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Ground States of Mixed Local and Nonlocal System of Choquard Type

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Abstract

This paper establishes the existence of solution for a family of weakly coupled nonlinear Choquard-type system. We apply variational methods and utilize the introduced manifold $\mathcal{N}_\omega$ to investigate this problem.

Keywords: Choquard; Local-nonlocal operators; Existence; Ground state

1. Introduction

This article seeks to address the following problem: the Choquard system with weak coupling, outlined below

$$\begin{cases} -\Delta m + (-\Delta)^s m = (I_\alpha * |m|^p)|m|^{p-2}m + \omega n & \text{in} \quad \mathbb{R}^N \\ -\Delta n + (-\Delta)^s n = (I_\alpha * |n|^p)|n|^{p-2}n + \omega m & \text{in} \quad \mathbb{R}^N \end{cases} \tag{1}$$

where $N \geq 3$, $0 < \alpha < N$, $0 < s < 1$, $\frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-2}$, $0 < \omega < 1$, I_α is the Riesz potential, defined as

$$I_\alpha = \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{\frac{N}{2}} 2^\alpha |x|^{N-\alpha}}.$$

Γ is the Gamma function, $(-\Delta)^s$ is defined by

$$(-\Delta)^s m = C(N,s) P.V. \int_{\mathbb{R}^N} \frac{m(x) - m(y)}{|x-y|^{N+2s}} \mathrm{d}y.$$

where $C(N,s)$ is a normalizing constant and $P.V.$ denotes the Cauchy principal value. The operator corresponding to system (1), discussed in this article, consists of the classical Laplacian operator and the fractional Laplacian operator , commonly referred to as the mixed local and nonlocal operator. In the past five years, mixed operators have attracted considerable interest from researchers, with applications spanning across various domains including physical phenomena, mathematical biology, and resource management. Luo and Hajaiej (2022), Luo and Xie (2023) studied existence and non-existence, Biagi et al. (2022) studied maximum principle results and Su et al. (2025) studied regularity results.

Pekar (1954) described the equation for polarons in a stationary state in solid-state quantum mechanics, referred to as the Choquard or Schrödinger-Newton equation.

$$-\Delta u + \lambda u = (I_\alpha * |u|^2)u, \quad u \in H^1(\mathbb{R}^N) \tag{2}$$

Since its inception, related research has gradually flourished, with numerous scholars conducting in-depth studies of the equation over the past few decades, and it has been widely applied to practical problems. Variational methods are one of the commonly used analytical tools in these studies. Anthal et al. (2023) investigated the existence of solutions to the Choquard equation combined with mixed operators.

$$\mathcal{L}u = \left(\int_{\Omega} \frac{|u(y)|^{2^*_\mu}}{|x-y|^\mu} \right) |u|^{2^*_\mu - 2}u + \lambda u^p \quad \text{in} \quad \Omega. \tag{3}$$

with $\mathcal{L} = -\Delta + (-\Delta)^s$, $2^*_\mu = (2N-\mu)/(N-2)$, $0 < \mu < N$, $p \in [1, 2^* - 1)$. The existence of the ground states for the local weakly coupled nonlinear Choquard system, has been investigated in Chen and Liu (2018).

$$\begin{cases} -\Delta u + u = (I_\alpha * |u|^p)|u|^{p-2}u + \lambda v & \text{in} \quad \mathbb{R}^N \\ -\Delta v + v = (I_\alpha * |v|^p)|v|^{p-2}v + \lambda u & \text{in} \quad \mathbb{R}^N \end{cases} \tag{4}$$

Inspired by the aforementioned literature, it is natural to consider whether we can generalize equations involving mixed operators to Choquard-type systems. As far as we are aware, no previous results exist for such a Choquard system, and in this paper, we address problem (1). At the same time, we observe when system (1) becomes a single Choquard equation, specifically when $\omega = 0$

$$-\Delta m + (-\Delta)^s m = (I_\alpha * |m|^p)|m|^{p-2}m \quad \text{in} \quad \mathbb{R}^N \tag{5}$$

system (1) can be interpreted as a perturbation with linear interaction relative to equation (5).

Theorem 1 *For any $0 < \omega < 1$, system (1) has a positive radial ground state $(m_\omega, n_\omega) \in \mathcal{N}_\omega$ with $J_\omega(m_\omega, n_\omega) = C_\omega > 0$. Furthermore, $(m_\omega, n_\omega) \to (m_1, n_1)$ in E as $\omega \to 0$, where m_1, n_1 are ground states of (5).*

2. Preliminaries

Let ,we definde

$$[m]_s = \left(\frac{C(N,s)}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|m(x) - m(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{2}},$$

where $m : \mathbb{R}^N \to \mathbb{R}$ is a measurable function. For a detailed study of the Hilbert space $X_0^1(\mathbb{R}^N)$, we refer to Biagi (2023). The Hilbert space is defined as

$$X_0^1(\mathbb{R}^N) = \{m \in L^{2^*}(\mathbb{R}^N) : \nabla m \in L^2(\mathbb{R}^N) \text{ and } [m]_s < \infty\}.$$

where $2^* = \frac{2N}{N-2}$. The embedding of $X_0^1(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ is continuous but not compact. $X_0^1(\mathbb{R}^N)$ with the norm defined by $||m||_{X_0^1} := (||m||^2 + [m]_s^2)^{\frac{1}{2}}$, and we denote by $||m||^2 = \int_{\mathbb{R}^N} |\nabla m|^2 \mathrm{d}x$. For convenience of notation, we define, if $m \in X_0^1(\mathbb{R}^N)$,

$$K(m,m) := \int_{\mathbb{R}^N} (I_\alpha * |m|^p)|m|^p \mathrm{d}x.$$

Let $E = X_0^1(\mathbb{R}^N) \times X_0^1(\mathbb{R}^N)$, endowed with the norm $||(m,n)||_E^2 = ||m||_{X_0^1}^2 + ||n||_{X_0^1}^2$. We now introduce the functional:

$$J_\omega(m,n) = \frac{1}{2}||m||_{X_0^1}^2 + ||n||_{X_0^1}^2 - \frac{1}{2p}K(m,m) - \frac{1}{2p}K(n,n) - \omega\int_{\mathbb{R}^N} mn\mathrm{d}x.$$

By Anthal et al. (2023), we observe that $J_\omega(m,v)$ is well defined. $J_\omega \in C^1(E,\mathbb{R})$, for any $\phi,\psi \in E$ we have

$$\begin{aligned}\langle J_\omega^{'}(m,n),(\phi,\psi)\rangle &= \int_{\mathbb{R}^N}\nabla m\nabla\phi\mathrm{d}x + \int_{\mathbb{R}^N}\nabla n\nabla\psi\mathrm{d}x \\ &+ \frac{C(N,s)}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(m(x)-m(y))(\phi(x)-\phi(y))}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y \\ &+ \frac{C(N,s)}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(n(x)-n(y))(\psi(x)-\psi(y))}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y \\ &- \int_{\mathbb{R}^N}(I_\alpha * |m|^p)|m|^{p-2}m\phi\mathrm{d}x - \int_{\mathbb{R}^N}(I_\alpha * |n|^p)|n|^{p-2}n\psi\mathrm{d}x - \omega\int_{\mathbb{R}^N}(m\phi + n\psi)\mathrm{d}x.\end{aligned}$$

We introduce the Nehari manifold associated with J_ω , that is,

$$\mathcal{N}_\omega = \left\{(m,n) \in E\setminus\{(0,0)\} : \langle J_\omega^{'}(m,n),(m,n)\rangle = 0\right\}.$$

Dedine

$$C_\omega = \inf\{J_\omega(m,n) : (m,n) \in \mathcal{N}_\omega\}.$$

Proposition 2 *(Hardy-Littlewood-Sobolev inequality. (Lieb and Loss, 2001)) Let* $p,q > 1$, $0 < \alpha < N$, $1 \le r < s < \infty$ *be such that* $\frac{1}{p}+\frac{1}{q}-\frac{\alpha}{N} = 1$, $\frac{1}{r}-\frac{1}{s} = \frac{\alpha}{N}$. *(i) For any* $f \in L^p(\mathbb{R}^N)$ *and* $g \in L^q(\mathbb{R}^N)$ *, there exists a constant* $C(N,p,\alpha)$, *independent of* f,h *such that*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{f(x)g(y)}{|x-y|^{N-\alpha}}\mathrm{d}x\mathrm{d}y \le C(t,n,\mu,r)\,\|f\|_{L^p(\mathbb{R}^N)}\,\|g\|_{L^q(\mathbb{R}^N)}\,.$$

(ii) For any $f \in L^r(\mathbb{R}^N)$ *,*

$$\left\|\frac{1}{|\centerdot|^{N-\alpha}}\right\|_{L^s(\mathbb{R}^N)} \le C(N,\alpha,r)||f||_{L^r(\mathbb{R}^N)}.$$

Lemma 3 *There exists* $\beta > 0$ *such that* $||(m,n)||_E > \beta$ *for any* $(m,n) \in \mathcal{N}_\omega$ *.*

Proof Since $(m,n) \in \mathcal{N}_\omega$, by Proposition 2 and the Minkowski inequality, we derive

$$\begin{aligned}||m||_{X_0^1}^2 + ||n||_{X_0^1}^2 = K(m,m) + K(n,n) + 2\omega\int_{\mathbb{R}^N} mn\mathrm{d}x &\le ||m||_{X_0^1}^{2p} + ||n||_{X_0^1}^{2p} + 2\omega||m||_{X_0^1}||n||_{X_0^1} \\ &\le (||m||_{X_0^1}^2 + ||n||_{X_0^1}^2)^p + \omega(||m||_{X_0^1}^2 + ||n||_{X_0^1}^2)\end{aligned}$$

And $0 < \omega < 1$, we get $||(m,n)||_E > \beta > 0$.

Lemma 4 *For all $0 < \omega < 1$ and $(m, n) \in \mathcal{N}_\omega$, such that $C_\omega > 0$.*

Proof Since $(m, n) \in \mathcal{N}_\omega$

$$\begin{aligned} J_\omega(m, n) &= \Big(\frac{1}{2} - \frac{1}{2p}\Big)(||m||^2_{X_0^1} + ||n||^2_{X_0^1}) - 2\omega \int_{\mathbb{R}^N} mn\mathrm{d}x \\ &\geq \Big(\frac{1}{2} - \frac{1}{2p}\Big)(||m||^2_{X_0^1} + ||n||^2_{X_0^1}) - 2\omega ||m||^2_{X_0^1} ||n||^2_{X_0^1} \\ &\geq \Big(\frac{1}{2} - \frac{1}{2p}\Big)(1 - \omega)(||m||^2_{X_0^1} + ||n||^2_{X_0^1}). \end{aligned} \tag{6}$$

By Lemma 3 $C_\omega \geq \Big(\frac{1}{2} - \frac{1}{2p}\Big)(1 - \omega)\beta^2 > 0$.

Lemma 5 *Let η_1, η_2 be positive solutions of (5), and let $t > 0$ such that $(t\eta_1, t\eta_2) \in \mathcal{N}_\omega$. Then $t \in (0, 1)$.*

Proof Since η_1, η_2 be positive solutions of (5), we obtain

$$||\eta_1||^2_{X_0^1} = K(\eta_1, \eta_1), \qquad ||\eta_2||^2_{X_0^1} = K(\eta_2, \eta_2). \tag{7}$$

Given that $t > 0$ satisfies $(t\eta_1, t\eta_2) \in \mathcal{N}_\omega$, we obtain

$$t^2(||\eta_1||^2_{X_0^1} + ||\eta_2||^2_{X_0^1} - 2\omega \int_{\mathbb{R}^N} \eta_1\eta_2\mathrm{d}x) = t^{2p}\big(K(\eta_1, \eta_1) + K(\eta_2, \eta_2)\big). \tag{8}$$

By combining (7), (8), we conclude

$$t^{2p-2} = \frac{||\eta_1||^2_{X_0^1} + ||\eta_2||^2_{X_0^1} - 2\omega \int_{\mathbb{R}^N} \eta_1\eta_2\mathrm{d}x}{||\eta_1||^2_{X_0^1} + ||\eta_2||^2_{X_0^1}} < 1.$$

Lemma 6 *Let $\{m_k\} \subset X^1_{0,r}(\mathbb{R}^N)$ and $m \in X^1_{0,r}(\mathbb{R}^N)$ such that $m_k \rightharpoonup m$ in $X^1_{0,r}(\mathbb{R}^N)$ as $k \to \infty$. Then, for any $\phi \in X^1_0(\mathbb{R}^N)$*

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} (I_\alpha * |m_k|^p)|m_k|^p = \int_{\mathbb{R}^N} (I_\alpha * |m|^p)|m|^p,$$
$$\lim_{k\to\infty} \int_{\mathbb{R}^N} (I_\alpha * |m_k|^p)|m_k|^{p-2}\phi = \int_{\mathbb{R}^N} (I_\alpha * |m|^p)|m|^{p-2}\phi.$$

Proof The proof is same as proposition 2.2. in Seok (2017).

3. Conclusion

We have presented an in-depth analysis of the ground state solutions to the system (1) in this paper. The analysis began by examining the existence of these solutions and then extended to the characterization of their asymptotic behavior. This study provides a foundation for extending to more complex nonlinear Choquard variational problem models, with similar energy minimization principles applicable to a broader range of cases. Future research will focus on generalizing these results to more scenarios, exploring the impact of different values of . We can investigate the system (1) in the critical Choquard nonlinear case, i.e., when $\frac{N+\alpha}{N-2}$.

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References

Gurdev C Anthal, Jacques Giacomoni, and Konijeti Sreenadh. A choquard type equation involving mixed local and nonlocal operators. *Journal of Mathematical Analysis and Applications*, 527(2): 127440, 2023.

Albert Baernstein et al. A unified approach to symmetrization. *Partial differential equations of elliptic type*, 35:47–91, 1994.

Stefano Biagi. The brezis-nirenberg problem for mixed local and nonlocal operators. In *Bruno Pini Mathematical Analysis Seminar*, volume 14, pages 15–37, 2023.

Stefano Biagi, Serena Dipierro, Enrico Valdinoci, and Eugenio Vecchi. Mixed local and nonlocal elliptic operators: regularity and maximum principles. *Communications in Partial Differential Equations*, 47(3):585–629, 2022.

Peng Chen and Xiaochun Liu. Ground states of linearly coupled systems of choquard type. *Applied Mathematics Letters*, 84:70–75, 2018.

Elliott H Lieb and Michael Loss. *Analysis*, volume 14. American Mathematical Soc., 2001.

Tingjian Luo and Hichem Hajaiej. Normalized solutions for a class of scalar field equations involving mixed fractional laplacians. *Advanced Nonlinear Studies*, 22(1):228–247, 2022.

Tingjian Luo and Qihuan Xie. Sharp existence of ground states solutions for a class of elliptic equations with mixed local and nonlocal operators and general nonlinearity. *Mathematics*, 11 (16):3464, 2023.

Solomon I Pekar. *Untersuchungen über die Elektronentheorie der Kristalle*. De Gruyter, 1954.

Jinmyoung Seok. Nonlinear choquard equations involving a critical local term. *Applied Mathematics Letters*, 63:77–87, 2017.

Xifeng Su, Enrico Valdinoci, Yuanhong Wei, and Jiwen Zhang. On some regularity properties of mixed local and nonlocal elliptic equations. *Journal of Differential Equations*, 416:576–613, 2025.

My Proof of Theorem 1

Proof First, we prove the existence of the ground state. Let $(m_k, n_k) \in \mathcal{N}_\omega$ be such that $J_\omega(m_k, n_k) \to C_\omega$. From (6), we conclude that $\{(m_k, n_k)\}$ is bounded in E . After selecting t_k we proceed with further variable substitution and optimization. Let $t_k > 0$ be such that $(t_k|m_k|, t_k|n_k|) \in \mathcal{N}_\omega$, we get

$$t_k^{2p-2} = \frac{||m_k||^2_{X_0^1} + ||n_k||^2_{X_0^1} - 2\omega \int_{\mathbb{R}^N} |m_k||n_k| \mathrm{d}x}{||\eta_1||^2_{X_0^1} + ||\eta_2||^2_{X_0^1}} \le \frac{||m_k||^2_{X_0^1} + ||n_k||^2_{X_0^1} - 2\omega \int_{\mathbb{R}^N} m_k n_k \mathrm{d}x}{||\eta_1||^2_{X_0^1} + ||\eta_2||^2_{X_0^1}} < 1. \tag{9}$$

This implies $0 < t_k \le 1$, then

$$\begin{aligned} J_\omega(t_k|m_k|, t_k|n_k|) &= t_k^{2p}\Big(\frac{1}{2} - \frac{1}{2p}\Big)(K(m_k, m_k) + K(n_k, n_k)) \\ &\le \Big(\frac{1}{2} - \frac{1}{2p}\Big)(K(m_k, m_k) + K(n_k, n_k)) = J_\omega(m_k, n_k). \end{aligned} \tag{10}$$

Thus, we assume that $m_k \ge 0$ and $n_k \ge 0$. Let $\widehat{m_k}$, $\widehat{n_k}$ be the symmetric decreasing rearrangement of m_k, n_k. From references Chen and Liu (2018), Lieb and Loss (2001), and Baernstein et al. (1994), we know that

$$\begin{aligned} &\int_{\mathbb{R}^N} |\nabla \widehat{m_k}|^2 \mathrm{d}x \le \int_{\mathbb{R}^N} |\nabla m_k|^2 \mathrm{d}x, \ \int_{\mathbb{R}^N} |\widehat{m_k}|^2 \mathrm{d}x = \int_{\mathbb{R}^N} |m_k|^2 \mathrm{d}x, \\ &[\widehat{m_k}]_s^2 \le [m_k]_s^2, \ \ K(\widehat{m_k}, \widehat{m_k}) \ge K(m_k, m_k), \ \int_{\mathbb{R}^N} \widehat{m_k}\widehat{n_k} \mathrm{d}x \ge \int_{\mathbb{R}^N} m_k n_k \mathrm{d}x. \end{aligned} \tag{11}$$

Let $\widehat{t_k} > 0$ be such that $(\widehat{t_k}\widehat{m_k}, \widehat{t_k}\widehat{n_k}) \in \mathcal{N}_\omega$, from (11)

$$\widehat{t_k}^{2p-2} = \frac{||\widehat{m_k}||^2_{X_0^1} + ||\widehat{n_k}||^2_{X_0^1} - 2\omega \int_{\mathbb{R}^N} \widehat{m_k}\widehat{n_k} \mathrm{d}x}{K(\widehat{m_k}, \widehat{m_k},) + K(\widehat{n_k}, \widehat{n_k})} \le \frac{||m_k||^2_{X_0^1} + ||n_k||^2_{X_0^1} - 2\omega \int_{\mathbb{R}^N} m_k n_k \mathrm{d}x}{K(m_k, m_k) + K(n_k, n_k)} < 1.$$

This implies $0 < \widehat{t_k} \le 1$,

$$\begin{aligned} J_\omega(\widehat{t_k}\widehat{m_k}, \widehat{t_k}\widehat{n_k}) &= \widehat{t_k}^{2p}\Big(\frac{1}{2} - \frac{1}{2p}\Big)(K(\widehat{m_k}, \widehat{m_k}) + K(\widehat{n_k}, \widehat{n_k})) \\ &\le \Big(\frac{1}{2} - \frac{1}{2p}\Big)(K(m_k, m_k) + K(n_k, n_k)) = J_\omega(m_k, n_k). \end{aligned}$$

Building on the previous analysis, we may assume that m_k, n_k are radial, $(m_k, n_k) \rightharpoonup (m_\omega, n_\omega)$ weakly in E . Further assume that $m_k \to m_\omega$ a.e. in $\mathbb{R}^N$, $n_k \to n_\omega$ a.e. in $\mathbb{R}^N$. m_ω, n_ω is radial . From Lemma 3 we obtain

$$\begin{aligned} K(m_k, m_k) + K(n_k, n_k) &= ||m_k||^2_{X_0^1} + ||n_k||^2_{X_0^1} - 2\omega \int_{\mathbb{R}^N} m_k n_k \mathrm{d}x \\ &\ge (1-\omega)(||m_k||^2_{X_0^1} + ||n_k||^2_{X_0^1}) \ge (1-\omega)\beta^2. \end{aligned}$$

From lemma 6, we obtain $K(m_\omega, m_\omega) + K(n_\omega, n_\omega) \ge (1-\omega)\beta^2 > 0$. From this, we deduce that $(m_\omega, n_\omega) \ne (0, 0)$ and m_ω, n_ω are not identically 0. Applying Fatou's lemma and Lemma 3, we obtain

$$||m_\omega||^2_{X_0^1} + ||n_\omega||^2_{X_0^1} - 2\omega \int_{\mathbb{R}^N} m_\omega n_\omega \mathrm{d}x \le K(m_\omega, m_\omega) + K(n_\omega, n_\omega).$$

Let $t > 0$, be such that $(tm_\omega, tn_\omega) \in \mathcal{N}_\omega$. By the same calculation as in (9) and (10), we obtain $0 < t \leq 1$ and

$$C_\omega \leq J_\omega(tm_\omega, tn_\omega) \leq J_\omega(m_\omega, n_\omega) = \lim_{k\to\infty} J_\omega(m_k, n_k) = C_\omega.$$

We can get $t = 1$, is achieved by $(m_\omega, n_\omega) \in \mathcal{N}_\omega$ and $m_\omega, n_\omega \geq 0$. By the strong maximum principle, $m_\omega, n_\omega > 0$. We analyze the asymptotic behavior of the ground state. Let $\{\omega_k\}$ be a sequence such that $\omega_k \in (0, \frac{1}{2})$ and $\omega_k \to 0$ as $k \to \infty$. Let $(m_{\omega_k}, n_{\omega_k})$ denote the positive radial ground state of (1), as derived earlier. Obviously, $\{(m_{\omega_k}, n_{\omega_k})\}$ is bounded. It's easy to calculate $(m_{\omega_k}, n_{\omega_k}) \to (m_1, n_1)$ in E as $k \to \infty$.

Let η_1, eta_2 , be positive solutions of (5) and let $t_k > 0$ be such that $(t_k\eta_1, t_k\eta_2) \in \mathcal{N}_{\omega_k}$. We claim $t_k \to 1$ as $k \to \infty$. The proof refers to Chen and Liu (2018). we have

$$J_0(\eta_1, \eta_2) \leq J_0(m_1, n_1) = \lim_{k\to\infty} J_{\omega_k}(m_{\omega_k}, n_{\omega_k}) \leq \lim_{k\to\infty} J_{\omega_k}(t_k\eta_1, t_k\eta_2) = J_0(\eta_1, \eta_2).$$

Given that $J_0(m_1, n_1)$ is the sum of the energy of m_1, n_1 to the equation (5) and since η_1, η_2 have the same least energy, m_1, n_1 are also the ground states of (5).