"All-Something-Nothing" Phase Transitions in Planted k-Factor Recovery

Julia Gaudio JULIA.GAUDIO@NORTHWESTERN.EDU
Northwestern University, Department of Industrial Engineering and Management Sciences

Colin Sandon COLIN.SANDON@EPFL.CH
École Polytechnique Fédérale de Lausanne

Jiaming Xu JIAMING.XU868@DUKE.EDU
Duke University, The Fuqua School of Business

Dana Yang XY374@CORNELL.EDU
Cornell University, Department of Statistics and Data Science

Editors: Nika Haghtalab and Ankur Moitra

Abstract

This paper studies the problem of inferring a k-factor, specifically a spanning k-regular graph, planted within an Erdős–Rényi random graph $\mathcal{G}(n, \lambda/n)$. We uncover an interesting "all-something-nothing" phase transition. Specifically, we show that as the average degree λ surpasses the critical threshold of $1/k$, the inference problem undergoes a transition from almost exact recovery ("all" phase) to partial recovery ("something" phase). Moreover, as λ tends to infinity, the accuracy of recovery diminishes to zero, leading to the onset of the "nothing" phase. This finding complements the recent result by Mossel, Niles-Weed, Sohn, Sun, and Zadik who established that for certain sufficiently dense graphs, the problem undergoes an "all-or-nothing" phase transition, jumping from near-perfect to near-zero recovery. In addition, we characterize the recovery accuracy of a linear-time iterative pruning algorithm and show that it achieves almost exact recovery when $\lambda < 1/k$. A key component of our analysis is a two-step cycle construction: we first build trees through local neighborhood exploration and then connect them by sprinkling using reserved edges. Interestingly, for proving impossibility of almost exact recovery, we construct $\Theta(n)$ many small trees of size $\Theta(1)$, whereas for establishing the algorithmic lower bound, a single large tree of size $\Theta(\sqrt{n \log n})$ suffices. [1]

Keywords: Planted subgraph recovery, phase transition, recovery thresholds, iterative pruning

Acknowledgments The authors thank Souvik Dhara, Anirudh Sridhar, and Miklós Rácz for helpful discussions. J. Xu is supported in part by an NSF CAREER award CCF-2144593 and J. Gaudio is supported in part by an NSF CAREER award CCF-2440539.

1. Extended abstract. Full version appears as [arXiv:2503.08984v2]

References

Vivek Bagaria, Jian Ding, David Tse, Yihong Wu, and Jiaming Xu. Hidden Hamiltonian cycle recovery via linear programming. *Operations Research*, 68(1):53–70, 2020.

Béla Bollobás. *Random graphs*. Cambridge Studies in Advanced Mathematics. 2 edition, 2001.

Amin Coja-Oghlan, Oliver Gebhard, Max Hahn-Klimroth, Alexander S Wein, and Ilias Zadik. Statistical and computational phase transitions in group testing. In *Conference on Learning Theory*, pages 4764–4781. PMLR, 2022.

Jian Ding, Yihong Wu, Jiaming Xu, and Dana Yang. Consistent recovery threshold of hidden nearest neighbor graphs. In *Conference on Learning Theory*, pages 1540–1553. PMLR, 2020.

Jian Ding, Yihong Wu, Jiaming Xu, and Dana Yang. The planted matching problem: Sharp threshold and infinite-order phase transition. *Probability Theory and Related Fields*, pages 1–71, 2023.

Richard Durrett. *Random graph dynamics*, volume 200. 2007.

Bruce Hajek, Yihong Wu, and Jiaming Xu. Recovering a hidden community beyond the Kesten–Stigum threshold in $O(|E| \log^\star |V|)$ time. *Journal of Applied Probability*, 55(2):325–352, 2018.

Anton Kotzig. Moves without forbidden transitions in a graph. *Matematický Časopis*, 18(1):76–80, 1968.

Luděk Kučera. Expected complexity of graph partitioning problems. *Discrete Applied Mathematics*, 57(2-3):193–212, 1995.

Cheng Mao, Alexander S Wein, and Shenduo Zhang. Detection-recovery gap for planted dense cycles. In *The Thirty Sixth Annual Conference on Learning Theory*, pages 2440–2481. PMLR, 2023.

Laurent Massoulié, Ludovic Stephan, and Don Towsley. Planting trees in graphs, and finding them back. In *Conference on Learning Theory*, pages 2341–2371. PMLR, 2019.

Henk Meijer, Yurai Núñez-Rodríguez, and David Rappaport. An algorithm for computing simple k-factors. *Information Processing Letters*, 109(12):620–625, 2009.

Mehrdad Moharrami, Cristopher Moore, and Jiaming Xu. The planted matching problem: Phase transitions and exact results. *The Annals of Applied Probability*, 31(6):2663–2720, 2021.

Elchanan Mossel, Joe Neeman, and Allan Sly. Reconstruction and estimation in the planted partition model. *Probability Theory and Related Fields*, 162:431–461, 2015.

Elchanan Mossel, Jonathan Niles-Weed, Youngtak Sohn, Nike Sun, and Ilias Zadik. Sharp thresholds in inference of planted subgraphs. In *The Thirty Sixth Annual Conference on Learning Theory*, pages 5573–5577. PMLR, 2023.

Jonathan Niles-Weed and Ilias Zadik. The all-or-nothing phenomenon in sparse tensor PCA. *Advances in Neural Information Processing Systems*, 33:17674–17684, 2020.

Pavel A. Pevzner. DNA physical mapping and alternating Eulerian cycles in colored graphs. *Algorithmica*, 13(1):77–105, 1995.

Galen Reeves, Jiaming Xu, and Ilias Zadik. The all-or-nothing phenomenon in sparse linear regression. *Mathematical Statistics and Learning*, 3(3):259–313, 2021.

Gabriele Sicuro and Lenka Zdeborová. The planted k-factor problem. *Journal of Physics A: Mathematical and Theoretical*, 54(17):175002, 2021.

Lan V Truong, Matthew Aldridge, and Jonathan Scarlett. On the all-or-nothing behavior of Bernoulli group testing. *IEEE Journal on Selected Areas in Information Theory*, 1(3):669–680, 2020.

Yihong Wu, Jiaming Xu, and H Yu Sophie. Settling the sharp reconstruction thresholds of random graph matching. *IEEE Transactions on Information Theory*, 68(8):5391–5417, 2022.