Proceedings of Machine Learning Research vol 278:1–7, 2025 2025 2nd International Conference on Machine Learning and Intelligent Computing

Dispersive optical solitons of the stochastic Fokas-Lenells equation with multiplicative white noise based on the trial equation method

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Editors: Nianyin Zeng, Ram Bilas Pachori and Dongshu Wang

Abstract

In this paper, we study the Fokas-Lenells (FL) equation with multiplicative white noise in the Itô sense for polarization-maintaining fibers, using the trial equation method and the polynomial complete discriminant system approach. All exact solutions of the equation in its general form are obtained, including solitary wave solutions, trigonometric function solutions, rational solutions, and Jacobian elliptic function doubly periodic solutions. Furthermore, several representative numerical simulation results are presented under given parameter conditions.

Keywords: Fokas-Lenells equation; test equation method;polynomial complete discriminant system;exact solution.

1. Introduction

In recent years, the propagation model of ultrashort pulses in polarization-maintaining fibers has attracted significant attention in the academic community. Due to its intrinsic mathematical structure and physical analogies, this model holds great importance across multiple fields including nonlinear optics (Wen et al., 2007), photonics, communication technologies (Wada, 2004), and precision measurements. In the field of nonlinear optics, the nonlinear Schrödinger equation (NLSE) serves as the fundamental model governing ultrashort pulse propagation in nonlinear media, providing crucial theoretical support. Consequently, numerous academic studies have emerged focusing on the NLSE, yielding substantial research achievements (Arshed and Arif, 2020; Lenells, 2010). Arshed and Arif (2020) obtained novel soliton solutions and various exact traveling wave solutions using two analytical approaches. Lenells (2010) developed an innovative application of the dressing method to generate previously unknown integrable NLSE systems, culminating in rigorous N-soliton solution formulae. This work offers significant guidance for applications of ultrashort pulses in fiber-optic communications and related domains. Moreover, as evidenced by Referencess (Pal et al., 2017; Xu et al., 2018; Biswas and Milovic, 2009), extensive systematic studies have been performed on the NLSE by various research groups.

To investigate wave dynamic problems involving higher-order dispersion and nonlinear effects, it is necessary to develop integrable generalizations of the nonlinear Schrödinger equation. In this paper, we study the Fokas-Lenells (FL) equation with multiplicative white noise in the Itô sense for polarization-maintaining fibers. This equation is generally employed to describe the propagation of nonlinear pulses in optical fibers, and its general form is given below (Zayed et al., 2022):

$$iu_t+\alpha_1 u_{xx}+\alpha_2 u_{xt}+|u|^2(au+ibu_x)+\mu(u-i\alpha_2 u_x)\frac{dW(t)}{dt}=i[cu_x+d(|u|^2u)_x+\lambda(|u|^2)_x u], \quad (1)$$

Here, $u(x,t)$ is a complex-valued function representing the wave profile, where $i=\sqrt{-1}$ denotes the imaginary unit, x and t represent the spatial and temporal variables respectively, and

$a, b, c, d, \alpha_1, \alpha_2, \mu, \lambda$ are real-valued coefficients with $a, b, \alpha_1, \alpha_2, \mu$ corresponding to self-phase modulation (SPM), nonlinear dispersion term, chromatic dispersion (CD), spatio-temporal dispersion (STD), and noise intensity correlation coefficient, respectively, while $W(t)$ represents the Wiener process and $\frac{dW(t)}{dt}$ stands for white noise.

Currently, integrable generalizations of the nonlinear Schrödinger equation have been extensively studied with numerous emerging models. Kartashov et al. (2011) investigated solitons in nonlinear lattices and successfully obtained soliton solutions. Triki and Wazwaz (2017) systematically solved the Fokas-Lenells (FL) equation by establishing an undetermined coefficient equation, obtaining a class of chirped soliton solutions under specific constraints. Gómez (2024) investigated a generalized form of the Fokas-Lenells equation (FLE) using a modified tanh-coth method and successfully derived standard FLE exact chirped soliton solutions.

While existing methods remain essential for solving nonlinear PDEs (NPDEs), their diversity precludes universal solutions. Current approaches typically require solution-form assumptions. This study seeks more effective alternatives to overcome these limitations. Notably, Liu (2011)'s trial equation method enables direct solutions without traditional assumptions (Cheng-Shi, 2006a; Liu, 2011; Cheng-Shi, 2006b). Liu (2011)'s method significantly enhances the systematicity and completeness of NPDE solutions, establishing a universal framework for complex nonlinear models (Li and Kai, 2023; Du, 2021; Yin, 2021).

2. Mathematical Analysis

This study adopts the traveling wave transformation approach by introducing the following wave variable:

$$u(x,t) = \rho(\xi)e^{i[-vx+kt+\mu W(t)-\mu^2 t]}, \xi = x - \phi t, \eta(x,t) = -vx + kt + \mu W(t) - \mu^2 t, \quad (2)$$

Here we employ the traveling wave transformation $\xi = x - \phi t$, where v, k, and ϕ are real constants representing the wavenumber, frequency, and soliton propagation velocity, respectively. The function $\rho(\xi)$ denotes the real-valued amplitude profile of the wave packet.By implementing the substitution of Eq. (2) in Eq. (1), we obtain the real part

$$-\phi - 2\alpha_1 v + \alpha_2 k - \alpha_2\mu^2 + \alpha_2\phi v - c + \rho^2(b - 3d - 2\lambda) = 0 \quad (3)$$

and the imaginary part

$$(\alpha_1 - \alpha_2\phi)\rho'' + [(k-\mu^2)(\alpha_2 v - 1) - \alpha_1 v^2 - cv]\rho + [a + v(b-d)]\rho^3 = 0 \quad (4)$$

Multiplying both sides of Eq.(4) by ρ and then integrating once about ξ, we acquire

$$(\rho')^2 = \frac{[a+v(b-d)]}{2(\alpha_2\phi - \alpha_1)}\rho^4 + \frac{(k-\mu^2)(\alpha_2 v - 1) - \alpha_1 v^2 - cv}{\alpha_2\phi - \alpha_1}\rho^2 + \frac{c}{\alpha_2\phi - \alpha_1}, \quad (5)$$

Let

$$a_4 = \frac{[a+v(b-d)]}{2(\alpha_2\phi - \alpha_1)}, a_2 = \frac{(k-\mu^2)(\alpha_2 v - 1) - \alpha_1 v^2 - cv}{\alpha_2\phi - \alpha_1}, a_0 = \frac{c}{\alpha_2\phi - \alpha_1}. \quad (6)$$

We get

$$(\rho')^2 = a_4\rho^4 + a_2\rho^2 + a_0 \quad (7)$$

Make a transformation

$$G = (4a_4)^{\frac{1}{3}}\rho^2, \xi_1 = (4a_4)^{\frac{1}{3}}\xi \tag{8}$$

equation (7) become

$$(G_{\xi_1})^2 = G^3 + Q_1 G^2 + Q_0 G \tag{9}$$

where

$$Q_1 = (\frac{16(\alpha_2\phi - \alpha_1)^2}{[a + v(b-d)]^2})^{\frac{1}{3}} \frac{(k-\mu^2)(\alpha_2 v - 1) - \alpha_1 v^2 - cv}{\alpha_2\phi - \alpha_1} \tag{10}$$

$$Q_0 = 4a_0 \frac{(\alpha_2\phi - \alpha_1)^{\frac{1}{3}}}{(2[a + v(b-d)])^{\frac{1}{3}}} \tag{11}$$

Reformulate the equation (9) as

$$\pm(\xi_1 - \xi_0) = \int \frac{dG}{\sqrt{F(G)}} \tag{12}$$

where

$$F(G) = G(G^2 + Q_1 G + Q_0) \tag{13}$$

The complete discriminant system of the second-order polynomial is

$$\Delta = Q_1^2 - 4Q_0 \tag{14}$$

The roots of the polynomial $G^2 + Q_1 G + Q_0$are classified through the discriminant system,and then all possible solutions of the integral(12) are pbtained.

3. Exact Solutions

Case 1. $\Delta > 0$ and $Q_0 = 0$. When $G > -Q_1$, if $Q_1 > 0$, we can obtain dark and singular solitons.

$$u_1 = \{(\frac{2[a+v(b-d)]}{\alpha_2\phi - \alpha_1})^{-\frac{1}{3}} \times (\frac{Q_1}{2} tanh^2(\frac{1}{2}\sqrt{\frac{Q_1}{2}}((\frac{2[a+v(b-d)]}{\alpha_2\phi - \alpha_1})^{\frac{1}{3}}\xi - \xi_0)) - Q_1)\}^{\frac{1}{2}} e^{i(-vx+kt+\mu W(t)-\mu^2 t)} \tag{15}$$

$$u_2 = \{(\frac{2[a+v(b-d)]}{\alpha_2\phi - \alpha_1})^{-\frac{1}{3}} \times (\frac{Q_1}{2} coth^2(\frac{1}{2}\sqrt{\frac{Q_1}{2}}((\frac{2[a+v(b-d)]}{\alpha_2\phi - \alpha_1})^{\frac{1}{3}}\xi - \xi_0)) - Q_1)\}^{\frac{1}{2}} e^{i(-vx+kt+\mu W(t)-\mu^2 t)} \tag{16}$$

If $Q_1 < 0$, a singular periodic wave can be obtained.

$$u_3 = \{(\frac{2[a+v(b-d)]}{\alpha_2\phi - \alpha_1})^{-\frac{1}{3}} \times (-\frac{Q_1}{2} tan^2(\frac{1}{2}\sqrt{-\frac{Q_1}{2}}((\frac{2[a+v(b-d)]}{\alpha_2\phi - \alpha_1})^{\frac{1}{3}}\xi - \xi_0)) - Q_1)\}^{\frac{1}{2}} e^{i(-vx+kt+\mu W(t)-\mu^2 t)} \tag{17}$$

Case 2. $\Delta > 0$ and $Q_0 \neq 0$. Suppose $\beta_1 < \beta_2 < \beta_3$. one of them is 0, and others two are roots of $G^2 + Q_1 G + Q_0$. When $\beta_1 < G < \beta_2$, we have the snoidal wave.

$$u_4 = \{(\frac{2[a+v(b-d)]}{\alpha_2\phi - \alpha_1})^{-\frac{1}{3}} \times [\beta_1 + (\beta_2 - \beta_1) sn^2(\frac{\sqrt{\beta_3 - \beta_1}}{2}((\frac{2[a+v(b-d)]}{\alpha_2\phi - \alpha_1})^{\frac{1}{3}}\xi - \xi_0), m)]\}^{\frac{1}{2}} e^{i(-vx+kt+\mu W(t)-\mu^2 t)} \tag{18}$$

When $G > \beta_3$, a combined elliptic cosione wave and elliptic sine wave can be obtained.

$$u_5 = \{(\frac{2[a+v(b-d)]}{\alpha_2\phi-\alpha_1})^{-\frac{1}{3}} \times [\frac{\beta_3 - \beta_2 sn^2[\frac{\sqrt{\beta_3-\beta_2}}{2}((\frac{2[a+v(b-d)]}{\alpha_2\phi-\alpha_1})^{\frac{1}{3}}\xi - \xi_0), m]}{cn^2[\frac{\sqrt{\beta_3-\beta_2}}{2}((\frac{2[a+v(b-d)]}{\alpha_2\phi-\alpha_1})^{\frac{1}{3}}\xi - \xi_0), m]}]\}^{\frac{1}{2}} e^{i(-vx+kt+\mu W(t)-\mu^2 t)} \tag{19}$$

where $m^2 = \frac{\beta_2-\beta_1}{\beta_3-\beta_1}$.

Case 3. $\Delta = 0$, For $G > 0$. if $Q_1 < 0$, we have dark solitons and singular solitons.

$$u_6 = \{(\frac{2[a+v(b-d)]}{\alpha_2\phi-\alpha_1})^{-\frac{1}{3}} \times (-\frac{Q_1}{2} tanh^2(\frac{1}{2}\sqrt{-\frac{Q_1}{2}}((\frac{2[a+v(b-d)]}{\alpha_2\phi-\alpha_1})^{\frac{1}{3}}\xi - \xi_0)))\}^{\frac{1}{2}} e^{i(-vx+kt+\mu W(t)-\mu^2 t)} \tag{20}$$

$$u_7 = \{(\frac{2[a+v(b-d)]}{\alpha_2\phi-\alpha_1})^{-\frac{1}{3}} \times (-\frac{Q_1}{2} coth^2(\frac{1}{2}\sqrt{-\frac{Q_1}{2}}((\frac{2[a+v(b-d)]}{\alpha_2\phi-\alpha_1})^{\frac{1}{3}}\xi - \xi_0)))\}^{\frac{1}{2}} e^{i(-vx+kt+\mu W(t)-\mu^2 t)} \tag{21}$$

If $Q_1 > 0$, a singular periodic wave can be obtained.

$$u_8 = \{(\frac{2[a+v(b-d)]}{\alpha_2\phi-\alpha_1})^{-\frac{1}{3}} \times (\frac{Q_1}{2} tan^2(\frac{1}{2}\sqrt{\frac{Q_1}{2}}((\frac{2[a+v(b-d)]}{\alpha_2\phi-\alpha_1})^{\frac{1}{3}}\xi - \xi_0)))\}^{\frac{1}{2}} e^{i(-vx+kt+\mu W(t)-\mu^2 t)} \tag{22}$$

If $Q_1 = 0$, a rational wave can be obtained.

$$u_9 = \{\frac{4(\frac{2[a+v(b-d)]}{\alpha_2\phi-\alpha_1})^{-\frac{1}{3}}}{((\frac{2[a+v(b-d)]}{\alpha_2\phi-\alpha_1})^{\frac{1}{3}}\xi - \xi_0)^2}\}^{\frac{1}{2}} e^{i(-vx+kt+\mu W(t)-\mu^2 t)} \tag{23}$$

Case 4. $\Delta < 0$, When $G > 0$, a cnoidal wave can be obtained.

$$u_{10} = \{(\frac{2[a+v(b-d)]}{\alpha_2\phi-\alpha_1})^{-\frac{1}{3}} \times (\frac{2\sqrt{Q_0}}{1 + cn(Q_0^{\frac{1}{4}}(\frac{2[a+v(b-d)]}{\alpha_2\phi-\alpha_1}\xi - \xi_0), m)} - \sqrt{Q_0})\}^{\frac{1}{2}} e^{i(-vx+kt+\mu W(t)-\mu^2 t)} \tag{24}$$

Where $m^2 = \frac{1-\frac{Q_1}{2\sqrt{Q_0}}}{2}$.

4. The physical manifestations of solutions

Example 1. Solitary wave solutions.

In case 3, Let $a_0 = \frac{1}{4}$, $a = c = d = v = k = \mu = \alpha_1 = \alpha_2 = Q_0 = 1$, $b = 2$, $\phi = 5$, $\xi_0 = 0$ and $Q_1 = -2$, we get

$$|u_6| = \sqrt{(\frac{1}{2})^{-\frac{1}{3}} \times tanh^2[(\frac{1}{2})^{\frac{4}{3}}\xi]} \tag{25}$$

$$|u_7| = \sqrt{(\frac{1}{2})^{-\frac{1}{3}} \times coth^2[(\frac{1}{2})^{\frac{4}{3}}\xi]} \tag{26}$$

Figure 1 and Figure 2 show the two-dimensional depictions of $|u_6|$ and $|u_7|$ respectively.

Example 2. Triangular function solutions

In case 1, Let $a = d = v = k = \mu = \alpha_2 = 1, b = 2, c = a_0 = Q_0 = \xi_0 = 0, \alpha_1 = Q_1 = -1$ and $\phi = -3$, we obtain

$$|u_3| = \sqrt{\frac{1}{2}tan^2(-\frac{1}{2}\sqrt{\frac{1}{2}}\xi) + 1} \tag{27}$$

Figure 3 show the two-dimensional depictions of $|u_3|$ respectively.

Example 3. Rational solution

In case 3, Let $a_0 = c = \alpha_1 = Q_1 = Q_0 = \xi_0 = 0$, $k = \mu = \alpha_2 = v = a = d = \phi = 1$ and $b = 2$, we get

$$|u_9| = \frac{4^{\frac{2}{3}}}{4^{\frac{1}{3}}\xi} \tag{28}$$

Figure 4 show the two-dimensional depictions of $|u_9|$ respectively.

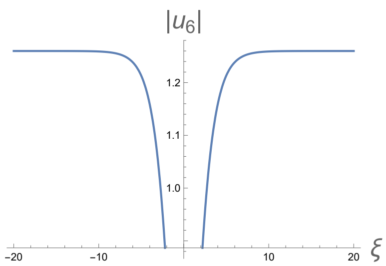

Figure 1: $|u_6|$.

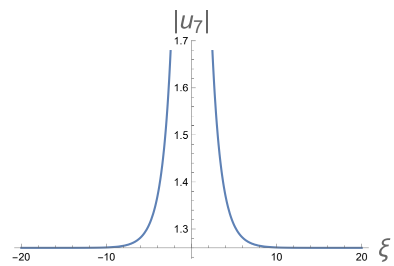

Figure 2: $|u_7|$.

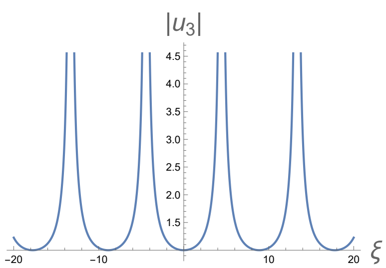

Figure 3: $|u_3|$.

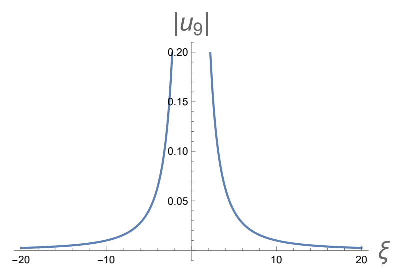

Figure 4: $|u_9|$.

Example 4. Jacobi elliptic function double periodic solutions

In case 2, Let $a = c = d = v = k = \mu = \alpha_2 = \beta_2 = 1$, $b = Q_0 = \beta_3 = 2$, $\xi_0 = \beta_1 = 0$,$\alpha_1 = 5$,$\phi = 9$, $a_0 = \frac{1}{4}$ and $Q_1 = -3$, we obtain

$$|u_4| = \sqrt{sn^2[\frac{\sqrt{2}}{2}(\xi), \sqrt{\frac{1}{2}}]} \tag{29}$$

when $G > 2$, we obtain

$$|u_5| = \sqrt{\frac{sn^2[\frac{\sqrt{2}}{2}(\xi)], \sqrt{\frac{1}{2}}}{cn^2[\frac{\sqrt{2}}{2}(\xi)], \sqrt{\frac{1}{2}}}} \tag{30}$$

Figure 5 and Figure 6 show the two-dimensional depictions of $|u_4|$ and $|u_5|$ respectively.

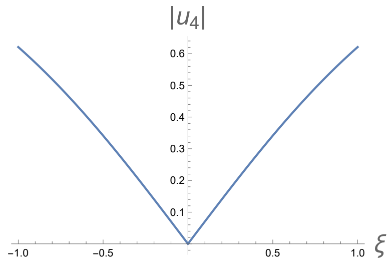

Figure 5: $|u_4|$.

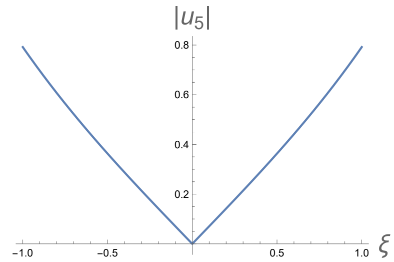

Figure 6: $|u_5|$.

5. Conclusion

This paper investigates the Fokas-Lenells (FL) equation with multiplicative white noise in the Itô sense for polarization-maintaining fibers. The FL equation is commonly used to describe ultrashort pulse propagation in polarization-maintaining fibers. By employing the trial equation method, we obtain ten classes of exact solutions with diverse forms, including solitary wave solutions, trigonometric solutions, rational solutions, and Jacobian elliptic function double-periodic solutions. Within the parameter ranges, we construct two-dimensional graphical representations of these solutions by specifying particular parameter values.

Compared with previous studies, the trial equation method combined with the polynomial complete discriminant system approach adopted in this work yields more comprehensive exact solutions. Notably, the proposed methodology provides crucial advantages for solving similar nonlinear differential equations and demonstrates significant value. Our research offers novel insights into the FL equation with Itô-type multiplicative noise in polarization-maintaining fibers while establishing a new framework for subsequent investigations.

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