How Likely Are Two Voting Rules Different?

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Abstract

We characterize the maximum likelihood that two voting rule outcomes are different and that the winner of one voting rule is the loser of the other (implying that they are *drastically different*) on positional scoring rules, Condorcet winner/loser, Copeland, Ranked Pairs, and STV (Single Transferable Vote) under any fixed number of alternatives. The most famous problem in this scope is strong Borda's paradox, in which the winner of the plurality rule is the Condorcet loser. Under mild assumptions, we show that the maximum likelihood that different rules are drastically different is $\Theta(1)$ except for a few special cases, demonstrating the difference between these rules. We also prove that two scoring rules with linear independent scoring vectors have different winners with probability $\Theta(1)$, no matter how similar they are. Our analysis adopts the *smoothed social choice framework* [Xia, 2020] and can be applied to a variety of statistical models, including the standard impartial culture (IC).

1 INTRODUCTION

In recent years, hot debates have emerged regarding electoral system reforms in the United States, as many localities substituted the traditional plurality voting rule with Ranked Choice Voting (RCV) [FairVote, 2023, Post, 2021, Party, 2020, Voting, 2023] or approval voting [Report., 2020] and the fifth president who lost the popular vote but won the election emerges [Revesz, 2016]. Supporters of the reform argue that new voting rules can give more opportunities to independent candidates and underrepresented groups, while the opponents claim that the chance that a different winner would be elected is minimal under the two-party system. This dispute inspires our question: **how likely is the outcome of different voting rules being (drastically) different?**

The impact of the question is double-sided. A confirmative answer supports the reforms and drives researchers and localities to discover, design, and deploy different voting rules to fit diverse scenarios. On the other hand, a dissenting answer justifies the legitimacy of the current plurality rule and eliminates the possibility of undermining democracy by campaigns on changing voting rules.

The significance of the different outcome problems gives rise to a spectrum of theoretical work on studying its existence and likelihood, which can be dated back to Borda's paradox [Borda, 1781]. When a (strong) Borda's paradox occurs, a plurality winner is a Condorcet loser. An even stronger version, named *strict* Borda's paradox, additionally requires the Condorcet winner to be in the last place under the plurality rule.

Example 1 (Borda's Paradox, [Gehrlein and Lepelley, 2010])**.** *Suppose that there are three candidates A, B, and C. The election contains one vote for $[A \succ B \succ C]$, seven votes for $[A \succ C \succ B]$, seven votes for $[B \succ C \succ A]$, and six votes for $[C \succ B \succ A]$. Under the plurality rule that counts the top ranks, A becomes the winner, while C is in the last place. However, the Condorcet winner/loser is elected by pairwise comparisons. Therefore, C becomes the Condorcet winner while A is the loser. Therefore, both strong Borda's paradox and strict Borda's paradox occur in the vote.*

There is a large literature [Tataru and Merlin, 1997, Cervone et al., 2005, Gehrlein and Lepelley, 2010, Diss and Gehrlein, 2012] on the likelihood for strong and strict Borda's paradox in a three-candidate election under the standard assumption of impartial culture (IC) or impartial anonymous culture (IAC). Nevertheless, previous work faces three aspects of challenges.

Firstly, most papers are restricted to the comparison between positional scoring rules and the Condorcet winner/loser. On

Proceedings of the 41st Conference on Uncertainty in Artificial Intelligence (UAI 2025), PMLR 244:4849–4868.

the other hand, (drastically) different outcomes occur among a wide range of common voting rules, including general positional scoring rules, ranked-choice voting, Copeland, Ranked Pairs, and Single Transferable vote (STV).

Secondly, the assumption of IC and IAC has widely been criticized as unrealistic (see, for example, [Lehtinen and Kuorikoski, 2007, Nurmi, 1999]) and fails to reflect real-world scenarios.

Finally, few previous analyses on the likelihood can be extended to settings with more than three candidates due to the restriction on the techniques or overburden calculations. This largely undermines their application to real-world scenarios.

Therefore, our question on how likely the outcome of different voting rules is (drastically) different remains largely unanswered.

1.1 OUR CONTRIBUTIONS

We analyze the asymptotic likelihood, with respect to n, the number of voters, that the outcomes between voting rules are different (named *different winner*, *DW*) and that the winner under one voting rule is the loser under the other rule (named *drastically different*, *DD*), including the strong Borda's paradox, where a loser is the opposite of a winner—roughly speaking, while a winner receives the most support among candidates, a loser receives the least. We conduct an analysis across positional scoring rules, Copeland, Ranked Pairs, STV, and the Condorcet winner/loser. We adopt the smoothed social choice framework in Xia [2020], which can be summarized as follows. Let Π be the set of distributions $\pi \in \mathbb{R}^{m!}$, where m is the number of alternatives and $m!$ is the number of linear orders among candidiates. For each vote, the adversary first picks a distribution $\pi \in \Pi$, and then a linear order is generated by sampling from π, which is treated as the vote.

Our work extends previous studies in three aspects. Firstly, our likelihood results cover a wider range of voting rules. Secondly, the semi-random analysis provides a more general and practical characterization than the standard IC. With the semi-random framework, our results can be applied for analysis of the likelihood under a variety of statistical models including IC. Finally, our results hold for any fixed number of alternatives, while most previous work focused on three or four alternatives.

We investigate the likelihood of DW and DD events under two assumption respectively. The first assumption requires the uniform distribution $\pi_{uni} = (\frac{1}{m!}, \cdots, \frac{1}{m!})$ belongs to the convex hull of Π. This assumption applies not only to the standard IC, but also to other ranking models that do not include the IC, such as the single-agent Mallows' model [Xia, 2020]. We characterize the likelihood of the drastically different problems between a wide range of voting rules. We also show that there is $\Theta(1)$ probability for STV and Plurality (Theorem 7) to have different outcomes. The second assumption requires Π is a single-element set (hence the votes are generated i.i.d.) and give dichotomy results for the likelihood of drastically different winners between two pairs of rules. A summary of these results is shown in Table 1 and Table 2.

Table 1: The likelihood of DD and DW problems under the assumption that IC lies within the convex hull. All but one of the STV results is for the drastically different problem (DD), while one of the STV results is for the different winner problem (DW). PSR is the abbreviation of the positional scoring rule.

Voting Rules		Likelihood	Results
Borda	Condorcet	0	Thm. 2 Coro. 1
Other PSR		$\Theta(1)$	
Linear Independent PSRs		$\Theta(1)$	Thm. 4
Any PSR	Copeland	$\Theta(1)$	Thm. 5 Coro. 4
Borda	Ranked Pairs	$\Theta(n^{-\frac{3}{2}})$	Thm. 6 Coro. 5
Other PSR		$\Theta(1)$	
STV	Plurality	DW: $\Theta(1)$ DD: $\Theta(n^{-\frac{1}{2}})$ or $\Theta(n^{-\frac{m-1}{2}})$	Thm. 7 Thm. 8 Coro. 6

Table 2: The likelihood of *DD* and *DW* under the assumption that the votes are generated i.i.d.

Voting Rules		Likelihood	Results
PSR	Condorcet	$\Theta(1)$ or $\exp(-\Theta(n))$	Coro. 2
Linear Independent PSRs		$\Theta(1)$ or $\exp(-\Theta(n))$	Coro. 3

Our results demonstrate that, under natural assumptions, these voting rules are indeed significantly different from each other except for a few very special cases. Specifically, Theorem 2 shows that strong Borda's paradox occurs with $\Theta(1)$ probability in positional scoring rules except for Borda's rule.

We also ran simulation experiments in Python to calculate numerically the likelihood of DD and DW for different voting rules under impartial culture and Mallows' model. Our experimental results demonstrate our theorems that these rules are indeed different.

1.2 RELATED WORKS AND DISCUSSIONS

There is a large literature characterizing the likelihood of Borda's paradox. Under settings of three alternatives, Tataru and Merlin [1997] and Cervone et al. [2005] characterized the likelihood of strong Borda's paradox under IC and IAC respectively, and Diss and Gehrlein [2012] study the likelihood of the strict Borda's Paradox under IC and IAC. Gehrlein and Lepelley [2010] examine the impact that degrees of mutual coherence among voters' preferences will have on the likelihood of the paradox. Diss and Tlidi [2018] characterize conditions for a profile to show Borda's paradox or not in weighted scoring rules. There are also many empirical studies of Borda's paradox in real-world elections, including the 2016 Republican presidential primaries [Kurrild-Klitgaard, 2018] and the 2017 Iran presidential election [Feizi et al., 2020]. The reader is referred to the work by Diss and Gehrlein [2012] for additional empirical literature.

Studies have also been conducted on paradoxes that are closely related to Borda's paradox. Diss et al. [2018] study the likelihood of a similar class of paradoxes named *absolute majority winner/loser paradox*, in which an alternative ranked top by more than half of the voters is not elected as the winner. Diss and Gehrlein [2015] study the likelihood that a Condorcet winner is elected by a scoring rule (named *Condorcet efficiency*) under a modified version of IAC. Brandt et al. [2022] leverage ILP techniques to identify the minimum number of voters and alternatives for multiple voting paradoxes (including Borda's paradoxes and absolute majority winner/loser paradox) to occur under a wide range of voting rules. Bruns et al. [2019] identify the likelihood of Borda's paradox and Condorcet efficiency of plurality with runoff under a four-alternative setting and IAC.

Smoothed analysis [Spielman and Teng, 2004, 2009] refers to a type of worst-average analysis in which an adversary (the "worst" component) first selects an instance, and then random noises (the "average" component) are added. Smoothed analysis has been widely applied to analyze the practical performance of algorithms, circumventing the hard worst-case. For example, Spielman and Teng [2004] use smoothed complexity analysis to explain why simplex algorithms take polynomial time most of the time despite their hardness in the worst case. Readers are referred to Spielman and Teng [2009] for more applications.

Recent studies introduce smoothed analysis into computational social choice [Baumeister et al., 2020, Xia, 2020] to study topics including the likelihood of axiom satisfactions [Xia, 2021b, 2023b], manipulations and paradoxes [Xia, 2020, Liu and Xia, 2022, Xia, 2023a], and ties [Xia, 2021a]. More specifically, Baumeister et al. [2020] first propose the blue-sky idea of incorporating smoothed analysis into social choice, and Xia [2020] first implements the idea by proposing the formal smoothed social choice framework and characterizing the smoothed likelihood of famous Condorcet's paradox and ANR impossibility. Xia and Zheng [2021, 2022] study the smoothed complexity of the winner determination for the NP-hard Kemeny, Dogson, and Young rule.

2 PRELIMINARY

For any positive integer $q \in \mathbb{N}$, let $[q] = \{1, 2, \cdots, q\}$. The set of alternatives is defined as $\mathcal{A} := [m]$, and the set of all linear orders over $\mathcal{A}$ is denoted by $\mathcal{L}(\mathcal{A})$. Let n be the number of voters. Each voter represents their preference using a linear order $R \in \mathcal{L}(\mathcal{A})$, which is called a *vote*. The vector of n agents' votes is a *(preference) profile*, denoted by P. Such a profile is also referred to as an n-profile. The set of n-profiles for all $n \in \mathbb{N}$ is denoted by $\mathcal{L}(\mathcal{A})^* := \cup_{n=1}^{\infty} \mathcal{L}(\mathcal{A})^n$. A fractional profile is a preference profile together with a weight vector $\vec{w_P} \in \mathbb{R}^n$. A profile can be considered as a fractional profile where all weights are 1. Specifically, a distribution π on $\mathcal{L}(\mathcal{A})$ can be represented as a fractional profile, where $P = \mathcal{L}(\mathcal{A})$ and each entry of $\vec{w_P}$ represents the likelihood of a linear order. Let $\text{Hist}(P) \in \mathbb{Z}_{\geq 0}^{m!}$ denote the *histogram* of P, whose R-element is the multiplicity of linear order $R \in \mathcal{L}(\mathcal{A})$ occurs in profile P.

An *(irresolute) voting rule* $r : \mathcal{L}(\mathcal{A})^* \to 2^{\mathcal{A}}$ maps every profile to its corresponding set of winners in $\mathcal{A}$. In this work, all voting rules are assumed to be irresolute unless stated otherwise. For a voting rule r, let $\hat{r}$ be the mapping from a profile to its corresponding set of losers (see the definition of each voting rule).

An *integer positional scoring rule* is defined by an integer scoring vector $\vec{s} = (s_1, \cdots, s_m) \in \mathbb{Z}^m$ where $s_1 \geq s_2 \geq \cdots \geq s_m$ and $s_1 > s_m = 0$. Given a (fractional) profile P, each alternative receives a score s_i from a vote if it is ranked at i-th position. The alternatives with the maximum overall score are the winners, while the alternatives with the minimum score are considered the losers. For example, *plurality* is characterized by $(1, 0, \cdots, 0)$ and *Borda's rule* is characterized by $(m-1, m-2, \cdots, 1, 0)$.

Given a profile P, let $P[a \succ b]$ be the number of votes in P which rank a higher than b. The *weighted majority graph* $\text{WMG}(P)$ is a weighted directed graph, where each vertex represents an alternative, and the weight of the edge $a \to b$ is $w_P(a, b) = P[a \succ b] - P[b \succ a]$. The *unweighted majority graph* $\text{UMG}(P)$ is the unweighted directed graph obtained from $\text{WMG}(P)$ by keeping the edges with strictly positive weights. We say a voting rule r is *weighted-majority-graph-based (WMG-based)*, if $r(P_1) = r(P_2)$ whenever $WMG(P_1) = WMG(P_2)$. In this paper, we consider the following three WMG-based rules.

- **Condorcet winner/loser.** Given profile P, an alternative a is a Condorcet winner if $w_P(a, b) > 0$ for all $b \neq a$. In other words, a loses no pairwise comparisons. Similarly, a is a Condorcet loser if and only if $w_P(a,b) < 0$ for all alternatives $b \neq a$. We will use $r_C(P)$ and $\hat{r}_C(P)$ to denote the set of Condorcet winner and loser in profile P. It can be verified that the size of winner or loser set is either 0 or 1.

- **Copeland.** The Copeland (Cd_α) rule with parameter $0 \leq \alpha \leq 1$, assigns 1 point to alternative a every time he/she wins a pairwise comparison and α points for both alternatives in each tie. Then the winners of $\mathrm{Copeland}_\alpha$ are the set of all alternatives with the highest total score. The losers are the alternatives with the lowest total score.
- **Ranked Pairs.** The ranked pairs rule (RP) sorts every pair of alternatives (a,b) according to $w_P(a,b)$ in the descending order. An acyclic graph G is then constructed by sequentially adding each pair as a directed edge (from the winner to the loser in the pairwise comparison) in the sorted order, provided it does not form a cycle. Whenever there is a tie, i.e. $w_P(a,b) = w_P(a',b')$, the order of adding edges can have different outcome. Then an alternative is a Ranked Pairs winner (loser) if there exists a tie-breaking sequence such that it is the winner (loser). This tie-breaking method is called *parallel-universe tie-breaking method [Conitzer et al., 2009]*.

We also consider the *single transferable vote (STV)* rule. The STV is a voting rule that selects the winner through $m-1$ rounds. In each round, the plurality rule is applied, and the alternative with the fewest plurality votes (the plurality loser) is eliminated from the election. Whenever there are more than one alternative that has the lowest plurality score, we arbitrarily break the tie and eliminate one of them. An alternative is an STV winner (loser) if there exists a sequence of tie-breaking such that it survives to the end (elminated in the first round). An illustrative example is presented in Appendix A.

2.1 SEMI-RANDOM LIKELIHOOD OF DIFFERENT WINNERS

In this paper, we consider two levels of difference between voting rules. We say two voting rules have *different winners* (*DW*) if their winners are different and *drastically different* (*DD*) if a winner of the first voting rule is a loser of the second rule.

Definition 1. *Given two voting rules r_1 and r_2 and a profile P, suppose the winner and the loser sets of both r_1 and r_2 are not empty. Two voting rules r_1 and r_2 are said to have different winners on a profile P, denoted by DW$(r_1, r_2, P) = 1$, if $r_1(P) \neq r_2(P)$. r_1 and r_2 are considered drastically different on P, denoted by DD$(r_1, r_2, P) = 1$, if $r_1(P) \cap \hat{r}_2(P) \neq \emptyset$.*

For certain voting rules, such as the Condorcet rule, the winner or loser may not exist. In such cases, both *DD* and *DW* are set to 0. In other words, if the Condorcet winner does not exist, we do not consider the Condorcet rule and another voting rule to have different winners or to be drastically different.

We then introduce the statistic model applied in the semi-random analysis model. A *single-agent preference model* [Xia, 2020] $\mathcal{M} = (\Theta, \mathcal{L}(\mathcal{A}), \Pi)$ is characterized by a parameter space Θ, the sample space $\mathcal{L}(\mathcal{A})$, and a set of distributions Π parameterized by Θ. The distribution set Π is said to be closed if it forms a closed subset of $\mathbb{R}^{m!}$. It is strictly positive if there exists a constant $\varepsilon > 0$ such that the probability of any linear order in $\mathcal{L}(\mathcal{A})$ under any distribution π in Π is at least ε. In this paper, we assume that Π is closed and strictly positive. An example of a model with closed and strictly positive Π is shown in Example 3.

We first formalize *DW* and *DD* in the context of the semi-random likelihood model.

Definition 2 (Max semi-random likelihood of different winner and drastically different problem)**.** *Given a single-agent preference model $\mathcal{M} = (\Theta, \mathcal{L}(\mathcal{A}), \Pi)$, and two voting rules r_1, r_2, the adversary aims to maximize the likelihood of DW (DD, respectively) by choosing the distribution $\pi \in \Pi$ for each agent, whose votes are generated independently. Let $\vec{\pi} \in \Pi^n$ be the vector of distributions for all agents. The max semi-random likelihood of DW and DD are defined as follows.*

$$\widetilde{DW}^{\max}_{r_1,r_2} := \sup_{\vec{\pi}\in\Pi^n} \Pr_{P\sim\vec{\pi}}(DW(r_1, r_2, P) = 1),$$
$$\widetilde{DD}^{\max}_{r_1,r_2} := \sup_{\vec{\pi}\in\Pi^n} \Pr_{P\sim\vec{\pi}}(DD(r_1, r_2, P) = 1).$$

Note that *DD* is not symmetric it specifically considers r_1's winner and r_2's loser. When r_1 is plurality and r_2 is Condorcet, *DD* is exactly strong Borda's paradox. In contrast, *DW* is symmetric.

3 PMV-IN-POLYHEDRON PROBLEM

We characterize the likelihood of different outcomes by converting them into PMV-in-polyhedron problems [Xia, 2021a]. Roughly speaking, the probability of each target event occurring also corresponds to the probability that a randomly generated profile (which is a Poisson multivariate variables) lies within a polyhedron defined by a set of linear constraints.

Definition 3 (Poisson multivariate variables (PMVs))**.** *Given any $q, n \in \mathbb{N}$ and any vector $\vec{\pi} = (\pi_1, \cdots, \pi_n)$ of n distributions over $[q]$, a (n,q)-PMV, denoted by $\vec{X}_{\vec{\pi}}$, is the histogram of n independent random variables $Y_1, Y_2, \cdots, Y_n$, of which Y_i follows distribution π_i.*

We clarify the notations used for π as follows. (1) π: A distribution over $\mathcal{L}(\mathcal{A})$, representing the probability of each linear order in the set of all linear orders $\mathcal{L}(\mathcal{A})$. (2) $\vec{\pi}$: A n-vector where each entry π_i is a distribution over $\mathcal{L}(\mathcal{A})$. (3) Π: The set of distributions. Intuitively, $\vec{\pi}$ is the collection of n random votes, and the PMV $\vec{X}_{\vec{\pi}}$ counts the occurrences of each linear order $R \in \mathcal{L}(\mathcal{A})$ across n votes. Sampling $\vec{\pi}$ generates a profile P, and $\text{Hist}(P)$ is the corresponding sample of $\vec{X}_{\vec{\pi}}$.

A *polyhedron* $\mathcal{H} \subseteq \mathbb{R}^q$ can be characterized by a matrix A and a vector $\vec{b}$, i.e. $\mathcal{H} := \{\vec{x} \in \mathbb{R}^q : A\vec{x} \leq \vec{b}\}$. The characteristic cone of $\mathcal{H}$ is $\mathcal{H}_{\leq 0} := \{\vec{x} \in \mathbb{R}^q : A\vec{x} \leq \vec{0}\}$. The dimension of the polyhedron $\mathcal{H}$, denoted by $\dim(\mathcal{H})$, is the dimension of the smallest affine subspace containing $\mathcal{H}$. A polyhedron $\mathcal{H} \subseteq \mathbb{R}^q$ is full-dimensional if $\dim(\mathcal{H}) = q$.

Definition 4 (PMV-in-polyhedron problem)**.** *Given a constant $q \in \mathbb{N}$, a polyhedron $\mathcal{H} \subseteq \mathbb{R}^q$, and a set Π of distributions over $[q]$, we are interested in the max likelihood $\sup_{\vec{\pi}\in\Pi^n} \Pr\left[\vec{X}_{\vec{\pi}} \in \mathcal{H}\right]$.*

Methodology For each different winner and drastically different problem, we represent the event by a polyhedron $\mathcal{H}$ or the union of several polyhedra, such that the event occurs in a profile P if and only if $\text{Hist}(P) \in \mathcal{H}$ or $\text{Hist}(P)$ lies in the union of polyhedra. This converts the *DW* or *DD* problem into a PMV-in-polyhedron problem. We then apply the main technical theorem from Xia [2021a] to characterize the max semi-random likelihood of the problem.

Let $\mathcal{H}_n := \{\vec{x} \in \mathcal{H} \cap \mathbb{R}^q_{\geq 0} : \vec{x} \cdot \vec{1} = n\}$, and $\mathcal{H}_n^{\mathbb{Z}} := \mathcal{H}_n \cap \mathbb{Z}^q$. Given the set of distributions Π, let $CH(\Pi)$ be the convex hull of set Π. The theorem is as follows.

Theorem 1 (Smooth Likelihood of PMV-in-polyhedron [Xia, 2021a])**.** *Given any $q \in N$, any closed and strictly positive Π over $[q]$, and any polyhedron $\mathcal{H}$ characterized by an integer matrix A, for any $n \in \mathbb{N}$,*

$$\sup_{\vec{\pi}\in\Pi^n} \Pr(\vec{X}_{\vec{\pi}} \in \mathcal{H}) =$$
$$\begin{cases} 0 & if \mathcal{H}_n^{\mathbb{Z}} = \emptyset \\ \exp(-\Theta(n)) & if \mathcal{H}_n^{\mathbb{Z}} \neq \emptyset \wedge \mathcal{H}_{\leq 0} \cap CH(\Pi) = \emptyset \\ \Theta\left(\sqrt{n}^{\dim(\mathcal{H}_{\leq 0}) - q}\right) & otherwise. \end{cases}$$

4 RESULTS

4.1 STRONG BORDA'S PARADOX

Our first result characterizes the likelihood that a winner of a positional scoring rule is also a Condorcet loser, which is a generalized case of the strong Borda's paradox. We consider two different choices of the distribution set Π, which are

1. $\pi_{uni} \in CH(\Pi)$, where $CH(\Pi)$ denotes the convex hull of Π,
2. $\Pi = \{\pi\}$, a single-element set, where each vote is generated by π i.i.d.

Let $r_{\vec{s}}$ denote the positional rule characterized by $\vec{s}$, and $\pi_{uni} = (\frac{1}{m!}, \cdots, \frac{1}{m!})$ denote the uniform distribution over $\mathcal{L}(\mathcal{A})$.

Specifically, these two choices of Π extend the IC assumption in two distinct ways. To be exact, IC is a special case of the second case, where we choose $\Pi = \{\pi_{uni}\}$. The first case generalizes the IC assumption differently, allowing the profile P is generated by a mixture of $\pi \in \Pi$ and π_{uni} is not necessarily an element of Π.

Theorem 2. *Suppose the set of distributions Π be strictly positive, closed and $\pi_{uni} \in CH(\Pi)$ then*

$$\widetilde{DD}^{\max}_{r_{\vec{s}}, r_C}(n) = \begin{cases} 0 & \text{if } r_{\vec{s}} \text{ is the Borda's rule} \\ \Theta(1) & otherwise. \end{cases}$$

Theorem 2 enables us to analyze the likelihood of (drastically) different winners under a wider range of statistical models, including IC and single-agent Mallows' (Example 3), bringing a deeper implication to the votings in practice. As shown in Example 4 in the appendix, the theorem works even when the uniform distribution is not an element of Π, but lies in its convex hull.

Proof Sketch of Theorem 2. The Borda's rule's case comes from the following theorem.

Theorem 3 (Fishburn and Gehrlein [1976])**.** *For every $m \geq 3$ and any scoring vector $\vec{s}$, there exists a profile P such that a winner of $r_{\vec{s}}$ is a Condorcet loser if and only if $r_{\vec{s}}$ is NOT the Borda's rule.*

Hence, it suffices to show the case of other positional scoring rules. The following lemma serves as an outline of the proofs in this paper. In most of the proofs, we first specify the event X and the sub-event Y. Then we show that Y satisfies the conditions in the lemma. Finally, we apply Theorem 1 and show that the likelihood of Y (consequently X) is $\Theta(1)$.

Lemma 1 (Main Techincal Lemma)**.** *Let X be the event we are interested in and*

$$\widetilde{X}^{\max}(n) := \sup_{\vec{\pi}\in\Pi^n} \Pr_{P\sim\vec{\pi}}(X \text{ happens when profile is } P),$$

where Π is closed and strictly positive. We say an event Y is a sub-event of X, if Y happens implies X happens. If we can find a sub-event Y of X, which can be characterized by a polyhedron $\mathcal{H} = \{A\vec{x} \leq \vec{b}\}$, and the polyhedron $\mathcal{H}$ satisfies (1) A is an integer matrix (2) $\dim(\mathcal{H}_{\leq 0}) = m!$ (3) $\mathcal{H}_n^{\mathbb{Z}} \neq \emptyset$ (4) $\mathcal{H}_{\leq 0} \cap CH(\Pi) \neq \emptyset$.

Then we can apply Theorem 1 and show that $\widetilde{X}^{\max}(n) = \Theta(1)$.

Proof. It suffices to show $\widetilde{Y}^{\max}(n) = \Theta(1)$ since Y is a sub-event of X, where $\widetilde{X}^{\max}(n) \geq \widetilde{Y}^{\max}(n)$. We verify all the conditions of Theorem 1. First, it is clear that A is an interger matrix and Π is strictly positive. Also, the conditions $\mathcal{H}_n^{\mathbb{Z}} \neq \emptyset$ and $\mathcal{H}_{\leq 0} \cap CH(\Pi) \neq \emptyset$ imply the problem falls into the $\Theta\left(\sqrt{n}^{\dim(\mathcal{H}_{\leq 0})-q}\right)$ case. The conclusion follows directly, since $\dim(\mathcal{H}_{\leq 0}) = m!$. □

Proof of Theorem 2. We give a proof sketch when $r_{\vec{s}}$ is not Borda's rule.

Step 1: Define the Sub-Event and the Polyhedron $\mathcal{H}^a$. The sub-event Y_a is "given profile P, alternative a is the unique winner in $r_{\vec{s}}$ and the Condorcet loser". Now we construct the polyhedron representing Y_a via the score difference vector $Score^{\vec{s}}_{x,y}$ and the pairwise difference vector $Pair_{x,y}$. For a profile P, $Score^{\vec{s}}_{x,y} \cdot \text{Hist}(P)$ is the score difference between x and y under $\vec{s}$, and $Pair_{x,y} \cdot \text{Hist}(P) > 0$ if and only if x wins the pairwise comparison with y. The full definition of these two vectors are presented in Appendix B. The polyhedron $\mathcal{H}^a$ is then defined by the combination of constraints $Score^{\vec{s}}_{b,a} \cdot \vec{x} \leq -\vec{1}$, and $Pair_{a,b} \cdot \vec{x} \leq -\vec{1}$ for all alternatives $b \neq a$. It is straightforward to verify that $\mathcal{H}^a$ represents the sub-event Y_a.

Step 2: Prove the conditions on $\mathcal{H}^a$.

We prove the properties as follows, which are derived from the fact that IC lying in the convex hull and the $\mathcal{H}^a$ is full-dimensional.

Claim 1. *For polyhedron $\mathcal{H}^a$, the conditions in Lemma 1 hold: (1) $\mathcal{H}^a_{\leq 0} \cap CH(\Pi) \neq \emptyset$, (2) $\dim(\mathcal{H}^a_{\leq 0}) = m!$, and (3) $(\mathcal{H}^a)^{\mathbb{Z}}_n \neq \emptyset$.*

(1) is guaranteed by the assumption that $\pi_{uni} \in CH(\Pi)$, and that $\pi_{uni} \in \mathcal{H}^a_{\leq 0}$ since $Score_{x,y} \cdot \vec{\pi}_{uni} = 0$ and $Pair_{x,y} \cdot \vec{\pi}_{uni} = 0$ for every pair of alternatives x, y.

(2) comes from applying the following lemma given that $\mathcal{H}^a$ is not empty is guaranteed by Theorem 3.

Lemma 2. *Any non-empty polyhedron $\mathcal{H} = \{\vec{x} \in \mathbb{R}^q : A\vec{x} \leq \vec{b}\}$ where $\vec{b} < \vec{0}$ satisfies $\dim(\mathcal{H}_{\leq 0}) = q$.*

For any $\vec{x}_0 \in \mathcal{H}$ and any $\lambda > 1$, $\vec{b} < \vec{0}$ implies $A \cdot \lambda\vec{x}_0 = \lambda A\vec{x}_0 \leq \lambda\vec{b} < \vec{0}$. Then we can show that $\lambda\vec{x}_0$ is an inner point of $\mathcal{H}_{\leq 0}$. The existence of an inner point implies $\dim(\mathcal{H}_{\leq 0}) = q$.

Returning to the proof of the claim, we observe that H^a is characterized by $Ax \leq b$, where $b = -\vec{1} < 0$. Since $\mathcal{H}^a$ is non-empty (ensured by Theorem 3), it follows from Lemma 2 that $\dim(\mathcal{H}^a_{\leq 0}) = m!$.

Finally, (3) comes from applying another lemma as follows.

Lemma 3. *Let $\mathcal{H}$ be the polyhedron characterized by $A, \vec{b}$, where $\vec{b} \leq \vec{0}$. Suppose $\mathcal{H}$ is full-dimensional, then there exists $N \in \mathbb{N}$, such that $\mathcal{H}^{\mathbb{Z}}_n$ is not empty for every $n > N$.*

Since $\mathcal{H}$ is full-dimensional, we can find an open ball in $\mathcal{H}_n$ (the intersection of $\mathcal{H}$ and hyperplane $\vec{x} \cdot 1^\top = n$) for any n. Then according to the fact that $A \cdot \lambda\vec{x}_0 = \lambda A\vec{x}_0 \leq \lambda\vec{b} < \vec{b}$, the radius of such open ball is proportional to n. The larger n is, the larger open ball we can find. Therefore, for all sufficiently large n, we can find an open ball in $\mathcal{H}_n$ that contains an integer point.

The full proof of Lemma 2 and 3 is in Appendix C.

Step 3: Apply Lemma 1.

By assumption, Π is a closed and strictly positive set of distributions over $\mathcal{L}(\mathcal{A})$. And we have proved that sub-event Y_a is characterized by $\mathcal{H}^a$ satisfying all the conditions. Hence, by applying Lemma 1 we have $\widetilde{DD}^{\max}_{r_{\vec{s}},r_C}(n) = \Theta(1)$. □

The following corollary computes the likelihood of the reverse case in Theorem 2. Specifically, for every pair of voting rules (r_1, r_2) (excluding STV), and a profile P such that one of the r_1 winners is a r_2 winner, we consider the complement of the profile, denoted by $\bar{P}$. In $\bar{P}$, one of the r_1 losers is a r_2 winner. The formal definition of a complement profile is provided in Appendix B, and the full proof is presented in Appendix D.1. Specifically, for $r_{\vec{s}}$ and r_C, the positional score vector and the weighted majority graph of a profile are the negations of those of its complement.

Corollary 1. *Suppose the set of distributions Π be strictly positive, closed and $\pi_{uni} \in CH(\Pi)$ then*

$$\widetilde{DD}^{\max}_{r_C,r_{\vec{s}}}(n) = \begin{cases} 0 & \textit{if } r_{\vec{s}} \textit{ is the Borda's rule} \\ \Theta(1) & \textit{otherwise} \end{cases}$$

We then analyze the max semi-likelihood under the other distribution set Π, where Π is a single-element set. Recall that a distribution π can be treated as a fractional profile, allowing us to calculate both the winner and loser sets for such a profile. This leads to the following corollary.

Corollary 2. *Suppose the distribution set $\Pi = \{\pi\}$ where π is strictly positive. Let*

$$DD_{r_{\vec{s}},r_C}(n) := \Pr_{P\sim\pi^n}(DD(r_{\vec{s}}, r_C, P) = 1),$$

where $\vec{s}$ is a score vector. Consider π as a fractional profile, we have

$$DD_{r_{\vec{s}},r_C}(n) = \begin{cases} \Theta(1) & r_{\vec{s}}(\pi) \cap \hat{r}_C(\pi) \neq \emptyset, \\ \exp(-\Theta(n)) & \textit{otherwise.} \end{cases}$$

Proof Sketch. We separately consider the $\Theta(1)$ and exponential cases. For the $\Theta(1)$ case, we identify a sub-case with the likelihood of $\Theta(1)$; for the exponential case, we construct both a super-case and a sub-case, each with a likelihood of $\exp(-\Theta(n))$. Then the conclusion is clear. We present the full proof in Appendix D.2. We also provide Example 5 in Appendix A, which computes the likelihood of *DD* under the single-agent Mallows' model with constant dispersion.

4.2 DRASTICALLY DIFFERENCE BETWEEN DIFFERENT POSITIONAL RULES

Theorem 4. *For fixed $m \geq 3$ and any two linear independent score vectors $\vec{s}^1$ and $\vec{s}^2$, suppose the set of distributions Π is strictly positive and closed, and $\pi_{uni} \in CH(\Pi)$, then*

$$\widetilde{DD}^{\max}_{r_{\vec{s}^1}, r_{\vec{s}^2}}(n) = \Theta(1).$$

Proof Sketch. We construct a sub-event Y_a and check it satisfies the conditions of Lemma 1, and then the $\Theta(1)$ result can be derived by the lemma. Given alternative a, we define the sub-event Y_a by "a is the winner under $r_{\vec{s}^1}$ and the loser under $r_{\vec{s}^2}$ in P". $\mathcal{H}^a$ will satisfy the conditions whenever it is not empty, and hence we construct a profile P such that $\text{Hist}(P) \in \mathcal{H}^a$. Finally we shown

$$\widetilde{DD}^{\max}_{r_{\vec{s}^1}, r_{\vec{s}^2}}(n) = \Theta(1).$$

The full proof is in Appendix D.3.

Similar to the Section 4.1, we have the following corollary.

Corollary 3. *Suppose the distribution set $\Pi = \{\pi\}$ where π is strictly positive. Let*

$$DD_{r_{\vec{s}^1}, r_{\vec{s}^2}}(n) := \Pr_{P \sim \pi^n}(DD(r_{\vec{s}^1}, r_{\vec{s}^2}, P) = 1),$$

where $\vec{s}^1, \vec{s}^2$ are two linearly independent score vectors. Consider π as a fractional profile, we have

$$DD_{r_{\vec{s}^1}, r_{\vec{s}^2}}(n) = \begin{cases} \Theta(1) & r_{\vec{s}^1}(\pi) \cap \hat{r}_{\vec{s}^2}(\pi) \neq \emptyset, \\ \exp(-\Theta(n)) & \textit{otherwise.} \end{cases}$$

4.3 COPELAND AND INTEGER POSITIONAL RULES ARE DRASTICALLY DIFFERENT

Theorem 5. *Let Cd_α denotes the copeland rule with parameter α. For fixed $m \geq 3$, $\alpha \in [0, 1]$, and $\vec{s}$ which is an integer score vector, suppose the set of distributions Π be strictly positive, closed, and $\pi_{uni} \in CH(\Pi)$. Then*

$$\widetilde{DD}^{\max}_{r_{\vec{s}}, \text{Cd}_\alpha}(n) = \Theta(1).$$

Proof Sketch. When $r_{\vec{s}}$ is not the Borda's rule, we claim that the Strong Borda Paradox is a special case of the desired event, hence its likelihood is $\Theta(1)$.

When $r_{\vec{s}}$ is the Borda's rule, we construct sub-event Y as follows. We re-index the alternatives as $\mathcal{A} = \{a, 1, \cdots, m-1\}$. Let G be the unweighted majority graph such that (1) a beats 1, (2) for $2 \leq j \leq m-1$, j beats a, and for $1 \leq i < j \leq m-1$, i beats j. We set the sub-event Y to be that a is the unique winner of $r_{\vec{s}}$ and $UMG(P) = G$. We define $\mathcal{H}^{a,G}$ as follows. Let $A^{a,G} := \begin{pmatrix} A^{a,\vec{s}^0} \\ S^G \end{pmatrix}$ and $\vec{b} = -\vec{1}$. Here $A^{a,\vec{s}^0}$ is the matrix with row vectors $\{Score^{\vec{s}^0}_{k,a} : k \in [m], k \neq a\}$. Let S^G be the matrix whose row vectors are $\{Pair_{y,x} : (x, y) \in Edge(G)\}$. Then following the reasoning as in the proof of Theorem 2, it suffices to construct a profile P to show that $\mathcal{H}^{a,G}$ is non-empty. We do this based on the following observation.

Observation 1. *Let $BC(a)$ be the Borda score of alternative a. For a profile P, we have*

$$\sum_{b \neq a} P[a \succ b] = BC(a) = \frac{(m-1)n}{2} + \frac{1}{2} \sum_{b \neq a} w_P(a, b).$$

The full proof is in Appendix D.4.

We also state the reverse version, which can be proved by similar reasoning and using the properties of the complement profile as in the proof of Corollary 1.

Corollary 4. *Suppose the set of distributions Π be strictly positive, closed, and $\pi_{uni} \in CH(\Pi)$. Then*

$$\widetilde{DD}^{\max}_{\text{Cd}_\alpha, r_{\vec{s}}}(n) = \Theta(1).$$

4.4 POSITIONAL RULES AND RANKED PAIRS ARE DRASTICALLY DIFFERENT

Theorem 6. *Let r_{RP} denote the ranked pairs rule. For fixed $m \geq 3$ and an integer score vector $\vec{s}$, suppose the set of distributions Π be strictly positive, closed, and $\pi_{uni} \in CH(\Pi)$. Then*

$$\widetilde{DD}^{\max}_{r_{\vec{s}}, r_{\text{RP}}}(n) = \begin{cases} \Theta(n^{-\frac{3}{2}}) & m = 3,\ r_{\vec{s}} \textit{ is Borda}, \\ \Theta(1) & otherwise. \end{cases}$$

Proof Sketch. On the one hand, when $r_{\vec{s}}$ is Borda and $m = 3$, the desired event (DD) happens if and only if $WMG(P)$ is a directed cycle where the weight of each edge is the same, and hence it can be described by polyhedron and the result is derived by applying Theorem 1. On the other hand, we follow a similar way of Theorem 2. The full proof is in Appendix D.5.

We also consider the reverse version. Let G and G' be the acyclic graph constructed by the Ranked Pairs rule under profiles P and $\bar{P}$, respectively. By the properties of the complement profile, the sink (source) of G is exactly the source (sink) of G'. Hence, we can follow the same reasoning in Corollary 1 and state the following corollary.

Corollary 5. *Suppose the set of distributions* Π *be strictly positive, closed, and* $\vec{\pi}_{uni} \in CH(\Pi)$*. Then*

$$\widetilde{DD}^{\max}_{r_{\mathrm{RP}}, r_{\vec{s}}}(n) = \begin{cases} \Theta(n^{-\frac{3}{2}}) & m = 3,\ r_{\vec{s}} \text{ is Borda} \\ \Theta(1) & \text{otherwise} \end{cases}$$

4.5 STV AND PLURALITY

Theorem 7. *Given* $m \geq 3$*,*

$$\widetilde{DW}^{max}_{STV,Plurality}(n) = \Theta(1).$$

Proof Sketch. Given a permutation σ of $[m]$, the sub-event Y is "$\sigma(j)$ is the only loser and being eliminated in j-th round, while the plurality winner is i". Given the sequence of eliminated alternatives, we can characterize Y by a polyhedron $\mathcal{H}^{\sigma,i}$. We then construct a profile in $\mathcal{H}^{\sigma,i}$ and hence the conditions of Lemma 1 holds. Finally, we apply Lemma 1 and derive the conclusion. Full proof is presented in Appendix D.6.

Theorem 8. *Given* $m \geq 3$*,*

$$\widetilde{DD}^{max}_{STV,Plurality}(n) = \Theta(n^{-\frac{1}{2}}).$$

Proof Sketch of Theorem D.7. We analyze the desired event by decomposing it into the union of sub-events defined by the elimination sequence and the tie-breaking sequence. Let $\mathcal{H}$ denote the corresponding polyhedron, and let $\mathcal{H}_{\leq 0}$ represent its characteristic cone. We prove that $\dim(\mathcal{H} \leq 0) \leq m! - 1$ in the case of a tie, which concludes our proof. The full proof is presented in Appendix D.7.

For the converse, the event "one of the plurality winners is the STV loser (eliminated in the first round)" can only occur when all candidates have the same plurality score. By following a similar approach to the proof of Theorem 8, we derive the following corollary.

Corollary 6.

$$\widetilde{DD}^{max}_{Plurality,STV}(n) = \Theta(n^{-\frac{m-1}{2}}).$$

5 EXPERIMENT RESULTS

We conducted numerical experiments to estimate the probability of *DW* and *DD* for different voting rules using synthetic ranking data generated from common statistical models.

5.1 EXPERIMENT SETUP

We set the number of alternatives to $m = 4$ and consider three settings. The first distribution set is $\Pi = \{\pi_{uni}\}$, which corresponds to our results under assumptions that IC lies within the convex hull and the $\Theta(1)$ case when Π is single-element. The other two settings use the single-agent Mallows' Model with fixed dispersion rate $\varphi = 0.8, 0.9$ (Example 3), which corresponds to our $\exp(-\Theta(n))$ results when the distribution set Π is single-element.

For each setting, voter preferences are sampled independently and identically (i.i.d.) from the corresponding distribution. Each experiment consists of 10^5 simulations, during which we compute the frequencies (probabilities) of *DW* and *DD* outcomes. We vary the number of voters n across the following values: $n = 100, 200, \ldots, 1000$ to observe how the likelihoods converge as n increases.

The experiment involves six common voting rules/criteria: Borda, plurality, Condorcet winner/loser, ranked pairs, STV, and Copeland ($\alpha = 0.5$), as defined in the preliminaries.

5.2 RESULTS FOR SINGLE-ELEMENT DISTRIBUTION

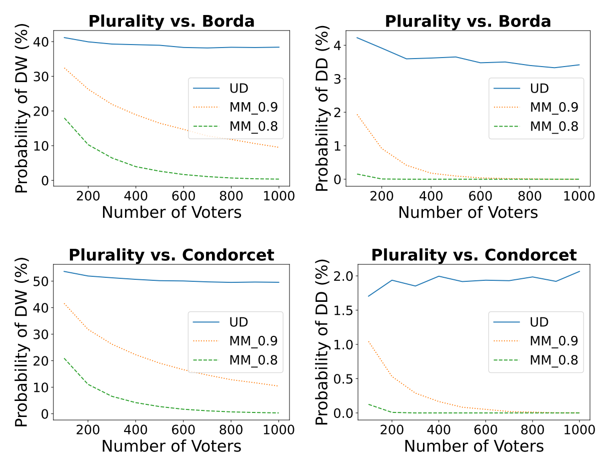

Figure 1: Two pairs of comparison. UD is the uniform distribution, and MM is the single-agent Mallows' model with fixed dispersion φ.

We show two pairs of comparison in Figure 1. The curve of uniform distribution converges to a value of $\Theta(1)$ while the single-agent Mallows' model curves converge to 0 rapidly, as predicted in Corollary 2 and Example 5.

5.3 STV VS. PLURALITY RULE

Figure 2 shows the change of *DW* and *DD* for STV and plurality rule as the number of voters increases. The curve of uniform distribution corroborates our result of $\Theta(1)$ for *DW* and $\Theta(n^{-\frac{1}{2}})$ for *DD* as revealed in Theorem 7 and Theorem 8.

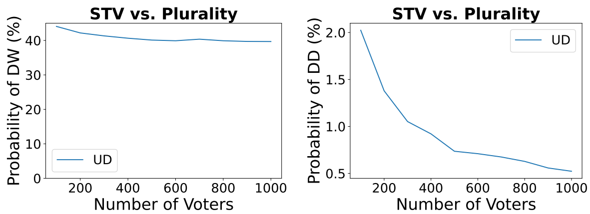

Figure 2: The probability of *DW* and *DD* for the STV and Plurality under uniform distribution.

5.4 POSITIONAL RULES VS. RANKED PAIRS WITH THREE CANDIDATES

When the number of candidates is 3, Figure 3 shows the change of *DD* as the number of voters increases. The curve of uniform distribution corroborates our result of $\Theta(n^{-\frac{3}{2}})$ for the Borda rule while the Plurality rule has a probability of $\Theta(1)$ to produce a drastically different outcome against Ranked Pairs.

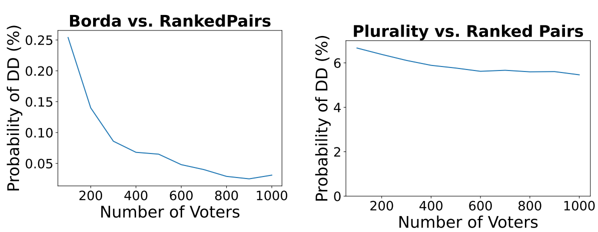

Figure 3: The probability of *DD* under uniform distribution when $m = 3$.

For additional experiments and results, see Appendix E.

6 FUTURE WORKS

We focus on studying the max semi-random likelihood of the different winner problem. Similar techniques of our work can be applied to analyze the min semi-random likelihood. This may require a finer characterization and case discussions. Another question that remains open is the smoothed likelihood of strict Borda's paradox. We are also interested in applying semi-random analysis to more voting paradoxes, including the no-show paradox and Condorcet's paradox. These analysis can brings us a more comprehensive understanding on how these paradoxes may impact votes.

ACKNOWLEDGMENTS

LX acknowledges support from the National Science Foundation under grants CCF-2450124, CCF-2517733, and CCF-2518373.

References

Dorothea Baumeister, Tobias Hogrebe, and Jörg Rothe. Towards reality: smoothed analysis in computational social choice. In *Proceedings of the 19th International Conference on Autonomous Agents and Multiagent Systems*, pages 1691–1695, 2020.

J-C de Borda. Mémoire sur les élections au scrutin: Histoire de l'académie royale des sciences. *Paris, France*, 12, 1781.

Felix Brandt, Marie Matthäus, and Christian Saile. Minimal voting paradoxes. *Journal of Theoretical Politics*, 34(4): 527–551, 2022.

Winfried Bruns, Bogdan Ichim, and Christof Söger. Computations of volumes and ehrhart series in four candidates elections. *Annals of Operations Research*, 280:241–265, 2019.

Davide P Cervone, William V Gehrlein, and William S Zwicker. Which scoring rule maximizes condorcet efficiency under iac? *Theory and Decision*, 58:145–185, 2005.

Vincent Conitzer, Matthew Rognlie, and Lirong Xia. Preference functions that score rankings and maximum likelihood estimation. In *Twenty-First International Joint Conference on Artificial Intelligence*, 2009.

Mostapha Diss and William V Gehrlein. Borda's paradox with weighted scoring rules. *Social Choice and Welfare*, 38:121–136, 2012.

Mostapha Diss and William V. Gehrlein. The true impact of voting rule selection on condorcet efficiency. *Economics Bulletin*, 35(4):2418–2426, 2015.

Mostapha Diss and Abdelmonaim Tlidi. Another perspective on borda's paradox. *Theory and Decision*, 84:99–121, 2018.

Mostapha Diss, Eric Kamwa, and Abdelmonaim Tlidi. A note on the likelihood of the absolute majority paradoxes. *Economics Bulletin*, 38(4):pp–1727, 2018.

FairVote. Where is ranked choice voting used?, 2023. URL `https://fairvote.org/our-reforms/ranked-choice-voting-information/#where-is-ranked-choice-voting-used`.

Mehdi Feizi, Rasoul Ramezanian, and Saeed Malek Sadati. Borda paradox in the 2017 iranian presidential election: empirical evidence from opinion polls. *Economics of Governance*, 21:101–113, 2020.

Peter C Fishburn and William V Gehrlein. Borda's rule, positional voting, and condorcet's simple majority principle. *Public Choice*, pages 79–88, 1976.

William V Gehrlein and Dominique Lepelley. On the probability of observing borda's paradox. *Social Choice and Welfare*, 35:1–23, 2010.

Peter Kurrild-Klitgaard. Trump, condorcet and borda: Voting paradoxes in the 2016 republican presidential primaries. *European Journal of Political Economy*, 55:29–35, 2018. ISSN 0176-2680.

Aki Lehtinen and Jaakko Kuorikoski. Unrealistic assumptions in rational choice theory. *Philosophy of the Social Sciences*, 37(2):115–138, 2007.

Ao Liu and Lirong Xia. The semi-random likelihood of doctrinal paradoxes. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 36, pages 5124–5132, 2022.

David C McGarvey. A theorem on the construction of voting paradoxes. *Econometrica: Journal of the Econometric Society*, pages 608–610, 1953.

Hannu Nurmi. *Voting paradoxes and how to deal with them*. Springer Science & Business Media, 1999.

The Indiana Republican Party. 2020 state convention, 2020. URL `https://www.indiana.gop/stateconvention`.

Washington Post. How ranked-choice voting saved the virginia gop from itself, 2021. URL `https://www.washingtonpost.com/outlook/how-ranked-choice-voting-saved-the-virginia-gop-from-itself/2021/11/05/6d1aa738-3dc2-11ec-8ee9-4f14a26749d1_story.html`.

U.S. News & World Report. St. louis voters approve nonpartisan elections, 2020. URL `https://www.usnews.com/news/best-states/missouri/articles/2020-11-04/st-louis-voters-approve-nonpartisan-elections`.

JRachael Revesz. Five presidential nominees who won popular vote but lost the election. In *Independent*. 2016. URL `https://www.independent.co.uk/news/world/americas/popular-vote-electoral-college-five-presidential-nominees-hillary-clinton-al-gore-a7420971.html`.

Daniel A. Spielman and Shang-Hua Teng. Smoothed analysis of algorithms: Why the simplex algorithm usually takes polynomial time. *J. ACM*, 51(3):385–463, May 2004. ISSN 0004-5411.

Daniel A Spielman and Shang-Hua Teng. Smoothed analysis: an attempt to explain the behavior of algorithms in practice. *Communications of the ACM*, 52(10):76–84, 2009.

Maria Tataru and Vincent Merlin. On the relationship of the condorcet winner and positional voting rules. *Mathematical Social Sciences*, 34(1):81–90, 1997.

Utah Ranked Choice Voting. Ranked choice voting in utah, 2023. URL `https://utahrcv.com/ranked-choice-voting-in-utah-1`.

Lirong Xia. The smoothed possibility of social choice. *Advances in Neural Information Processing Systems*, 33: 11044–11055, 2020.

Lirong Xia. How likely are large elections tied? In *Proceedings of the 22nd ACM Conference on Economics and Computation*, pages 884–885, 2021a.

Lirong Xia. The semi-random satisfaction of voting axioms. *Advances in Neural Information Processing Systems*, 34: 6075–6086, 2021b.

Lirong Xia. The impact of a coalition: Assessing the likelihood of voter influence in large elections. In *Proceedings of the 24th ACM Conference on Economics and Computation*, pages 1156–1156, 2023a.

Lirong Xia. Semi-random impossibilities of condorcet criterion. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 37, pages 5867–5875, 2023b.

Lirong Xia and Weiqiang Zheng. The smoothed complexity of computing kemeny and slater rankings. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 35, pages 5742–5750, 2021.

Lirong Xia and Weiqiang Zheng. Beyond the worst case: Semi-random complexity analysis of winner determination. In *International Conference on Web and Internet Economics*, pages 330–347. Springer, 2022.

A EXAMPLES

Example 2. *We consider a three-alternative profile as follows. The profile contains seven votes for $[A \succ B \succ C]$, six votes for $[B \succ A \succ C]$ and six votes for $[C \succ B \succ A]$. In the first round of the Single Transferable Vote (STV) method, either alternative B or C is eliminated from the election. If B is eliminated, the winner will be A and the loser will be B. If C is eliminated, the winner will be B and the loser will be C. Hence, the STV winner set of this profile is $\{A, B\}$, and the loser set is $\{B, C\}$.*

Example 3. *Here we take a single-agent Mallows' model as an example [Xia, 2020, Example 2 in the appendix]. For any $0 \leq \underline{\varphi} < \overline{\varphi} \leq 1$, we let $\mathcal{M}_{Ma}^{[\underline{\varphi}, \overline{\varphi}]}$ denote the Mallows model whose parameter space $\Theta = \mathcal{L}(\mathcal{A}) \times [\underline{\varphi}, \overline{\varphi}]$. For each $(R, \varphi) \in \Theta$, R is the central ranking, and φ is the parameter of dispersion. For distribution $\pi_{(R,\varphi)}$, any ranking R' is generated with probability $\varphi^{KT(R,R')}/Z_\varphi$, where $KT(R, R')$ is the Kendall Tau distance between R and R', defined by the number of pairwise disagreements between R and R', and $Z_\varphi = \sum_{R' \in \mathcal{L}(A)} \varphi^{KT(R,R')}$ is the normalization constant. The distribution set Π in single-agent Mallows' model is closed and strictly positive.*

Example 4. *We consider a single-agent Mallows' model $\mathcal{M}_{Ma}^{[\underline{\varphi}, \overline{\varphi}]}$ mentioned in Example 3. Note that the $\pi_{uni} \notin \Pi$ when $\overline{\varphi} < 1$ because for any $\pi_{(R,\varphi)}$, the likelihood of generating any ranking R' other than the central ranking R is strictly lower than that of R itself. However, for a fixed $\underline{\varphi} \leq \varphi \leq \overline{\varphi}$, let $\pi_0 = \frac{1}{m!} \sum_{R \in \mathcal{L}(\mathcal{A})} \pi_{(R,\varphi)}$ be the average of all $\pi_{(R,\varphi)}$ for all ranking R. Then due the symmetricity, $\pi_0 = \pi_{uni}$. Therefore, We have $\pi_{uni} \in CH(\Pi)$. And by applying Theorem 2, we show that under single-agent Mallows' model, and for any $r_{\vec{s}}$ not being a Borda's rule, the max semi-random likelihood that the winner of $r_{\vec{s}}$ is a Condorcet loser is $\Theta(1)$.*

Example 5. *We use the single-agent Mallows' model with constant dispersion as an example. Given the central ranking $R_0 = [1 \succ 2 \succ \cdots \succ m]$ and dispersion parameter $\varphi < 1$, $\Pi = \{\pi_{(R_0,\varphi)}\}$ is strictly positive. Additionally, $r_{\vec{s}}(\pi) \cap \hat{r}_C(\pi) = \emptyset$, since 1 is the expected winner of both $r_{\vec{s}}$ and r_C. Hence,*

$$DD_{r_{\vec{s}}, r_C}(n) = \exp(-\Theta(n))$$

under this Mallows' model when $\varphi < 1$.

B DEFINITIONS

Definition 5 (Score difference vector [Xia, 2021a])**.** *For any scoring vector $\vec{s} = (s_1, \cdots, s_m)$ and any pair of different alternatives x, y, let $Score_{x,y}^{\vec{s}}$ denote the $m!$-dimensional vector indexed by rankings in $\mathcal{L}(\mathcal{A})$, the R-element of $Score_{x,y}^{\vec{s}}$ is $s_{j_1} - s_{j_2}$, where j_1 and j_2 are the ranks of x and y in R, respectively.*

Definition 6 (Pairwise difference vector [Xia, 2020])**.** *For any pair of different alternatives x, y, $Pair_{x,y}$ denotes the $m!$-dimensional vector indexed by $R \in \mathcal{L}(\mathcal{A})$, whose R-element is 1 if x ranks higher than y in R and otherwise -1.*

In our analysis, it is often necessary to combine multiple profiles. The combination of two profiles P_1 and P_2 is a profile with $|P_1| + |P_2|$ votes. Since the contribution of individual profiles to the positional scores and the weights of edges in the WMG are linear, we define the combination of these two profiles as $P = P_1 + P_2$. However, the subtraction of profiles, $P_1 - P_2$, cannot be trivially defined, as a negative number of votes does not make sense. To address this, we define the *complement* of a profile P as follows.

Definition 7 (Complement of Profiles)**.** *Given a profile P where k is the maximum entry in $\text{Hist}(P)$. Let A be the profile whose histogram is $(1, \cdots, 1)$. Then the complement of P, denoted by $\bar{P}$, is defined as removing votes in P from kA.*

The histogram of $\bar{P}$ has all non-negative entries, making $\bar{P}$ well-defined. Also, the contribution of P and $\bar{P}$ to the positional scores and weights of edges in the WMG are opposite. Sometimes we denote $\bar{P}$ as $-P$ for the simplicity of terminology.

C PROOF OF LEMMAS

Lemma 2. *Suppose a polyhedron $\mathcal{H} = \{\vec{x} \in \mathbb{R}^q : A\vec{x} \leq \vec{b}\}$ where $\vec{b} < \vec{0}$. Then $\mathcal{H}$ is either empty or full-dimensional. Furthermore, $\dim(\mathcal{H}_{\leq 0}) = q$ when $\mathcal{H} \neq \emptyset$.*

Proof. Suppose $\vec{x}_0 \in \mathcal{H}$. It suffices to show $\dim(\mathcal{H}) = q$. For any $\lambda \geq 1$, we have

$$A \cdot \lambda \vec{x}_0 = \lambda A \vec{x}_0 \leq \lambda \vec{b} < \vec{b}.$$

Hence, $\lambda \vec{x}_0 \in \mathcal{H}$ for any $\lambda \geq 1$. In Euclidean space, a polyhedron $\mathcal{H}$ contains an interior point that implies $\mathcal{H}$ is full-dimensional. Then it suffices to find an interior point of $\mathcal{H}$. We consider the $\mathcal{L}^\infty$ norm for vector $\vec{n} = (n_1, \cdots, n_q)$ and matrix $A = (a_{ij})_{L \times q}$, where

$$\|\vec{n}\|_\infty := \max_{i \in [q]} |n_i|.$$

$$\|A\|_\infty := \max_{i,j} |a_{ij}|.$$

Suppose $\vec{b} = (b_1, \cdots, b_L)$, let $\|\vec{b}\|_0 := \min_{k \in [L]} |b_k|$. We have $\|\vec{b}\|_0 > 0$ since $\vec{b} < \vec{0}$. Then there exists $\lambda_1 \geq 1$ such that $\lambda_1 \|\vec{b}\|_0 \geq 2\|\vec{b}\|_\infty$. For each row of A, denoted by $\vec{a}_i$, we have $\vec{a}_i \cdot \vec{x}_0 \leq b_i \leq -\|\vec{b}\|_0$. Hence, for any vector $d \in \mathbb{R}^q$ satisfy $\|d\|_\infty \leq \frac{\|b\|_\infty}{2\|A\|_\infty}$, we have

$$\begin{aligned}
\vec{a}_i(\lambda_1 \vec{x}_0 + \vec{d}) &= \vec{a}_i \cdot \lambda_1 \vec{x}_0 + \vec{a}_i \cdot \vec{d} \\
&\leq -\lambda_1 \|\vec{b}\|_0 + \|\vec{a}_i\|_\infty \cdot \|\vec{d}\|_\infty \\
&\leq -\lambda_1 \|\vec{b}\|_0 + \|A\|_\infty \|\vec{d}\|_\infty \\
&\leq -\lambda_1 \|\vec{b}\|_0 + \frac{1}{2}\|\vec{b}\|_\infty \\
&\leq -\frac{3}{2}\|\vec{b}\|_\infty \leq b_i.
\end{aligned}$$

Hence, $\lambda_i \vec{x}_0 + \vec{d} \in \mathcal{H}$. Since the $\mathcal{L}^\infty$ norm is equivalent to that of $\mathcal{L}^2$, $\lambda_i \vec{x}_0$ is an interior point of $\mathcal{H}$. Also, since $\vec{b} < \vec{0}$, $\mathcal{H}$ is a subset of the characteristic cone $\mathcal{H}_{\leq 0}$. Hence, $\mathcal{H}_{\leq 0}$ is also full-dimensional. □

Lemma 4. *Let F_n denote the hyperplane*

$$\{\vec{x} \in \mathbb{R}^q : \vec{x} \cdot \vec{1}^\top = n\}.$$

Let $B_{q-1}(\vec{x}_0, r)$ be a $q-1$ dimensional ball, which centered at $\vec{x}_0$ with radius r and contained in F_n. Then there exists an integer point $\alpha_n \in B_{q-1}(\vec{x}_0, r)$ when $r > \sqrt{q}$.

Proof of Lemma 4: Let $\vec{x}_0 = (x_0^1, \cdots, x_0^q)$, and $k := \sum_{i=1}^q \lfloor x_0^i \rfloor$. Clearly, $n - q < k \leq n$ and $k \in \mathbb{Z}$, and the equality holds if and only if all components of $\vec{x}_0$ are integers. Hence, $\alpha_0 = x_0$ will be the desired integer point on F_n when $k = n$.

If not so, then $n - k \geq 1$. We define α_n by

$$\begin{aligned}
\alpha_n := (&\lfloor x_0^1 \rfloor + 1, \lfloor x_0^2 \rfloor + 1, \cdots, \\
&\lfloor x_0^{n-k} \rfloor + 1, \lfloor x_0^{n-k+1} \rfloor, \lfloor x_0^{n-k+2} \rfloor, \cdots, \lfloor x_0^q \rfloor).
\end{aligned}$$

It is easy to verify $\alpha_n \in F_n$, since $\alpha_n \cdot \vec{1}^\top = n - k + \sum_{i=1}^q \lfloor x_0^i \rfloor = n$. Also,

$$\mathrm{Dist}(x_0, \alpha_n) = \left(\sum_{i=1}^q (x_0^i - \lfloor x_0^i \rfloor - 1)^2 \right)^{1/2} < \sqrt{\sum_{i=1}^q 1^2} = \sqrt{q}.$$

Notice that $r > \sqrt{q}$, then show the existence of the integer point α_n in a constructive way.

Lemma 3. *Let $\mathcal{H}$ be the polyhedron characterized by $A, \vec{b}$, where $\vec{b} \leq \vec{0}$. Suppose $\dim(\mathcal{H}) = q$, then there exists $N \in \mathbb{N}$, such that*

$$\mathcal{H}_n^{\mathbb{Z}} := \mathcal{H} \cap \{\vec{x} \in \mathbb{Z}^q : \vec{x} \cdot 1^\top = n\}$$

is not empty for every $n > N$.

Proof. Since $\dim(\mathcal{H}) = q$, there exists $\vec{x}_0 \in \text{int}(\mathcal{H})$. In other word, there exists an open ball $B(\vec{x}_0, r), r > 0$, such that $B(\vec{x}_0, r) \subseteq \mathcal{H}$. We claim that $B(\lambda\vec{x}_0, \lambda r) \in \mathcal{H}$ for any $\lambda > 1$.This is because $A\vec{x} \leq \vec{b} \implies A(\lambda\vec{x}) \leq \lambda\vec{b} \leq \vec{b}$, since $\lambda > 1$. Let $\lambda_n := n/(\vec{x}_0 \cdot \vec{1}^\top) \in \mathbb{R}$, then $\lambda_n \vec{x}_0 \in F_n$. Clearly, there exists N, such that

$$n > N \implies \lambda_n r > \sqrt{q}.$$

Then for any $n > N$, $B_{q-1}(\lambda_n \vec{x}_0, \lambda_n r) \cap F_n$ is a ball in hyperplane F_n. Since its radius is greater than $\sqrt{q}$, by Lemma 4 there exists an integer point α_n such that $\alpha_n \in B_{q-1}(\lambda_n \vec{x}_0, \lambda_n r) \subseteq \mathcal{H}$ and $\alpha_n \cdot 1^\top = n$. Hence, $\mathcal{H}_n^{\mathbb{Z}} \neq \emptyset$ for any $n > N$. $\square$

D FULL PROOF OF MAIN RESULTS

D.1 PROOF OF COROLLARY 1

The proof follows a similar route as Theorem 2. Let $\mathcal{H}^a$ characterized by A and $\vec{b}$ be the polyhedron of sub-event Y (a is the unique winner of $r_{\vec{s}}$ and the Condorcet loser) in Theorem 2. Then the reverse event $\hat{Y}$ (a is the unique loser of $r_{\vec{s}}$ and the Condorcet winner) can be characterized by the polyhedron $\hat{\mathcal{H}}^a := \{-A\vec{x} \leq \vec{b}\}$. Once we show that $\hat{\mathcal{H}}^a$ is non-empty, we can follow the reasoning in the proof for Theorem 2 and apply Lemma 1 to achieve the desired result. It can be verified that $-P$ (the complement of profile P) is the desired profile, where P is the profile for which $\text{Hist}(P) \in \mathcal{H}^a$, and is constructed in the proof of Theorem 2.

D.2 PROOF OF COROLLARY 2

We separately consider the $\Theta(1)$ and exponential cases. For the $\Theta(1)$ case, we identify a sub-case with the likelihood of $\Theta(1)$; for the exponential case, we construct both a super-case and a sub-case, each with a likelihood of $\exp(-\Theta(n))$. Then the conclusion is clear.

When $r_{\vec{s}}(\vec{\pi}) \cap \hat{r}_C(\vec{\pi}) \neq \emptyset$, suppose a is one of its elements and let Y'_a be the event 'a is one of the winners of $r_{\vec{s}}$ and one of the losers of r_C'. This event can be characterized by $\mathcal{H}^a_\star := (\mathcal{H}^a)_{\leq 0}$ in the proof of Theorem 2, and $\mathcal{H}^a \subseteq (\mathcal{H}^a)_{\leq 0}$. Since the conditions in Lemma 1 hold for Y_a, they also hold for Y'_a since Y'_a is a super-case of Y_a. Then we obtain $\mathit{DD}_{r_{\vec{s}}, r_C}(n) = \Theta(1)$ by applying Lemma 1.

Otherwise, we construct the super-case and sub-case as follows. On the one hand, given any $i \in [m]$, we have (1) $\dim((\mathcal{H}^i_\star)_{\leq 0}) = m!$ (2) $(\mathcal{H}^i_\star)^{\mathbb{Z}}_n \neq \emptyset$. Also, $CH(\Pi) \cap (\mathcal{H}^i_\star)_{\leq 0} = \emptyset$ since $r_{\vec{s}}(\vec{\pi}) \cap \hat{r}_C(\vec{\pi}) = \emptyset$, hence

$$\Pr_{P \sim \vec{\pi}^n}(\mathit{DD}(r_{\vec{s}}, r_C, P) = 1) \leq \sum_{i \in [m]} \Pr_{P \sim \vec{\pi}^n}(Y'_i) = \exp(-\Theta(n))$$

by Theorem 1. On the other hand, at least one of Y'_i happens given $\mathit{DD}(r_{\vec{s}}, r_C, P) = 1$, hence

$$\Pr_{P \sim \vec{\pi}^n}(\mathit{DD}(r_{\vec{s}}, r_C, P) = 1) \geq \inf_{i \in [m]} \Pr_{P \sim \vec{\pi}^n}(Y'_i) = \exp(-\Theta(n)),$$

which completes the proof of the exponential case. $\square$

D.3 PROOF OF THEOREM 4

Step 1: Sub-Event and Polyhedron $\mathcal{H}^a$

Given alternative a, we define the sub-event Y_a by "a is the unique winner under $r_{\vec{s}^1}$ and the unique loser under $r_{\vec{s}^2}$ in P". Let $A^{\vec{s}^1, a}$ be the matrix whose row vectors are $\{Score^{\vec{s}^1}_{k,a} : k \in [m], k \neq a\}$ and $A^{\vec{s}^2, a}$ be the matrix with row vectors $\{Score^{\vec{s}^2}_{a,k} : k \in [m], k \neq a\}$. Let $A^a = \begin{pmatrix} A^{\vec{s}^1, a} \\ A^{\vec{s}^2, a} \end{pmatrix}$ and $\vec{b} = -\vec{1}$. Then $\mathcal{H}^a := \{\vec{x} \in \mathbb{R}^{m!} : A^a \vec{x} \leq \vec{b}\}$. Then it's not hard to verify that $\mathcal{H}^a$ represents sub-event Y_a.

Step 2: Prove the conditions on $\mathcal{H}^a$

It suffices to show that $\mathcal{H}^a$ is non-empty since the rest of the reasoning is similar to the proof of Theorem 2. We then construct a profile P with $\text{Hist}(P) \in \mathcal{H}^a$.

We consider a series of profiles $\{P_i\}_{i\in[m]}$ defined as follows. Reindex the set of alternatives by $\mathcal{A} = \{a, c_1, \cdots, c_{m-1}\}$. Given $i \in [m]$, let

$$P_i = \{v_j : (v_j)_{-i} = (c_j, \cdots, c_{m-1}, \cdots, c_{j-1}), (v_j)_i = a\},$$

be the profile of $m-1$ votes. Let $\vec{s}^1 = (s_1^1, s_2^1, \cdots, s_m^1)$, $\vec{s}^2 = (s_1^2, s_2^2, \cdots, s_m^2)$, and $S_k := \sum_{j=1}^m s_k^j$ where $k \in \{1, 2\}$. We have

$$Score_{a,c_j}^{\vec{s}^k}(P_i) = ms_i^k - S_k, k \in \{1, 2\},$$

for any $j \in \{1, \cdots, m-1\}$, since the score of c_j are all the same.

We then show that there exists $(t_1, t_2, \cdots, t_m) \in \mathbb{Z}^m$ such that $P = \sum_{i=1}^m t_i P_i$ satisfies $\text{Hist}(P) \in \mathcal{H}^a$. Given profile P, $-P$ is defined as the profile that reverses each vote's rank in P, and an easy observation is $Score_{a,b}^{\vec{s}}(P) = -Score_{a,b}^{\vec{s}}(-P)$ for any alternatives a, b and score vector $\vec{s}$. Hence, it suffices to find an integer vector $\vec{t} = (t_1, \cdots, t_m)$, such that

$$\sum_{i=1}^m t_i(ms_i^1 - S_1) = \vec{t} \cdot (m\vec{s}^1 - S_1\vec{1}) > 0,$$

$$\sum_{i=1}^m t_i(ms_i^2 - S_2) = \vec{t} \cdot (m\vec{s}^2 - S_2\vec{1}) < 0.$$

Since $\vec{s}^1$ and $\vec{s}^2$ are linear independent, we claim that $m\vec{s}^1 - S_1\vec{1}$ and $m\vec{s}^2 - S_2\vec{1}$ are linear independent. If not so, there exists $\lambda \in \mathbb{R}$ such that for all $i \in [m]$, $ms_i^1 - S_1 = \lambda(ms_i^2 - S_2)$. Since $s_m^1 = s_m^2 = 0$, we have $S_1 = \lambda S_2$, hence $ms_i^1 = ms_i^2$ for all $i \in [m]$, which contradicts to the fact that $\vec{s}^1$ and $\vec{s}^2$ are independent. Then there exists $\vec{t'} \in \mathbb{R}^m$ satisfies the inequalities, since $m\vec{s}^1 - S_1\vec{1}$ and $m\vec{s}^2 - S_2\vec{1}$ are independent. Suppose $\vec{t'} \cdot (m\vec{s}^1 - S_1\vec{1}) = x_1$ and $\vec{t'} \cdot (m\vec{s}^2 - S_2\vec{1}) = -x_2$ where $x_1, x_2 > 0$. Let $x := \min\{x_1, x_2\}$ and $M := \max\{\sum_{i=1}^m |ms_i^1 - S_1|, \sum_{i=1}^m |ms_i^2 - S_2|\}$. There exists $\vec{t^*} \in \mathbb{Q}^m$ such that $\sum_{i=1}^m |t_i' - t_i^*| < \frac{x}{2M}$ since $\mathbb{Q}^m$ are dense in $\mathbb{R}^m$. It can be verified that t^* also satisfies these inequalities and the integer vector $\vec{t}$ is obtained by multiplying an appropriate integer to t^*. Hence we have constructed a profile $P = \sum_{i=1}^m t_i P_i$ where $\text{Hist}(P) \in \mathcal{H}^a$.

Step 3: Apply Lemma 1. Finally, we apply Lemma 1 and get

$$\widetilde{DD}^{\max}_{r_{\vec{s}^1}, r_{\vec{s}^2}}(n) = \Theta(1).$$

□

D.4 PROOF OF THEOREM 5

We prove this theorem through the case study of $\vec{s}$ is a Borda's rule or not.

Case 1: $\vec{s}$ is not a Borda's rule.

By Theorem 3, there exists a profile P such that the winner of positional rule under $\vec{s}$ is the Condorcet loser with probability $\Theta(1)$. Since a Condorcet loser always ranks the last place in Cd_α for any $\alpha \in [0, 1]$, P is also a profile where the positional rule winner is the Cd_α loser. Recall that r_C is the Condorcet rule, hence,

$$\begin{aligned}
\widetilde{DD}^{\max}_{r_{\vec{s}}, r_{\text{Cd}_\alpha}}(n) &= \sup_{\vec{\pi}\in\Pi^n} \sum_{a\in[m]} \Pr_{P\sim\vec{\pi}}[a \in r_{\vec{s}}(P) \cap \hat{r}_{\text{Cd}_\alpha}(P)] \\
&\geq \sup_{\vec{\pi}\in\Pi^n} \sum_{a\in[m]} \Pr_{P\sim\vec{\pi}}[a \in r_{\vec{s}}(P) \cap \hat{r}_C(P)] \\
&= \widetilde{DD}^{\max}_{r_{\vec{s}}, r_C}(n) = \Theta(1) \quad \text{by Theorem 2.}
\end{aligned}$$

Case 2: $\vec{s}$ is a Borda's rule.

Step 1: Find the Sub-event and Characterize Polyhdron $\mathcal{H}^{a,G}$.

We use $\vec{s}^0$ denotes the Borda's rule. For a profile P, we consider the *unweighted majority graph*, denoted by $\mathrm{UMG}(P)$ where each vertex represents an alternative and a directed edge $(a,b) \in E(G)$ if and only if $w_P(a,b) = P[a \succ b] - P[b \succ a] > 0$. We say there is a *tie* between a and b if $w_P(a,b) = 0$. A tournament graph is a simple (no loop or bidirected edges) and complete (each pair of vertex is joined by at least one directed edge) directed graph.

We consider the following event: a is the unique winner of $r_{\vec{s}}$ and $\mathrm{UMG}(P) = G$, where G is a tournament graph such that (1) a beats 1, (2) for $2 \le j \le m-1$, j beats a, and for $1 \le i < j \le m-1$, i beats j. We denote this event by Y.

We define $\mathcal{H}^{a,G}$ as follows. Let $A^{a,G} := \begin{pmatrix} A^{a,\vec{s}^0} \\ S^G \end{pmatrix}$ and $\vec{b} = -\vec{1}$. Here $A^{a,\vec{s}^0}$ is the matrix with row vectors $\{Score^{\vec{s}^0}_{k,a} : k \in [m], k \neq a\}$. Let S^G be the matrix whose row vectors are $\{Pair_{y,x} : (x,y) \in Edge(G)\}$.

Claim 2. $\mathrm{Hist}(P) \in \mathcal{H}^{a,G}$ *if and only if* Y *happens.*

Proof of Claim 2 As we have discussed previously, $A^{a,\vec{s}^0} \cdot \mathrm{Hist}(P) \le -1$ implies a is the unique $r_{\vec{s}^0}$ winner. Also, $Pair_{y,x} \cdot \mathrm{Hist}(P) \le -1$ if and only if x wins y in the pairwise comparison. Hence, the claim holds.

Step 2: Show the conditions on Y and $\mathcal{H}^{a,G}$.

Claim 3. $Y \neq \emptyset$.

We give a construction later.

Claim 4. $\mathcal{H}_{\le 0} \cap CH(\Pi) \neq \emptyset$.

Our assumption guarantees that $\pi_{uni} \in CH(\Pi)$. And $\pi_{uni} \in \mathcal{H}_{\le 0}$ since $Score_{x,y} \cdot \vec{\pi}_{uni} = 0$ and $Pair_{x,y} \cdot \vec{\pi}_{uni} = 0$ for every pair of alternatives x, y.

Claim 5. $\dim(\mathcal{H}^{a,G}) = m!$

Since $\mathcal{H}^{a,G}$ can be characterized by an integer matrix A and $\vec{b} < \vec{0}$, we have $\dim(\mathcal{H}^{a,G}) = m!$ if $\mathcal{H}^{a,G}$ is not empty by Lemma 2.

Step 3: Apply Lemma 1 We check the conditions in Lemma 1. Clearly, Y is a subset of $r_{\vec{s}^0} \cap r^{-1}_{\mathrm{Cd}_\alpha} \neq \emptyset$, which is the desired event in this theorem. Also, $\mathcal{H}^{a,G}$ can be characterized by an integer matirx A and $\vec{b}$. Finally, $\mathcal{H}^{a,G}_{\le 0} \cap CH(\Pi) \neq \emptyset$ and $\dim(\mathcal{H}^{a,G}) = m!$. Then by Lemma 1, it suffices to construct such a profile P, such that $r_{\vec{s}^0}(P) = \{a\}$, $a \in \hat{r}_{\mathrm{Cd}_\alpha}$m and $\mathrm{UMG}(P)$ is a tournament graph.

We then construct a profile P such that $a \in r_{\vec{s}^0} \cap \hat{r}_{\mathrm{Cd}_\alpha}$. We start with several observations. We use $BC(a)$ to represent the Borda's rule of a.

Observation 2. *We consider a profile P and the corresponding weighted majority graph* $\mathrm{WMG}(P)$*, then for any alternative $a \in [m]$,*

1. $$\sum_{b \neq a} P[a \succ b] = BC(a).$$

2. $$BC(a) = \frac{(m-1)n}{2} + \frac{1}{2}\sum_{b \neq a} w_P(a,b).$$

Proof. We first prove the first equality. For each vote v_i, we consider its contribution to both left hand side (LHS) and right hand side (RHS). Suppose a ranks higher than k other alternative in this vote. Then clearly the contribution of v_i to the RHS is k. Also, for every agent b ranked lower than a, v_i contributes one point to $P[a \succ b]$ in LHS, which is in total k points. Hence we have equality.

Then we prove the second equality. Recall that $w_P(a,b) = P[a \succ b] - P[b \succ a]$. By definition, we have $P[a \succ b] = (n + w_P(a,b))/2$. Then we take the sum of every $b \neq a$, and we have

$$BC(a) = \sum_{b \neq a} P[a \succ b] = \frac{(m-1)n}{2} + \frac{1}{2}\sum_{b \neq a} w_P(a,b)$$

□ □

By this observation, $BC(x) > BC(y)$ if and only if $\sum_{k\neq x} w_P(x,k) > \sum_{k\neq y} w_P(y,k)$. We then construct P by constructing a weighted directed graph G as follows. Reindex the set of alternatives by $\mathcal{A} = \{a, 1, \cdots, m-1\}$. For every $1 \leq i < j \leq m-1$, let $w(i,j) = 2$. Let $w(a,1) = 2m^2$ and $w(j,a) = 2$ for $2 \leq j \leq m-1$. Mcgarvey McGarvey [1953] guarantee that there exists a profile P such that $\text{WMG}(P) = G$.

Then it suffices to show P is the desired profile. First, a is the unique $\vec{s}_0$ winner, since $\sum_{b\neq a} w_P(a,b) \geq 2m(m-1) > \sum_{j\neq i} w_P(i,j)$ for any $i \in [m-1]$. Also, $a \in r^{-1}_{\text{Cd}_\alpha}$, since its Cd_α score is one and any other agent win at least one pairwise comparison. Hence, P is the desired profile and we can conclude $\widetilde{DD}^{\max}_{r_{\vec{s}^0}, r_{\text{Cd}_\alpha}}(n) = \Theta(1)$.

D.5 PROOF OF THEOREM 6

When $r_{\vec{s}}$ is not Borda's rule, the statement holds with the same reason on Case 1 in the proof of Theorem 5. This is because a Condorcet winner (loser) is always a Ranked Pair winner (loser). Hence, we follow the same processes and have

$$\widetilde{DD}^{\max}_{r_{\vec{s}}, r_{\text{RP}}}(n) = \Theta(1).$$

We separately consider the case when $m \geq 4$ and $m = 3$ when $r_{\vec{s}}$ is Borda's rule.

Case 1: $m \geq 4$**.** When $r_{\vec{s}}$ is Borda's rule and $m \geq 4$, we follow a similar reasoning in Theorem 5. And it suffices to specify a profile P, such that a is both the unique winner of $r_{\vec{s}}$ and the loser (not necessarily unique) of ranked pairs and there are no ties in any pairwise comparison.

We then construct P as follows. Reindex the set of alternatives by $\mathcal{A} = \{a, 1, \cdots, m-1\}$. Consider the weighted majority graph G as follows. For $i \in [m-2]$, let $w(a,i) = 2m^2$, and $-w(m-1,a) = w(a,m-1) = -2m^2 - 2$. For $j \in [m-2]$, let $w(j,j+1) = 4m^2 + 2m - 2j$, and $w(m-1,1) = 2m^2 + 2$. For any other undefined edges (i,j), let $w(i,j) = 2$ if $i < j$. We can calculate that $\sum_{i\neq a} w_P(a,i) = 2m^3 - 6m^2 - 2$, and for all $j \in [m-1]$, $\sum_{k\neq j} w_P(j,k) \leq 4m$. Since $m \geq 4$, we have

$$2(m^3 - 3m^2 - 1) = 2[(m-3)m^2 - 1] \geq 2m^2 - 2 > 4m.$$

The existence of the P such that $WMG(P) = G$ is guaranteed by McGarvey [1953]. Hence, a is the unique $r_{\vec{s}^0}$ winner under this profile. However, if we apply the Ranked Pair rule, we will first fix $(1,2), \cdots, (m-2, m-1)$ and finally fix $(m-1, a)$. Hence, a is the Ranked Pair loser.

Then we define the sub-event Y_a by "a being the Borda winner in P, and the weighted majority graph of P is G". When $m \geq 4$ and $r_{\vec{s}}$ is Borda's rule, we have

$$\widetilde{DD}^{\max}_{r_{\vec{s}}, \text{Cd}_\alpha}(n) = \Theta(1),$$

followed by the similar reasoning in the proof of Theorem 5.

Case 2: $m = 3$**.** First, let WMG^* be the set of all $\text{WMG}(P)$, we have

$$\begin{aligned}
&\Pr[r_{\vec{s}^0}(P) \cap \hat{r}_{\text{RP}}(P) \neq \emptyset] \\
&= \sum_{G \in \text{WMG}^*} \Pr[r_{\vec{s}^0}(P) \cap \hat{r}_{\text{RP}}(P) \neq \emptyset, \text{WMG}(P) = G] (*)
\end{aligned}$$

Let $G_k = (V, E_k)$ be the directed graph defined as $V = [m] = [3]$ and $E_k = \{1 \to 2, 2 \to 3, 3 \to 1\}$, where the weight of each edge is k. Without loss of generality suppose a is one of the Ranked Pair losers. It can be verified that $w_P(a,b) + w_P(a,c) \leq 0$ for any given profile P. If $w_P(a,b) + w_P(a,c) = 0$, then $BC(a) = n$ by Observation 1. Hence, $a \in r_{\vec{s}^0}(P)$ implies $BC(b) = BC(c) = n$, since $BC(a) + BC(b) + BC(c) = 3n$, and it can be verified that $WMG(P) = G_k$ for some integers $0 \leq k \leq n/3$. Whenever $w_P(a,b) + w_P(a.c) < 0$, Observation 1 gives $BC(a) = n + \frac{1}{2}[w_P(a,b) + w_P(a,c)] < n$. This excludes the possibility of a having the highest Borda Score. Hence, let $\mathcal{G} := \{G_k : 0 \leq k \leq n/3\}$, we have

$$
\begin{aligned}
&\widetilde{DD}^{\max}_{r_{\vec{s}^0}, r_{\mathrm{RP}}}(n)\\
&= \sup_{\vec{\pi}\in\Pi^n} \Pr_{P\sim\vec{\pi}}[r_{\vec{s}^0}(P) \cap \hat{r}_{\mathrm{RP}}(P) \neq \emptyset]\\
&= \sup_{\vec{\pi}\in\Pi^n} \Pr_{P\sim\vec{\pi}}[r_{\vec{s}^0}(P) \cap \hat{r}_{\mathrm{RP}}(P) \neq \emptyset, WMG(P) \in \mathcal{G}]\\
&= \sup_{\vec{\pi}\in\Pi^n} \Pr_{P\sim\vec{\pi}}[WMG(P) \in \mathcal{G}].
\end{aligned}
$$

Step 1: Characterize $\mathrm{WMG}(P) \in \mathcal{G}$ **by** $\mathcal{H}^0$**.** Let $\mathcal{H}^0 := \{A\vec{x} \leq \vec{b}\}$ where A is the matrix whose six rows are $\{Pair_{i,j} - Pair_{j,i} : i \neq j\}$ and $\vec{b} = \vec{0}$.

Step 2: Prove Properties of $\mathcal{H}^0$**.**

Claim 6. $\dim((\mathcal{H}^0)_{\leq 0}) = m! - 3$.

This is because $(\mathcal{H}^0)_{\leq 0} = \mathcal{H}^0$, and $\mathcal{H}^0$ is defined by three linear independent equations. Therefore, the dimension of the null space of $(\mathcal{H}^0)_{\leq 0}$ is 3, and $\dim(\mathcal{H}^0)_{\leq 0}) = m! - 3$.

Claim 7. $CH(\Pi) \cap (\mathcal{H}^0)_{\leq 0} \neq \emptyset$.

This is because $\vec{\pi}_{uni} \in CH(\Pi)$ and $\vec{\pi}_{uni} \in (\mathcal{H}^0)_{\leq 0}$.

Claim 8. $(\mathcal{H}^0)^{\mathbb{Z}}_n \neq \emptyset$ *for every integer* $n \geq 2$.

Let P_0 be a profile with two votes $\{[1 \succ 2 \succ 3], [3 \succ 2 \succ 1]\}$, and P_1 be the profile with three votes $\{[1 \succ 2 \succ 3], [2 \succ 3 \succ 1], [3 \succ 1 \succ 2]\}$. $WMG(k_0P_0 + k_1P_1) \in \mathcal{G}$ for any integers k_1, k_2, and hence we prove the claim.

Step 3: Apply Theorem 1. We check all conditions of Theorem 1. First, Π is closed and strictly positive, and A is an integer matrix. Then, $CH(\Pi) \cap (\mathcal{H}^0)_{\leq 0} \neq \emptyset$ and $(\mathcal{H}^0)^{\mathbb{Z}}_n \neq \emptyset$, then it suffices to consider the $\Theta(n^{\frac{\dim(\mathcal{H}_{\leq 0}) - q}{2}})$ case. Finally, we have $\dim((\mathcal{H}^0)_{\leq 0}) = m! - 3$. Hence, by Theorem 1, we have

$$
\begin{aligned}
\widetilde{DD}^{\max}_{r_{\vec{s}^0}, r_{\mathrm{RP}}}(n) &= \sup_{\vec{\pi}\in\Pi^n} \Pr_{P\sim\vec{\pi}}[\mathrm{WMG}(P) \in \mathcal{G}]\\
&= \Theta(n^{\frac{m! - 3 - m!}{2}}) = \Theta(n^{-\frac{3}{2}}),
\end{aligned}
$$

when $\vec{s}^0$ represents the Borda's rule. □

D.6 PROOF OF THEOREM 7.

Step 1: Sub-Event and Polyhedron We characterize the sub-event Y by a permutation of $[m]$ (denoted by $\sigma([m])$), and an alternative $i \neq \sigma(m)$. The sub-event Y is "$\sigma(j)$ is eliminated in j-th round, while the plurality winner is i". We further suppose that $\sigma(j)$ is the only loser in the j-th round, where it is unnecessary to break the tie.

We consider the polyhedron $\mathcal{H}^{\sigma,i}$ defined as follows. Let

$$
A = \begin{pmatrix} A^{\sigma}_1 \\ A^{\sigma}_2 \\ \vdots \\ A^{\sigma}_{m-1} \\ A^i \end{pmatrix}, \vec{b} = -\vec{1},
$$

where $A^i \cdot \mathrm{Hist}(P) \leq -\vec{1}$ characterizes "i" is the plurality winner and $A^{\sigma}_j \cdot \mathrm{Hist}(P) \leq -\vec{1}$ characterizes "given $\sigma(1), \cdots, \sigma(j-1)$ are eliminated in former rounds, $\sigma(j)$ is eliminated in j-th round". It can be verified these events can be exactly described by linear constraints.

Step 2: Prove the conditions on $\mathcal{H}^{\sigma,i}$**.** We consider a special case where $\sigma_0(1, 2, \cdots, m) = (1, 2, \cdots, m)$ and $i_0 = m - 1$. It suffices to show that $\mathcal{H}^{\sigma_0, i_0}$ is not empty and the rest of the proof is similar to that of Theorem 2. Consider the voting

Num of votes	2	4	$\cdots$	$2(m-1)$	$2m-3$
First place	1	2	$\cdots$	$m-1$	m
Second place	m	m	$\cdots$	m	1

Table 3: Example: STV winner but not plurality winner.

profile in Table 3. Since $m \geq 3$, $2m-3 > 2$ and 1 will be the first alternative being eliminated, while the score of m in the second round becomes $2m-1$, and hence 2 will be eliminated in the second round. It can be verified that m is the STV winner and $m-1$ is the plurality winner, which implies $\mathcal{H}^{\sigma_0, m-1}$ is not empty.

Step 3: Apply Lemma 1. Finally, we apply Lemma 1 and obtain

$$\widetilde{DW}^{max}_{STV,Plurality}(n) = \Theta(1).$$

$\square$

D.7 PROOF OF THEOREM 8.

Proof. The desired event is "one of the STV winners being the plurality loser", and this event can be further divided by the union of sub-events which is characterized by the elimination sequence and the tie-breaking sequence. An elimination sequence, denoted by σ, is a permutation of the alternative set $\mathcal{A} = [m]$. Given an elimination sequence and a profile P, the tie-breaking sequence, denoted by $\vec{t}$, is an m-vector whose i-th component is the set of alternatives who have the lowest plurality score in the i-th round. Let $\sigma(\mathcal{A})$ be the set of permutations and T be the set of all tie sequences. Let $\sigma(i)$ and $\vec{t}(i)$ be the i-th component of the corresponding vector. It can be verified σ and $\vec{t}$ can decide the outcome of STV for any given profile P. We have

$$\begin{aligned}
&\Pr[r_{STV}(P) \cap \hat{r}_{Plur}(P) \neq \emptyset] \\
&= \sum_{\sigma \in \sigma(\mathcal{A})} \sum_{\vec{t} \in T} \Pr\left[\sigma, \vec{t}, r_{STV}(P) \cap \hat{r}_{Plur}(P) \neq \emptyset\right] \\
&= \sum_{\sigma \in \sigma(\mathcal{A})} \sum_{\vec{t} \in T} \Pr_P[\sigma, \vec{t}, \sigma(m) \in \vec{t}(1), |\vec{t}(1)| \geq 2].
\end{aligned} \tag{*}$$

The last equality is followed by the observation as follows. If the STV winner $\sigma(m)$ is a plurality loser, both $\sigma(m)$ and $\sigma(1)$ have the lowest plurality score and $\sigma(1)$ is decided to be eliminated in the first round after tie-breaking.

Step 1: Define the Polyhedron $\mathcal{H}^{\sigma,\vec{t}}$. Let

$$A = \begin{pmatrix} A_1^{\sigma,\vec{t}} \\ A_2^{\sigma,\vec{t}} \\ \vdots \\ A_{m-1}^{\sigma,\vec{t}} \end{pmatrix}, \vec{b} = \begin{pmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vdots \\ \vec{b}_{m-1} \end{pmatrix},$$

where we use $A_i^{\sigma,\vec{t}} x \leq \vec{b}_i$ to describe the event "$\sigma(i)$ is eliminated and alternatives in $\vec{t}(i)$ have the lowest plurality score" for each i. When $|\vec{t}(i)| \geq 2$, the tie between alternatives in $\vec{t}(i)$ can be described by $Score_{a,b}^{\vec{s}_{Plur}}(P) \leq 0$ and $Score_{b,a}^{\vec{s}_{Plur}}(P) \leq 0$ for each pair $a, b \in \vec{t}(i)$ and $a \neq b$. Clearly $Ax \leq b$ precisely describes the event "the elimination sequence is σ, tie sequence is $\vec{t}$".

Step 2: Prove the properties of $\mathcal{H}^{\sigma,\vec{t}}$. Given σ, t which satisfies two conditions $\sigma(1) \in \vec{t}(1)$ and $|\vec{t}(1)| \geq 2$, we claim that dimension of the corresponding characteristic cone $\mathcal{H}_{\leq 0}^{\sigma,\vec{t}}$ is at most $m!-1$. This is because $|\vec{t}(1)| \geq 2$ and hence the number of equalities in $Ax \leq \vec{0}$ is at least 1.

Also, we show that there exists a profile P_0 such that its corresponding $\sigma_0, \vec{t_0}$ satisfies 1) $\sigma_0(1) \in \vec{t_0}(1)$, 2) one of the STV winners is a Plurality loser, 3) $\dim(\mathcal{H}_{\leq 0}^{\sigma_0,t_0}) = m!-1$. Let v_i be the vote defined as $(v_i)_{-2} = (i, i+1, \cdots, m-1, \cdots, i-1), (v_i)_2 = m$ for every $i \in [m-1]$. Let $v_0 = (m, 1, \cdots, m-1)$, and the desired profile $P_0 := 2v_0 + 2v_1 + \sum_{i=2}^{m-1}(i+1)v_i$.

We now apply the STV rule to P_0. In the first round, alternative $1, m$ have the lowest plurality score, and alternative 1 is eliminated in tie-breaking. In the i-th round, alternative i is eliminated, and the alternative m is the final winner. Hence the corresponding $\sigma_0 = (1, 2, \cdots, m)$ and $\vec{t} = (\{1, m\}, \{2\}, \cdots, \{m\})$. The first two conditions is clear, and $\dim(\mathcal{H}_{\le 0}^{\sigma_0, \vec{t_0}}) = m! - 1$ because $\mathcal{H}_{\le 0}^{\sigma_0, \vec{t_0}}$ is the intersection of a full-dimensional polyhedron and a hyperplane.

Step 3: Apply Theorem 1. By Theorem 1 and properties of $\mathcal{H}^{\sigma, \vec{t}}$ shown in Step 2,

$$\Pr\left[\sigma, \vec{t}, r_{STV}(P) \cap \hat{r}_{Plur}(P) \neq \emptyset\right] \le \Theta(n^{-\frac{m!-1-m!}{2}}) = \Theta(n^{-\frac{1}{2}}),$$

and

$$\Pr\left[\sigma_0, \vec{t_0}, r_{STV}(P) \cap \hat{r}_{Plur}(P) \neq \emptyset\right] = \Theta(n^{-\frac{1}{2}}).$$

Hence, we obtain

$$\widetilde{DD}_{STV, Plurality}^{max}(n) = \Theta(n^{-\frac{1}{2}}).$$

□

E SIMULATION RESULTS

We ran experiments in Python to calculate numerically the probability of *DW* and *DD* for different voting rules, comparing the convergence rate of uniform distribution and Mallows' model intuitively by using line chart.

E.1 OVERVIEW UNDER UNIFORM DISTRIBUTION

Additionally, for the uniform distribution, we extend the analysis to $n = 2000$ and compute their 95% confidence intervals to investigate further convergence behavior.

A summary of results and corresponding theoretical results under uniform distribution when $m = 4$ and $n = 2000$ is presented in Table 4. The experimental findings align closely with the theoretical results. The likelihood that Borda and Condorcet are drastically different is 0, as predicted. Our experiments also implies that DD and DW can serve as criteria to evaluate the similarity between voting rules, such as Borda is similar to WMG-based rules than plurality.

Rule 1	Rule 2	Probability		Confidence Interval		Corresponding Theoretic Results
		DW	DD	DW	DD	
Plurality	Condorcet	48.66%	1.967%	(48.35%, 48.97%)	(1.881%, 2.053%)	Thm. 2
Borda	Condorcet	30.25%	0.0%	(29.97%, 30.54%)	− − −	
Plurality	Borda	38.24%	3.371%	(37.94%, 38.54%)	(3.259%, 3.483%)	Thm. 4
Plurality	Copeland	47.13%	7.227%	(46.82%, 47.43%)	(7.067%, 7.388%)	Thm. 5
Borda	Copeland	27.72%	0.741%	(27.45%, 28.00%)	(0.688%, 0.794%)	
Plurality	Ranked Pairs	40.18%	3.940%	(39.88%, 40.48%)	(3.819%, 4.061%)	Thm. 6
Borda	Ranked Pairs	18.26%	0.021%	(18.02%, 18.50%)	(0.012%, 0.030%)	
STV	Plurality	38.63%	0.404%	(38.33%, 38.93%)	(0.365%, 0.443%)	Thm. 7 Thm. 8

Table 4: DW and DD under uniform distribution with 4 alternatives and 2000 voters.

Other pairs of comparison are listed below. The simulation results under uniform distribution align with our IC results (to be $\Theta(1)$). We also conduct the experiment under Mallows model and observed an exponential rate of convergence.

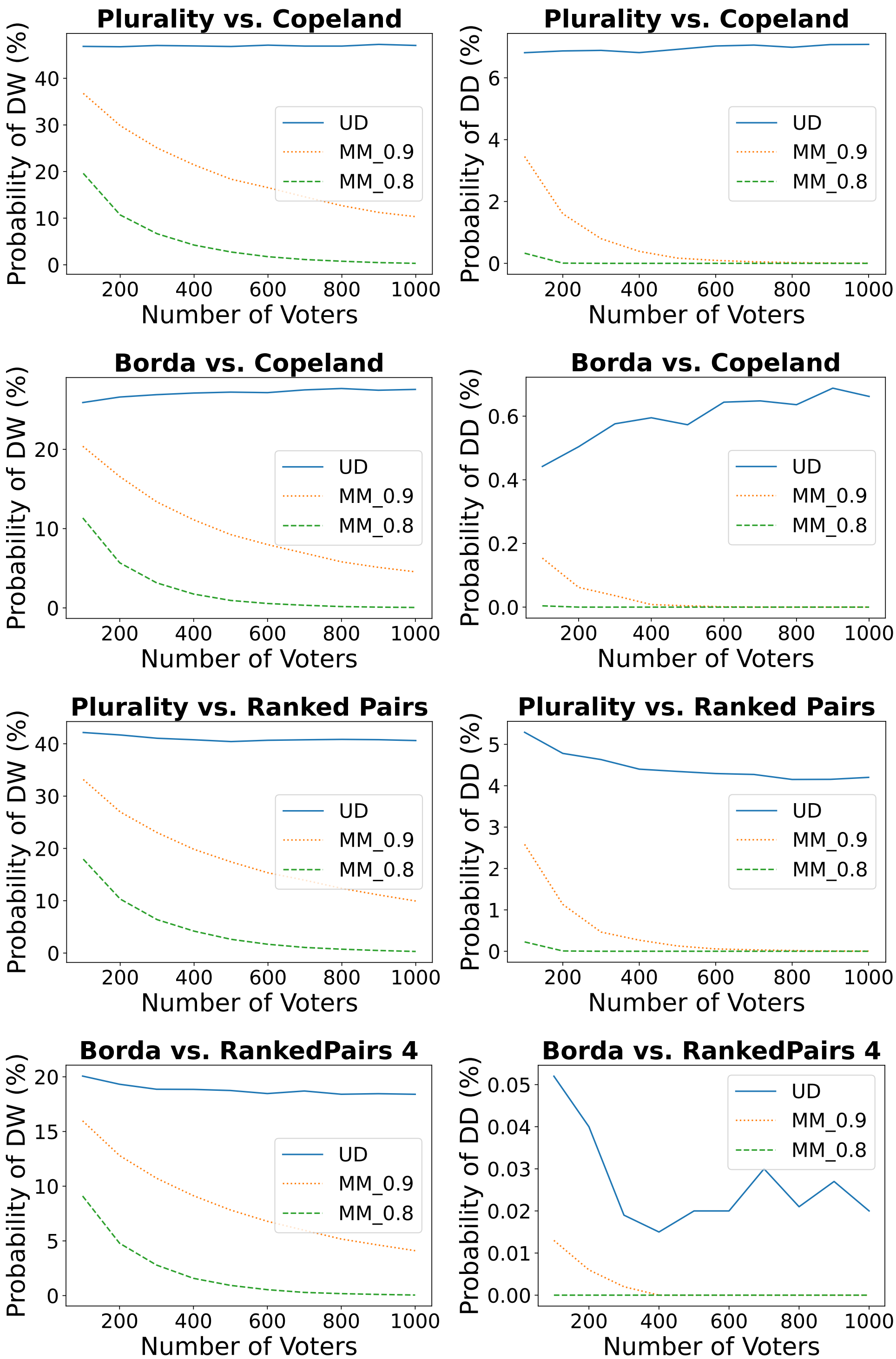

Figure 4: the probability of *DW* and *DD*.