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Polynomial low degree hardness for Broadcasting on Trees (Extended Abstract)

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Broadcasting on trees is a fundamental model from statistical physics that plays an important role in information theory, noisy computation and phylogenetic reconstruction within computational biology and linguistics. While this model permits efficient linear-time algorithms for the inference of the root from the leaves, recent work suggests that non-trivial computational complexity may be required for inference.

The inference of the root state can be performed using the celebrated Belief Propagation (BP) algorithm, which achieves Bayes-optimal performance. Although BP runs in linear time using real arithmetic operations, recent research indicates that it requires non-trivial computational complexity using more refined complexity measures.

Moitra, Mossel, and Sandon demonstrated such complexity by constructing a Markov chain for which estimating the root better than random guessing (for typical inputs) is NC^1-complete. Kohler and Mossel constructed chains where, for trees with N leaves, achieving better-than-random root recovery requires polynomials of degree $N^{\Omega(1)}$. The papers above raised the question of whether such complexity bounds hold generally below the celebrated Kesten-Stigum bound.

The low degree method has emerged as a predictive tool for establishing computational hardness for statistical problems. This method postulate that *if all low degree functions are not good estimators, then the problem is computationally hard* see e.g. Barak et al. (2019); Hopkins and Steurer (2017); Hopkins (2018); Schramm and Wein (2020). (see also Kunisky et al. (2019) for a survey.)

Previously, the authors established a general degree lower bound of $\Omega(\log N)$ below the Kesten-Stigum bound. Specifically, they proved that any function expressed as a linear combination of functions of at most $O(logN)$ leaves has vanishing correlation with the root. In this work, we get an exponential improvement of this lower bound by establishing an $N^{\Omega(1)}$ degree lower bound, for any broadcast process in the whole regime below the Kesten-Stigum bound. We confirmed that the Kesten-Stigum bound is a sharp transition from feasibility via degree-1 estimators (simply counting leaf types) *above* KS to exponential-degree requirements *below* it. This result makrs the BOT model as an interesting counterexample to the "computational-hardness" belief on low-degree analysis.

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