Complete Characterization for Adjustment in Summary Causal Graphs of Time Series

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Abstract

The identifiability problem for interventions aims at assessing whether the total causal effect can be written with a do-free formula, and thus be estimated from observational data only. We study this problem, considering multiple interventions, in the context of time series when only an abstraction of the true causal graph, in the form of a summary causal graph, is available. We propose in particular both necessary and sufficient conditions for the adjustment criterion, which we show is complete in this setting, and provide a pseudo-linear algorithm to decide whether the query is identifiable or not.

1 INTRODUCTION

Knowing the effect of interventions is key to understanding the effect of a treatment in medicine or the effect of a maintenance operation in IT monitoring systems for example. When one cannot perform interventions in practice, for example when these interventions may endanger people's life or when they may disrupt a critical process or be too costly, one can try and identify do-free formulas which allow one to estimate the effects of interventions using only observational data.

Finding such do-free formulas is referred to as the identifiability problem for interventions in causal graphs. Solving the identifiability problem usually amounts to providing a graphical criterion under which the total effect can be identified, and in providing a do-free formula for its estimation on observational data. The problem is, under causal sufficiency, relatively easy for simple graphs, like DAGs (directed acyclic graphs) for static variables (Pearl, 1995) or FTCGs (full time causal graphs) for time series (Blondel et al., 2016), where the backdoor criterion is sound and complete for monovariate interventions. It becomes much harder when the graphs considered are abstractions of simple graphs, like CPDAGs (completed partially directed acyclic graphs) and MPDAGs (maximally oriented partially directed) for static variables (Maathuis and Colombo, 2013; Perkovic et al., 2016) or SCGs (summary causal graphs) for time series (Assaad et al., 2024). This is due to the fact that, for interventions to be identifiable, one needs to prove that the same do-free formula holds in all the simpler causal graphs corresponding to the abstraction considered.

Despite this increased complexity, Perkovic (2020) was able to propose, under causal sufficiency, a sound and complete graphical criterion to the identifiability problem for CPDAGs and MPDAGs, namely the general adjustment criterion. However, for SCGs, only a sufficient condition for identifiability has been proposed so far (Assaad et al., 2024), using the backdoor criterion which is sound but not complete and under causal sufficiency.

We study in this work the identifiability problem in SCGs under causal sufficiency. In particular:

- We introduce a common adjustment criterion, which we show is both sound and complete for the adjustment formulae,
- We propose both necessary and sufficient conditions which further characterize conditions for identifiability by adjustment in SCGs,
- Based on these conditions, we derive an algorithm of limited (pseudo-linear) complexity to decide whether the problem is identifiable or not.

These results are established here for a single effect and multiple interventions, and hold, on different forms, whether the consistency through time assumption is made or not. They furthermore rely on novel concepts and tools.

The remainder of the paper is structured as follows: related work is discussed in Section 2; Section 3 introduces the main notions while Section 4 presents our main result regarding identifiability with the adjustment criterion without assuming *consistency through time*; Section 5 presents a

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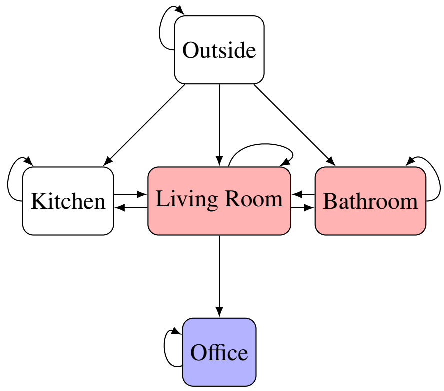


Figure 1: Thermoregulation (Assaad et al., 2024; Peters et al., 2013). Only the living room and bathroom have radiators on which we can intervene, highlighted in red. In both scenarios, we are interested in the temperature in the office, highlighted in blue. **Scenario 1**: $P(\mathrm{Of}_t \mid do(L_{t-1}, L_t, B_{t-1}, B_t))$. **Scenario 2**: $P(\mathrm{Of}_t \mid do(L_{t-1}, B_{t-1}))$.

similar result when *consistency through time* holds and a numerical experiment to illustrate the estimation; lastly, Section 6 concludes the paper. All proofs are provided in the Supplementary Material.

Running Example. As a running example throughout this paper, we consider the SCG in Figure 1, which models thermoregulation in a house where only the living room and bathroom have radiators. For notational simplicity, let L_t, K_t, B_t, Of_t, and Out_t denote the temperatures in the living room, kitchen, bathroom, office, and outside at time t, respectively. In Scenario 1, we aim to predict the office temperature at time t, assuming interventions that set the living-room and bathroom thermostats at times $t-1$ and t: $P(\mathrm{Of}_t \mid do(L_{t-1}, L_t, B_{t-1}, B_t))$. In Scenario 2, we assume interventions only at time $t-1$, giving $P(\mathrm{Of}_t \mid do(L_{t-1}, B_{t-1}))$. The Python implementation is available at this repository.[1]

2 STATE OF THE ART

The identifiability problem for DAGs and under causal sufficiency can be solved with the backdoor criterion, which is sound and complete for total effects with single interventions (Pearl, 1995). However, Shpitser et al. (2010) have shown that this criterion does not allow one to identify all possible adjustment sets. When the backdoor is not complete, *e.g.*, with hidden confounders or multiple interventions, one may relate to the do-calculus (Pearl, 1995) and the associated ID algorithm, which are sound and complete (Shpitser et al., 2010).

For CPDAGs, Maathuis and Colombo (2013); Perkovic et al. (2016) provided both necessary and sufficient conditions of identifiability for single interventions, which are nevertheless only sufficient for multiple interventions. Perkovic (2020) later developed necessary and sufficient conditions under causal sufficiency and the adjustment criterion for MPDAGs, which encompass DAGs, CPDAGs and CPDAGs with background knowledge. When considering latent confounding, Jaber et al. (2022); Wang et al. (2023) provided sufficient conditions of identifiability for PAGs (partial ancestral graphs). Cluster DAGs (Anand et al., 2023) constitute another interesting abstraction of simple graphs as they encode partially understood causal relationships between variables grouped into predefined clusters, within which internal causal dependencies remain unspecified. They extended do-calculus to establish necessary and sufficient conditions for identifying total effects in these structures.

Fewer studies have however been devoted to the identifiability problem on causal graphs defined over time series, like FTCGs, and abstractions one can define over them (Assaad et al., 2022), like ECGs (extended summary causal graphs) and SCGs. As mentioned before, if the problem can be solved relatively easily for FTCGs (Blondel et al., 2016), it is more complex for SCGs. Eichler and Didelez (2007) provided sufficient conditions for identifiability of the total effect on graphs based on time series which can directly be generalized to SCGs with no instantaneous relations. With possible instantaneous relations, Assaad et al. (2023) demonstrated that the total effect is always identifiable on SCGs under causal sufficiency and in the absence of cycles larger than one in the SCG (allowing only self-causes). Another assumption one can make to simplify the problem is to consider that the underlying causal model is linear. This allowed Ferreira and Assaad (2024) to propose both necessary and sufficient conditions for identifying direct effects in SCGs. On a slightly different line, Assaad (2025) provided sufficient conditions based on the front-door criterion when causal sufficiency is not satisfied. The most general result proposed so far on SCGs is the one presented by Assaad et al. (2024), who showed that, under causal sufficiency, the total effect is always identifiable in ECGs and exhibited sufficient conditions for identifiability by common backdoor assuming consistency through time and considering single interventions (but without making assumptions on the form of the SCG or the underlying causal model).

Our work fits within this line of research as it also addresses the identifiability problem in SCGs and goes further than previous studies by introducing a graphical criterion, shown to be both sound and complete, for identifiability in SCGs, together with necessary and sufficient conditions allowing one to efficiently decide on identifiability, without other assumptions than causal sufficiency. These results furthermore hold for both single and multiple interventions, with and without consistency through time.

[1]https://gricad-gitlab.univ-grenoble-alpes.fr/yvernesc/multivariateicainscg

3 CONTEXT

3.1 NOTATIONS AND ELEMENTARY NOTIONS

For a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, if $X \rightarrow Y$, then X is a *parent* of Y and Y is a *child* of X. A *path* is a sequence of distinct vertices in which each vertex is connected to its successor by an edge in $\mathcal{G}$. A *directed path*, or a *causal path*, is a path in which all edges are pointing towards the last vertex. A *non-causal* path refers to any path that is not causal. If there is a directed path from X to Y, then X is an *ancestor* of Y, and Y is a *descendant* of X. The sets of parents, children, ancestors and descendants of X in $\mathcal{G}$ are denoted by $\mathrm{Pa}(X, \mathcal{G})$, $\mathrm{Ch}(X, \mathcal{G})$, $\mathrm{Anc}(X, \mathcal{G})$ and $\mathrm{Desc}(X, \mathcal{G})$ respectively. We write $X \rightsquigarrow Y$ (or equivalently $Y \leftsquigarrow X$) to indicate that the graph contains a directed path from X to Y consisting of at least one edge. Furthermore, the *mutilated graph* $\mathcal{G}_{\overline{\mathbf{X}}\underline{\mathbf{Y}}}$ represents the graph obtained by removing from $\mathcal{G}$ all incoming edge on $\mathbf{X}$ and all outgoing edges from $\mathbf{Y}$. The *skeleton* of $\mathcal{G}$ is the undirected graph given by forgetting all arrow orientations in $\mathcal{G}$. The *subgraph* $\mathcal{G}_{|S}$ of a graph $\mathcal{G}$ induced by a vertex set S includes all nodes in S and all edges in $\mathcal{G}$ with both endpoints in S. For two disjoint subsets $\mathbf{X}, \mathbf{Y} \subseteq \mathcal{V}$, a *path* from $\mathbf{X}$ to $\mathbf{Y}$ is a path from some $X \in \mathbf{X}$ to some $Y \in \mathbf{Y}$. A path from $\mathbf{X}$ to $\mathbf{Y}$ is *proper* if only its first node is in $\mathbf{X}$. A *backdoor path* between X and Y is a path between X and Y in which the first arrow is pointing to X. A *directed cycle* is a circular list of distinct vertices in which each vertex is a parent of its successor. If a path π contains $X_i \rightarrow X_j \leftarrow X_k$ as a subpath, then X_j is a *collider* on π. A path π is *blocked* by a subset of vertices $\mathbf{Z}$ if a non-collider in π belongs to $\mathbf{Z}$ or if π contains a collider of which no descendant belongs to $\mathbf{Z}$. Otherwise, $\mathbf{Z}$ *d-connects* π.

Let $\mathbf{X}, \mathbf{Y}$ and $\mathbf{Z}$ be pairwise distinct sets of variables in a DAG $\mathcal{G}$. $\mathbf{Z}$ is an *adjustment set* relative to $(\mathbf{X}, \mathbf{Y})$ in $\mathcal{G}$ if for a distribution P compatible with $\mathcal{G}$ (Pearl, 2000, Def. 1.2.2) we have[2]:

$$P(\mathbf{y} \mid \mathrm{do}(\mathbf{x})) = \begin{cases} P(\mathbf{y} \mid \mathbf{x}) & \text{if } \mathbf{Z} = \emptyset, \\ \sum_{\mathbf{z}} P(\mathbf{y} \mid \mathbf{x}, \mathbf{z}) P(\mathbf{z}) & \text{otherwise.} \end{cases}$$

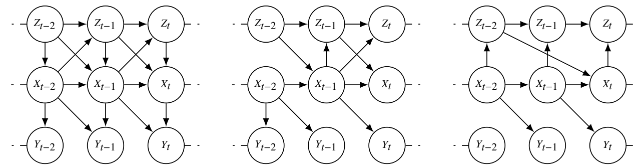
Lastly, following Perkovic et al. (2016), we make use of the forbidden set in the adjustment criterion.

Definition 1 (Adjustment criterion)**.** Let $\mathbf{X}, \mathbf{Y}$ and $\mathbf{Z}$ be pairwise distinct sets of variables in a DAG $\mathcal{G}$. $\mathbf{Z}$ is said to satisfy the *adjustment criterion* relative to $\mathbf{X}$ and $\mathbf{Y}$ in $\mathcal{G}$ if:

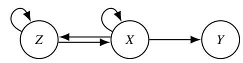
1. $\mathrm{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G}) \cap \mathbf{Z} = \emptyset$; and
2. $\mathbf{Z}$ blocks all proper non-causal paths from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathcal{G}$,

where the *forbidden set* $\mathrm{Forb}(\mathbf{X}, \mathbf{Y}, \mathcal{G})$ is the set of all descendants of any $W \notin \mathbf{X}$ which lies on a proper causal path from $\mathbf{X}$ to $\mathbf{Y}$.

[2]As standard in causality studies, $do(\mathbf{x})$ denotes the intervention setting $\mathbf{X}$ to $\mathbf{x}$.



(a) Three FTCGs, $\mathcal{G}^f_1$, $\mathcal{G}^f_2$ and $\mathcal{G}^f_3$.



(b) The SCG $\mathcal{G}^s$, reduced from any FTCG in (a).

Figure 2: Illustration: (a) three FTCGs; (b) the SCG which can be derived from any FTCG in (a).

3.2 CAUSAL GRAPHS IN TIME SERIES

Consider $\mathcal{V}$ a set of p observational time series and $\mathcal{V}^f = \{\mathcal{V}_t | t \in \mathbb{Z}\}$ the set of temporal instances of $\mathcal{V}$ observed over discrete time, where $\mathcal{V}_t$ corresponds to the variables of the time series at time t. We suppose that the discrete time observations $\mathcal{V}^f$ are generated from an *unknown* structural causal model, which defines an FTCG which we call the *true* FTCG and a joint distribution P over its vertices which we call the *true* probability distribution, which is compatible with, or Markov relative to, the true FTCG by construction.

As common in causality studies on time series, we consider in the remainder acyclic FTCGs with potential self-causes, *i.e.*, the fact that, for any time series X, $X_{t-\ell}$ ($\ell \in \mathbb{N}^*$) may cause X_t. Note that acyclicity is guaranteed for relations between variables at different time stamps and that self-causes are present in most time series. As a result, FTCGs are DAGs in which descendant relationships are constrained by the fact that causality cannot go backward in time, and all causal notions extend directly to FTCGs.

Experts are used to working with abstractions of causal graphs which summarize the information into a smaller graph that is interpretable, often with the omission of precise temporal information. We consider in this study a known causal abstraction for time series, namely *summary causal graphs* (Peters et al., 2013; Meng et al., 2020), which represents causal relationships among time series, regardless of the time delay between the cause and its effect.

Definition 2 (Summary causal graph (SCG), Figure 2b)**.** Let $\mathcal{G}^f = (\mathcal{V}^f, \mathcal{E}^f)$ be an FTCG built from the set of time series $\mathcal{V}$. The *summary causal graph* (SCG) $\mathcal{G}^s = (\mathcal{V}^s, \mathcal{E}^s)$ associated to $\mathcal{G}^f$ is such that:

- $\mathcal{V}^s$ corresponds to the set of time series $\mathcal{V}$,
- $X \rightarrow Y \in \mathcal{E}^s$ if and only if there exists at least one timepoint t and one temporal lag $0 \leq \gamma$ such that $X_{t-\gamma} \rightarrow Y_t \in \mathcal{E}^f$.

In that case, we say that $\mathcal{G}^s$ is *reduced from* $\mathcal{G}^f$.

SCGs may include directed cycles and even self-loops. For example, the three FTCGs in Figure 2a are acyclic, while the SCG in Figure 2b has a cycle. We use the notation $X \rightleftarrows Y$ to indicate situations where there exist time instants in which X causes Y and Y causes X. It is furthermore worth noting that if there is a single SCG reduced from a given FTCG, different FTCGs, with possibly different orientations and skeletons, can yield the same SCG. For example, the SCG in Figure 2b can be reduced from any FTCG in Figure 2a, even though they may have different skeletons or different orientations. In the remainder, we refer to any FTCG from which a given SCG $\mathcal{G}^s$ can be reduced as a *candidate FTCG* for $\mathcal{G}^s$. For example, in Figure 2, $\mathcal{G}^f_1$, $\mathcal{G}^f_2$ and $\mathcal{G}^f_3$ are all candidate FTCGs for $\mathcal{G}^s$. The class of all candidate FTCGs for $\mathcal{G}^s$ is denoted by $C(\mathcal{G}^s)$.

3.3 PROBLEM SETUP

We focus in this paper on identifying total effects (Pearl, 2000) of multiple interventions on single effects, written $P\left(Y_t = y_t \middle| (\mathrm{do}(X^i_{t_i} = x^i_{t_i}))_i\right)$ (as well as $P\left(y_t \mid \mathrm{do}\left((x^i_{t_i})_i\right)\right)$ by a slight abuse of notation) when only the SCG reduced from the true FTCG is known, using the common adjustment criterion defined below.

Definition 3 (Common adjustment criterion)**.** Let $\mathcal{G}^s = (\mathcal{V}^s, \mathcal{E}^s)$ be an SCG. Let $\mathbf{X}, \mathbf{Y}$ and $\mathbf{Z}$ be pairwise distinct subsets of $\mathcal{V}^f$. $\mathbf{Z}$ satisfies the *common adjustment criterion* relative to $\mathbf{X}$ and $\mathbf{Y}$ in $\mathcal{G}^s$ if for all $\mathcal{G}^f \in C(\mathcal{G}^s)$, $\mathbf{Z}$ satisfies the adjustment criterion relative to $\mathbf{X}$ and $\mathbf{Y}$ in $\mathcal{G}^f$.

This criterion is sound and complete for the adjustment formulae, meaning that:

Proposition 1. Let $\mathcal{G}^s = (\mathcal{V}^s, \mathcal{E}^s)$ be an SCG and let $\mathbf{X}, \mathbf{Y}$ and $\mathbf{Z}$ be pairwise distinct subsets of $\mathcal{V}^f$. We say that a probability distribution P is *compatible* with $\mathcal{G}^s$ if there exists $\mathcal{G}^f \in C(\mathcal{G}^s)$ such that P is compatible with $\mathcal{G}^f$. The two following propositions are equivalent:

(i) $\mathbf{Z}$ satisfies the common adjustment criterion relative to $\mathbf{X}$ and $\mathbf{Y}$,

(ii) for all P compatible with $\mathcal{G}^s$:

$$P(\mathbf{y} \mid \mathrm{do}(\mathbf{x})) = \begin{cases} P(\mathbf{y} \mid \mathbf{x}) & \text{if } \mathbf{Z} = \emptyset \\ \sum_{\mathbf{z}} P(\mathbf{y} \mid \mathbf{x}, \mathbf{z})\, P(\mathbf{z}) & \text{otherwise.} \end{cases} \tag{1}$$

When either (i) or (ii) hold, we say that the total effect $P(\mathbf{y} \mid \mathrm{do}(\mathbf{x}))$ is *identifiable in $\mathcal{G}^s$ by adjustment criterion.*

Finally, our problem takes the form:

Problem 1. Consider an SCG $\mathcal{G}^s$. We aim to find out operational[3] necessary and sufficient conditions to identify the total effect $P\left(y_t \mid \mathrm{do}\left((x^i_{t_i})_i\right)\right)$ by common adjustment when having access solely to the SCG $\mathcal{G}^s$.

Note that if Y is not a descendant of one of the intervening variables X^i in $\mathcal{G}^s$ or if $\gamma_i := t - t_i < 0$, then $X^i_{t_i}$ can be removed from the conditioning set through, *e.g.*, the adjustment for direct causes (Pearl, 2000). In the extreme case where Y is not a descendant of any element of $\{X^i\}_i$, then $P\left(y_t \mid \mathrm{do}(x^1_{t-\gamma_1}), \ldots, \mathrm{do}(x^n_{t-\gamma_n})\right) = P(y_t)$. In the remainder, we thus assume that Y is a descendant of each element in $\{X^i\}_i$ in $\mathcal{G}^s$ and that $\gamma_i \geq 0$ for all i, and will use the following notations: $\mathcal{X}^f := \{X^i_{t-\gamma_i}\}_i$ and $\mathcal{X}^s := \{X^i\}_i$.

4 IDENTIFIABILITY BY COMMON ADJUSTMENT

We provide in this section the main results of this paper, which is a graphical necessary and sufficient condition for identifiability of the causal effect by common adjustment, and a solution to compute it in practice. The classical *consistency through time*, assuming that causal relations are the same at different time instants, is not assumed here and its discussion is postponed to Section 5. All the proofs are deferred to Section D in the Supplementary Material.

4.1 NECESSARY AND SUFFICIENT CONDITION BASED ON THE COMMON FORBIDDEN SET

We first introduce the *common forbidden set*, the set of vertices that belong to $\mathrm{Forb}\left(\mathcal{X}^f, Y_t, \mathcal{G}^f\right)$ in at least one candidate FTCG $\mathcal{G}^f$. The common forbidden set, and the related notion of non-conditionable set defined below, define a set of variables which cannot be elements of a common adjustment set as they violate the first condition in Definition 1. As such, they cannot be used as conditioning variables in the do-free formula rewriting the interventions (Equation 1 in Proposition 1).

Definition 4. Let $\mathcal{G}^s = (\mathcal{V}^s, \mathcal{E}^s)$ be an SCG and $P(y_t \mid \mathrm{do}(x^1_{t-\gamma_1}), \ldots, \mathrm{do}(x^n_{t-\gamma_n}))$ be the considered effect. We define the *common forbidden set* as follows:

$$\mathcal{CF} := \bigcup_{\mathcal{G}^f \in C(\mathcal{G}^s)} \mathrm{Forb}\left(\mathcal{X}^f, Y_t, \mathcal{G}^f\right).$$

The *set of non-conditionable variables* is defined by

$$\mathcal{NC} := \mathcal{CF} \setminus \mathcal{X}^f.$$

Running Example. In the first scenario, we have $\mathcal{NC} = \{\mathrm{Of}_{t-1}, \mathrm{Of}_t\}$, whereas, in the second scenario, we have $\mathcal{NC} = \{K_{t-1}, K_t, B_t, L_t, \mathrm{Of}_{t-1}, \mathrm{Of}_t\}$. In the second scenario, K_{t-1} cannot belong to a common adjustment set as there exists a candidate FTCG which contains the path $L_{t-1} \to K_{t-1} \to L_t \to \mathrm{Of}_t$. Similarly, L_t cannot belong to a common adjustment set as there exists a candidate FTCG which contains the path $L_{t-1} \to L_t \to \mathrm{Of}_t$.

[3] That is, conditions one can rely on in practice. In particular the number of candidate FTCGs in $C(\mathcal{G}^s)$ is usually too costly to enumerate (it may even be infinite); operational conditions should thus not rely on the enumeration of all FTCGs.

Theorem 1 below shows that identifiability by common adjustment is directly related to the existence of collider-free backdoor path remaining in this set.

Theorem 1. Let $\mathcal{G}^s = (\mathcal{V}^s, \mathcal{E}^s)$ be an SCG and $P(y_t \mid \mathrm{do}(x^1_{t-\gamma_1}), \ldots, \mathrm{do}(x^n_{t-\gamma_n}))$ be the considered effect. Then the two statements are equivalent:

1. The effect is identifiable by common adjustment in $\mathcal{G}^s$.

2. For all intervention $X^i_{t-\gamma_i}$ and candidate FTCG $\mathcal{G}^f \in C(\mathcal{G}^s)$, $\mathcal{G}^f$ does not contain a collider-free backdoor path going from $X^i_{t-\gamma_i}$ to Y_t that remains in $\mathcal{NC} \cup \{X^i_{t-\gamma_i}\}$.

In that case, a common adjustment set is given by $\mathcal{C} := \left(\mathcal{V}^f \setminus \mathcal{NC}\right) \setminus \mathcal{X}^f$, and we have

$$P(y_t \mid \mathrm{do}((x^i_{t-\gamma_i})_i)) = \sum_{\mathbf{c}} P\left(y_t \mid (x^i_{t-\gamma_i})_i, \mathbf{c}\right) P(\mathbf{c}).$$

Proof Sketch. Let $\mathcal{G}^f$ be a candidate FTCG. For any $X^i_{t-\gamma_i} \in \mathcal{X}^f$, consider any proper non-causal path π^f from $\mathcal{X}^f$ to Y_t that starts at $X^i_{t-\gamma_i}$. Then π^f either:

- leaves $\mathcal{CF} \cup \{X^i_{t-\gamma_i}\}$, in which case it contains a non-collider in $\mathcal{C}$ (see C_{t_c} in Figure 3) and is blocked by $\mathcal{C}$,
- remains in $\mathcal{CF} \cup \{X^i_{t-\gamma_i}\}$ and contains a collider, in which case it is also blocked by $\mathcal{C}$, since the collider and its descendants remain in $\mathcal{NC}$,
- remains in $\mathcal{CF} \cup \{X^i_{t-\gamma_i}\}$ and contains no collider (i.e., it is a collider-free backdoor path), in which case it cannot be blocked.

Thus, collider-free backdoor paths entirely contained in $\mathcal{NC} \cup \{X^i_{t-\gamma_i}\}$ are therefore the only proper non-causal paths from $\mathcal{X}^f$ to Y_t starting at $X^i_{t-\gamma_i}$ that cannot be blocked by $\mathcal{C}$. Moreover, such paths cannot be blocked by any common adjustment set, as they remain in $\mathcal{NC} \cup \mathcal{X}^f$. As a result, the effect is identifiable by common adjustment if and only if no such path exists for any $X^i_{t-\gamma_i} \in \mathcal{X}^f$. □

Running Example. In the first scenario, we have $\mathcal{NC} = \{\mathrm{Of}_{t-1}, \mathrm{Of}_t\}$. No candidate FTCG contains a non-causal path that remains within $\mathcal{NC}$. As a result, $P(\mathrm{Of}_t \mid \mathrm{do}(L_{t-1}, L_t, B_{t-1}, B_t))$ is identifiable by common adjustment. In the second scenario,we have $\mathcal{NC} = \{K_{t-1}, K_t, B_t, L_t, \mathrm{Of}_{t-1}, \mathrm{Of}_t\}$ and we know that both K_{t-1} and L_t cannot be part of a common adjustment set. Since a candidate FTCG contains the path $L_{t-1} \leftarrow K_{t-1} \rightarrow L_t \rightarrow \mathrm{Of}_t$, $P(\mathrm{Of}_t \mid \mathrm{do}(L_{t-1}, B_{t-1}))$ is not identifiable by common adjustment.

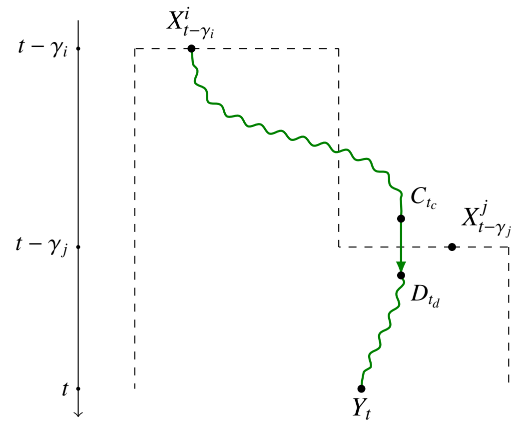


Figure 3: Proof idea of Theorem 1. The green path represents a proper non-causal path π^f from $\mathcal{X}^f$ to Y_t starting at $X^i_{t-\gamma_i}$. The node $X^j_{t-\gamma_j}$ represents another intervention (if any). The dashed lines depict the set $\mathcal{CF} \cup \{X^i_{t-\gamma_i}\}$. The vertex C_{t_c} is the last node on π^f outside of $\mathcal{CF} \cup \{X^i_{t-\gamma_i}\}$, and D_{t_d} is its successor on π^f. Necessarily, π^f must contain the arrow $C_{t_c} \rightarrow D_{t_d}$; otherwise, C_{t_c} would belong to $\mathcal{CF}$.

4.2 AN EFFICIENT WAY TO DECIDE ON IDENTIFIABILITY

To determine whether the causal effect is identifiable, we propose an algorithm that efficiently tests the existence of collider-free backdoor paths that remain in $\mathcal{NC}$, except perhaps for their first vertices. Instead of enumerating all FTCGs and such paths in $\mathcal{NC}$, which would be computationally prohibitive, we introduce a more refined approach to characterize their existence. Specifically, we distinguish between those with and without forks. Paths without forks can be easily and efficiently identified. The situation is more complex for those that contain forks, but they can still be efficiently handled via a divide-and-conquer strategy. All these elements are detailed in the following subsections.

4.2.1 Additional Characterizations

Characterization of $\mathcal{NC}$ We first introduce another characterization of $\mathcal{NC}$ based on the time instant a time series first arrives in this set.

Definition 5. Let $\mathcal{G}^s = (\mathcal{V}^s, \mathcal{E}^s)$ be an SCG and $P(y_t \mid \mathrm{do}(x^1_{t-\gamma_1}), \ldots, \mathrm{do}(x^n_{t-\gamma_n}))$ be the considered effect. For a time series $F \in \mathcal{V}^s$, we define

$$t_{\mathcal{NC}}(F) := \min\{t_1 \mid F_{t_1} \in \mathcal{NC}\},$$

as the first time step at which F enters the non-conditionable set $\mathcal{NC}$, with the convention that $\min\{\emptyset\} = +\infty$.

Running Example. In the second scenario, we have $\mathcal{NC} = \{K_{t-1}, K_t, B_t, L_t, \mathrm{Of}_{t-1}, \mathrm{Of}_t\}$. As a result, $t_{\mathcal{NC}}(\mathrm{Kitchen}) = t-1$ and $t_{\mathcal{NC}}(\mathrm{Outside}) = +\infty$.

Algorithm 1: Computation of $(t_{\mathcal{NC}}(S))_{S\in\mathcal{V}^s}$

Input: $\mathcal{G}^s = (\mathcal{V}^s, \mathcal{E}^s)$ an SCG and $\mathcal{X}^f$.
Output: $(t_{\mathcal{NC}}(S))_{S\in\mathcal{V}^s}$

// Compute t_C $\forall$ $C \in \mathrm{Ch}(\mathcal{X}^s)$. (cf. Lemma 7)
$AncY \leftarrow \left(\max\left\{t_1 \mid \exists \mathcal{G}^f \text{ s.t } S_{t_1} \in \mathrm{Anc}(Y_t, \mathcal{G}^f \setminus \mathcal{X}^f)\right\}\right)_{S\in\mathcal{V}^s}$;
foreach $C \in Ch(\mathcal{X}^s)$ **do**
    $t_{\min} \leftarrow \min\{t - \gamma_i \mid X^i \in \mathrm{Pa}(C, \mathcal{G}^s)\}$;
    $t_C \leftarrow \min\left\{t_1 \in [t_{\min}, AncY[C]] \mid C_{t_1} \notin \mathcal{X}^f\right\}$;
    $d(C) \leftarrow \#\{i \mid X^i \in \mathrm{Pa}(C, \mathcal{G}^s) \text{ and } t - \gamma_i < t_C\} \geq 1$
        or $\#\{i \mid X^i \in \mathrm{Pa}(C, \mathcal{G}^s) \text{ and } t - \gamma_i = t_C\} \geq 2$;

// Avoid extra computations. (cf. Lemma 8)
$L \leftarrow [(C, t_C)]_{C\in\mathrm{Desc}(\mathcal{X}^f),\text{with } t_C<+\infty}$;
Sort L using $(t_C, not\ d(C))$ lexicographically;
// Compute $(t_{\mathcal{NC}}(S))_{S\in\mathcal{V}^s}$. (cf. Lemma 11)
$(t_{\mathcal{NC}}(S)) \leftarrow +\infty \quad \forall S \in \mathcal{V}^s$;
$S.seen \leftarrow False \quad \forall S \in \mathcal{V}^s$;
for $(C, t_C) \in L$ **do**
    if $d(C)$ **then**
        foreach *unseen* $D \in Desc(C, \mathcal{G}^s)$ **do**
            $t_{\mathcal{NC}}(D) \leftarrow \min\{t_1 \mid t_1 \geq t_C \text{ and } D_{t_1} \notin \mathcal{X}^f\}$;
            $D.seen \leftarrow$ true ;
    else
        foreach *unseen* $D \in Desc(C, \mathcal{G}^s \setminus \mathcal{X}^s)$ **do**
            $t_{\mathcal{NC}}(D) \leftarrow \min\{t_1 \mid t_1 \geq t_C \text{ and } D_{t_1} \notin \mathcal{X}^f\}$;
            $D.seen \leftarrow$ true ;
        foreach *unseen* $D \in Desc(C, \mathcal{G}^s)$ **do**
            $t_{\mathcal{NC}}(D) \leftarrow \min\{t_1 \mid t_1 \geq t_C + 1 \text{ and } D_{t_1} \notin \mathcal{X}^f\}$;
            $D.seen \leftarrow$ true ;

In the next lemma, we show that $\{t_{\mathcal{NC}}(F)\}_{F\in\mathcal{V}^s}$ gives a simple characterization of these sets.

Lemma 1. (Characterization of $\mathcal{NC}$) Let $\mathcal{G}^s = (\mathcal{V}^s, \mathcal{E}^s)$ be an SCG and let $P(y_t \mid \mathrm{do}(x^1_{t-\gamma_1}), \ldots, \mathrm{do}(x^n_{t-\gamma_n}))$ be the considered effect. With the convention $\{F_{t_1}\}_{t_1\geq+\infty} = \emptyset$, we have:

$$\mathcal{NC} = \bigcup_{Z\in\mathcal{V}^S} \{Z_{t_1}\}_{t_1\geq t_{\mathcal{NC}}(Z)} \setminus \mathcal{X}^f.$$

Moreover, $(t_{\mathcal{NC}}(F))_{F\in\mathcal{V}^s}$ can be computed through Algorithm 1, detailed in Appendix D, which complexity is pseudo-linear with respect to $\mathcal{G}^s$ and $\mathcal{X}^f$.

The above characterization, based on $t_{\mathcal{NC}}(F)$, slightly departs from standard, purely graphical characterizations often used in the identifiability literature. This is due to the complexity of the class of candidate FTCGs and the difficulty to explore this class efficiently.

Collider-free backdoor paths without fork in $\mathcal{NC}$ First, Lemma 2 characterizes efficiently the existence of collider-free backdoor paths that do not contain a fork.

Lemma 2. (Characterization of collider-free backdoor paths without fork) Let $\mathcal{G}^s = (\mathcal{V}^s, \mathcal{E}^s)$ be an SCG and $P(y_t \mid \mathrm{do}(x^1_{t-\gamma_1}), \ldots, \mathrm{do}(x^n_{t-\gamma_n}))$ be the considered effect. The following statements are equivalent:

1. There exists an intervention $X^i_{t-\gamma_i}$ and a candidate FTCG $\mathcal{G}^f \in \mathcal{C}(\mathcal{G}^s)$ which contains $X^i_{t-\gamma_i} \leftsquigarrow Y_t$ which remains in $\mathcal{NC} \cup \{X^i_{t-\gamma_i}\}$.
2. There exists an intervention $X^i_{t-\gamma_i}$ such that $\gamma_i = 0$ and $X^i \in \mathrm{Desc}\left(Y, \mathcal{G}^s_{|\mathcal{S}}\right)$, where $\mathcal{S} := \{S \in \mathcal{V}^s \mid t_{\mathcal{NC}}(S) \leq t\} \cup \{X^i \in \mathcal{X}^s \mid \gamma_i = 0\}$.

Fork collider-free backdoor paths in $\mathcal{NC}$ We first introduce an accessibility concept essential to the enumeration of fork paths.

Definition 6 ($\mathcal{NC}$**-accessibility**)**.** Let $\mathcal{G}^s = (\mathcal{V}^s, \mathcal{E}^s)$ be an SCG, $P(y_t \mid \mathrm{do}(x^1_{t-\gamma_1}), \ldots, \mathrm{do}(x^n_{t-\gamma_n}))$ be the considered effect and $V_{t_v} \in \mathcal{V}^f$. We say that $F_{t_1} \in \mathcal{V}^f \setminus \{V_{t_v}\}$ *is V_{t_v}-$\mathcal{NC}$-accessible* if there exists a candidate FTCG which contains a directed path from F_{t_1} to V_{t_v} which remains in $\mathcal{NC}$ except perhaps for V_{t_v}. We denote

$$t^{\mathcal{NC}}_{V_{t_v}}(F) := \max\{t_1 \mid F_{t_1} \text{is } V_{t_v}\text{-}\mathcal{NC}\text{-accessible}\},$$

with the convention $\max\{\emptyset\} = -\infty$.

Running Example. In both scenarios, L_t is Of_t-$\mathcal{NC}$-accessible, since there exists a candidate FTCG containing the path $L_t \rightarrow \mathrm{Of}_t$. Although there is also a candidate FTCG containing the path $B_t \rightarrow L_t \rightarrow \mathrm{Of}_t$, B_t is Of_t-$\mathcal{NC}$-accessible only in the second scenario, because in the first scenario B_t is itself an intervention. Consequently, in both scenarios, we have $t^{\mathcal{NC}}_{\mathrm{Of}_t}(\text{Living Room}) = t$. However, $t^{\mathcal{NC}}_{\mathrm{Of}_t}(\text{Bathroom}) = -\infty$ in Scenario 1 and $t^{\mathcal{NC}}_{\mathrm{Of}_t}(\text{Bathroom}) = t$ in Scenario 2.

This leads to characterize efficiently the existence of a collider-free backdoor path with a fork that remains in $\mathcal{NC}$, as proposed in Lemma 3.

Lemma 3. (Characterization of collider-free backdoor paths with fork) Let $\mathcal{G}^s = (\mathcal{V}^s, \mathcal{E}^s)$ be an SCG and $P(y_t \mid \mathrm{do}(x^1_{t-\gamma_1}), \ldots, \mathrm{do}(x^n_{t-\gamma_n}))$ be the considered effect such that for all $\mathcal{G}^f$ belonging to $\mathcal{C}(\mathcal{G}^s)$, $\mathcal{G}^f$ does not contain a directed path from Y_t to an intervention $X^i_{t-\gamma_i} \leftsquigarrow Y_t$ which remains in $\mathcal{NC} \cup \{X^i_{t-\gamma_i}\}$. The following statements are equivalent:

1. There exists an intervention $X^i_{t-\gamma_i}$, $F_{t_f} \in \mathcal{V}^f$ and a candidate FTCG $\mathcal{G}^f \in \mathcal{C}(\mathcal{G}^s)$ which contains the path $X^i_{t-\gamma_i} \leftsquigarrow F_{t_f} \rightsquigarrow Y_t$ which remains in $\mathcal{NC} \cup \{X^i_{t-\gamma_i}\}$.
2. There exists an intervention $X^i_{t-\gamma_i}$ and $F_{t_f} \in \mathcal{V}^f$ such that F_{t_f} is $X^i_{t-\gamma_i}$-$\mathcal{NC}$-accessible and Y_t-$\mathcal{NC}$-accessible.

The intuition behind Lemma 3 relies on a divide-and-conquer strategy to avoid searching for collider-free backdoor paths with a fork directly. In an FTCG, any such path decomposes into two directed subpaths. The lemma shows that it suffices to exhibit the first subpath in one candidate FTCG and the second subpath in another. This ensures that some candidate FTCG realizes the entire fork path, without resorting to an explicit reconstruction argument. Since testing for a single directed path in a candidate FTCG can be done efficiently, this reduction renders the overall existence check more tractable. The formal proof appears in the Supplementary Material.

Condition 2 in Lemma 3 is not efficiently tractable, since it requires checking each $F_{t_f} \in \mathcal{V}^f$, and the set $\mathcal{V}^f$ is infinite. Fortunately, the set $\{t_1 \mid F_{t_1} \text{ is } V_{t_v}\text{-}\mathcal{NC}\text{-accessible}\}$ is bounded by $t^{\mathcal{NC}}_{V_{t_v}}(F)$ and $t_{\mathcal{NC}}(F)$ (see Lemma 13 in the appendix), and testing only the single time point $F_{t_{\mathcal{NC}}(F)}$ for each time series F is sufficient (see Corollary 1 in the appendix). Consequently, Condition 2 in Lemma 3 reduces to the following:

- There exists an intervention $X^i_{t-\gamma_i}$ and a time series F such that $t_{\mathcal{NC}}(F) \leq t^{\mathcal{NC}}_{X^i_{t-\gamma_i}}(F)$ and $t_{\mathcal{NC}}(F) \leq t^{\mathcal{NC}}_{Y_t}(F)$,

improving further the tractability of the check of existence of a collider-free backdoor path with a fork.

4.2.2 An Efficient Algorithm

The above results show that identifiability by common adjustment in $\mathcal{G}^s$ is equivalent to the following two conditions:

1. There does not exist an intervention $X^i_{t-\gamma_i}$ such that $\gamma_i = 0$ and $X^i \in \text{Desc}\left(Y, \mathcal{G}^s_{|\mathcal{S}}\right)$, where $\mathcal{S} := \{S \in \mathcal{V}^s \mid t_{\mathcal{NC}}(S) \leq t\} \cup \{X^i \mid \gamma_i = 0\}$ (see Lemma 2),
2. And, there does not exist an intervention $X^i_{t-\gamma_i}$ and a time series $F \in \mathcal{V}^s$ such that $t_{\mathcal{NC}}(F) \leq t^{\mathcal{NC}}_{X^i_{t-\gamma_i}}(F)$ and $t_{\mathcal{NC}}(F) \leq t^{\mathcal{NC}}_{Y_t}(F)$ (see discussion below Lemma 3)

Condition 2 from Lemma 2 can be verified efficiently. Indeed, it suffices to compute the set of all descendants of Y in $\mathcal{G}^s_{|\mathcal{S}}$ using a single breadth- or depth-first search in time $O(|\mathcal{V}^s| + |\mathcal{E}^s|)$ (Cormen et al., 2009, Chapter 22), and then check if there exists an intervention $X^i_{t-\gamma_i}$ with $\gamma_i = 0$ such that $X^i \in \text{Desc}_{\mathcal{G}^s_{|\mathcal{S}}}(Y)$.

Having ruled out directed-paths, we now focus on the fork-path condition. To this end, we present Algorithm 2, which computes $\{t^{\mathcal{NC}}_{V_{t_v}}(F) \mid F \in \mathcal{V}^S\}$ in pseudo-linear time (see Lemma 14 in the appendix).

As a result, the second statement of Lemma 3 can be checked by executing Algorithm 2 twice, knowing that Algorithm 1 has already been run. Consequently, the overall complexity is $O\left(\left|\mathcal{X}^f\right| \log \left|\mathcal{X}^f\right| + (|\mathcal{E}^s| + |\mathcal{V}^s|) \log |\mathcal{V}^s|\right)$.

By combining the previous results as in Algorithm 3, one can directly assess whether the effect is identifiable or not:

Algorithm 2: Computation of $(t^{\mathcal{NC}}_{V_{t_v}}(S))_{S \in \mathcal{V}^s}$

Input: $\mathcal{G}^s = (\mathcal{V}^s, \mathcal{E}^s)$ an SCG, $\mathcal{X}^f$ and $V_{t_v} \in \mathcal{V}^f$
Output: $(t^{\mathcal{NC}}_{V_{t_v}}(S))_{S \in \mathcal{V}^s}$
$Q \leftarrow$ PriorityQueue(V_{t_v}) ;
$t^{\mathcal{NC}}_{V_{t_v}}(S) \leftarrow -\infty \quad \forall S \in \mathcal{V}^s$;
$S.seen \leftarrow False \quad \forall S \in \mathcal{V}^s$;
while $Q \neq \emptyset$ **do**
    $S_{t_s} \leftarrow Q$.pop_element_with_max_time_index();
    foreach *unseen* $P \in Pa(S, \mathcal{G}^s)$ **do**
        $t^{\mathcal{NC}}_{V_{t_v}}(P) \leftarrow \max\{t_1 \mid t_1 \leq t_s \text{ and } P_{t_1} \in \mathcal{NC} \setminus \{V_{t_v}\}\}$;
        if $t^{\mathcal{NC}}_{V_{t_v}}(P) \neq -\infty$ **then** $Q.insert(P_{t^{\mathcal{NC}}_{V_{t_v}}(P)})$;
        $P.seen \leftarrow true$;

Algorithm 3: Identifiability by common adjustment.

Input: $\mathcal{G}^s = (\mathcal{V}^s, \mathcal{E}^s)$ an SCG and $\mathcal{X}^f$.
Output: A boolean indicating whether the effect is identifiable by common adjustment or not.
$(t_{\mathcal{NC}}(S))_{S \in \mathcal{V}^s} \leftarrow$ Algorithm 1 ;
`// Enumeration of directed paths.`
$\mathcal{S} \leftarrow \{S \in \mathcal{V}^s \mid t_{\mathcal{NC}}(S) \leq 0\} \cup \{X^i \mid t - \gamma_i = 0\}$;
if $\exists i \in \{1, \ldots, n\}$ *s.t.* $X^i \in Desc\left(Y, \mathcal{G}^s_{|\mathcal{S}}\right)$ *and* $\gamma_i = 0$ **then**
    return *False*
`// Enumeration of fork paths.`
foreach $V_{t_v} \in \{Y_t, X^1_{t-\gamma_1}, \cdots, X^n_{t-\gamma_n}\}$ **do**
    $(t^{\mathcal{NC}}_{V_{t_v}}(S))_{S \in \mathcal{V}^s} \leftarrow$ Algorithm 2 ;
foreach $F \in \mathcal{V}^s, X^i_{t-\gamma_i} \in (X^j_{t-\gamma_j})_j$ **do**
    if $t_{\mathcal{NC}}(F) \leq t^{\mathcal{NC}}_{X^i_{t-\gamma_i}}(F)$ *and* $t_{\mathcal{NC}}(F) \leq t^{\mathcal{NC}}_{Y_t}(F)$ **then**
        return *False*
return *True*

Theorem 2. Let $\mathcal{G}^s = (\mathcal{V}^s, \mathcal{E}^s)$ be an SCG and $P(y_t \mid \text{do}(x^1_{t-\gamma_1}), \ldots, \text{do}(x^n_{t-\gamma_n}))$ be the considered effect. Then the two statements are equivalent:

- The effect is identifiable by common adjustment in $\mathcal{G}^s$.
- Algorithm 3 outputs True.

Moreover Algorithm 3 has a polynomial complexity of $O\left(\left|\mathcal{X}^f\right| \left(\log \left|\mathcal{X}^f\right| + (|\mathcal{E}^s| + |\mathcal{V}^s|) \log |\mathcal{V}^s|\right)\right)$.

The complexity of Algorithm 3 can be further reduced to pseudo-linear time, as detailed in Section F of the Supplementary Material. There is little interest in replacing the efficient implementation of Algorithm 3 with a formula.[4] Indeed, we can not expect having a complexity better than $O\left(\left|\mathcal{X}^f\right| + |\mathcal{E}^s| + |\mathcal{V}^s|\right)$ because in the worst case, it is necessary to traverse $\mathcal{G}^s$ and consider all interventions.

[4]We refer to a formula as a condition involving a combination of descendant, ancestor, and cycle sets, ...

5 WITH CONSISTENCY THROUGH TIME

In practice, it is usually impossible to work with general FTCGs in which causal relations may change from one time instant to another, and people have resorted to the *consistency through time* assumption (also referred to as Causal Stationarity in Runge (2018)), to obtain a simpler class of FTCGs.

Assumption 1 (*Consistency through time*)**.** An FTCG $\mathcal{G}^f$ is said to be *consistent through time* if all the causal relationships remain constant in direction through time.

Under this assumption, the number of candidate FTCGs for a fixed SCG $\mathcal{G}^s$ is smaller, meaning that conditions to be identifiable are weaker and thus that more effects should be identifiable. We detail in Section 5.1 necessary and sufficient conditions to be identifiable. All the proofs are deferred to Section E in the Supplementary Material.

5.1 IDENTIFIABILITY

Theorem 1 remains valid under Assumption 1. Lemma 2 also holds because Assumption 1 only affects paths that traverse different time indices. The enumeration of collider-free backdoor paths containing a fork that remains within $\mathcal{NC}$, except perhaps at their first vertices, is however more complex, as detailed below.

Lemma 4. Let $\mathcal{G}^s = (\mathcal{V}^s, \mathcal{E}^s)$ be an SCG and $P(y_t \mid \mathrm{do}(x^1_{t-\gamma_1}), \ldots, \mathrm{do}(x^n_{t-\gamma_n}))$ be the considered effect such that for all $\mathcal{G}^f$ belonging to $\mathcal{C}(\mathcal{G}^s)$, $\mathcal{G}^f$ does not contain a directed path from Y_t to an intervention $X^i_{t-\gamma_i} \leftsquigarrow Y_t$ which remains in $\mathcal{NC} \cup \{X^i_{t-\gamma_i}\}$. The following statements are equivalent:

1. There exist an intervention $X^i_{t-\gamma_i}$, $F_{t'} \in \mathcal{V}^f$ and an FTCG $\mathcal{G}^f \in \mathcal{C}(\mathcal{G}^s)$ containing the path $X^i_{t-\gamma_i} \leftsquigarrow F_{t'} \rightsquigarrow Y_t$ which remains in $\mathcal{NC} \cup \{X^i_{t-\gamma_i}\}$.
2. At least one of the following conditions is satisfied:
   (a) There exist an intervention $X^i_{t-\gamma_i}$ and $F \in \mathcal{V}^s$ such that $F_{t_{\mathcal{NC}}(F)}$ is well defined, $X^i_{t-\gamma_i}$-$\mathcal{NC}$-accessible and Y_t-$\mathcal{NC}$-accessible, and $\begin{cases} F \neq Y, \text{ or} \\ t-\gamma_i \neq t_{\mathcal{NC}(F)}. \end{cases}$
   (b) There exists an intervention $X^i_{t-\gamma_i}$ such that $t-\gamma_i = t_{\mathcal{NC}(Y)}$ and at least one of the following properties is satisfied:
      i. $Y_{t_{\mathcal{NC}(Y)}}$ is $X^i_{t-\gamma_i}$ - $\mathcal{NC}$-accessible without using $X^i_{t-\gamma_i} \leftarrow Y_{t-\gamma_i}$ and Y_t - $\mathcal{NC}$-accessible.
      ii. $Y_{t_{\mathcal{NC}(Y)}}$ is $X^i_{t-\gamma_i}$ - $\mathcal{NC}$-accessible and Y_t - $\mathcal{NC}$-accessible without using $X^i_t \rightarrow Y_t$.

Lemma 4 characterizes the existence of a collider-free backdoor path containing a fork. While the conditions outlined are more complex than those in Corollary 1, they play the same role, and still require only a small number of calls to $\mathcal{NC}$-accessibility. Consequently, one can replace the conditions in the final loop of Algorithm 3 with conditions 2.(a) and 2.(b) of Lemma 4 to derive an algorithm for identifiability by common adjustment in $\mathcal{G}^s$ under *consistency through time*, as stated in the following theorem which is the counterpart of Theorem 2.

Theorem 3. Let $\mathcal{G}^s = (\mathcal{V}^s, \mathcal{E}^s)$ be an SCG that satisfies Assumption 1 and $P(y_t \mid \mathrm{do}(x^1_{t-\gamma_1}), \ldots, \mathrm{do}(x^n_{t-\gamma_n}))$ be the considered effect. Then the two statements are equivalent:

- The effect is identifiable by common adjustment in $\mathcal{G}^s$.
- An adaptation of Algorithm 3 outputs True.

In that case, a common adjustment set is given by $\mathcal{C} := \left(\mathcal{V}^f \setminus \mathcal{NC}\right) \setminus \mathcal{X}^f$.

In its simpler form, the adaptation of Algorithm 3 still has a polynomial complexity of $\mathcal{O}\left(\left|\mathcal{X}^f\right| \cdot (|\mathcal{E}^s| + |\mathcal{V}^s| \log |\mathcal{V}^s|)\right)$. A pseudo-linear algorithm is discussed in Appendix G.

The main difference between the two algorithms (with and without consistency through time) lies in how they test for collider-free backdoor paths with forks. Without consistency through time, this check is based on Lemma 3, while with consistency through time, this check relies on Lemma 4. Note that assuming consistency through time reduces the number of candidate FTCGs: any candidate FTCG under consistency through time is also candidate without this assumption. As a result, the algorithm in Theorem 2 is sound (but not complete) under consistency through time, while the algorithm from Theorem 3 is complete (but not sound) without this assumption.

5.2 EXPERIMENTAL ILLUSTRATION

Although this article is primarily theoretical, we have conducted experiments to demonstrate the practical relevance of the results in terms of computation time and estimation. The Python implementation is available at this repository.[5]

Execution time Algorithm 3 has been implemented in Python with some speed ups discussed in Appendix F. We measure its execution speed, on a standard laptop, as a function of the graph size. For each graph size, 20 random SCGs are generated, and 5 interventions are selected at random. The average execution time of the algorithm is then measured over 5 runs and presented in Figure 4. As one can note, even for very large graphs (with up to 100,000 vertices), the execution time remains reasonable, around 1 second, showing that the theoretical complexity of the algorithm translates into an acceptable computation time.

[5] https://gricad-gitlab.univ-grenoble-alpes.fr/yvernesc/multivariateicainscg

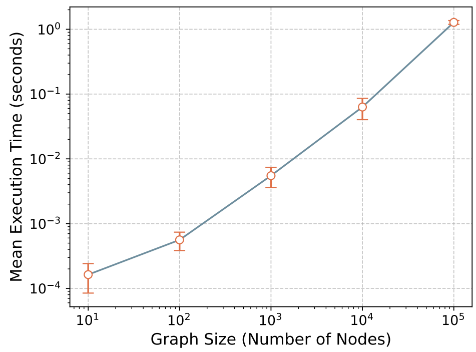


Figure 4: Average execution time of the implementation (in seconds) as a function of the number of vertices in the graph, with error bars representing standard deviation over 5 runs.

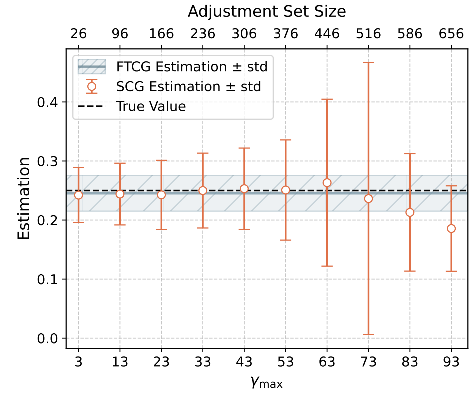


Figure 5: Estimated total causal effect $P(y_t \mid \mathrm{do}(x_t))$ using the adjustment set from the SCG, as a function of $\gamma_{\max}$ and as a function of the size of the adjustment set. Error bars represent the standard deviation over 100 repetitions. As a baseline, we provide the true total effect (back dashed) and the one estimated by the backdoor formulae when knowing the true FTCG, with standard deviation (gray hatched).

Estimation We further considered a fixed FTCG under a linear Structural Causal Model (SCM) with additive standard Gaussian noises with a lag of 1. We designed the SCM so that the total effect is 0.25. For each choice of $\gamma_{\max}$, which defines the farthest time horizon up to which past information is considered, we estimated the total causal effect $P(y_t \mid \mathrm{do}(x_t))$ by using the adjustment set given by Theorem 1 up to the time index $t-\gamma_{\max}$ over 500 data points (non overlapping windows). This estimation procedure was repeated 100 times for each $\gamma_{\max}$. The estimated total effect and its standard deviation across these repetitions are given in Figure 5.

For comparison, we estimated the total effect using a backdoor set from the true FTCG, yielding an estimate of 0.245 with a standard deviation of 0.036 over 100 runs. The bias and variance of the SCG-based estimator remain comparable to those of the FTCG-based estimator for $\gamma \leq 50$. However, as the adjustment set expands already to 376 variables at $\gamma = 53$, variance increases beyond large γ, and for even larger γ a non-negligible bias emerges. This suggests that the adjustment set proposed in this work is particularly useful when one can assume a maximal latency of reasonable size.

6 CONCLUSION

This work has established complete conditions for identifying the effect of multiple interventions on single effects by common adjustment in summary causal graphs of time series, a criterion we have shown is both sound and complete in this setting. Specifically, Theorem 1 shows that, both with and without *consistency through time*, the problem reduces to testing the existence of collider-free backdoor paths from interventions to effects that remain within a specified set. This complete characterization allowed us to derive efficient algorithms to determine whether an effect is identifiable by common adjustment, again both with and without *consistency through time*. All the provided proofs are constructive, meaning that whenever an effect is not identifiable by common backdoor, it is possible to explicitly exhibit a collider-free backdoor path that remains within $\mathcal{NC}$. Future work will focus on multiple effects and completeness results for global identifiability, *i.e.*, not restricted to adjustment.

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Complete Characterization for Adjustment in Summary Causal Graphs of Time Series
(Supplementary Material)

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CONTENTS

A NOTATIONS

Notation $\mathcal{G} \models$. When the underlying graph is not obvious from the context, we write $\mathcal{G} \models \phi$ to indicate that the graphical property ϕ holds in the graph $\mathcal{G}$. For example, $\mathcal{G} \models X \rightarrow Y$ means that X is a parent of Y in $\mathcal{G}$.

B PROOFS OF SECTION 3

Proposition 1. Let $\mathcal{G}^s = (\mathcal{V}^s, \mathcal{E}^s)$ be an SCG and let $\mathbf{X}, \mathbf{Y}$ and $\mathbf{Z}$ be pairwise distinct subsets of $\mathcal{V}^f$. We say that a probability distribution P is *compatible* with $\mathcal{G}^s$ if there exists $\mathcal{G}^f \in C(\mathcal{G}^s)$ such that P is compatible with $\mathcal{G}^f$. The two following propositions are equivalent:

(i) $\mathbf{Z}$ satisfies the common adjustment criterion relative to $\mathbf{X}$ and $\mathbf{Y}$,

(ii) for all P compatible with $\mathcal{G}^s$:

$$P(\mathbf{y} \mid \mathrm{do}(\mathbf{x})) = \begin{cases} P(\mathbf{y} \mid \mathbf{x}) & \text{if } \mathbf{Z} = \emptyset \\ \sum_{\mathbf{z}} P(\mathbf{y} \mid \mathbf{x}, \mathbf{z})\, P(\mathbf{z}) & \text{otherwise.} \end{cases} \tag{1}$$

When either (i) or (ii) hold, we say that the total effect $P(\mathbf{y} \mid \mathrm{do}(\mathbf{x}))$ is *identifiable in $\mathcal{G}^s$ by adjustment criterion*.

Proof. Let us prove the two implications.

- (i) $\Rightarrow$ (ii): If $\mathbf{Z}$ satisfies the common adjustment criterion relative to $\mathbf{X}$ and $\mathbf{Y}$. Then for all $\mathcal{G}^f \in C(\mathcal{G}^s)$, $\mathbf{Z}$ satisfies the adjustment criterion relative to $\mathbf{X}$ and $\mathbf{Y}$ in $\mathcal{G}^f$. Therefore, by Theorem 56 from (Perkovic et al., 2016), Equation (1) holds.
- (ii) $\Rightarrow$ (i): Let us show the contrapositive. If $\mathbf{Z}$ does not satisfy the common adjustment criterion relative to $\mathbf{X}$ and $\mathbf{Y}$. Then there exists $\mathcal{G}^f \in C(\mathcal{G}^s)$ such that $\mathbf{Z}$ does not satisfies the adjustment criterion relative to $\mathbf{X}$ and $\mathbf{Y}$ in $\mathcal{G}^f$. By Theorem 57 from (Perkovic et al., 2016), there exists P compatible with $\mathcal{G}^f$ such that Equation (1) does not hold. By definition P is compatible with $\mathcal{G}^s$ and Equation (1) does not hold.

□

C HOW TO BUILD AN FTCG BELONGING TO $C(\mathcal{G}^s)$?

Many proofs in this paper use constructive arguments to demonstrate the existence of an FTCG belonging to $C(\mathcal{G}^s)$ that contains a given structure π^f[1]. To facilitate the understanding of these arguments, Lemma 5 shows how these FTCGs are constructed by adding missing arrows.

C.1 WITHOUT ASSUMPTION 1

Lemma 5. Let $\mathcal{G}^s$ be an SCG and π^f be a graph over $\mathcal{V}^f$. If π^f is a DAG, all its arrows respect time orientation and its reduction is a subgraph of $\mathcal{G}^s$, then there exists an FTCG $\mathcal{G}^f$ in $C(\mathcal{G}^s)$ which contains π^f.

Proof. $\mathcal{G}^s$ is a SCG, by definition it is the reduction of an FTCG $\mathcal{G}^f_\star = (\mathcal{V}^f_\star, \mathcal{E}^f_\star)$. Let $t_{\min}$ be the minimum time index seen by π^f. Let us consider the graph $\mathcal{G}^f = (\mathcal{V}^f := \mathcal{V}^f_\star, \mathcal{E}^f)$ whose edges are constructed as follows:

1. All edges from π^f are set in $\mathcal{E}^f$.
2. For all the edges $A \rightarrow B$ in $\mathcal{G}^s$ that are not reductions of arrows in π^f, add the edge $A_{t_{\min}-1} \rightarrow B_t$ into $\mathcal{E}^f$.

We have the following properties:

- $\mathcal{G}^f$ contains π^f because all arrows of π^f are set in $\mathcal{E}^f$.
- $\mathcal{G}^f$ is a DAG. Indeed, there is no cycle due to instantaneous arrows because all instantaneous arrows come from π^f and π^f is a DAG. Moreover, there is no cycle due to delayed arrows because all of them follow the flow of time.

[1]In most of the proofs π^f is a path.

- $\mathcal{G}^f$ belongs to $C(\mathcal{G}^s)$: Let us consider $\mathcal{G}^r = (\mathcal{V}^r, \mathcal{E}^r)$ the reduction of $\mathcal{G}^f$. By definition, $\mathcal{V}^r = \mathcal{V} = \mathcal{V}^s$. Les us prove that $\mathcal{E}^r = \mathcal{E}^s$ by showing the two inclusions:
  - $\mathcal{E}^r \subseteq \mathcal{E}^s$: Let us consider an arrow a^r of G^r and a^f a corresponding arrow in $\mathcal{G}^f$. We distinguish two cases:
    * If a^f is in π^f then a^r is in π^r and π^r is a subgraph of $\mathcal{G}^s$. Therefore $a^r \in \mathcal{E}^s$.
    * Otherwise, a^r have been added to $\mathcal{E}^f$ during step 2. Therefore $a^r \in \mathcal{E}^s$.

    In both cases $a^r \in \mathcal{E}^s$, therefore $\mathcal{E}^r \subseteq \mathcal{E}^s$.
  - $\mathcal{E}^s \subseteq \mathcal{E}^r$: Let us consider an arrow $A \to B$ of G^s. we distinguish two cases:
    * If $A \to B$ is the reduction of an arrow a^f of π^f, then $a^f \in \mathcal{E}^f$. Therefore $A \to B$ is an arrow of $\mathcal{G}^r$
    * Otherwise, $A_{t_{\min}-1} \to B_t$ is in $\mathcal{E}^f$. Therefore $A \to B$ is an arrow of $\mathcal{G}^r$

    In both cases $A \to B$ is an arrow of $\mathcal{G}^r$, therefore $\mathcal{E}^s \subseteq \mathcal{E}^r$.

  Therefore, $\mathcal{G}^r = \mathcal{G}^s$ and thus $\mathcal{G}^f$ belongs to $C(\mathcal{G}^s)$.

Therefore, $\mathcal{G}^f$ is an FTCG belonging to $C(\mathcal{G}^s)$ which contains π^f. □

C.2 WITH ASSUMPTION 1

The construction given in Lemma 5 does not work to build an FTCG that satisfies Assumption 1. Indeed, some arrows are missing. By copying the construction of lemma 5 at each time step, if π^f verify the correct properties, it is possible to construct an FTCG belonging to $C(\mathcal{G}^s)$ which contains π^f and satisfy assumption 1. This reasoning is encapsulated by Lemmma 6.

Lemma 6. Let $\mathcal{G}^s$ be an SCG and π^f be a graph over $\mathcal{V}^f$. If π^f is a DAG, all its arrows respect time orientation, its reduction is a subgraph of $\mathcal{G}^s$ and all its instantaneous arrows respect Assumption 1, then there exists an FTCG $\mathcal{G}^f$ in $C(\mathcal{G}^s)$ which contains π^f and satisfies Assumption 1.

Proof. $\mathcal{G}^s$ is a SCG, by definition it is the reduction of an FTCG $\mathcal{G}^f_\star = (\mathcal{V}^f_\star, \mathcal{E}^f_\star)$. Let $t_{\min}$ be the minimum time index seen by π^f. Let us consider the graph $\mathcal{G}^f = (\mathcal{V}^f := \mathcal{V}^f_\star, \mathcal{E}^f)$ whose edges are constructed as follows:

1. All edges from π^f are set in $\mathcal{E}^f$.
2. For all the edges $A \to B$ in $\mathcal{G}^s$ that are not reductions of arrows in π^f, add the edge $A_{t_{\min}-1} \to B_t$ into $\mathcal{E}^f$.
3. Copy these arrows at each time step.

We have the following properties:

- $\mathcal{G}^f$ contains π^f because all arrows of π^f are set in $\mathcal{E}^f$.
- $\mathcal{G}^f$ is a DAG: All instantaneous arrows come from π^f and its copies. Since π^f is a DAG and all its instantaneous arrows respect Assumption 1, we know that there is no cycle made of instantaneous arrows. Moreover, delayed arrows follow the flow of time, therefore, they cannot form a cycle. Therefore, $\mathcal{G}^f$ does not contain any cycle.

  Moreover, there is no cycle due to delayed arrows because all of them follow the flow of time.
- $\mathcal{G}^f$ belongs to $C(\mathcal{G}^s)$: By the same reasoning of Lemma 5, at Step 2, the reduction $\mathcal{G}^r_{\text{at Step 2}}$ of $\mathcal{G}^f_{\text{at Step 2}}$ is equal to $\mathcal{G}^s$. Adding copies of arrows does not change the reduction. Therefore, the reduction of $\mathcal{G}^f$ is equal to $\mathcal{G}^s$ i.e. $\mathcal{G}^f$ belongs to $C(\mathcal{G}^s)$.
- $\mathcal{G}^f$ satisfies Assumption 1 because copying all arrows ensure that all causal relationships remain constant throughout time

Therefore, $\mathcal{G}^f$ is an FTCG belonging to $C(\mathcal{G}^s)$ which contains π^f and satisfies Assumption 1. □

D PROOFS OF SECTION 4

D.1 PROOFS OF SECTION 4.1

D.1.1 Proof of Lemma 1

Lemma 1. (Characterization of $\mathcal{NC}$) Let $\mathcal{G}^s = (\mathcal{V}^s, \mathcal{E}^s)$ be an SCG and let $P(y_t \mid \mathrm{do}(x^1_{t-\gamma_1}), \dots, \mathrm{do}(x^n_{t-\gamma_n}))$ be the considered effect. With the convention $\{F_{t_1}\}_{t_1 \geq +\infty} = \emptyset$, we have:

$$\mathcal{NC} = \bigcup_{Z \in \mathcal{V}^S} \{Z_{t_1}\}_{t_1 \geq t_{\mathcal{NC}}(Z)} \setminus \mathcal{X}^f.$$

Moreover, $(t_{\mathcal{NC}}(F))_{F \in \mathcal{V}^s}$ can be computed through Algorithm 1, detailed in Appendix D, which complexity is pseudo-linear with respect to $\mathcal{G}^s$ and $\mathcal{X}^f$.

Proof. **We start by proving the characterizations of $\mathcal{NC}$.**
By definition, we have:

$$\mathcal{NC} := \bigcup_{\mathcal{G}^f \in C(\mathcal{G}^s)} \mathrm{Forb}\left(\mathcal{X}^f, Y_t, \mathcal{G}^f\right) \setminus \mathcal{X}^f$$

Le us denote $\mathcal{S} := \bigcup_{Z \in \mathcal{V}^S} \{Z_{t_1}\}_{t_1 \geq t_{\mathcal{NC}}(Z)} \setminus \mathcal{X}^f$. We prove $\mathcal{NC} = \mathcal{S}$ by double inclusion:

- $\mathcal{NC} \subseteq \mathcal{S}$: Let $Z_{t_1} \in \mathcal{NC}$. By definition, $t_1 \geq t_{\mathcal{NC}}(Z)$ and $Z_{t_1} \notin \mathcal{X}^f$. Therefore $\mathcal{NC} \subseteq \mathcal{S}$.
- $\mathcal{S} \subseteq \mathcal{NC}$: Let $Z_{t_1} \in \mathcal{S}$. By definition $Z_{t_1} \notin \mathcal{X}^f$. Necessarily $t_{\mathcal{NC}}(Z) < +\infty$ thus there exists an FTCG $\mathcal{G}^f \in C(\mathcal{G}^s)$ in which $Z_{t_{\mathcal{NC}}} \in \mathrm{Forb}\left(\mathcal{X}^f, Y_t, \mathcal{G}^f\right)$. Thus, in $\mathcal{G}^f$, there exists a vertex $W_{t_w} \notin \mathcal{X}^f$ which lies on a proper causal path π^f_1 from $\mathcal{X}^f$ to Y_t and π^f_2 a directed path from W_{t_w} to $Z_{t_{\mathcal{NC}}(Z)}$. We can construct another FTCG belonging to $C(\mathcal{G}^s)$ containing π^f_1 and π^f_2 except that the last arrow of π^f_2 points to Z_{t_1} instead of $Z_{t_{\mathcal{NC}}(Z)}$. Thus, $Z_{t_1} \in \mathcal{NC}$. Therefore $\mathcal{S} \subseteq \mathcal{NC}$.

Therefore, $\mathcal{NC} = \bigcup_{Z \in \mathcal{V}^S} \{Z_{t_1}\}_{t_1 \geq t_{\mathcal{NC}}(Z)} \setminus \mathcal{X}^f$.

Lemma 11 proves that Algorithm 1 computes computes $(t_{\mathcal{NC}}(S))_{S \in \mathcal{V}^s}$ in pseudo-linear complexity. The proof requires several additional lemmas. It is discussed in the following paragraphs. □

To compute $(t_{\mathcal{NC}}(F))_{F \in \mathcal{V}^s}$ in pseudo-linear complexity, we need to use an efficient characterization of $\mathcal{CF}$ given by Lemma 7.

Lemma 7 (Characterization of $\mathcal{CF}$.)**.** Let $\mathcal{G}^s = (\mathcal{V}^s, \mathcal{E}^s)$ be an SCG and let $P(y_t \mid \mathrm{do}(x^1_{t-\gamma_1}), \dots, \mathrm{do}(x^n_{t-\gamma_n}))$ be the considered effect. For each $C \in \mathrm{Ch}(\mathcal{X}^s)$, we define:

$$t_C := \min\{t_1 \mid C_{t_1} \notin \mathcal{X}^f \text{ and } \exists i \exists \mathcal{G}^f \in C(\mathcal{G}^s) \text{ s.t } \mathcal{G}^f \models X^i_{t-\gamma_i} \to C_{t_1} \text{ and } C_{t_1} \in \mathrm{Anc}(Y_t, \mathcal{G}^f \setminus \mathcal{X}^f)\},$$

with the convention that $\min \emptyset = +\infty$. We have:

$$\mathcal{CF} = \bigcup_{\substack{C \in \mathrm{Ch}(\mathcal{X}^s) \\ t_C < +\infty}} \bigcup_{\substack{\mathcal{G}^f \in C(\mathcal{G}^s) \\ \exists i\ \mathcal{G}^f \models X^i_{t-\gamma_i} \to C_{t_C}}} \mathrm{Desc}\left(C_{t_C}, \mathcal{G}^f\right).$$

Proof. We prove this lemma by double inclusion. Let $S = \bigcup_{\substack{C \in \mathrm{Ch}(\mathcal{X}^s) \\ t_C < +\infty}} \bigcup_{\substack{\mathcal{G}^f \in C(\mathcal{G}^s) \\ \exists i\ \mathcal{G}^f \models X^i_{t-\gamma_i} \to C_{t_C}}} \mathrm{Desc}\left(C_{t_C}, \mathcal{G}^f\right)$:

- Let $D_{t_d} \in \mathcal{CF}$. By definition there exists an FTCG $\mathcal{G}^f \in C(\mathcal{G}^s)$ such that $D_{t_d} \in \mathrm{Forb}\left(\mathcal{X}^f, Y_t, \mathcal{G}^f\right)$. Thus, $\mathcal{G}^f$ contains a proper causal path π^f from $\mathcal{X}^f$ to Y_t and there exists $W_{t_w} \in \pi^f \setminus \mathcal{X}^f$ such that $D_{t_d} \in \mathrm{Desc}(W_{t_w}, \mathcal{G}^f)$. Let C_{t_1} be the second vertex of π^f. Let X^i be the parent of C in $\mathcal{G}^s$ with the smallest corresponding $t - \gamma_i$. Let $t_\star$ be the minimal time greater than $t - \gamma_i$ such that $C_{t_\star}$ is not an intervention. By changing the first two vertices of π^f, we can construct an FTCG $\mathcal{G}^{f\prime}$ that contains $\pi^{f\prime} = X^i_{t-\gamma_i} \to C_{t_\star} \rightsquigarrow Y_t$ such that $W_{t_w} \in \pi^{f\prime} \setminus \mathcal{X}^f$ and $D_{t_d} \in \mathrm{Desc}(W_{t_w}, \mathcal{G}^{f\prime})$. By construction,

$C_{t_\star} \notin \mathcal{X}^f$, $\mathcal{G}^{f\prime} \models X^i_{t-\gamma_i} \to C_{t_\star}$ and $C_{t_\star} \in \mathrm{Anc}(Y_t, \mathcal{G}^{f\prime} \setminus \mathcal{X}^f)$. Thus $t_C \leq t_\star < +\infty$. X^i is the parent of C in $\mathcal{G}^s$ with the smallest corresponding $t - \gamma_i$ thus $t_C = t_\star$. Thus, $W_{t_w} \in \mathrm{Desc}(C_{t_c}, \mathcal{G}^{f\prime})$ and we already know that $D_{t_d} \in \mathrm{Desc}(W_{t_w}, \mathcal{G}^{f\prime})$. Thus $D_{t_d} \in \mathrm{Desc}(C_{t_C}, \mathcal{G}^{f\prime})$. Therefore $D_{t_d} \in S$.

- Let $D_{t_d} \in S$. By definition, $t_c < +\infty$, thus there exist an intervention $X^i_{t-\gamma_i}$ and an FTCG $\mathcal{G}^f_1$ such that $\mathcal{G}^f_1 \models X^i_{t-\gamma_i} \to C_{t_C}$ and $C_{t_C} \in \mathrm{Anc}(Y_t, \mathcal{G}^f_1 \setminus \mathcal{X}^f)$. Let π^f_1 be this path. Without loss of generality, we can assume that π^f_1 does not pass twice through the same time series. Moreover, there exist an intervention $X^j_{t-\gamma_j}$ and an FTCG $\mathcal{G}^f_2$ such that $\mathcal{G}^f_2 \models X^j_{t-\gamma_j} \to C_{t_C} \rightsquigarrow D_{t_d}$. Let π^f_2 be this path. We construct $\mathcal{G}^f_3$ with the following procedure:
  1. $\mathcal{G}^f_3$ is the graph without edges on $\mathcal{V}^f$.
  2. Insert π^f_1 and $X^i_{t-\gamma_i} \to C_{t_C}$ into $\mathcal{G}^f_3$.
  3. At this point of the construction, we know that C_{t_C} is on a proper causal path from $\mathcal{X}^f$ to Y_t in $\mathcal{G}^f_3$.
  4. Let $W_{t_w} \in \pi^f_2$ be such that W is the last time series seen by π^f_2 that is also seen by π^f_1. Let $\langle V^1_{t_1} = C_{t_C}, \ldots, V^k_{t_k}\rangle$ denote the vertex in π^f_1 until it reaches W. For all m from 1 to $k-1$, insert $V^m_{t_C} \to V^{m+1}_{t_C}$ into $\mathcal{G}^f_3$. Insert $V^{k-1}_{t_C} \to W_{t_w}$ into $\mathcal{G}^f_3$. Insert all the arrows from π^f_2 after W_{t_w}.
  5. At this point of the construction, D_{t_d} is a descendant of C_{t_C} in $\mathcal{G}^f_3$.
  6. Add all missing arrow to a greater time lag so that $\mathcal{G}^f_3 \in \mathcal{C}(\mathcal{G}^s)$.

  Therefore, $D_{t_d} \in \mathcal{CF}$.

Therefore, $\mathcal{CF} = S$. One can check that indeed $\mathcal{G}^f_3 \in \mathcal{C}(\mathcal{G}^s)$ and that $\mathcal{G}^f_3$ satisfies Assumption 1 if both $\mathcal{G}^f_1$ and $\mathcal{G}^f_2$ satisfy Assumption 1. □

Lemma 8 specifies Lemma 7.

Lemma 8. Let $\mathcal{G}^s = (\mathcal{V}^s, \mathcal{E}^s)$ be an SCG and let $P(y_t \mid \mathrm{do}(x^1_{t-\gamma_1}), \ldots, \mathrm{do}(x^n_{t-\gamma_n}))$ be the considered effect. Let $C \in \mathrm{Ch}(\mathcal{X}^s)$ such that $t_C < +\infty$. We have:

$$\bigcup_{\substack{\mathcal{G}^f \in \mathcal{C}(\mathcal{G}^s) \\ \exists i\ \mathcal{G}^f \models X^i_{t-\gamma_i} \to C_{t_C}}} \mathrm{Desc}\left(C_{t_C}, \mathcal{G}^f\right) = \begin{cases} \bigcup_{\mathcal{G}^f \in \mathcal{C}(\mathcal{G}^s)} \mathrm{Desc}\left(C_{t_C}, \mathcal{G}^f\right) & \text{if } d(C), \\ \bigcup_{\mathcal{G}^f \in \mathcal{C}(\mathcal{G}^s)} \mathrm{Desc}\left(C_{t_C}, \mathcal{G}^f \setminus \{X^i_{t-\gamma_i}\}\right) & \text{otherwise.} \end{cases}$$

Where $d(C) := \#\{i \mid X^i \in \mathrm{Pa}(C, \mathcal{G}^s) \text{ and } t - \gamma_i < t_C\} \geq 1$ or $\#\{i \mid X^i \in \mathrm{Pa}(C, \mathcal{G}^s) \text{ and } t - \gamma_i = t_C\} \geq 2$, and, in the second case, X^i is the only element of $\{X^i \in \mathrm{Pa}(C, \mathcal{G}^s) \text{ and } t - \gamma_i \leq t_C\} = \{X^i \in \mathrm{Pa}(C, \mathcal{G}^s) \text{ and } t - \gamma_i = t_C\}$.

Proof. $t_C < +\infty$ thus $\#\{i \mid X^i \in \mathrm{Pa}(C, \mathcal{G}^s) \text{ and } t - \gamma_i \leq t_C\} \geq 1$. We distinguish two cases:

- If $\#\{i \mid X^i \in \mathrm{Pa}(C, \mathcal{G}^s) \text{ and } t - \gamma_i < t_C\} \geq 1$. Let us show the two inclusions:
  - By definition, $\bigcup_{\substack{\mathcal{G}^f \in \mathcal{C}(\mathcal{G}^s) \\ \exists i\ \mathcal{G}^f \models X^i_{t-\gamma_i} \to C_{t_C}}} \mathrm{Desc}\left(C_{t_C}, \mathcal{G}^f\right) \subseteq \bigcup_{\mathcal{G}^f \in \mathcal{C}(\mathcal{G}^s)} \mathrm{Desc}\left(C_{t_C}, \mathcal{G}^f\right)$.
  - Let $X^i_{t-\gamma_i}$ be an element of $\{X^i \in \mathrm{Pa}(C, \mathcal{G}^s) \mid t - \gamma_i < t_C\}$. The arrow from $X^i_{t-\gamma_i}$ to C_{t_C} cannot contradict any other arrow. Therefore, $\bigcup_{\mathcal{G}^f \in \mathcal{C}(\mathcal{G}^s)} \mathrm{Desc}\left(C_{t_C}, \mathcal{G}^f\right) \subseteq \bigcup_{\substack{\mathcal{G}^f \in \mathcal{C}(\mathcal{G}^s) \\ \exists i\ \mathcal{G}^f \models X^i_{t-\gamma_i} \to C_{t_C}}} \mathrm{Desc}\left(C_{t_C}, \mathcal{G}^f\right)$.

  Therefore, $\bigcup_{\substack{\mathcal{G}^f \in \mathcal{C}(\mathcal{G}^s) \\ \exists i\ \mathcal{G}^f \models X^i_{t-\gamma_i} \to C_{t_C}}} \mathrm{Desc}\left(C_{t_C}, \mathcal{G}^f\right) = \bigcup_{\mathcal{G}^f \in \mathcal{C}(\mathcal{G}^s)} \mathrm{Desc}\left(C_{t_C}, \mathcal{G}^f\right)$.
- Otherwise, $\#\{i \mid X^i \in \mathrm{Pa}(C, \mathcal{G}^s) \text{ and } t - \gamma_i = t_C\} \geq 1$. We distinguish two cases:
  - If $\#\{i \mid X^i \in \mathrm{Pa}(C, \mathcal{G}^s) \text{ and } t - \gamma_i = t_C\} = 1$. Let $X^i_{t-\gamma_i}$ be the only element of $\{X^i_{t-\gamma_i} \mid X^i \in \mathrm{Pa}(C, \mathcal{G}^s) \mid t - \gamma_i = t_C\}$. In this case, $\bigcup_{\substack{\mathcal{G}^f \in \mathcal{C}(\mathcal{G}^s) \\ \exists i\ \mathcal{G}^f \models X^i_{t-\gamma_i} \to C_{t_C}}} \mathrm{Desc}\left(C_{t_C}, \mathcal{G}^f\right) = \bigcup_{\substack{\mathcal{G}^f \in \mathcal{C}(\mathcal{G}^s) \\ \mathcal{G}^f \models X^i_{t-\gamma_i} \to C_{t-\gamma_i}}} \mathrm{Desc}\left(C_{t-\gamma_i}, \mathcal{G}^f\right)$. Let us show the two inclusions:
    - Let $D_{t_d} \in \bigcup_{\substack{\mathcal{G}^f \in \mathcal{C}(\mathcal{G}^s) \\ \mathcal{G}^f \models X^i_{t-\gamma_i} \to C_{t-\gamma_i}}}$. There exists an FTCG $\mathcal{G}^f$ which contains a directed path π^f from $C_{t-\gamma_i}$ to D_{t_d} and such that $\mathcal{G}^f \models X^i_{t-\gamma_i} \to C_{t-\gamma_i}$. π^f does not pass through $X^i_{t-\gamma_i}$ because, otherwise, $\mathcal{G}^f \models X^i_{t-\gamma_i} \leftarrow C_{t-\gamma_i}$. Thus $D_{t_d} \in \bigcup_{\mathcal{G}^f \in \mathcal{C}(\mathcal{G}^s)} \mathrm{Desc}\left(C_{t_C}, \mathcal{G}^f \setminus \{X^i_{t-\gamma_i}\}\right)$.

* Let $D_{t_d} \in \bigcup_{\mathcal{G}^f \in C(\mathcal{G}^s)} \mathrm{Desc}\left(C_{t_C}, \mathcal{G}^f \setminus \{X^i_{t-\gamma_i}\}\right)$. Thus, there exists an FTCG containing a directed path π^f from $C_{t-\gamma_i}$ to D_{t_d} without passing through $X^i_{t-\gamma_i}$. Without loss of generality, we can assume that π^f does not pass twice through the same time series. We distinguish two cases:
  · If π^f does not use an arrow from $C_{t-\gamma_i}$ to some $X^i_{t'} \in X^i$. Then, we can construct an FTCG containing π^f and $X^i_{t-\gamma_i} \to C_{t-\gamma_i}$. Thus, $D_{t_d} \in \bigcup_{\substack{\mathcal{G}^f \in C(\mathcal{G}^s) \\ \mathcal{G}^f \models X^i_{t-\gamma_i} \to C_{t-\gamma_i}}}$.
  · Otherwise, π^f does not pass through $X^i_{t-\gamma_i}$, thus $t_d < t - \gamma_i$. We can construct an FTCG containing $X^i_{t-\gamma_i} \to C_{t-\gamma_i}$ and a path $\pi^{f\prime}$ from $C_{t-\gamma_i}$ to D_{t_d} where the first arrow is $C_{t-\gamma_i} \to X^i_{t_d}$ and all other arrows are the arrows from π^f but at time t_d. Thus, $D_{t_d} \in \bigcup_{\substack{\mathcal{G}^f \in C(\mathcal{G}^s) \\ \mathcal{G}^f \models X^i_{t-\gamma_i} \to C_{t-\gamma_i}}}$.

  Therefore, $D_{t_d} \in \bigcup_{\substack{\mathcal{G}^f \in C(\mathcal{G}^s) \\ \mathcal{G}^f \models X^i_{t-\gamma_i} \to C_{t-\gamma_i}}}$.

 Therefore, $\bigcup_{\substack{\mathcal{G}^f \in C(\mathcal{G}^s) \\ \mathcal{G}^f \models X^i_{t-\gamma_i} \to C_{t-\gamma_i}}} = \bigcup_{\mathcal{G}^f \in C(\mathcal{G}^s)} \mathrm{Desc}\left(C_{t_C}, \mathcal{G}^f \setminus \{X^i_{t-\gamma_i}\}\right)$.

– Otherwise, $\#\{i \mid X^i \in \mathrm{Pa}(C, \mathcal{G}^s) \text{ and } t - \gamma_i = t_C\} \geq 2$. Let us show the two inclusions:
  * By definition, $\bigcup_{\substack{\mathcal{G}^f \in C(\mathcal{G}^s) \\ \exists i\ \mathcal{G}^f \models X^i_{t-\gamma_i} \to C_{t_C}}} \mathrm{Desc}\left(C_{t_C}, \mathcal{G}^f\right) \subseteq \bigcup_{\mathcal{G}^f \in C(\mathcal{G}^s)} \mathrm{Desc}\left(C_{t_C}, \mathcal{G}^f\right)$.
  * Let $D_{t_d} \in \bigcup_{\mathcal{G}^f \in C(\mathcal{G}^s)} \mathrm{Desc}\left(C_{t_C}, \mathcal{G}^f\right)$. There exist an FTCG which contains a directed path π^f from C_{t_C} to D_{t_d}. Without loss of generality, we can assume that π^f does not pass twice through the same time series. Thus, π^f contains only one arrow a coming from C_{t_C}. $\#\{i \mid X^i \in \mathrm{Pa}(C, \mathcal{G}^s) \text{ and } t - \gamma_i = t_C\} \geq 2$ thus there exists an intervention $X^i_{t-\gamma_i} \in \{X^i_{t-\gamma_i} \mid X^i \in \mathrm{Pa}(C, \mathcal{G}^s) \text{ and } t - \gamma_i = t_C\}$ such that $X^i_{t-\gamma_i} \to C_{t_C}$ does not contradict a. Thus we can construct another FTCG containing $X^i_{t-\gamma_i} \to C_{t_C}$ and π^f. Thus, $\bigcup_{\mathcal{G}^f \in C(\mathcal{G}^s)} \mathrm{Desc}\left(C_{t_C}, \mathcal{G}^f\right) \subseteq \bigcup_{\substack{\mathcal{G}^f \in C(\mathcal{G}^s) \\ \exists i\ \mathcal{G}^f \models X^i_{t-\gamma_i} \to C_{t_C}}} \mathrm{Desc}\left(C_{t_C}, \mathcal{G}^f\right)$

 Therefore, $\bigcup_{\substack{\mathcal{G}^f \in C(\mathcal{G}^s) \\ \exists i\ \mathcal{G}^f \models X^i_{t-\gamma_i} \to C_{t_C}}} \mathrm{Desc}\left(C_{t_C}, \mathcal{G}^f\right) = \bigcup_{\mathcal{G}^f \in C(\mathcal{G}^s)} \mathrm{Desc}\left(C_{t_C}, \mathcal{G}^f\right)$.

□

Lemm 9 shows how to compute t_C.

Lemma 9. Let $\mathcal{G}^s = (\mathcal{V}^s, \mathcal{E}^s)$ be an SCG and let $P(y_t \mid \mathrm{do}(x^1_{t-\gamma_1}), \ldots, \mathrm{do}(x^n_{t-\gamma_n}))$ be the considered effect. Let $C \in \mathrm{Ch}(\mathcal{X}^s)$. let $t_{\max} := \max\left\{t_1 \mid \exists \mathcal{G}^f \text{ s.t } C_{t_1} \in \mathrm{Anc}(Y_t, \mathcal{G}^f \setminus \mathcal{X}^f)\right\}$ and $t_{\min} := \min\{t - \gamma_i \mid X^i \in \mathrm{Pa}(C, \mathcal{G}^s)\}$. We have the following identity:

$$\left\{t_1 \mid C_{t_1} \notin \mathcal{X}^f \text{ and } \exists i \exists \mathcal{G}^f \in C(\mathcal{G}^s) \text{ s.t } \mathcal{G}^f \models X^i_{t-\gamma_i} \to C_{t_1} \text{ and } C_{t_1} \in \mathrm{Anc}(Y_t, \mathcal{G}^f \setminus \mathcal{X}^f)\right\} = \left\{t_1 \in [t_{\min}, t_{\max}] \mid C_{t_1} \notin \mathcal{X}^f\right\}.$$

Proof. Let us prove the two inclusions.

- Let $t_1 \in \{t_1 \mid C_{t_1} \notin \mathcal{X}^f \text{ and } \exists i \exists \mathcal{G}^f \in C(\mathcal{G}^s) \text{ s.t } \mathcal{G}^f \models X^i_{t-\gamma_i} \to C_{t_1} \text{ and } C_{t_1} \in \mathrm{Anc}(Y_t, \mathcal{G}^f \setminus \mathcal{X}^f)\}$. By definition, there exist $\mathcal{G}^f$ and an intervention $X^i_{t-\gamma_i}$ such that $\mathcal{G}^f \models X^i_{t-\gamma_i} \to C_{t_1}$. Thus $\mathcal{G}^s \models X^i \to C$. Thus $t_1 \geq t - \gamma_i \geq t_{\min}$ because causality does not move backwards in time. Moreover, $C_{t_1} \in \mathrm{Anc}(Y_t, \mathcal{G}^f \setminus \mathcal{X}^f)$. Thus $t_1 \leq t_{\max}$. Therefore $\{t_1 \mid C_{t_1} \notin \mathcal{X}^f \text{ and } \exists i \exists \mathcal{G}^f \in C(\mathcal{G}^s) \text{ s.t } \mathcal{G}^f \models X^i_{t-\gamma_i} \to C_{t_1} \text{ and } C_{t_1} \in \mathrm{Anc}(Y_t, \mathcal{G}^f \setminus \mathcal{X}^f)\} \subseteq \left\{t_1 \in [t_{\min}, t_{\max}] \mid C_{t_1} \notin \mathcal{X}^f\right\}$.
- Let $t_1 \in \left\{t_1 \in [t_{\min}, t_{\max}] \mid C_{t_1} \notin \mathcal{X}^f\right\}$. By definition $t_1 \leq t_{\max}$. Thus there exist an FTCG containing π^f, a directed path from $C_{t_{\max}}$ to Y_t that does not go through $\mathcal{X}^f$. By definition $t_1 \geq t_{\min}$, thus there exists an intervention $X^i_{t-\gamma_i}$ such that $t - \gamma_i = t_{\min} \leq t_1$ and $X^i \in \mathrm{Pa}(C, \mathcal{G}^s)$. Since π^f does not go through $\mathcal{X}^f$, by changing its first arrow we can construct an FTCG $\mathcal{G}^f \in C(\mathcal{G}^s)$ such that $\mathcal{G}^f \models X^i_{t-\gamma_i} \to C_{t_1}$ and $C_{t_1} \in \mathrm{Anc}(Y_t, \mathcal{G}^f \setminus \mathcal{X}^f)$. Therefore $\left\{t_1 \in [t_{\min}, t_{\max}] \mid C_{t_1} \notin \mathcal{X}^f\right\} \subseteq \{t_1 \mid C_{t_1} \notin \mathcal{X}^f \text{ and } \exists i \exists \mathcal{G}^f \in C(\mathcal{G}^s) \text{ s.t } \mathcal{G}^f \models X^i_{t-\gamma_i} \to C_{t_1} \text{ and } C_{t_1} \in \mathrm{Anc}(Y_t, \mathcal{G}^f \setminus \mathcal{X}^f)\}$.

Therefore, $\{t_1 \mid C_{t_1} \notin \mathcal{X}^f \text{ and } \exists i \exists \mathcal{G}^f \in C(\mathcal{G}^s) \text{ s.t } \mathcal{G}^f \models X^i_{t-\gamma_i} \to C_{t_1} \text{ and } C_{t_1} \in \mathrm{Anc}(Y_t, \mathcal{G}^f \setminus \mathcal{X}^f)\} = \left\{t_1 \in [t_{\min}, t_{\max}] \mid C_{t_1} \notin \mathcal{X}^f\right\}$. □

Lemma 10. Let $\mathcal{G}^s = (\mathcal{V}^s, \mathcal{E}^s)$ be an SCG and let $P(y_t \mid \mathrm{do}(x^1_{t-\gamma_1}), \ldots, \mathrm{do}(x^n_{t-\gamma_n}))$ be the considered effect. Let $C \in \mathrm{Ch}(\mathcal{X}^s)$ and S be a time series. We have the following identity:

$$\min\left\{t_1 \mid S_{t_1} \in \bigcup_{\mathcal{G}^f \in C(\mathcal{G}^s)} \mathrm{Desc}(C_{t_C}, \mathcal{G}^f) \setminus \mathcal{X}^f\right\} = \begin{cases} \min\{t_1 \mid t_1 \geq t_C \text{ and } S_{t_1} \notin \mathcal{X}^f\} & \text{if } S \in \mathrm{Desc}(C, \mathcal{G}^s), \\ +\infty & \text{otherwise.} \end{cases} \tag{2}$$

When $t_C = t - \gamma_i$, we also have:

$$\min\left\{t_1 \mid S_{t_1} \in \bigcup_{\mathcal{G}^f \in C(\mathcal{G}^s)} \mathrm{Desc}(C_{t_C}, \mathcal{G}^f \setminus \{X^i_{t-\gamma_i}\}) \setminus \mathcal{X}^f\right\} = \begin{cases} \min\{t_1 \mid t_1 \geq t_C \text{ and } S_{t_1} \notin \mathcal{X}^f\} & \text{if } S \in \mathrm{Desc}(C, \mathcal{G}^s \setminus \{X^i\}), \\ \min\{t_1 \mid t_1 \geq t_C + 1 \text{ and } S_{t_1} \notin \mathcal{X}^f\} & \text{else if } S \in \mathrm{Desc}(C, \mathcal{G}^s), \\ +\infty & \text{otherwise.} \end{cases} \tag{3}$$

Proof. Let us prove Equation (2). We distinguish two cases:

- If $S \in \mathrm{Desc}(C, \mathcal{G}^s)$, then there exists a directed path π^s from C to S in $\mathcal{G}^s$. For all $t_1 \geq t_C$, there exists an FTCG which contains a directed path from C_{t_C} to S_{t_1}. Therefore $\min\left\{t_1 \mid S_{t_1} \in \bigcup_{\mathcal{G}^f \in C(\mathcal{G}^s)} \mathrm{Desc}(C_{t_C}, \mathcal{G}^f) \setminus \mathcal{X}^f\right\} = \min\{t_1 \mid t_1 \geq t_C \text{ and } S_{t_1} \notin \mathcal{X}^f\}$.
- Otherwise, necessarily, for all t_1, S_{t_1} cannot be a descendant of C_{t_C} in any FTCG because otherwise S would be a descendant of C in $\mathcal{G}^s$. Thus $\min\left\{t_1 \mid S_{t_1} \in \bigcup_{\mathcal{G}^f \in C(\mathcal{G}^s)} \mathrm{Desc}(C_{t_C}, \mathcal{G}^f) \setminus \mathcal{X}^f\right\} = +\infty$.

Therefore, Equation (2) holds.

Let us prove Equation (3). We distinguish three cases:

- If $S \in \mathrm{Desc}(C, \mathcal{G}^s \setminus \{X^i\})$, then there exists a directed path π^s from C to S in $\mathcal{G}^s \setminus \{X^i\}$. For all $t_1 \geq t_C$, there exists an FTCG $\mathcal{G}^f$ such that $\mathcal{G}^f \setminus \{X^i_{t-\gamma_i}\}$ contains a directed path from C_{t_C} to S_{t_1}. Therefore $\min\left\{t_1 \mid S_{t_1} \in \bigcup_{\mathcal{G}^f \in C(\mathcal{G}^s)} \mathrm{Desc}(C_{t_C}, \mathcal{G}^f \setminus \{X^i_{t-\gamma_i}\}) \setminus \mathcal{X}^f\right\} = \min\{t_1 \mid t_1 \geq t_C \text{ and } S_{t_1} \notin \mathcal{X}^f\}$.
- Else if $S \in \mathrm{Desc}(C, \mathcal{G}^s)$, then there exist directed paths from C to S in $\mathcal{G}^s$ but all of them goes through X^i. Thus for all $t_1 \geq t_C + 1$, there exists an FTCG $\mathcal{G}^f$ such that $\mathcal{G}^f \setminus \{X^i_{t-\gamma_i}\}$ contains a directed path from C_{t_C} to S_{t_1}. But there is no FTCG which contains a directed path from C_{t_C} to S_{t_C} because this path would need to go through $X^i_{t-\gamma_i}$. Therefore, $\min\left\{t_1 \mid S_{t_1} \in \bigcup_{\mathcal{G}^f \in C(\mathcal{G}^s)} \mathrm{Desc}(C_{t_C}, \mathcal{G}^f \setminus \{X^i_{t-\gamma_i}\}) \setminus \mathcal{X}^f\right\} = \min\{t_1 \mid t_1 \geq t_C + 1 \text{ and } S_{t_1} \notin \mathcal{X}^f\}$.
- Otherwise, necessarily, for all t_1, S_{t_1} cannot be a descendant of C_{t_C} in any FTCG because otherwise S would be a descendant of C in $\mathcal{G}^s$. Thus $\min\left\{t_1 \mid S_{t_1} \in \bigcup_{\mathcal{G}^f \in C(\mathcal{G}^s)} \mathrm{Desc}(C_{t_C}, \mathcal{G}^f \setminus \{X^i_{t-\gamma_i}\}) \setminus \mathcal{X}^f\right\} = +\infty$.

Therefore, Equation (3) holds. □

Lemma 11. Let $\mathcal{G}^s = (\mathcal{V}^s, \mathcal{E}^s)$ be an SCG and let $P(y_t \mid \mathrm{do}(x^1_{t-\gamma_1}), \ldots, \mathrm{do}(x^n_{t-\gamma_n}))$ be the considered effect. Algorithm 1 computes $(t_{\mathcal{NC}}(S))_{S \in \mathcal{V}^s}$ in pseudo-linear complexity.

Proof. **Let us prove that Algorithm 1 is correct:**

Algorithm 1 is split into two parts, the first one computes t_C for all $C \in \mathrm{Ch}(\mathcal{X}^s)$ as defined in Lemma 7. The second part computes $(t_{\mathcal{NC}}(S))_{S \in \mathcal{V}^s}$. Lemma 9 shows that Algorithm 1 computes $(t_{\mathcal{NC}}(S))_{S \in \mathcal{V}^s}$ correctly.

Let us prove that the second part of the algorithm computes $(t_{\mathcal{NC}}(S))_{S \in \mathcal{V}^s}$ correctly. Let S be a time series. By Lemma 7, we know that:

$$\mathcal{NC} = \bigcup_{\substack{C \in \mathrm{Ch}(\mathcal{X}^s) \\ t_C < +\infty}} \bigcup_{\substack{\mathcal{G}^f \in C(\mathcal{G}^s) \\ \exists i \ \mathcal{G}^f \models X^i_{t-\gamma_i} \to C_{t_C}}} \mathrm{Desc}\left(C_{t_C}, \mathcal{G}^f\right) \setminus \mathcal{X}^f.$$

Thus, we have:

$$t_{\mathcal{NC}}[S] = \min_{\substack{C \in \mathrm{Ch}(\mathcal{X}^s) \\ t_C < +\infty}} \min \left\{ t_1 \mid S_{t_1} \in \bigcup_{\substack{\mathcal{G}^f \in \mathcal{C}(\mathcal{G}^s) \\ \exists i \; \mathcal{G}^f \models X^i_{t-\gamma_i} \to C_{t_C}}} \mathrm{Desc}(C_{t_C}, \mathcal{G}^f) \setminus \mathcal{X}^f \right\}$$

By Lemma 8 we have:

$$t_{\mathcal{NC}}[S] = \min_{\substack{C \in \mathrm{Ch}(\mathcal{X}^s) \\ t_C < +\infty}} \min \left\{ t_1 \mid S_{t_1} \in \begin{cases} \bigcup_{\mathcal{G}^f \in \mathcal{C}(\mathcal{G}^s)} \mathrm{Desc}\left(C_{t_C}, \mathcal{G}^f\right) & \text{if } d(C), \\ \bigcup_{\mathcal{G}^f \in \mathcal{C}(\mathcal{G}^s)} \mathrm{Desc}\left(C_{t_C}, \mathcal{G}^f \setminus \{X^i_{t-\gamma_i}\}\right) & \text{otherwise.} \end{cases} \right\}$$

Where $d(C) := \#\{i \mid X^i \in \mathrm{Pa}(C, \mathcal{G}^s) \text{ and } t - \gamma_i < t_C\} \geq 1$ or $\#\{i \mid X^i \in \mathrm{Pa}(C, \mathcal{G}^s) \text{ and } t - \gamma_i = t_C\} \geq 2$, and, in the second case, X^i is the only element of $\{X^i \in \mathrm{Pa}(C, \mathcal{G}^s) \text{ and } t - \gamma_i \leq t_C\} = \{X^i \in \mathrm{Pa}(C, \mathcal{G}^s) \text{ and } t - \gamma_i = t_C\}$. Thus, we have:

$$t_{\mathcal{NC}}[S] = \min_{\substack{C \in \mathrm{Ch}(\mathcal{X}^s) \\ t_C < +\infty}} \begin{cases} \min\left\{t_1 \mid S_{t_1} \in \bigcup_{\mathcal{G}^f \in \mathcal{C}(\mathcal{G}^s)} \mathrm{Desc}\left(C_{t_C}, \mathcal{G}^f\right)\right\} & \text{if } d(C), \\ \min\left\{t_1 \mid S_{t_1} \in \bigcup_{\mathcal{G}^f \in \mathcal{C}(\mathcal{G}^s)} \mathrm{Desc}\left(C_{t_C}, \mathcal{G}^f \setminus \{X^i_{t-\gamma_i}\}\right)\right\} & \text{otherwise.} \end{cases} \tag{4}$$

We can use Equation (3) from Lemma 10 to specify the first case of Equation (4). For the second case, we know that X^i is the only element of $\{X^i \in \mathrm{Pa}(C, \mathcal{G}^s) \text{ and } t - \gamma_i \leq t_C\} = \{X^i \in \mathrm{Pa}(C, \mathcal{G}^s) \text{ and } t - \gamma_i = t_C\}$. Thus $t_C = t - \gamma_i$, and we can use Equation (3) to specify this case. Algorithm 1 uses these specifications to compute Equation (4). It enumerates the $C \in \mathrm{Ch}(\mathcal{X}^s)$ such that $t_C < +\infty$ in the correct order to avoid unnecessary operations. Therefore, Algorithm 1 is correct.

We now prove that Algorithm 1 works in pseudo linear complexity:
We will go through the lines of Algorithm 1 and we will show that they run in pseudo-linear complexity:

- The first line of Algorithm 1 is $AncY \leftarrow \left(\max\left\{t_1 \mid \exists \mathcal{G}^f \text{ s.t } S_{t_1} \in \mathrm{Anc}(Y_t, \mathcal{G}^f \setminus \mathcal{X}^f)\right\}\right)_{S \in \mathcal{V}^s}$. By Lemma 12, this line is computed by Algorithm 5 in $O((|\mathcal{E}^s| + |\mathcal{V}^s|)\log|\mathcal{V}^s|)$.
- Then, the algorithm computes t_C for all $C \in \mathrm{Ch}(\mathcal{X}^s)$. While enumerating all the $C \in \mathrm{Ch}(\mathcal{X}^s)$, only the arrows from $\mathcal{X}^s$ are seen. Thus, this enumeration is done in $O(|\mathcal{E}^s|)$ time. With the appropriate pre-computations all the computations for each C are done in $O(1)$ time. Thus, this loop takes at most $O(|\mathcal{E}^s|)$ time.
- The sort of L can be done in a complexity of $O(|\mathcal{V}^s|\log|\mathcal{V}^s|)$. Indeed, L contains at most $|\mathcal{V}^s|$ elements.
- Then, the algorithm computes $(t_{\mathcal{NC}}(S))_{S \in \mathcal{V}^s}$. For each D computed by the algorithm, the algorithm computes $t_{\mathcal{NC}}(D) \leftarrow \min\{t_1 \mid t_1 \geq t_\star \text{ and } D_{t_1} \notin \mathcal{X}^f\}$, where $t_\star \in \{t_C, t_C + 1\}$ depending on the context. These lines can be computed in $O(1)$. Indeed, we need to distinguish two cases:
  - If $D_{t_\star} \notin \mathcal{X}^f$, the computations becomes $t_{\mathcal{NC}}(D) \leftarrow t_\star$.
  - Otherwise, we pick the value from $\{t_1 \mid t_1 \geq t - \gamma_i \text{ and } X^i_{t_1} \notin \mathcal{X}^f\}_{i \in \{1,\cdots,n\}}$, already pre-computed by Algorithm 4.

  Since Algorithm 4 runs in $O(\left|\mathcal{X}^f\right|\log\left|\mathcal{X}^f\right|)^2$, all computations in all foreach loops are done in $O(1)$ by adding $O(\left|\mathcal{X}^f\right|\log\left|\mathcal{X}^f\right|)$ to the overall complexity of Algorithm 1. Moreover, by sharing the unseen set among all calculations of descendant sets, all time series that are not a child of some $X^i \in \mathcal{X}^s$ are seen at most one time and all $C \in \mathrm{Ch}(\mathcal{X}^s)$ are seen at most three times. Similarly, arrows from a time series that is not a child of some $X^i \in \mathcal{X}^s$ are seen at most one time and arrows from all $C \in \mathrm{Ch}(\mathcal{X}^s)$ are seen at most two times. Thus, the computation of $(t_{\mathcal{NC}}(S))_{S \in \mathcal{V}^s}$ is done in $O(|\mathcal{E}^s| + |\mathcal{V}^s|)$.

Therefore,the overall complexity of Algorithm 1 is $O\left(\left|\mathcal{X}^f\right|\log\left|\mathcal{X}^f\right| + (|\mathcal{E}^s| + |\mathcal{V}^s|)\log|\mathcal{V}^s|\right)$. It is indeed a pseudo linear complexity with respect to $\mathcal{G}^s$ and $\mathcal{X}^f$.

[2] The $\log n$ comes from sorting interventions by time index.

Algorithm 5: Computation of $\left(\max\left\{t_1 \mid \exists \mathcal{G}^f \text{ s.t } S_{t_1} \in \text{Anc}(Y_t, \mathcal{G}^f \setminus \mathcal{X}^f)\right\}\right)_{S \in \mathcal{V}^s}$

Input: $\mathcal{G}^s = (\mathcal{V}^s, \mathcal{E}^s)$ an SCG, $\mathcal{X}^f$ and $V_{t_v} \in \mathcal{V}^f$

Output: $\left(\max\left\{t_1 \mid \exists \mathcal{G}^f \text{ s.t } S_{t_1} \in \text{Anc}(Y_t, \mathcal{G}^f \setminus \mathcal{X}^f)\right\}\right)_{S \in \mathcal{V}^s}$ written as $(AncY[S])_{S \in \mathcal{V}^s}$.

$Q \leftarrow \text{PriorityQueue}(Y_t)$;
$AncY[S] \leftarrow -\infty \quad \forall S \in \mathcal{V}^s$;
$S.seen \leftarrow False \quad \forall S \in \mathcal{V}^s$;
$AncY[Y] \leftarrow 0$;
$Y.seen \leftarrow True$ **while** $Q \neq \emptyset$ **do**
  $S_{t_s} \leftarrow Q$.pop_element_with_max_time_index();
  foreach *unseen* $P \in Pa(S, \mathcal{G}^s)$ **do**
    $AncY[P] \leftarrow \max\{t_1 \mid t_1 \leq t_s \text{ and } P_{t_1} \notin \mathcal{X}^f\}$;
    if $AncY[P] \neq -\infty$ **then** $Q.insert(P_{AncY[P]})$;
    $P.seen \leftarrow true$;

Algorithm 4: Computation of $\{t_1 \mid t_1 \geq t - \gamma_i \text{ and } X^i_{t_1} \notin \mathcal{X}^f\}_{i \in \{1, \cdots, n\}}$

Input: List of interventions, sorted by decreasing time indices.

Output: $\{\min\{t_1 \mid t_1 \geq t - \gamma_i \text{ and } X^i_{t_1} \notin \mathcal{X}^f\}\}_{i \in \{1, \cdots, n\}}$, denoted as $\{t_{X^i_{t-\gamma_i}}\}_i$

$L \leftarrow$ List of lists of interventions, grouped by time series, preserving the time index ordering;
for $l \in L$ **do**
  $X^i_{t-\gamma_i} \leftarrow l[0]$;
  $t_{X^i_{t-\gamma_i}} \leftarrow t - \gamma_i + 1$;
  foreach $X^j_{t-\gamma_j}$ *in* $l[1:]$ **do**
    Let $X^i_{t-\gamma_i}$ be the predecessor of $X^j_{t-\gamma_j}$ in l;
    if $t - \gamma_i + 1 = t - \gamma_j$ **then**
      $t_{X^j_{t-\gamma_j}} \leftarrow t_{X^i_{t-\gamma_i}}$;
    else
      $t_{X^j_{t-\gamma_j}} \leftarrow t - \gamma_j + 1$;

□

Lemma 12. Let $\mathcal{G}^s = (\mathcal{V}^s, \mathcal{E}^s)$ be an SCG and let $P(y_t \mid \text{do}(x^1_{t-\gamma_1}), \ldots, \text{do}(x^n_{t-\gamma_n}))$ be the considered effect. $\left(\max\left\{t_1 \mid \exists \mathcal{G}^f \text{ s.t } S_{t_1} \in \text{Anc}(Y_t, \mathcal{G}^f \setminus \mathcal{X}^f)\right\}\right)_{S \in \mathcal{V}^s}$ is computed by Algorithm 5 in $\mathcal{O}((|\mathcal{E}^s| + |\mathcal{V}^s|) \log |\mathcal{V}^s|)$.

Proof. The proof is identical to the proof of Lemma 14 except that we compute the Y_t-$(\mathcal{V}^s \setminus \mathcal{X}^f)$-accessibility instead of the Y_t-$(\mathcal{CF} \setminus \mathcal{X}^f)$-accessibility. Precisely, in this case, Y_t is considered to be Y_t-$\mathcal{V}^s \setminus \mathcal{X}^f$-accessible. □

D.1.2 Proof of Theorem 1

Theorem 1. Let $\mathcal{G}^s = (\mathcal{V}^s, \mathcal{E}^s)$ be an SCG and $P(y_t \mid \text{do}(x^1_{t-\gamma_1}), \ldots, \text{do}(x^n_{t-\gamma_n}))$ be the considered effect. Then the two statements are equivalent:

1. The effect is identifiable by common adjustment in $\mathcal{G}^s$.
2. For all intervention $X^i_{t-\gamma_i}$ and candidate FTCG $\mathcal{G}^f \in \mathcal{C}(\mathcal{G}^s)$, $\mathcal{G}^f$ does not contain a collider-free backdoor path going from $X^i_{t-\gamma_i}$ to Y_t that remains in $\mathcal{NC} \cup \{X^i_{t-\gamma_i}\}$.

In that case, a common adjustment set is given by $\mathcal{C} := \left(\mathcal{V}^f \setminus \mathcal{NC}\right) \setminus X^f$, and we have

$$P(y_t \mid \text{do}((x^i_{t-\gamma_i})_i)) = \sum_{\mathbf{c}} P\left(y_t \mid (x^i_{t-\gamma_i})_i, \mathbf{c}\right) P(\mathbf{c}).$$

Proof. **We start by proving that every proper non-causal paths from $\mathcal{X}^f$ to Y_t that leaves $\mathcal{CF}$, except perhaps for its first vertex, are blocked by $\mathcal{C} := \left(\mathcal{V}^f \setminus \mathcal{NC}\right) \setminus \mathcal{X}^f$:**

Let us consider an FTCG $\mathcal{G}^f$ and a proper non-causal path π^f from $X^i_{t-\gamma_i} \in \mathcal{X}^f$ to Y_t that leaves $\mathcal{CF} \cup \{X^i_{t-\gamma_i}\}$. Let C_{t_c} be the last vertex of π^f in $\mathcal{V}^f \setminus \left(\mathcal{CF} \cup \{X^i_{t-\gamma_i}\}\right)$. We know that Y is a descendant of an intervention in $\mathcal{G}^s$. Thus, $Y_t \in \mathcal{CF}$. Thus, $C_{t_c} \neq Y_t$ and C_{tC} is not the last vertex of π^f. Let us consider D_{t_d}, the successor of C_{t_c} in π^f. By definition, $D_{t_d} \in \mathcal{CF}$. Necessarily $\pi^f_{|\{C_{t_c},D_{t_d}\}} = D_{t_d} \leftarrow C_{t_c}$. Indeed, otherwise C_{t_c} would be in $\mathcal{CF}$. Hence, C_{t_c} is not a collider on π^f. Moreover, π^f is a proper path thus, $C_{t_c} \in \left(\mathcal{V}^f \setminus \mathcal{CF}\right) \setminus \mathcal{X}^f = \left(\mathcal{V}^f \setminus \mathcal{NC}\right) \setminus \mathcal{X}^f$. Therefore $\mathcal{C} = \left(\mathcal{V}^f \setminus \mathcal{NC}\right) \setminus \mathcal{X}^f$ blocks π^f.

Let us prove the two implications of the theorem:

- $1 \Rightarrow 2$: Let us prove the contrapositive: Let $\mathcal{G}^f$ be an FTCG that contains π^f, a collider-free backdoor path from $X^i_{t-\gamma_i}$ to Y_t that remains within $\mathcal{NC} \cup \{X^i_{t-\gamma_i}\}$. Thus π^f is a proper non-causal path from $\mathcal{X}^f$ to Y_t. Moreover, π^f does not have a collider, so the only way to block it is by conditioning on one of its vertices. However, all its vertices are within $\mathcal{NC} \cup \{X^i_{t-\gamma_i}\} \subseteq \mathcal{NC} \cup \mathcal{X}^f$. Thus, π^f cannot be blocked. Therefore, the effect is not identifiable by common adjustment.
- $2 \Rightarrow 1$: Let us assume that condition 2 holds. We will show that $\mathcal{C} = \left(\mathcal{V}^f \setminus \mathcal{NC}\right) \setminus \mathcal{X}^f$ is a valid common adjustment set. Let $\mathcal{G}^f \in \mathcal{C}(\mathcal{G}^s)$ be an FTCG. Let us check that $\mathcal{C}$ is a valid adjustment set in $\mathcal{G}^f$:
  - By definition, $\mathcal{C} \cap \mathcal{CF} = \emptyset$. Thus $\mathcal{C} \cap \text{Forb}(\mathcal{X}^f, Y_t, \mathcal{G}^f) = \emptyset$.
  - Let π^f be a proper non-causal path from $\mathcal{X}^f$ to Y_t in $\mathcal{G}^f$. Let $X^i_{t-\gamma_i}$ be the first vertex of π^f. We distinguish two cases:
    - If π^f leaves $\mathcal{CF} \cup \{X^i_{t-\gamma_i}\}$, then $\mathcal{C}$ blocks π^f.
    - Otherwise, π^f remains within $\mathcal{CF} \cup \{X^i_{t-\gamma_i}\}$. π^f is a proper path, hence π^f remains within $\mathcal{NC} \cup \{X^i_{t-\gamma_i}\}$. By contradiction, we show that π^f contains a collider. Let us assume that π^f is collider-free. We distinguish two cases:
      - If π^f starts by $X^i_{t-\gamma_i} \rightarrow$, then π^f would be a causal path, which contradicts the definition of π^f.
      - If π^f starts by $X^i_{t-\gamma_i} \leftarrow$, then π^f would be a collider-free backdoor path from $X^i_{t-\gamma_i}$ to Y_t that remains in $\mathcal{NC} \cup \{X^i_{t-\gamma_i}\}$, which contradicts condition 2.

      Therefore, π^f contains a collider denoted C_{t_c}. $C_{t_c} \in \mathcal{CF}$, thus every descendant of C_{t_c} belongs to $\mathcal{CF}$. Thus $\mathcal{C} \cap \text{Desc}(C_{t_c}, \mathcal{G}^f) \subseteq \mathcal{C} \cap \mathcal{CF} = \emptyset$. Thus $\mathcal{C}$ blocks π^f.

    In all cases, $\mathcal{C}$ blocks π^f.

  Therefore, $\mathcal{C}$ is a valid adjustment set in $\mathcal{G}^f$. Therefore, $\mathcal{C} = \left(\mathcal{V}^f \setminus \mathcal{NC}\right) \setminus \mathcal{X}^f$ is a valid common adjustment set.

□

D.2 PROOFS OF SECTION 4.2

D.2.1 Proofs of Section 4.2.1

Lemma 2. (Characterization of collider-free backdoor paths without fork) Let $\mathcal{G}^s = (\mathcal{V}^s, \mathcal{E}^s)$ be an SCG and $P(y_t \mid \text{do}(x^1_{t-\gamma_1}), \dots, \text{do}(x^n_{t-\gamma_n}))$ be the considered effect. The following statements are equivalent:

1. There exists an intervention $X^i_{t-\gamma_i}$ and a candidate FTCG $\mathcal{G}^f \in \mathcal{C}(\mathcal{G}^s)$ which contains $X^i_{t-\gamma_i} \leftsquigarrow Y_t$ which remains in $\mathcal{NC} \cup \{X^i_{t-\gamma_i}\}$.
2. There exists an intervention $X^i_{t-\gamma_i}$ such that $\gamma_i = 0$ and $X^i \in \text{Desc}\left(Y, \mathcal{G}^s_{|\mathcal{S}}\right)$, where $\mathcal{S} := \{S \in \mathcal{V}^s \mid t_{\mathcal{NC}}(S) \leq t\} \cup \{X^i \in \mathcal{X}^s \mid \gamma_i = 0\}$.

Proof. **Let us prove the two implications:**

- $1 \Rightarrow 2$: Let us consider $X^i_{t-\gamma_i}$ and an FTCG $\mathcal{G}^f$ belonging to $\mathcal{C}(\mathcal{G}^s)$ such that $\mathcal{G}^f$ contains $\pi^f := X^i_{t-\gamma_i} \leftsquigarrow Y_t$ which remains in $\mathcal{NC} \cup \{X^i_{t-\gamma_i}\}$. By definition, we already know that $t - \gamma_i \leq t$. Causality does not move backwards in time thus $t - \gamma_i \geq t$. Therefore, $t - \gamma_i = t$ and all vertices of π^f are a time t. Therefore the reduction π^s of π^f is still a path in $\mathcal{G}^s$. Moreover, π^f remains in $\mathcal{NC} \cup \{X^i_{t-\gamma_i}\}$ thus $\pi^s \subseteq \mathcal{S}$. Therefore, $X^i_{t-\gamma_i} \in \text{Desc}(Y, \mathcal{G}^s_{|\mathcal{S}})$ and the intervention is at time t.

- $2 \Rightarrow 1$: Let us consider $\pi^s = X^i \leftarrow V^2 \leftarrow \cdots \leftarrow V^{n-1} \leftarrow Y$, the smallest path from Y to an interventional time series at time t in $\mathcal{G}^s_{|\mathcal{S}}$. For all $i \in \{2, \cdots, n-1\}$, V^i_t is not an intervention because otherwise we could find a smaller path that π^f. We can construct an FTCG $\mathcal{G}^f$ which contains the path $\pi^f = X^i_t \leftarrow V^2_t \leftarrow \cdots \leftarrow V^{n-1}_t \leftarrow Y_t$. π^f remains in $\mathcal{NC} \cup \{X^i_{t-\gamma_i}\}$ because for all i, $t_{\mathcal{NC}}(V^i) \leq 0$. Therefore, there exist an intervention $X^i_{t-\gamma_i}$ and an FTCG $\mathcal{G}^f$ belonging to $C(\mathcal{G}^s)$ which contains $X^i_{t-\gamma_i} \leftsquigarrow Y_t$ which remains in $\mathcal{NC} \cup \{X^i_{t-\gamma_i}\}$.

$\square$

D.2.2 Proofs of Section 4.2.1

Lemma 3. (Characterization of collider-free backdoor paths with fork) Let $\mathcal{G}^s = (\mathcal{V}^s, \mathcal{E}^s)$ be an SCG and $P(y_t \mid \mathrm{do}(x^1_{t-\gamma_1}), \ldots, \mathrm{do}(x^n_{t-\gamma_n}))$ be the considered effect such that for all $\mathcal{G}^f$ belonging to $C(\mathcal{G}^s)$, $\mathcal{G}^f$ does not contain a directed path from Y_t to an intervention $X^i_{t-\gamma_i} \leftsquigarrow Y_t$ which remains in $\mathcal{NC} \cup \{X^i_{t-\gamma_i}\}$. The following statements are equivalent:

1. There exists an intervention $X^i_{t-\gamma_i}$, $F_{t_f} \in \mathcal{V}^f$ and a candidate FTCG $\mathcal{G}^f \in C(\mathcal{G}^s)$ which contains the path $X^i_{t-\gamma_i} \leftsquigarrow F_{t_f} \rightsquigarrow Y_t$ which remains in $\mathcal{NC} \cup \{X^i_{t-\gamma_i}\}$.
2. There exists an intervention $X^i_{t-\gamma_i}$ and $F_{t_f} \in \mathcal{V}^f$ such that F_{t_f} is $X^i_{t-\gamma_i}$-$\mathcal{NC}$-accessible and Y_t-$\mathcal{NC}$-accessible.

Proof. **Let us prove the two implications of the theorem:**

- $1 \Rightarrow 2$: Let $X^i_{t-\gamma_i}$ be the intervention and $\mathcal{G}^f$ the FTCG belonging to $C(\mathcal{G}^s)$ which contains the path $X^i_{t-\gamma_i} \leftsquigarrow F_{t_f} \rightsquigarrow Y_t$ which remains in $\mathcal{NC} \cup \{X^i_{t-\gamma_i}\}$. $\mathcal{G}^f$ proves that F_{t_f} is $X^i_{t-\gamma_i}$-$\mathcal{NC}$-accessible and Y_t-$\mathcal{NC}$-accessible.
- $2 \Rightarrow 1$: Let $X^i_{t-\gamma_i}$ and F_{t_f} be such that F_{t_f} is $X^i_{t-\gamma_i}$-$\mathcal{NC}$-accessible and Y_t-$\mathcal{NC}$-accessible. Therefore there exists $\mathcal{G}^f_1$ in which there is $\pi^f_1 := X^i_{t-\gamma_i} \leftsquigarrow F_{t_f}$ which remains in $\mathcal{NC} \cup \{X^i_{t-\gamma_i}\}$and there exists $\mathcal{G}^f_2$ in which there is $\pi^f_2 := F_{t_f} \rightsquigarrow Y_t$ which remains in $\mathcal{NC}$. We distinguish two cases:
  - If $\pi^f_1 \cap \pi^f_2 = \{F_{t_f}\}$, then we can build an FTCG $\mathcal{G}^f_3 \in C(\mathcal{G}^s)$ which contains $\pi^f_3 := X^i_{t-\gamma_i} \leftsquigarrow F_{t_f} \rightsquigarrow Y_t$ the concatenation of π^f_1 and π^f_2. π^f_3 is a path without cycle because $\pi^f_1 \cap \pi^f_2 = \{F_{t_f}\}$. It remains in $\mathcal{NC} \cup \{X^i_{t-\gamma_i}\}$ because its vertices come from π^f_1 and π^f_2 which remain in $\mathcal{NC} \cup \{X^i_{t-\gamma_i}\}$.
  - Otherwise, $\pi^f_1 \cap \pi^f_2$ contains at least two element. We know that $X^i_{t-\gamma_i} \notin \pi^f_1 \cap \pi^f_2$ because π^f_2 remains in $\mathcal{NC}$. Similarly, we know that $Y_t \notin \pi^f_1 \cap \pi^f_2$ because there would be a directed path from Y_t to $X^i_{t-\gamma_i}$. Let us consider V_{t_v} the latest element of π^f_1 in $\pi^f_1 \cap \pi^f_2$. In $\mathcal{G}^f_1$ there is $\pi^{f\prime}_1 := X^i_{t-\gamma_i} \leftsquigarrow V_{t_v}$, in $\mathcal{G}^f_2$ there is $\pi^{f\prime}_2 := V_{t_v} \rightsquigarrow Y_t$ and $\pi^{f\prime}_1 \cap \pi^{f\prime}_2 = \{V_{t_v}\}$.Therefore, with the same reasoning as in the first case, we can construct $\mathcal{G}^f_3$ which contains the path $X^i_{t-\gamma_i} \leftsquigarrow V_{t_v} \rightsquigarrow Y_t$ which remains in $\mathcal{NC} \cup \{X^i_{t-\gamma_i}\}$.

  In all cases, there exist an intervention $X^i_{t-\gamma_i}$ and an FTCG $\mathcal{G}^f$ belonging to $C(\mathcal{G}^s)$ which contains the path $X^i_{t-\gamma_i} \leftsquigarrow F_{t_f} \rightsquigarrow Y_t$ which remains in $\mathcal{NC} \cup \{X^i_{t-\gamma_i}\}$.

$\square$

As shown by Lemma 13, for any time series $F \in \mathcal{V}^s$, knowledge of $t^{\mathcal{NC}}_{V_{t_v}}(F)$ and $t_{\mathcal{NC}}(F)$ is sufficient to characterize efficiently the set $\{t_1 \mid F_{t_1} \text{is } V_{t_v}\text{-}\mathcal{NC}\text{-accessible}\}$.

Lemma 13. Let $\mathcal{G}^s = (\mathcal{V}^s, \mathcal{E}^s)$ be an SCG, $P(y_t \mid \mathrm{do}(x^1_{t-\gamma_1}), \ldots, \mathrm{do}(x^n_{t-\gamma_n}))$ be the considered effect, $V_{t_v} \in \mathcal{V}^f$ and $F \in \mathcal{V}^s$. Then the following statements are equivalent:

1. F_{t_f} is V_{t_v}-$\mathcal{NC}$-accessible.
2. $t_{\mathcal{NC}}(F) \leq t_f \leq t^{\mathcal{NC}}_{V_{t_v}}(F)$ and $F_{t_f} \notin \mathcal{X}^f$.

Proof. **Let us prove the two implications:**

- $1 \Rightarrow 2$: Let F_{t_f} and V_{t_v} be such that F_{t_f} is V_{t_v}-$\mathcal{NC}$-accessible. Thus $F_{t_f} \in \mathcal{NC}$. Hence, $F_{t_f} \notin \mathcal{X}^f$ and $t_{\mathcal{NC}}(F) \leq t_f$ by definition of $t_{\mathcal{NC}}(F)$. By definition of $t^{\mathcal{NC}}_{V_{t_v}}(F)$, $t_f \leq t^{\mathcal{NC}}_{V_{t_v}}(F)$. Therefore $t_{\mathcal{NC}}(F) \leq t_f \leq t^{\mathcal{NC}}_{V_{t_v}}(F)$ and $F_{t_f} \notin \mathcal{X}^f$.

- $1 \Rightarrow 2$: Let F_{t_f} and V_{t_v} be such that $t_{\mathcal{NC}}(F) \leq t_f \leq t^{\mathcal{NC}}_{V_{t_v}}(F)$ and $F_{t_f} \notin \{X^i_{t-\gamma_i}\}_i$. Since $t_{\mathcal{NC}}(F) \leq t_f$, it follows that $t_{\mathcal{NC}}(F) \neq +\infty$. By Lemma 1, we have $\mathcal{NC} = \bigcup_{Z \in \mathcal{V}^s} \{Z_{t_1}\}_{t_1 \geq t_{\mathcal{NC}}(Z)} \setminus \{X^i_{t-\gamma_i}\}_i$, and given that $F_{t_f} \notin \{X^i_{t-\gamma_i}\}_i$, it follows that $F_{t_f} \in \mathcal{NC}$. Since $t_f \leq t^{\mathcal{NC}}_{V_{t_v}}(F)$, it follows that $t^{\mathcal{NC}}_{V_{t_v}}(F) \neq -\infty$. Thus, $F_{t^{\mathcal{NC}}_{V_{t_v}}(F)}$ is V_{t_v}-$\mathcal{NC}$-accessible. Hence, there exists an FTCG in which there is a path $\pi^f : V_{t_v} \leftsquigarrow F_{t^{\mathcal{NC}}_{V_{t_v}}(F)}$ that remains in $\mathcal{NC}$. By changing the last arrow of this path, we can construct an FTCG which contains the path $V_{t_v} \leftsquigarrow F_{t_f}$. This path remains in $\mathcal{NC}$ because $F_{t_f} \in \mathcal{NC}$ and all other vertices of the paths are in π^f which remains in $\mathcal{NC}$. Therefore F_{t_f} is V_{t_v}-$\mathcal{NC}$-accessible.

□

By setting $t_f = t_{\mathcal{NC}}(F)$ in Lemma 13, we observe that it suffices to test only the single time point $F_{t_{\mathcal{NC}}(F)}$ in Condition 2 of Lemma 3.

Corollary 1. Let $\mathcal{G}^s = (\mathcal{V}^s, \mathcal{E}^s)$ be an SCG, $P(y_t \mid \text{do}(x^1_{t-\gamma_1}), \ldots, \text{do}(x^n_{t-\gamma_n}))$ be the considered effect, $V_{t_v} \in \mathcal{V}^f$ and $F \in \mathcal{V}^s$. Let $X^i_{t-\gamma_i}$ be a fixed intervention. The following statements are equivalent:

1. There exists t_f such that F_{t_f} is $X^i_{t-\gamma_i}$-$\mathcal{NC}$-accessible and Y_t-$\mathcal{NC}$-accessible.
2. $t_{\mathcal{NC}}(F) \leq t^{\mathcal{NC}}_{X^i_{t-\gamma_i}}(F)$ and $t_{\mathcal{NC}}(F) \leq t^{\mathcal{NC}}_{Y_t}(F)$.

Proof. Let us show the two implications of the corollary:

- $1 \Rightarrow 2$: If there exists t_f such that F_{t_f} is $X^i_{t-\gamma_i}$-$\mathcal{NC}$-accessible and Y_t-$\mathcal{NC}$-accessible then $F_{t_{\mathcal{NC}}(F)}$ is $X^i_{t-\gamma_i}$-$\mathcal{NC}$-accessible and Y_t-$\mathcal{NC}$-accessible. Thus, by Lemma 13, $t_{\mathcal{NC}}(F) \leq t_{\mathcal{NC}}(F) \leq t^{\mathcal{NC}}_{X^i_{t-\gamma_i}}(F)$ and $F_{t_{\mathcal{NC}}(F)} \notin \mathcal{X}^f$ and $t_{\mathcal{NC}}(F) \leq t_{\mathcal{NC}}(F) \leq t^{\mathcal{NC}}_{Y_t}(F)$ and $F_{t_{\mathcal{NC}}(F)} \notin \mathcal{X}^f$. Hence $t_{\mathcal{NC}}(F) \leq t^{\mathcal{NC}}_{X^i_{t-\gamma_i}}(F)$ and $t_{\mathcal{NC}}(F) \leq t^{\mathcal{NC}}_{Y_t}(F)$.
- $2 \Rightarrow 1$: If $t_{\mathcal{NC}}(F) \leq t^{\mathcal{NC}}_{X^i_{t-\gamma_i}}(F)$ and $t_{\mathcal{NC}}(F) \leq t^{\mathcal{NC}}_{Y_t}(F)$, then $t_{\mathcal{NC}}(F) \neq +\infty$ and $t^{\mathcal{NC}}_{X^i_{t-\gamma_i}}(F) \neq -\infty$. Therefore, $F_{t_{\mathcal{NC}}(F)} \notin \mathcal{X}^f$ and $t_{\mathcal{NC}}(F) \leq t_{\mathcal{NC}}(F) \leq t^{\mathcal{NC}}_{X^i_{t-\gamma_i}}(F)$ and $F_{t_{\mathcal{NC}}(F)} \notin \mathcal{X}^f$ and $t_{\mathcal{NC}}(F) \leq t_{\mathcal{NC}}(F) \leq t^{\mathcal{NC}}_{Y_t}(F)$. Therefore, by Lemma 13, $F_{t_{\mathcal{NC}}(F)}$ is $X^i_{t-\gamma_i}$-$\mathcal{NC}$-accessible and Y_t-$\mathcal{NC}$-accessible. Therefore, there exists t_f such that F_{t_f} is $X^i_{t-\gamma_i}$-$\mathcal{NC}$-accessible and Y_t-$\mathcal{NC}$-accessible.

□

D.2.3 Proofs of Section 4.2.2

Lemma 14. Let $\mathcal{G}^s = (\mathcal{V}^s, \mathcal{E}^s)$ be an SCG, $P(y_t \mid \text{do}(x^1_{t-\gamma_1}), \ldots, \text{do}(x^n_{t-\gamma_n}))$ be the considered effect and $V_{t_v} \in \mathcal{V}^f$. The set $\{t^{\mathcal{NC}}_{V_{t_v}}(F) \mid F \in \mathcal{V}^S\}$ can be computed by Algorithm 2 which complexity is $\mathcal{O}(|\mathcal{E}^s| + |\mathcal{V}^s| \log |\mathcal{V}^s|)$[3].

Proof. Let V_{t_v} be a temporal variable.

Let us prove that Algorithm 2 terminates:

Algorithm 2 **terminates** because at each step of the **while** loop, (number of unseen times series, length of Q) is strictly decreasing with respect to the lexicographic ordering.

Let us prove that Algorithm 2 is correct:

Firstly, by induction, **we show that the algorithm computes the correct value for each seen time series:**

- At the first step of the while loop, for each parent P of V in $\mathcal{G}^s$, the algorithm will compute $t^{\mathcal{NC}}_{V_{t_v}}(P) \leftarrow \max\{t_1 \mid t_1 \leq t_v \text{ and } P_{t_1} \in \mathcal{NC} \setminus \{V_{t_v}\}\}$.

  Let P be a parent of V in $\mathcal{G}^s$. We will show that $\{t_1 \mid P_{t_1} \text{is } V_{t_v}\text{-}\mathcal{NC}\text{-accessible}\} = \{t_1 \mid t_1 \leq t_v \text{ and } P_{t_1} \in \mathcal{NC} \setminus \{V_{t_v}\}\}$ by showing the two inclusions:

[3] In amortized time. Using a binary heap, the algorithm runs in $\mathcal{O}((|\mathcal{E}^s| + |\mathcal{V}^s|) \log |\mathcal{V}^s|)$.

- For all $t_1 \in \{t_1 \mid t_1 \leq t_v$ and $P_{t_1} \in \mathcal{NC} \setminus \{V_{t_v}\}\}$, we can construct an FTCG $\mathcal{G}^f_{t_1} \in C(\mathcal{G}^s)$ which contains $P_{t_1} \rightarrow V_{t_v}$. Therefore, all $P_{t_1} \in \{P_{t_1} \mid t_1 \leq t_v$ and $P_{t_1} \in \mathcal{NC} \setminus \{V_{t_v}\}\}$ are V_{t_v}-$\mathcal{NC}$-accessible, *i.e.*, $\{t_1 \mid t_1 \leq t_v$ and $P_{t_1} \in \mathcal{NC} \setminus \{V_{t_v}\}\} \subseteq \{t_1 \mid P_{t_1}$ is V_{t_v}-$\mathcal{NC}$-accessible$\}$
- Let t_1 be such that P_{t_1} is V_{t_v}-$\mathcal{NC}$-accessible. Causality does not move backwards in time thus $t_1 \leq t_v$. Moreover, $P_{t_1} \in \mathcal{NC} \setminus \{V_{t_v}\}$. Indeed, P_{t_1} is V_{t_v}-$\mathcal{NC}$-accessible thus $P_{t_1} \in \mathcal{NC}$ and $P_{t_1} \neq V_{t_v}$ because no FTCG can contain a self loop. Therefore $\{t_1 \mid P_{t_1}$ is V_{t_v}-$\mathcal{NC}$-accessible$\} \subseteq \{t_1 \mid t_1 \leq t_v$ and $P_{t_1} \in \mathcal{NC} \setminus \{V_{t_v}\}\}$

Therefore $\{t_1 \mid F_{t_1}$ is V_{t_v}-$\mathcal{NC}$-accessible$\} = \{t_1 \mid t_1 \leq t_v$ and $P_{t_1} \in \mathcal{NC} \setminus \{V_{t_v}\}\}$, thus $t^{\mathcal{NC}}_{V_{t_v}}(P) = \max\{t_1 \mid t_1 \leq t_v$ and $P_{t_1} \in \mathcal{NC} \setminus \{V_{t_v}\}\}$.

Therefore the algorithm computes correct values at the first step of the loop.

- Let us suppose that the algorithm is correct until the $(n-1)$-th loop step. It pops $S_{t_s} = S_{t^{\mathcal{NC}}_{V_{t_v}}(S)}$ with $t^{\mathcal{NC}}_{V_{t_v}}(S)$ being the element of Q with maximum time index. Let P be an unseen parent of S in $\mathcal{G}^s$, the algorithm will compute $t^{\mathcal{NC}}_{V_{t_v}}(P) \leftarrow \max\{t_1 \mid t_1 \leq t_s$ and $P_{t_1} \in \mathcal{NC} \setminus \{V_{t_v}\}\}$. We will show that $\{t_1 \mid P_{t_1}$ is V_{t_v}-$\mathcal{NC}$-accessible$\} = \{t_1 \mid t_1 \leq t_v$ and $P_{t_1} \in \mathcal{NC} \setminus \{V_{t_v}\}\}$ by showing the two inclusions:
  - $S_{t^{\mathcal{NC}}_{V_{t_v}}(S)}$ is $\mathcal{NC}$-accessible thus there exists an FTCG $\mathcal{G}^f \in C(\mathcal{G}^s)$ which contains $S_{t^{\mathcal{NC}}_{V_{t_v}}(S)} \rightsquigarrow V_{t_v}$. For all $t_1 \in \{t_1 \mid t_1 \leq t_s$ and $P_{t_1} \in \mathcal{NC} \setminus \{V_{t_v}\}\}$, we can construct an FTCG $\mathcal{G}^f_{t_1} \in C(\mathcal{G}^s)$ which contains $P_{t_1} \rightarrow S_{t^{\mathcal{NC}}_{V_{t_v}}(S)} \rightsquigarrow V_{t_v}$. Therefore, all $P_{t_1} \in \{P_{t_1} \mid t_1 \leq t_s$ and $P_{t_1} \in \mathcal{NC} \setminus \{V_{t_v}\}\}$ are V_{t_v}-$\mathcal{NC}$-accessible, *i.e.*, $\{t_1 \mid t_1 \leq t_v$ and $P_{t_1} \in \mathcal{NC} \setminus \{V_{t_v}\}\} \subseteq \{t_1 \mid P_{t_1}$ is V_{t_v}-$\mathcal{NC}$-accessible$\}$
  - Let t_1 be such that P_{t_1} is V_{t_v}-$\mathcal{NC}$-accessible. Thus, $P_{t_1} \in \mathcal{NC} \setminus \{V_{t_v}\}$. Let us show that $t_1 \leq t_s$: there exists an FTCG $\mathcal{G}^f$ which contains $\pi^f := P_{t_1} \rightsquigarrow V_{t_v}$ which remains in $\mathcal{NC}$. Let U_{t_u} be the successor of P_{t_1} in π^f. Causality does not move backwards in time thus $t_1 \leq t_u$. We distinguish two cases:
    - ∗ If $U = S$, then we already have $t_1 \leq t_s$.
    - ∗ Otherwise, $t_u \leq t_s$. Indeed, otherwise, thanks to the priority queue, U would have been seen before S and P would not be an unseen vertex. Therefore $t_1 \leq t_u \leq t_s$.

    In all cases $t_1 \leq t_s$. Therefore, $\{t_1 \mid P_{t_1}$ is V_{t_v}-$\mathcal{NC}$-accessible$\} \subseteq \{t_1 \mid t_1 \leq t_s$ and $P_{t_1} \in \mathcal{NC} \setminus \{V_{t_v}\}\}$.

Therefore $\{t_1 \mid P_{t_1}$ is V_{t_v}-$\mathcal{NC}$-accessible$\} = \{t_1 \mid t_1 \leq t_s$ and $P_{t_1} \in \mathcal{NC} \setminus \{V_{t_v}\}\}$, thus $t^{\mathcal{NC}}_{V_{t_v}}(P) = \max\{t_1 \mid t_1 \leq t_v$ and $P_{t_1} \in \mathcal{NC} \setminus \{V_{t_v}\}\}$.

Therefore the algorithm computes correct values at the n-th step of the loop.

Therefore, by induction principle, the algorithm computes the correct value for each seen time series.

Secondly, **we show that the algorithm sees all time series S such that $t^{\mathcal{NC}}_{V_{t_v}}(S) \neq -\infty$:**

Let S be a time series such that $t^{\mathcal{NC}}_{V_{t_v}}(S) \neq -\infty$. By definition, $S_{t^{\mathcal{NC}}_{V_{t_v}}(S)}$ is V_{t_v}-$\mathcal{NC}$-accessible. Therefore there exists an FTCG in which there exists a path $\pi^f := S_{t^{\mathcal{NC}}_{V_{t_v}}(S)} \rightsquigarrow V_{t_v}$ which remains in $\mathcal{NC}$. By cutting unnecessary parts and using Lemma 5 or 6, we can assume that π^f does not pass twice by the same time series except perhaps for V if $S = V$. Thus there is a directed path π^S from S to V in $\mathcal{G}^s$ in which all vertices U have a finite $t^{\mathcal{NC}}_{V_{t_v}}(U)$. Since, if the algorithm sees U and $t^{\mathcal{NC}}_{V_{t_v}}(U) \neq -\infty$ then all parents of U are seen by the algorithm and since all the parents of V are seen by the algorithm, by induction on π^s, we can conclude that S is seen.

Therefore, **the algorithm sees all time series S such that $t^{\mathcal{NC}}_{V_{t_v}}(S) \neq -\infty$.**

Finally, **we show that Algorithm 2 is correct:** Let S be a time series. We distinguish two cases:

- If S is seen by the algorithm, then the correct value is computed by the algorithm.
- Otherwise, the algorithm computes $t^{\mathcal{NC}}_{V_{t_v}}(S) \leftarrow -\infty$ thanks to the initialisation step, which is the correct value.

Therefore, Algorithm 2 is correct.

Let us prove that Algorithm 2 has a complexity of $O(|\mathcal{E}^s| + |\mathcal{V}^s| \cdot \log|\mathcal{V}^s|)$:

$t^{\mathcal{NC}}_{V_{t_v}}(P) \leftarrow \max\{t_1 \mid t_1 \leq t_s$ and $P_{t_1} \in \mathcal{NC} \setminus \{V_{t_v}\}\}$ can be computed in $O(1)$ during the algorithm. Indeed, to do so we need to run first an algorithm similar to Algorithm 4.

Let us assume that the priority queue is implemented using a Fibonacci heap. In this case, inserting an element takes $O(1)$ amortized time, and extracting the max element takes $O(\log \mid Q \mid)$ amortized time. During the execution of the algorithm, the priority queue Q contains at most $|\mathcal{V}^s|$ elements, where $|\mathcal{V}^s|$ is the number of time series (or nodes) in $\mathcal{G}^s$. Therefore, each extraction costs $O(\log |\mathcal{V}^s|)$, and since there are at most $|\mathcal{V}^s|$ extractions, the total cost of all extractions is $O(|\mathcal{V}^s| \log |\mathcal{V}^s|)$. Additionally, each edge in $\mathcal{G}^s$ is processed at most once during the computation of the unseen parents of S in $\mathcal{G}^s$ and the update of $t^{NC}Vt_v(P)$. These computations are performed in $O(1)$ time. Thus, the total cost of processing all edges is $O(|\mathcal{E}^s|)$.

Therefore, the overall complexity of the algorithm is $O(|\mathcal{E}^s| + |\mathcal{V}^s| \log |\mathcal{V}^s|)$.

□

Theorem 2. Let $\mathcal{G}^s = (\mathcal{V}^s, \mathcal{E}^s)$ be an SCG and $P(y_t \mid \mathrm{do}(x^1_{t-\gamma_1}), \ldots, \mathrm{do}(x^n_{t-\gamma_n}))$ be the considered effect. Then the two statements are equivalent:

- The effect is identifiable by common adjustment in $\mathcal{G}^s$.
- Algorithm 3 outputs True.

Moreover Algorithm 3 has a polynomial complexity of $O\left(\left|\mathcal{X}^f\right|\left(\log\left|\mathcal{X}^f\right| + (|\mathcal{E}^s| + |\mathcal{V}^s|) \log |\mathcal{V}^s|\right)\right)$.

Proof. Algorithm 3 uses directly the characterizations of Lemma 2 and Lemma 3 via Corollary 1. It outputs $False$ if and only if there exists an FTCG $\mathcal{G}^f$ in which there is a backdoor path from $X^i_{t-\gamma_i}$ to Y_t, otherwise it outputs $True$. Therefore, by Theorem 1, Algorithm 3 is correct.

Algorithm 3 runs Algorithm 1 in $O\left(\left|\mathcal{X}^f\right| \log\left|\mathcal{X}^f\right| + (|\mathcal{E}^s| + |\mathcal{V}^s|) \log |\mathcal{V}^s|\right)$, tests the existence of directed paths in $O\left(|\mathcal{V}^s| + |\mathcal{E}^s| + |\mathcal{X}^f|\right)$, calls Algorithm 2 $\left(\left|\mathcal{X}^f\right| + 1\right)$ times in $O\left(\left|\mathcal{X}^f\right| (|\mathcal{E}^s| + |\mathcal{V}^s| \log |\mathcal{V}^s|)\right)$ and does $O\left(|\mathcal{X}^f| \cdot |\mathcal{V}^s|\right)$ comparisons. Therefore, its overall complexity is $O\left(\left|\mathcal{X}^f\right|\left(\log\left|\mathcal{X}^f\right| + (|\mathcal{E}^s| + |\mathcal{V}^s|) \log |\mathcal{V}^s|\right)\right)$. □

E PROOFS OF SECTION 5

Lemma 4. Let $\mathcal{G}^s = (\mathcal{V}^s, \mathcal{E}^s)$ be an SCG and $P(y_t \mid \mathrm{do}(x^1_{t-\gamma_1}), \ldots, \mathrm{do}(x^n_{t-\gamma_n}))$ be the considered effect such that for all $\mathcal{G}^f$ belonging to $C(\mathcal{G}^s)$, $\mathcal{G}^f$ does not contain a directed path from Y_t to an intervention $X^i_{t-\gamma_i} \leftsquigarrow Y_t$ which remains in $\mathcal{NC} \cup \{X^i_{t-\gamma_i}\}$. The following statements are equivalent:

1. There exist an intervention $X^i_{t-\gamma_i}$, $F_{t'} \in \mathcal{V}^f$ and an FTCG $\mathcal{G}^f \in C(\mathcal{G}^s)$ containing the path $X^i_{t-\gamma_i} \leftsquigarrow F_{t'} \rightsquigarrow Y_t$ which remains in $\mathcal{NC} \cup \{X^i_{t-\gamma_i}\}$.
2. At least one of the following conditions is satisfied:
   (a) There exist an intervention $X^i_{t-\gamma_i}$ and $F \in \mathcal{V}^s$ such that $F_{t_{\mathcal{NC}}(F)}$ is well defined, $X^i_{t-\gamma_i}$-$\mathcal{NC}$-accessible and Y_t-$\mathcal{NC}$-accessible, and $\begin{cases} F \neq Y, \text{ or} \\ t - \gamma_i \neq t_{\mathcal{NC}(F)}. \end{cases}$
   (b) There exists an intervention $X^i_{t-\gamma_i}$ such that $t - \gamma_i = t_{\mathcal{NC}(Y)}$ and at least one of the following properties is satisfied:
      i. $Y_{t_{\mathcal{NC}}(Y)}$ is $X^i_{t-\gamma_i}$ - $\mathcal{NC}$-accessible without using $X^i_{t-\gamma_i} \leftarrow Y_{t-\gamma_i}$ and Y_t - $\mathcal{NC}$-accessible.
      ii. $Y_{t_{\mathcal{NC}}(Y)}$ is $X^i_{t-\gamma_i}$ - $\mathcal{NC}$-accessible and Y_t - $\mathcal{NC}$-accessible without using $X^i_t \rightarrow Y_t$.

Proof. Let us prove the direct implication ($1 \Rightarrow 2$). Let F and $X^i_{t-\gamma_i}$ be such that there exists an FTCG $\mathcal{G}^f \in C(\mathcal{G}^s)$ that contains the path $X^i_{t-\gamma_i} \leftsquigarrow F_{t'} \rightsquigarrow Y_t$ which remains in $\mathcal{NC} \cup \{X^i_{t-\gamma_i}\}$. Two cases arise:

- If $F \neq Y$ or $t - \gamma_i \neq t_{\mathcal{NC}}(F)$, the accessibility in $\mathcal{G}^f$ ensures that $F_{t_{\mathcal{NC}}(F)}$ is both $X^i_{t-\gamma_i}$-$\mathcal{NC}$-accessible and Y_t-$\mathcal{NC}$-accessible. Thus, we have proven Proposition 2a in this case.
- Otherwise, $Y = F$ and $t - \gamma_i = t_{\mathcal{NC}}(F) = t_{\mathcal{NC}}(Y)$. Let us prove by contradiction that Lemma 4 2b holds. If it does not hold, then $\mathcal{G}^f$ must necessarily contain the edges $X^i_{t-\gamma_i} \leftarrow Y_{t_{\mathcal{NC}}(Y)}$ and $X^i_t \rightarrow Y_t$, which contradicts Assumption 1.

Conversely, let us prove the indirect direction ($2 \Rightarrow 1$). To prove this implication, we must show that Lemma 4 2a implies Lemma 4 1, and that Lemma 4 2b implies Lemma 4 1. For each of these proofs, we are given $\mathcal{G}^f_1$, which contains $\pi^f_1 := X^i_{t-\gamma_i} \leftsquigarrow F_{t_{\mathcal{NC}}(F)}$, which remains in $\mathcal{NC} \cup \{X^i_{t-\gamma_i}\}$, and $\mathcal{G}^f_2$, which contains $\pi^f_2 := F_{t_{\mathcal{NC}}(F)} \rightsquigarrow Y_t$, which remains in $\mathcal{NC}$. To prove Proposition 1, it suffices to construct $\mathcal{G}^f_3$ that contains $\pi^f_3 := X^i_{t-\gamma_i} \leftsquigarrow F_{t'} \rightsquigarrow Y_t$, which remains in $\mathcal{NC} \cup \{X^i_{t-\gamma_i}\}$.

Furthermore, without loss of generality, we can assume that the only intersection between π^f_1 and π^f_2 is $F_{t_{\mathcal{NC}}(F)}$. Indeed, consider V_{t_v}, the last element of π^f_2 in $\pi^f_1 \cap \pi^f_2$. By contradiction, we show that $V_{t_v} \neq Y_t$. If $V_{t_v} = Y_t$, then $\mathcal{G}^f_1$ contains $X^i_{t-\gamma_i} \leftsquigarrow Y_t$. However, by assumption, no FTCG contains a backdoor path without a fork from an intervention to Y_t. Thus, $V_{t_v} \neq Y_t$, and we can therefore work on $\pi^{f\prime}_1 := X^i_{t-\gamma_i} \leftsquigarrow V_{t_{\mathcal{NC}}(V)}$ and $\pi^{f\prime}_2 := V_{t_{\mathcal{NC}}(V)} \rightsquigarrow Y_t$. Therefore, concatenating π^f_1 and π^f_2 to create an FTCG does not create a cycle. Only Assumption 1 can be violated.

If it is possible to concatenate π^f_1 and π^f_2 while maintaining Assumption 1, then we prove Proposition 1. If this is not the case, we show that, thanks to the assumptions of 2a or 2b, we can construct a suitable π^f_3. We thus merge the proofs of (2a $\Rightarrow$ 1) and (2b $\Rightarrow$ 1) as the reasoning is identical.

Therefore, we assume that it is not possible to concatenate π^f_1 and π^f_2 because Assumption 1 would be violated. We denote by $V^1_{t_1} \rightarrow V^2_{t_1}$ the last arrow of π^f_1 that contradicts π^f_2, and $V^1_{t_2} \leftarrow V^2_{t_2}$ the last arrow of π^f_2 that contradicts $V^1_{t_1} \rightarrow V^2_{t_1}$. We proceed by case distinction:

- If $V^1_{t_1} \neq F_{t_{\mathcal{NC}}(F)}$ and $V^2_{t_1} \neq X^i_{t-\gamma_i}$, we further distinguish three cases:
  - If $t_1 < t_2$, then by adding the arrow $V^2_{t_1} \rightarrow V^1_{t_2}$, we can construct $\pi^f_3 = X^i_{t-\gamma_i} \leftsquigarrow V^2_{t_1} \rightarrow V^1_{t_2} \rightsquigarrow Y_t$ (see Figure 6a).
  - If $t_1 > t_2$, then by adding the arrow $V^1_{t_2} \rightarrow V^2_{t_1}$, we can construct $\pi^f_3 = X^i_{t-\gamma_i} \leftsquigarrow V^2_{t_1} \leftarrow V^1_{t_2} \rightsquigarrow Y_t$ (see Figure 6b).
  - The case $t_1 = t_2$ is excluded because the only intersection between π^f_1 and π^f_2 is $F_{t_{\mathcal{NC}}(F)}$.
- If $V^1_{t_1} \neq F_{t_{\mathcal{NC}}(F)}$ and $V^2_{t_1} = X^i_{t-\gamma_i}$. $\mathcal{G}^f_2$ contains $V^1_{t_2} \rightsquigarrow Y_t$, which can be transformed into $V^1_{t-\gamma_i} \rightsquigarrow Y_t$ by changing the first arrow, yielding $\pi^f_3 = X^i_{t-\gamma_i} \leftarrow V^1_{t-\gamma_i} \rightsquigarrow Y_t$ (see Figure 6c).
- If $V^1_{t_1} = F_{t_{\mathcal{NC}}(F)}$ and $V^2_{t_1} \neq X^i_{t-\gamma_i}$. $\mathcal{G}^f_2$ contains $F_{t_2} \rightsquigarrow Y_t$, which can be transformed into $F_{t_{\mathcal{NC}}(F)} \rightsquigarrow Y_t$ by only changing the first arrow, yielding $\pi^f_3 = X^i_{t-\gamma_i} \leftsquigarrow V^2_{t_{\mathcal{NC}}(F)} \leftarrow F_{t_{\mathcal{NC}}(F)} \rightsquigarrow Y_t$ (see Figure 6d).

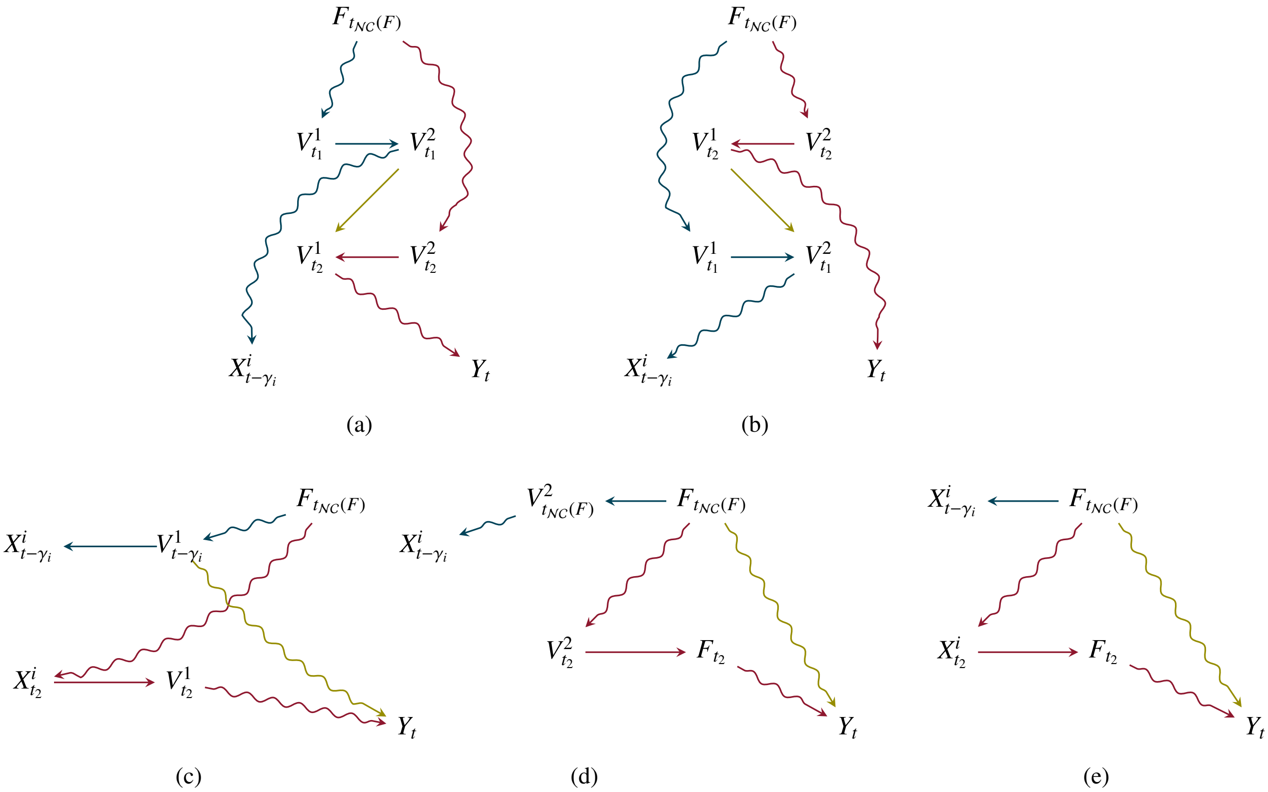

Figure 6: π_1^f is represented in blue, π_2^f is represented in red, and the modification to be made to construct π_3^f in $\mathcal{G}_3^f$ is represented in green.

- If $V_{t_1}^1 = F_{t_{NC(F)}}$ and $V_{t_1}^2 = X_{t-\gamma_i}^i$, we further distinguish two cases:
  - If $F \neq Y$, then $t - \gamma_i = t_{NC(F)}$, because the case $t - \gamma_i \neq t_{NC(F)}$ does not contradict Assumption 1. $\mathcal{G}_2^f$ contains $F_{t_2} \rightsquigarrow Y_t$, which can be transformed into $F_{t_{NC(F)}} \rightsquigarrow Y_t$, yielding $\pi_3^f = X_{t-\gamma_i}^i \leftarrow V_{t_{NC(F)}} \leftarrow F_{t_{NC(F)}} \rightsquigarrow Y_t$ (see Figure 6e).
  - Otherwise, if $F = Y$, the only case that can contradict Assumption 1 is $t - \gamma_i = t_{NC(F)}$, i.e., we are strictly under the assumptions of Proposition 2b. According to Proposition 2b, $Y_{t_{NC(Y)}}$ is Y_t and is $\mathcal{NC}$-accessible without using $X_t^i \to Y_t$. Therefore, we can construct π_3^f without contradicting Assumption 1.

□

F SPEED UP OF ALGORITHM 3

It is possible to compute $\left(\max_{X^i_{t-\gamma_i}}\left\{t^{\mathcal{NC}}_{X^i_{t-\gamma_i}}(S)\right\}\right)_{S\in\mathcal{V}^s}$ in a single traversal of the graph $\mathcal{G}^s$. To achieve this, one only needs to modify the initialization of the priority queue Q in Algorithm 2. In this optimized strategy, Q is initialized with $\left\{X^i_{t-\gamma_i}\right\}_{i\in\{1,\dots,n\}}$. This is equivalent to computing $\left(t^{\mathcal{NC}}_{M_\star}(S)\right)_{S\in\mathcal{V}^s}$, where $M_\star$ is a fictitious vertex corresponding to the merging of all interventions in on variable . The complexity remains in $O\left(|\mathcal{E}^s|+|\mathcal{V}^s|\log|\mathcal{V}^s|\right)$, as the single traversal avoids redundant checks.

During this computation, it is also possible to efficiently test for the existence of collider-free backdoor paths involving a fork that remains within $\mathcal{NC}$, except perhaps for its first vertex. Since $(t^{\mathcal{NC}}_{Y_t}(S))_{S\in\mathcal{V}^s}$ is already computed, the algorithm can, for each parent P encountered, directly verify whether $t_{\mathcal{NC}}(P)\leq t^{\mathcal{NC}}_{M_\star}(P)=\max_{X^i_{t-\gamma_i}}\left\{t^{\mathcal{NC}}_{X^i_{t-\gamma_i}}(P)\right\}$ and whether $t_{\mathcal{NC}}(P)\leq t^{\mathcal{NC}}_{Y_t}(F)$. This characterizes the existence of a collider-free backdoor path with a fork remaining in $\mathcal{CF}$ from any intervention to Y_t. Therefore, it is possible to test for the existence of a backdoor path with a fork in $O\left(|\mathcal{E}^s|+|\mathcal{V}^s|\log|\mathcal{V}^s|\right)$.

Thus, the need for separate loops to test the existence of backdoor paths with forks is eliminated and identifiability by common adjustment can be computed in $O\left(\left|\mathcal{X}^f\right|\log\left|\mathcal{X}^f\right|+|\mathcal{E}^s|+|\mathcal{V}^s|\log|\mathcal{V}^s|\right)$.

G AN EFFICIENT ALGORITHM FOR IDENTIFIABILITY BY COMMON ADJUSTMENT UNDER ASSUMPTION 1

The algorithm presented in Algorithm 6 aims to determine whether the causal effect $P(y_t \mid \mathrm{do}(x^1_{t-\gamma_1}),\dots,\mathrm{do}(x^n_{t-\gamma_n}))$ is identifiable by common adjustment under Assumption 1. Its correctness arises from Theorem 1. Algorithm 6 tests the existence of an FTCG which contains a collider-free backdoor path from an intervention to Y_t which remains in $\mathcal{CF}$. It starts by checking the existence of a directed path and then test the existence of backdoor paths with a fork using the characterisation of Lemma 4.

The algorithm checks the existence of a directed collider-free backdoor path that remains in $\mathcal{CF}$ as in Algorithm 3. The rest of the algorithm then focuses on checking the existence of backdoor path with a fork. To do so, it uses the characterisation of Lemma 4. On line 1, it checks the backdoor path whose fork reduces to $F\neq Y$. Thanks to the strategy presented in Subsection F, this can be done in a single traversing. Then, on line 2, the algorithm checks that last case of case 2a. It works on path whose fork reduces to Y and whose interventions are at time $t-\gamma_i\neq t_{\mathcal{NC}}(Y)$. Thanks to the strategy presented in Subsection F, this can be done in a single traversing. Then, the algorithm checks the cases 2b. It starts by the case 2(b)i on line 3. Since $\forall X^i_{t-\gamma_i}\in\mathcal{X}, t-\gamma_i=t_{\mathcal{NC}}(Y)$, finding a directed path from $Y_{t_{\mathcal{NC}}(Y)}$ to $X^i_{t_{\mathcal{NC}}(Y)}$ without using $X^i_{t_{\mathcal{NC}}(Y)}\leftarrow Y_{t_{\mathcal{NC}}(Y)}$ is equivalent of finding a directed path from Y to X^i in $\mathcal{G}^s$ without using $Y\rightarrow X^i$. This can be done by a BFS algorithm in $\mathcal{G}^s$. The algorithm finishes by checking the case 2(b)ii. $\mathcal{X}'$ represents the set of interventions at time $t_{\mathcal{NC}}(Y)$ for which $Y_{t_{\mathcal{NC}}(Y)}$ is $X^i_{t-\gamma_i}$-$\mathcal{NC}$-accessible. If $|\mathcal{X}'|\geq 2$, since $Y_{t_{\mathcal{NC}}(Y)}$ is Y_t-$\mathcal{NC}$-accessible, there is at least one $X^i_{t-\gamma_i}\in\mathcal{X}'$ that $Y_{t_{\mathcal{NC}}(Y)}$ is Y_t -$\mathcal{NC}$-accessible without using $X^i_t\rightarrow Y_t$. Indeed, $Y_{t_{\mathcal{NC}}(Y)}$ is Y_t-$\mathcal{NC}$-accessible, therefore, there exists an FTCG which contains a directed path from $Y_{t_{\mathcal{NC}}(Y)}$ to Y_t. This path cannot use simultaneously an arrow $X^i_t\rightarrow Y_t$ and an arrow $X^j_t\rightarrow Y_t$. This reasoning explains the fourth line of the algorithm. The line 5 is reached when $|\mathcal{X}'|=1$. In this case, a traversing algorithm starting from $Y_{t_{\mathcal{NC}}(Y)}$ can test if $Y_{t_{\mathcal{NC}}(Y)}$ is Y_t - $\mathcal{NC}$- accessible without using $X^i_t\rightarrow Y_t$. All the subcases of case 2(b)ii have been checked by the algorithm

Algorithm 6 calls 5 traversing of $\mathcal{G}^s$. Therefore, its complexity is indeed pseudo-linear.

Algorithm 6: Computation of identifiability by common adjustment under Assumption 1.

Input: $\mathcal{G}^s$ an SCG and $P(y_t \mid \mathrm{do}(x^1_{t-\gamma_1}), \dots, \mathrm{do}(x^n_{t-\gamma_n}))$ the considered effect.
Output: A boolean indicating whether the effect is identifiable by common adjustment or not.

$(t_{\mathcal{NC}}(S))_{S \in \mathcal{V}^s} \leftarrow$ Algorithm 1 ;
`// Enumeration of directed paths.`
$\mathcal{S} \leftarrow \{S \in \mathcal{V}^s \mid t_{\mathcal{NC}}(S) \leq 0\} \cup \{X^i \mid t - \gamma_i = 0\}$;
if $\exists i \in \{1, \dots, n\}$ *s.t.* $X^i \in Desc\left(Y, \mathcal{G}^s_{|\mathcal{S}}\right)$ *and* $\gamma_i = 0$ **then**
    return *False*
`// Enumeration of fork paths.`
$(t^{\mathcal{NC}}_{Y_t}(S))_{S \in \mathcal{V}^s} \leftarrow$ Algorithm 2 ;
`// Test all all forks` $F \neq Y$:
1. **if** *there exist* $X^i_{t-\gamma_i}$ *and* F *such that* $F_{t_{\mathcal{NC}}(F)}$ *is well defined,* $X^i_{t-\gamma_i}$*-*$\mathcal{NC}$*-accessible and* Y_t*-*$\mathcal{NC}$*-accessible* **then** `/* F */`
    return *False*
`// Test for` $F = Y$:
if $Y_{t_{\mathcal{NC}}(Y)}$ *is* Y_t*-*$\mathcal{NC}$*-accessible* **then**
    `// Last case of 2a`
2.   **if** *there exist* $X^i_{t-\gamma_i}$ *such that* $t - \gamma_i \neq t_{\mathcal{NC}}(Y)$ *and* $Y_{t_{\mathcal{NC}}(Y)}$ *is* $X^i_{t-\gamma_i}$*-*$\mathcal{NC}$*-accessible* **then** `/* F */`
        return *False*
    `// Test for 2b:`
    $\mathcal{X} \leftarrow \{X^i_{t-\gamma_i} \mid t - \gamma_i = t_{\mathcal{NC}}(Y)\}$;
    `// Case 2(b)i:`
3.   **if** $\exists X^i_{t-\gamma_i} \in \mathcal{X}$ *such that* $Y_{t_{\mathcal{NC}}(Y)}$ *is* $X^i_{t-\gamma_i}$ *-* $\mathcal{NC}$*-accessible without using* $X^i_{t-\gamma_i} \leftarrow Y_{t-\gamma_i}$ **then** `/* BFS */`
        return *False*
    `// Case 2(b)ii:`
    $\mathcal{X}' \leftarrow \{X^i_{t-\gamma_i} \mid t - \gamma_i = t_{\mathcal{NC}}(Y)$ and $X^i \in \mathrm{Desc}(Y, \mathcal{G}^s)\}$; `/* BFS */`
4.   **if** $|\mathcal{X}'| \geq 2$ **then**
        return *False*
    if $\mathcal{X}' = \{X^i_{t-\gamma_i}\}$ ***and*** $Y_{t_{\mathcal{NC}}(Y)}$ *is* Y_t *-* $\mathcal{NC}$*- accessible without using* $X^i_t \to Y_t$. **then** `/* BFS */`
        return *False*
return *True*