CSI - 3105 Design & Analysis of Algorithms Course 14

Jean-Lou De Carufel

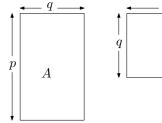
Fall 2019

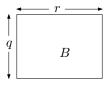
Matrix Chain Multiplication

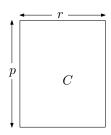
 $A: p \times q \text{ matrix}$

 $B: q \times r \text{ matrix}$

 $C = A B : p \times r \text{ matrix}$







C has pr entries, each of which can be computed in O(q) time. So C can be computed in O(pqr) time. We define the cost of multiplying A and B to be pqr.

 $\textit{A}_1:10\times100$

 $A_2: 100 \times 5$

 $A_3:5\times50$

How to compute $A_1A_2A_3$:

 $\textit{A}_1:10\times100$

 $A_2: 100 \times 5$

 $A_3:5\times50$

How to compute $A_1A_2A_3$:

•

• Compute A_1A_2 , $cost = 10 \times 100 \times 5 = 5000$

 $A_1:10 \times 100$

 $A_2:100\times 5$

 $A_3:5\times50$

How to compute $A_1A_2A_3$:

•

- Compute A_1A_2 , $cost = 10 \times 100 \times 5 = 5000$
- Compute $(A_1A_2)A_3$, $cost = 10 \times 5 \times 50 = 2500$.

 $A_1:10 \times 100$

 $A_2:100\times 5$

 $A_3:5\times50$

How to compute $A_1A_2A_3$:

0

- Compute A_1A_2 , $cost = 10 \times 100 \times 5 = 5000$
- Compute $(A_1A_2)A_3$, $cost = 10 \times 5 \times 50 = 2500$.

For a total cost of 5000 + 2500 = 7500.

 $A_1:10 \times 100$

 $A_2: 100 \times 5$

 $A_3:5\times50$

How to compute $A_1A_2A_3$:

•

- Compute A_1A_2 , $cost = 10 \times 100 \times 5 = 5000$
- Compute $(A_1A_2)A_3$, $cost = 10 \times 5 \times 50 = 2500$.

For a total cost of 5000 + 2500 = 7500.

•

• Compute A_2A_3 , $cost = 100 \times 5 \times 50 = 25000$

 $A_1:10 \times 100$

 $A_2: 100 \times 5$

 $A_3:5\times50$

How to compute $A_1A_2A_3$:

0

- Compute A_1A_2 , $cost = 10 \times 100 \times 5 = 5000$
- Compute $(A_1A_2)A_3$, $cost = 10 \times 5 \times 50 = 2500$.

For a total cost of 5000 + 2500 = 7500.

•

- Compute A_2A_3 , $cost = 100 \times 5 \times 50 = 25000$
- Compute $A_1(A_2A_3)$, $cost = 10 \times 100 \times 50 = 50000$.

 $A_1:10 \times 100$

 $A_2: 100 \times 5$

 $A_3:5\times50$

How to compute $A_1A_2A_3$:

0

- Compute A_1A_2 , $cost = 10 \times 100 \times 5 = 5000$
- Compute $(A_1A_2)A_3$, $cost = 10 \times 5 \times 50 = 2500$.

For a total cost of 5000 + 2500 = 7500.

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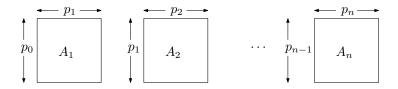
- Compute A_2A_3 , $cost = 100 \times 5 \times 50 = 25000$
- Compute $A_1(A_2A_3)$, $cost = 10 \times 100 \times 50 = 50000$.

For a total cost of 25000 + 50000 = 75000.

Which one is better?

In general,

- $p_0, p_1, ..., p_n$: positive integers
- $A_1A_2,...,A_n$: matrices such that A_i has p_{i-1} rows and p_i columns.



Compute the best order to compute $A_1A_2 \cdot ... \cdot A_n$, i.e., minimize the total cost.

Consider the best order to compute $A_iA_{i+1} \cdot ... \cdot A_j$. In the **last** step, we multiply

$$\underbrace{(A_i \cdot \ldots \cdot A_k)}_{\text{already computed}} \underbrace{(A_{k+1} \cdot \ldots \cdot A_j)}_{\text{already computed}}$$

for some k such that $i \le k \le j-1$.

Consider the best order to compute $A_iA_{i+1} \cdot ... \cdot A_j$. In the **last** step, we multiply

$$\underbrace{(A_i \cdot \ldots \cdot A_k)}_{\text{already computed}} \underbrace{(A_{k+1} \cdot \ldots \cdot A_j)}_{\text{already computed}}$$

for some k such that $i \le k \le j-1$.

How did we compute $A_i \cdot ... \cdot A_k$?

Consider the best order to compute $A_iA_{i+1} \cdot ... \cdot A_j$. In the **last** step, we multiply

$$\underbrace{(A_i \cdot \ldots \cdot A_k)}_{\text{already computed}} \underbrace{(A_{k+1} \cdot \ldots \cdot A_j)}_{\text{already computed}}$$

for some k such that $i \le k \le j-1$.

How did we compute $A_i \cdot ... \cdot A_k$? In the best order.

5 / 20

Consider the best order to compute $A_iA_{i+1} \cdot ... \cdot A_j$. In the **last** step, we multiply

$$\underbrace{(A_i \cdot \ldots \cdot A_k)}_{\text{already computed}} \underbrace{(A_{k+1} \cdot \ldots \cdot A_j)}_{\text{already computed}}$$

for some k such that $i \le k \le j-1$.

How did we compute $A_i \cdot ... \cdot A_k$? In the best order. How did we compute $A_{k+1} \cdot ... \cdot A_i$?

5 / 20

Consider the best order to compute $A_iA_{i+1} \cdot ... \cdot A_j$. In the **last** step, we multiply

$$\underbrace{(A_i \cdot \ldots \cdot A_k)}_{\text{already computed}} \underbrace{(A_{k+1} \cdot \ldots \cdot A_j)}_{\text{already computed}}$$

for some k such that $i \le k \le j-1$.

How did we compute $A_i \cdot ... \cdot A_k$? In the best order. How did we compute $A_{k+1} \cdot ... \cdot A_i$? In the best order!

Consider the best order to compute $A_i A_{i+1} \cdot ... \cdot A_i$. In the **last** step, we multiply

$$\underbrace{(A_i \cdot \ldots \cdot A_k)}_{\text{already computed}} \underbrace{(A_{k+1} \cdot \ldots \cdot A_j)}_{\text{already computed}}$$

for some k such that $i \leq k \leq j-1$.

How did we compute $A_i \cdot ... \cdot A_k$? In the best order. How did we compute $A_{k+1} \cdot ... \cdot A_i$? In the best order!

```
minimum cost to compute A_i \cdot ... \cdot A_i
```

minimum cost to compute
$$A_i \cdot ... \cdot A_k$$

minimum cost to compute
$$A_{k+1} \cdot ... \cdot A_i$$

$$p_{i-1}p_kp_i$$

J.-L. De Carufel (U. of O.)

For $1 \le i \le j \le n$, define

 $m(i,j) = \text{minimum cost to compute } A_i \cdot ... \cdot A_j$.

We want to compute m(1, n).

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We want to compute m(1, n).

If we know k, then

$$m(i,j) = m(i,k) + m(k+1,j) + p_{i-1}p_kp_j.$$

6 / 20

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But we do not know k, so try all values of k, $i \le k \le j-1$ and take the best one.

6 / 20

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$$m(i,j) = m(i,k) + m(k+1,j) + p_{i-1}p_kp_j.$$

But we do not know k, so try all values of k, $i \le k \le j-1$ and take the best one.

Recurrence:

• For $1 \le i \le n$: m(i, i) = 0.

For $1 \le i \le j \le n$, define

$$m(i,j) = \text{minimum cost to compute } A_i \cdot ... \cdot A_j$$
.

We want to compute m(1, n).

If we know k, then

$$m(i,j) = m(i,k) + m(k+1,j) + p_{i-1}p_kp_j.$$

But we do not know k, so try all values of k, $i \le k \le j-1$ and take the best one.

Recurrence:

- For $1 \le i \le n$: m(i, i) = 0.
- For $1 \le i < j \le n$:

$$m(i,j) = \min_{i \le k \le j-1} \{m(i,k) + m(k+1,j) + p_{i-1}p_kp_j\}$$

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Step 3: Solve the Recurrence Bottom-Up

Compute, in this order,

$$m(1,1), m(2,2), ..., m(n, n)$$

 $m(1,2), m(2,3), ..., m(n-1, n)$
 $m(1,3), m(2,4), ..., m(n-2, n)$
 $m(1,4), m(2,5), ..., m(n-3, n)$
 \vdots
 $m(1, n-1), m(2, n)$
 $m(1, n)$

Algorithm

Algorithm Matrix Chain Multiplication

```
1: for i = 1 to n do
      m(i, i) = 0
 3: end for
 4: for \ell = 2 to n do
    // Compute m(1, \ell), m(2, \ell + 1), ..., m(n - \ell + 1, n)
 6:
     for i = 1 to n - \ell + 1 do
 7:
      // Compute m(i, i + \ell - 1)
 8:
      i = i + \ell - 1
       // Compute m(i,j) using the recurrence
10:
    m(i, j) = \infty
11:
          for k = i to i - 1 do
12:
             m(i, j) = \min \{ m(i, j), m(i, k) + m(k + 1, j) + p_{i-1} p_k p_i \}
13:
          end for
14:
       end for
15: end for
16: return m(1, n)
```

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$$m(i,j) = \begin{cases} 0 & i = j \\ \min_{i \le k < j} \{ m(i,k) + m(k+1,j) + p_{i-1}p_kp_j \} & 1 \le i < j \le 6 \end{cases}$$

-	i∖j	1	2	3	4	5	6
	1	0	15750				
	2		0				
	3			0			
	4				0		
	5					0	
	6						0

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$$m(i,j) = \begin{cases} 0 & i = j \\ \min_{i \le k < j} \{ m(i,k) + m(k+1,j) + p_{i-1}p_kp_j \} & 1 \le i < j \le 6 \end{cases}$$

i\	$j \mid$	1	2	3	4	5	6
1		0	15750				
2	l		0	2625			
3				0			
4					0		
5	ı					0	
6	l						0

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$$m(i,j) = \begin{cases} 0 & i = j \\ \min_{i \le k < j} \{ m(i,k) + m(k+1,j) + p_{i-1}p_kp_j \} & 1 \le i < j \le 6 \end{cases}$$

$i \setminus j$	1	2	3	4	5	6
1	0	15750				
2		0	2625			
3			0	750		
4				0		
5					0	
6						0

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$$m(i,j) = \begin{cases} 0 & i = j \\ \min_{i \le k < j} \{ m(i,k) + m(k+1,j) + p_{i-1}p_kp_j \} & 1 \le i < j \le 6 \end{cases}$$

$i \setminus j$	1	2	3	4	5	6
1	0	15750				
2		0	2625			
3			0	750		
4				0	1000	
5					0	
6						0

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$$m(i,j) = \begin{cases} 0 & i = j \\ \min_{i \le k < j} \{ m(i,k) + m(k+1,j) + p_{i-1}p_kp_j \} & 1 \le i < j \le 6 \end{cases}$$

$i \setminus j$	1	2	3	4	5	6
1	0	15750				
2		0	2625			
3			0	750		
4				0	1000	
5					0	5000
6						0

$$m(i,j) = \begin{cases} 0 & i = j \\ \min_{i \le k < j} \{ m(i,k) + m(k+1,j) + p_{i-1}p_kp_j \} & 1 \le i < j \le 6 \end{cases}$$

$i \setminus j$	1	2	3	4	5	6
1	0	15750	7875			
2		0	2625			
3			0	750		
4				0	1000	
5					0	5000
6						0

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$$m(i,j) = \begin{cases} 0 & i = j \\ \min_{i \le k < j} \{ m(i,k) + m(k+1,j) + p_{i-1}p_kp_j \} & 1 \le i < j \le 6 \end{cases}$$

$i \setminus j$	1	2	3	4	5	6
1	0	15750	7875			
2		0	2625	4375		
3			0	750		
4				0	1000	
5					0	5000
6						0

40 14 14 14 14 1 1 100

$$m(i,j) = \begin{cases} 0 & i = j \\ \min_{i \le k < j} \{ m(i,k) + m(k+1,j) + p_{i-1}p_kp_j \} & 1 \le i < j \le 6 \end{cases}$$

$i \setminus j$	1	2	3	4	5	6
1	0	15750	7875			
2		0	2625	4375		
3			0	750	2500	
4				0	1000	
5					0	5000
6						0

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$i \setminus j$	1	2	3	4	5	6
1	0	15750	7875			
2		0	2625	4375		
3			0	750	2500	
4				0	1000	3500
5					0	5000
6						0

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$$m(i,j) = \begin{cases} 0 & i = j \\ \min_{i \le k < j} \{ m(i,k) + m(k+1,j) + p_{i-1}p_kp_j \} & 1 \le i < j \le 6 \end{cases}$$

i∖j	1	2	3	4	5	6
1	0	15750	7875	9375		
2		0	2625	4375		
3			0	750	2500	
4				0	1000	3500
5					0	5000
6						0

40 1 40 1 4 2 1 4 2 1 9 9 0

Matrices

$$A_1$$
 A_2
 A_3
 A_4
 A_5
 A_6

 Dimensions
 30×35
 35×15
 15×5
 5×10
 10×20
 20×25
 $p_0 \times p_1$
 $p_1 \times p_2$
 $p_2 \times p_3$
 $p_3 \times p_4$
 $p_4 \times p_5$
 $p_5 \times p_6$

$$m(i,j) = \begin{cases} 0 & i = j \\ \min_{i \le k < j} \{ m(i,k) + m(k+1,j) + p_{i-1}p_kp_j \} & 1 \le i < j \le 6 \end{cases}$$

$i \setminus j$	1	2	3	4	5	6
1	0	15750	7875	9375		
2		0	2625	4375	7125	
3			0	750	2500	
4				0	1000	3500
5					0	5000
6						0

40 1 40 1 4 2 1 4 2 1 9 9 0

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Matrices

$$A_1$$
 A_2
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 Dimensions
 30×35
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 $p_0 \times p_1$
 $p_1 \times p_2$
 $p_2 \times p_3$
 $p_3 \times p_4$
 $p_4 \times p_5$
 $p_5 \times p_6$

$$m(i,j) = \begin{cases} 0 & i = j \\ \min_{i \le k < j} \{ m(i,k) + m(k+1,j) + p_{i-1}p_kp_j \} & 1 \le i < j \le 6 \end{cases}$$

i∖j	1	2	3	4	5	6
1	0	15750	7875	9375		
2		0	2625	4375	7125	
3			0	750	2500	5375
4				0	1000	3500
5					0	5000
6						0

$$m(i,j) = \begin{cases} 0 & i = j \\ \min_{i \le k < j} \{ m(i,k) + m(k+1,j) + p_{i-1}p_kp_j \} & 1 \le i < j \le 6 \end{cases}$$

i∖j	1	2	3	4	5	6
1	0	15750	7875	9375	11875	
2		0	2625	4375	7125	
3			0	750	2500	5375
4				0	1000	3500
5					0	5000
6						0

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$$m(i,j) = \begin{cases} 0 & i = j \\ \min_{i \le k < j} \{ m(i,k) + m(k+1,j) + p_{i-1}p_kp_j \} & 1 \le i < j \le 6 \end{cases}$$

$i \setminus j$	1	2	3	4	5	6
1	0	15750	7875	9375	11875	
2		0	2625	4375	7125	10500
3			0	750	2500	5375
4				0	1000	3500
5					0	5000
6						0

40 1 40 1 4 2 1 4 2 1 9 9 0

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Design & Analysis of Algorithms

Matrices

$$A_1$$
 A_2
 A_3
 A_4
 A_5
 A_6

 Dimensions
 30×35
 35×15
 15×5
 5×10
 10×20
 20×25
 $p_0 \times p_1$
 $p_1 \times p_2$
 $p_2 \times p_3$
 $p_3 \times p_4$
 $p_4 \times p_5$
 $p_5 \times p_6$

$$m(i,j) = \begin{cases} 0 & i = j \\ \min_{i \le k < j} \{ m(i,k) + m(k+1,j) + p_{i-1}p_kp_j \} & 1 \le i < j \le 6 \end{cases}$$

$i \setminus j$	1	2	3	4	5	6
1	0	15750	7875	9375	11875	15125
2		0	2625	4375	7125	10500
3			0	750	2500	5375
4				0	1000	3500
5					0	5000
6						0

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Matrices						
Dimensions	30 × 35	35 × 15	15 × 5	5 × 10	10 × 20	20 × 25
	$p_0 \times p_1$	$p_1 \times p_2$	$p_2 \times p_3$	$p_3 \times p_4$	$p_4 \times p_5$	$p_5 \times p_6$
	<i>(</i>					
ma(; ;)	0				i = j	
$m(i,j) = \left. \left\{ \right. \right.$	$\min_{i \le k < j} \{ m($	(i,k)+m((k+1,j) +	$-p_{i-1}p_kp_j$	$1 \le i < 1$	$(j \leq 6)$
$i \setminus j \mid 1$	2 3	4 5	6	$i \setminus i \mid 1$	2 3 4	5 6

2 3 4 5 6 3 4

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			A_3				A_6
Dimensions	30 × 35	35 × 15	15 × 5	5 × 10	10	× 20	20 × 25
	$p_0 \times p_1$	$p_1 \times p_2$	$p_2 \times p_3$	$p_3 \times p_4$	p_4	$\times p_5$	$p_5 \times p_6$
	· _						
m(i, i) =	0				i	= j	
$m(i,j) = \langle$	$\min_{i \le k < j} \{ m \}$	(i,k)+m((k+1,j)	$-p_{i-1}p_k p_i$	$ o_j $ 1	≤ i <	$j \leq 6$
1 0 15	2 3 750	4 5	0	- 1 J	0 1	3 4	5 0
$\begin{array}{c cccc} i \searrow j & 1 & 2 \\ \hline 1 & 0 & 15 \\ 2 & & & \end{array}$	2625			2	0	2	

0

750 0

0

0

0

0

Matrice	:S	A_1	A_2	2	A_3	A_4		,	4 5		F	4_6
Dimensio					15 × 5						20	× 25
	p_0	$p_1 \times p_1$	$p_1 \times$	p_2	$p_2 \times p_3$	$p_3 \times p$	04	p_4	$\times p_{i}$	5	<i>p</i> ₅	$\times p_6$
<i>(: :</i>)	∫0							i	= j			
m(1, J)	$= \left\{ \begin{array}{l} m \\ i \leq k \end{array} \right.$	in { <i>m</i> (k< <i>j</i>	(i, k) +	- m(k	(i+1,j) -	$+ p_{i-1}p_k$, p j }	1	$\leq i$	<	$j \leq$	6
$i \setminus j \mid 1$	2	3	4	5	6	$i \setminus j$	1	2	3	4	5	6
1 0	15750					1	0	1				
2	0	2625				2		0	2			
1 0 2 3		0	750			3			0	3		
4			0	1000)	4				0	4	
5				0		5					0	
5 6					0	6						0

5000

0

0

5000

$$\frac{\text{Matrices}}{\text{Dimensions}} \begin{vmatrix} A_1 & A_2 & A_3 & A_4 & A_5 & A_6 \\ 30 \times 35 & 35 \times 15 & 15 \times 5 & 5 \times 10 & 10 \times 20 & 20 \times 25 \\ p_0 \times p_1 & p_1 \times p_2 & p_2 \times p_3 & p_3 \times p_4 & p_4 \times p_5 & p_5 \times p_6 \end{vmatrix}$$

$$m(i,j) = \begin{cases} 0 & i = j \\ \min \{m(i,k) + m(k+1,j) + p_{i-1}p_kp_i\} & 1 < i < j < 6 \end{cases}$$

$$m(i,j) = \begin{cases} 0 & i = j \\ \min_{i \le k < j} \{ m(i,k) + m(k+1,j) + p_{i-1}p_kp_j \} & 1 \le i < j \le 6 \end{cases}$$

1 \ \ J	I	2	3	4	5	6	1 \]	1	2	3	4	5	6
1	0	15750	7875				1	0	1	1			
2		0	2625	4375			2		0	2	3		
3			0	750			3			0	3		
4				0	1000		4				0	4	
5					0	5000	5					0	5
6						0	6						0
2 3 4 5 6		0			1000 0	5000 0	2 3 4 5 6		0	2	3 3 0	4 0	5 0

$$m(i,j) = \begin{cases} 0 & i = j \\ \min_{i \le k < j} \{ m(i,k) + m(k+1,j) + p_{i-1}p_kp_j \} & 1 \le i < j \le 6 \end{cases}$$

i∖j	1	2	3	4	5	6	i∖j	1	2	3	4	5	6
1	0	15750	7875				1	0	1	1			
2		0	2625	4375			2		0	2	3		
3			0	750	2500		3			0	3	3	
4				0	1000		4				0	4	
5					0	5000	5					0	5
6						0	6						0

40 > 40 > 42 > 42 > 2 > 900

$$m(i,j) = \begin{cases} 0 & i = j \\ \min_{i \le k < j} \{ m(i,k) + m(k+1,j) + p_{i-1}p_kp_j \} & 1 \le i < j \le 6 \end{cases}$$

1 0 15750 7875 1 0 1 1	
2 0 2625 4375 2 0 2 3	
3 0 750 2500 3 0 3 3	
4 0 1000 3500 4 0 4	5
5 0 5000 5 0	5
6 0 6	0

4 D > 4 A > 4 B > 4 B > B 9 9 0

$$m(i,j) = \begin{cases} 0 & i = j \\ \min_{i \le k < j} \{ m(i,k) + m(k+1,j) + p_{i-1}p_kp_j \} & 1 \le i < j \le 6 \end{cases}$$

1 \]	I	2	3	4	5	О	1 \ J	1	2	3	4	5	О
1	0	15750	7875	9375			1	0	1	1	3		
2		0	2625	4375			2		0	2	3		
3			0	750	2500		3			0	3	3	
4				0	1000	3500	4				0	4	5
5					0	5000	5					0	5
6						0	6						0

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$$m(i,j) = \begin{cases} 0 & i = j \\ \min_{i \le k < j} \{ m(i,k) + m(k+1,j) + p_{i-1}p_kp_j \} & 1 \le i < j \le 6 \end{cases}$$

$I \setminus J$	1	2	3	4	5	О	1 \ J	I	2	3	4	5	О
1	0	15750	7875	9375			1	0	1	1	3		
2		0	2625	4375	7125		2		0	2	3	3	
3			0	750	2500		3			0	3	3	
4				0	1000	3500	4				0	4	5
5					0	5000	5					0	5
6						0	6						0

4 D > 4 D >

$$m(i,j) = \begin{cases} 0 & i = j \\ \min_{i \le k < j} \{ m(i,k) + m(k+1,j) + p_{i-1}p_kp_j \} & 1 \le i < j \le 6 \end{cases}$$

i\j	1	2	3	4	5	6	i\j	1	2	3	4	5	6
1	0	15750	7875	9375			1	0	1	1	3		
2		0	2625	4375	7125		2		0	2	3	3	
3			0	750	2500	5375	3			0	3	3	3
4				0	1000	3500	4				0	4	5
5					0	5000	5					0	5
6						0	6						0

4 D > 4 A > 4 B > 4 B > B 9 9 0

Matrices	A_1	A_2	A_3	A_4	A_5	A_6
Dimensions						
	$p_0 \times p_1$	$p_1 \times p_2$	$p_2 \times p_3$	$p_3 \times p_4$	$p_4 \times p_5$	$p_5 \times p_6$

$$m(i,j) = \begin{cases} 0 & i = j \\ \min_{i \le k < j} \{ m(i,k) + m(k+1,j) + p_{i-1}p_kp_j \} & 1 \le i < j \le 6 \end{cases}$$

i\	\ j	1	2	3	4	5	6	i∖j	1	2	3	4	5	6
1		0	15750	7875	9375	11875		1	0	1	1	3	3	
2	2		0	2625	4375	7125		2		0	2	3	3	
3	}			0	750	2500	5375	3			0	3	3	3
4	ļ				0	1000	3500	4				0	4	5
5	,					0	5000	5					0	5
6	j						0	6						0
		1					-	-	1					

$$m(i,j) = \begin{cases} 0 & i = j \\ \min_{i \le k < j} \{ m(i,k) + m(k+1,j) + p_{i-1}p_kp_j \} & 1 \le i < j \le 6 \end{cases}$$

i∖j	1	2	3	4	5	6	i∖j	1	2	3	4	5	6
1	0	15750	7875	9375	11875		1	0	1	1	3	3	
2		0	2625	4375	7125	10500	2		0	2	3	3	3
3			0	750	2500	5375	3			0	3	3	3
4				0	1000	3500	4				0	4	5
5					0	5000	5					0	5
6						0	6						0
Ū	1					Ü	Ū	l					U

$$m(i,j) = \begin{cases} 0 & i = j \\ \min_{i \le k < j} \{ m(i,k) + m(k+1,j) + p_{i-1}p_kp_j \} & 1 \le i < j \le 6 \end{cases}$$

i∖j	1	2	3	4	5	6	i∖j	1	2	3	4	5	6
1	0	15750	7875	9375	11875	15125	1	0	1	1	3	3	3
2		0	2625	4375	7125	10500	2		0	2	3	3	3
3			0	750	2500	5375	3			0	3	3	3
4				0	1000	3500	4				0	4	5
5					0	5000	5					0	5
6						0	6						0
	ı												

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There are 3 nested loops, so this algorithm takes $O(n^3)$ time.

More careful counting:

- *l*: 2 to *n*
 - For each ℓ , we have i: 1 to $n \ell + 1$
 - For each i, we have k: i to $i + \ell 2$

Algorithm

Algorithm Matrix Chain Multiplication

```
1: for i = 1 to n do
      m(i, i) = 0
 3: end for
 4: for \ell = 2 to n do
    // Compute m(1, \ell), m(2, \ell + 1), ..., m(n - \ell + 1, n)
 6:
     for i = 1 to n - \ell + 1 do
 7:
       // Compute m(i, i + \ell - 1)
 8:
      i = i + \ell - 1
         // Compute m(i,j) using the recurrence
10:
    m(i, j) = \infty
11:
          for k = i to i - 1 do
12:
             m(i, j) = \min \{ m(i, j), m(i, k) + m(k + 1, j) + p_{i-1} p_k p_i \}
13:
          end for
14:
       end for
15: end for
16: return m(1, n)
```

$$\sum_{\ell=2}^{n} \sum_{i=1}^{n-\ell+1} \sum_{k=i}^{i+\ell-2} 1$$

$$\sum_{\ell=2}^{n} \sum_{i=1}^{n-\ell+1} \sum_{k=i}^{i+\ell-2} 1$$

$$= \sum_{\ell=2}^{n} \sum_{i=1}^{n-\ell+1} (\ell-1)$$

$$\sum_{\ell=2}^{n} \sum_{i=1}^{n-\ell+1} \sum_{k=i}^{i+\ell-2} 1$$

$$= \sum_{\ell=2}^{n} \sum_{i=1}^{n-\ell+1} (\ell-1)$$

$$= \sum_{\ell=2}^{n} (n-\ell+1)(\ell-1)$$

Total time:

$$\sum_{\ell=2}^{n} \sum_{i=1}^{n-\ell+1} \sum_{k=i}^{i+\ell-2} 1$$

$$= \sum_{\ell=2}^{n} \sum_{i=1}^{n-\ell+1} (\ell-1)$$

$$= \sum_{\ell=2}^{n} (n-\ell+1)(\ell-1)$$

$$= \sum_{\ell=2}^{n} (n-\ell+1)(\ell-1)$$

 $=\sum (n-\ell+1)(\ell-1)$ since the summand is 0 when $\ell=0$

$$\begin{split} &\sum_{\ell=2}^{n} \sum_{i=1}^{n-\ell+1} \sum_{k=i}^{i+\ell-2} 1 \\ &= \sum_{\ell=2}^{n} \sum_{i=1}^{n-\ell+1} (\ell-1) \\ &= \sum_{\ell=2}^{n} (n-\ell+1)(\ell-1) \\ &= \sum_{\ell=1}^{n} (n-\ell+1)(\ell-1) \quad \text{ since the summand is 0 when } \ell=0 \\ &= \frac{n^3-n}{6} \\ &= \Theta(n^3). \end{split}$$