CSI - 3105 Design & Analysis of Algorithms
Course 15

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Fall 2019

§5.3 Longest Common Subsequence

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$$Y = (b, d, c, a, b, a).$$

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$LCS(X, Y)$ is the longest common subsequence of X and Y:

$$(b, c, b, a) \qquad \text{or} \qquad (b, d, a, b).$$

Both have length 4.

Problem:

Input : Sequences

- $X = (x_1, x_2, ..., x_m)$
- $Y = (y_1, y_2, ..., y_n)$

Output : A longest sequence that is a subsequence of X and a subsequence of Y, i.e., $Z = LCS(X, Y)$.

Step 1: Structure of the Optimal Solution

$$X = (x_1, x_2, ..., x_m) \qquad Y = (y_1, y_2, ..., y_n)$$

Consider

$$Z = (z_1, z_2, ..., z_k) = LCS(X, Y)$$

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Then $z_k = x_m = y_n$ and

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Case 2 b): $z_k \neq y_n$

$$(z_1, z_2, ..., z_k) = LCS(x_1 x_2 \cdots x_m, y_1 y_2 \cdots y_{n-1})$$

But we do not know if we are in Case 2 a) or 2 b).

If $x_m \neq y_n$, then $(z_1, z_2, \ldots, z_k)$ is the longest of

$$LCS(x_1 x_2 \cdots x_{m-1}, y_1 y_2 \cdots y_n)$$

and

$$LCS(x_1 x_2 \cdots x_m, y_1 y_2 \cdots y_{n-1}).$$

Step 2: Set Up a recurrence for the Optimal Solution

For $0 \leq i \leq m$ and $0 \leq j \leq n$, define

$$c(i, j) = \text{length of } LCS(x_1 x_2 \cdots x_i, y_1 y_2 \cdots y_j).$$

We want to compute $c(m, n)$.

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- If $i \geq 1$, $j \geq 1$ and $x_i \neq y_j$,

$$c(i,j) = \max\{c(i-1,j), c(i,j-1)\}$$

Step 3: Solve the Recurrence Bottom-Up

Fill in the matrix $c(i, j)$ for $0 \leq i \leq m$ and $0 \leq j \leq n$.

First row:

$$c(0, 0) = c(0, 1) = ... = c(0, n) = 0$$

First column:

$$c(0, 0) = c(1, 0) = ... = c(m, 0) = 0$$

Then fill in the matrix, row by row, in each row from left to right.

Algorithm

Algorithm Longest Common Subsequence

```
for i = 0 to m do
    c(i, 0) = 0
end for
for j = 0 to n do
    c(0, j) = 0
end for
for i = 1 to m do
    for j = 1 to n do
        if x_i = y_j then
            c(i, j) = 1 + c(i − 1, j − 1)
        else
            c(i, j) = max {c(i − 1, j), c(i, j − 1)}
        end if
    end for
end for
return c(m, n)
```

$$
\begin{aligned}
X &= ABCBDAB \\
Y &= BDCABA \\
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\end{aligned}
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	j	0	1	2	3	4	5	6
i		y_j	B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0						
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1	A	0	0	0	0	1	1	1
2	B	0	1	1	1	1	2	2
3	C	0	1	1	2	2	2	2
4	B	0	1					
5	D	0						
6	A	0						
7	B	0						

$$X = ABCBDAB$$
$$Y = BDCABA$$
$$c(i,j) = \begin{cases} 0 & i = 0 \text{ or } j = 0 \\ 1 + c(i-1, j-1) & i \geq 1, j \geq 1 \text{ and } x_i = y_j \\ \max\{c(i, j-1), c(i-1, j)\} & i \geq 1, j \geq 1 \text{ and } x_i \neq y_j \end{cases}$$

	j	0	1	2	3	4	5	6
i		y_j	B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	0	0	0	1	1	1
2	B	0	1	1	1	1	2	2
3	C	0	1	1	2	2	2	2
4	B	0	1	1				
5	D	0						
6	A	0						
7	B	0						

$$
\begin{aligned}
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\end{aligned}
$$

	j	0	1	2	3	4	5	6
i		y_j	B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	0	0	0	1	1	1
2	B	0	1	1	1	1	2	2
3	C	0	1	1	2	2	2	2
4	B	0	1	1	2			
5	D	0						
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$$

	j	0	1	2	3	4	5	6
i		y_j	B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	0	0	0	1	1	1
2	B	0	1	1	1	1	2	2
3	C	0	1	1	2	2	2	2
4	B	0	1	1	2	2		
5	D	0						
6	A	0						
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	j	0	1	2	3	4	5	6
i		y_j	B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	0	0	0	1	1	1
2	B	0	1	1	1	1	2	2
3	C	0	1	1	2	2	2	2
4	B	0	1	1	2	2	3	
5	D	0						
6	A	0						
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	j	0	1	2	3	4	5	6
i		y_j	B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	0	0	0	1	1	1
2	B	0	1	1	1	1	2	2
3	C	0	1	1	2	2	2	2
4	B	0	1	1	2	2	3	3
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	j	0	1	2	3	4	5	6
i		y_j	B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	0	0	0	1	1	1
2	B	0	1	1	1	1	2	2
3	C	0	1	1	2	2	2	2
4	B	0	1	1	2	2	3	3
5	D	0	1					
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$$

	j	0	1	2	3	4	5	6
i		y_j	B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	0	0	0	1	1	1
2	B	0	1	1	1	1	2	2
3	C	0	1	1	2	2	2	2
4	B	0	1	1	2	2	3	3
5	D	0	1	2				
6	A	0						
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$$

	j	0	1	2	3	4	5	6
i		y_j	B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	0	0	0	1	1	1
2	B	0	1	1	1	1	2	2
3	C	0	1	1	2	2	2	2
4	B	0	1	1	2	2	3	3
5	D	0	1	2	2			
6	A	0						
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	j	0	1	2	3	4	5	6
i		y_j	B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	0	0	0	1	1	1
2	B	0	1	1	1	1	2	2
3	C	0	1	1	2	2	2	2
4	B	0	1	1	2	2	3	3
5	D	0	1	2	2	2		
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	j	0	1	2	3	4	5	6
i		y_j	B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	0	0	0	1	1	1
2	B	0	1	1	1	1	2	2
3	C	0	1	1	2	2	2	2
4	B	0	1	1	2	2	3	3
5	D	0	1	2	2	2	3	
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	j	0	1	2	3	4	5	6
i		y_j	B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	0	0	0	1	1	1
2	B	0	1	1	1	1	2	2
3	C	0	1	1	2	2	2	2
4	B	0	1	1	2	2	3	3
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	j	0	1	2	3	4	5	6
i		y_j	B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	0	0	0	1	1	1
2	B	0	1	1	1	1	2	2
3	C	0	1	1	2	2	2	2
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	j	0	1	2	3	4	5	6
i		y_j	B	D	C	A	B	A
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1	A	0	0	0	0	1	1	1
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	j	0	1	2	3	4	5	6
i		y_j	B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	0	0	0	1	1	1
2	B	0	1	1	1	1	2	2
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4	B	0	1	1	2	2	3	3
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	j	0	1	2	3	4	5	6
i		y_j	B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	0	0	0	1	1	1
2	B	0	1	1	1	1	2	2
3	C	0	1	1	2	2	2	2
4	B	0	1	1	2	2	3	3
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	j	0	1	2	3	4	5	6
i		y_j	B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	0	0	0	1	1	1
2	B	0	1	1	1	1	2	2
3	C	0	1	1	2	2	2	2
4	B	0	1	1	2	2	3	3
5	D	0	1	2	2	2	3	3
6	A	0	1	2	2	3	3	
7	B	0						

$$
\begin{aligned}
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	j	0	1	2	3	4	5	6
i		y_j	B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	0	0	0	1	1	1
2	B	0	1	1	1	1	2	2
3	C	0	1	1	2	2	2	2
4	B	0	1	1	2	2	3	3
5	D	0	1	2	2	2	3	3
6	A	0	1	2	2	3	3	4
7	B	0						

$$
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$$

	j	0	1	2	3	4	5	6
i		y_j	B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	0	0	0	1	1	1
2	B	0	1	1	1	1	2	2
3	C	0	1	1	2	2	2	2
4	B	0	1	1	2	2	3	3
5	D	0	1	2	2	2	3	3
6	A	0	1	2	2	3	3	4
7	B	0	1					

$$\begin{aligned} X &= ABCBDAB \\ Y &= BDCABA \\ c(i,j) &= \begin{cases} 0 & i = 0 \text{ or } j = 0 \\ 1 + c(i-1, j-1) & i \geq 1, j \geq 1 \text{ and } x_i = y_j \\ \max\{c(i, j-1), c(i-1, j)\} & i \geq 1, j \geq 1 \text{ and } x_i \neq y_j \end{cases} \end{aligned}$$

i	j	0	1	2	3	4	5	6
		y_j	B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	0	0	0	1	1	1
2	B	0	1	1	1	1	2	2
3	C	0	1	1	2	2	2	2
4	B	0	1	1	2	2	3	3
5	D	0	1	2	2	2	3	3
6	A	0	1	2	2	3	3	4
7	B	0	1	2				

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	j	0	1	2	3	4	5	6
i		y_j	B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	0	0	0	1	1	1
2	B	0	1	1	1	1	2	2
3	C	0	1	1	2	2	2	2
4	B	0	1	1	2	2	3	3
5	D	0	1	2	2	2	3	3
6	A	0	1	2	2	3	3	4
7	B	0	1	2	2			

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$$

	j	0	1	2	3	4	5	6
i		y_j	B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	0	0	0	1	1	1
2	B	0	1	1	1	1	2	2
3	C	0	1	1	2	2	2	2
4	B	0	1	1	2	2	3	3
5	D	0	1	2	2	2	3	3
6	A	0	1	2	2	3	3	4
7	B	0	1	2	2	3		

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	j	0	1	2	3	4	5	6
i		y_j	B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	0	0	0	1	1	1
2	B	0	1	1	1	1	2	2
3	C	0	1	1	2	2	2	2
4	B	0	1	1	2	2	3	3
5	D	0	1	2	2	2	3	3
6	A	0	1	2	2	3	3	4
7	B	0	1	2	2	3	4	

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$$

	j	0	1	2	3	4	5	6
i		y_j	B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	0	0	0	1	1	1
2	B	0	1	1	1	1	2	2
3	C	0	1	1	2	2	2	2
4	B	0	1	1	2	2	3	3
5	D	0	1	2	2	2	3	3
6	A	0	1	2	2	3	3	4
7	B	0	1	2	2	3	4	4

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$$

	j	0	1	2	3	4	5	6
i		y_j	B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	0	0	0	1	1	1
2	B	0	1	1	1	1	2	2
3	C	0	1	1	2	2	2	2
4	B	0	1	1	2	2	3	3
5	D	0	1	2	2	2	3	3
6	A	0	1	2	2	3	3	4
7	B	0	1	2	2	3	4	4

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$$

	j	0	1	2	3	4	5	6
i		y_j	B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	0	0	0	1	1	1
2	B	0	1	1	1	1	2	2
3	C	0	1	1	2	2	2	2
4	B	0	1	1	2	2	3	3
5	D	0	1	2	2	2	3	3
6	A	0	1	2	2	3	3	4
7	B	0	1	2	2	3	4	4

$$
\begin{aligned}
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0	x_i	0	0	0	0	0	0	0
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2	B	0	1	1	1	1	2	2
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Running time: $O(m\,n)$

Space: $O(m\,n)$

But if we only want to compute $C(m,n)$, we only need the current row and the previous row. Hence,

Space: $O(m+n)$.

§5.4 General Structure of Dynamic Programming

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Do **not** use a recursive algorithm!

A subsequence of a given sequence is *palindromic* if it is the same whether it is read from left to right, or from right to left. For instance, the sequence

$$A, C, G, T, G, T, C, A, A, A, A, T, C, G$$

has many palindromic subsequences, including A, C, G, C, A and A, A, A, A. On the other hand, the subsequence A, C, T is not palindromic.

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Question: Design a deterministic algorithm to solve the following problem **using dynamic programming**.

input: A sequence $X = (x_1, x_2, ..., x_n)$.

output: A longest palindromic subsequence of X.

What do you think of the following solution?

- Let $Reverse(X)$ be the sequence X in reverse order.
- Return $LCS(X, Reverse(X))$.

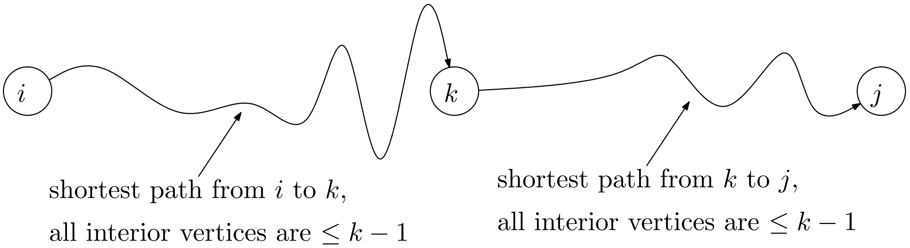
§5.5 All-Pairs Shortest Paths

Let $G = (V, E)$ be a directed graph, where $V = \{1, 2, ..., n\}$. Each edge (i, j) has a weight $wt(i, j) > 0$.

For all i and j, compute the weight of a shortest path in G from i to j, which we denote by $\delta_G(i, j)$.

Step 1: Structure of the Optimal Solution

Consider the shortest path from i to j, and assume this path has at least one interior vertex. Let k be the largest interior vertex.



Step 2: Set Up a recurrence for the Optimal Solution

For

$$1 \le i \le n \qquad 1 \le j \le n \qquad 0 \le k \le n,$$

let $dist(i, j, k)$ be the length of a shortest path from i to j, all of whose interior vertices are $\le k$.

Step 2: Set Up a recurrence for the Optimal Solution

For

$$1 \le i \le n \qquad 1 \le j \le n \qquad 0 \le k \le n,$$

let $dist(i, j, k)$ be the length of a shortest path from i to j, all of whose interior vertices are $\le k$.

We want to compute

$$dist(i, j, n) = \delta_G(i, j)$$

for all $1 \le i \le n$, $1 \le j \le n$.

Recurrence:

- For $1 \leq i \leq n$, $dist(i, i, 0) = 0$.

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- For $1 \leq i \leq n$, $dist(i, i, 0) = 0$.
- For $1 \leq i \leq n$, $1 \leq j \leq n$, $i \neq j$,

$$dist(i, j, 0) = \begin{cases} wt(i, j) & \text{if } (i, j) \text{ is an edge,} \\ \infty & \text{otherwise.} \end{cases}$$

Recurrence:

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- For $1 \leq i \leq n$, $1 \leq j \leq n$, $1 \leq k \leq n$,

$$dist(i, j, k) = \min \{dist(i, j, k-1), dist(i, k, k-1) + dist(k, j, k-1)\} .$$

Step 3: Solve the Recurrence Bottom-Up

Algorithm Floyd-Warshall

```
for i = 1 to n do
    for j = 1 to n do
        if i = j then
            dist(i, j, 0) = 0
        else
            dist(i, j, 0) = ∞
        end if
    end for
end for
for all edges (i, j) do
    dist(i, j, 0) = wt(i, j)
end for
for k = 1 to n do
    for i = 1 to n do
        for j = 1 to n do
            dist(i, j, k) = min {dist(i, j, k − 1), dist(i, k, k − 1) + dist(k, j, k − 1)}
        end for
    end for
end for
```

Running time: $O(n^3)$