CSI - 3105 Design & Analysis of Algorithms
Course 14

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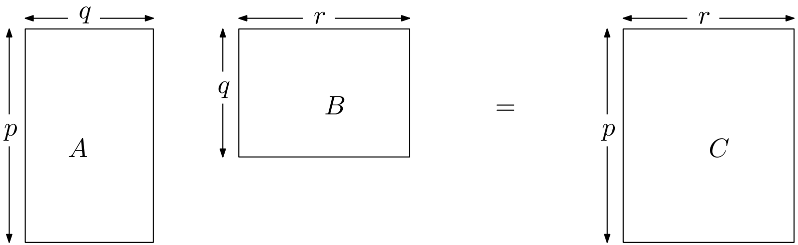
Fall 2019

Matrix Chain Multiplication

$$A : p \times q \text{ matrix}$$
$$B : q \times r \text{ matrix}$$
$$C = A\,B : p \times r \text{ matrix}$$



C has pr entries, each of which can be computed in $O(q)$ time. So C can be computed in $O(pqr)$ time. We define the *cost* of multiplying A and B to be pqr.

Consider 3 matrices

$$A_1 : 10 \times 100$$
$$A_2 : 100 \times 5$$
$$A_3 : 5 \times 50$$

How to compute $A_1 A_2 A_3$:

Consider 3 matrices

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$$A_2 : 100 \times 5$$
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How to compute $A_1 A_2 A_3$:

-
    - Compute $A_1 A_2$, $cost = 10 \times 100 \times 5 = 5000$

Consider 3 matrices

$$A_1 : 10 \times 100$$
$$A_2 : 100 \times 5$$
$$A_3 : 5 \times 50$$

How to compute $A_1 A_2 A_3$:

-
    - Compute $A_1 A_2$, $cost = 10 \times 100 \times 5 = 5000$
    - Compute $(A_1 A_2) A_3$, $cost = 10 \times 5 \times 50 = 2500$.

Consider 3 matrices

$$A_1 : 10 \times 100$$
$$A_2 : 100 \times 5$$
$$A_3 : 5 \times 50$$

How to compute $A_1 A_2 A_3$:

-
  - Compute $A_1 A_2$, $cost = 10 \times 100 \times 5 = 5000$
  - Compute $(A_1 A_2) A_3$, $cost = 10 \times 5 \times 50 = 2500$.

  For a total cost of $5000 + 2500 = 7500$.

Consider 3 matrices

$$A_1 : 10 \times 100$$
$$A_2 : 100 \times 5$$
$$A_3 : 5 \times 50$$

How to compute $A_1 A_2 A_3$:

-
    - Compute $A_1 A_2$, $cost = 10 \times 100 \times 5 = 5000$
    - Compute $(A_1 A_2) A_3$, $cost = 10 \times 5 \times 50 = 2500$.

  For a total cost of $5000 + 2500 = 7500$.
-
    - Compute $A_2 A_3$, $cost = 100 \times 5 \times 50 = 25000$

Consider 3 matrices

$$A_1 : 10 \times 100$$
$$A_2 : 100 \times 5$$
$$A_3 : 5 \times 50$$

How to compute $A_1A_2A_3$:

-
  - Compute A_1A_2, $cost = 10 \times 100 \times 5 = 5000$
  - Compute $(A_1A_2)A_3$, $cost = 10 \times 5 \times 50 = 2500$.

  For a total cost of $5000 + 2500 = 7500$.
-
  - Compute A_2A_3, $cost = 100 \times 5 \times 50 = 25000$
  - Compute $A_1(A_2A_3)$, $cost = 10 \times 100 \times 50 = 50000$.

Consider 3 matrices

$$A_1 : 10 \times 100$$
$$A_2 : 100 \times 5$$
$$A_3 : 5 \times 50$$

How to compute $A_1 A_2 A_3$:

-
  - Compute $A_1 A_2$, $cost = 10 \times 100 \times 5 = 5000$
  - Compute $(A_1 A_2) A_3$, $cost = 10 \times 5 \times 50 = 2500$.

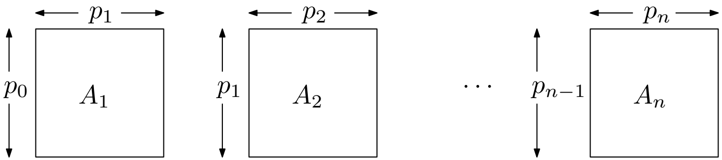
  For a total cost of $5000 + 2500 = 7500$.
-
  - Compute $A_2 A_3$, $cost = 100 \times 5 \times 50 = 25000$
  - Compute $A_1 (A_2 A_3)$, $cost = 10 \times 100 \times 50 = 50000$.

  For a total cost of $25000 + 50000 = 75000$.

Which one is better?

In general,

- $p_0, p_1, \ldots, p_n$: positive integers
- $A_1 A_2, \ldots, A_n$: matrices such that A_i has p_{i-1} rows and p_i columns.



Compute the best order to compute $A_1 A_2 \cdot \ldots \cdot A_n$, i.e., minimize the total cost.

Step 1: Structure of the Optimal Solution

Consider the best order to compute $A_i A_{i+1} \cdot \ldots \cdot A_j$. In the **last** step, we multiply

$$\underbrace{(A_i \cdot \ldots \cdot A_k)}_{\text{already computed}} \; \underbrace{(A_{k+1} \cdot \ldots \cdot A_j)}_{\text{already computed}}$$

for some k such that $i \leq k \leq j - 1$.

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How did we compute $A_i \cdot \ldots \cdot A_k$?

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How did we compute $A_i \cdot \ldots \cdot A_k$? In the best order.

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How did we compute $A_i \cdot \ldots \cdot A_k$? In the best order.
How did we compute $A_{k+1} \cdot \ldots \cdot A_j$? In the best order!

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How did we compute $A_i \cdot \ldots \cdot A_k$? In the best order.
How did we compute $A_{k+1} \cdot \ldots \cdot A_j$? In the best order!

minimum cost to compute $A_i \cdot \ldots \cdot A_j$

$=$

minimum cost to compute $A_i \cdot \ldots \cdot A_k$

$+$

minimum cost to compute $A_{k+1} \cdot \ldots \cdot A_j$

$+$

$p_{i-1} p_k p_j$

Step 2: Set Up a Recurrence for the Optimal Solution

For $1 \leq i \leq j \leq n$, define

$$m(i,j) = \text{minimum cost to compute } A_i \cdot \ldots \cdot A_j.$$

We want to compute $m(1,n)$.

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If we know k, then

$$m(i,j) = m(i,k) + m(k+1,j) + p_{i-1}p_kp_j.$$

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But we do not know k, so try all values of k, $i \leq k \leq j-1$ and take the best one.

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Recurrence:

- For $1 \leq i \leq n$: $m(i,i) = 0$.

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If we know k, then

$$m(i,j) = m(i,k) + m(k+1,j) + p_{i-1}p_kp_j.$$

But we do not know k, so try all values of k, $i \leq k \leq j-1$ and take the best one.

Recurrence:

- For $1 \leq i \leq n$: $m(i,i) = 0$.
- For $1 \leq i < j \leq n$:

$$m(i,j) = \min_{i \leq k \leq j-1} \{m(i,k) + m(k+1,j) + p_{i-1}p_kp_j\}$$

Step 3: Solve the Recurrence Bottom-Up

Compute, in this order,

$$\begin{array}{l}
m(1,1), m(2,2), \ldots, m(n,n) \\
m(1,2), m(2,3), \ldots, m(n-1,n) \\
m(1,3), m(2,4), \ldots, m(n-2,n) \\
m(1,4), m(2,5), \ldots, m(n-3,n) \\
\qquad\qquad \vdots \\
m(1,n-1), m(2,n) \\
m(1,n)
\end{array}$$

Algorithm

Algorithm Matrix Chain Multiplication

for $i = 1$ to n **do**
  $m(i,i) = 0$
end for
for $\ell = 2$ to n **do**
  // Compute $m(1,\ell), m(2,\ell+1), \ldots, m(n-\ell+1, n)$
  for $i = 1$ to $n-\ell+1$ **do**
    // Compute $m(i, i+\ell-1)$
    $j = i + \ell - 1$
    // Compute $m(i,j)$ using the recurrence
    $m(i,j) = \infty$
    for $k = i$ to $j-1$ **do**
      $m(i,j) = \min\{m(i,j), m(i,k) + m(k+1,j) + p_{i-1}p_k p_j\}$
    end for
  end for
end for
return $m(1,n)$

Example

Matrices	A_1	A_2	A_3	A_4	A_5	A_6
Dimensions	30×35	35×15	15×5	5×10	10×20	20×25
	$p_0 \times p_1$	$p_1 \times p_2$	$p_2 \times p_3$	$p_3 \times p_4$	$p_4 \times p_5$	$p_5 \times p_6$

$$m(i,j) = \begin{cases} 0 & i = j \\ \min_{i \leq k < j} \{m(i,k) + m(k+1,j) + p_{i-1} p_k p_j\} & 1 \leq i < j \leq 6 \end{cases}$$

$i \diagdown j$	1	2	3	4	5	6
1	0					
2		0				
3			0			
4				0		
5					0	
6						0

Example

Matrices	A_1	A_2	A_3	A_4	A_5	A_6
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$i \setminus j$	1	2	3	4	5	6
1	0	15750				
2		0				
3			0			
4				0		
5					0	
6						0

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Matrices	A_1	A_2	A_3	A_4	A_5	A_6
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$i \diagdown j$	1	2	3	4	5	6
1	0	15750				
2		0	2625			
3			0	750		
4				0		
5					0	
6						0

Example

Matrices	A_1	A_2	A_3	A_4	A_5	A_6
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$i \diagdown j$	1	2	3	4	5	6
1	0	15750	7875			
2		0	2625			
3			0	750		
4				0	1000	
5					0	5000
6						0

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$i \diagdown j$	1	2	3	4	5	6
1	0	15750	7875			
2		0	2625	4375		
3			0	750		
4				0	1000	
5					0	5000
6						0

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4				0	1000	3500
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1	0	15750	7875	9375		
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3			0	750	2500	
4				0	1000	3500
5					0	5000
6						0

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1	0	15750	7875	9375		
2		0	2625	4375	7125	
3			0	750	2500	
4				0	1000	3500
5					0	5000
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$i \diagdown j$	1	2	3	4	5	6
1	0	15750	7875	9375	11875	
2		0	2625	4375	7125	
3			0	750	2500	5375
4				0	1000	3500
5					0	5000
6						0

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2		0	2625	4375	7125	10500
3			0	750	2500	5375
4				0	1000	3500
5					0	5000
6						0

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2		0	2625	4375	7125	10500
3			0	750	2500	5375
4				0	1000	3500
5					0	5000
6						0

Example

Matrices	A_1	A_2	A_3	A_4	A_5	A_6
Dimensions	30×35	35×15	15×5	5×10	10×20	20×25
	$p_0 \times p_1$	$p_1 \times p_2$	$p_2 \times p_3$	$p_3 \times p_4$	$p_4 \times p_5$	$p_5 \times p_6$

$$m(i,j) = \begin{cases} 0 & i = j \\ \min\limits_{i \le k < j} \{m(i,k) + m(k+1,j) + p_{i-1}p_kp_j\} & 1 \le i < j \le 6 \end{cases}$$

$i \diagdown j$	1	2	3	4	5	6
1	0					
2		0				
3			0			
4				0		
5					0	
6						0

$i \diagdown j$	1	2	3	4	5	6
1	0					
2		0				
3			0			
4				0		
5					0	
6						0

Example

Matrices	A_1	A_2	A_3	A_4	A_5	A_6
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$i \diagdown j$	1	2	3	4	5	6
1	0	15750				
2		0				
3			0			
4				0		
5					0	
6						0

$i \diagdown j$	1	2	3	4	5	6
1	0	1				
2		0				
3			0			
4				0		
5					0	
6						0

Example

Matrices	A_1	A_2	A_3	A_4	A_5	A_6
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$i \diagdown j$	1	2	3	4	5	6
1	0	15750				
2		0	2625			
3			0			
4				0		
5					0	
6						0

$i \diagdown j$	1	2	3	4	5	6
1	0	1				
2		0	2			
3			0			
4				0		
5					0	
6						0

Example

Matrices	A_1	A_2	A_3	A_4	A_5	A_6
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$$m(i,j) = \begin{cases} 0 & i = j \\ \min\limits_{i \leq k < j} \{m(i,k) + m(k+1,j) + p_{i-1}p_kp_j\} & 1 \leq i < j \leq 6 \end{cases}$$

$i \diagdown j$	1	2	3	4	5	6
1	0	15750				
2		0	2625			
3			0	750		
4				0		
5					0	
6						0

$i \diagdown j$	1	2	3	4	5	6
1	0	1				
2		0	2			
3			0	3		
4				0		
5					0	
6						0

Example

Matrices	A_1	A_2	A_3	A_4	A_5	A_6
Dimensions	30×35	35×15	15×5	5×10	10×20	20×25
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$$m(i,j) = \begin{cases} 0 & i = j \\ \min\limits_{i \leq k < j} \{m(i,k) + m(k+1,j) + p_{i-1}p_kp_j\} & 1 \leq i < j \leq 6 \end{cases}$$

$i \setminus j$	1	2	3	4	5	6
1	0	15750				
2		0	2625			
3			0	750		
4				0	1000	
5					0	
6						0

$i \setminus j$	1	2	3	4	5	6
1	0	1				
2		0	2			
3			0	3		
4				0	4	
5					0	
6						0

Example

Matrices	A_1	A_2	A_3	A_4	A_5	A_6
Dimensions	30×35	35×15	15×5	5×10	10×20	20×25
	$p_0 \times p_1$	$p_1 \times p_2$	$p_2 \times p_3$	$p_3 \times p_4$	$p_4 \times p_5$	$p_5 \times p_6$

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$i \diagdown j$	1	2	3	4	5	6
1	0	15750				
2		0	2625			
3			0	750		
4				0	1000	
5					0	5000
6						0

$i \diagdown j$	1	2	3	4	5	6
1	0	1				
2		0	2			
3			0	3		
4				0	4	
5					0	5
6						0

Example

Matrices	A_1	A_2	A_3	A_4	A_5	A_6
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$i \diagdown j$	1	2	3	4	5	6
1	0	15750	7875			
2		0	2625			
3			0	750		
4				0	1000	
5					0	5000
6						0

$i \diagdown j$	1	2	3	4	5	6
1	0	1	1			
2		0	2			
3			0	3		
4				0	4	
5					0	5
6						0

Example

Matrices	A_1	A_2	A_3	A_4	A_5	A_6
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$i \diagdown j$	1	2	3	4	5	6
1	0	15750	7875			
2		0	2625	4375		
3			0	750		
4				0	1000	
5					0	5000
6						0

$i \diagdown j$	1	2	3	4	5	6
1	0	1	1			
2		0	2	3		
3			0	3		
4				0	4	
5					0	5
6						0

Example

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Dimensions	30×35	35×15	15×5	5×10	10×20	20×25
	$p_0 \times p_1$	$p_1 \times p_2$	$p_2 \times p_3$	$p_3 \times p_4$	$p_4 \times p_5$	$p_5 \times p_6$

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$i \setminus j$	1	2	3	4	5	6
1	0	15750	7875			
2		0	2625	4375		
3			0	750	2500	
4				0	1000	
5					0	5000
6						0

$i \setminus j$	1	2	3	4	5	6
1	0	1	1			
2		0	2	3		
3			0	3	3	
4				0	4	
5					0	5
6						0

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$i \diagdown j$	1	2	3	4	5	6
1	0	15750	7875			
2		0	2625	4375		
3			0	750	2500	
4				0	1000	3500
5					0	5000
6						0

$i \diagdown j$	1	2	3	4	5	6
1	0	1	1			
2		0	2	3		
3			0	3	3	
4				0	4	5
5					0	5
6						0

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$i \diagdown j$	1	2	3	4	5	6
1	0	15750	7875	9375		
2		0	2625	4375		
3			0	750	2500	
4				0	1000	3500
5					0	5000
6						0

$i \diagdown j$	1	2	3	4	5	6
1	0	1	1	3		
2		0	2	3		
3			0	3	3	
4				0	4	5
5					0	5
6						0

Example

Matrices	A_1	A_2	A_3	A_4	A_5	A_6
Dimensions	30×35	35×15	15×5	5×10	10×20	20×25
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$i \setminus j$	1	2	3	4	5	6
1	0	15750	7875	9375		
2		0	2625	4375	7125	
3			0	750	2500	
4				0	1000	3500
5					0	5000
6						0

$i \setminus j$	1	2	3	4	5	6
1	0	1	1	3		
2		0	2	3	3	
3			0	3	3	
4				0	4	5
5					0	5
6						0

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Matrices	A_1	A_2	A_3	A_4	A_5	A_6
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$i \diagdown j$	1	2	3	4	5	6
1	0	15750	7875	9375		
2		0	2625	4375	7125	
3			0	750	2500	5375
4				0	1000	3500
5					0	5000
6						0

$i \diagdown j$	1	2	3	4	5	6
1	0	1	1	3		
2		0	2	3	3	
3			0	3	3	3
4				0	4	5
5					0	5
6						0

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$i \diagdown j$	1	2	3	4	5	6
1	0	15750	7875	9375	11875	
2		0	2625	4375	7125	
3			0	750	2500	5375
4				0	1000	3500
5					0	5000
6						0

$i \diagdown j$	1	2	3	4	5	6
1	0	1	1	3	3	
2		0	2	3	3	
3			0	3	3	3
4				0	4	5
5					0	5
6						0

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$i \diagdown j$	1	2	3	4	5	6
1	0	15750	7875	9375	11875	
2		0	2625	4375	7125	10500
3			0	750	2500	5375
4				0	1000	3500
5					0	5000
6						0

$i \diagdown j$	1	2	3	4	5	6
1	0	1	1	3	3	
2		0	2	3	3	3
3			0	3	3	3
4				0	4	5
5					0	5
6						0

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$i \diagdown j$	1	2	3	4	5	6
1	0	15750	7875	9375	11875	15125
2		0	2625	4375	7125	10500
3			0	750	2500	5375
4				0	1000	3500
5					0	5000
6						0

$i \diagdown j$	1	2	3	4	5	6
1	0	1	1	3	3	3
2		0	2	3	3	3
3			0	3	3	3
4				0	4	5
5					0	5
6						0

Running Time

There are 3 nested loops, so this algorithm takes $O(n^3)$ time.

More careful counting:

- ℓ: 2 to n
- For each ℓ, we have i: 1 to $n - \ell + 1$
- For each i, we have k: i to $i + \ell - 2$

Running Time

Total time:

$$\sum_{\ell=2}^{n} \sum_{i=1}^{n-\ell+1} \sum_{k=i}^{i+\ell-2} 1$$

Running Time

Total time:

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$$= \sum_{\ell=2}^{n} \sum_{i=1}^{n-\ell+1} (\ell - 1)$$

Running Time

Total time:

$$\sum_{\ell=2}^{n} \sum_{i=1}^{n-\ell+1} \sum_{k=i}^{i+\ell-2} 1$$

$$= \sum_{\ell=2}^{n} \sum_{i=1}^{n-\ell+1} (\ell - 1)$$

$$= \sum_{\ell=2}^{n} (n - \ell + 1)(\ell - 1)$$

Running Time

Total time:

$$
\begin{aligned}
&\sum_{\ell=2}^{n} \sum_{i=1}^{n-\ell+1} \sum_{k=i}^{i+\ell-2} 1 \\
&= \sum_{\ell=2}^{n} \sum_{i=1}^{n-\ell+1} (\ell - 1) \\
&= \sum_{\ell=2}^{n} (n-\ell+1)(\ell-1) \\
&= \sum_{\ell=1}^{n} (n-\ell+1)(\ell-1) \qquad \text{since the summand is 0 when } \ell = 0
\end{aligned}
$$

Running Time

Total time:

$$\begin{aligned}
&\sum_{\ell=2}^{n} \sum_{i=1}^{n-\ell+1} \sum_{k=i}^{i+\ell-2} 1 \\
&= \sum_{\ell=2}^{n} \sum_{i=1}^{n-\ell+1} (\ell - 1) \\
&= \sum_{\ell=2}^{n} (n - \ell + 1)(\ell - 1) \\
&= \sum_{\ell=1}^{n} (n - \ell + 1)(\ell - 1) \qquad \text{since the summand is 0 when } \ell = 0 \\
&= \frac{n^3 - n}{6} \\
&= \Theta(n^3).
\end{aligned}$$

Longest Common Subsequence

We have two sequences:

$$X = (a, b, c, b, d, a, b)$$
$$Y = (b, d, c, a, b, a).$$

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$$X = (a, \underline{b}, \underline{c}, b, \underline{d}, a, \underline{b})$$
$$Y = (b, d, c, a, b, a).$$

The sequence $Z = (b, c, d, b)$ is a subsequence of X, but not of Y.

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We have two sequences:

$$X = (a, b, c, b, d, a, b)$$
$$Y = (b, d, c, a, b, a).$$

The sequence $Z = (b, c, d, b)$ is a subsequence of X, but not of Y.

$LCS(X, Y)$ is the longest common subsequence of X and Y:

$$(b, c, b, a) \qquad \text{or} \qquad (b, d, a, b).$$

Both have length 4.

Problem:

Input : Sequences

- $X = (x_1, x_2, ..., x_m)$
- $Y = (y_1, y_2, ..., y_n)$

Output : A longest sequence that is a subsequence of X and a subsequence of Y, i.e., $Z = LCS(X, Y)$.

Step 1: Structure of the Optimal Solution

$$X = (x_1, x_2, ..., x_m) \qquad\qquad Y = (y_1, y_2, ..., y_n)$$

Consider

$$Z = (z_1, z_2, ..., z_k) = LCS(X, Y)$$

Step 1: Structure of the Optimal Solution

$$X = (x_1, x_2, ..., x_m) \qquad Y = (y_1, y_2, ..., y_n)$$

Consider

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Case 1: $x_m = y_n$

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Consider

$$Z = (z_1, z_2, ..., z_k) = LCS(X, Y)$$

Case 1: $x_m = y_n$

Then $z_k = x_m = y_n$ and

$$(z_1, z_2, ..., z_{k-1}) = LCS(x_1 x_2 \cdots x_{m-1}, y_1 y_2 \cdots y_{n-1})$$

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$$X = (x_1, x_2, ..., x_m) \qquad Y = (y_1, y_2, ..., y_n)$$

Consider

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Case 1: $x_m = y_n$

Then $z_k = x_m = y_n$ and

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Case 2: $x_m \neq y_n$

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Case 1: $x_m = y_n$
Then $z_k = x_m = y_n$ and

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Case 2: $x_m \neq y_n$
Then $z_k \neq x_m$ or $z_k \neq y_n$ (or both).

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Case 1: $x_m = y_n$

Then $z_k = x_m = y_n$ and

$$(z_1, z_2, ..., z_{k-1}) = LCS(x_1 x_2 \cdots x_{m-1}, y_1 y_2 \cdots y_{n-1})$$

Case 2: $x_m \neq y_n$

Then $z_k \neq x_m$ or $z_k \neq y_n$ (or both).

Case 2 a): $z_k \neq x_m$

$$(z_1, z_2, ..., z_k) = LCS(x_1 x_2 \cdots x_{m-1}, y_1 y_2 \cdots y_n)$$

Step 1: Structure of the Optimal Solution

$$X = (x_1, x_2, ..., x_m) \qquad Y = (y_1, y_2, ..., y_n)$$

Consider

$$Z = (z_1, z_2, ..., z_k) = LCS(X, Y)$$

Case 1: $x_m = y_n$
Then $z_k = x_m = y_n$ and

$$(z_1, z_2, ..., z_{k-1}) = LCS(x_1 x_2 \cdots x_{m-1}, y_1 y_2 \cdots y_{n-1})$$

Case 2: $x_m \neq y_n$
Then $z_k \neq x_m$ or $z_k \neq y_n$ (or both).

Case 2 a): $z_k \neq x_m$

$$(z_1, z_2, ..., z_k) = LCS(x_1 x_2 \cdots x_{m-1}, y_1 y_2 \cdots y_n)$$

Case 2 b): $z_k \neq y_n$

$$(z_1, z_2, ..., z_k) = LCS(x_1 x_2 \cdots x_m, y_1 y_2 \cdots y_{n-1})$$

But we do not know if we are in Case 2 a) or 2 b).

If $x_m \neq y_n$, then $(z_1, z_2, ..., z_k)$ is the longest of

$$LCS(x_1 x_2 \cdots x_{m-1}, y_1 y_2 \cdots y_n)$$

and

$$LCS(x_1 x_2 \cdots x_m, y_1 y_2 \cdots y_{n-1}).$$

Step 2: Set Up a recurrence for the Optimal Solution

For $0 \leq i \leq m$ and $0 \leq j \leq n$, define

$$c(i,j) = \text{length of } LCS(x_1x_2\cdots x_i, y_1y_2\cdots y_j).$$

We want to compute $c(m,n)$.

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- If $i \geq 1$, $j \geq 1$ and $x_i \neq y_j$,

$$c(i, j) = \max\{c(i - 1, j), c(i, j - 1)\}$$

Step 3: Solve the Recurrence Bottom-Up

Fill in the matrix $c(i,j)$ for $0 \leq i \leq m$ and $0 \leq j \leq n$.

First row:

$$c(0,0) = c(0,1) = \ldots = c(0,n) = 0$$

First column:

$$c(0,0) = c(1,0) = \ldots = c(m,0) = 0$$

Then fill in the matrix, row by row, in each row from left to right.

Algorithm

Algorithm Longest Common Subsequence

```
for i = 0 to m do
    c(i, 0) = 0
end for
for j = 0 to n do
    c(0, j) = 0
end for
for i = 1 to m do
    for j = 1 to n do
        if x_i = y_j then
            c(i, j) = 1 + c(i − 1, j − 1)
        else
            c(i, j) = max {c(i − 1, j), c(i, j − 1)}
        end if
    end for
end for
return c(m, n)
```

Example