Time-Uniform, Self-Normalized Concentration for Vector-Valued Processes (Extended Abstract)

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Self-normalized processes arise naturally in many learning-related tasks, such as bandit learning (Kaufmann et al., 2016; Abbasi-Yadkori et al., 2011; Lattimore and Szepesvári, 2020) and online convex optimization (Jun and Orabona, 2019; Li and Orabona, 2020). While self-normalized concentration has been extensively studied for scalar-valued processes (Howard et al., 2020, 2021), there are few results for multivariate processes outside of the sub-Gaussian setting (de la Peña et al., 2009). In this work, we construct a general, self-normalized inequality for $\mathbb{R}^d$-valued processes that satisfy a simple yet broad "sub-ψ" tail condition, which generalizes assumptions based on cumulant generating functions (CGFs). We say a multivariate process $(S_t)_{t\geq 0}$ is sub-ψ with variance proxy $(V_t)_{t\geq 0}$ if, for any $\lambda \in \mathrm{dom}(\psi)$ and $\nu \in \mathbb{S}^{d-1}$, there is a non-negative supermartingale $(L_t^{\lambda\cdot\nu})_{t\geq 0}$ such that

$$M_t^{\lambda\cdot\nu} := \exp\left\{\lambda\langle\nu, S_t\rangle - \psi(\lambda)\langle\nu, V_t\nu\rangle\right\} \leq L_t^{\lambda\cdot\nu}, \qquad \text{for all } t \geq 0.$$

As an example, if $S_t = \sum_s X_s$ and $\mathbb{E}_{t-1}|\langle\nu, X_t\rangle|^k \leq \frac{k!}{2}c^{k-2}\mathbb{E}_{t-1}\langle\nu, X_t\rangle^2$ for all directions $\nu \in \mathbb{S}^{d-1}$ and some constant $c > 0$, one can show that S_t is sub-$\psi_{G,c}$ with variance proxy $V_t := \sum_{s=1}^t \mathbb{E}_{s-1}X_sX_s^\top$. Here, $\psi_{G,c}(\lambda) = \frac{\lambda^2}{2(1-c\lambda)}$ is a bound on the CGF of a centered Gamma random variable. For a more complicated example, if $\mathbb{E}_{t-1}\langle\nu, X_t\rangle^2 < \infty$ for all t and $\nu \in \mathbb{S}^{d-1}$, then one can show S_t is sub-ψ_N with variance proxy $V_t = \frac{1}{3}\sum_{s=1}^t X_sX_s^\top + \frac{2}{3}\sum_{s=1}^t \mathbb{E}_{s-1}X_sX_s^\top$. Under the assumption that $(S_t)_{t\geq 0}$ is sub-ψ with variance proxy $(V_t)_{t\geq 0}$, one can show that, with high-probability

$$\left\|V_t^{-1/2}S_t\right\| \lesssim \sqrt{\gamma_{\min}(V_t)} \cdot (\psi^*)^{-1}\left(\frac{1}{\gamma_{\min}(V_t)}\left[\log\log(\gamma_{\max}(V_t)) + d\log\kappa(V_t)\right]\right) \tag{1}$$

for all $t \geq 0$ simultaneously, where ψ^* is the convex conjugate of ψ, and $\gamma_{\min}(V), \gamma_{\max}(V)$, and $\kappa(V)$ denote respectively the minimum eigenvalue, maximum eigenvalue, and the condition number of a matrix V. In the case $\psi(\lambda) = \psi_N(\lambda) := \frac{\lambda^2}{2}$, our bound becomes $\|V_t^{-1/2}S_t\| \lesssim \sqrt{\log\log\gamma_{\max}(V_t) + d\log\kappa(V_t)}$, which is generally incomparable to the $O(\sqrt{\log\det(V_t)})$ rate method of mixtures bounds commonly leveraged in bandit optimization (de la Peña et al., 2009; Abbasi-Yadkori et al., 2011). In particular, when the variance proxy V_t is well-conditioned, our bound may be significantly tighter. Likewise, in the case $\psi(\lambda) = \psi_{G,c}(\lambda)$, our bound becomes

$$\|V_t^{-1/2}S_t\| \lesssim \sqrt{\log\log\gamma_{\max}(V_t) + d\log\kappa(V_t)} + \frac{c}{\sqrt{\gamma_{\min}(V_t)}}\left[\log\log\gamma_{\max}(V_t) + d\log\kappa(V_t)\right].$$

While we omit dependence on constants, tuning parameters, and failure probability for the sake of exposition, we note that all constants are small, and point to our main paper for a more thorough exposition.[1] From the general inequality in Equation (1), we derive an upper law of the iterated logarithm for sub-ψ vector-valued processes that is unimprovable up to small constants. Further, we provide applications in prototypical statistical tasks, such as parameter estimation in online linear regression and bounded mean estimation via a new (multivariate) empirical Bernstein concentration inequality.

Keywords: Stochastic Processes, Martingales, Concentration Inequalities, Online Estimation

1. Extended Abstract. Full version available at `https://arxiv.org/abs/2310.09100v2`.

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