CSI - 3105 Design & Analysis of Algorithms
Course 3

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Matrix Multiplication

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \qquad B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n,1} & b_{n2} & \cdots & b_{nn} \end{pmatrix}$$

$$
\begin{aligned}
C &= AB \\
c_{ij} &= \sum_{k=1}^{n} a_{ik} b_{kj}
\end{aligned}
$$

Matrix Multiplication

Example

$$\begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{-1} \\ \mathbf{0} & \mathbf{2} & \mathbf{3} \\ 1 & 5 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{4} & 0 & \mathbf{7} \\ \mathbf{-2} & 1 & \mathbf{0} \\ \mathbf{6} & 1 & \mathbf{-1} \end{pmatrix} = \begin{pmatrix} -2 & -1 & \mathbf{8} \\ \mathbf{14} & 5 & -3 \\ -6 & 5 & 7 \end{pmatrix}$$

Matrix Multiplication

Input: Two square matrices $A_{n \times n}$ and $B_{n \times n}$.

Output: AB.

 1: Initialize matrix $C_{n \times n}$
 2: **for** $i = 1$ to n **do**
 3:     **for** $j = 1$ to n **do**
 4:         $c_{ij} = 0$
 5:         **for** $k = 1$ to n **do**
 6:             $c_{ij} = c_{ij} + a_{ik} b_{kj}$
 7:         **end for**
 8:     **end for**
 9: **end for**
10: **return** C

Matrix Multiplication

Input: Two square matrices $A_{n \times n}$ and $B_{n \times n}$.
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1: Initialize matrix $C_{n \times n}$
2: **for** $i = 1$ to n **do**
3:     **for** $j = 1$ to n **do**
4:         $c_{ij} = 0$
5:         **for** $k = 1$ to n **do**
6:             $c_{ij} = c_{ij} + a_{ik} b_{kj}$
7:         **end for**
8:     **end for**
9: **end for**
10: **return** C

$$T(n) = \Theta\left(n^3\right)$$

How can we Divide and Conquer?

Assume $n = 2^k$.

$$\begin{pmatrix} 1 & 0 & -1 & 2 \\ 3 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 2 & 4 & 7 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 & 3 \\ -2 & 1 & -1 & 1 \\ 1 & 0 & 1 & 0 \\ 5 & 1 & -2 & -1 \end{pmatrix} = \begin{pmatrix} 9 & 3 & -3 & 1 \\ 4 & 5 & 4 & 9 \\ 5 & 1 & -2 & -1 \\ 4 & 7 & 5 & 9 \end{pmatrix}$$

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Assume $n = 2^k$.

$$\begin{pmatrix} 1 & 0 & -1 & 2 \\ 3 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 2 & 4 & 7 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 & 3 \\ -2 & 1 & -1 & 1 \\ 1 & 0 & 1 & 0 \\ 5 & 1 & -2 & -1 \end{pmatrix} = \begin{pmatrix} 9 & 3 & -3 & 1 \\ 4 & 5 & 4 & 9 \\ 5 & 1 & -2 & -1 \\ 4 & 7 & 5 & 9 \end{pmatrix}$$

$$\begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} & \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 2 & 4 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 7 & 1 \end{pmatrix} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ -2 & 1 \end{pmatrix} & \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ -2 & -1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 9 & 3 \\ 4 & 5 \end{pmatrix} & \begin{pmatrix} -3 & 1 \\ 4 & 9 \end{pmatrix} \\ \begin{pmatrix} 5 & 1 \\ 4 & 7 \end{pmatrix} & \begin{pmatrix} -2 & -1 \\ 5 & 9 \end{pmatrix} \end{pmatrix}$$

How can we Divide and Conquer?

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$$\begin{pmatrix} 1 & 0 & -1 & 2 \\ 3 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 2 & 4 & 7 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 & 3 \\ -2 & 1 & -1 & 1 \\ 1 & 0 & 1 & 0 \\ 5 & 1 & -2 & -1 \end{pmatrix} = \begin{pmatrix} 9 & 3 & -3 & 1 \\ 4 & 5 & 4 & 9 \\ 5 & 1 & -2 & -1 \\ 4 & 7 & 5 & 9 \end{pmatrix}$$

$$\begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} & \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 2 & 4 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 7 & 1 \end{pmatrix} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ -2 & 1 \end{pmatrix} & \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ -2 & -1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 9 & 3 \\ 4 & 5 \end{pmatrix} & \begin{pmatrix} -3 & 1 \\ 4 & 9 \end{pmatrix} \\ \begin{pmatrix} 5 & 1 \\ 4 & 7 \end{pmatrix} & \begin{pmatrix} -2 & -1 \\ 5 & 9 \end{pmatrix} \end{pmatrix}$$

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \qquad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

How can we Divide and Conquer?

Assume $n = 2^k$.

$$\begin{pmatrix} 1 & 0 & -1 & 2 \\ 3 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 2 & 4 & 7 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 & 3 \\ -2 & 1 & -1 & 1 \\ 1 & 0 & 1 & 0 \\ 5 & 1 & -2 & -1 \end{pmatrix} = \begin{pmatrix} 9 & 3 & -3 & 1 \\ 4 & 5 & 4 & 9 \\ 5 & 1 & -2 & -1 \\ 4 & 7 & 5 & 9 \end{pmatrix}$$

$$\left(\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 7 & 1 \end{pmatrix} \right) \left(\begin{pmatrix} 0 & 1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2 & -1 \end{pmatrix} \right) = \left(\begin{pmatrix} 9 & 3 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} -3 & 1 \\ 4 & 9 \end{pmatrix} \\ \begin{pmatrix} 5 & 1 \\ 4 & 7 \end{pmatrix} \begin{pmatrix} -2 & -1 \\ 5 & 9 \end{pmatrix} \right)$$

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \qquad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

$$C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

$$
\begin{aligned}
C_{11} &= A_{11}B_{11} + A_{12}B_{21} \\
C_{12} &= A_{11}B_{12} + A_{12}B_{22} \\
C_{21} &= A_{21}B_{11} + A_{22}B_{21} \\
C_{22} &= A_{21}B_{12} + A_{22}B_{22}
\end{aligned}
$$

How can we Divide and Conquer?

Assume $n = 2^k$.

$$\begin{pmatrix} 1 & 0 & -1 & 2 \\ 3 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 2 & 4 & 7 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 & 3 \\ -2 & 1 & -1 & 1 \\ 1 & 0 & 1 & 0 \\ 5 & 1 & -2 & -1 \end{pmatrix} = \begin{pmatrix} 9 & 3 & -3 & 1 \\ 4 & 5 & 4 & 9 \\ 5 & 1 & -2 & -1 \\ 4 & 7 & 5 & 9 \end{pmatrix}$$

$$\begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} & \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 2 & 4 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 7 & 1 \end{pmatrix} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ -2 & 1 \end{pmatrix} & \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ -2 & -1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 9 & 3 \\ 4 & 5 \end{pmatrix} & \begin{pmatrix} -3 & 1 \\ 4 & 9 \end{pmatrix} \\ \begin{pmatrix} 5 & 1 \\ 4 & 7 \end{pmatrix} & \begin{pmatrix} -2 & -1 \\ 5 & 9 \end{pmatrix} \end{pmatrix}$$

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \qquad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

$$C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

$$\begin{aligned} C_{11} &= A_{11}B_{11} + A_{12}B_{21} \\ C_{12} &= A_{11}B_{12} + A_{12}B_{22} \\ C_{21} &= A_{21}B_{11} + A_{22}B_{21} \\ C_{22} &= A_{21}B_{12} + A_{22}B_{22} \end{aligned}$$

$$T(n) = 8\,T(n/2) + 4\,(n/2)^2 \quad = \quad 8\,T(n/2) + n^2$$

$$T(n) = \begin{cases} 1 & \text{if } n=1 \\ 8T(n/2) + n^2 & \text{if } n \geq 2 \end{cases}$$

Assume $n = 2^K$.

$$T(n) = 8T(n/2) + n^2$$

$$= 8\left(8T(n/2^2) + \left(\frac{n}{2}\right)^2\right) + n^2$$

$$= 8^2 T(n/2^2) + 2n^2 + n^2$$

$$= 8^2\left(8 \cdot T(n/2^3) + \left(\frac{n}{2^2}\right)^2\right) + 2n^2 + n^2$$

$$= 8^3 T(n/2^3) + 4n^2 + 2n^2 + n^2$$

$$= 8^3\left(8 \cdot T(n/2^4) + \left(\frac{n}{2^3}\right)^2\right) + 4n^2 + 2n^2 + n^2$$

$$= 8^4 \cdot T(n/2^4) + 8n^2 + 4n^2 + 2n^2 + n^2$$

$$= 8^4 \cdot T(n/2^4) + \left(2^0 + 2^1 + 2^2 + 2^3\right)n^2$$

$$\vdots$$

$$T(1) = 1$$

$$= \underbrace{8^K}_{(2^3)^K = (2^K)^3 = n^3} \cdot \overbrace{T(n/2^K)} + \underbrace{\left(2^0 + 2^1 + \ldots + 2^{K-1}\right)}_{2^K - 1 = n - 1} n^2$$

$$= n^3 + (n-1)n^2$$

$$= O(n^3) \qquad , \qquad \text{no improvement !}$$

Strassen Algorithm (1969)

$S_1 = B_{12} - B_{22}$ $\quad S_2 = A_{11} + A_{12}$ $\quad S_3 = A_{21} + A_{22}$ $\quad S_4 = B_{21} - B_{11}$ $\quad S_5 = A_{11} + A_{22}$

$S_6 = B_{11} + B_{22}$ $\quad S_7 = A_{12} - A_{22}$ $\quad S_8 = B_{21} + B_{22}$ $\quad S_9 = A_{11} - A_{21}$ $\quad S_{10} = B_{11} + B_{12}$

Strassen Algorithm (1969)

$S_1 = B_{12} - B_{22}$ $S_2 = A_{11} + A_{12}$ $S_3 = A_{21} + A_{22}$ $S_4 = B_{21} - B_{11}$ $S_5 = A_{11} + A_{22}$
$S_6 = B_{11} + B_{22}$ $S_7 = A_{12} - A_{22}$ $S_8 = B_{21} + B_{22}$ $S_9 = A_{11} - A_{21}$ $S_{10} = B_{11} + B_{12}$

$P_1 = A_{11}S_1 = A_{11}B_{12} - A_{11}B_{22}$
$P_2 = S_2B_{22} = A_{11}B_{22} + A_{12}B_{22}$
$P_3 = S_3B_{11} = A_{21}B_{11} + A_{22}B_{11}$
$P_4 = A_{22}S_4 = A_{22}B_{21} - A_{22}B_{11}$
$P_5 = S_5S_6 = A_{11}B_{11} + A_{11}B_{22} + A_{22}B_{11} + A_{22}B_{22}$
$P_6 = S_7S_8 = A_{12}B_{21} + A_{12}B_{22} - A_{22}B_{21} - A_{22}B_{22}$
$P_7 = S_9S_{10} = A_{11}B_{11} + A_{11}B_{12} - A_{21}B_{11} - A_{21}B_{12}$

Strassen Algorithm (1969)

$$S_1 = B_{12} - B_{22} \quad S_2 = A_{11} + A_{12} \quad S_3 = A_{21} + A_{22} \quad S_4 = B_{21} - B_{11} \quad S_5 = A_{11} + A_{22}$$
$$S_6 = B_{11} + B_{22} \quad S_7 = A_{12} - A_{22} \quad S_8 = B_{21} + B_{22} \quad S_9 = A_{11} - A_{21} \quad S_{10} = B_{11} + B_{12}$$

$$P_1 = A_{11}S_1 = A_{11}B_{12} - A_{11}B_{22}$$
$$P_2 = S_2B_{22} = A_{11}B_{22} + A_{12}B_{22}$$
$$P_3 = S_3B_{11} = A_{21}B_{11} + A_{22}B_{11}$$
$$P_4 = A_{22}S_4 = A_{22}B_{21} - A_{22}B_{11}$$
$$P_5 = S_5S_6 = A_{11}B_{11} + A_{11}B_{22} + A_{22}B_{11} + A_{22}B_{22}$$
$$P_6 = S_7S_8 = A_{12}B_{21} + A_{12}B_{22} - A_{22}B_{21} - A_{22}B_{22}$$
$$P_7 = S_9S_{10} = A_{11}B_{11} + A_{11}B_{12} - A_{21}B_{11} - A_{21}B_{12}$$

$$P_5 + P_4 - P_2 + P_6$$

Strassen Algorithm (1969)

$S_1 = B_{12} - B_{22}$ $S_2 = A_{11} + A_{12}$ $S_3 = A_{21} + A_{22}$ $S_4 = B_{21} - B_{11}$ $S_5 = A_{11} + A_{22}$

$S_6 = B_{11} + B_{22}$ $S_7 = A_{12} - A_{22}$ $S_8 = B_{21} + B_{22}$ $S_9 = A_{11} - A_{21}$ $S_{10} = B_{11} + B_{12}$

$P_1 = A_{11}S_1 = A_{11}B_{12} - A_{11}B_{22}$

$P_2 = S_2B_{22} = A_{11}B_{22} + A_{12}B_{22}$

$P_3 = S_3B_{11} = A_{21}B_{11} + A_{22}B_{11}$

$P_4 = A_{22}S_4 = A_{22}B_{21} - A_{22}B_{11}$

$P_5 = S_5S_6 = A_{11}B_{11} + A_{11}B_{22} + A_{22}B_{11} + A_{22}B_{22}$

$P_6 = S_7S_8 = A_{12}B_{21} + A_{12}B_{22} - A_{22}B_{21} - A_{22}B_{22}$

$P_7 = S_9S_{10} = A_{11}B_{11} + A_{11}B_{12} - A_{21}B_{11} - A_{21}B_{12}$

$$
\begin{array}{lllll}
& P_5 + P_4 - P_2 + P_6 & & & \\
= & A_{11}B_{11} & + \quad A_{11}B_{22} & + \quad A_{22}B_{11} & + \quad A_{22}B_{22} \\
& & & - \quad A_{22}B_{11} & \quad\quad\quad\quad + \quad A_{22}B_{21} \\
& - \quad A_{11}B_{22} & & & \quad\quad\quad\quad\quad - \quad A_{12}B_{22} \\
& & & - \quad A_{22}B_{22} \quad - \quad A_{22}B_{21} & + \quad A_{12}B_{22} \quad + \quad A_{12}B_{21}
\end{array}
$$

Strassen Algorithm (1969)

$$S_1 = B_{12} - B_{22} \quad S_2 = A_{11} + A_{12} \quad S_3 = A_{21} + A_{22} \quad S_4 = B_{21} - B_{11} \quad S_5 = A_{11} + A_{22}$$
$$S_6 = B_{11} + B_{22} \quad S_7 = A_{12} - A_{22} \quad S_8 = B_{21} + B_{22} \quad S_9 = A_{11} - A_{21} \quad S_{10} = B_{11} + B_{12}$$

$$P_1 = A_{11}S_1 = A_{11}B_{12} - A_{11}B_{22}$$
$$P_2 = S_2B_{22} = A_{11}B_{22} + A_{12}B_{22}$$
$$P_3 = S_3B_{11} = A_{21}B_{11} + A_{22}B_{11}$$
$$P_4 = A_{22}S_4 = A_{22}B_{21} - A_{22}B_{11}$$
$$P_5 = S_5S_6 = A_{11}B_{11} + A_{11}B_{22} + A_{22}B_{11} + A_{22}B_{22}$$
$$P_6 = S_7S_8 = A_{12}B_{21} + A_{12}B_{22} - A_{22}B_{21} - A_{22}B_{22}$$
$$P_7 = S_9S_{10} = A_{11}B_{11} + A_{11}B_{12} - A_{21}B_{11} - A_{21}B_{12}$$

$$
\begin{aligned}
& \quad P_5 + P_4 - P_2 + P_6 \\
= & \quad A_{11}B_{11} \;+\; A_{11}B_{22} \;+\; A_{22}B_{11} \;+\; A_{22}B_{22} \\
& \qquad\qquad\qquad\qquad - A_{22}B_{11} \qquad\qquad + A_{22}B_{21} \\
& \qquad - A_{11}B_{22} \qquad\qquad\qquad\qquad\qquad\qquad - A_{12}B_{22} \\
& \qquad\qquad\qquad\qquad - A_{22}B_{22} - A_{22}B_{21} + A_{12}B_{22} + A_{12}B_{21} \\
\hline
= & \quad A_{11}B_{11} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + A_{12}B_{21}
\end{aligned}
$$

Strassen Algorithm (1969)

$S_1 = B_{12} - B_{22}$ $S_2 = A_{11} + A_{12}$ $S_3 = A_{21} + A_{22}$ $S_4 = B_{21} - B_{11}$ $S_5 = A_{11} + A_{22}$
$S_6 = B_{11} + B_{22}$ $S_7 = A_{12} - A_{22}$ $S_8 = B_{21} + B_{22}$ $S_9 = A_{11} - A_{21}$ $S_{10} = B_{11} + B_{12}$

$P_1 = A_{11}S_1 = A_{11}B_{12} - A_{11}B_{22}$
$P_2 = S_2B_{22} = A_{11}B_{22} + A_{12}B_{22}$
$P_3 = S_3B_{11} = A_{21}B_{11} + A_{22}B_{11}$
$P_4 = A_{22}S_4 = A_{22}B_{21} - A_{22}B_{11}$
$P_5 = S_5S_6 = A_{11}B_{11} + A_{11}B_{22} + A_{22}B_{11} + A_{22}B_{22}$
$P_6 = S_7S_8 = A_{12}B_{21} + A_{12}B_{22} - A_{22}B_{21} - A_{22}B_{22}$
$P_7 = S_9S_{10} = A_{11}B_{11} + A_{11}B_{12} - A_{21}B_{11} - A_{21}B_{12}$

$$
\begin{array}{rllllll}
& P_5 + P_4 - P_2 + P_6 & & & & & \\
= & A_{11}B_{11} & + & A_{11}B_{22} & + & A_{22}B_{11} & + & A_{22}B_{22} \\
& & & & - & A_{22}B_{11} & & & + & A_{22}B_{21} \\
& & - & A_{11}B_{22} & & & & & - & A_{12}B_{22} \\
& & & & - & A_{22}B_{22} & - & A_{22}B_{21} & + & A_{12}B_{22} & + & A_{12}B_{21} \\
\hline
= & A_{11}B_{11} & & & & & & & & & + & A_{12}B_{21} \\
= & C_{11} & & & & & & & & & &
\end{array}
$$

Strassen Algorithm (1969)

$$S_1 = B_{12} - B_{22} \quad S_2 = A_{11} + A_{12} \quad S_3 = A_{21} + A_{22} \quad S_4 = B_{21} - B_{11} \quad S_5 = A_{11} + A_{22}$$
$$S_6 = B_{11} + B_{22} \quad S_7 = A_{12} - A_{22} \quad S_8 = B_{21} + B_{22} \quad S_9 = A_{11} - A_{21} \quad S_{10} = B_{11} + B_{12}$$

$$P_1 = A_{11}S_1 = A_{11}B_{12} - A_{11}B_{22}$$
$$P_2 = S_2B_{22} = A_{11}B_{22} + A_{12}B_{22}$$
$$P_3 = S_3B_{11} = A_{21}B_{11} + A_{22}B_{11}$$
$$P_4 = A_{22}S_4 = A_{22}B_{21} - A_{22}B_{11}$$
$$P_5 = S_5S_6 = A_{11}B_{11} + A_{11}B_{22} + A_{22}B_{11} + A_{22}B_{22}$$
$$P_6 = S_7S_8 = A_{12}B_{21} + A_{12}B_{22} - A_{22}B_{21} - A_{22}B_{22}$$
$$P_7 = S_9S_{10} = A_{11}B_{11} + A_{11}B_{12} - A_{21}B_{11} - A_{21}B_{12}$$

$$
\begin{aligned}
& \quad P_5 + P_4 - P_2 + P_6 \\
=& \quad A_{11}B_{11} \;+\; A_{11}B_{22} \;+\; A_{22}B_{11} \;+\; A_{22}B_{22} \\
& \qquad\qquad\qquad\qquad -\; A_{22}B_{11} \qquad\qquad +\; A_{22}B_{21} \\
& \qquad -\; A_{11}B_{22} \qquad\qquad\qquad\qquad\qquad\qquad -\; A_{12}B_{22} \\
& \qquad\qquad\qquad\qquad -\; A_{22}B_{22} \;-\; A_{22}B_{21} \;+\; A_{12}B_{22} \;+\; A_{12}B_{21} \\
\hline
=& \quad A_{11}B_{11} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad +\; A_{12}B_{21} \\
=& \quad C_{11}
\end{aligned}
$$

$$C_{11} \quad = \quad P_5 + P_4 - P_2 + P_6$$

Strassen Algorithm (1969)

$$S_1 = B_{12} - B_{22} \quad S_2 = A_{11} + A_{12} \quad S_3 = A_{21} + A_{22} \quad S_4 = B_{21} - B_{11} \quad S_5 = A_{11} + A_{22}$$
$$S_6 = B_{11} + B_{22} \quad S_7 = A_{12} - A_{22} \quad S_8 = B_{21} + B_{22} \quad S_9 = A_{11} - A_{21} \quad S_{10} = B_{11} + B_{12}$$

$$P_1 = A_{11}S_1 = A_{11}B_{12} - A_{11}B_{22}$$
$$P_2 = S_2B_{22} = A_{11}B_{22} + A_{12}B_{22}$$
$$P_3 = S_3B_{11} = A_{21}B_{11} + A_{22}B_{11}$$
$$P_4 = A_{22}S_4 = A_{22}B_{21} - A_{22}B_{11}$$
$$P_5 = S_5S_6 = A_{11}B_{11} + A_{11}B_{22} + A_{22}B_{11} + A_{22}B_{22}$$
$$P_6 = S_7S_8 = A_{12}B_{21} + A_{12}B_{22} - A_{22}B_{21} - A_{22}B_{22}$$
$$P_7 = S_9S_{10} = A_{11}B_{11} + A_{11}B_{12} - A_{21}B_{11} - A_{21}B_{12}$$

$$
\begin{array}{rllllll}
 & P_5 + P_4 - P_2 + P_6 & & & & & \\
= & A_{11}B_{11} & + & A_{11}B_{22} & + & A_{22}B_{11} & + & A_{22}B_{22} \\
 & & & & - & A_{22}B_{11} & & + A_{22}B_{21} \\
 & & - & A_{11}B_{22} & & & & - A_{12}B_{22} \\
 & & & & - & A_{22}B_{22} & - A_{22}B_{21} & + A_{12}B_{22} & + A_{12}B_{21} \\
\hline
= & A_{11}B_{11} & & & & & & + A_{12}B_{21} \\
= & C_{11} & & & & & &
\end{array}
$$

$$
\begin{aligned}
C_{11} &= P_5 + P_4 - P_2 + P_6 \\
C_{12} &= P_1 + P_2 \\
C_{21} &= P_3 + P_4 \\
C_{22} &= P_5 + P_1 - P_3 - P_7
\end{aligned}
$$

Strassen Algorithm (1969)

$S_1 = B_{12} - B_{22}$ $\quad S_2 = A_{11} + A_{12}$ $\quad S_3 = A_{21} + A_{22}$ $\quad S_4 = B_{21} - B_{11}$ $\quad S_5 = A_{11} + A_{22}$

$S_6 = B_{11} + B_{22}$ $\quad S_7 = A_{12} - A_{22}$ $\quad S_8 = B_{21} + B_{22}$ $\quad S_9 = A_{11} - A_{21}$ $\quad S_{10} = B_{11} + B_{12}$

$P_1 = A_{11}S_1 = A_{11}B_{12} - A_{11}B_{22}$

$P_2 = S_2B_{22} = A_{11}B_{22} + A_{12}B_{22}$

$P_3 = S_3B_{11} = A_{21}B_{11} + A_{22}B_{11}$

$P_4 = A_{22}S_4 = A_{22}B_{21} - A_{22}B_{11}$

$P_5 = S_5S_6 = A_{11}B_{11} + A_{11}B_{22} + A_{22}B_{11} + A_{22}B_{22}$

$P_6 = S_7S_8 = A_{12}B_{21} + A_{12}B_{22} - A_{22}B_{21} - A_{22}B_{22}$

$P_7 = S_9S_{10} = A_{11}B_{11} + A_{11}B_{12} - A_{21}B_{11} - A_{21}B_{12}$

$$
\begin{aligned}
& \quad P_5 + P_4 - P_2 + P_6 \\
= & \quad A_{11}B_{11} \;+\; A_{11}B_{22} \;+\; A_{22}B_{11} \;+\; A_{22}B_{22} \\
& \qquad\qquad\qquad\qquad\quad -\; A_{22}B_{11} \qquad\qquad +\; A_{22}B_{21} \\
& \qquad -\; A_{11}B_{22} \qquad\qquad\qquad\qquad\qquad\qquad -\; A_{12}B_{22} \\
& \qquad\qquad\qquad\qquad\qquad -\; A_{22}B_{22} \;-\; A_{22}B_{21} \;+\; A_{12}B_{22} \;+\; A_{12}B_{21} \\
\hline
= & \quad A_{11}B_{11} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad +\; A_{12}B_{21} \\
= & \quad C_{11}
\end{aligned}
$$

$$
\begin{aligned}
C_{11} &= P_5 + P_4 - P_2 + P_6 \\
C_{12} &= P_1 + P_2 \\
C_{21} &= P_3 + P_4 \\
C_{22} &= P_5 + P_1 - P_3 - P_7 \\
T(n) &= 7\,T(n/2) + 18(n/2)^2 = 7\,T(n/2) + (9/2)n^2
\end{aligned}
$$

$$T(n) = \begin{cases} 1 & \text{if } n=1 \\ 7\,T(n/2) + \frac{9}{2} n^2 & \text{if } n \geq 2 \end{cases}$$

Assume $n = 2^K$.

$$T(n) = 7\,T(n/2) + \frac{9}{2} n^2$$

$$= 7 \left(7\,T(n/2^2) + \frac{9}{2}(n/2)^2 \right) + \frac{9}{2} n^2$$

$$= 7^2\,T(n/2^2) + \frac{7}{4} \cdot \frac{9}{2} \cdot n^2 + \frac{9}{2} n^2$$

$$= 7^2 \left(7\,T(n/2^3) + \frac{9}{2}\left(\frac{n}{2^2}\right)^2 \right) + \frac{7}{4} \cdot \frac{9}{2} n^2 + \frac{9}{2} n^2$$

$$= 7^3\,T\left(\frac{n}{2^3}\right) + \left(\frac{7}{4}\right)^2 \cdot \frac{9}{2} n^2 + \frac{7}{4} \cdot \frac{9}{2} n^2 + \frac{9}{2} n^2$$

$$= 7^3 \left(7\,T\left(\frac{n}{2^4}\right) + \frac{9}{2}\left(\frac{n}{2^3}\right)^2 \right) + \left(\frac{7}{4}\right)^2 \frac{9}{2} n^2 + \frac{7}{4} \cdot \frac{9}{2} n^2 + \frac{9}{2} n^2$$

$$= 7^4\,T\left(\frac{n}{2^4}\right) + \left(\frac{7}{4}\right)^3 \frac{9}{2} n^2 + \left(\frac{7}{4}\right)^2 \frac{9}{2} n^2 + \frac{7}{4} \cdot \frac{9}{2} n^2 + \frac{9}{2} n^2$$

$$= 7^4\,T\left(\frac{n}{2^K}\right) + \frac{9}{2}\left(\left(\frac{7}{4}\right)^0 + \left(\frac{7}{4}\right)^1 + \left(\frac{7}{4}\right)^2 + \left(\frac{7}{4}\right)^3 \right) n^2$$

$$\vdots \qquad \to T(1) = 1$$

$$= 7^K \boxed{T\left(\frac{n}{2^K}\right)} + \frac{9}{2}\left(\left(\frac{7}{4}\right)^0 + \left(\frac{7}{4}\right)^1 + \cdots + \left(\frac{7}{4}\right)^{K-1} \right) n^2$$

Recall:

$$\sum_{i=0}^{K-1} x^i = 1 + x + x^2 + \cdots + x^{K-1} = \frac{x^K - 1}{x-1} \qquad (x \neq 1)$$

$$= 7^k + \frac{9}{2} \cdot \frac{(7/4)^k - 1}{\frac{7}{4} - 1} \, n^2$$

$$= 7^k + 6 \left((7/4)^k - 1 \right) n^2$$

$$= \left(2^{\log_2(7)} \right)^k + 6 \left(\left(2^{\log_2(7/4)} \right)^k - 1 \right) n^2$$

$$= \left(2^k \right)^{\log_2(7)} + 6 \left(\left(2^k \right)^{\log_2(7/4)} - 1 \right) n^2$$

$$= n^{\log_2(7)} + 6 \left(n^{\log_2(7/4)} - 1 \right) n^2$$

$$= 7 \, n^{\log_2(7)} - 6 \, n^2$$

$$= O\left(n^{\log_2(7)} \right),$$

$$\text{where} \quad \log_2(7) \approx 2.81$$

Strassen Algorithm (1969)

$$T(n) = 7\,T(n/2) + (9/2)n^2$$
$$T(n) = O\left(n^{\log_2(7)}\right),$$

where $\log_2(7) \approx 2.81$.

Strassen Algorithm (1969)

$$T(n) = 7\, T(n/2) + (9/2)n^2$$
$$T(n) = O\left(n^{\log_2(7)}\right),$$

where $\log_2(7) \approx 2.81$.

Coppersmith & Winograd (1990): $O\left(n^{2.38}\right)$

Theorem (Master Theorem)

Let $T : \mathbb{N} \longrightarrow \mathbb{R}^+$ be a function. Suppose that

$$T(n) = \begin{cases} 1 & \text{if } n = 1, \\ a \cdot T(n/b) + n^d & \text{if } n > 1, \end{cases}$$

where $a \geq 1$, $b > 1$ and $d \geq 0$.

- If $d > \log_b(a)$, then $T(n) = O\left(n^d\right)$.

- If $d = \log_b(a)$, then $T(n) = O\left(n^d \log(n)\right)$.

- If $d < \log_b(a)$, then $T(n) = O\left(n^{\log_b(a)}\right)$.

Merge sort:
$$T(n) = 2T(n/2) + n^1$$

Merge sort:
$$T(n) = 2T(n/2) + n^1$$

$a = 2,\ b = 2,\ d = 1$

Merge sort:

$$T(n) = 2T(n/2) + n^1$$

$a = 2,\ b = 2,\ d = 1$

$$d = \log_b(a)$$

Merge sort:
$$T(n) = 2T(n/2) + n^1$$

$a = 2,\ b = 2,\ d = 1$ $\qquad\qquad d = \log_b(a)$

$$T(n) = O(n^1 \log(n)) = O(n \log(n))$$

Merge sort:
$$T(n) = 2\,T(n/2) + n^1$$

$a = 2,\ b = 2,\ d = 1$ $\qquad\qquad d = \log_b(a)$

$$T(n) = O(n^1 \log(n)) = O(n \log(n))$$

Strassen:
$$T(n) = 7\,T(n/2) + n^2$$

Merge sort:
$$T(n) = 2T(n/2) + n^1$$

$a = 2$, $b = 2$, $d = 1$ $\qquad d = \log_b(a)$

$$T(n) = O(n^1 \log(n)) = O(n \log(n))$$

Strassen:
$$T(n) = 7T(n/2) + n^2$$

$a = 7$, $b = 2$, $d = 2$

Merge sort:
$$T(n) = 2\,T(n/2) + n^1$$

$a = 2,\ b = 2,\ d = 1$ $\qquad d = \log_b(a)$

$$T(n) = O(n^1 \log(n)) = O(n \log(n))$$

Strassen:
$$T(n) = 7\,T(n/2) + n^2$$

$a = 7,\ b = 2,\ d = 2$ $\qquad d < \log_b(a)$

Merge sort:

$$T(n) = 2T(n/2) + n^1$$

$a = 2$, $b = 2$, $d = 1$

$$d = \log_b(a)$$

$$T(n) = O(n^1 \log(n)) = O(n \log(n))$$

Strassen:

$$T(n) = 7T(n/2) + n^2$$

$a = 7$, $b = 2$, $d = 2$

$$d < \log_b(a)$$

$$T(n) = O\left(n^{\log_2(7)}\right)$$

Merge sort: $$T(n) = 2T(n/2) + n^1$$

$a = 2$, $b = 2$, $d = 1$ $\qquad d = \log_b(a)$

$$T(n) = O(n^1 \log(n)) = O(n \log(n))$$

Strassen: $$T(n) = 7T(n/2) + n^2$$

$a = 7$, $b = 2$, $d = 2$ $\qquad d < \log_b(a)$

$$T(n) = O\left(n^{\log_2(7)}\right)$$

Binary Search: $$T(n) = T(n/2) + n^0$$

Merge sort:
$$T(n) = 2T(n/2) + n^1$$

$a = 2$, $b = 2$, $d = 1$
$$d = \log_b(a)$$

$$T(n) = O(n^1 \log(n)) = O(n \log(n))$$

Strassen:
$$T(n) = 7T(n/2) + n^2$$

$a = 7$, $b = 2$, $d = 2$
$$d < \log_b(a)$$

$$T(n) = O\left(n^{\log_2(7)}\right)$$

Binary Search:
$$T(n) = T(n/2) + n^0$$

$a = 1$, $b = 2$, $d = 0$

Merge sort: $$T(n) = 2T(n/2) + n^1$$

$a = 2$, $b = 2$, $d = 1$ $\qquad d = \log_b(a)$

$$T(n) = O(n^1 \log(n)) = O(n \log(n))$$

Strassen: $$T(n) = 7T(n/2) + n^2$$

$a = 7$, $b = 2$, $d = 2$ $\qquad d < \log_b(a)$

$$T(n) = O\left(n^{\log_2(7)}\right)$$

Binary Search: $$T(n) = T(n/2) + n^0$$

$a = 1$, $b = 2$, $d = 0$ $\qquad d = \log_b(a)$

Merge sort:
$$T(n) = 2T(n/2) + n^1$$

$a = 2$, $b = 2$, $d = 1$
$$d = \log_b(a)$$

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Strassen:
$$T(n) = 7T(n/2) + n^2$$

$a = 7$, $b = 2$, $d = 2$
$$d < \log_b(a)$$

$$T(n) = O\left(n^{\log_2(7)}\right)$$

Binary Search:
$$T(n) = T(n/2) + n^0$$

$a = 1$, $b = 2$, $d = 0$
$$d = \log_b(a)$$

$$T(n) = O(n^0 \log(n)) = O(\log(n))$$