CSI - 3105 Design & Analysis of Algorithms
Course 16

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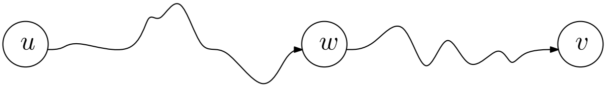
Fall 2019

Final Remark About Dynamic Programming

Consider a directed graph $G = (V, E)$, where all edges have weight 1. Let $u, v \in V$ be two vertices.

Assume we know a vertex w on the shortest path from u to v.



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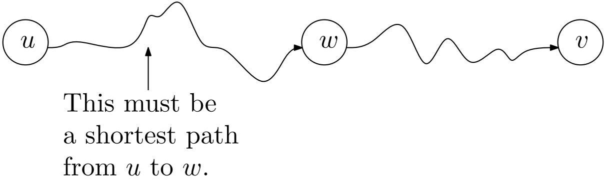

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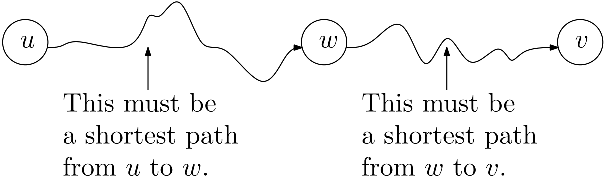

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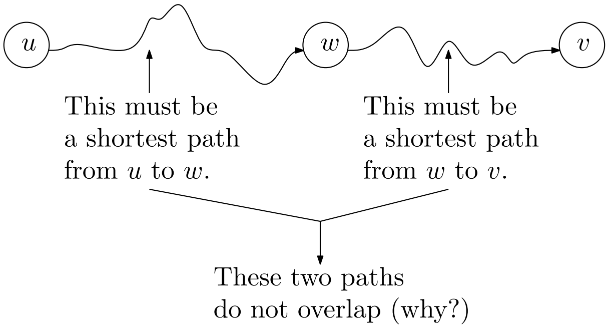

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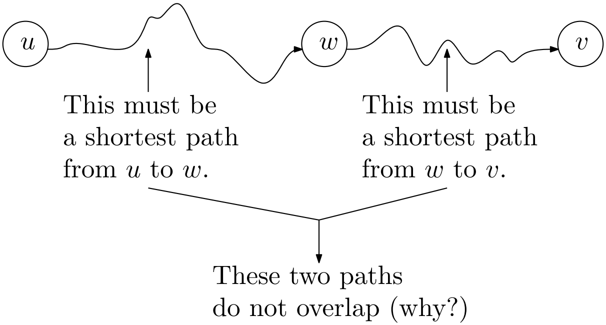

Optimal substructure!

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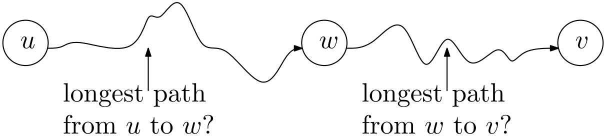

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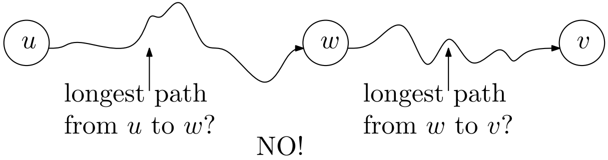

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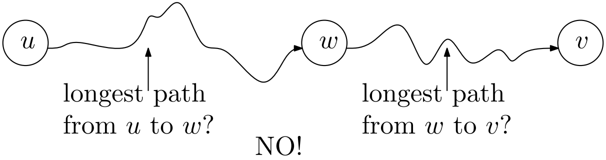
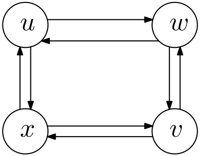


Example:



Longest path from u to v:

$u \longrightarrow w \longrightarrow v$

longest path from u to w? longest path from w to v?

NO!

In fact, computing the longest path is NP-Hard...

Chapter 6: P vs NP

§6.1 Introduction

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Intuitively speaking,

polynomial means good, fast, efficient, easy, ...

exponential means bad, slow, "try all possible solutions", difficult, ...

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$P =$ set of all decision problems that can be solved in polynomial time

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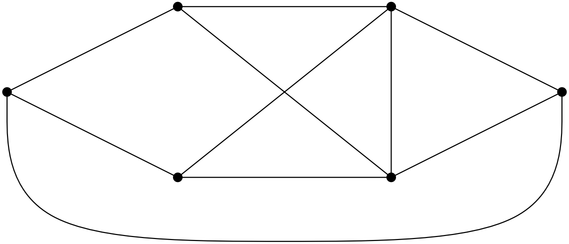
Examples of problems that are in P:

- Is a given input graph connected?
- Is a given input graph bipartite?
- Is a given input sequence sorted?
- Does a given input graph contain an Euler cycle? An *Euler cycle* is a cycle that traverses each edge exactly once.

Other Problems
HAM-CYCLE

input: An undirected graph $G = (V, E)$ stored using adjacency lists.

question: Does G contain a Hamiltonian cycle? A *Hamiltonian cycle* is a cycle that traverses each vertex exactly once.



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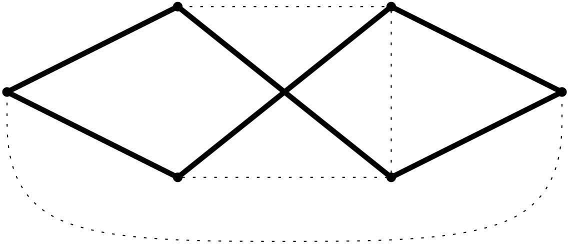

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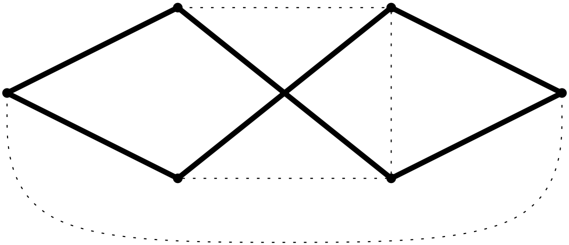

Not known if this problem is in P!

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Traveling Salesman Problem (TSP)

input:

- A complete directed graph $G = (V, E)$, where each edge $(u, v) \in E$ has a weight $wt(u, v) > 0$.
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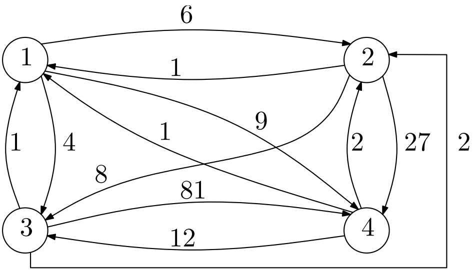

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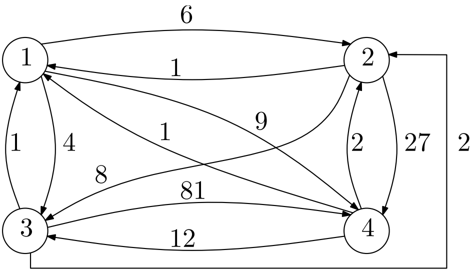

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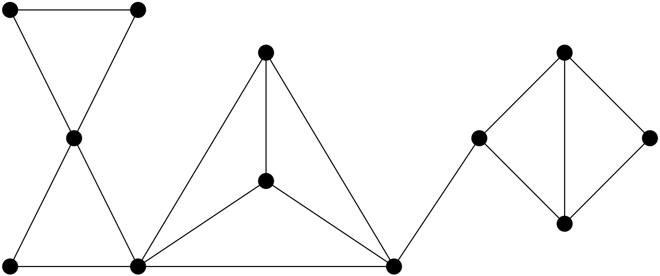

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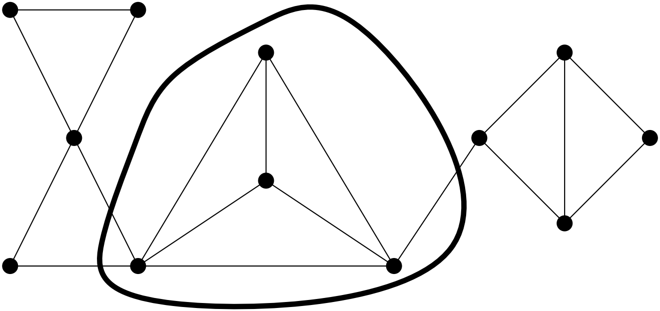

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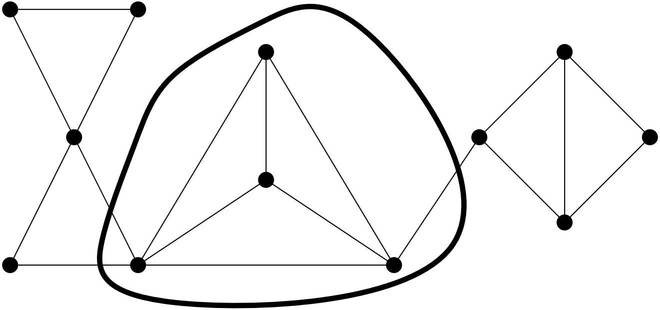

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- Do we have

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- For each $u, v \in V'$ such that $u \neq v$, is $\{u, v\}$ an edge in E?