + A_{11} B_{22} + A_{22} B_{11} + A_{22} B_{22} \\
P_6 &= S_7 S_8 = A_{12} B_{21} + A_{12} B_{22} - A_{22} B_{21} - A_{22} B_{22} \\
P_7 &= S_9 S_{10} = A_{11} B_{11} + A_{11} B_{12} - A_{21} B_{11} - A_{21} B_{12}
\end{aligned}$$

$$P_5 + P_4 - P_2 + P_6$$

Strassen Algorithm (1969)

$$
\begin{array}{lllll}
S_1 = B_{12} - B_{22} & S_2 = A_{11} + A_{12} & S_3 = A_{21} + A_{22} & S_4 = B_{21} - B_{11} & S_5 = A_{11} + A_{22} \\
S_6 = B_{11} + B_{22} & S_7 = A_{12} - A_{22} & S_8 = B_{21} + B_{22} & S_9 = A_{11} - A_{21} & S_{10} = B_{11} + B_{12}
\end{array}
$$

$$
\begin{aligned}
P_1 &= A_{11} S_1 = A_{11} B_{12} - A_{11} B_{22} \\
P_2 &= S_2 B_{22} = A_{11} B_{22} + A_{12} B_{22} \\
P_3 &= S_3 B_{11} = A_{21} B_{11} + A_{22} B_{11} \\
P_4 &= A_{22} S_4 = A_{22} B_{21} - A_{22} B_{11} \\
P_5 &= S_5 S_6 = A_{11} B_{11} + A_{11} B_{22} + A_{22} B_{11} + A_{22} B_{22} \\
P_6 &= S_7 S_8 = A_{12} B_{21} + A_{12} B_{22} - A_{22} B_{21} - A_{22} B_{22} \\
P_7 &= S_9 S_{10} = A_{11} B_{11} + A_{11} B_{12} - A_{21} B_{11} - A_{21} B_{12}
\end{aligned}
$$

$$
\begin{array}{cccccccccccccc}
 & \multicolumn{13}{l}{P_5 + P_4 - P_2 + P_6} \\
= & A_{11} B_{11} & + & A_{11} B_{22} & + & A_{22} B_{11} & + & A_{22} B_{22} & & & & & & \\
 & & & & - & A_{22} B_{11} & & & + & A_{22} B_{21} & & & & \\
 & & - & A_{11} B_{22} & & & & & & & - & A_{12} B_{22} & & \\
 & & & & & & - & A_{22} B_{22} & - & A_{22} B_{21} & + & A_{12} B_{22} & + & A_{12} B_{21} \\
\hline
\end{array}
$$

Strassen Algorithm (1969)

$$
\begin{array}{lllll}
S_1 = B_{12} - B_{22} & S_2 = A_{11} + A_{12} & S_3 = A_{21} + A_{22} & S_4 = B_{21} - B_{11} & S_5 = A_{11} + A_{22} \\
S_6 = B_{11} + B_{22} & S_7 = A_{12} - A_{22} & S_8 = B_{21} + B_{22} & S_9 = A_{11} - A_{21} & S_{10} = B_{11} + B_{12}
\end{array}
$$

$$
\begin{aligned}
P_1 &= A_{11}S_1 = A_{11}B_{12} - A_{11}B_{22} \\
P_2 &= S_2B_{22} = A_{11}B_{22} + A_{12}B_{22} \\
P_3 &= S_3B_{11} = A_{21}B_{11} + A_{22}B_{11} \\
P_4 &= A_{22}S_4 = A_{22}B_{21} - A_{22}B_{11} \\
P_5 &= S_5S_6 = A_{11}B_{11} + A_{11}B_{22} + A_{22}B_{11} + A_{22}B_{22} \\
P_6 &= S_7S_8 = A_{12}B_{21} + A_{12}B_{22} - A_{22}B_{21} - A_{22}B_{22} \\
P_7 &= S_9S_{10} = A_{11}B_{11} + A_{11}B_{12} - A_{21}B_{11} - A_{21}B_{12}
\end{aligned}
$$

$$
\begin{array}{rccccccccccccc}
 & P_5 + P_4 - P_2 + P_6 & & & & & & & & & & & & \\
= & A_{11}B_{11} & + & A_{11}B_{22} & + & A_{22}B_{11} & + & A_{22}B_{22} & & & & & & \\
 & & & & - & A_{22}B_{11} & & & + & A_{22}B_{21} & & & & \\
 & & - & A_{11}B_{22} & & & & & & & - & A_{12}B_{22} & & \\
 & & & & & & - & A_{22}B_{22} & - & A_{22}B_{21} & + & A_{12}B_{22} & + & A_{12}B_{21} \\
\hline
= & A_{11}B_{11} & & & & & & & & & & & + & A_{12}B_{21}
\end{array}
$$

Strassen Algorithm (1969)

$$
\begin{array}{lllll}
S_1 = B_{12} - B_{22} & S_2 = A_{11} + A_{12} & S_3 = A_{21} + A_{22} & S_4 = B_{21} - B_{11} & S_5 = A_{11} + A_{22} \\
S_6 = B_{11} + B_{22} & S_7 = A_{12} - A_{22} & S_8 = B_{21} + B_{22} & S_9 = A_{11} - A_{21} & S_{10} = B_{11} + B_{12}
\end{array}
$$

$$
\begin{aligned}
P_1 &= A_{11} S_1 &&= A_{11} B_{12} - A_{11} B_{22} \\
P_2 &= S_2 B_{22} &&= A_{11} B_{22} + A_{12} B_{22} \\
P_3 &= S_3 B_{11} &&= A_{21} B_{11} + A_{22} B_{11} \\
P_4 &= A_{22} S_4 &&= A_{22} B_{21} - A_{22} B_{11} \\
P_5 &= S_5 S_6 &&= A_{11} B_{11} + A_{11} B_{22} + A_{22} B_{11} + A_{22} B_{22} \\
P_6 &= S_7 S_8 &&= A_{12} B_{21} + A_{12} B_{22} - A_{22} B_{21} - A_{22} B_{22} \\
P_7 &= S_9 S_{10} &&= A_{11} B_{11} + A_{11} B_{12} - A_{21} B_{11} - A_{21} B_{12}
\end{aligned}
$$

$$
\begin{array}{rlllllllllllll}
 & P_5 + P_4 - P_2 + P_6 \\
= & A_{11}B_{11} & + & A_{11}B_{22} & + & A_{22}B_{11} & + & A_{22}B_{22} \\
 & & & & - & A_{22}B_{11} & & & + & A_{22}B_{21} \\
 & & - & A_{11}B_{22} & & & & & & & - & A_{12}B_{22} \\
 & & & & & & - & A_{22}B_{22} & - & A_{22}B_{21} & + & A_{12}B_{22} & + & A_{12}B_{21} \\
\hline
= & A_{11}B_{11} & & & & & & & & & & & + & A_{12}B_{21} \\
= & C_{11}
\end{array}
$$

Strassen Algorithm (1969)

$$
\begin{array}{lllll}
S_1 = B_{12} - B_{22} & S_2 = A_{11} + A_{12} & S_3 = A_{21} + A_{22} & S_4 = B_{21} - B_{11} & S_5 = A_{11} + A_{22} \\
S_6 = B_{11} + B_{22} & S_7 = A_{12} - A_{22} & S_8 = B_{21} + B_{22} & S_9 = A_{11} - A_{21} & S_{10} = B_{11} + B_{12}
\end{array}
$$

$$
\begin{aligned}
P_1 &= A_{11} S_1 = A_{11} B_{12} - A_{11} B_{22} \\
P_2 &= S_2 B_{22} = A_{11} B_{22} + A_{12} B_{22} \\
P_3 &= S_3 B_{11} = A_{21} B_{11} + A_{22} B_{11} \\
P_4 &= A_{22} S_4 = A_{22} B_{21} - A_{22} B_{11} \\
P_5 &= S_5 S_6 = A_{11} B_{11} + A_{11} B_{22} + A_{22} B_{11} + A_{22} B_{22} \\
P_6 &= S_7 S_8 = A_{12} B_{21} + A_{12} B_{22} - A_{22} B_{21} - A_{22} B_{22} \\
P_7 &= S_9 S_{10} = A_{11} B_{11} + A_{11} B_{12} - A_{21} B_{11} - A_{21} B_{12}
\end{aligned}
$$

$$
\begin{array}{rcccccccccccccc}
 & \multicolumn{14}{l}{P_5 + P_4 - P_2 + P_6} \\
= & A_{11}B_{11} & + & A_{11}B_{22} & + & A_{22}B_{11} & + & A_{22}B_{22} & & & & & & \\
 & & & & - & A_{22}B_{11} & & & + & A_{22}B_{21} & & & & \\
 & & - & A_{11}B_{22} & & & & & & & - & A_{12}B_{22} & & \\
 & & & & & & - & A_{22}B_{22} & - & A_{22}B_{21} & + & A_{12}B_{22} & + & A_{12}B_{21} \\
\hline
= & A_{11}B_{11} & & & & & & & & & & & + & A_{12}B_{21} \\
= & C_{11} & & & & & & & & & & & &
\end{array}
$$

$$C_{11} = P_5 + P_4 - P_2 + P_6$$

Strassen Algorithm (1969)

$$
\begin{array}{lllll}
S_1 = B_{12} - B_{22} & S_2 = A_{11} + A_{12} & S_3 = A_{21} + A_{22} & S_4 = B_{21} - B_{11} & S_5 = A_{11} + A_{22} \\
S_6 = B_{11} + B_{22} & S_7 = A_{12} - A_{22} & S_8 = B_{21} + B_{22} & S_9 = A_{11} - A_{21} & S_{10} = B_{11} + B_{12}
\end{array}
$$

$$
\begin{aligned}
P_1 &= A_{11} S_1 = A_{11} B_{12} - A_{11} B_{22} \\
P_2 &= S_2 B_{22} = A_{11} B_{22} + A_{12} B_{22} \\
P_3 &= S_3 B_{11} = A_{21} B_{11} + A_{22} B_{11} \\
P_4 &= A_{22} S_4 = A_{22} B_{21} - A_{22} B_{11} \\
P_5 &= S_5 S_6 = A_{11} B_{11} + A_{11} B_{22} + A_{22} B_{11} + A_{22} B_{22} \\
P_6 &= S_7 S_8 = A_{12} B_{21} + A_{12} B_{22} - A_{22} B_{21} - A_{22} B_{22} \\
P_7 &= S_9 S_{10} = A_{11} B_{11} + A_{11} B_{12} - A_{21} B_{11} - A_{21} B_{12}
\end{aligned}
$$

$$
\begin{array}{rlllllllllllll}
 & P_5 + P_4 - P_2 + P_6 \\
= & A_{11} B_{11} & + & A_{11} B_{22} & + & A_{22} B_{11} & + & A_{22} B_{22} \\
 & & & & - & A_{22} B_{11} & & & + & A_{22} B_{21} \\
 & & - & A_{11} B_{22} & & & & & & & - & A_{12} B_{22} \\
 & & & & & & - & A_{22} B_{22} & - & A_{22} B_{21} & + & A_{12} B_{22} & + & A_{12} B_{21} \\
\hline
= & A_{11} B_{11} & & & & & & & & & & & + & A_{12} B_{21} \\
= & C_{11}
\end{array}
$$

$$
\begin{aligned}
C_{11} &= P_5 + P_4 - P_2 + P_6 \\
C_{12} &= P_1 + P_2 \\
C_{21} &= P_3 + P_4 \\
C_{22} &= P_5 + P_1 - P_3 - P_7
\end{aligned}
$$

Strassen Algorithm (1969)

$$\begin{array}{lllll}
S_1 = B_{12} - B_{22} & S_2 = A_{11} + A_{12} & S_3 = A_{21} + A_{22} & S_4 = B_{21} - B_{11} & S_5 = A_{11} + A_{22} \\
S_6 = B_{11} + B_{22} & S_7 = A_{12} - A_{22} & S_8 = B_{21} + B_{22} & S_9 = A_{11} - A_{21} & S_{10} = B_{11} + B_{12}
\end{array}$$

$$\begin{aligned}
P_1 &= A_{11} S_1 = A_{11} B_{12} - A_{11} B_{22} \\
P_2 &= S_2 B_{22} = A_{11} B_{22} + A_{12} B_{22} \\
P_3 &= S_3 B_{11} = A_{21} B_{11} + A_{22} B_{11} \\
P_4 &= A_{22} S_4 = A_{22} B_{21} - A_{22} B_{11} \\
P_5 &= S_5 S_6 = A_{11} B_{11} + A_{11} B_{22} + A_{22} B_{11} + A_{22} B_{22} \\
P_6 &= S_7 S_8 = A_{12} B_{21} + A_{12} B_{22} - A_{22} B_{21} - A_{22} B_{22} \\
P_7 &= S_9 S_{10} = A_{11} B_{11} + A_{11} B_{12} - A_{21} B_{11} - A_{21} B_{12}
\end{aligned}$$

$$\begin{array}{rlllllllllllll}
 & P_5 + P_4 - P_2 + P_6 \\
= & A_{11}B_{11} & + & A_{11}B_{22} & + & A_{22}B_{11} & + & A_{22}B_{22} \\
 & & & & - & A_{22}B_{11} & & & + & A_{22}B_{21} \\
 & & - & A_{11}B_{22} & & & & & & & - & A_{12}B_{22} \\
 & & & & & & - & A_{22}B_{22} & - & A_{22}B_{21} & + & A_{12}B_{22} & + & A_{12}B_{21} \\
\hline
= & A_{11}B_{11} & & & & & & & & & & & + & A_{12}B_{21} \\
= & C_{11}
\end{array}$$

$$\begin{aligned}
C_{11} &= P_5 + P_4 - P_2 + P_6 \\
C_{12} &= P_1 + P_2 \\
C_{21} &= P_3 + P_4 \\
C_{22} &= P_5 + P_1 - P_3 - P_7 \\
T(n) &= 7\,T(n/2) + 18(n/2)^2 = 7\,T(n/2) + (9/2)n^2
\end{aligned}$$

Strassen Algorithm (1969)

$$T(n) = 7\,T(n/2) + (9/2)n^2$$
$$T(n) = O\left(n^{\log_2(7)}\right),$$

where $\log_2(7) \approx 2.81$.

Strassen Algorithm (1969)

$$T(n) = 7\,T(n/2) + (9/2)n^2$$
$$T(n) = O\left(n^{\log_2(7)}\right),$$

where $\log_2(7) \approx 2.81$.

Coppersmith & Winograd (1990): $O\left(n^{2.38}\right)$

Theorem (Master Theorem)

Let $T : \mathbb{N} \longrightarrow \mathbb{R}^+$ *be a function. Suppose that*

$$T(n) = \begin{cases} 1 & \text{if } n = 1, \\ a \cdot T(n/b) + n^d & \text{if } n > 1, \end{cases}$$

where $a \geq 1$, $b > 1$ *and* $d \geq 0$.

- *If* $d > \log_b(a)$, *then* $T(n) = O\left(n^d\right)$.
- *If* $d = \log_b(a)$, *then* $T(n) = O\left(n^d \log(n)\right)$.
- *If* $d < \log_b(a)$, *then* $T(n) = O\left(n^{\log_b(a)}\right)$.

Merge sort: $T(n) = 2T(n/2) + n^1$

Merge sort: $T(n) = 2T(n/2) + n^1$

$a = 2,\ b = 2,\ d = 1$

Merge sort: $T(n) = 2T(n/2) + n^1$

$a = 2,\ b = 2,\ d = 1$ $d = \log_b(a)$

Merge sort: $T(n) = 2T(n/2) + n^1$

$a = 2,\ b = 2,\ d = 1$ $\quad d = \log_b(a)$

$$T(n) = O(n^1 \log(n)) = O(n \log(n))$$

Merge sort: $T(n) = 2T(n/2) + n^1$

$a = 2,\ b = 2,\ d = 1$ $d = \log_b(a)$

$$T(n) = O(n^1 \log(n)) = O(n \log(n))$$

Strassen: $T(n) = 7T(n/2) + n^2$

Merge sort: $T(n) = 2T(n/2) + n^1$

$a = 2,\ b = 2,\ d = 1$ $d = \log_b(a)$

$$T(n) = O(n^1 \log(n)) = O(n \log(n))$$

Strassen: $T(n) = 7T(n/2) + n^2$

$a = 7,\ b = 2,\ d = 2$

Merge sort: $T(n) = 2T(n/2) + n^1$

$a = 2,\ b = 2,\ d = 1$ $d = \log_b(a)$

$$T(n) = O(n^1 \log(n)) = O(n \log(n))$$

Strassen: $T(n) = 7T(n/2) + n^2$

$a = 7,\ b = 2,\ d = 2$ $d < \log_b(a)$

Merge sort: $T(n) = 2T(n/2) + n^1$

$a = 2,\ b = 2,\ d = 1$ $\qquad d = \log_b(a)$

$$T(n) = O(n^1 \log(n)) = O(n \log(n))$$

Strassen: $T(n) = 7T(n/2) + n^2$

$a = 7,\ b = 2,\ d = 2$ $\qquad d < \log_b(a)$

$$T(n) = O\left(n^{\log_2(7)}\right)$$

Merge sort: $T(n) = 2T(n/2) + n^1$

$a = 2,\ b = 2,\ d = 1$ $d = \log_b(a)$

$$T(n) = O(n^1 \log(n)) = O(n \log(n))$$

Strassen: $T(n) = 7T(n/2) + n^2$

$a = 7,\ b = 2,\ d = 2$ $d < \log_b(a)$

$$T(n) = O\left(n^{\log_2(7)}\right)$$

Binary Search: $T(n) = T(n/2) + n^0$

Merge sort: $T(n) = 2T(n/2) + n^1$

$a = 2,\ b = 2,\ d = 1$ $\qquad d = \log_b(a)$

$$T(n) = O(n^1 \log(n)) = O(n \log(n))$$

Strassen: $T(n) = 7T(n/2) + n^2$

$a = 7,\ b = 2,\ d = 2$ $\qquad d < \log_b(a)$

$$T(n) = O\left(n^{\log_2(7)}\right)$$

Binary Search: $T(n) = T(n/2) + n^0$

$a = 1,\ b = 2,\ d = 0$

Merge sort: $T(n) = 2T(n/2) + n^1$

$a = 2,\ b = 2,\ d = 1$ $d = \log_b(a)$

$$T(n) = O(n^1 \log(n)) = O(n \log(n))$$

Strassen: $T(n) = 7T(n/2) + n^2$

$a = 7,\ b = 2,\ d = 2$ $d < \log_b(a)$

$$T(n) = O\left(n^{\log_2(7)}\right)$$

Binary Search: $T(n) = T(n/2) + n^0$

$a = 1,\ b = 2,\ d = 0$ $d = \log_b(a)$

Merge sort: $T(n) = 2T(n/2) + n^1$

$a = 2,\ b = 2,\ d = 1$ $\quad d = \log_b(a)$

$$T(n) = O(n^1 \log(n)) = O(n \log(n))$$

Strassen: $T(n) = 7T(n/2) + n^2$

$a = 7,\ b = 2,\ d = 2$ $\quad d < \log_b(a)$

$$T(n) = O\left(n^{\log_2(7)}\right)$$

Binary Search: $T(n) = T(n/2) + n^0$

$a = 1,\ b = 2,\ d = 0$ $\quad d = \log_b(a)$

$$T(n) = O(n^0 \log(n)) = O(\log(n))$$