
A Study on the Ramanujan Graph Property of Winning Lottery Tickets

Bithika Pal¹ Arindam Biswas² Sudeshna Kolay¹ Pabitra Mitra¹ Biswajit Basu³

Abstract

Winning lottery tickets refer to sparse subgraphs of deep neural networks which have classification accuracy close to the original dense networks. Resilient connectivity properties of such sparse networks play an important role in their performance. The attempt is to identify a sparse and yet well-connected network to guarantee unhindered information flow. Connectivity in a graph is best characterized by its spectral expansion property. Ramanujan graphs are robust expanders which lead to sparse but highly-connected networks, and thus aid in studying the winning tickets. A feed-forward neural network consists of a sequence of bipartite graphs representing its layers. We analyze the Ramanujan graph property of such bipartite layers in terms of their spectral characteristics using the Cheeger's inequality for irregular graphs. It is empirically observed that the winning ticket networks preserve the Ramanujan graph property and achieve a high accuracy even when the layers are sparse. Accuracy and robustness to noise start declining as many of the layers lose the property. Next we find a robust winning lottery ticket by pruning individual layers while retaining their respective Ramanujan graph property. This strategy is observed to improve the performance of existing network pruning algorithms.

1. Introduction

Neural Network (NN) and its recent advancements have made a significant contribution to solve various machine learning applications. The power of an over-parameterized NN lies in its capability to learn simple patterns and mem-

^{*}Equal contribution ¹Department of Computer Science and Engineering, Indian Institute of Technology, Kharagpur, India

²Department of Mathematical Sciences, University of Copenhagen, Denmark

³School of Civil, Structural and Environmental Engineering, Trinity College, Dublin, Ireland. Correspondence to: Bithika Pal <bithika.pal2014@gmail.com>.

Proceedings of the 39th International Conference on Machine Learning, Baltimore, Maryland, USA, PMLR 162, 2022. Copyright 2022 by the author(s).

orize the noise in the data (Neyshabur et al., 2018). However, the training of such networks requires enormous computational resources, and often the deployment onto low-resource environments such as mobile devices, or embedded systems becomes difficult. Recent trend in research to reduce training time of deep neural networks has shifted towards pre-training following the introduction of a remarkable contribution, named the *Lottery Ticket Hypothesis (LTH)*, which hypothesize the existence of a highly sparse subnetwork and weight initialization to reduce the training resources as well (Frankle & Carbin, 2019). It uses a simple iterative, magnitude-based pruning algorithm, and empirically shows that even after removing approximately 90% of the weights, the subnetwork can preserve the original generalization error. In the subsequent studies, the focus goes on finding this lottery ticket for more competitive tasks by pruning with weight rewinding(Frankle et al., 2019), fine tuning the learning rates (Renda et al., 2020), more efficient training (You et al., 2019; Brix et al., 2020; Girish et al., 2021).

Various analysis for explaining the LTH have been attempted in the past. Researchers (Evci et al., 2020b) explain empirically why the LTH works through gradient flow at different stages of the training. Despite previous attempts to explain why the Lottery Ticket Hypothesis works, the underlying phenomenon associated with the hypothesis still remains ill-understood. All of these studies related to LTH identify that a sparse sub-network can be trained instead of a complete network and the network needs to be connected from input to output layers. However, none of them try to explain the LTH and the properties of the pruned network through the lens of spectral graph theory. The network connectivity can be described from the *graph expansion* point of view, where any subset of vertices of size less than or equal to half of the number of vertices in a graph, is adjacent to at least a fraction of the number of vertices in that set; for details, see (Lubotzky, 2010). Graphs satisfying this property are known as expander graphs. The *Ramanujan Graph* is a special graph in a bounded degree expander family, where the eigenbound is maximal (Nilli, 1991). This leads to a maximum possible sparsity of a network while preserving the connectivity.

In this paper, we initiate a study to observe the characteristics of a pruned sub-network from the spectral properties

of its adjacency matrix, which, has not been reported previously. We represent a feed-forward neural network as a series of connected bipartite graphs. Both weighted and unweighted bi-adjacency matrices are considered. The Ramanujan graph properties of each of the bipartite layers are studied. We use the results of (Hoory, 2005) on the bound of spectral gap for the weight matrix of a pruned network. It is empirically observed that existence of winning tickets in a pruned network is dependent on the Ramanujan graph properties of the bipartite layers. As network sparsity increases with more aggressive pruning, we obtain regions where test set accuracy does not decrease much and the bipartite layers satisfy Ramanujan graph property. Subsequently we obtain regions where the Ramanujan graph properties are lost for all the layers and test accuracy decreases sharply. Also, we observe that the impact of noise in the data, on test set accuracy is more adverse when some of the layers lose their Ramanujan graph properties. Experimental results are presented for the Lenet architecture on the MNIST dataset and the Conv4 architecture on the CIFAR10 dataset. Results for other popular feed-forward network are presented in the Appendix.

We suggest that preservation of Ramanujan graph properties may benefit existing network pruning algorithms. We propose a modified pruning algorithm that uses the spectral bounds. The algorithm identifies network layers that may still be pruned further, while avoiding pruning in layers that have already lost their Ramanujan graph property. Neural network weight score functions suggested by various pruning algorithms are used to represent the bipartite layers as weighted graphs. Spectral bounds for these graphs are used to verify the Ramanujan property. For a number of popular pruning algorithms, we experimentally demonstrate significant improvement in accuracy for sparse networks by using these connectivity criteria. **Contributions:** The contributions of the paper are the following:

- We propose a methodology for analyzing winning lottery tickets with respect to their spectral properties.
- We empirically observe that winning lottery tickets often satisfy layerwise bipartite Ramanujan graph property representing a sparse but resiliently connected global network. The property is checked using spectral bounds that generalize to irregular networks. We also notice better noise robustness when all the layers of the pruned sparse networks preserve this property.
- Based on the above property we propose a modified iterative network pruning algorithm that attempts to preserve Ramanujan graph property for all the layers even at low network densities. It identifies layers that are still amenable to pruning while avoiding further pruning in layers that have lost their Ramanujan graph property.

Section 2 and 3 brief the lottery ticket hypothesis and the expander graphs, respectively. Section 4 and 5 discuss the proposed approach and the results, respectively. Section 6 reviews related literature and Section 7 concludes the work.

2. The Lottery Ticket Hypothesis and Network Pruning

The lottery ticket hypothesis states that a randomly initialized, dense neural network contains a subnetwork which when trained independently using the same initialization achieves a test accuracy close to the original network after training for less or at most the same number of iterations (Frankle & Carbin, 2019). These subnetworks, denoted as ‘winning tickets’, can be uncovered by network pruning algorithms. Weight pruning is one such simple strategy. Let the original dense network be represented as the function $\mathcal{N}(x; \theta)$, where x is the input and θ are the weights. The weights have an initialization of θ_0 . Weight pruning generates a mask $m \in \{0, 1\}^{|\theta|}$ such that the pruned network can be represented by $\mathcal{N}(x; m \odot \theta)$ with initialization $m \odot \theta_0$.

Pruning algorithms that are used to obtain the winning tickets can be either one-shot or iterative. In one-shot pruning the original network is trained to convergence, then $p\%$ of weights are pruned and the surviving weights are re-initialized to their values in θ_0 followed retraining/fine tuning the subnetwork. Here, the network training and pruning are not simultaneously performed and pruning occurs only after convergence is reached. Iterative pruning repeats one-shot pruning over several iteration. This often leads to higher pruning percentage while retaining test set accuracy. However, iterative pruning is more time consuming than one-shot pruning. After pruning the surviving weights may alternately be initialized to their values in θ_i , i denoting a small iteration number, rather than their initial values in θ_0 . This process is effective for large networks. We adopt this in our study. Various scoring functions are used to prioritize the weights for pruning. They may be based on weight magnitudes, gradient, information flow (Blalock et al., 2020; Hoefer et al., 2021) or saliency (Tanaka et al., 2020). Magnitude pruning provides a simple method for obtaining the pruning mask by retaining the top $p\%$ weights $w_i \in \theta$ that have the highest values of $|w_i|$. The role of the weights in local computation in the network layers is not considered. A higher pruning efficiency may be achieved by algorithms that account for connectivity structures in the individual layers. In the next section we describe graph parameters of the network that determine such connectivity.

3. Expanders and Ramanujan Graphs

Expanders are highly connected, and yet sparse graphs. In this work, we shall be considering finite, connected, undi-

rected, but not necessarily regular graphs. Recall that the degree of a vertex v in a graph is the number of half edges emanating from v .

Definition 3.1 $((n, d, \varepsilon)$ -expander). Let $\varepsilon > 0$. An $((n, d, \varepsilon)$ -expander is a graph $\mathcal{G} = (V, E)$ on $|V| = n$ vertices, having maximal degree d , such that for every set $\emptyset \neq U \subseteq V$ satisfying $|U| \leq \frac{n}{2}$, $|\delta(U)| \geq \varepsilon|U|$ holds.

Here, $\delta(U)$ denotes the vertex boundary of U . The quantity $\frac{|\delta(U)|}{|U|}$ measures the rate of expansion and the infimum $\frac{|\delta(U)|}{|U|}$ as U varies among the non-empty subsets of V with $|U| \leq \frac{n}{2}$ is called the (vertex) Cheeger constant $h(\mathcal{G})$ of the graph \mathcal{G} . The higher the value of $h(\mathcal{G})$, the more expansion property it exhibits and vice versa. Expansion and the Cheeger constant quantifies the connectivity properties of a graph as a high value of $h(\mathcal{G})$ signifies that the graph is strongly connected. This ensures that information can flow freely without much bottlenecks.

Definition 3.2 (Expander family). A sequence of finite, connected graphs $\{\mathcal{G}_i = (V_i, E_i)\}_{i=1,2,\dots}$ on V_i vertices and E_i edges is called an expander family if there exists an uniform $\epsilon > 0$ such that each graph in the sequence is an $(|V_i|, d_i, \epsilon)$ expander for some d_i 's.

The study of expansion properties of graphs is closely related to the study of the spectrum (distribution of eigenvalues) of the adjacency operator defined on them. Given a finite r -regular graph of size $|V| = n$, the eigenvalues t_i of the adjacency matrix are all real and they satisfy, $-r \leq t_n \leq \dots \leq t_2 \leq t_1 = r$. The graph is connected iff $t_2 < t_1$ and is bipartite iff t_i 's are symmetric about 0 (in particular $t_n = -r$). The quantity $t_1 - t_2$ is known as the *spectral gap* and it is related to the Cheeger constant via the discrete Cheeger-Buser inequality, discovered independently by (Dodziuk, 1984) and by (Alon & Milman, 1985). In our context, we consider a stronger notion of the spectral gap (but it is equivalent for bipartite graphs). Let $t := \max\{|t_i| : 1 < i \leq n, |t_i| < t_1\}$. In here, the quantity $t_1 - t$ will denote the spectral gap.

The more connected a graph is, the larger is the spectral gap and ideally, a graph with strong expansion properties has a very large spectral gap. However, for a bounded degree expander family, this spectral gap cannot be arbitrarily large. This brings us to the notion of Ramanujan graphs.

Definition 3.3 (Ramanujan graph). Let \mathcal{G} be a r -regular graph on n vertices, with adjacency eigenvalues $\{t_i\}_{i=1,2,\dots,n}$, satisfying $-r \leq t_n \leq \dots \leq t_2 \leq t_1 = r$. Let $t(\mathcal{G}) := \max\{|t_i| : 1 < i \leq n, |t_i| < t_1\}$. Then \mathcal{G} is a Ramanujan graph if $t(\mathcal{G}) \leq 2\sqrt{r-1} = 2\sqrt{t_1 - 1}$.

The fact that in a bounded degree expander family, the eigenvalue bound in Ramanujan graphs is maximal can be deduced from the following result due to Alon, see (Nilli,

1991), $t(\mathcal{G}) \geq 2\sqrt{r-1} - \frac{2\sqrt{r-1}-1}{\lfloor m/2 \rfloor}$ where m denotes the diameter of the graph. As $m \rightarrow +\infty$ and r is bounded, we obtain the result (this also follows from the Alon-Boppana theorem).

For our applications, we shall encounter not necessarily regular graphs, thus we need a notion of irregular version of Ramanujan graphs. The two ways we shall exploit Definition 3.3 for irregular graphs will be to: (i) Use the average degree d_{avg} in place of the regularity; (ii) For weighted graphs, use t_1 , the largest eigenvalue of the adjacency matrix. Note that a motivation for considering the above bounds comes from the following generalisation of the definition of Ramanujan graphs to irregular graphs. For a finite, connected graph \mathcal{G} (not necessarily regular) consider its universal cover $\tilde{\mathcal{G}}$, for details see (Hoory et al., 2006, sec. 6). A Ramanujan graph is a graph satisfying $t(\mathcal{G}) \leq \rho(\tilde{\mathcal{G}})$ where $\rho(\tilde{\mathcal{G}})$ denotes the spectral radius of $\tilde{\mathcal{G}}$. See also Marcus–Spielman–Srivastava (Marcus et al., 2015, sec. 2.3). A result of Hoory, see (Hoory, 2005) implies that $2\sqrt{d_{avg} - 1} \leq \rho(\tilde{\mathcal{G}})$. Thus, it makes sense to consider $t(\mathcal{G}) \leq 2\sqrt{d_{avg} - 1} \leq \rho(\tilde{\mathcal{G}})$. The above consideration also result in extremal families, (Hoory, 2005). Further, for irregular bipartite graphs, with minimal degree at least two and an average degree d_{avgL} on the left and d_{avgR} on the right, we can further consider the sharper estimate $t(\mathcal{G}) \leq \sqrt{d_{avgL} - 1} + \sqrt{d_{avgR} - 1} \leq \rho(\tilde{\mathcal{G}})$ see (Hoory, 2005).

Upto now, we had only discussed about unweighted graphs. A weighted graph is a graph with a weight function $w : E \rightarrow \mathbf{R}_{\geq 0}$ attached on the edges. It can be looked upon as a generalisation of unweighted graphs, in the sense that in the unweighted case, the function w takes values in the set $\{0, 1\}$. In the case of weighted one, we shall use the absolute values of the weight functions according to the architecture for the corresponding dataset. It also means that in the case of weighted graphs, we need to modify the definition of the edge set of the graph to incorporate multiple (as well as fractional) edges. The theory of characterisation of weighted Ramanujan graphs is not well developed. However, characterisation of weighted expanders (with positive weights) exist, due to the Cheeger inequality for such graphs, see (Chung, 1996, sec. 5) and we use the largest eigenvalue of the adjacency matrix in place of the regularity. In the case of regular graphs, it coincides with the notion of Ramanujan graphs and even in the general case, by Cheeger inequality, it ensures a large expansion which in turn supports free flow of information. It forms the theoretical basis of our work.

4. Ramanujan Graph Characterization of Neural Networks

We can represent any neural network using graphs (possibly weighted, depending on the context) and in this article, we

will be dealing with a sequence of finite, bipartite graphs. This is because if \mathbf{N} denotes a neural network having l layers $\mathbf{N}_1, \mathbf{N}_2, \dots, \mathbf{N}_l$ respectively, then each \mathbf{N}_i ($i = 1, 2, \dots, l$) is a complete bipartite graph to start with. A pruned subnetwork results in edge sparsification of the underlying graphs. For that purpose, we need to approximate complete, bipartite graphs by sparse, proper subgraphs. The motivation to study pruning based on expander characteristics stems from the fact that complete graphs can be approximated using expanders, see (Spielman, 2018). The notion of Ramanujan graphs allow us to quantify the pruning limit, and we empirically justify our technique.

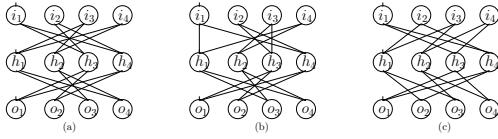


Figure 1. Examples of small feed-forward networks having same sparsity but different connectivity properties; (a) disconnected network, (b) partially connected (c) strongly connected. When all of them are scaled to larger networks, (c) has large spectral gap compared to (a) and (b) signifying a high rate of expansion.

In Figure 1, we present examples of three small feed-forward networks with a single hidden layer and four neurons in each layer. The networks consists of two bipartite graphs corresponding to the input-hidden and hidden-output layers. The sparsity of all the three networks, as measured by the number of edges present, are the same. However, in the first network none of the layer-wise bipartite graphs satisfy the expander property and thus information flow from a significant number of input nodes to output nodes are disrupted. For the second network, the input-hidden layer bipartite graph is an expander, while the hidden-output layer bipartite graph is not an expander.

Here few of the flow paths from input to the output nodes are disconnected. Both the bipartite graphs for the third network are expanders, thus all the inputs nodes are connected to all the output nodes. We denote the first network as a disconnected one, the second network as partially connected, and the third one as fully connected. The example illustrates that layer-wise sparse bipartite expander graphs ensures global information flow across layers. While connected but non-expander sparse bipartite layers do not necessarily lead to global connectivity across layers. It is known in literature that sparse but resiliently connected neural networks not only have good generalization performance but also achieve noise robustness (Liu et al., 2018).

4.1. Bipartite Graph Structure

In this work, we focus on the fully-connected layers, and convolution layers of the feed-forward neural network only.

Since, pruning is a part of network compression, we only consider the trainable layers here. We consider both unweighted and weighted representations of the bipartite graphs. We ignore signs of the weight values and consider only the magnitudes for the weighted graph representation. Even though sign of the weights are important for determining the neural network functionality, we argue that for studying their connectivity properties only the magnitudes need to be considered.

Fully Connected Layers (FC): For a fully connected layer \mathbf{N}_i , with n^{i-1} number of inputs and n^i number of outputs the weighted bi-adjacency matrix of the corresponding graph is $\mathbf{W}_i \in \mathbb{R}^{[n^{i-1} \times n^i]}$, and the corresponding pruning mask is $\mathbf{M}_i \in \{0, 1\}^{[n^{i-1} \times n^i]}$.

Convolution Layers (Conv): Here, we consider the kernel size, the number of input and output channels to unfold the layer into a complete bipartite graph. For a convolution layer \mathbf{N}_i with the kernel of size k , n^{i-1} input, and n^i output channels, the weighted bi-adjacency matrix of the corresponding graph will be $\mathbf{W}_i \in \mathbb{R}^{[(n^{i-1}.k.k) \times n^i]}$, and the corresponding pruning mask is $\mathbf{M}_i \in \{0, 1\}^{[(n^{i-1}.k.k) \times n^i]}$. An example is shown in Figure 2.

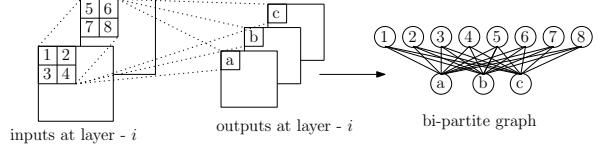


Figure 2. An example of bipartite graph computation for a particular convolution layer with kernel size 2×2 , 2 input channels, and 3 output channels

Spectral Bounds for the Adjacency Matrix: Now, the bounds are analyzed for both unweighted (\mathbf{M}_i) and weighted (\mathbf{W}_i) bipartite graphs at each layer \mathbf{N}_i . Table 1 describes different bounding constraint depending on the type of the considered graph and bound type. We consider the bound differences (Δ_S and Δ_R) for the eigenvalue and average degree respectively. The differences are divided by t_2 to make it comparable for different sized of bipartite graphs. A transition of the Δ values from positive to negative denotes a violation of the bounds, and thus loss of Ramanujan graph property of the bipartite graph for the corresponding pair of layers in the feed-forward network.

4.2. Ramanujan Graph Property Preserving Pruning Algorithm

In this section, we describe a modification of the iterative pruning algorithm that preserves the Ramanujan graph property of a pruned neural network. In iterative pruning algo-

Table 1. Different bound criteria on the second largest eigenvalue t_2 of the bi-adjacency matrices

BI-ADJACENCY	EIGENVALUE (eb)	AVERAGE DEGREE (db)
Unweighted (\mathbf{M}_i)	$t_2(\mathbf{M}_i) \leq 2\sqrt{t_1(\mathbf{M}_i) - 1}$	$t_2(\mathbf{M}_i) \leq \sqrt{d_{avgL}(\mathbf{M}_i) - 1} + \sqrt{d_{avgR}(\mathbf{M}_i) - 1}$
Weighted (\mathbf{W}_i)	$t_2(\mathbf{W}_i) \leq 2\sqrt{t_1(\mathbf{W}_i) - 1}$	
difference on bound	$\Delta_S = (2\sqrt{t_1 - 1} - t_2)/t_2$	$\Delta_R = (\sqrt{d_{avgL} - 1} + \sqrt{d_{avgR} - 1} - t_2)/t_2$

rithms, pruning is employed once the training is complete (i.e. after it converged to an optimum or it reached certain epochs). The pruned network is trained again with its initial weight values to perform the next pruning iteration. The magnitude of a score function θ is used to identify the victim weights. The weights are made to zero if their score function values lie in the bottom p percentile. In the Iterative Magnitude Pruning (IMP) (Frankle & Carbin, 2019), magnitude of the weight is used as the score function. More sophisticated score functions are used in SynFlow (Tanaka et al., 2020), SNIP (Lee et al., 2018).

We propose a modification to the above method. If a weight is identified as a victim based on the above score function based scheme; we check the Ramanujan property of the layer \mathbf{N}_i to which this weight belongs. First, we verify the connectivity property of the layer using the bounds on the unweighted graph as defined in Table 1($\Delta_S, \Delta_R > 0$). If the unweighted graph bounds are not satisfied, we consider a weighted graph with score function values as the edge weights. The bounds using the eigenvalues of this weighted graph is used to verify the spectral expansion property for a particular layer. If the pruned network \mathbf{N}_i^{temp} follows the Ramanujan graph property given by the bounds, then the algorithm simply proceeds to the next level of pruning operation. If the bound is not satisfied for a particular layer i , the weights are simply reset to the values before pruning and the pruning percentile p is halved and search for a better approximation of \mathbf{N}_i is resumed. The target pruning level $k \in [K]$ is a controlling parameter to limit the search iteration. The proposed algorithm attempts to preserve Ramanujan property for as many layers as possible by using the spectral bound criteria to determine the layers which are still amenable to pruning. It avoids further pruning of the layers that have lost their connectivity property. We experimentally show that the approach is effective when used with IMP, SNIP, SynFlow pruning algorithms with their designated score functions.

The method is described in Algorithm 1. It takes the trained network \mathbf{N} , score function θ , pruning percentile p , pruning level K as inputs and returns the pruned network \mathbf{N}^* as output. This algorithm is called at each pruning iteration, and continued till the stopping criterion is met, i.e., $\mathbf{N}_i^* = \mathbf{N}_i, \forall i$. As discussed in Section 3, we mainly study two types of bounds on the second largest eigenvalue of the bipartite graph $t_2 = t(\mathcal{G})$, i.e., (i) **eb**- based on the largest

Algorithm 1 Layer-wise Connectivity Based Pruning

Require: Trained Network \mathbf{N} , Score function θ , Pruning percentile p , pruning level K

- 1: $\mathbf{N}^* \leftarrow \mathbf{N}$ {Initialize the pruned network}
- 2: **for** $i \leftarrow 1$ **to** l **do**
- 3: $k \leftarrow 0$
- 4: **while** $k < K$ **do**
- 5: $\mathbf{N}_i^{temp} \leftarrow \text{PRUNE BY PERCENTILE}(p/2^k, \mathbf{N}_i, \theta_i)$
 {PRUNE USING THE SCORE FUNCTION}
- 6: CALCULATE d_{avgL} AND d_{avgR} OF \mathbf{N}_i^{temp}
- 7: **IF** $\min(d_{avgL}, d_{avgR}) < 2$ **THEN**
- 8: $\mathbf{N}_i^* \leftarrow \mathbf{N}_i$ {TO AVOID LAYER COLLAPSE}
- 9: **BREAK**
- 10: CALCULATE Δ_S, Δ_R FOR \mathbf{N}_i^{temp}
- 11: **IF** $\max(\Delta_S, \Delta_R) > 0$ **THEN**
- 12: $\mathbf{N}_i^* \leftarrow \mathbf{N}_i^{temp}$ {SATISFY RAMANUJAN PROPERTY ON THE UNWEIGHTED GRAPH}
- 13: **BREAK**
- 14: **ELSE**
- 15: CALCULATE Δ_S^θ FOR \mathbf{N}_i^{temp} USING θ_i
- 16: **IF** $\Delta_S^\theta \geq 0$ AND $t_1^\theta \geq 1$ **THEN**
- 17: $\mathbf{N}_i^* \leftarrow \mathbf{N}_i^{temp}$ {SATISFY EXPANSION PROPERTY \mathbf{N}_i USING θ_i AS WEIGHTS}
- 18: **BREAK**
- 19: $k \leftarrow k + 1$
- 20: **Return** \mathbf{N}^*

eigenvalue(t_1) and (ii) **db**- based on the average degree (d_{avgL} and d_{avgR}). We have considered $\max(\Delta_S, \Delta_R)$ to be greater than zero to define loss of the Ramanujan graph property (Hoory, 2005). To allow further pruning, we have checked whether the weighted graph follows the Ramanujan graph property using eigenvalue-bound Δ_S^θ and the largest eigenvalue t_1^θ for the network weights. In order to avoid ‘layer collapse’ we stop pruning for a layer if the bipartite graph has a node with degree less than two.

The top-2 largest magnitude eigenvalues at layer i are computed from the symmetric matrix $\mathbf{N}_i^T \mathbf{N}_i$ with implicitly restarted Arnoldi methods (Lehoucq et al., 1998) and, the average degree is computed from the connected component of the bipartite-graph. Hence, the computational complexity is mainly driven by the pruning level K , the number of input and output neurons at each layer, and the number of layers. The additional space will be required to store the bipartite

Table 2. Hyper-parameter settings for experimenting LTH using iterative magnitude based pruning

	Lenet/MNIST	Conv4/CIFAR10
Optimizer	Adam	Adam
Train-Iterations	50000	25000
Batch size	60	60
Learning Rate	0.0012	0.0003
Pruning epochs	50	50
Initialization	Kaiming Normal	Kaiming Normal
	64,64, pool	
Conv Layers		128, 128, pool
FC layers	300, 100, 10	256, 256, 10

graph of a layer. For the convolutional layers the size of the adjacency matrix is effectively same as the size of the convolution kernel which is usually small. Also, usually the number of fully connected layers are not very high.

5. Experimental Results

We have used the MNIST and CIFAR10 datasets in our study. As an evaluation measure we use the classification accuracy for both clean and noisy test sets. Noisy test sets were generated by adding zero mean σ variance Gaussian noise to the image pixels. We report results for the Lenet, and Conv4 network architectures following the methodology adopted in (Frankle & Carbin, 2019). We also perform the studies on other networks which are presented in the Appendix. Hyper-parameter values are reported in Table 2.

5.1. Experimental Setup

We study the variation of classification accuracy with the network density as measured by the remaining weights percentile $100 - p$, where p is the pruning percentile. For different layers the percentile can be different, as denoted by p_{fc} for fully connected (FC) layers, p_{conv} for convolution layers, and p_{out} for output layers. We study the two cases for the bound specific to unweighted (M_i) and weighted graph (W_i). For each of the studies, we plot four parameters; (i) eigenbound bound difference (Δ_S), (ii) average degree bound difference (Δ_R), (iii) network density for each layer, and (iv) test accuracy on both clean and noisy data. The degree and eigenbound differences are defined in Table 1.

5.2. Relationship between the Ramanujan Graph Property and LTH

The primary goal of our study is to identify the LTH regimes, determined by the spectral properties of the bipartite-network layers. The existence of LTH regime is illustrated for two representative established networks. The results for Lenet on MNIST dataset is shown in Figure 3; and for Conv4 on CIFAR10 is shown in Figure 4. In each of the plots, we show the variation of classification accuracy on

clean and noisy test sets (having various noise levels (σ)), with network density. The spectral properties are characterized by eigenbound difference Δ_S for both the weighted and unweighted graph representations, and the degree bound difference Δ_R for only the unweighted graph representation. A transition of Δ_S and Δ_R from positive to negative values denotes the loss of Ramanujan graph property of the corresponding layer bipartite graph. Accordingly, the plot is divided into three regimes - (i) fully Ramanujan, where all the layers maintain the Ramanujan property i.e., $\Delta > 0, \forall N_i$, (ii) partially Ramanujan, where some of the layers maintain the property, and (iii) non-Ramanujan, where none of the layers retain the property. For the Conv4 network which has more number of layers we show results for the unweighted graph representation only. Similar results for other networks and different pruning settings are discussed in the Appendix.

In all the figures, we observe that the accuracy values start dropping sharply beyond the partially Ramanujan graph property boundary (pink shade) when all the layers loose their Ramanujan graph property. Accuracy also starts reducing slowly from the fully Ramanujan boundary (gray shade). The accuracy on clean data falls sharply when the layers do not satisfy the bounds for the weighted graphs. The bounds for the unweighted graph allow only the robust winning tickets, having high accuracy even on noisy data. For very low network densities it is seen that the Δ values become positive again. This is a boundary effect owing to the in-applicability of Ramanujan graph bounds for disconnected graphs. We also add the fact that the accuracy doesn't drop sharply can be attributed to two reasons. (1) The Ramanujan bound is optimal for a sequence of finite graphs going to infinity. Thus, the larger each bipartite layer is, the more accurately it is depicted. (2) Instead of considering the whole network as a Ramanujan graph, we are considering each bipartite layer as one and then computing the stopping criteria based on this. A drop in accuracy is indeed detected from the start of the pink region (where one layer loses the Ramanujan graph property) to the end (where all the layers lose it). The statistical significance of the relation between the Ramanujan Graph Property and LTH is discussed in Appendix.

5.3. Comparison of Pruning Approaches

We show that consideration Ramanujan graph properties benefits network pruning algorithms. We consider three popular network pruning algorithms - (i) Iterative Magnitude Pruning (IMP) (Frankle & Carbin, 2019), (ii) the iterative verion of Single-shot Network Pruning based on Connection Sensitivity (SNIP) (Lee et al., 2018), (iii) Synaptic flow based pruning SynFlow (Tanaka et al., 2020) and show the results of test accuracy achieved with similar densities with and without the bound condition. To do this experimentation, we choose the a fix compression ratio α

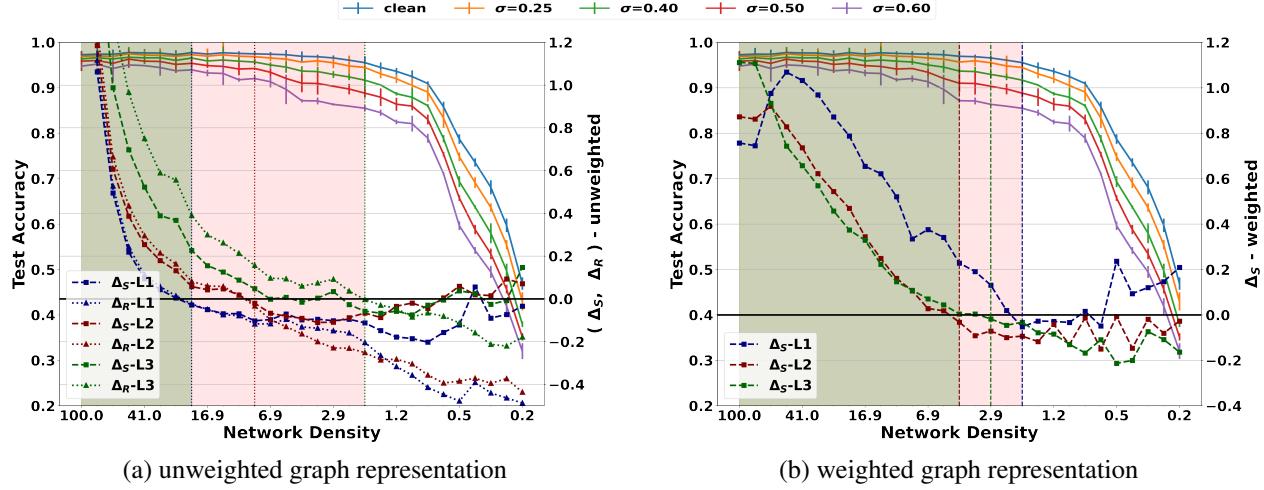


Figure 3. Results for MNIST dataset on Lenet architecture; (a) considering unweighted bi-adjacency matrices, (b) considering weighted bi-adjacency matrices. Variation of accuracy with network density is plotted for the clean and noisy test sets with increasing noise variances σ . Error bars for the accuracy values computed over 5 runs are shown. For the layers L1, L2, and L3 the values of Δ_S are plotted for both unweighted and weighted representations, and Δ_R for only the unweighted representation. As mentioned in Table 1, Δ_S and Δ_R denote the difference in bounds of the eigenvalues and average degrees. Transition of the Δ values from positive to negative denote the loss of Ramanujan graph property. The plot is divided into three regimes- *fully Ramanujan* (gray shade), where the Ramanujan graph property holds for all the layers, *partially Ramanujan* (pink shade), where the property holds for some of the layers, and *non-Ramanujan* (no shade) where none of the layers retain the property.

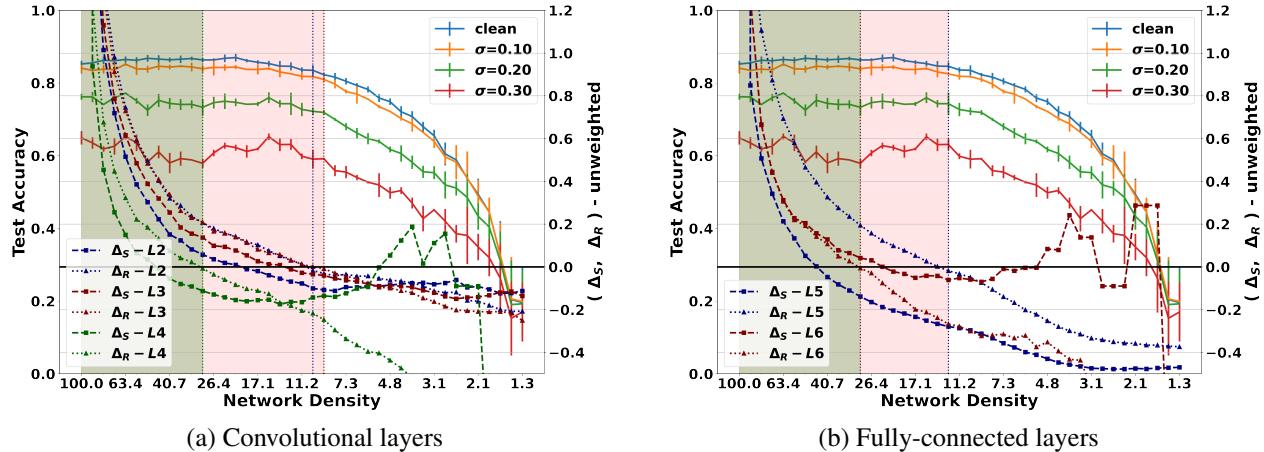


Figure 4. Results for CIFAR10 dataset on Conv4 architecture considering unweighted graph. Results for the convolution layers (L2,3,4) is shown in (a), while for the FC layers (L5,6) is shown in (b). We exclude the first and last layers in this study, due to the low cardinality of one of the parts in the bipartite graph for these layers. These layers are usually not pruned by the pruning algorithms.

to achieve the desired density of the pruned network as $10^{-\alpha} \times 100$, and observe the difference in accuracy due to the network connectivity. The results are presented in Table 3 for Lenet/MNIST and Conv4/CIFAR10. For each of the algorithm we consider three cases - (i) the network is pruned to very low densities using existing algorithms, (ii) network pruning is stopped when there is loss of Ramanujan graph properties (#AlgoName-Bound), and (iii) the existing algorithm is compared with the target density achieved by applying case-ii (#AlgoName*). The bounds are consid-

ered as two types of bounds - first it tries to maintain the bounds on the unweighted graph. Next, if the unweighted graph loses the Ramanujan property it tries to preserve the information flow by considering the spectral bounds for the weighted graphs with score function. We use layer-wise pruning for all the algorithms.

Here, adding the bound criteria guides the pruning algorithm to stop at a density where the accuracy is comparable with its original network. For all the algorithms at same density(typically very low) the addition of bounding cri-

Table 3. Results for network density versus test accuracy using different pruning algorithms under the Ramanujan Graph property preservation. The existing algorithms (#AlgoName- α) are compared at low density region with $\alpha \in [1.0, 1.5, 2.0]$ to capture the accuracy drops. The best result after applying the bound is reported in #AlgoName-Bound with bold font. The accuracy of the existing algorithm at same density is reported in #AlgoName* and highlighted in italic font.

Pruning Algorithm	Lenet/MNIST		Conv4/CIFAR10	
	Density	Accuracy	Density	Accuracy
No Pruning	100.0	97.16	100.0	85.86
IMP-1.0	10.0	97.17	10.0	81.91
IMP-1.5	3.16	93.88	3.16	60.61
IMP-2.0	1.0	45.39	1.0	10
IMP-Bound	3.6	96.74	2.54	70.58
IMP*	3.6	95.07	2.54	28.13
SNIP-1.0	10.0	97.35	10.0	80.3
SNIP-1.5	3.16	79	3.16	72.95
SNIP-2.0	1.0	49.8	1.0	64.26
SNIP-Bound	7.64	95.41	2.24	72.06
SNIP*	7.64	95.18	2.24	68.7
SynFlow-1.0	10.0	97.22	10.0	82.5
SynFlow-1.5	3.16	95.92	3.16	77.18
SynFlow-2.0	1.0	49.11	1.0	69.2
SynFlow-Bound	1.33	93.82	1.01	69.34
SynFlow*	1.33	67.21	-	-

teria improves the test accuracy. In MNIST dataset, the improvement is highest for SynFlow algorithm. While the usual SynFlow algorithm results in an accuracy of 67.21% at the density of 1.33, the SynFlow-Bound algorithm has an accuracy of 93.82%. In the case of SNIP, the proposed algorithm stops pruning at higher density due to its underlying pruned network structure and the score function. For Conv4/CIFAR10, a significant improvement is observed for both IMP and SNIP. However, the scope of improvement in Synflow with bound is limited, as it preserves the unweighted graph connectivity property for most of the layers. By analyzing the weights, we observe that IMP does not preserve Ramanujan property at low density whereas SynFlow algorithm preserves the same (owing to the nature of the pruning score θ) for many of the layers. As an example, we see the result for Conv4/CIFAR10. With density of 3.16 the IMP achieves accuracy of 60.61% while SynFlow meets 77.18% accuracy. It has been found that the Ramanujan property is lost in FC layer-1 only for SynFlow where the same is observed in IMP for both Conv Layer-4 and FC layer-1. The detailed result is given in Appendix. Ramanujan property thus remains a decisive factor in designing new pruning algorithms and constructing better winning ticket.

6. Related Work

Neural network pruning involves sparsification of the network (LeCun et al., 1990; Blalock et al., 2020). It identifies the weight parameters, removal of which incurs minimal

effect on the generalization error. There exists different categories of pruning based on (i) how the pruning is performed, for instance based on the weight magnitude (Han et al., 2015; Zhu & Gupta, 2017), gradient in the backpropagation, hessian of the weight (Hassibi et al., 1993; Dong et al., 2017; Lee et al., 2018), etc; (ii) whether the pruning is global or local; (iii) how often pruning should be applied like one-shot (Lee et al., 2018; Wang et al., 2020), iterative (Tanaka et al., 2020). One of the primary goals in the literature has been to reduce the computational footprint at the time of prediction, i.e., during post-training. In recent LTH studies, the victim weights are determined by their value at the initialization, gradient of the error, and network topology (Lee et al., 2019; Tanaka et al., 2020). To understand weight initialization, (Malach et al., 2020) show that pruning makes a stronger hypothesis with bounded weight distribution. The sparsity of the network is reduced from polynomial to a logarithmic factor of the number of training variables (Orseau et al., 2020). Recently, the deep-expander networks and their block sparse approximations are introduced for getting better sparse network (Prabhu et al., 2018; Vooturi et al., 2021). (Mocanu et al., 2018) suggest to consider the topology of the network from a network science point of view. The pruning algorithm starts from a random Erdős Rényi graph and returns a scale-free network of a high sparsity factor based on the number of neurons in each layer. The method is further evolved for CNN layers considering both the magnitude and gradient of the weights(Evci et al., 2020a).

7. Conclusion

In this work, we study the validity of the lottery ticket hypothesis (LTH) based on structural connectivity properties of the neural network. Ramanujan graph properties of the bipartite layers are studied in terms of certain spectral bounds. As test accuracy varies with decreasing network density as a result of pruning, three distinct regions are demarcated using these bounds. In the first region, all the bipartite layers are Ramanujan graphs, in the second region some of them are, and in the third low network density regions none of the layers are Ramanujan graphs. We empirically demonstrate the validity of the lottery ticket hypothesis robustly in the first region and partially in the second region. We propose a modification of existing iterative pruning algorithms that preserves Ramanujan graph property. Further refinement of this approach of pruning can result in more efficient winning ticket search, which will be the basis of future research.

Acknowledgements

A. Biswas acknowledges the support of the grant ISF 1226/19 of C. Meiri at the Technion and the ERC grant 716424-CASe of K. Adiprasito at the Univ. of Copenhagen.

References

- Alon, N. and Milman, V. D. λ_1 , isoperimetric inequalities for graphs, and superconcentrators. *J. Combin. Theory Ser. B*, 38(1):73–88, 1985. ISSN 0095-8956. doi: 10.1016/0095-8956(85)90092-9. URL [https://doi.org/10.1016/0095-8956\(85\)90092-9](https://doi.org/10.1016/0095-8956(85)90092-9).
- Blalock, D., Ortiz, J. J. G., Frankle, J., and Guttag, J. What is the state of neural network pruning? *arXiv preprint arXiv:2003.03033*, 2020.
- Brix, C., Bahar, P., and Ney, H. Successfully applying the stabilized lottery ticket hypothesis to the transformer architecture. *arXiv preprint arXiv:2005.03454*, 2020.
- Chung, F. R. K. Laplacians of graphs and Cheeger’s inequalities. In *Combinatorics, Paul Erdős is eighty, Vol. 2 (Keszthely, 1993)*, volume 2 of *Bolyai Soc. Math. Stud.*, pp. 157–172. János Bolyai Math. Soc., Budapest, 1996.
- Dodziuk, J. Difference equations, isoperimetric inequality and transience of certain random walks. *Trans. Amer. Math. Soc.*, 284(2):787–794, 1984. ISSN 0002-9947. doi: 10.2307/1999107. URL <https://doi.org/10.2307/1999107>.
- Dong, X., Chen, S., and Pan, S. J. Learning to prune deep neural networks via layer-wise optimal brain surgeon. *arXiv preprint arXiv:1705.07565*, 2017.
- Evci, U., Gale, T., Menick, J., Castro, P. S., and Elsen, E. Rigging the lottery: Making all tickets winners. In *International Conference on Machine Learning*, pp. 2943–2952. PMLR, 2020a.
- Evci, U., Ioannou, Y. A., Keskin, C., and Dauphin, Y. Gradient flow in sparse neural networks and how lottery tickets win. *arXiv preprint arXiv:2010.03533*, 2020b.
- Frankle, J. and Carbin, M. The lottery ticket hypothesis: Training pruned neural networks. In *International Conference on Learning Representations (ICLR)*, 2019.
- Frankle, J., Dziugaite, G. K., Roy, D. M., and Carbin, M. Stabilizing the lottery ticket hypothesis. *arXiv preprint arXiv:1903.01611*, 2019.
- Girish, S., Maiya, S. R., Gupta, K., Chen, H., Davis, L. S., and Shrivastava, A. The lottery ticket hypothesis for object recognition. In *Proceedings of the IEEE/CVF Conference on Computer Vision and Pattern Recognition*, pp. 762–771, 2021.
- Han, S., Pool, J., Tran, J., and Dally, W. Learning both weights and connections for efficient neural network. *Advances in neural information processing systems*, 28: 1135–1143, 2015.
- Hassibi, B., Stork, D. G., and Wolff, G. J. Optimal brain surgeon and general network pruning. In *IEEE international conference on neural networks*, pp. 293–299. IEEE, 1993.
- Hoefer, T., Alistarh, D., Ben-Nun, T., Dryden, N., and Peste, A. Sparsity in deep learning: Pruning and growth for efficient inference and training in neural networks, 2021.
- Hoory, S. A lower bound on the spectral radius of the universal cover of a graph. *J. Combin. Theory Ser. B*, 93(1):33–43, 2005. ISSN 0095-8956. doi: 10.1016/j.jctb.2004.06.001. URL <https://doi.org/10.1016/j.jctb.2004.06.001>.
- Hoory, S., Linial, N., and Wigderson, A. Expander graphs and their applications. *Bull. Amer. Math. Soc. (N.S.)*, 43(4):439–561, 2006. ISSN 0273-0979. doi: 10.1090/S0273-0979-06-01126-8. URL <https://doi.org/10.1090/S0273-0979-06-01126-8>.
- LeCun, Y., Denker, J. S., and Solla, S. A. Optimal brain damage. In *Advances in neural information processing systems*, pp. 598–605, 1990.
- Lee, N., Ajanthan, T., and Torr, P. H. Snip: Single-shot network pruning based on connection sensitivity. *arXiv preprint arXiv:1810.02340*, 2018.
- Lee, N., Ajanthan, T., Gould, S., and Torr, P. H. A signal propagation perspective for pruning neural networks at initialization. *arXiv preprint arXiv:1906.06307*, 2019.
- Lehoucq, R. B., Sorensen, D. C., and Yang, C. *ARPACK users’ guide: solution of large-scale eigenvalue problems with implicitly restarted Arnoldi methods*. SIAM, 1998.
- Liu, M., Liu, S., Su, H., Cao, K., and Zhu, J. Analyzing the noise robustness of deep neural networks. *CoRR*, abs/1810.03913, 2018. URL <http://arxiv.org/abs/1810.03913>.
- Lubotzky, A. *Discrete groups, expanding graphs and invariant measures*. Springer Science & Business Media, 2010.
- Malach, E., Yehudai, G., Shalev-Shwartz, S., and Shamir, O. Proving the lottery ticket hypothesis: Pruning is all you need. *arXiv preprint arXiv:2002.00585*, 2020.
- Marcus, A. W., Spielman, D. A., and Srivastava, N. Interlacing families I: Bipartite Ramanujan graphs of all degrees. *Ann. of Math.* (2), 182(1):307–325, 2015. ISSN 0003-486X. doi: 10.4007/annals.2015.182.1.7. URL <https://doi.org/10.4007/annals.2015.182.1.7>.

- Mocanu, D. C., Mocanu, E., Stone, P., Nguyen, P. H., Gibescu, M., and Liotta, A. Scalable training of artificial neural networks with adaptive sparse connectivity inspired by network science. *Nature communications*, 9(1):1–12, 2018.
- Neyshabur, B., Li, Z., Bhojanapalli, S., LeCun, Y., and Srebro, N. Towards understanding the role of over-parametrization in generalization of neural networks. *arXiv preprint arXiv:1805.12076*, 2018.
- Nilli, A. On the second eigenvalue of a graph. *Discrete Math.*, 91(2):207–210, 1991. ISSN 0012-365X. doi: 10.1016/0012-365X(91)90112-F. URL [https://doi.org/10.1016/0012-365X\(91\)90112-F](https://doi.org/10.1016/0012-365X(91)90112-F).
- Orseau, L., Hutter, M., and Rivasplata, O. Logarithmic pruning is all you need. *Advances in Neural Information Processing Systems*, 33, 2020.
- Prabhu, A., Varma, G., and Namboodiri, A. Deep expander networks: Efficient deep networks from graph theory. In *Proceedings of the European Conference on Computer Vision (ECCV)*, pp. 20–35, 2018.
- Renda, A., Frankle, J., and Carbin, M. Comparing rewinding and fine-tuning in neural network pruning. *arXiv preprint arXiv:2003.02389*, 2020.
- Spielman, D. A. Properties of expander graphs. <http://www.cs.yale.edu/homes/spielman/561/lect17-18.pdf>, 2018.
- Tanaka, H., Kunin, D., Yamins, D. L., and Ganguli, S. Pruning neural networks without any data by iteratively conserving synaptic flow. *arXiv preprint arXiv:2006.05467*, 2020.
- Vooturi, D. T., Varma, G., and Kothapalli, K. Ramanujan bipartite graph products for efficient block sparse neural networks. *Concurrency and Computation: Practice and Experience*, pp. e6363, 2021.
- Wang, C., Zhang, G., and Grosse, R. Picking winning tickets before training by preserving gradient flow. *arXiv preprint arXiv:2002.07376*, 2020.
- You, H., Li, C., Xu, P., Fu, Y., Wang, Y., Chen, X., Baraniuk, R. G., Wang, Z., and Lin, Y. Drawing early-bird tickets: Towards more efficient training of deep networks. *arXiv preprint arXiv:1909.11957*, 2019.
- You, J., Leskovec, J., He, K., and Xie, S. Graph structure of neural networks. In III, H. D. and Singh, A. (eds.), *Proceedings of the 37th International Conference on Machine Learning*, volume 119 of *Proceedings of Machine Learning Research*, pp. 10881–10891. PMLR, 13–18 Jul 2020. URL <https://proceedings.mlr.press/v119/you20b.html>.
- Zhu, M. and Gupta, S. To prune, or not to prune: exploring the efficacy of pruning for model compression. *arXiv preprint arXiv:1710.01878*, 2017.

A. Appendix

We present additional results to supplement those given in the paper. We have used the code from https://github.com/facebookresearch/open_lth to generate all the results for LTH explanation. For the experiment of comparison of different pruning algorithms, we have modified the code from <https://github.com/ganguli-lab/Synaptic-Flow>. The results for corresponding to the LTH hypothesis is presented. Then detailed results concerning the pruning algorithms are described.

A.1. More Results for Lenet Architecture on MNIST Dataset

Here, we study different pruning scheme of IMP to understand the Ramanujan characteristic of the bipartite graph in each layer and its relation to the classification performance in terms of accuracy.

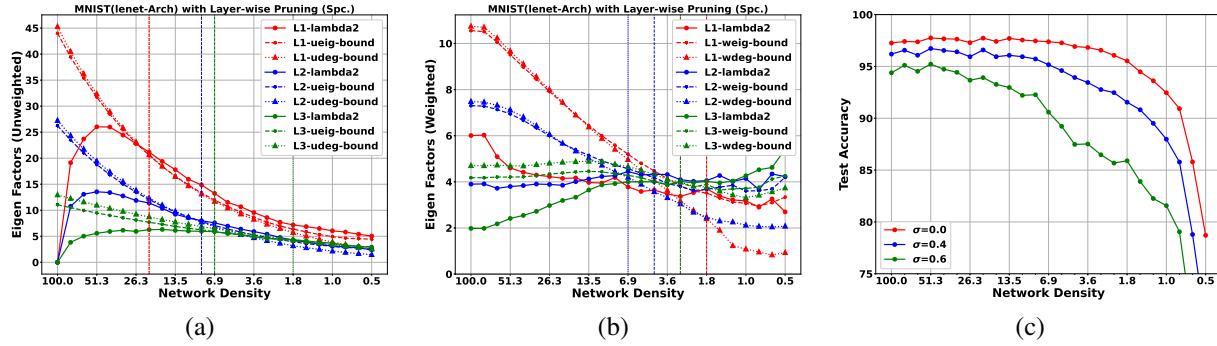


Figure 5. The results for Lenet using MNIST dataset with pruning percentile $p_{fc} = 0.2$ and $p_{out} = 0.1$ (layer-wise-pruning); (a) eigen factors for the unweighted graph, (b) eigen factors for the weighted graph, (c) test accuracy

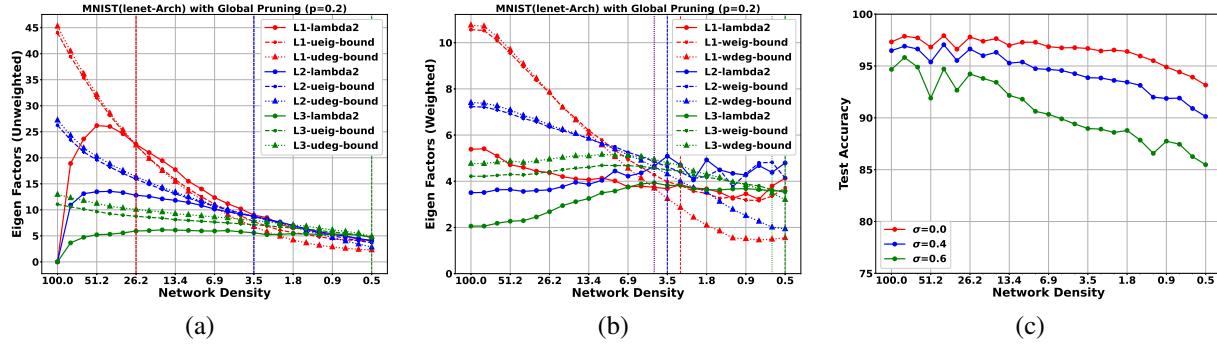


Figure 6. The eigen factors for global pruning with pruning percentile $p = 0.2$ on MNIST dataset using Lenet architecture

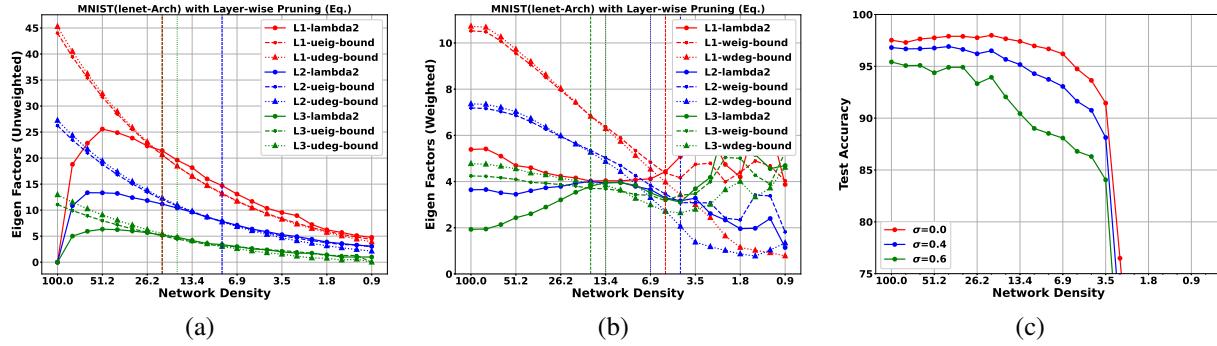


Figure 7. The eigen factors for layer-wise pruning with pruning percentile $p = 0.2$ on MNIST dataset using Lenet architecture

A.2. More Results for Conv4 Architecture on CIFAR10 Dataset

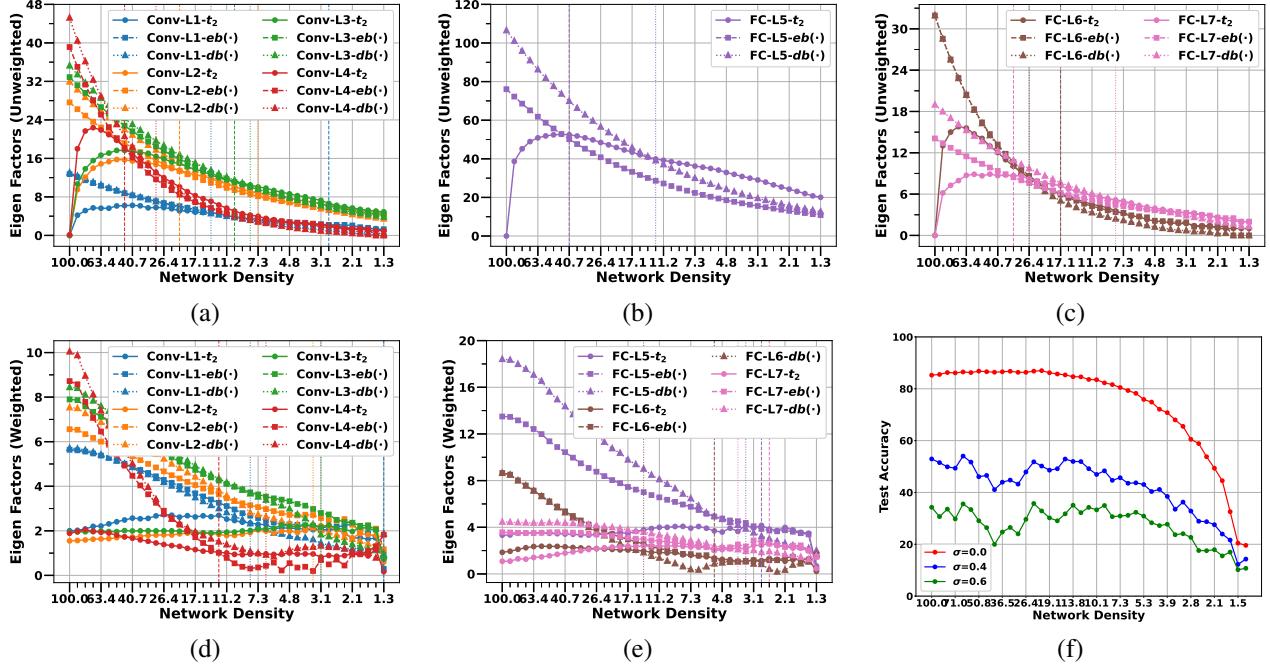


Figure 8. The eigen factors for the IMP pruning algorithm on CIFAR10 dataset using Conv4 architecture; (a-e) Layer-wise Pruning with $p_{conv} = 0.1$, $p_{fc} = 0.2$, and $p_{out} = 0.1$, (f) test accuracy

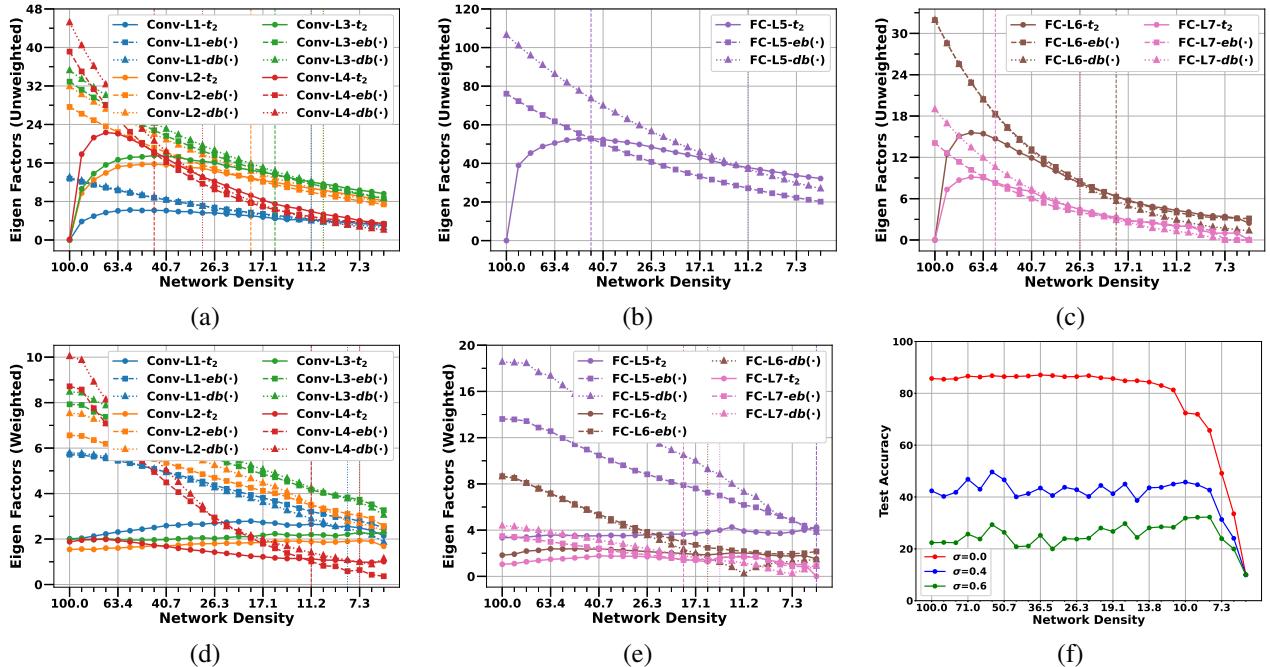


Figure 9. The eigen factors for the IMP pruning algorithm on CIFAR10 dataset using Conv4 architecture; (a-e) Layer-wise Pruning with $p_{conv} = 0.1$, $p_{fc} = 0.2$, and $p_{out} = 0.2$, (f) test accuracy

A.3. Results for Conv6 Architecture on CIFAR10 Dataset

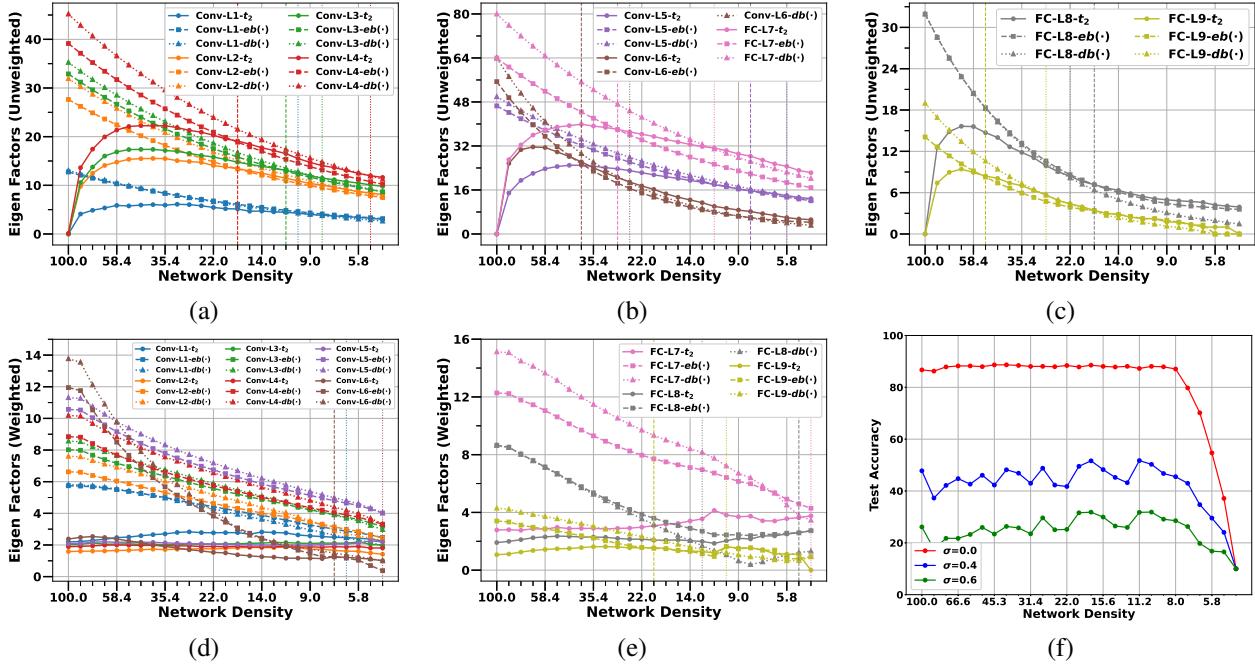


Figure 10. The eigen factors for the IMP pruning algorithm on CIFAR10 dataset using Conv6 architecture; (a-e) Layer-wise Pruning with $p_{conv} = 0.1$, $p_{fc} = 0.2$, and $p_{out} = 0.2$

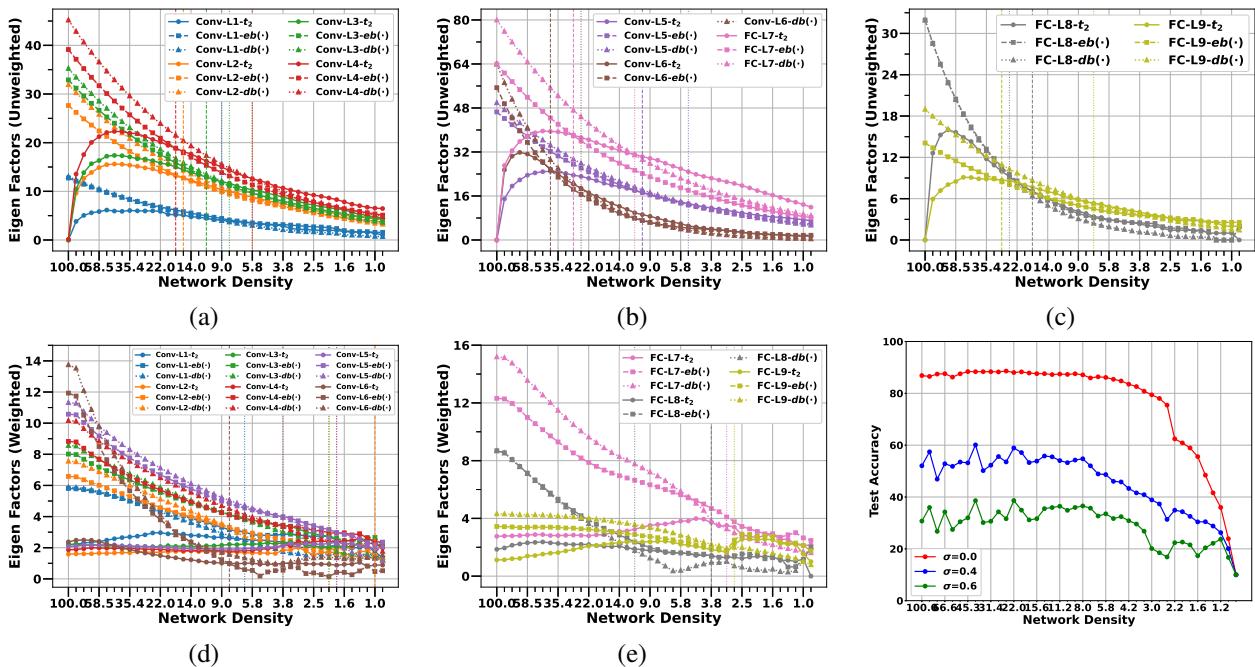


Figure 11. The eigen factors for the IMP pruning algorithm on CIFAR10 dataset using Conv6 architecture; (a-e) Layer-wise Pruning with $p_{conv} = 0.1$, $p_{fc} = 0.2$, and $p_{out} = 0.1$

A.4. Results for VGG19 on CIFAR10 Dataset

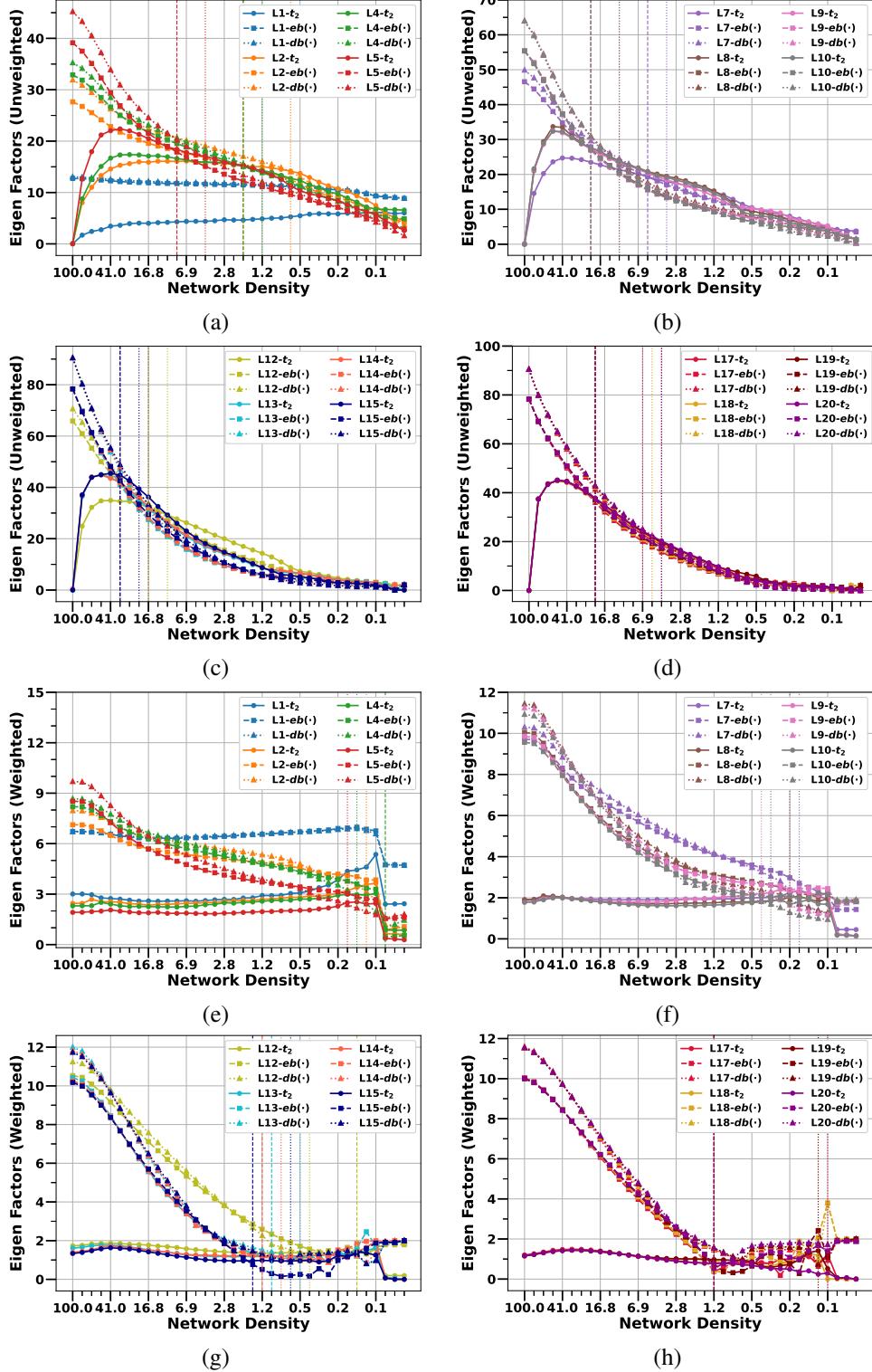


Figure 12. The eigen factors for the IMP pruning algorithm on CIFAR10 dataset using VGG19 architecture and Global Pruning with $p = 0.2$ and $lr = 0.01$

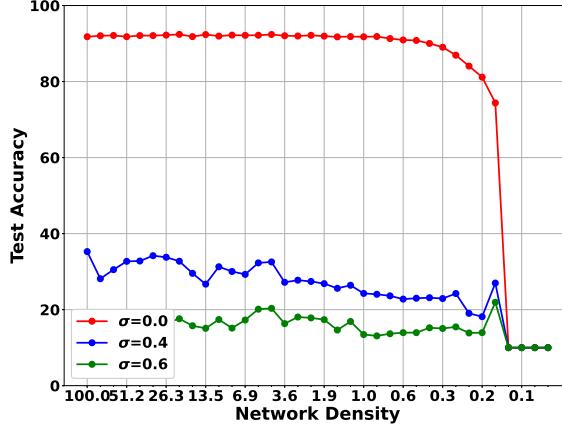


Figure 13. Results for the test accuracy of VGG19 on different noise level

A.5. The Statistical Significance of the Relation between the Ramanujan Graph Property and LTH

We have performed statistical significance tests for the zero crossing of Δ values. At every pruning iteration, we compute the Δ values at each layer for 10 different runs. We test the null hypothesis $H_0 : \max(\Delta_S, \Delta_R) \geq 0$ and the alternate hypothesis $H_1 : \max(\Delta_S, \Delta_R) < 0$. A t-test is performed with a confidence level 0.95, and the upper and lower mean value are computed for both the architectures in Table 4 and 5, for MNIST/LeNet and CIFAR10/Conv4, respectively. As the variance of Δ_R or Δ_S depends largely on the network size and the weights, the upper and lower bound of the mean value will help in deciding the stopping point of further pruning for different layers. In Figures 3 and 4, the gray region corresponds to the pruning iterations where for all the layers the null hypothesis H_0 is significant. For the pink and the white regions the null hypothesis is true for some and none of the layers respectively. Similarly, at every pruning iteration we test the drop of accuracy as compared to the unpruned network. Let a_0 and a_t be the test accuracy of the unpruned network and the network at pruning iteration t respectively. We test the null hypothesis $H_0 : \frac{a_0 - a_t}{a_0} \leq k$. A t-test is again performed. We observe that for $k = 0.001$ the null hypothesis is significant in the gray region, and for $k = 0.01$ in the pink region for MNIST/LeNet. We also observe that same for CIFAR/conv4 with different values of k . Here, for $k = 0.01$ the null hypothesis is significant in the gray region, and for $k = 0.05$ in the pink region for CIFAR/conv4. Thus the statistical significance of the relation between Ramanujan graph property and LTH is established.

Table 4. The quantity mean- Δ_R at the loss of Ramanujan Graph property and its upper and lower bound for confidence level 0.95 - for MNIST/LeNet

Layer	Density	sample mean	Upper mean	Lower Mean
FC-L1	26.31	-0.000868211	0.01904843	-0.020784853
FC-L2	10.82	-0.023357875	-0.010215451	-0.036500298
FC-L3	1.84	0.016654836	0.068376088	-0.035066416

Table 5. The quantity mean- Δ_R at the loss of Ramanujan Graph property and its upper and lower bound for confidence level 0.95 - for CIFAR10/Conv4

Layer	Density	sample mean	Upper mean	Lower Mean
Conv-L1	11.1879	-0.0163	0.0153	-0.0439
Conv-L2	10.0586	-0.0175	0.0445	-0.0467
Conv-L3	9.0443	-0.0087	0.0321	-0.0114
Conv-L4	29.3766	-0.0074	-0.0017	-0.0275
FC-L1	29.3766	-0.0059	0.0114	-0.0063
FC-L2	12.4454	-0.0184	-0.0123	-0.0295
FC-L3	8.1330	-0.0228	0.0158	-0.0371

A.6. Importance of the Ramanujan Graph Property in Designing Pruning Algorithms

Here, we show the result in support of the importance of the Ramanujan Graph property in the sparse sub-graph of the lottery ticket. As more number of layers starts losing this property the network is prone to observe impact in its performance. We show the comparison of accuracy and the number of layers lose the Ramanujan Graph property (in same density) for two pruning Algorithms, IMP, and Synflow in Figure 14. We see the result for Conv4/CIFAR10. With density of 3.16 the IMP achieves accuracy of 60.61% while SynFlow meets 77.18% accuracy. By analyzing the detail it has been found that the Ramanujan property is lost in FC layer-1 only for SynFlow where the same is observed in IMP for both Conv Layer-4 and FC layer-1. Hence, we can infer that a better pruning algorithm or a winning ticket must have the maximum possible layers in preserving the Ramanujan graph property.

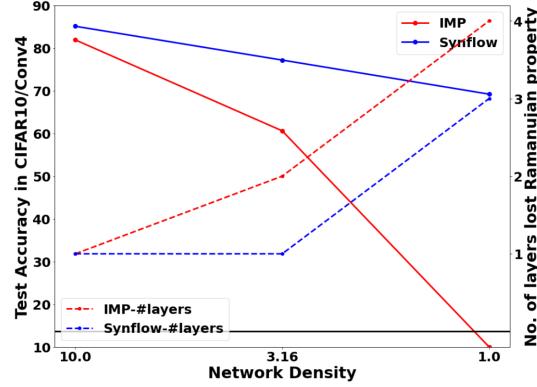


Figure 14. Results for two pruning algorithms IMP, and Synflow on CIFAR10/Conv4