

Decision Making in Hybrid Environments: A Model Aggregation Approach

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Editors: Nika Haghtalab and Ankur Moitra

Abstract

Recent work by [Foster et al. \(2021, 2022, 2023b\)](#) and [Xu and Zeevi \(2023\)](#) developed the framework of *decision estimation coefficient* (DEC) that characterizes the complexity of general online decision making problems and provides a general algorithm design principle. These works, however, either focus on the pure stochastic regime where the world remains fixed over time, or the pure adversarial regime where the world arbitrarily changes over time. For the hybrid regime where the dynamics of the world is fixed while the reward arbitrarily changes, they only give pessimistic bounds on the decision complexity. In this work, we propose a general extension of DEC that more precisely characterizes this case. Besides applications in special cases, our framework leads to a flexible algorithm design where the learner learns over subsets of the hypothesis set, trading estimation complexity with decision complexity, which could be of independent interest. Our work covers model-based learning and model-free learning in the hybrid regime, with a newly proposed extension of the bilinear classes ([Du et al., 2021](#)) to the adversarial-reward case. In addition, our method improves the best-known regret bounds for linear Q^*/V^* MDPs in the pure stochastic regime.

1. Introduction

The study of decision making problems based on the decision-estimation coefficient (DEC) complexity measure ([Foster et al., 2021](#)) has been shown to characterize the sample complexity of decision making problems. Unlike previous studies on general function approximation that focus more on upper bounds ([Jiang et al., 2017](#); [Sun et al., 2019](#); [Jin et al., 2021](#); [Du et al., 2021](#); [Zhong et al., 2022](#); [Xie et al., 2023](#)), DEC not only provides a general algorithm design principle that interleaves model estimation and decision making, but also characterizes the regret lower bound for decision making problems.

Since being proposed by [Foster et al. \(2021\)](#), the DEC framework has been refined or extended in different directions: [Chen et al. \(2022\)](#) extended the framework to various learning targets beyond no-regret learning, [Foster et al. \(2023c\)](#) proposed a related complexity measure for model-free learning, [Foster et al. \(2023b\)](#) refined the gap between the regret upper and lower bounds, [Foster et al. \(2023a\)](#) applied the approach to multi-agent learning, and [Wagenmaker and Foster \(2023\)](#) established instance-optimal guarantees. Another highly related line of research by [Foster et al. \(2022\)](#) and [Xu and Zeevi \(2023\)](#) focus on characterizing the complexity of *adversarial* decision making, where the environment may change its underlying model arbitrarily across time.

Yet, there are still important open questions remaining in this area. One of them is how to close the gap between the upper and lower bounds in Foster et al. (2023b). The lower bound only captures the complexity of decision making (i.e., the cost of exploration), while the upper bound includes both the complexity of decision making and model estimation. The nature of the algorithm in Foster et al. (2023b) is a model-based algorithm, so the model estimation error naturally appears in the regret bound. To mitigate such gap, Chen et al. (2024) introduced a novel model-estimation complexity measure and provide a complete characterization for bandit learnability. However, their approach still exhibits a significant gap in reinforcement learning. For general decision making, a common route to reduce estimation errors is to avoid fine-grained model estimation, but only perform *value* estimation, as is the case of model-free value-based learning. However, such a coarser estimation may lead to loss of information that affect the complexity of decision making. It remains open how to characterize such trade-off.

Another open direction is the study of intermediate environments that sit between pure stochastic environments (Foster et al., 2021; Chen et al., 2022; Foster et al., 2023b,c) and pure adversarial environments (Foster et al., 2022; Xu and Zeevi, 2023). For learning in Markov decision processes (MDPs), it is well-known that when the transition changes arbitrarily, the worst-case sample complexity could be exponential in the horizon length. Such bound gives a very pessimistic indication for learning in adversarial MDPs. However, it has been shown in some special cases, when only the reward function is adversarial and transition remains fixed, polynomial regret or sample complexity is possible (Neu et al., 2013; Rosenberg and Mansour, 2019; Jin et al., 2020). This setting has practical significance, as it models the scenario where the learner continues to learn new skills while the underlying world remains static (Abel et al., 2024; Kumar et al., 2023). Can we characterize the complexity of such hybrid setting general cases? Being raised since Foster et al. (2022), this question still remains open.

In this paper, we propose a general framework that contributes to the above two directions. Compared to pure model-based learning approach in Foster et al. (2021, 2023b) that leads to large estimation complexity, or pure policy-based learning approach in Foster et al. (2022) and Xu and Zeevi (2023) that leads to large decision complexity, our framework allows learning over flexible partitions in the joint model and policy space, and trade-off estimation complexity with decision complexity. Under this framework, each partition may aggregate a subset of models or policies within which the learner does not perform finer-grained estimation. It naturally captures settings such as model-free value learning (aggregating models that correspond to the same value functions), or fixed-transition adversarial-reward settings (aggregating models that have the same transition function), or a combination of them. We expect that such a general framework may find its more uses beyond the applications in this paper.

We apply this general framework to the following concrete settings, obtaining new results or recovering existing results from a different viewpoint. Below, hybrid setting refers to MDPs with fixed transition and adversarial reward.

1. **Statistical complexity for the hybrid setting.** We show that the hybrid setting has essentially the same complexity as the fully stochastic setting, as long as the set of reward functions is convex (which is usually the case). The only overhead for adversarial reward is a $\log(|\Pi||\mathcal{P}|)$ estimation error as opposed to the $\log(|\mathcal{M}|) = \log(|\mathcal{R}||\mathcal{P}|)$ in the stochastic setting, where $\Pi, \mathcal{M}, \mathcal{P}, \mathcal{R}$ are the policy space, model space, transition space, and reward space, respectively. See Section 4.

In scenarios where the adversarial reward function is revealed at the end of each episode (full-information), the same complexity measure applies. However, $\log(|\Pi||\mathcal{P}|)$ can be improved in MDPs to $\log(|\mathcal{A}||\mathcal{P}|)$, where \mathcal{A} is the action space. See [Section 4](#).

2. **Model-free guarantee for the hybrid setting.** We propose a natural extension of the bilinear class ([Du et al., 2021](#)) to the hybrid setting. We derive regret bounds that only scale with $\log(|\mathcal{F}|)$ for when there is full-information loss feedback, where \mathcal{F} is the set of value functions, and $|\mathcal{F}|$ could be much smaller than $|\mathcal{P}|$. See [Section 6](#).
3. **New results for the stochastic setting.** For model-free learning in stochastic environments, our framework achieves \sqrt{T} regret for linear Q^*/V^* MDPs ([Du et al., 2021](#)). To the best of our knowledge, this is the first algorithm that achieves \sqrt{T} regret in this setting without additional assumptions, improving over the previously known $T^{\frac{2}{3}}$ regret attained by the algorithm in [Du et al. \(2021\)](#).¹ This advances the broader goal of adapting the DEC framework for model-free learning, an objective shared by [Foster et al. \(2023c\)](#). See [Section 5](#).

Related work in the hybrid setting There is a rich literature in learning fixed-transition adversarial-reward MDPs. Most results are for the tabular case ([Neu et al., 2013](#); [Rosenberg and Mansour, 2019](#); [Jin et al., 2020](#); [Shani et al., 2020](#); [Chen and Luo, 2021](#)), or the case of linear function approximation ([Luo et al., 2021](#); [Dai et al., 2023](#); [Sherman et al., 2023](#); [Liu et al., 2023](#); [Kong et al., 2023](#); [Zhao et al., 2024](#); [Li et al., 2024](#)). Extensions beyond linear function approximation are scarce, as only seen in [Zhao et al. \(2024\)](#) and [Liu et al. \(2024\)](#) for low-rank MDPs. We note that the work by [Foster et al. \(2022\)](#) and [Xu and Zeevi \(2023\)](#) for general adversarial decision making can also be applied here, but only achieves tight guarantees when the transition function is known.

Concurrent work Independently and concurrently, [Chen and Rakhlin \(2025\)](#) proposed a closely related extension to the DEC framework. Their work and ours, however, focus on rather different applications.

2. Preliminaries

We consider the general decision making problem. Let \mathcal{M} be a model class, Π be a policy class, and \mathcal{O} be an observation space. Every model $M : \Pi \rightarrow \Delta(\mathcal{O})$ specifies a distribution over observations when executing policy π in model M . At each round $t = 1, 2, \dots, T$, the adversary first chooses a model $M_t \in \mathcal{M}$ without revealing it to the learner. Then the learner chooses a policy $\pi_t \in \Pi$, and observes an observation $o_t \sim M_t(\cdot|\pi_t)$. Every model M is associated with a value function $V_M : \Pi \rightarrow [0, 1]$ that specifies the expected reward of policy π in model M . We also define π_M as the policy that maximize V_M .

1. Although [Du et al. \(2021\)](#) established an $O(1/\epsilon^2)$ sample complexity bound for linear Q^*/V^* , their algorithm does not maintain a confidence set that shrinks with ϵ . Therefore, the standard policy-elimination reduction from $O(1/\epsilon^2)$ sample complexity to \sqrt{T} -regret cannot be applied. The strongest regret bound we are able to obtain from their method is $O(T^{2/3})$ by using an explore-then-commit approach.

The regret with respect to the policy sequence $\pi_{1:T}^* = (\pi_1^*, \pi_2^*, \dots, \pi_T^*)$ is defined as

$$\text{Reg}(\pi_{1:T}^*) = \sum_{t=1}^T (V_{M_t}(\pi_t^*) - V_{M_t}(\pi_t)).$$

For simplicity, in this work, we assume $|\mathcal{M}|$ and $|\Pi|$ are finite.

2.1. Complexity Measures in Prior Work

DEC Let $\text{co}(\mathcal{M})$ be the convex hull of \mathcal{M} . The offset DEC with KL-divergence (Foster et al., 2021; Xu and Zeevi, 2023) is defined as

$$\text{DEC}_{\eta}^{\text{KL}}(\mathcal{M}) = \max_{\bar{M} \in \text{co}(\mathcal{M})} \min_{p \in \Delta(\Pi)} \max_{M \in \mathcal{M}} \mathbb{E}_{\pi \sim p} \left[V_M(\pi_M) - V_M(\pi) - \frac{1}{\eta} D_{\text{KL}}(M(\cdot|\pi), \bar{M}(\cdot|\pi)) \right]. \quad (1)$$

The DEC aims to achieve the optimal balance between exploitation (minimizing suboptimality $V_M(\pi_M) - V_M(\pi)$) and exploration (maximizing information gain $D_{\text{KL}}(M(\cdot|\pi), \bar{M}(\cdot|\pi))$) under the worst possible environment.

AIR Xu and Zeevi (2023) introduced AIR as a framework to analyse the regret of arbitrary algorithms and derive upper bounds on the complexity of online learning problems. Given $\rho \in \Delta(\Pi)$ and $\eta > 0$, $\text{AIR} : \Delta(\Pi) \times \Delta(\Pi \times \mathcal{M}) \rightarrow \mathbb{R}$ is defined as

$$\text{AIR}_{\rho, \eta}(p, \nu) = \mathbb{E}_{\pi \sim p} \mathbb{E}_{(M, \pi^*) \sim \nu} \mathbb{E}_{o \sim M(\cdot|\pi)} \left[V_M(\pi^*) - V_M(\pi) - \frac{1}{\eta} D_{\text{KL}}(\nu_{\pi^*}(\cdot|\pi, o), \rho) \right].$$

Here, ν is a distribution over $\mathcal{M} \times \Pi$, and $\nu_{\pi^*}(\cdot|\pi, o)$ is a posterior distribution over π^* given (π, o) , which can be calculated as

$$\nu_{\pi^*}(\pi^*|\pi, o) = \frac{\sum_M \nu(M, \pi^*) M(o|\pi)}{\sum_{M, \pi'} \nu(M, \pi') M(o|\pi)}. \quad (2)$$

The subscript π^* in the notation $\nu_{\pi^*}(\cdot|\pi, o)$ specifies what variable the ‘ \cdot ’ represents. We place it in the subscript and make it bold to distinguish it from a realized value of that random variable. When the variable is clear, we omit this subscript. For example, the left-hand side of Eq. (2) can be simply written as $\nu(\pi^*|\pi, o)$. Similar to DEC in Eq. (1), AIR also seeks the optimal balance between minimizing sub-optimality gap and maximizing information gain. The key difference is that AIR considers an arbitrary comparator policy π^* instead of π_M as in DEC, with exploration measured by the information gain associated with π^* . The flexibility of the comparator policy enables AIR to handle the adversarial setting.

MAIR For model based stochastic environments, given $\rho \in \Delta(\mathcal{M})$ and η , Xu and Zeevi (2023) further define $\text{MAIR} : \Delta(\Pi) \times \Delta(\mathcal{M}) \rightarrow \mathbb{R}$ as

$$\text{MAIR}_{\rho, \eta}(p, \nu) = \mathbb{E}_{\pi \sim p} \mathbb{E}_{M \sim \nu} \mathbb{E}_{o \sim M(\cdot|\pi)} \left[V_M(\pi_M) - V_M(\pi) - \frac{1}{\eta} D_{\text{KL}}(\nu_M(\cdot|\pi, o), \rho) \right].$$

Here ν is only a distribution over \mathcal{M} . This is because in stochastic setting, it suffices to compare against $V_M(\pi_M)$, making it unnecessary to consider a joint distribution over $\mathcal{M} \times \Pi$. [Xu and Zeevi \(2023\)](#) also shows that MAIR is tightly related to DEC.

EXO by [Lattimore and Szepesvári \(2020\)](#) and [Foster et al. \(2022\)](#) is closely related to AIR, and also serves as a useful complexity measure and algorithm design principle for adversarial decision making, with the additional benefit of getting high-probability bounds. In this work, we base our presentation in AIR due to its conciseness, but everything could be translated to the EXO framework.

2.2. Markov Decision Processes

A Markov decision process (MDP) is defined by a tuple $(\mathcal{S}, \mathcal{A}, P, R, H, s_1)$, where \mathcal{S} is the state space, \mathcal{A} is the action space, $P : \mathcal{S} \times \mathcal{A} \rightarrow \Delta(\mathcal{S})$ the transition kernel, $R : \mathcal{S} \times \mathcal{A} \rightarrow [0, 1]$ the reward function, H the horizon and s_1 the starting state. A learner interacts with an MDP for H rounds $h = 1, \dots, H$. In every round, the learner observes the current state s_h and selects an action $a_h \in \mathcal{A}$. The state is transitioning into next state via the transition kernel $s_{h+1} \sim P(\cdot | s_h, a_h)$ and receives the reward $R(a_h, s_h)$. We assume that the reward functions are constraint such that $\sum_{h=1}^H R(s_h, a_h) \leq 1$ for any policy almost surely. Given a policy $\pi : \mathcal{S} \rightarrow \mathcal{A}$, the Q -function and value function are defined by $Q_h^\pi(s, a) = \mathbb{E}^\pi[\sum_{h'=h}^{H-1} R(s_{h'}, a_{h'}) | s_h = s, a_h = a]$, $V_h^\pi(s) = Q_h^\pi(s, \pi(s))$ respectively. The value and Q -function of an optimal policy π^* are abbreviated with V^*/Q^* . We use $Q_h^\pi(s, a; M)$ and $Q_h^*(s, a; M)$ where $M = (P, R)$ to denote the value functions under model M .

3. A General Framework

In order to formalize mixed-adversarial and stochastic regimes, we propose the following framework. Let Φ be a collection of disjoint subsets of $\mathcal{M} \times \Pi$. That is, 1) each element $\phi \in \Phi$ is a subset of $\mathcal{M} \times \Pi$, and 2) for any $\phi_1, \phi_2 \in \Phi$, if $\phi_1 \neq \phi_2$, then $\phi_1 \cap \phi_2 = \emptyset$.²

Definition 1 *A Φ -restricted environment is an (adversarial) decision making problem in which the environment commits to $\phi^* \in \Phi$ at the beginning of the game and henceforth selects $(M_t, \pi_t^*) \in \phi^*$ in every round t arbitrarily based on the history.*

Example Let $\mathcal{M} = \mathcal{P} \times \mathcal{R}$ decomposes into the set of transitions and rewards of a class of MDPs respectively. Set $\Phi = \{\phi_{P, \pi^*} | (P, \pi^*) \in \mathcal{P} \times \Pi\}$, where $\phi_{P, \pi} = \{((P, R), \pi) | R \in \mathcal{R}\}$. The Φ -restricted environment is exactly the setting of an MDP with fixed transition, fixed comparator and adversarial rewards.

To characterize the complexity of Φ -restricted environments, we define for $\rho \in \Delta(\Phi)$, $\eta > 0$,

$$\text{AIR}_{\rho, \eta}^\Phi(p, \nu) = \mathbb{E}_{\pi \sim p} \mathbb{E}_{(M, \pi^*) \sim \nu} \mathbb{E}_{o \sim M(\cdot | \pi)} \left[V_M(\pi^*) - V_M(\pi) - \frac{1}{\eta} D_{\text{KL}}(\nu_\phi(\cdot | \pi, o), \rho) \right],$$

2. The requirement that subsets in Φ be disjoint is not really necessary, as models can be duplicated if needed. However, we adopt this assumption as it holds in all our applications and simplifies the notation.

Algorithm 1 General Algorithm

 Let $\rho_1(\phi) = \frac{1}{|\Phi|}$ for all $\phi \in \Phi$.

for $t = 1, 2, \dots, T$ **do**

 Find distributions $p_t \in \Delta(\Pi)$ and $\nu_t \in \Delta(\Psi)$ that solve the saddle point of

$$\min_{p \in \Delta(\Pi)} \max_{\nu \in \Delta(\Psi)} \text{AIR}_{\rho_t, \eta}^{\Phi}(p, \nu).$$

 Sample decision $\pi_t \sim p_t$ and observe $o_t \sim M_t(\cdot | \pi_t)$.

 Update $\rho_{t+1}(\phi) = \nu_t(\phi | \pi_t, o_t)$ for all $\phi \in \Phi$.

end

where $\nu_{\phi}(\cdot | \pi, o)$ is the posterior over Φ given (π, o) , which can be calculated as

$$\nu_{\phi}(\phi | \pi, o) = \frac{\sum_{(M, \pi^*) \in \phi} \nu(M, \pi^*) M(o | \pi)}{\sum_{\phi' \in \Phi} \sum_{(M, \pi^*) \in \phi'} \nu(M, \pi^*) M(o | \pi)}.$$

Again, this follows our convention that the ϕ in the subscript of $\nu_{\phi}(\cdot | \pi, o)$ is only an indicator of the variable represented by ‘ \cdot ’, rather than a realized value, and it will be omitted when it is clear.

AIR $^{\Phi}$ recovers AIR and MAIR. Specifically, AIR $^{\Phi}$ recovers AIR when every ϕ corresponds to a single $\pi^* \in \Pi$ and all $M \in \mathcal{M}$, i.e., $\Phi = \{\phi_{\pi^*} : \pi^* \in \Pi\}$ where $\phi_{\pi^*} = \{(M, \pi^*) : M \in \mathcal{M}\}$. AIR $^{\Phi}$ recovers MAIR when every ϕ corresponds to a single model M and the associated best policy π_M , i.e., $\Phi = \{\phi_M : M \in \mathcal{M}\}$ where $\phi_M = \{(M, \pi_M)\} = \{(M, \arg\max_{\pi \in \Pi} V_M(\pi))\}$. We define $\Psi = \left(\bigcup_{\phi \in \Phi} \phi\right) \subset \mathcal{M} \times \Pi$.

With this definition, we propose the general algorithm in [Algorithm 1](#). Similar to [Xu and Zeevi \(2023\)](#), we can decompose the objective in [Algorithm 1](#) as the following terms:

$$\underbrace{\mathbb{E}_{\pi \sim p} \mathbb{E}_{(M, \pi^*) \sim \nu} [V_M(\pi^*) - V_M(\pi)]}_{\text{expected regret}} - \frac{1}{\eta} \underbrace{\mathbb{E}_{\pi \sim p} \mathbb{E}_{(M, \pi^*) \sim \nu} \mathbb{E}_{o \sim M(\cdot | \pi)} [D_{\text{KL}}(\nu_{\phi}(\cdot | \pi, o), \nu_{\phi})]}_{\text{information gain}} - \frac{1}{\eta} \underbrace{D_{\text{KL}}(\nu_{\phi}, \rho_t)}_{\text{regularization}}$$

where $\nu_{\phi}(\phi) = \sum_{(M, \pi^*) \in \phi} \nu(M, \pi^*)$. The algorithm proceeds as follows. Upon receiving a reference distribution $\rho_t \in \Delta(\Phi)$ (an estimation from the previous round about ϕ^*), the algorithm finds a policy distribution p_t that simultaneously tries to minimize the expected regret and maximize the information gain about ϕ^* against the worst case choice of the world distribution $\nu_t \in \Delta(\mathcal{M} \times \Pi)$ that is not too far from the previous estimation ρ_t . At the end of round t , the algorithm calculates ρ_{t+1} as a posterior of ν_t with the new observation (π_t, o_t) , only stores this information, and forgets everything else.

The algorithm AIR by [Xu and Zeevi \(2023\)](#) only maintains an estimation of the optimal policy π^* , while their MAIR maintains an estimation of the underlying model M^* . Our framework provides a natural generalization of them by allowing the target of estimation be subsets of $\mathcal{M} \times \Pi$. For example, in MDPs with fixed transition and adversarial reward, we jointly estimate (P^*, π^*) , while for model-free value learning in stochastic MDPs, we estimate (f^*, π_{f^*}) where f^* is the true optimal value function and π_{f^*} is its corresponding optimal policy. These are achievable by properly choosing Φ , as will be elaborated in the following sections.

The regret bound achieved by [Algorithm 1](#) is stated in the following theorem.

Theorem 2 *For a Φ -restricted environment, there is an algorithm such that*

$$\mathbb{E} [\text{Reg}(\pi_{1:T}^*)] \leq \frac{\log |\Phi|}{\eta} + T \max_{\rho \in \Delta(\Phi)} \max_{\nu \in \Delta(\Psi)} \min_{p \in \Delta(\Pi)} \text{AIR}_{\rho, \eta}^{\Phi}(p, \nu).$$

The bound in [Theorem 2](#) consists of two parts. The first part $\frac{\log |\Phi|}{\eta}$ captures the *estimation complexity* over Φ , while the other part $T \cdot \text{AIR}^{\Phi}$ captures the *decision complexity* that quantifies the cost of exploitation-exploration tradeoff. A finer partition Φ leads to larger estimation complexity but smaller decision complexity, and a coarser partition leads to the opposite. Since a coarser partition makes the environment strictly more powerful, the learner has the freedom to find the best trade-off over coarser partitions, while still ensuring a bound on the environment at hand.

Remark 3 *All concrete applications studied in this paper actually consider the standard regret $\text{Reg}(\pi^*)$ where the regret comparator π^* is fixed over time. That requires each $\phi \in \Phi$ to have a unique π^* . The generality of our framework in allowing changing comparators $\pi_{1:T}^*$ may be of independent interest and is left for future investigation. We note that this theorem does not violate the impossibility result for general time-varying comparator $\pi_{1:T}^*$, as in that case the AIR^{Φ} term in [Theorem 2](#) could be of order $\Omega(1)$. This is discussed more deeply in [Lemma 5](#).*

3.1. Relation with DEC

As noted in [Foster et al. \(2022\)](#) and [Xu and Zeevi \(2023\)](#), for pure adversarial decision making, the decision complexity is governed by the DEC of the convexified model class $\text{CO}(\mathcal{M})$. In the Φ -restricted environment ([Definition 1](#)) that we consider, we show that the complexity is upper bounded by the DEC of a refined convexified model class taking the partition into account.

Definition 4 *The Φ -aware convexification of the model class \mathcal{M} is defined as*

$$\overline{\mathcal{M}}(\Phi) := \bigcup_{\phi \in \Phi} \text{CO}(\{M \in \mathcal{M} \mid \exists \pi \text{ such that } (M, \pi) \in \phi\}).$$

In other words, models are convexified within each subset ϕ . This describes the complexity of the Φ -restricted environment as follows.

Lemma 5 *It holds that $\max_{\rho \in \Delta(\Phi)} \max_{\nu \in \Delta(\Psi)} \min_{p \in \Delta(\Pi)} \text{AIR}_{\rho, \eta}^{\Phi}(p, \nu) \leq \text{DEC}_{\eta}^{\text{KL}}(\overline{\mathcal{M}}(\Phi))$, unless Φ is an environment with adaptive comparator where sublinear regret is impossible. In this case $\max_{\rho \in \Delta(\Phi)} \max_{\nu \in \Delta(\Psi)} \min_{p \in \Delta(\Pi)} \text{AIR}_{\rho, \eta}^{\Phi}(p, \nu) = \Omega(1)$ independently of η .*

While this is only providing an upper bound, it is actually tight in many environments of interest. Let a *fixed comparator game* be such that all $\phi \in \Phi$ contain a single policy π^{ϕ} .

Lemma 6 *In all fixed comparator games if the choice of model is independent of the choice of comparator then $\max_{\rho \in \Delta(\Phi)} \max_{\nu \in \Delta(\Psi)} \min_{p \in \Delta(\Pi)} \text{AIR}_{\rho, \eta}^{\Phi}(p, \nu) = \text{DEC}_{\eta}^{\text{KL}}(\overline{\mathcal{M}}(\Phi))$.*

The choice of model being independent of the choice of comparator means that $\Phi = \Theta \times \Pi$ for an arbitrary disjoint partition $\mathcal{M} = \bigcup_{\theta \in \Theta} \theta$ of models. This is in fact the case for most of our applications except for the application in [Section 6](#).

Note that in stochastic regimes, we often trim the set to pairs of model and its optimal policy (see [Section 5](#)). This trimming neither impacts the maxmin value of AIR^{Φ} nor the convexification (see [Lemma 18](#) in the appendix), so equality still holds.

3.2. Informed Comparator Case

In this section, we consider the fixed-comparator regime where the choice of model is independent of the comparator, that means there is a partition Θ of \mathcal{M} , such that $\Phi = \Theta \times \Pi$.

The complexity $|\Phi|$ is $|\Theta| \cdot |\Pi|$, which can be highly sub-optimal when $|\Theta| \ll |\Pi|$. To overcome this limitation, we propose to split the online learning problem into two modified subgames defined as follows.

Simultaneous learning game: The environment mainly follows the same protocol as before, but the learner selects a meta-policy $(\pi_t^\theta)_{\theta \in \Theta} \in \Pi^{|\Theta|}$ in every round instead. Essentially, the learner is allowed to choose a dedicated policy for *each* model subset $\theta \in \Theta$. However, the learner loses control over the observation policy π_t . The regret is only evaluated on the choice of $\pi_t^{\theta^*}$, where θ^* is the unknown ground truth decided by the environment at the beginning (see Eq. (3)).

Informed comparator game: At the beginning of the game, the environment still secretly decides $\theta^* \in \Theta$. In each round, additionally to deciding $M_t \in \theta^*$, the environment also chooses $(\pi_t^\theta)_{\theta \in \Theta} \in \Pi^{|\Theta|}$, which is revealed to the learner before they make their decision π_t . The comparator in round t is $\pi_t^{\theta^*}$.

These games are motivated by the following regret decomposition.

$$\mathbb{E} [\text{Reg}(\pi^*)] = \underbrace{\mathbb{E} \left[\sum_{t=1}^T V_{M_t}(\pi^*) - V_{M_t}(\pi_t^{\theta^*}) \right]}_{\text{Regret in Simultaneous learning game}} + \underbrace{\mathbb{E} [\text{Reg}(\pi_{1:T}^{\theta^*})]}_{\text{Regret in Informed comparator game}}. \quad (3)$$

In general, the loss of control over the observation policy makes the *Simultaneous learning game* hard to solve. However, when the observation is independent of the observation policy, this game is strictly easier than the vanilla setting. We present a specific example at the end of the section.

Let us first show that the *Informed comparator game* can be solved independently of the size of the policy space. Denote $\pi^\Theta = (\pi^\theta)_{\theta \in \Theta}$ and define for a distribution $\nu \in \Delta(\mathcal{M})$

$$\text{InfoAIR}_{\rho, \eta}^\Theta(p, \nu, \pi^\Theta) = \mathbb{E}_{\pi \sim p} \mathbb{E}_{\theta^* \sim \nu} \mathbb{E}_{M \sim \nu(\cdot | \theta^*)} \mathbb{E}_{o \sim M(\cdot | \pi)} \left[V_M(\pi^{\theta^*}) - V_M(\pi) - \frac{1}{\eta} D_{\text{KL}}(\nu_\theta(\cdot | \pi, o), \rho) \right].$$

The following theorem refers to the use of Algorithm 2.

Theorem 7 *In the informed comparator setting, there exists an algorithm with regret*

$$\begin{aligned} \mathbb{E} [\text{Reg}(\pi_{1:T}^{\theta^*})] &\leq \frac{\log |\Theta|}{\eta} + T \max_{\pi^\Theta \in \Pi^\Theta} \max_{\rho \in \Delta(\Theta)} \max_{\nu \in \Delta(\mathcal{M})} \min_{p \in \Delta(\Pi)} \text{InfoAIR}_{\rho, \eta}^\Phi(p, \nu, \pi^\Theta) \\ &= \frac{\log |\Theta|}{\eta} + T \cdot \text{DEC}_\eta^{\text{KL}}(\overline{\mathcal{M}}(\Theta)). \end{aligned}$$

Since $\overline{\mathcal{M}}(\Theta) = \overline{\mathcal{M}}(\Phi)$, this is the same bound as Theorem 2, except that we only suffer the penalty $\log(|\Theta|)$ instead of $\log(|\Theta||\Pi|)$.

Finally, let us present a case in which the *Simultaneous learning game* is efficiently solvable. We define full-information feedback such that the learner receives a candidate model M_t^θ for every $\theta \in \Theta$ after round t , with the guarantee that $M_t = M_t^{\theta^*}$.

Algorithm 2 General Algorithm with Informed Comparator

Let $\rho_1(\theta) = \frac{1}{|\Theta|}$ for all $\theta \in \Theta$.

for $t = 1, 2, \dots, T$ **do**

 Receive $\pi_t^\Theta = (\pi_t^\theta)_{\theta \in \Theta}$ from the environment.

 Find a distribution p_t of π and a distribution ν_t of \mathcal{M} that solve the saddle-point of

$$\min_{p \in \Delta(\Pi)} \max_{\nu \in \Delta(\mathcal{M})} \text{InfoAIR}_{\rho_t, \eta}^\Theta(p, \nu, \pi_t^\Theta)$$

 Sample decision $\pi_t \sim p_t$ and observe $o_t \sim M_t(\cdot | \pi_t)$.

 Update $\rho_{t+1}(\theta) = \nu_t(\theta | \pi_t, o_t)$ for all $\theta \in \Theta$.

end

Theorem 8 *Let Θ describe an environment over H -staged MDPs with action set \mathcal{A} and full-information reward feedback. For any $\gamma > 0$ there exists an algorithm in the Simultaneous learning game with regret bounded by $\left(\frac{\log |\mathcal{A}|}{\gamma} + \gamma T\right) H$.*

In H -staged MDPs with state set \mathcal{S} , the policy space we consider can be as large as $|\Pi| = |\mathcal{A}|^{|\mathcal{S}|}$. In cases where \mathcal{S} is very large, i.e. $|\mathcal{S}| \gg \max\{|\mathcal{A}|, |\Theta|\}$, combining [Theorem 7](#) and [Theorem 8](#) yield significantly better bounds than applying [Theorem 2](#) directly.

4. Model-Based RL in Adversarial MDPs with Fixed Transition

In this section, we consider the setting where the reward is adversarially chosen but the transition is fixed. We decompose the model space $\mathcal{M} = \mathcal{R} \times \mathcal{P}$ where \mathcal{R} is the reward space and \mathcal{P} is the transition space.

This corresponds to the partition $\Phi_A = \{\phi_{\pi^*, P^*} : \pi^* \in \Pi, P^* \in \mathcal{P}\}$ where $\phi_{\pi^*, P^*} = \{((P^*, R), \pi^*) : R \in \mathcal{R}\}$. Moreover, now $\Psi_A = \left(\bigcup_{\phi \in \Phi_A} \phi\right) = \mathcal{M} \times \Pi$. By definition, we have $|\Phi_A| = |\Pi| \cdot |\mathcal{P}|$ and $\overline{\mathcal{M}}(\Phi_A) = \mathcal{P} \times \text{co}(\mathcal{R})$. Note that for typical applications, \mathcal{R} is convex and hence $\overline{\mathcal{M}}(\Phi_A) = \mathcal{M}$.

Corollary 9 *If the reward space \mathcal{R} is convex, there exists an algorithm for the hybrid problem with regret*

$$\mathbb{E}[\text{Reg}(\pi^*)] \leq \frac{\log(|\Pi||\mathcal{P}|)}{\eta} + T \cdot \text{DEC}_\eta^{\text{KL}}(\mathcal{M}).$$

The proof follows directly from combining [Theorem 2](#) and [Lemma 5](#). Furthermore, when there is full-information reward feedback,

Corollary 10 *If the reward space \mathcal{R} is convex and the learner observe R_t after round t , then there exists an algorithm with regret*

$$\mathbb{E}[\text{Reg}(\pi^*)] \leq \left(\frac{\log(|\mathcal{A}|)}{\gamma} + \gamma T\right) H + \frac{\log |\mathcal{P}|}{\eta} + T \cdot \text{DEC}_\eta^{\text{KL}}(\mathcal{M}).$$

This is a direct corollary of [Theorem 7](#), [Theorem 8](#) and [Lemma 5](#).

Example For the model-based low-rank MDP setting with fixed transition and adversarial linear rewards studied in [Liu et al. \(2024\)](#), these corollaries directly improve the best-known regret guarantees from $T^{\frac{2}{3}}$ to \sqrt{T} for both full-information and bandit feedback.

5. Model-Free RL in Stochastic MDPs

In this section, we apply the general framework to model-free RL in stochastic MDPs. In model-free RL with value function approximation, the learner is provided with a function set \mathcal{F} that contains possible value functions of the world. We define Φ as $\Phi = \{\phi_f : f \in \mathcal{F}\}$ where $\phi_f = \{(M, \pi_M) : M \text{ induces } f\}$. With [Algorithm 1](#), we are able to obtain the regret bound

$$\mathbb{E} [\text{Reg}(\pi^*)] \leq \frac{\log |\mathcal{F}|}{\eta} + T \cdot \text{DEC}_\eta^{\text{KL}}(\overline{\mathcal{M}}(\Phi)).$$

In fact, model-free learning in stochastic MDPs is easier than the Φ -restricted environment defined in [Definition 1](#), as the adversary cannot modify the underlying model in every round. However, the general framework offers a means to reduce estimation complexity from $\log |\mathcal{M}|$ to $\log |\mathcal{F}|$. If we can show that aggregation does not significantly increase decision complexity, this approach could provide a viable path to improving the regret bound.

Existing literature has identified general classes where model-free RL allows for $\log |\mathcal{F}|$ estimation complexity and bounded decision complexity. One of the most general classes is the *bilinear class* ([Du et al., 2021](#)), where the *bilinear rank* serves as the decision complexity. While we are currently unable to relate $\text{DEC}_\eta^{\text{KL}}(\overline{\mathcal{M}}(\Phi))$ to bilinear rank in full generality, we identify special cases where such a connection holds. Specifically, in the linear Q^*/V^* setting—a subclass of bilinear class—we show that $\text{DEC}_\eta^{\text{KL}}(\overline{\mathcal{M}}(\Phi))$ can be upper bounded by the bilinear rank. We formally define the linear Q^*/V^* setting as the following:

Definition 11 A class of MDPs satisfies linear Q^*/V^* if there are known feature vectors $\varphi : \mathcal{S} \times \mathcal{A} \rightarrow \mathbb{R}^d$ and $\psi : \mathcal{S} \rightarrow \mathbb{R}^d$ and known function class $\mathcal{F} = \{(\theta, w)\} \subset \mathbb{R}^{d+d}$ such that $Q^*(s, a) = \varphi(s, a)^\top \theta^*$ and $V^*(s) = \psi(s)^\top w^*$ for some $(\theta^*, w^*) \in \mathcal{F}$.

We assume $|\mathcal{F}|$ is finite for simplicity. It is straightforward to extend the result to the case where \mathcal{F} lies in a bounded region in \mathbb{R}^{d+d} . With this, we let $\Phi = \{\phi_{\theta, w} : (\theta, w) \in \mathcal{F}\}$, where

$$\phi_{\theta, w} = \{(M, \pi_M) : Q^*(s, a; M) = \varphi(s, a)^\top \theta, \quad V^*(s; M) = \psi(s)^\top w\}.$$

Then we have the following bound for $\text{DEC}_\eta^{\text{KL}}(\overline{\mathcal{M}}(\Phi))$.

Theorem 12 In the linear Q^*/V^* setting, $\text{DEC}_\eta^{\text{KL}}(\overline{\mathcal{M}}(\Phi)) \leq 4\eta d H^2$.

From [Theorem 12](#), we can get \sqrt{T} regret for linear Q^*/V^* MDPs for the first time. Previous works on this setting either only get sample complexity bound ([Du et al., 2021](#)) or require additional Bellman completeness assumption ([Jin et al., 2021](#)). Establishing similar results for general bilinear classes or other existing complexity measures for model-free RL is left as future work. We provide more discussions in [Section 7](#).

6. Model-Free RL in Adversarial MDPs with Fixed Transition and Full Information

In this section, we study model-free RL in the hybrid setting with full-information reward feedback. The algorithm used in this section deviates from the general algorithm in [Algorithm 1](#), but it is still based on the idea of learning over partitions of $\mathcal{M} \times \Pi$. To the best of our knowledge, there is no general value function approximation scheme defined for the hybrid setting in the literature. Therefore, we propose the following framework.

Definition 13 (Value function approximation for the hybrid setting) *Assume for each policy $\pi \in \Pi$, there is a function class \mathcal{F}^π . Each function $f \in \mathcal{F}^\pi$ is a ternary function mapping $\mathcal{S} \times \mathcal{A} \times \mathcal{R} \rightarrow \mathbb{R}$. We say a transition P induces $f \in \mathcal{F}^\pi$ if $Q^\pi(s, a; (P, R)) = f(s, a, R)$ ³ for any s, a, R . With abuse of notation, for any $\pi \in \Pi$ and $f \in \mathcal{F}^\pi$, we write $f(\pi; R) = f(s_0, \pi(s_0), R)$, indicating that it is the value of policy π predicted by function f under reward R . Define $\mathcal{F} = \cup_{\pi \in \Pi} \mathcal{F}^\pi$. We assume $f(\pi, R) \in [0, 1]$ for any f, π, R , which matches the assumption on value function V_M .*

Compared to standard value function approximation for the stochastic setting (e.g., [Du et al. \(2021\)](#)), the framework in [Definition 13](#) is different in two ways: 1) In the usual stochastic setting, it is a full model $M = (P, R)$ that underlies (or induces) a function. In other words, given a model (P, R) , there is a unique function f corresponding to it. However, in the hybrid setting, it is a transition P that induces a function. The reward R , on the other hand, is an input to the function. 2) In the usual stochastic setting, the functions only represent the optimal Q -functions, while in the hybrid setting, the functions represent Q -functions of all policies. This is also natural because even under a fixed transition, every policy has the possibility to become the optimal policy when the reward can vary.

In the model-free hybrid setting where the learner is equipped with a function class \mathcal{F} described in [Definition 13](#), we make the partition be $\Phi = \{\phi_{\pi, f} : \pi \in \Pi, f \in \mathcal{F}^\pi\}$, where $\phi_{\pi, f} = \{(P, R, \pi) : P \text{ induces } f, R \in \mathcal{R}\}$ for any $\pi \in \Pi$ and $f \in \mathcal{F}^\pi$. Since every partition corresponds to a single f and policy π , for any partition $\phi \in \Phi$, we denote its corresponding f as f^ϕ , and π as π^ϕ .

Note that, unlike the applications in previous sections, here we have a *policy-dependent* partition over models. This is why, in [Section 3](#), we construct Φ based on the joint model and policy class rather than solely on the model class.

Next, we further make structural assumptions on the model and function classes. Specifically, we provide a natural extension of the bilinear class ([Du et al., 2021](#)) to the hybrid setting described in [Definition 13](#).

Definition 14 (Bilinear class for the hybrid setting) *A hybrid function class \mathcal{F} (defined in [Definition 13](#)) induced by a model class $\mathcal{M} = \mathcal{P} \times \mathcal{R}$ has bilinear rank d if there exists functions $X_h : \Pi \times \mathcal{P} \rightarrow \mathbb{R}^d$ and $W_h : \mathcal{F} \times \mathcal{R} \times \mathcal{P} \rightarrow \mathbb{R}^d$ for all $h \in [H]$ such that for any $\pi \in \Pi$, $f \in \mathcal{F}^\pi$, $R \in \mathcal{R}$, and $P \in \mathcal{P}$, we have*

$$|f(\pi; R) - V_{P, R}(\pi)| \leq \sum_{h=1}^H |\langle X_h(\pi; P), W_h(f, R; P) \rangle|.$$

3. Similar to [Foster et al. \(2023c\)](#), for the sake of simplicity, we adopt a less general definition where “functions” only represent Q -functions. The results can be readily extended to more general settings where functions may additionally represent e.g. V -functions.

Moreover, there exists an estimation policy mapping function $\text{est} : \Pi \rightarrow \Pi$, and for every $h \in [H]$, there exists a discrepancy function $\ell_h^{\text{est}} : \mathcal{F} \times \mathcal{O} \times \mathcal{R} \rightarrow \mathbb{R}$ such that for any $\pi \in \Pi$, $f \in \mathcal{F}^\pi$, and $f' \in \mathcal{F}$,

$$|\langle X_h(\pi; P), W_h(f', R; P) \rangle| = \left| \mathbb{E}^{\pi \circ_h \text{est}(\pi), P} [\ell_h^{\text{est}}(f', o_h, R)] \right|$$

where \mathcal{O} is the space of per-step observation (s, a, s') , and $o_h = (s_h, a_h, s_{h+1})$. $\mathbb{E}^{\pi, P}[\cdot]$ denotes the expectation over the occupancy measure generated by policy π and transition P . Let $\pi \circ_h \text{est}(\pi)$ denote playing π for the first $h - 1$ steps and play policy $\text{est}(\pi)$ at the h -th step.

Definition 14 encompasses numerous interesting instances, such as low-rank MDPs and low-occupancy MDPs (Du et al., 2021). Their learnability in the hybrid setting remains largely unexplored. Further discussion on these specific instances can be found in Appendix D.

We will show that MDPs that satisfy Definition 14 is learnable in the hybrid setting with model-free guarantees. To emphasize more on the idea of partition, we use the partition notation to describe our Algorithm 3. For partition $\phi = \phi_{\pi, f}$ where $\pi \in \Pi$ and $f \in \mathcal{F}^\pi$, we write $X_h(\phi; P) = X_h(\pi; P)$, $W_h(\phi; P) = W_h(f, R; P)$, $\ell_h^{\text{est}}(\phi, o_h, R) = \ell_h^{\text{est}}(f, o_h, R)$, and $\pi_{\text{est}}^\phi = \text{est}(\pi)$. We define the bilinear divergence

$$D_{bi}^\pi(\phi || P, R) = \sum_{h=1}^H \left(\mathbb{E}^{\pi, P} [\ell_h^{\text{est}}(\phi, o_h, R)] \right)^2.$$

For this setting, we devise Algorithm 3, which is inspired by Foster et al. (2023c) who studied the problem in the stochastic setting. The episodes are divided into epochs. At the beginning of each epoch, a reference distribution $\rho \in \Delta(\Phi)$ is selected based on optimistic posterior sampling (Zhang, 2022) with estimated bilinear divergence (Eq. (4)). Subsequently, the algorithm computes a behavior policy distribution p_t through a minimax optimization Eq. (5) analogous to the DEC objective defined in Eq. (1). However, unlike the standard DEC that uses the optimal value $V_M(\pi_M)$ as the comparator where M is the worst-case model, Eq. (5) adopts $f^\phi(\pi^\phi, R)$ as the comparator, where ϕ is drawn from the optimistic posterior determined in Eq. (4). As in Foster et al. (2023c), such change of comparator enables the effective use of optimism to derive improved model-free guarantee. With the policy distribution obtained from Eq. (5), a policy is sampled from it and executed throughout the entire epoch to gather data for the next iteration. The guarantee of Algorithm 3 is presented in Theorem 15. In Appendix D.2, besides proving Theorem 15, we introduce the optimistic DEC notion, which is an extension from Foster et al. (2023c) to the hybrid setting. The result in Theorem 15 can thus also be generalized to learnable classes under optimistic DEC.

Theorem 15 For the hybrid bilinear class defined in Definition 14, Algorithm 3 ensures with probability at least $1 - \delta$:

- If for all $\pi \in \Pi$, $\text{est}(\pi) = \pi$, then $\text{Reg} \leq O(\sqrt{d \log(|\mathcal{F}||\Pi|/\delta)} T^{\frac{3}{4}})$.
- If there exists $\pi \in \Pi$ such that $\text{est}(\pi) \neq \pi$, then $\text{Reg} \leq O((d \log(|\mathcal{F}||\Pi|/\delta))^{\frac{1}{3}} T^{\frac{5}{6}})$.

Algorithm 3 Model-free learning for the hybrid setting with full information

Input: Partition set $\Phi = \{\phi_{\pi,f} : \pi \in \Pi, f \in \mathcal{F}^\pi\}$, $p_1(\phi) = \frac{1}{|\Phi|}$, $\forall \phi \in \Phi$, epoch length τ , and learning rate γ, η .

for epoch $k = 1, 2, \dots, \frac{T}{\tau}$ **do**

 Sample $\pi_k \sim p_k$.

for $i = 1, \dots, \tau$ **do**

 Execute π_k , obtain trajectory $(o_{k,1}^i, \dots, o_{k,H}^i)$ where $o_{k,h}^i = (s_{k,h}^i, a_{k,h}^i, s_{k,h+1}^i)$, and observe the full reward function R_k^i .

end

 Define $R_k = \frac{1}{\tau} \sum_{i \in [\tau]} R_k^i$ as the average reward in the k -th epoch.

 Compute $\rho_{k+1} \in \Delta(\Phi)$ as

$$\rho_{k+1}(\phi) \propto \rho_k(\phi) \exp \left(\gamma \eta f^\phi(\pi^\phi; R_k) - \gamma \sum_{h=1}^H \left(\frac{1}{\tau} \sum_{i \in [\tau]} \ell_h^{est}(\phi, o_{k,h}^i, R_k) \right)^2 \right). \quad (4)$$

 Solve the following minimax optimization for p_{k+1}

$$p_{k+1} = \min_{p \in \Delta(\Pi)} \max_{(P,R) \in \mathcal{M}} \mathbb{E}_{\pi' \sim p} \mathbb{E}_{\phi \sim \rho_{k+1}} \left[f^\phi(\pi^\phi; R) - V_{(P,R)}(\pi') - \frac{1}{8\eta} D_{bi}^{\pi'}(\phi || P, R) \right]. \quad (5)$$

end

7. Discussions and Future Work

In this work, we showed that the general framework developed in [Section 3](#) is able to characterize the statistical complexity of hybrid MDPs with fixed transitions and adversarial rewards, and leads to near-optimal regret under model-based learning ([Section 4](#)).

The framework also extends to model-free value learning, as shown in [Section 5](#) and [Section 6](#). However, additional challenges arise as it does not fully recover the result for the general bilinear class in the stochastic regime, and requires modifying the objective to an alternative DEC formulation in the hybrid regime. The latter poses a barrier to extending the framework to the model-free bandit feedback case. One future direction would be to more deeply investigate the role of the divergence measure in AIR^Φ and see if there are better alternatives for model-free value learning.

We believe that the framework has the potential to characterize the complexity of more settings that lie between model learning and policy learning. Another avenue for future work is leveraging the framework we provide to derive efficient algorithms that do not have to maintain memory expensive priors or solve computationally complex saddle-point problems.

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Appendix A. Proof of Theorem 2

Proof of Theorem 2

$$\begin{aligned}
 & \mathbb{E}[\text{Reg}(\pi_{1:T}^*)] - \frac{\log |\Phi|}{\eta} \\
 & \leq \mathbb{E} \left[\sum_{t=1}^T \left(V_{M_t}(\pi_t^*) - V_{M_t}(\pi_t) - \frac{1}{\eta} \log \frac{\rho_{t+1}(\phi^*)}{\rho_t(\phi^*)} \right) \right] \\
 & = \mathbb{E} \left[\sum_{t=1}^T \mathbb{E}_{\pi \sim p_t} \mathbb{E}_{o \sim M_t(\cdot|\pi)} \left[V_{M_t}(\pi_t^*) - V_{M_t}(\pi) - \frac{1}{\eta} \log \frac{\nu_t(\phi^*|\pi, o)}{\rho_t(\phi^*)} \right] \right] \\
 & \leq \mathbb{E} \left[\sum_{t=1}^T \max_{\mu \in \Delta(\Psi)} \mathbb{E}_{\phi^* \sim \mu} \mathbb{E}_{(M, \pi^*) \sim \mu(\cdot|\phi^*)} \mathbb{E}_{\pi \sim p_t} \mathbb{E}_{o \sim M(\cdot|\pi)} \left[V_M(\pi^*) - V_M(\pi) - \frac{1}{\eta} \log \frac{\nu_t(\phi^*|\pi, o)}{\rho_t(\phi^*)} \right] \right] \\
 & = \mathbb{E} \left[\sum_{t=1}^T \max_{\nu \in \Delta(\Psi)} \mathbb{E}_{\pi \sim p_t} \mathbb{E}_{(M, \pi^*) \sim \nu} \mathbb{E}_{o \sim M(\cdot|\pi)} \left[V_M(\pi^*) - V_M(\pi) - \frac{1}{\eta} D_{\text{KL}}(\nu_\phi(\cdot|\pi, o), \rho_t) \right] \right] \\
 & \hspace{25em} \text{(Lemma 24)} \\
 & = \mathbb{E} \left[\sum_{t=1}^T \min_{p \in \Delta(\Pi)} \max_{\nu \in \Delta(\Psi)} \mathbb{E}_{\pi \sim p} \mathbb{E}_{(M, \pi^*) \sim \nu} \mathbb{E}_{o \sim M(\cdot|\pi)} \left[V_M(\pi^*) - V_M(\pi) - \frac{1}{\eta} D_{\text{KL}}(\nu_\phi(\cdot|\pi, o), \rho_t) \right] \right] \\
 & = \mathbb{E} \left[\sum_{t=1}^T \min_{p \in \Delta(\Pi)} \max_{\nu \in \Delta(\Psi)} \text{AIR}_{\rho_t, \eta}^\Phi(p, \nu) \right] \\
 & = \mathbb{E} \left[\sum_{t=1}^T \max_{\nu \in \Delta(\Psi)} \min_{p \in \Delta(\Pi)} \text{AIR}_{\rho_t, \eta}^\Phi(p, \nu) \right] \\
 & \hspace{15em} (\text{AIR}_{\rho_t, \eta}^\Phi(p, \nu) \text{ is convex in } p \text{ and concave in } \nu \text{ and Lemma 21}) \\
 & \leq T \max_{\rho \in \Delta(\Phi)} \max_{\nu \in \Delta(\Psi)} \min_{p \in \Delta(\Pi)} \text{AIR}_{\rho, \eta}^\Phi(p, \nu).
 \end{aligned}$$

■

Appendix B. Relation with DEC

We introduce a useful notion of a DEC like complexity that equivalently describes the complexity of AIR^Φ .

Definition 16 *The Φ -dependent DEC is defined as*

$$\begin{aligned}
 & \text{DEC}_\eta^{\text{KL}}(\mathcal{M}, \Phi) = \\
 & \max_{\overline{M} \in \text{co}(\mathcal{M})} \min_{p \in \Delta(\Pi)} \max_{\phi \in \Phi} \max_{\nu \in \Delta(\phi)} \mathbb{E}_{\pi \sim p} \mathbb{E}_{\phi \sim \nu} \left[\mathbb{E}_{(M, \pi^*) \sim \nu} [V_M(\pi^*)] - V_{M^\nu}(\pi) - \frac{1}{\eta} D_{\text{KL}}(M^\nu(\cdot|\pi), \overline{M}(\cdot|\pi)) \right].
 \end{aligned}$$

We first establish that the maxmin of AIR^Φ is bounded by the new notion of DEC.

Lemma 17 $\max_{\rho \in \Delta(\Phi)} \max_{\nu \in \Delta(\Psi)} \min_{p \in \Delta(\Pi)} \text{AIR}_{\rho, \eta}^{\Phi}(p, \nu) = \text{DEC}_{\eta}^{\text{KL}}(\mathcal{M}, \Phi).$

Proof of Lemma 17

$$\begin{aligned} \text{AIR}_{\rho, \eta}^{\Phi}(p, \nu) &= \mathbb{E}_{\pi \sim p} \mathbb{E}_{(M, \pi^*) \sim \nu} \mathbb{E}_{o \sim M(\cdot|\pi)} \left[V_M(\pi^*) - V_M(\pi) - \frac{1}{\eta} D_{\text{KL}}(\nu_{\phi}(\cdot|\pi, o), \rho) \right] \\ &= \mathbb{E}_{\pi \sim p} \mathbb{E}_{(M, \pi^*) \sim \nu} \mathbb{E}_{o \sim M(\cdot|\pi)} \left[V_M(\pi^*) - V_M(\pi) - \frac{1}{\eta} D_{\text{KL}}(\nu_{\phi}(\cdot|\pi, o), \nu_{\phi}) - \frac{1}{\eta} D_{\text{KL}}(\nu_{\phi}, \rho) \right] \end{aligned}$$

is jointly concave over (ν, ρ) , hence by Lemma 21

$$\max_{\rho \in \Delta(\Phi)} \max_{\nu \in \Delta(\Psi)} \min_{p \in \Delta(\Pi)} \text{AIR}_{\rho, \eta}^{\Phi}(p, \nu) = \min_{p \in \Delta(\Pi)} \max_{\nu \in \Delta(\Psi)} \max_{\rho \in \Delta(\Phi)} \text{AIR}_{\rho, \eta}^{\Phi}(p, \nu).$$

Let's assume ν and π are fixed, we have for the ρ -dependent part

$$\max_{\rho \in \Delta(\Phi)} \mathbb{E}_{(M, \pi^*) \sim \nu} \mathbb{E}_{o \sim M(\cdot|\pi)} [-D_{\text{KL}}(\nu_{\phi}, \rho)] = 0,$$

with the argmax at $\rho = \nu_{\phi}$, hence

$$\max_{\rho \in \Delta(\Phi)} \text{AIR}_{\rho, \eta}^{\Phi}(p, \nu) = \text{AIR}_{\nu_{\phi}, \eta}^{\Phi}(p, \nu).$$

We continue with the KL term in this expression. In an overload of notation, let ν conditioned on π define a distribution over $\Delta(\Psi) \times \mathcal{O}$ via $\nu(o, M, \pi^*|\pi) = \nu(M, \pi^*)M(o|\pi)$. Note that for fixed π ,

$$\begin{aligned} \mathbb{E}_{(M, \pi^*) \sim \nu} \mathbb{E}_{o \sim M(\cdot|\pi)} D_{\text{KL}}(\nu_{\phi}(\cdot|\pi, o), \nu_{\phi}) &= \mathbb{E}_{o \sim \nu_{\mathcal{O}}(\cdot|\pi)} D_{\text{KL}}(\nu_{\phi}(\cdot|\pi, o), \nu_{\phi}) \\ &= \mathbb{E}_{o \sim \nu_{\mathcal{O}}(\cdot|\pi)} \mathbb{E}_{\phi \sim \nu_{\phi}(\cdot|\pi, o)} \log \left(\frac{\nu(\phi|\pi, o)}{\nu(\phi)} \right) \\ &= \mathbb{E}_{\phi \sim \nu_{\phi}} \mathbb{E}_{o \sim \nu_{\mathcal{O}}(\cdot|\phi, \pi)} \log \left(\frac{\nu(\phi|\pi, o)}{\nu(\phi)} \right) \\ &= \mathbb{E}_{\phi \sim \nu_{\phi}} \mathbb{E}_{o \sim \nu_{\mathcal{O}}(\cdot|\phi, \pi)} \log \left(\frac{\nu(o|\phi, \pi)}{\nu(o|\pi)} \right) \quad (\text{Bayes-rule}) \\ &= \mathbb{E}_{\phi \sim \nu} D_{\text{KL}}(\nu_{\mathcal{O}}(\cdot|\phi, \pi), \nu_{\mathcal{O}}(\cdot|\pi)). \end{aligned}$$

We now consider $\mu \in \Delta(\mathcal{M})$, which again is overloaded to denote a distribution over $\Delta(\mathcal{M}) \times \mathcal{O}$.

$$\begin{aligned} -\mathbb{E}_{\phi \sim \nu} D_{\text{KL}}(\nu_{\mathcal{O}}(\cdot|\phi, \pi), \nu_{\mathcal{O}}(\cdot|\pi)) &= \max_{\mu \in \Delta(\mathcal{M})} \mathbb{E}_{\phi \sim \nu} [-D_{\text{KL}}(\nu_{\mathcal{O}}(\cdot|\phi, \pi), \nu_{\mathcal{O}}(\cdot|\pi)) - D_{\text{KL}}(\nu_{\mathcal{O}}(\cdot|\pi), \mu_{\mathcal{O}}(\cdot|\pi))] \\ &= \max_{\mu \in \Delta(\mathcal{M})} \mathbb{E}_{\phi \sim \nu} [-D_{\text{KL}}(\nu_{\mathcal{O}}(\cdot|\phi, \pi), \mu_{\mathcal{O}}(\cdot|\pi))] \end{aligned}$$

Combining everything, we have shown that

$$\begin{aligned} &\max_{\rho \in \Delta(\Phi)} \max_{\nu \in \Delta(\Psi)} \min_{p \in \Delta(\Pi)} \text{AIR}_{\rho, \eta}^{\Phi}(p, \nu) \\ &= \min_{p \in \Delta(\Pi)} \max_{\nu \in \Delta(\Psi)} \max_{\mu \in \Delta(\mathcal{M})} \mathbb{E}_{\pi \sim p} \mathbb{E}_{(M, \pi^*) \sim \nu} \mathbb{E}_{o \sim M(\cdot|\pi)} \left[V_M(\pi^*) - V_M(\pi) - \frac{1}{\eta} D_{\text{KL}}(\nu_{\mathcal{O}}(\cdot|\phi, \pi), \mu_{\mathcal{O}}(\cdot|\pi)) \right]. \end{aligned}$$

Again this is jointly concave over (ν, μ) and convex concave in (p, ν) given μ , hence by [Lemma 21](#)

$$\begin{aligned}
 &= \max_{\mu \in \Delta(\mathcal{M})} \min_{p \in \Delta(\Pi)} \max_{\nu \in \Delta(\Psi)} \mathbb{E}_{\pi \sim p} \mathbb{E}_{(M, \pi^*) \sim \nu} \mathbb{E}_{o \sim M(\cdot|\pi)} \left[V_M(\pi^*) - V_M(\pi) - \frac{1}{\eta} D_{\text{KL}}(\nu_o(\cdot|\phi, \pi), \mu_o(\cdot|\pi)) \right] \\
 &= \max_{\overline{M} \in \text{CO}(\mathcal{M})} \min_{p \in \Delta(\Pi)} \max_{\nu \in \Delta(\Psi)} \mathbb{E}_{\pi \sim p} \mathbb{E}_{(M, \pi^*) \sim \nu} \mathbb{E}_{o \sim M(\cdot|\pi)} \left[V_M(\pi^*) - V_M(\pi) - \frac{1}{\eta} D_{\text{KL}}(M^{\nu(\cdot|\phi)}(\cdot|\pi), \overline{M}(\cdot|\pi)) \right] \\
 &= \text{DEC}_{\eta}^{\text{KL}}(\mathcal{M}, \Phi).
 \end{aligned}$$

■

Proof of [Lemma 5](#) By [Lemma 17](#), we only need to compare $\text{DEC}_{\eta}^{\text{KL}}(\mathcal{M}, \Phi)$ and $\text{DEC}_{\eta}^{\text{KL}}(\overline{\mathcal{M}}(\Phi))$. Define

$$C(\Phi) := \max_{\phi \in \Phi} \max_{\nu \in \Delta(\phi)} \mathbb{E}_{(M, \pi^*) \sim \nu} [V_M(\pi^*)] - V_{M^{\nu}}(\pi_{M^{\nu}}),$$

where M^{ν} is the mixture model $M^{\nu}(\cdot|\pi) := \mathbb{E}_{M \sim \nu} [M(\cdot|\pi)]$, and $\pi_{M^{\nu}} := \arg\max_{\pi \in \Pi} V_{M^{\nu}}(\pi)$. In the fixed-comparator case, we have

$$\begin{aligned}
 &\max_{\phi \in \Phi} \max_{\nu \in \Delta(\phi)} \mathbb{E}_{(M, \pi^*) \sim \nu} [V_M(\pi^*)] - V_{M^{\nu}}(\pi_{M^{\nu}}) \\
 &= \max_{\phi \in \Phi} \max_{\nu \in \Delta(\phi)} V_{M^{\nu}}(\pi^{\phi}) - V_{M^{\nu}}(\pi_{M^{\nu}}) \leq 0.
 \end{aligned}$$

We have

$$\begin{aligned}
 &\max_{\phi \in \Phi} \max_{\nu \in \Delta(\phi)} \mathbb{E}_{\pi \sim p} \mathbb{E}_{\phi \sim \nu} \left[\mathbb{E}_{(M, \pi^*) \sim \nu} [V_M(\pi^*)] - V_{M^{\nu}}(\pi) - \frac{1}{\eta} D_{\text{KL}}(M^{\nu}(\cdot|\pi), \overline{M}(\cdot|\pi)) \right] \\
 &\leq \max_{\phi \in \Phi} \max_{\nu \in \Delta(\phi)} \mathbb{E}_{\pi \sim p} \mathbb{E}_{\phi \sim \nu} \left[V_{M^{\nu}}(\pi_{M^{\nu}}) - V_{M^{\nu}}(\pi) - \frac{1}{\eta} D_{\text{KL}}(M^{\nu}(\cdot|\pi), \overline{M}(\cdot|\pi)) \right] + C(\Phi) \\
 &= \max_{M^{\nu} \in \overline{\mathcal{M}}(\Phi)} \mathbb{E}_{\pi \sim p} \mathbb{E}_{\phi \sim \nu} \left[V_{M^{\nu}}(\pi_{M^{\nu}}) - V_{M^{\nu}}(\pi) - \frac{1}{\eta} D_{\text{KL}}(M^{\nu}(\cdot|\pi), \overline{M}(\cdot|\pi)) \right] + C(\Phi)
 \end{aligned}$$

Hence

$$\text{DEC}_{\eta}^{\text{KL}}(\mathcal{M}, \Phi) \leq \text{DEC}_{\eta}^{\text{KL}}(\overline{\mathcal{M}}(\Phi)) + C(\Phi).$$

If $C(\Phi) \leq 0$, which includes all fixed-comparator environments, this implies the Lemma.

Finally, assume $C(\Phi) > 0$. We show that these environments are unlearnable and that $\text{DEC}_{\eta}^{\text{KL}}(\mathcal{M}, \Phi) = \Omega(1)$ independent of η . Fix $(\phi', \nu') = \arg\max_{\phi \in \Phi, \nu \in \Delta(\phi)} \mathbb{E}_{(M, \pi^*) \sim \nu} [V_M(\pi^*)] - V_{M^{\nu}}(\pi_{M^{\nu}})$.

$$\text{DEC}_{\eta}^{\text{KL}}(\mathcal{M}, \Phi)$$

$$\begin{aligned}
 &\geq \min_{p \in \Delta(\Pi)} \max_{\phi \in \Phi} \max_{\nu \in \Delta(\phi)} \mathbb{E}_{\pi \sim p} \mathbb{E}_{\phi \sim \nu} \left[\mathbb{E}_{(M, \pi^*) \sim \nu} [V_M(\pi^*)] - V_{M^{\nu}}(\pi) - \frac{1}{\eta} D_{\text{KL}}(M^{\nu}(\cdot|\pi), M^{\nu'}(\cdot|\pi)) \right] \\
 &\geq \min_{p \in \Delta(\Pi)} \mathbb{E}_{\pi \sim p} \left[\mathbb{E}_{(M, \pi^*) \sim \nu'} [V_M(\pi^*)] - V_{M^{\nu'}}(\pi) - \frac{1}{\eta} D_{\text{KL}}(M^{\nu'}(\cdot|\pi), M^{\nu'}(\cdot|\pi)) \right]
 \end{aligned}$$

$$= C(\Phi) = \Omega(1).$$

To show that the problem is indeed unlearnable, let the environment sample i.i.d. $(M_t, \pi_t^*) \sim \nu'$ in every round. For any policy of the agent, the expected regret in every round is at least $C(\Phi)$, so the expected regret is lower bounded by $C(\Phi)T$. \blacksquare

Proof of Lemma 6 Using Lemma 17, we only need to compare the two notions of DEC. Recall we consider the case where $\Phi = \Theta \times \Pi$. We have for fixed p and \overline{M}

$$\begin{aligned} & \max_{\nu \in \Delta(\Psi)} \mathbb{E}_{\pi \sim p} \mathbb{E}_{\phi \sim \nu} \mathbb{E}_{(M, \pi^*) \sim \nu(\cdot|\phi)} \left[V_M(\pi^*) - V_M(\pi) - \frac{1}{\eta} D_{\text{KL}} \left(M^{\nu(\cdot|\phi)}(\cdot|\pi), \overline{M}(\cdot|\pi) \right) \right] \\ &= \max_{\theta \in \Theta} \max_{\nu \in \Delta(\theta)} \max_{\mu \in \Delta(\Pi)} \mathbb{E}_{\pi \sim p} \mathbb{E}_{(M, \pi^*) \sim (\nu, \mu)} \left[V_M(\pi^*) - V_M(\pi) - \frac{1}{\eta} D_{\text{KL}} \left(M^{\nu}(\cdot|\pi), \overline{M}(\cdot|\pi) \right) \right] \\ &= \max_{\theta \in \Theta} \max_{\nu \in \Delta(\theta)} \mathbb{E}_{\pi \sim p} \mathbb{E}_{M \sim \nu} \left[V_M(\pi_{M^\nu}) - V_M(\pi) - \frac{1}{\eta} D_{\text{KL}} \left(M^\nu(\cdot|\pi), \overline{M}(\cdot|\pi) \right) \right] \\ &= \max_{M^\nu \in \overline{\mathcal{M}}(\Phi)} \mathbb{E}_{\pi \sim p} \left[V_{M^\nu}(\pi_{M^\nu}) - V_{M^\nu}(\pi) - \frac{1}{\eta} D_{\text{KL}} \left(M^\nu(\cdot|\pi), \overline{M}(\cdot|\pi) \right) \right]. \end{aligned}$$

Since the first and last line are equal, taking the max over \overline{M} and min over p is still equal, which shows the Lemma. \blacksquare

Lemma 18 Let Φ describe a trimmed and aggregated stochastic game, i.e. for all $\phi \in \Theta$ we have that $\phi = \{(M, \phi_\theta^*) | M \in \mathcal{M}_\phi\}$ and such that $\pi_M = \pi_\theta$ for all $M \in \mathcal{M}_\phi$.

Further denote $\overline{\Phi}$ the unaggregated version, $\overline{\Phi} := \{(M, \pi) | M \in \mathcal{M}_\phi, \phi \in \Phi, \pi \in \Pi\}$. Then $\text{DEC}_\eta^{\text{KL}}(\Phi) = \text{DEC}_\eta^{\text{KL}}(\overline{\Phi})$ and $\overline{\mathcal{M}}(\Phi) = \overline{\mathcal{M}}(\overline{\Phi})$.

Proof The second statement is trivial since the convexification is policy-unaware. For the first statement, observe that for any function $F : \bigcup_{\phi \in \Phi} \Delta(\mathcal{M}_\phi) \times \Pi \rightarrow \mathbb{R}$, such that for any $\pi_\phi^* = \arg\max_{\pi \in \Pi} F(\nu_\phi, \pi)$ for any $\nu_\phi \in \Delta(\phi)$, we have

$$\begin{aligned} \max_{(\mathcal{M}_\phi, \pi) \in \overline{\Phi}} \max_{\nu_\phi \in \Delta(\mathcal{M}_\phi)} F(\nu_\phi, \pi) &= \max_{\phi \in \Phi} \max_{\nu_\phi \in \Delta(\mathcal{M}_\phi)} \max_{\pi \in \Pi} F(\nu_\phi, \pi) \\ &= \max_{\phi \in \Phi} \max_{\nu_\phi \in \Delta(\mathcal{M}_\phi)} F(\nu_\phi, \pi_\phi^*). \end{aligned}$$

Finally, notice that $\text{DEC}_\eta^{\text{KL}}(\mathcal{M}, \Phi)$ for a fixed \overline{M} and π is exactly such a function F . Using the equality above shows the equality between the complexities. \blacksquare

Lemma 19 For fixed comparator games, i.e. $\Phi = \Theta \times \Pi$, we have

$$\max_{\nu \in \Delta(\mathcal{M})} \max_{\pi \in \Pi^\Theta} \text{InfoAIR}_{\rho_t, \eta}^\Theta(p, \nu, \pi_t^\Theta) = \max_{\nu \in \Delta(\Psi)} \text{AIR}_{\rho, \eta}^\Phi(p, \nu).$$

Proof The RHS can be seen as a function

$$\max_{\nu \in \Delta(\Psi)} \mathbb{E}_{\phi^* \sim \nu} \left[F(\nu_{\mathbf{M}}(\cdot | \phi^*), \pi^{\phi^*}; \rho, \eta, p) \right]. \quad (6)$$

For any $\theta \in \Theta$, define

$$\phi^*(\theta) := \operatorname{argmax}_{\phi \in \Theta \times \Pi} F(\nu_{\mathbf{M}}(\cdot | \phi), \pi^{\phi}; \rho, \eta, p).$$

The objective Eq. (6) is maximized when $\forall \phi \in \Phi$: $\nu_{\mathbf{M}}(\cdot | \phi) = \operatorname{argmax}_{\mu \in \Delta(\phi)} F(\mu_{\mathbf{M}}, \pi^{\phi}; \rho, \eta, p)$. Also, we never decrease the objective by shifting probability mass from $\phi = (\theta, \pi)$ towards $\phi^*(\theta)$ by the definition of $\phi^*(\theta)$. Hence

$$\begin{aligned} \max_{\nu \in \Delta(\Psi)} \text{AIR}_{\rho, \eta}^{\Phi}(p, \nu) &= \max_{\nu \in \Delta(\Psi)} \mathbb{E}_{\phi \sim \nu} \left[F(\nu_{\mathbf{M}}(\cdot | \phi), \pi^{\phi}; \rho, \eta, p) \right] \\ &= \max_{\mu \in \Delta(\mathcal{M})} \mathbb{E}_{\theta \sim \mu} \left[F(\mu(\cdot | \theta), \pi^{\phi^*(\theta)}; \rho, \eta, p) \right] \\ &= \max_{\mu \in \Delta(\mathcal{M})} \max_{\pi^{\Theta} \in \Pi^{\Theta}} \mathbb{E}_{\theta \sim \mu} \left[F(\mu(\cdot | \theta), \pi^{\theta}; \rho, \eta, p) \right] \\ &= \max_{\nu \in \Delta(\mathcal{M})} \max_{\pi^{\Theta} \in \Pi^{\Theta}} \text{InfoAIR}_{\rho, \eta}^{\Theta}(p, \nu, \pi^{\Theta}). \end{aligned}$$

■

Proof of Theorem 7 The proof is almost the same as Theorem 2 but changing π^* to the revealed π^{θ} and not requiring a joint distribution over $\Theta \times \Pi$. We give a full proof here for completion.

$$\begin{aligned} &\mathbb{E}[\text{Reg}(\pi_{1:T}^{\theta^*})] - \frac{\log |\Theta|}{\eta} \\ &\leq \mathbb{E} \left[\sum_{t=1}^T \left(V_{M_t}(\pi_t^{\theta^*}) - V_{M_t}(\pi_t) - \frac{1}{\eta} \log \frac{\rho_{t+1}(\theta^*)}{\rho_t(\theta^*)} \right) \right] \\ &= \mathbb{E} \left[\sum_{t=1}^T \mathbb{E}_{\pi \sim p_t} \mathbb{E}_{o \sim M_t(\cdot | \pi)} \left[V_{M_t}(\pi_t^{\theta^*}) - V_{M_t}(\pi) - \frac{1}{\eta} \log \frac{\nu_t(\theta^* | \pi, o)}{\rho_t(\theta^*)} \right] \right] \\ &\leq \mathbb{E} \left[\sum_{t=1}^T \max_{\mu \in \Delta(\mathcal{M})} \mathbb{E}_{\theta^* \sim \mu} \mathbb{E}_{M \sim \mu(\cdot | \theta^*)} \mathbb{E}_{\pi \sim p_t} \mathbb{E}_{o \sim M(\cdot | \pi)} \left[V_M(\pi_t^{\theta^*}) - V_M(\pi) - \frac{1}{\eta} \log \frac{\nu_t(\theta^* | \pi, o)}{\rho_t(\theta^*)} \right] \right] \\ &= \mathbb{E} \left[\sum_{t=1}^T \max_{\nu \in \Delta(\mathcal{M})} \mathbb{E}_{\pi \sim p_t} \mathbb{E}_{\theta \sim \nu} \mathbb{E}_{M \sim \nu(\cdot | \theta)} \mathbb{E}_{o \sim M(\cdot | \pi)} \left[V_M(\pi_t^{\theta}) - V_M(\pi) - \frac{1}{\eta} D_{\text{KL}}(\nu_{\theta}(\cdot | \pi, o), \rho_t) \right] \right] \\ &\quad \text{(Lemma 24)} \\ &= \mathbb{E} \left[\sum_{t=1}^T \min_{p \in \Delta(\Pi)} \max_{\nu \in \Delta(\mathcal{M})} \mathbb{E}_{\pi \sim p} \mathbb{E}_{\theta \sim \nu} \mathbb{E}_{M \sim \nu(\cdot | \theta)} \mathbb{E}_{o \sim M(\cdot | \pi)} \left[V_M(\pi_t^{\theta}) - V_M(\pi) - \frac{1}{\eta} D_{\text{KL}}(\nu_{\theta}(\cdot | \pi, o), \rho_t) \right] \right] \\ &= \mathbb{E} \left[\sum_{t=1}^T \min_{p \in \Delta(\Pi)} \max_{\nu \in \Delta(\mathcal{M})} \text{InfoAIR}_{\rho_t, \eta}^{\Theta}(p, \nu, \pi_t^{\Theta}) \right] \end{aligned}$$

$$\begin{aligned}
 &= \mathbb{E} \left[\sum_{t=1}^T \max_{\nu \in \Delta(\mathcal{M})} \min_{p \in \Delta(\Pi)} \text{InfoAIR}^\Theta_{\rho_t, \eta}(p, \nu, \pi_t^\Theta) \right] \quad (\text{Lemma 21}) \\
 &\leq T \max_{\pi^\Theta \in \Pi^\Theta} \max_{\rho \in \Delta(\Theta)} \max_{\nu \in \Delta(\mathcal{M})} \min_{p \in \Delta(\Pi)} \text{InfoAIR}^\Theta_{\rho, \eta}(p, \nu, \pi^\Theta).
 \end{aligned}$$

Finally, by Lemma 21 and Lemma 19, we have

$$\max_{\pi^\Theta \in \Pi^\Theta} \max_{\rho \in \Delta(\Theta)} \max_{\nu \in \Delta(\mathcal{M})} \min_{p \in \Delta(\Pi)} \text{InfoAIR}^\Theta_{\rho, \eta}(p, \nu, \pi^\Theta) = \max_{\rho \in \Delta(\Phi)} \max_{\nu \in \Delta(\Psi)} \min_{p \in \Delta(\Pi)} \text{AIR}^\Phi_{\rho, \eta}(p, \nu).$$

Using Lemma 6 and noting $\overline{\mathcal{M}}(\Phi) = \overline{\mathcal{M}}(\Theta)$ finishes the proof. \blacksquare

Proof of Theorem 8 Define the players (randomized) meta-policy as

$$\pi_t^\theta(a|s) \propto \exp \left(\gamma \sum_{i=1}^{t-1} Q^{\pi_i^\theta}(s, a; M_i^\theta) \right),$$

where $Q^\pi(\cdot, \cdot; M)$ is the Q function of policy π under model M . That this policy is computable relies crucially on the availability of M_t^θ for all $\theta \in \Theta$. We have

$$\begin{aligned}
 &\mathbb{E} \left[\sum_{t=1}^T \left(V_{M_t}(\pi^*) - V_{M_t}(\pi_t^{\theta^*}) \right) \right] \\
 &\leq \mathbb{E} \left[\mathbb{E}_s^{M_t^{\theta^*}, \pi^*} \left[\sum_{t=1}^T \sum_a \left(\pi^*(a|s) - \pi_t^{\theta^*}(a|s) \right) Q^{\pi_t^{\theta^*}}(s, a; M_t^{\theta^*}) \right] \right] \quad (\text{Lemma 22}) \\
 &\leq \mathbb{E} \left[\mathbb{E}_s^{M_t^{\theta^*}, \pi^*} \left[\frac{\log |\mathcal{A}|}{\gamma} + \gamma T \right] \right] \quad (Q^\pi(s, a; M) \leq 1 \text{ and Lemma 23}) \\
 &\leq \left(\frac{\log |\mathcal{A}|}{\gamma} + \gamma T \right) H. \quad (H\text{-stage MDP})
 \end{aligned}$$

\blacksquare

Appendix C. Omitted Proofs in Section 5

Proof of Theorem 12 Notice that this is a fixed comparator game where every ϕ has a single policy. By Lemma 6 and Lemma 17, it suffices to bound

$$\text{DEC}_\eta^{\text{KL}}(\overline{\mathcal{M}}(\Phi)) = \text{DEC}_\eta^{\text{KL}}(\mathcal{M}, \Phi).$$

Let $\pi^\phi, \theta^\phi, w^\phi$ denote the π, θ, w for the group $\phi \in \Phi$. Let p be the marginal distribution of ν . Then

$$\begin{aligned}
 &\mathbb{E}_{\pi \sim p} \mathbb{E}_{(M, \pi^*) \sim \nu} [V_M(\pi^*) - V_M(\pi)] \\
 &= \mathbb{E}_{(M, \pi) \sim \nu} \mathbb{E}_{(M', \pi') \sim \nu} [V_M(\pi) - V_{M'}(\pi)] \\
 &= \mathbb{E}_{\phi \sim \nu} \mathbb{E}_{(M, \pi) \sim \nu(\cdot|\phi)} \mathbb{E}_{(M', \pi') \sim \nu} [V_M(\pi^\phi) - V_{M'}(\pi^\phi)]
 \end{aligned}$$

$$\begin{aligned}
 &= \mathbb{E}_{\phi \sim \nu} \mathbb{E}_{(M, \pi) \sim \nu(\cdot|\phi)} \mathbb{E}_{(M', \pi') \sim \nu} \mathbb{E}^{\pi^\phi, M'} \left[\sum_{h=1}^H (Q_h(s_h, a_h; M) - r_h - V_{h+1}(s_{h+1}; M)) \right] \\
 &= \mathbb{E}_{\phi \sim \nu} \mathbb{E}_{\phi' \sim \nu} \mathbb{E}_{(M, \pi) \sim \nu(\cdot|\phi)} \mathbb{E}_{(M', \pi') \sim \nu(\cdot|\phi')} \mathbb{E}^{\pi^\phi, M'} \left[\sum_{h=1}^H (Q_h(s_h, a_h; M) - V_{h+1}(s_{h+1}; M) \right. \\
 &\quad \left. - Q_h(s_h, a_h; M') + V_{h+1}(s_{h+1}; M')) \right] \\
 &= \mathbb{E}_{\phi \sim \nu} \mathbb{E}_{\phi' \sim \nu} \mathbb{E}_{(M', \pi') \sim \nu(\cdot|\phi')} \mathbb{E}^{\pi^\phi, M'} \left[\sum_{h=1}^H \left\langle \varphi(s_h, a_h), \theta^\phi - \theta^{\phi'} \right\rangle - \left\langle \psi(s_{h+1}), w^\phi - w^{\phi'} \right\rangle \right] \\
 &= \mathbb{E}_{\phi \sim \nu} \mathbb{E}_{\phi' \sim \nu} \mathbb{E}_{(M', \pi') \sim \nu(\cdot|\phi')} \sum_{h=1}^H \left\langle X_h(\pi^\phi; M'), W(\phi) - W(\phi') \right\rangle \tag{7}
 \end{aligned}$$

where we define for all h

$$\begin{aligned}
 X_h(\phi; M') &= \mathbb{E}^{\pi^\phi, M'} [\varphi(s_h, a_h), -\psi(s_{h+1})], \\
 W(\phi) &= [\theta^\phi, w^\phi].
 \end{aligned}$$

Let M^μ denote the mixture model such that $M^\mu(\cdot|\pi) = \mathbb{E}_{M \sim \mu}[M(\cdot|\pi)]$. Furthermore, in this proof, we denote $M^{\nu(\cdot|\phi)}$ as M^ϕ . Then the last expression in Eq. (7) can be written as

$$\sum_{h=1}^H \mathbb{E}_{\phi \sim \nu} \mathbb{E}_{\phi' \sim \nu} \left\langle X_h(\phi; M^{\nu(\cdot|\phi')}), W(\phi) - W(\phi') \right\rangle = \sum_{h=1}^H \mathbb{E}_{\phi \sim \nu} \mathbb{E}_{\phi' \sim \nu} \left\langle X_h(\phi; M^{\phi'}), W(\phi) - W(\phi') \right\rangle. \tag{8}$$

For each h and ϕ' , define

$$\Sigma_h^{\phi'} = \mathbb{E}_{\phi \sim \nu} [X_h(\phi; M^{\phi'}) X_h(\phi; M^{\phi'})^\top].$$

Then we have

$$\begin{aligned}
 &\mathbb{E}_{\phi \sim \nu} \mathbb{E}_{\phi' \sim \nu} \left\langle X_h(\phi; M^{\phi'}), W(\phi) - W(\phi') \right\rangle \\
 &\leq \mathbb{E}_{\phi' \sim \nu} \left[\sqrt{\mathbb{E}_{\phi \sim \nu} [\|X_h(\phi; M^{\phi'})\|_{(\Sigma_h^{\phi'})^{-1}}^2]} \sqrt{\mathbb{E}_{\phi \sim \nu} [\|W(\phi) - W(\phi')\|_{\Sigma_h^{\phi'}}^2]} \right] \\
 &= \mathbb{E}_{\phi' \sim \nu} \left[\sqrt{2d} \sqrt{\mathbb{E}_{\phi \sim \nu} \mathbb{E}_{\tilde{\phi} \sim \nu} \left[\left\langle X_h(\tilde{\phi}; M^{\phi'}), W(\phi) - W(\phi') \right\rangle^2 \right]} \right]. \tag{9}
 \end{aligned}$$

We continue to bound the square term in Eq. (9):

$$\begin{aligned}
 &\mathbb{E}_{\phi \sim \nu} \mathbb{E}_{\tilde{\phi} \sim \nu} \left[\left\langle X_h(\tilde{\phi}; M^{\phi'}), W(\phi) - W(\phi') \right\rangle^2 \right] \\
 &= \mathbb{E}_{\phi \sim \nu} \mathbb{E}_{\tilde{\phi} \sim \nu} \left[\left(\mathbb{E}^{\pi^{\tilde{\phi}}, M^{\phi'}} [\varphi(s_h, a_h)^\top \theta^{\tilde{\phi}} - \psi(s_{h+1})^\top w^{\tilde{\phi}}] - \mathbb{E}^{\pi^{\tilde{\phi}}, M^{\phi'}} [\varphi(s_h, a_h)^\top \theta^{\phi'} - \psi(s_{h+1})^\top w^{\phi'}] \right)^2 \right]
 \end{aligned}$$

$$\begin{aligned}
 &= \mathbb{E}_{\phi \sim \nu} \mathbb{E}_{\tilde{\phi} \sim \nu} \left[\left(\mathbb{E}^{\pi^{\tilde{\phi}}, M^{\phi'}} \left[\varphi(s_h, a_h)^\top \theta^\phi - r_h - \psi(s_{h+1})^\top w^\phi \right] \right)^2 \right] \\
 &= \mathbb{E}_{\phi \sim \nu} \mathbb{E}_{\tilde{\phi} \sim \nu} \left[\left(\mathbb{E}^{\pi^{\tilde{\phi}}, M^{\phi'}} \left[\varphi(s_h, a_h)^\top \theta^\phi - r_h - \psi(s_{h+1})^\top w^\phi \right] \right. \right. \\
 &\quad \left. \left. - \underbrace{\mathbb{E}^{\pi^{\tilde{\phi}}, M^\phi} \left[\varphi(s_h, a_h)^\top \theta^\phi - r_h - \psi(s_{h+1})^\top w^\phi \right]}_{=0} \right)^2 \right] \\
 &\leq \mathbb{E}_{\phi \sim \nu} \mathbb{E}_{\tilde{\phi} \sim \nu} \left[D_{\text{TV}}^2 \left(M^{\phi'}(\cdot | \pi^{\tilde{\phi}}), M^\phi(\cdot | \pi^{\tilde{\phi}}) \right) \right] \\
 &\leq 2 \mathbb{E}_{\phi \sim \nu} \mathbb{E}_{\tilde{\phi} \sim \nu} \left[D_{\text{H}}^2 \left(M^{\phi'}(\cdot | \pi^{\tilde{\phi}}), M^\phi(\cdot | \pi^{\tilde{\phi}}) \right) \right].
 \end{aligned}$$

Combining this with [Eq. \(9\)](#), we get

$$\begin{aligned}
 &\mathbb{E}_{\phi \sim \nu} \mathbb{E}_{\phi' \sim \nu} \left\langle X_h(\phi; M^{\phi'}), W(\phi) - W(\phi') \right\rangle \\
 &\leq 2 \sqrt{d \mathbb{E}_{\phi' \sim \nu} \mathbb{E}_{\phi \sim \nu} \mathbb{E}_{\tilde{\phi} \sim \nu} \left[D_{\text{H}}^2 \left(M^{\phi'}(\cdot | \pi^{\tilde{\phi}}), M^\phi(\cdot | \pi^{\tilde{\phi}}) \right) \right]} \\
 &\leq 2 \sqrt{2d \mathbb{E}_{\phi' \sim \nu} \mathbb{E}_{\phi \sim \nu} \mathbb{E}_{\tilde{\phi} \sim \nu} \left[D_{\text{H}}^2 \left(M^{\nu(\cdot | \phi')}(\cdot | \pi^{\tilde{\phi}}), M^{\nu(\cdot | \pi^{\tilde{\phi}})} \right) + D_{\text{H}}^2 \left(M^{\nu(\cdot | \pi^{\tilde{\phi}})}, M^{\nu(\cdot | \phi)}(\cdot | \pi^{\tilde{\phi}}) \right) \right]} \\
 &\leq 4 \sqrt{d \mathbb{E}_{\phi \sim \nu} \mathbb{E}_{\tilde{\phi} \sim \nu} \left[D_{\text{H}}^2 \left(M^{\nu(\cdot | \phi)}(\cdot | \pi^{\tilde{\phi}}), M^{\nu(\cdot | \pi^{\tilde{\phi}})} \right) \right]} \\
 &= 4 \sqrt{d \mathbb{E}_{\pi \sim p} \mathbb{E}_{\phi \sim \nu} \left[D_{\text{KL}} \left(M^{\nu(\cdot | \phi)}(\cdot | \pi), M^{\nu(\cdot | \pi)} \right) \right]}.
 \end{aligned}$$

Further combining this with [Eq. \(8\)](#) and [Eq. \(7\)](#), we get

$$\begin{aligned}
 &\mathbb{E}_{\pi \sim p} \mathbb{E}_{\phi \sim \nu} \left[V_{M^{\nu(\cdot | \phi)}}(\pi^\phi) - V_{M^{\nu(\cdot | \phi)}}(\pi) - \frac{1}{\eta} D_{\text{KL}} \left(M^{\nu(\cdot | \phi)}(\cdot | \pi), M^{\nu(\cdot | \pi)} \right) \right] \\
 &= \mathbb{E}_{\pi \sim p} \mathbb{E}_{(M, \pi^*) \sim \nu} \left[V_M(\pi^*) - V_M(\pi) - \frac{1}{\eta} D_{\text{KL}} \left(M^{\nu(\cdot | \phi)}(\cdot | \pi), M^{\nu(\cdot | \pi)} \right) \right] \\
 &\leq 4H \sqrt{d \mathbb{E}_{\pi \sim p} \mathbb{E}_{\phi \sim \nu} \left[D_{\text{KL}} \left(M^{\nu(\cdot | \phi)}(\cdot | \pi), M^{\nu(\cdot | \pi)} \right) \right]} - \frac{1}{\eta} \mathbb{E}_{\pi \sim p} \mathbb{E}_{\phi \sim \nu} \left[D_{\text{KL}} \left(M^{\nu(\cdot | \phi)}(\cdot | \pi), M^{\nu(\cdot | \pi)} \right) \right] \\
 &\leq 4\eta d H^2.
 \end{aligned}$$

Finally, notice that the choice of p in the definition of $\text{DEC}_\eta^{\text{KL}}(\mathcal{M}, \Phi)$ allows it to achieve a smaller value than the choice of p as a marginal distribution of ν . This gives the desired bound. ■

Appendix D. Omitted Proofs in [Section 6](#)

D.1. Specific instances that satisfies [Definition 14](#)

In this section, we provide examples that satisfy [Definition 14](#). Let $d_{P,h}^\pi$ be the occupancy measure induced by policy π , transition P at step h . Analog to Bellman residual decomposition, for any π and $f \in \mathcal{F}^\pi$, we have

$$f(\pi; R) - V_{P,R}(\pi)$$

$$\begin{aligned}
 &= \sum_{h=1}^H \left(\mathbb{E}_{(s_h, a_h) \sim d_{P,h}^\pi} [f(s_h, a_h; R)] - \mathbb{E}_{(s_{h+1}, a_{h+1}) \sim d_{P,h}^\pi} [f(s_{h+1}, a_{h+1}; R)] \right) - \sum_{h=1}^H \mathbb{E}_{(s_h, a_h) \sim d_{P,h}^\pi} [R(s_h, a_h)] \\
 &= \sum_{h=1}^H \mathbb{E}^{\pi, P} [f(s_h, a_h; R) - R(s_h, a_h) - \mathbb{E}_{a_{h+1} \sim \pi(\cdot|s_{h+1})} [f(s_{h+1}, a_{h+1}; R)]]
 \end{aligned}$$

Low-rank MDPs For low-rank MDPs, there exists feature φ and ψ such that the transition $P(s'|s, a) = \varphi(s, a)^\top \psi(s')$. We have

$$\begin{aligned}
 d_{P,h}^\pi(s) &= \sum_{s', a'} d_{P,h-1}^\pi(s', a') P(s|s', a') \\
 &= \sum_{s', a'} d_{P,h-1}^\pi(s', a') P(s|s', a') \\
 &= \sum_{s', a'} d_{P,h-1}^\pi(s', a') \langle \varphi(s', a'), \psi_h(s) \rangle \\
 &= \left\langle \underbrace{\sum_{s', a'} d_{P,h-1}^\pi(s', a') \varphi(s', a')}_{\varphi_h(\pi)}, \psi_h(s) \right\rangle
 \end{aligned}$$

Thus,

$$\begin{aligned}
 &\mathbb{E}^{\pi, P} [f(s_h, a_h; R) - R(s_h, a_h) - \mathbb{E}_{a_{h+1} \sim \pi(\cdot|s_{h+1})} [f(s_{h+1}, a_{h+1}; R)]] \\
 &= \sum_{s_h, a_h} d_{P,h}^\pi(s_h) \pi(a_h|s_h) (f(s_h, a_h; R) - R(s_h, a_h) - \mathbb{E}_{s_{h+1} \sim P(\cdot|s_h, a_h)} \mathbb{E}_{a_{h+1} \sim \pi(\cdot|s_{h+1})} [f(s_{h+1}, a_{h+1}; R)]) \\
 &= \left\langle \underbrace{\varphi_h(\pi)}_{X_h(\pi, P)}, \underbrace{\sum_{s_h, a_h} \psi_h(s_h) \pi(a_h|s_h) (f(s_h, a_h; R) - R(s_h, a_h) - \mathbb{E}_{s_{h+1} \sim P(\cdot|s_h, a_h)} \mathbb{E}_{a_{h+1} \sim \pi(\cdot|s_{h+1})} [f(s_{h+1}, a_{h+1}; R)])}_{W_h(f, R; P)} \right\rangle
 \end{aligned}$$

For any $\pi' \in \Pi$ and $f' \in \mathcal{F}^{\pi'}$, any $o_h = (s_h, a_h, s_{h+1})$, define

$$\ell_h^{est}(f', o_h, R) = |\mathcal{A}| \pi'(a_h|s_h) (f'(s_h, a_h; R) - R(s_h, a_h) - \mathbb{E}_{a_{h+1} \sim \pi'(\cdot|s_{h+1})} [f'(s_{h+1}, a_{h+1}; R)]).$$

Let π_{unif} be the uniform policy over \mathcal{A} , and $\text{est}(\pi) = \pi_{\text{unif}}$ for every π , we have

$$\begin{aligned}
 &\langle X_h(\pi; P), W_h(f', R, P) \rangle \\
 &= \sum_{s_h, a_h} d_{P,h}^\pi(s_h) \pi'(a_h|s_h) (f'(s_h, a_h; R) - R(s_h, a_h) - \mathbb{E}_{s_{h+1} \sim P(\cdot|s_h, a_h)} \mathbb{E}_{a_{h+1} \sim \pi'(\cdot|s_{h+1})} [f'(s_{h+1}, a_{h+1}; R)]) \\
 &= \mathbb{E}^{\pi \circ_h \pi_{\text{unif}}, P} [\ell_h^{est}(f', o_h, R)] \\
 &= \mathbb{E}^{\pi \circ_h \text{est}(\pi), P} [\ell_h^{est}(f', o_h, R)].
 \end{aligned}$$

Low-occupancy MDPs For low-occupancy MDPs, there exists features φ and ψ such that the state-action occupancy $d_{P,h}^\pi(s, a) = \varphi_h(\pi)^\top \psi_h(s, a)$. We have

$$\mathbb{E}^{\pi, P} [f(s_h, a_h; R) - R(s_h, a_h) - \mathbb{E}_{a_{h+1} \sim \pi(\cdot|s_{h+1})} [f(s_{h+1}, a_{h+1}; R)]]$$

$$\begin{aligned}
 &= \sum_{s_h, a_h} d_{P,h}^\pi(s_h, a_h) (f(s_h, a_h; R) - R(s_h, a_h) - \mathbb{E}_{s_{h+1} \sim P(\cdot|s_h, a_h)} \mathbb{E}_{a_{h+1} \sim \pi(\cdot|s_{h+1})} [f(s_{h+1}, a_{h+1}; R)]) \\
 &= \left\langle \underbrace{\varphi_h(\pi)}_{X_h(\pi, P)}, \underbrace{\sum_{s_h, a_h} \psi_h(s_h, a_h) (f(s_h, a_h; R) - R(s_h, a_h) - \mathbb{E}_{s_{h+1} \sim P(\cdot|s_h, a_h)} \mathbb{E}_{a_{h+1} \sim \pi(\cdot|s_{h+1})} [f(s_{h+1}, a_{h+1}; R)])}_{W_h(f, R; P)} \right\rangle
 \end{aligned}$$

For any $\pi' \in \Pi$ and $f' \in \mathcal{F}^{\pi'}$, any $o_h = (s_h, a_h, s_{h+1})$, define

$$\ell_h^{est}(f', o_h, R) = (f'(s_h, a_h; R) - R(s_h, a_h) - \mathbb{E}_{a_{h+1} \sim \pi'(\cdot|s_{h+1})} [f'(s_{h+1}, a_{h+1}; R)]).$$

We have

$$\begin{aligned}
 &\langle X_h(\pi; P), W_h(f', R, P) \rangle \\
 &= \sum_{s_h, a_h} d_{P,h}^\pi(s_h, a_h) (f'(s_h, a_h; R) - R(s_h, a_h) - \mathbb{E}_{s_{h+1} \sim P(\cdot|s_h, a_h)} \mathbb{E}_{a_{h+1} \sim \pi'(\cdot|s_{h+1})} [f'(s_{h+1}, a_{h+1}; R)]) \\
 &= \mathbb{E}^{\pi, P} [\ell_h^{est}(f', o_h, R)].
 \end{aligned}$$

In this instance, $\text{est}(\pi) = \pi$ for every π .

D.2. Proof of Theorem 15

In this section, we give a proof to Theorem 15, which is similar to Foster et al. (2023c) with adaptation to more general partition. To start, we first define an extension to model-free optimistic DEC with bilinear divergence introduced in Foster et al. (2023c). Given the definition of $f^\phi, \pi^\phi, \pi_{est}^\phi, \Phi$ and D_{bi}^π discussed in Section 6. We define

$$\text{ODEC}_\eta^{bi}(\mathcal{M}, \Phi) = \max_{\rho \in \Delta(\Phi)} \min_{p \in \Delta(\Pi)} \max_{(P, R) \in \mathcal{M}} \mathbb{E}_{\pi \sim p} \mathbb{E}_{\phi \sim \rho} \left[f^\phi(\pi^\phi, R) - V_{P, R}(\pi) - \frac{1}{\eta} D_{bi}^\pi(\phi || P, R) \right].$$

This complexity have the following bounds.

Lemma 20 *If $\pi_{est}^\phi = \pi^\phi$ for all ϕ , we have*

$$\text{ODEC}_\eta^{bi}(\mathcal{M}, \Phi) \leq \frac{\eta d H}{4}.$$

If $\pi_{est}^\phi \neq \pi^\phi$ for some ϕ , we have

$$\text{ODEC}_\eta^{bi}(\mathcal{M}, \Phi) \leq H \sqrt{\frac{\eta d}{2}}.$$

Proof For any $\rho \in \Delta(\Phi)$, we will bound

$$\begin{aligned}
 &\min_{p \in \Delta(\Pi)} \max_{(P, R) \in \mathcal{M}} \mathbb{E}_{\pi' \sim p} \mathbb{E}_{\phi \sim \rho} \left[f^\phi(\pi^\phi; R) - V_{(P, R)}(\pi') - \frac{1}{\eta} D_{bi}^{\pi'}(\phi || P, R) \right] \\
 &= \min_{p \in \Delta(\Pi)} \max_{\nu \in \Delta(\mathcal{M})} \mathbb{E}_{(P, R) \sim \nu} \mathbb{E}_{\pi' \sim p} \mathbb{E}_{\phi \sim \rho} \left[f^\phi(\pi^\phi; R) - V_{(P, R)}(\pi') - \frac{1}{\eta} D_{bi}^{\pi'}(\phi || P, R) \right]
 \end{aligned}$$

$$= \max_{\nu \in \Delta(\mathcal{M})} \min_{p \in \Delta(\Pi)} \mathbb{E}_{(P,R) \sim \nu} \mathbb{E}_{\pi' \sim p} \mathbb{E}_{\phi \sim \rho} \left[f^\phi(\pi^\phi; R) - V_{(P,R)}(\pi') - \frac{1}{\eta} D_{bi}^{\pi'}(\phi || P, R) \right] \quad (\text{Lemma 21})$$

For any $\nu \in \Delta(\mathcal{M})$, we choose p such that $\pi' \sim p$ is equivalent to sampling $\phi' \sim \rho$ and play $\pi_\alpha^{\phi'}$ where $\pi_\alpha^{\phi'}$ is the randomized policy such that for every step h , play $\pi^{\phi'}$ with probability $1 - \alpha/H$ and play $\pi_{est}^{\phi'}$ with probability α/H . The policy choice for any step h is independent. Thus, with probability $(1 - \frac{\alpha}{H})^H \geq 1 - \alpha$, $\pi^{\phi'}$ is played for every H . Using such p , since $V_M(\pi) \in [0, 1]$ for any M, π , we only need to bound

$$\begin{aligned} & \mathbb{E}_{\pi' \sim p} \mathbb{E}_{\phi \sim \rho} \left[f^\phi(\pi^\phi; R) - V_{(P,R)}(\pi') \right] \\ &= \mathbb{E}_{\phi' \sim \rho} \mathbb{E}_{\phi \sim \rho} \left[f^\phi(\pi^\phi; R) - V_{(P,R)}(\pi_\alpha^{\phi'}) \right] \\ &\leq \alpha + \mathbb{E}_{\phi' \sim \rho} \mathbb{E}_{\phi \sim \rho} \left[f^\phi(\pi^\phi; R) - V_{(P,R)}(\pi^{\phi'}) \right]. \end{aligned} \quad (10)$$

For any P, R , we have

$$\begin{aligned} & \mathbb{E}_{\phi' \sim \rho} \mathbb{E}_{\phi \sim \rho} \left[f^\phi(\pi^\phi; R) - V_{(P,R)}(\pi^{\phi'}) \right] \\ &= \mathbb{E}_{\phi \sim \rho} \left[f^\phi(\pi^\phi; R) - V_{(P,R)}(\pi^\phi) \right] \\ &\leq \sum_{h=1}^H \mathbb{E}_{\phi \sim \rho} [\langle X_h(\phi; P), W_h(\phi, R; P) \rangle]. \end{aligned} \quad (\text{Bilinear property Definition 14})$$

Define $\Sigma_h = \mathbb{E}_{\phi \sim \rho} [X_h(\phi; P) X_h(\phi; P)^\top]$ and let Σ_h^\dagger be its pseudo-inverse, we have

$$\begin{aligned} & \mathbb{E}_{\phi \sim \rho} [\langle X_h(\phi; P), W_h(\phi, R; P) \rangle] \\ &= \mathbb{E}_{\phi \sim \rho} \left[\left\langle \left(\Sigma_h^\dagger \right)^{\frac{1}{2}} X_h(\phi; P), \Sigma_h^{\frac{1}{2}} W_h(\phi, R; P) \right\rangle \right] \\ &\leq \sqrt{\mathbb{E}_{\phi \sim \rho} \left[\left\| \left(\Sigma_h^\dagger \right)^{\frac{1}{2}} X_h(\phi; P) \right\|_2^2 \right]} \sqrt{\mathbb{E}_{\phi \sim \rho} \left[\left\| \Sigma_h^{\frac{1}{2}} W_h(\phi, R; P) \right\|_2^2 \right]} \end{aligned}$$

We have

$$\begin{aligned} & \mathbb{E}_{\phi \sim \rho} \left[\left\| \Sigma_h^{\frac{1}{2}} W_h(\phi, R; P) \right\|_2^2 \right] \\ &= \mathbb{E}_{\phi \sim \rho} [\langle \Sigma W_h(\phi, R; P), W_h(\phi, R; P) \rangle] \\ &= \mathbb{E}_{\phi \sim \rho} \left[\left\langle \mathbb{E}_{\phi' \sim \rho} [X_h(\phi'; P) X_h(\phi'; P)^\top] W_h(\phi, R; P), W_h(\phi, R; P) \right\rangle \right] \\ &= \mathbb{E}_{\phi \sim \rho} \mathbb{E}_{\phi' \sim \rho} [\langle X_h(\phi'; P), W_h(\phi, R; P) \rangle^2] \end{aligned}$$

Moreover,

$$\mathbb{E}_{\phi \sim \rho} \left[\left\| \left(\Sigma_h^\dagger \right)^{\frac{1}{2}} X_h(\phi; P) \right\|_2^2 \right] \leq d$$

Thus, by Cauchy-Schwarz inequality, for any P, R , we have

$$\begin{aligned}
 & \mathbb{E}_{\phi' \sim \rho} \mathbb{E}_{\phi \sim \rho} \left[f^\phi(\pi^\phi; R) - V_{(P,R)}(\pi^{\phi'}) \right] \\
 & \leq \sqrt{dH \sum_{h=1}^H \mathbb{E}_{\phi \sim \rho} \mathbb{E}_{\phi' \sim \rho} \left[\langle X_h(\phi'; P), W_h(\phi, R; P) \rangle^2 \right]} \\
 & = \sqrt{dH \sum_{h=1}^H \mathbb{E}_{\phi \sim \rho} \mathbb{E}_{\phi' \sim \rho} \left[\left(\mathbb{E}^{\pi^{\phi'} \circ_h \pi_{est}^{\phi'}, P} [\ell_h^{est}(\phi, o_h; R)] \right)^2 \right]} \tag{11}
 \end{aligned}$$

(1) If $\pi_{est}^\phi = \pi^\phi$ for all ϕ , we can set $\alpha = 0$ and combine Eq. (11) and Eq. (10) to conclude that for any P, R

$$\begin{aligned}
 & \mathbb{E}_{\pi' \sim p} \mathbb{E}_{\phi \sim \rho} \left[f^\phi(\pi^\phi; R) - V_{(P,R)}(\pi') - \frac{1}{\eta} D_{bi}^{\pi'}(\phi || P, R) \right] \\
 & \leq \sqrt{dH \sum_{h=1}^H \mathbb{E}_{\phi \sim \rho} \mathbb{E}_{\phi' \sim \rho} \left[\left(\mathbb{E}^{\pi^{\phi'} \circ_h \pi_{est}^{\phi'}, P} [\ell_h^{est}(\phi, o_h; R)] \right)^2 \right]} - \frac{1}{\eta} \mathbb{E}_{\phi' \sim \rho} \mathbb{E}_{\phi \sim \rho} \left[D_{bi}^{\pi^{\phi'}}(\phi || P, R) \right] \\
 & \leq \sqrt{dH \mathbb{E}_{\phi' \sim \rho} \mathbb{E}_{\phi \sim \rho} \left[D_{bi}^{\pi^{\phi'}}(\phi || P, R) \right]} - \frac{1}{\eta} \mathbb{E}_{\phi' \sim \rho} \mathbb{E}_{\phi \sim \rho} \left[D_{bi}^{\pi^{\phi'}}(\phi || P, R) \right] \\
 & \leq \frac{\eta dH}{4}.
 \end{aligned}$$

(2) If $\pi_{est}^\phi \neq \pi^\phi$ for some ϕ , we need additional analysis. For any h , $\pi_\alpha^{\phi'}$ select $\pi^{\phi'}$ with probability $1 - \frac{\alpha}{H}$ and select $\pi_{est}^{\phi'}$ with probability $\frac{\alpha}{H}$ independently. We define the event \mathcal{E}_h as choosing $\pi^{\phi'}$ before step $h-1$ and choosing $\pi_{est}^{\phi'}$ at step h . We have $\mathbb{P}(\mathcal{E}_h) = (1 - \frac{\alpha}{H})^{h-1} \frac{\alpha}{H} \geq (1 - \frac{\alpha}{H})^{H-1} \frac{\alpha}{H}$ from the independence. We have for any $h \in [H]$,

$$\begin{aligned}
 & \mathbb{E}_{\phi' \sim \rho} \mathbb{E}_{\phi \sim \rho} \left[\left(\mathbb{E}^{\pi_\alpha^{\phi'}, P} [\ell_h^{est}(\phi, o_h; R)] \right)^2 \right] \\
 & \geq \mathbb{P}(\mathcal{E}_h) \cdot \mathbb{E}_{\phi' \sim \rho} \mathbb{E}_{\phi \sim \rho} \left[\left(\mathbb{E}^{\pi^{\phi'} \circ_h \pi_{est}^{\phi'}, P} [\ell_h^{est}(\phi, o_h; R)] \right)^2 \right] \\
 & \geq \left(1 - \frac{\alpha}{H}\right)^{H-1} \frac{\alpha}{H} \mathbb{E}_{\phi' \sim \rho} \mathbb{E}_{\phi \sim \rho} \left[\left(\mathbb{E}^{\pi^{\phi'} \circ_h \pi_{est}^{\phi'}, P} [\ell_h^{est}(\phi, o_h; R)] \right)^2 \right] \\
 & \geq \frac{\alpha}{2H} \mathbb{E}_{\phi' \sim \rho} \mathbb{E}_{\phi \sim \rho} \left[\left(\mathbb{E}^{\pi^{\phi'} \circ_h \pi_{est}^{\phi'}, P} [\ell_h^{est}(\phi, o_h; R)] \right)^2 \right]. \tag{\alpha \leq \frac{1}{2}}
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 \mathbb{E}_{\pi' \sim p} \mathbb{E}_{\phi \sim \rho} \left[D_{bi}^{\pi'}(\phi || P, R) \right] & = \mathbb{E}_{\phi' \sim \rho} \mathbb{E}_{\phi \sim \rho} \left[\sum_{h=1}^H \left(\mathbb{E}^{\pi_\alpha^{\phi'}, P} [\ell_h^{est}(\phi, o_h; R)] \right)^2 \right] \\
 & \geq \frac{\alpha}{2H} \mathbb{E}_{\phi' \sim \rho} \mathbb{E}_{\phi \sim \rho} \left[\sum_{h=1}^H \left(\mathbb{E}^{\pi^{\phi'} \circ_h \pi_{est}^{\phi'}, P} [\ell_h^{est}(\phi, o_h; R)] \right)^2 \right].
 \end{aligned}$$

Combining this equation with [Eq. \(11\)](#) and [Eq. \(10\)](#), we have

$$\begin{aligned}
 & \mathbb{E}_{\pi' \sim p} \mathbb{E}_{\phi \sim \rho} \left[f^\phi(\pi^\phi; R) - V_{(P,R)}(\pi') - \eta D_{bi}^{\pi'}(\phi || P, R) \right] \\
 & \leq \alpha + \sqrt{dH \sum_{h=1}^H \mathbb{E}_{\phi \sim \rho} \mathbb{E}_{\phi' \sim \rho} \left[\left(\mathbb{E}^{\pi^{\phi'} \circ_h \pi_{est}^{\phi'}, P} [\ell_h^{est}(\phi, o_h; R)] \right)^2 \right]} \\
 & \quad - \frac{1}{\eta} \cdot \frac{\alpha}{2H} \mathbb{E}_{\phi' \sim \rho} \mathbb{E}_{\phi \sim \rho} \left[\sum_{h=1}^H \left(\mathbb{E}^{\pi^{\phi'} \circ_h \pi_{est}^{\phi'}, P} [\ell_h^{est}(\phi, o_h; R)] \right)^2 \right] \\
 & \leq \alpha + \frac{\eta d H^2}{2\alpha}
 \end{aligned}$$

Thus, by setting $\alpha = H\sqrt{\eta d}$, we have

$$\mathbb{E}_{\pi' \sim p} \mathbb{E}_{\phi \sim \rho_k} \left[f^\phi(\pi^\phi; R) - V_{(P,R)}(\pi') - \frac{1}{\eta} D_{bi}^{\pi'}(\phi || P, R) \right] \leq H\sqrt{\frac{\eta d}{2}}.$$

■

We restate [Theorem 15](#) here with more detailed dependencies with the language of partition defined in [Section 6](#).

Theorem 10 Assume $|\ell_h^{est}(\cdot, \cdot, \cdot)| \leq L$. For bilinear class defined in [Definition 14](#), [Algorithm 3](#) ensures with probability $1 - \delta$:

- If $\pi_{est}^\phi = \pi^\phi$ for any ϕ , then $\text{Reg} \leq O\left(HL\sqrt{d \log(T|\Phi|/\delta)} T^{\frac{3}{4}}\right)$.
- If $\pi_{est}^\phi \neq \pi^\phi$ for some ϕ , then $\text{Reg} \leq O\left(HL^{\frac{2}{3}} d^{\frac{1}{3}} \log(T|\Phi|/\delta)^{\frac{1}{3}} T^{\frac{5}{6}}\right)$.

Proof We first analyze the regret of optimistic exponential weights, the proof is almost the same as [Foster et al. \(2023c\)](#). Choosing comparator as $\phi^* = \phi_{f^*, \pi^*}$ where $f^* = Q_{P^*}^{\pi^*}$. The reward function used in exponential weights is

$$g_k(\phi) = \eta f^\phi(\pi^\phi; R_k) - \sum_{h=1}^H \left(\frac{1}{\tau} \sum_{i \in [\tau]} \ell_h^{est}(\phi, o_{k,h}^i, R_k) \right)^2.$$

Since $f(\pi, R) \leq [0, 1]$ for any f, π, R , and $|\ell_h^{est}(\cdot, \cdot, \cdot)| \leq L$, we have $|g_k(\phi)| \leq \eta + HL^2$. From [Lemma 23](#), if $\gamma \leq \frac{1}{4\eta + 4HL^2}$ we have

$$\begin{aligned}
 & \sum_{k=1}^K \mathbb{E}_{\phi \sim \rho_k} \left[\eta f^{\phi^*}(\pi^{\phi^*}; R_k) - \eta f^\phi(\pi^\phi; R_k) \right. \\
 & \quad \left. + \sum_{h=1}^H \left(\frac{1}{\tau} \sum_{i \in [\tau]} \ell_h^{est}(\phi, o_{k,h}^i, R_k) \right)^2 - \sum_{h=1}^H \left(\frac{1}{\tau} \sum_{i \in [\tau]} \ell_h^{est}(\phi^*, o_{k,h}^i, R_k) \right)^2 \right] \\
 & \leq \frac{\log(|\Phi|)}{\gamma} + \gamma \sum_{k=1}^K \mathbb{E}_{\phi \sim \rho_k} [g_k(\phi)^2]
 \end{aligned}$$

$$\leq \frac{\log(|\Phi|)}{\gamma} + \frac{1}{2} \sum_{k=1}^K \sum_{h=1}^H \mathbb{E}_{\phi \sim \rho_k} \left[\left(\frac{1}{\tau} \sum_{i \in [\tau]} \ell_h^{est}(\phi, o_{k,h}^i, R_k) \right)^2 \right] + 2\eta^2 \gamma K. \quad (\gamma \leq \frac{1}{4HL^2})$$

Rearranging the inequality, we have

$$\begin{aligned} & \sum_{k=1}^K \mathbb{E}_{\phi \sim \rho_k} \left[\eta f^{\phi^*}(\pi^{\phi^*}; R_k) - \eta f^{\phi}(\pi^{\phi}; R_k) + \frac{1}{2} \sum_{h=1}^H \left(\frac{1}{\tau} \sum_{i \in [\tau]} \ell_h^{est}(\phi, o_{k,h}^i, R_k) \right)^2 \right] \\ & \leq \sum_{k=1}^K \sum_{h=1}^H \left(\frac{1}{\tau} \sum_{i \in [\tau]} \ell_h^{est}(\phi^*, o_{k,h}^i, R_k) \right)^2 + \frac{\log(|\Phi|)}{\gamma} + 2\eta^2 \gamma K \end{aligned} \quad (12)$$

By Hoeffding's inequality and union bound, with probability at least $1 - \delta$, for any $k \in [K]$, $h \in [H]$, and any $\phi \in \Phi$, we have

$$\left| \frac{1}{\tau} \sum_{i \in [\tau]} \ell_h^{est}(\phi, o_{k,h}^i, R_k) - \mathbb{E}^{\pi_k, P^*} [\ell_h^{est}(\phi, o_h, R_k)] \right| \leq L \sqrt{\frac{2 \log(KH|\Phi|/\delta)}{\tau}} = \epsilon_{conc}.$$

Thus, given $D_{bi}^{\pi'}(\phi || P, R) = \sum_{h=1}^H \left(\mathbb{E}^{\pi', P} [\ell_h^{est}(\phi, o_h, R)] \right)^2$, we have

$$\sum_{h=1}^H \left(\frac{1}{\tau} \sum_{i \in [\tau]} \ell_h^{est}(\phi, o_{k,h}^i, R_k) \right)^2 \leq 2D_{bi}^{\pi_k}(\phi || P^*, R_k) + 2H\epsilon_{conc}^2. \quad (13)$$

$$\sum_{h=1}^H \left(\frac{1}{\tau} \sum_{i \in [\tau]} \ell_h^{est}(\phi, o_{k,h}^i, R_k) \right)^2 \geq \frac{1}{2} D_{bi}^{\pi_k}(\phi || P^*, R_k) - H\epsilon_{conc}^2. \quad (14)$$

From Lemma A.3 in [Foster et al. \(2021\)](#), with probability at least $1 - \delta$, for any $\phi \in \Phi$, we have

$$\frac{1}{2} \sum_{k=1}^K \mathbb{E}_{\pi \sim p_k} [D_{bi}^{\pi}(\phi || P^*, R_k)] \leq \sum_{k=1}^K D_{bi}^{\pi_k}(\phi || P^*, R_k) + 4HL^2 \log(|\Phi|/\delta).$$

Combining [Eq. \(12\)](#), [Eq. \(13\)](#) and [Eq. \(14\)](#), and the fact that $D_{bi}^{\pi}(\phi^* || P^*, R) = 0$ for any π and R , we have

$$\begin{aligned} & \sum_{k=1}^K \mathbb{E}_{\phi \sim \rho_k} \left[\eta f^{\phi^*}(\pi^{\phi^*}; R_k) - \eta f^{\phi}(\pi^{\phi}; R_k) + \frac{1}{8} \mathbb{E}_{\pi \sim p_k} [D_{bi}^{\pi}(\phi || P^*, R_k)] \right] \\ & \leq \frac{\log(|\Phi|)}{\gamma} + \frac{2\gamma K}{\eta^2} + \frac{6L^2 KH \log(KH|\Phi|/\delta)}{\tau} + 4HL^2 \log(|\Phi|/\delta) \end{aligned}$$

Choose $\gamma = \min\{\eta \sqrt{\frac{\log(|\Phi|)}{K}}, \frac{1}{4\eta + 4HL^2}\}$, we have

$$\sum_{k=1}^K \mathbb{E}_{\phi \sim \rho_k} \left[f^{\phi^*}(\pi^{\phi^*}; R_k) - f^{\phi}(\pi^{\phi}; R_k) + \frac{1}{8\eta} \mathbb{E}_{\pi \sim p_k} [D_{bi}^{\pi}(\phi || P^*, R_k)] \right]$$

$$\leq O\left(\sqrt{K \log(|\Phi|)} + HL^2 \log(|\Phi|) + \frac{L^2 KH \log(KH|\Phi|/\delta)}{\eta\tau} + \frac{1}{\eta} HL^2 \log(|\Phi|/\delta)\right). \quad (15)$$

Since $f^{\phi^*} = Q_{P^*}^{\pi^*}$ and $\pi^{\phi^*} = \pi^*$, we have $V_{(P^*, R)}(\pi^*) = f^{\phi^*}(\pi^{\phi^*}; R)$ for any $(P, R, \pi^*) \in \phi^*$. Regret can be decomposed to

$$\begin{aligned} & \frac{1}{\tau} \text{Reg} \\ &= \sum_{k=1}^K V_{(P^*, R_k)}(\pi^*) - \sum_{k=1}^K \mathbb{E}_{\pi \sim p_k} [V_{(P^*, R_k)}(\pi)] \\ &= \sum_{k=1}^K f^{\phi^*}(\pi^{\phi^*}, R_k) - \sum_{k=1}^K \mathbb{E}_{\phi \sim \rho_k} [f^{\phi}(\pi^{\phi}; R_k)] + \sum_{k=1}^K \mathbb{E}_{\phi \sim \rho_k} [f^{\phi}(\pi^{\phi}; R_k)] - \sum_{k=1}^K \mathbb{E}_{\pi \sim p_k} [V_{(P^*, R_k)}(\pi)] \\ &= \underbrace{\sum_{k=1}^K \mathbb{E}_{\phi \sim \rho_k} \left[f^{\phi^*}(\pi^{\phi^*}; R_k) - f^{\phi}(\pi^{\phi}; R_k) + \frac{1}{8\eta} \mathbb{E}_{\pi' \sim p_k} [D_{bi}^{\pi'}(\phi || P^*, R_k)] \right]}_{\text{term1}} \\ & \quad + \underbrace{\sum_{k=1}^K \mathbb{E}_{\pi' \sim p_k} \mathbb{E}_{\phi \sim \rho_k} \left[f^{\phi}(\pi^{\phi}; R_k) - V_{(P^*, R_k)}(\pi') - \frac{1}{8\eta} D_{bi}^{\pi'}(\phi || P^*, R_k) \right]}_{\text{term2}} \end{aligned} \quad (16)$$

term1 is bounded through [Eq. \(15\)](#). For **term2**, we have

$$\begin{aligned} \text{term2} &\leq \sum_{k=1}^K \max_{(P, R) \in \mathcal{M}} \mathbb{E}_{\pi' \sim p_k} \mathbb{E}_{\phi \sim \rho_k} \left[f^{\phi}(\pi^{\phi}; R) - V_{(P, R)}(\pi') - \frac{1}{8\eta} D_{bi}^{\pi'}(\phi || P, R) \right] \\ &= \sum_{k=1}^K \underbrace{\min_{p \in \Delta(\Pi)} \max_{(P, R) \in \mathcal{M}} \mathbb{E}_{\pi' \sim p} \mathbb{E}_{\phi \sim \rho_k} \left[f^{\phi}(\pi^{\phi}; R) - V_{(P, R)}(\pi') - \frac{1}{8\eta} D_{bi}^{\pi'}(f, \pi || P, R) \right]}_{\leq \text{ODEC}_{8\eta}^{bi}(\mathcal{M}, \Phi)} \end{aligned}$$

From [Lemma 20](#), we have

(1) If $\pi_{est}^{\phi} = \pi^{\phi}$, **term2** $\leq O(\eta d H)$. Combining with [Eq. \(15\)](#) and regret decomposition [Eq. \(16\)](#), we have

$$\frac{1}{\tau} \text{Reg} \leq O\left(\eta d H K + \sqrt{K \log(|\Phi|)} + \frac{L^2 KH \log(KH|\Phi|/\delta)}{\eta\tau} + \frac{1}{\eta} L^2 H \log(|\Phi|/\delta)\right)$$

and thus (recall $T = K\tau$)

$$\text{Reg} \leq O\left(\eta d H T + \sqrt{T \log(|\Phi|)} + \frac{L^2 T H \log(TH|\Phi|/\delta)}{\eta\tau} + \frac{\tau}{\eta} L^2 H \log(|\Phi|/\delta)\right).$$

Setting $\eta = Ld^{-\frac{1}{2}}(\log(|\Phi|/\delta))^{\frac{1}{2}}T^{-\frac{1}{4}}$, $\tau = \sqrt{T}$, then $K = \frac{T}{\tau} = \sqrt{T}$, we have

$$\text{Reg} \leq \tilde{O}\left(HL\sqrt{d \log(|\Phi|/\delta)}T^{\frac{3}{4}}\right).$$

(2) If $\pi_{est}^{\phi'} \neq \pi^{\phi'}$, **term2** $\leq O(H\sqrt{\eta d})$. Combining with Eq. (15) and regret decomposition Eq. (16), we have

$$\frac{1}{\tau} \text{Reg} \leq O \left(H\sqrt{\eta d}K + \sqrt{K \log(|\Phi|)} + \frac{L^2 K H \log(KH|\Phi|/\delta)}{\eta\tau} + \frac{1}{\eta} L^2 H \log(|\Phi|/\delta) \right)$$

and thus

$$\text{Reg} \leq O \left(H\sqrt{\eta d}T + \sqrt{T\tau \log(|\Phi|)} + \frac{L^2 T H \log(TH|\Phi|/\delta)}{\eta\tau} + \frac{\tau}{\eta} L^2 H \log(|\Phi|/\delta) \right).$$

Setting $\eta = L^{\frac{4}{3}} d^{-\frac{1}{3}} (\log(|\Phi|/\delta))^{\frac{2}{3}} T^{-\frac{1}{3}}$, $\tau = \sqrt{T}$, then $K = \frac{T}{\tau} = \sqrt{T}$, we have

$$\text{Reg} \leq O \left(H L^{\frac{2}{3}} (d \log(|\Phi|/\delta))^{\frac{1}{3}} T^{\frac{5}{6}} \right).$$

■

Appendix E. Support Lemmas

Lemma 21 (Sion's minimax theorem (Sion, 1958)) *Let \mathcal{X} and \mathcal{Y} be convex and compact sets, and a function $F : \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{R}$ such that (1) For all $y \in \mathcal{Y}$, $F(\cdot, y)$ is convex and continuous over \mathcal{X} and (2) For all $x \in \mathcal{X}$, $F(x, \cdot)$ is concave and continuous over \mathcal{Y} . Then*

$$\min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} F(x, y) = \max_{y \in \mathcal{Y}} \min_{x \in \mathcal{X}} F(x, y)$$

Lemma 22 (Performance Difference Lemma) *For any finite-horizon MDPs M with state space \mathcal{S} , action space \mathcal{A} , reward function R , transition P , and horizon length H , we have*

$$V_{(P,R)}(\pi') - V_{(P,R)}(\pi) = \sum_{h=1}^H \mathbb{E}_{s \sim d_{P,h}^{\pi}} \left[\sum_{a \in \mathcal{A}} (\pi'_h(a|x) - \pi_h(a|x)) Q_h^{\pi'}(x, a; R) \right].$$

Lemma 23 (Exponential Weights) *Given a sequence of reward functions $\{g^t\}_{t=1}^T$ over a decision set Π , $\{p^t\}_{t=1}^T$ is a distribution sequence with $p^t \in \Delta(\Pi)$, $\forall t \in [T]$ such that*

$$p^{t+1}(\pi) \propto \exp \left(\eta \sum_{i=1}^t g^i(\pi) \right).$$

If p^1 is a uniform distribution over $|\Pi|$ and $\eta g^t(\pi) \leq 1$ for all $t \in [T]$ and $\pi \in \Pi$. Then

$$\max_{p \in \Delta(\Pi)} \left\{ \sum_{t=1}^T \langle g^t, p - p^t \rangle \right\} \leq \frac{\log(|\Pi|)}{\eta} + \eta \sum_{t=1}^T \sum_{\pi \in \Pi} p^t(\pi) g^t(\pi)^2.$$

Here we provide an extension of Lemma 5.2 in [Xu and Zeevi \(2023\)](#). The proof follows [Xu and Zeevi \(2023\)](#) with adaptation to general Φ .

Lemma 24 (Lemma 5.2 in [Xu and Zeevi \(2023\)](#) for general Φ) *Given $\rho \in \Delta(\Phi)$, $\eta, p \in \Delta(\Pi)$, and let $\nu \in \Delta(\Psi)$ be a maximizer of*

$$\nu \in \operatorname{argmax}_{\nu \in \Delta(\Psi)} \mathbb{E}_{\pi \sim p} \mathbb{E}_{(M, \pi^*) \sim \nu} \mathbb{E}_{o \sim M(\cdot|\pi)} \left[V_M(\pi^*) - V_M(\pi) - \frac{1}{\eta} D_{\text{KL}}(\nu_\phi(\cdot|\pi, o), \rho) \right],$$

then we have

$$\begin{aligned} & \max_{\mu \in \Delta(\Psi)} \mathbb{E}_{\pi \sim p} \mathbb{E}_{\phi^* \sim \mu} \mathbb{E}_{(M, \pi^*) \sim \mu(\cdot|\phi^*)} \mathbb{E}_{o \sim M(\cdot|\pi)} \left[V_M(\pi^*) - V_M(\pi) - \frac{1}{\eta} \log \frac{\nu(\phi^*|\pi, o)}{\rho(\phi^*)} \right] \\ &= \mathbb{E}_{\pi \sim p} \mathbb{E}_{(M, \pi^*) \sim \nu} \mathbb{E}_{o \sim M(\cdot|\pi)} \left[V_M(\pi^*) - V_M(\pi) - \frac{1}{\eta} D_{\text{KL}}(\nu_\phi(\cdot|\pi, o), \rho) \right] \end{aligned}$$

If the environment reveals $\pi^\Theta = (\pi^\theta)_{\theta \in \Theta}$, we also have

$$\begin{aligned} & \max_{\mu \in \Delta(\mathcal{M})} \mathbb{E}_{\pi \sim p} \mathbb{E}_{\theta^* \sim \mu} \mathbb{E}_{M \sim \mu(\cdot|\theta^*)} \mathbb{E}_{o \sim M(\cdot|\pi)} \left[V_M(\pi^{\theta^*}) - V_M(\pi) - \frac{1}{\eta} \log \frac{\nu(\theta^*|\pi, o)}{\rho_t(\theta^*)} \right] \\ &= \mathbb{E}_{\pi \sim p} \mathbb{E}_{\theta^* \sim \nu} \mathbb{E}_{M \sim \nu(\cdot|\theta^*)} \mathbb{E}_{o \sim M(\cdot|\pi)} \left[V_M(\pi^{\theta^*}) - V_M(\pi) - \frac{1}{\eta} D_{\text{KL}}(\nu_{\theta^*}(\cdot|\pi, o), \rho_t) \right], \end{aligned}$$

as long as ν is a maximizer of the RHS.

Proof Let \mathcal{Z} be the space of all mappings from $\Pi \times \mathcal{O}$ to $\Delta(\Phi)$. For every mapping $Z \in \mathcal{Z}$, denote $Z(\cdot|\pi, o) \in \Delta(\Phi)$ as the image of (π, o) . For a fixed p , define $B : \Delta(\Psi) \times \mathcal{Z} \rightarrow \mathbb{R}$ by

$$B(\mu, Z) = \mathbb{E}_{\phi^* \sim \mu} \mathbb{E}_{(M, \pi^*) \sim \mu(\cdot|\phi^*)} \mathbb{E}_{\pi \sim p} \mathbb{E}_{o \sim M(\cdot|\pi)} \left[V_M(\pi^*) - V_M(\pi) - \frac{1}{\eta} \log \frac{Z(\phi^*|\pi, o)}{q(\phi^*)} \right]$$

For any μ , $B(\mu, Z)$ is strongly convex with respect to Z . For any $\mu \in \Delta(\Psi)$, denote $Z_\mu = \operatorname{argmin}_{Z \in \mathcal{Z}} B(\mu, Z)$ as the unique minimizer of $B(\mu, z)$ given μ . Solve the first order optimality condition for the Lagrangian of $B(\mu, Z)$ constrained on Z as a valid distribution, we have $Z_\mu(\phi|\pi, o) = \mu(\phi|\pi, o)$ for every ϕ, π, o as the posterior of μ . Thus, we have

$$\begin{aligned} & \min_{Z \in \mathcal{Z}} B(\mu, Z) \\ &= B(\mu, Z_\mu) \\ &= \mathbb{E}_{\phi^* \sim \mu} \mathbb{E}_{(M, \pi^*) \sim \mu(\cdot|\phi^*)} \mathbb{E}_{\pi \sim p} \mathbb{E}_{o \sim M(\cdot|\pi)} \left[V_M(\pi^*) - V_M(\pi) - \frac{1}{\eta} \log \frac{\mu(\phi^*|\pi, o)}{q(\phi^*)} \right] \end{aligned}$$

Define the worst-case prior belief

$$\nu \in \operatorname{argmax}_{\mu} B(\mu, Z_\mu) = \operatorname{argmax}_{\mu} \min_{Z \in \mathcal{Z}} B(\mu, Z).$$

From a constructive minimax theorem (Lemma 5.1 in [Xu and Zeevi \(2023\)](#)), (ν, Z_ν) is a Nash equilibrium of function B , and Z_ν is a construction solution to the minimax optimization problem $\min_Z \max_\mu B(\mu, Z)$. Thus,

$$\begin{aligned}
 & \mathbb{E}_{\pi \sim p} \mathbb{E}_{(M, \pi^*) \sim \nu} \mathbb{E}_{o \sim M(\cdot | \pi)} \left[V_M(\pi^*) - V_M(\pi) - \frac{1}{\eta} D_{\text{KL}}(\nu_\phi(\cdot | \pi, o), \rho) \right] \\
 &= \mathbb{E}_{\phi^* \sim \nu} \mathbb{E}_{(M, \pi^*) \sim \nu(\cdot | \phi^*)} \mathbb{E}_{\pi \sim p} \mathbb{E}_{o \sim M(\cdot | \pi)} \left[V_M(\pi^*) - V_M(\pi) - \frac{1}{\eta} \log \frac{\nu(\phi^* | \pi, o)}{q(\phi^*)} \right] \\
 &= B(\nu, Z_\nu) \\
 &= \max_{\mu} B(\mu, Z_\nu) \quad \text{(Lemma 5.1 in [Xu and Zeevi \(2023\)](#))} \\
 &= \max_{\mu \in \Delta(\Psi)} \mathbb{E}_{\pi \sim p} \mathbb{E}_{\phi^* \sim \mu} \mathbb{E}_{(M, \pi^*) \sim \mu(\cdot | \phi^*)} \mathbb{E}_{o \sim M(\cdot | \pi)} \left[V_M(\pi^*) - V_M(\pi) - \frac{1}{\eta} \log \frac{\nu(\phi^* | \pi, o)}{\rho(\phi^*)} \right].
 \end{aligned}$$

The proof for the revealed $\pi^\Theta = (\pi^\theta)_{\theta \in \Theta}$ is similar. ■