Chapter 6

——P vs NP——

1. Prove or disprove: the following problem is in P.

   Input: an array $A[1..n]$ of numbers and a number x

   Question: is x an element of A?

2. Prove or disprove: the following problem is in P.

   Input: an array $A[1..n]$ of numbers

   Question: what is the maximum element in A?

3. Prove or disprove: the following problem is in P.

   Input: a connected and directed graph $G = (V, E)$ and an integer k

   Question: is it true that for all pairs of vertices $u, v \in V$, there is a path from u to v whose length is at most k?

4. Prove or disprove: the following problem is in NP.

   Input: an array $A[1..n]$ of numbers and a number x

   Question: is x an element of A?

5. Prove or disprove: the following problem is in NP.

   Input: an array $A[1..n]$ of numbers and two indices $1 \leq i \leq n$ and $1 \leq j \leq n$

   Question: is it true that $A[j] > A[i]$ and $A[j] - A[i]$ is minimum?

6. Prove or disprove: the following problem is in NP.

   Input: a connected and directed graph $G = (V, E)$

   Question: what is the vertex with smallest degree in G?

7. Prove or disprove: if a language L is in P, then it is in NP.

8. Prove or disprove: if a language L is not in NP, then it is in P.

9. Prove that VERTEX-COVER is in NP.

10. Prove that CLIQUE is in NP.

11. Define 2SAT and show that 2SAT is in P.

——Reductions——

12. Prove that $CLIQUE \leq_P INDEP - SET$

13. In class, we showed that $CLIQUE \leq_P VERTEX - COVER$ by providing a function f which transforms any input (G, k) to $CLIQUE$ into an input (G', k') to $VERTEX - COVER$. We showed that the function f satisfies the *famous* condition 2. Show that the function f satisfies conditions 1 and 3.

14. Prove that $VERTEX - COVER \leq_P CLIQUE$.

15. Consider the problem MIN, where

    Input: an array $A[1..n]$ of numbers together with a number x.

    Question: Is x the minimum element in A?

    Consider also the problem MAX, where

    Input: an array $A[1..n]$ of numbers together with a number x.

    Question: Is x the maximum element in A?

    (a) Show that $MIN \leq_P MAX$.
    (b) Show that $MAX \leq_P MIN$.

——Design & analysis of algorithms——

16. Suppose you have a polynomial-time algorithm to solve CLIQUE. Explain how to solve the following problem in polynomial time.

    Input: A graph $G = (V, E)$.

    Question: What is the size of a largest clique in G (we want as many vertices as possible)?

17. Suppose you have a polynomial-time algorithm to solve VERTEX-COVER. Explain how to solve the following problem in polynomial time.

    Input: A graph $G = (V, E)$.

    Question: What is the size of a smallest vertex cover in G (we want as few vertices as possible)?

——NP-Complete——

18. Prove that for any L and L' that are NP-Complete, we have $L \leq_P L'$ and $L' \leq_P L$.

19. Let L' be an NP-Complete problem. Prove that for all $L \in P$ we have $L \leq_P L'$.

20. In this problem, we want to answer the following question: is there a problem in NP that is not in NP-Complete? Consider the problem $WEIRD$, where

    Input: an array $A[1..n]$ of numbers

    Question: is $1 = 1$?

    (a) Show that $WEIRD$ is in P.

    (b) Conclude that $WEIRD$ is in NP.

    (c) Show that $WEIRD$ is not in NP-Complete.

    (d) Can you think of another problem that is in NP but not in NP-Complete?
    Hint: it is a problem very similar to WEIRD

21. Prove that $HAMCYCLE \leq_P TSP$.

22. (a) Find an example of a 3-SAT Boolean formula, with exactly 2 clauses, that is not satisfiable.

    (b) Find an example of a 3-SAT Boolean formula, with exactly 3 variables, that is not satisfiable.