CSI - 3105 Design & Analysis of Algorithms
Course 20

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Fall 2019

Theorem

The relation $\leq_P$ is transitive:

$$L \leq_P L' \quad \text{and} \quad L' \leq_P L'' \quad \Longrightarrow \quad L \leq_P L''$$

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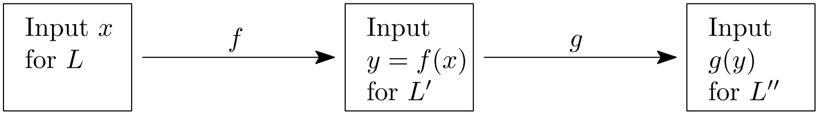

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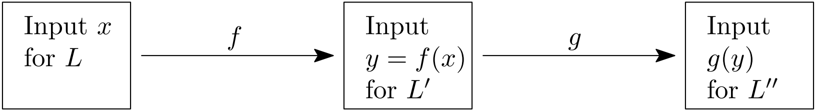

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The reduction from L to L'' is given by the function $g \circ f$. Given x, $(g \circ f)(x) = g(f(x))$ can be computed in time that is polynomial in the length of x (do you see why?)

§6.4 NP-Hard and NP-Complete

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Intuitively, this means that L belongs to the most difficult problems in NP.

This is what we were looking for in §6.2.

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We will show that CIRCUIT-SAT is NP-Complete.

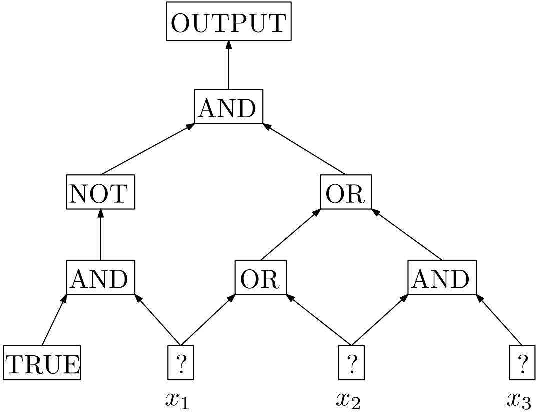
CIRCUIT-SAT

input: A Boolean circuit.

- Directed acyclic graph, where vertices are gates
- AND-gates and OR-gates have indegree 2
- NOT-gates have indegree 1
- Known input gates have indegree 0 and are labeled TRUE or FALSE.
- Unknown input gates have indegree 0 and are labeled "?".
- There is one output gate (whose outdegree is 0).

question: Is it possible to assign a truth-value to each unknown input gate, such that the output of the circuit is TRUE?

CIRCUIT-SAT



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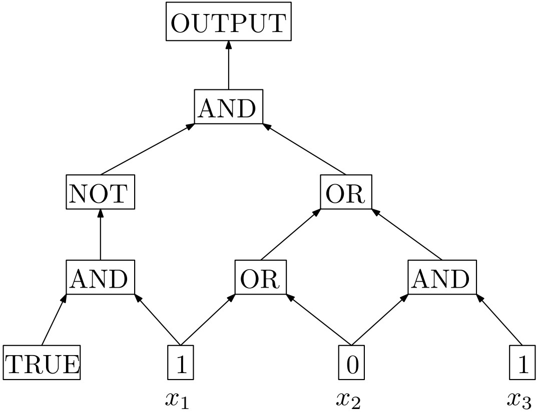

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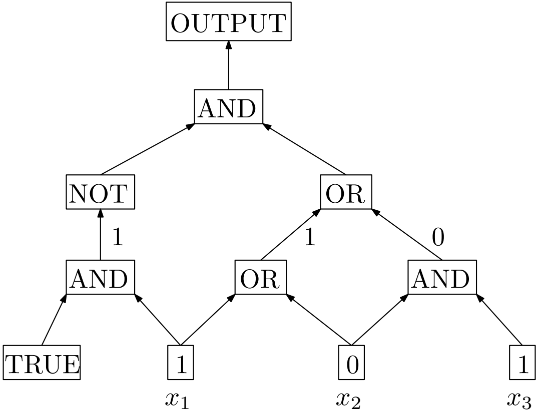

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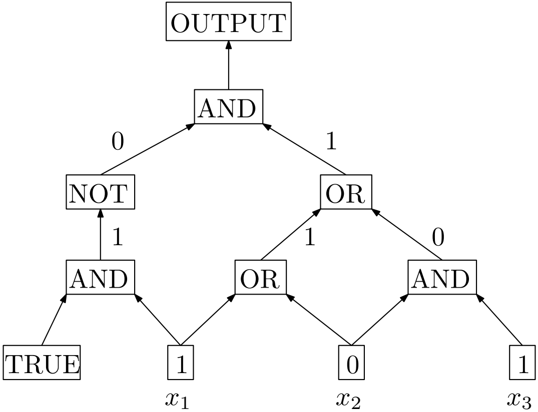

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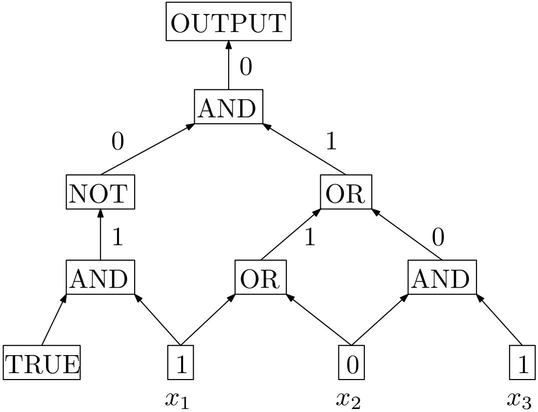

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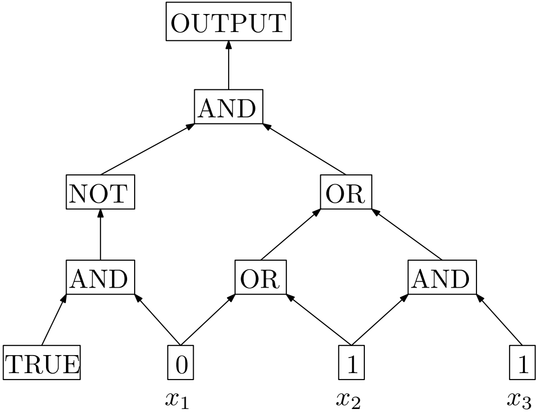

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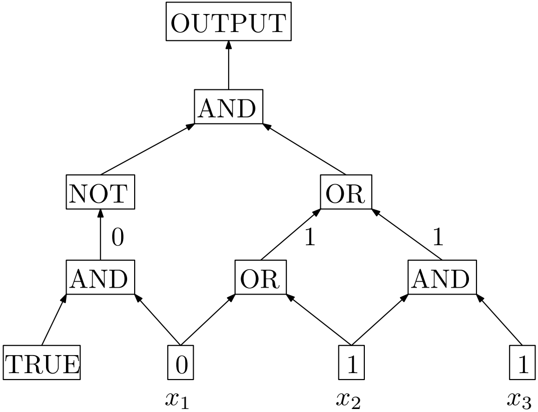

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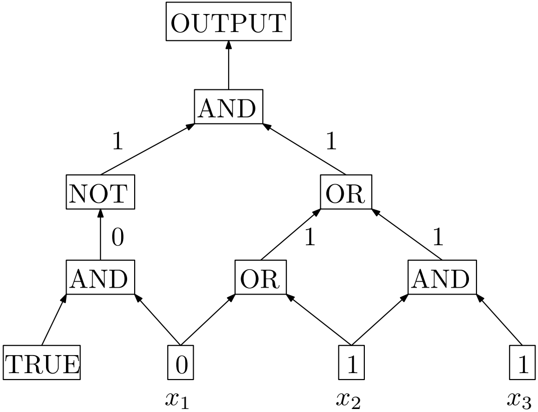

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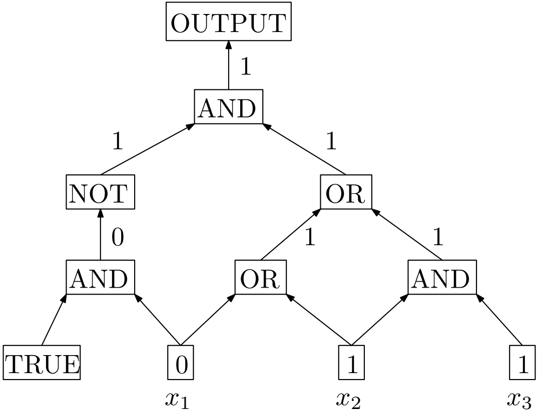

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To show that CIRCUIT-SAT is NP-Complete, we have to show two things:

1. CIRCUIT-SAT is in NP.
2. Show that for all L in NP, $L \leq_P CIRCUIT-SAT$

$$CIRCUIT - SAT = \{B \mid B \text{ is a Boolean circuit for which there exist truth values for the unknown input gates such that the ouput of } B \text{ is TRUE}\}$$

To show that CIRCUIT-SAT is NP-Complete, we have to show two things:

1. CIRCUIT-SAT is in NP.
2. Show that for all L in NP, $L \leq_P CIRCUIT - SAT$

The first item is easy:

CIRCUIT-SAT is in NP:

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Certificate: sequence of truth values for the unknown input gates.

Verification: evaluate the circuit (evaluate the gates in topological order.)

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How do we show that for all L in NP, $L \leq_P CIRCUIT - SAT$?

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2. $x \in L \iff B \in CIRCUIT - SAT$

How do we show that for all L in NP, $L \leq_P CIRCUIT - SAT$?

Let $L \in NP$. We need a function f such that

1. f takes any input x for L and produces an input $B = f(x)$ for $CIRCUIT - SAT$.
2. $x \in L \quad \Longleftrightarrow \quad B \in CIRCUIT - SAT$
3. The time time to compute B is poylnomial in the length of x.

We know that $L \in NP$. So there is a verification algorithm V such that

We know that $L \in NP$. So there is a verification algorithm V such that

- The input to V is (x, y), where x is an input for L and y is a certificate.
- For every input x to L,

$$
\begin{aligned}
x \in L \Longleftrightarrow & \text{ there exists a certificate } y \text{ such that} \\
& \cdot\ |y| \leq |x|^c, \\
& \cdot\ V(x, y) \text{ returns YES} \\
& \cdot\ \text{and the running time of } V(x, y) \text{ is} \\
& \quad \text{at most } |x|^{c'}.
\end{aligned}
$$

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- input is a string y of length at most $|x|^c$
- $V_x(y)$ runs $V(x, y)$
- If $V(x, y)$ terminates in at most $|x|^{c'}$ steps, then $V_x(y)$ terminates and returns the output of $V(x, y)$.
- If $V(x, y)$ has not terminated after $|x|^{c'}$ steps, then $V_x(y)$ terminates and returns NO.

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Observe:

- Running time of Algorithm V_x is at most $|x|^{c'}$.
- $$x \in L \iff \text{there exists an input } y \text{ for Algorithm } V_x \text{ such that } V_x(y) \text{ returns YES}$$

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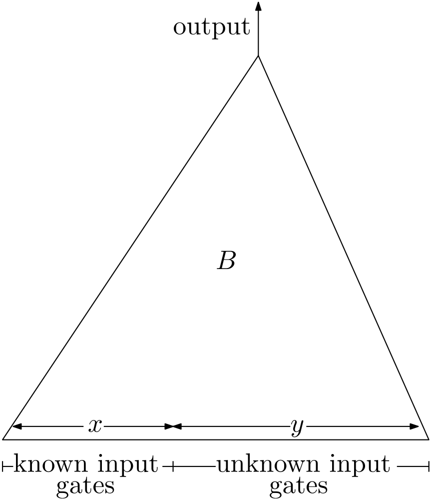

size of B:

$$O\left(|x| + |y| + |x|^{c'}\right)$$

$$= O\left(|x| + |x|^{c} + |x|^{c'}\right)$$

$$= \text{polynomial in } x!$$

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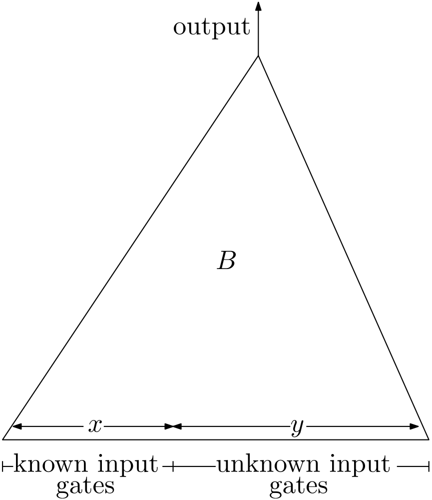

size of B:

$O\left(|x| + |y| + |x|^{c'}\right)$

$= O\left(|x| + |x|^c + |x|^{c'}\right)$

$=$ polynomial in x!

The functions f maps x to B!

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$$x \in L \iff \text{There exists } y \text{ such that } V_x(y) \text{ returns TRUE} \quad \text{(definition of } V_x\text{)}$$

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Conclusion: CIRCUIT-SAT is NP-COMPLETE! □