

Robustly Learning Monotone Generalized Linear Models via Data Augmentation

Nikos Zarifis*

Puqian Wang*

Ilias Diakonikolas

Jelena Diakonikolas

Department of Computer Sciences, University of Wisconsin, Madison

ZARIFIS@WISC.EDU

PWANG333@WISC.EDU

ILIAS@CS.WISC.EDU

JELENA@CS.WISC.EDU

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Abstract

We study the task of learning Generalized Linear models (GLMs) in the agnostic model under the Gaussian distribution. We give the first polynomial-time algorithm that achieves a constant-factor approximation for *any* monotone Lipschitz activation. Prior constant-factor GLM learners succeed for a substantially smaller class of activations. Our work resolves a well-known open problem, by developing a robust counterpart to the classical GLMtron algorithm (Kakade et al., 2011). Our robust learner applies more generally, encompassing all monotone activations with bounded $(2+\zeta)$ -moments, for any fixed $\zeta > 0$ —a condition that is essentially necessary. To obtain our results, we leverage a novel data augmentation technique with decreasing Gaussian noise injection and prove a number of structural results that may be useful in other settings.

Keywords: PAC Learning, Agnostic Learning, Generalized Linear Models

1. Introduction

A Generalized Linear Model (GLM) is any function of the form $\sigma(\mathbf{w}^* \cdot \mathbf{x})$, where $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ is a known activation function and \mathbf{w}^* is a hidden vector. GLMs constitute one of the most basic supervised learning models capturing hidden low-dimensional structure in high-dimensional labeled data. As such, GLMs have been studied over the course of several decades (Nelder and Wedderburn, 1972; Dobson and Barnett, 2008). Specifically, the special case where σ is the sign function corresponds to Linear Threshold Functions (LTFs) whose study goes back to Rosenblatt (1958).

In the realizable setting, the learning problem is as follows: given labeled examples $(\mathbf{x}, y) \in \mathbb{R}^d \times \mathbb{R}$ from an unknown distribution \mathcal{D} , whose labels are consistent with a GLM, i.e., $y = \sigma(\mathbf{w}^* \cdot \mathbf{x})$ where σ is known and \mathbf{w}^* unknown, the goal is to approximate the underlying function (and/or the hidden direction \mathbf{w}^*) with respect to the square loss. A classical work (Kakade et al., 2011) gave a simple and efficient gradient-based algorithm (GLMtron) for this problem when the data is supported on the unit ball, under the assumption that the activation function is monotone and Lipschitz. The GLMtron algorithm also succeeds in the presence of zero-mean random label noise.

We point out that for GLM learning to even be information-theoretically solvable, some regularity assumptions on the activation σ are necessary. Moreover, even if σ is sufficiently well-behaved so that no information-theoretic impediment exists, computational hardness results rule out efficient algorithms even for Gaussian data and a small amount of random label noise (Song et al., 2021).

* Equal contribution.

Over the past five years, there has been a resurgence of research interest on learning GLMs in the more challenging *agnostic* (or adversarial label noise) model (Haussler, 1992; Kearns et al., 1994), where no assumptions are made on the labels and the goal is to compute a hypothesis that is competitive with the *best-fit* function in the class. The ideal result in this setting would be an efficient agnostic learner that succeeds for all marginal distributions and achieves optimal error. Such a goal appears unattainable, due to known computational hardness. Specifically, even for Gaussian marginals and a ReLU activation, there is strong evidence that any such algorithm requires super-polynomial time (Diakonikolas et al., 2020b; Goel et al., 2020; Diakonikolas et al., 2021, 2023). Moreover, even if we relax our goal to any constant factor approximation, distributional assumptions are necessary (Manurangsi and Reichman, 2018; Diakonikolas et al., 2022a). Thus, research has focused on constant factor approximate learners in the distribution-specific setting. Denoting $\mathcal{L}(\mathbf{w}) := \mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}}[(\sigma(\mathbf{w} \cdot \mathbf{x}) - y)^2]$, our agnostic learning problem is defined as follows.

Problem 1 (Robustly Learning GLMs) *Let $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ be a known activation and \mathcal{D} be a distribution of $(\mathbf{x}, y) \in \mathbb{R}^d \times \mathbb{R}$ such that its \mathbf{x} -marginal $\mathcal{D}_{\mathbf{x}}$ is the standard normal. We say that an algorithm is a C -approximate proper GLM learner, for some $C \geq 1$, if given $\epsilon > 0$, $W > 0$, and i.i.d. samples from \mathcal{D} , the algorithm outputs a vector $\hat{\mathbf{w}} \in \mathbb{R}^d$ such that with high probability it holds $\mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}}[(\sigma(\hat{\mathbf{w}} \cdot \mathbf{x}) - y)^2] \leq C \text{OPT} + \epsilon$, where $\text{OPT} \triangleq \min_{\|\mathbf{w}\|_2 \leq W} \mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}}[(\sigma(\mathbf{w} \cdot \mathbf{x}) - y)^2]$.*

Motivated by the setting introduced in (Kakade et al., 2011), a major algorithmic goal in this area has been to obtain an efficient constant-factor approximate learner that succeeds for *any* monotone Lipschitz activation function. A line of recent work (Diakonikolas et al., 2020a, 2022b; Awasthi et al., 2023; Wang et al., 2023; Gollakota et al., 2023a; Zarifis et al., 2024; Guo and Vijayaraghavan, 2024) has made algorithmic progress on various special cases of this question. This progress notwithstanding, the general case remained open, prompting the following question:

Is there an efficient constant-factor approximate learner for monotone Lipschitz GLMs under Gaussian marginals?

As a special case of our main result, we answer this question in the affirmative.

Theorem 2 (Robustly Learning Monotone & Lipschitz GLMs) *There exists an algorithm with the following performance guarantee: For any known monotone and b -Lipschitz activation σ , given $\epsilon > 0$, $W > 0$, and $N = \tilde{\Theta}(d(bW)^2/\epsilon + d/\epsilon^2)$ samples, the algorithm runs in $\text{poly}(d, N)$ time and returns a vector $\hat{\mathbf{w}}$ such that with high probability, $\mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}}[(\sigma(\hat{\mathbf{w}} \cdot \mathbf{x}) - y)^2] \leq C \text{OPT} + \epsilon$, where C is an absolute constant independent of ϵ, d, b, W .*

We emphasize that the approximation ratio of our algorithm is a universal constant—independent of the dimension, the desired accuracy, the Lipschitz constant, and the radius of the space.

The key qualitative difference between prior work and Theorem 2 is in the assumptions on the activation. Specifically, prior constant-factor GLM learners succeed for a much smaller subclass of activations. In fact, our main algorithmic result (Theorem 12) applies more generally, encompassing all monotone activations with bounded $(2 + \zeta)$ moment, for any $\zeta > 0$ (Corollary 14). This in particular implies that the case of LTFs fits in our setting. We stress here that some assumption on top of monotonicity is information-theoretically necessary, even for realizable learning (Theorem 40).

Comparison to Prior Work Gollakota et al. (2023a) gave an efficient GLM learner for monotone Lipschitz activations and marginal distribution with bounded second moment. However, the error

of their algorithm scales linearly with W and the Lipschitz constant. Wang et al. (2023); Zarifis et al. (2024) studied Problem 1 under ‘well-behaved’ distributions, where σ is monotone and (a, b) unbounded, meaning that $|\sigma'(z)| \leq b$ and $\sigma'(z) \geq a$ when $z \geq 0$. They provided an efficient algorithm with error $O(\text{poly}(b/a))\text{OPT} + \epsilon$. Note that when $a = 0$, this error guarantee is vacuous. More recently, Wang et al. (2024) studied the same problem under Gaussian marginals for activations with bounded information-exponent. The approximation ratio of their method inherently scales polynomially with the radius W of the space. See Appendix A for more details.

Remark In the sequel, we assume that the scale of the target vector \mathbf{w}^* , $\|\mathbf{w}^*\|_2$, is known and, by, rescaling the space, we optimize \mathbf{w} on the unit sphere. The unknown scale of \mathbf{w}^* can be resolved by a simple grid search. For our approach, this rescaling is w.l.o.g. because—unlike in prior work (Wang et al., 2023, 2024; Zarifis et al., 2024)—the approximation ratio of our algorithm is **independent** of any problem parameters. For a formal justification, see Remark 30 and Lemma 31.

Organization In Section 1.1, we summarize our algorithmic ideas and techniques. In Section 2, we analyze the landscape of the augmented loss. Our main algorithm and its analysis for learning Gaussian GLMs of general activations is presented in Section 3. In Section 4, we focus on monotone activations and show that our algorithm achieves error $O(\text{OPT}) + \epsilon$ under very mild assumptions. Due to space limitations, several proofs have been deferred to the Appendix.

1.1. Technical Overview

Our work relies on three main technical ingredients: (1) data augmentation, which we use as a method to mitigate the effect of the adversarial label noise, (2) an optimization-theoretic local error bound, which in our work is a structural result that identifies the ‘signal’ vector field that guides the algorithm toward the set of target solutions, (3) a suite of structural results for (B, L) -regular monotone activations (see Definition 10), leveraging their piecewise-constant approximations, smoothing through data augmentation, and representation via Hermite polynomials.

Data Augmentation Data augmentation encompasses a broad set of techniques for modifying or artificially generating data to enhance learning and estimation tasks. In the context of our work, data augmentation refers to the injection of Gaussian noise into the data vectors \mathbf{x} while retaining the same labels. In particular, given any labeled example $(\mathbf{x}, y) \sim \mathcal{D}$ and a parameter $\rho \in (0, 1)$, the considered data augmentation process generates labeled examples $(\tilde{\mathbf{x}}, y)$, where $\tilde{\mathbf{x}} = \rho\mathbf{x} + \sqrt{1 - \rho^2}\mathbf{z}$ and \mathbf{z} is an independently generated sample from the standard normal distribution. While this type of data augmentation is a common empirical technique in machine learning, it is considered to be a wild card: although sometimes helpful, it can also be detrimental to learning guarantees (see, e.g., Yin et al. (2019); Lin et al. (2024)). Thus, on a conceptual level, one of our contributions is showing that for the considered GLM learning task, data augmentation is provably beneficial.

The effect of data augmentation on the considered GLM learning task is that it simulates the Ornstein–Uhlenbeck semigroup $T_\rho f(t) := \mathbf{E}_{\mathbf{z} \sim \mathcal{N}(0,1)}[f(\rho t + \sqrt{1 - \rho^2}\mathbf{z})]$ applied to any function $f(\mathbf{w} \cdot \mathbf{x})$. This process smoothens the function f and induces other regularity properties. Unlike the common use of smoothing in the optimization literature, where the key utilized properties are continuity and smoothness of the smoothed objective function (see, e.g., Nesterov and Spokoiny (2017); Duchi et al. (2012); Bubeck et al. (2019); Diakonikolas and Guzmán (2024)), in our work, the key feature is the effect of injected noise on enhancing the signal in the data, as explained below.

Suppose we were given a GLM learning task. Since the goal of a learning algorithm is to minimize the mean squared loss $\mathcal{L}(\mathbf{w}) = \mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}}[(\sigma(\mathbf{w} \cdot \mathbf{x}) - y)^2]$, a natural approach is to

follow a gradient field associated with the error $\sigma(\mathbf{w} \cdot \mathbf{x}) - y$. Indeed, all prior work for this task proceeds by applying (stochastic) gradient-based algorithms to either the original squared loss $\mathcal{L}(\mathbf{w})$ or its surrogate $\mathcal{L}_{\text{sur}}(\mathbf{w}) = \mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}} [\int_0^{\mathbf{w} \cdot \mathbf{x}} (\sigma(t) - y) dt]$. In either case, the associated gradient field can be represented by $\mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}} [(\sigma(\mathbf{w} \cdot \mathbf{x}) - y)h(\mathbf{w} \cdot \mathbf{x})\mathbf{x}]$ for some function h (in particular, for the squared loss $h(\mathbf{w} \cdot \mathbf{x}) = 2\sigma'(\mathbf{w} \cdot \mathbf{x})$, while for the surrogate loss, $h \equiv 2$). Since we are considering optimizing \mathbf{w} over the unit sphere (see [Remark 30](#)), the relevant information for updating \mathbf{w} is in its orthogonal complement (as we are not changing its length), so it suffices to consider $\mathbf{g}(\mathbf{w}, h) := \mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}} [(\sigma(\mathbf{w} \cdot \mathbf{x}) - y)h(\mathbf{w} \cdot \mathbf{x})\mathbf{x}^\perp \mathbf{w}]$.

Intuitively, if we can show that $-\mathbf{g}(\mathbf{w}, h)$ strongly correlates with \mathbf{w}^* , then this information can be used to update \mathbf{w} to better align with \mathbf{w}^* , until we reach the target approximation error. Observe first that, as the Gaussian distribution is independent across orthogonal directions, we have $-\mathbf{g}(\mathbf{w}, h) \cdot \mathbf{w}^* = \mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}} [yh(\mathbf{w} \cdot \mathbf{x})\mathbf{x}^\perp \mathbf{w} \cdot \mathbf{w}^*]$. Writing $y = \sigma(\mathbf{w}^* \cdot \mathbf{x}) + y - \sigma(\mathbf{w}^* \cdot \mathbf{x})$, the quantity $-\mathbf{g}(\mathbf{w}, h) \cdot \mathbf{w}^*$ can be decomposed into two parts: (i) corresponding to “clean” labels $\sigma(\mathbf{w}^* \cdot \mathbf{x})$, and (ii) corresponding to label noise $y - \sigma(\mathbf{w}^* \cdot \mathbf{x})$. Letting θ denote the angle between \mathbf{w} and \mathbf{w}^* , it is possible to argue (using Stein’s lemma, see [Fact 26](#)) that the “clean label” portion of $-\mathbf{g}(\mathbf{w}, h) \cdot \mathbf{w}^*$ equals $\mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}} [\sigma'(\mathbf{w}^* \cdot \mathbf{x})h(\mathbf{w} \cdot \mathbf{x})] \sin^2 \theta$. For the “noisy” label portion, by the Cauchy-Schwarz inequality and the definition of OPT, we can write

$$-\mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}} [(y - \sigma(\mathbf{w}^* \cdot \mathbf{x}))h(\mathbf{w} \cdot \mathbf{x})(\mathbf{w}^* \cdot \mathbf{x}^\perp \mathbf{w})] \leq \sqrt{\text{OPT}} \|h\|_{L_2} \sin(\theta(\mathbf{w}, \mathbf{w}^*)),$$

where $\|h\|_{L_2} := (\mathbf{E}_{z \sim \mathcal{N}(0, 1)} [h^2(z)])^{1/2}$. Since labels are adversarial, the inequality can in fact be made to hold with equality. Thus, summarizing the above discussion, we have

$$-\mathbf{g}(\mathbf{w}, h) \cdot \mathbf{w}^* \geq \mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}} [\sigma'(\mathbf{w}^* \cdot \mathbf{x})h(\mathbf{w} \cdot \mathbf{x})] \sin^2 \theta - \sqrt{\text{OPT}} \|h\|_{L_2} \sin \theta. \quad (1)$$

We can assume w.l.o.g. that $\|h\|_{L_2} = 1$, since dividing both sides by $\|h\|_{L_2}$ would give us the same conclusion. For $-\mathbf{g}(\mathbf{w}, h)$ to contain a useful signal guiding the algorithm towards target solutions, we need that $-\mathbf{g}(\mathbf{w}, h) \cdot \mathbf{w}^* > 0$, for which we ought to argue that $G(h) := \mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}} [\sigma'(\mathbf{w}^* \cdot \mathbf{x})h(\mathbf{w} \cdot \mathbf{x})] > 0$. It is possible to argue that $G(h)$ is maximized for the “ideal” choice of $h(\mathbf{w} \cdot \mathbf{x}) \propto \sigma'(\cos \theta \mathbf{w} \cdot \mathbf{x} + \sin \theta \mathbf{z} \cdot \mathbf{x})$ with independently sampled $\mathbf{z} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$. This can equivalently be seen as applying the Ornstein–Uhlenbeck semi-group with parameter $\rho = \cos \theta$ to σ' , which motivates its use in our work. Of course, since $\cos \theta$ is not known to the algorithm, the smoothing parameter ρ needs to be carefully chosen and adjusted between the algorithm updates.

Alignment and Optimization Local error bounds have long history in optimization and represent some of the most important technical tools for establishing iterate convergence to target solutions, especially in the context of gradient-based algorithms (see, e.g., [Pang \(1997\)](#)). Broadly speaking, local error bounds are inequalities that bound below some measure of the problem “residual” or error by a measure of distance to the target solution set. “Local” in the name refers to such inequalities being valid only in a local region around the target solution set. Within learning theory and in the context of GLM learning, they have played a crucial role in the analysis of (stochastic) gradient-based algorithms ([Mei et al., 2018](#); [Wang et al., 2023](#); [Zarifis et al., 2024](#); [Wang et al., 2024](#)).

Our main structural result, stated in [Proposition 5](#), is a local error bound for which the residual is $-\mathbf{g}(\mathbf{w}) \cdot \mathbf{w}^*$ for the gradient field $\mathbf{g}(\mathbf{w})$ corresponding to the data augmented squared loss function, as discussed above. This residual has the meaning of the “alignment” between $-\mathbf{g}(\mathbf{w})$ and \mathbf{w}^* .

Specifically, we prove that in a local region around a certain set \mathcal{S} , the following inequality holds for any $\rho \in (0, 1)$ and θ being the angle between \mathbf{w}, \mathbf{w}^* :

$$-\mathbf{g}(\mathbf{w}) \cdot \mathbf{w}^* \geq (2/3) \|\mathbf{T}_{\sqrt{\rho \cos \theta}} \sigma'\|_{L_2}^2 \sin^2 \theta. \quad (2)$$

Observe that, since we are optimizing over the unit sphere, $\|\mathbf{w} - \mathbf{w}^*\|_2^2 \approx \sin^2(\theta)$. This structural result allows us to argue that, provided an initial parameter vector \mathbf{w}_0 for which (2) holds, we can update iterates \mathbf{w} to contract the angle θ , until the set \mathcal{S} is reached. While this general idea seems relatively simple, making it work requires a rather technical argument to (i) ensure we can initialize the algorithm in the region where (2) holds, (ii) adjust the value of ρ between the algorithm updates to ensure we remain in the region where (2) applies, and (iii) argue that all parameter vectors in \mathcal{S} are $O(\text{OPT})$ approximate solutions. Part (ii) is handled using an intricate inductive argument. Parts (i) and (iii) are addressed by proving a series of structural results for the class of (B, L) -regular monotone activations, discussed below.

Approximation and Regularity of Monotone Functions While handling arbitrary monotone functions is provably impossible, we show that fairly minimal assumptions suffice for our approach. In particular, we handle all (B, L) -regular monotone activations, which we show can be well-approximated by monotone piecewise-constant (staircase) functions. In more detail, instead of directly proving the desired properties of monotone (B, L) -regular activations, we consider the class of staircase functions, which only increase within a compact interval (and are constant outside it). For this class of staircase functions, we prove that the high-degree terms in their Hermite expansion (see [Appendix B](#) for relevant definitions)—namely, terms with degree $> 1/\theta^2$ for θ sufficiently small—are bounded by $\|\mathbf{T}_{\cos \theta} \sigma'\|_{L_2}^2 \sin^2 \theta$, and, further, this result extends to all (B, L) -regular functions ([Proposition 16](#)). Proving this structural result relies on auxiliary results relating Ornstein–Uhlenbeck semigroups of activations and their derivatives that may be of independent interest. [Proposition 16](#) is then used to argue that the target set \mathcal{S} to which the iterates of the algorithm converge only contains vectors with L_2^2 error $O(\text{OPT})$, addressing the aforementioned issue (iii).

Since the result from [Proposition 16](#) only applies for sufficiently small θ , we need to argue that the algorithm can be appropriately initialized. In particular, random initialization is insufficient since we need roughly that $\theta_0 \leq \pi/2 - \Omega((\log(1/\epsilon))^{-1/2})$. To address this requirement, we apply a label transformation $\tilde{y} = \mathbb{1}\{y \geq t\}$ for a carefully chosen threshold t , where $\mathbb{1}$ denotes the indicator function. In particular, to select t , we leverage the staircase approximation of monotone functions discussed above. We argue that the problem reduces to learning $\text{sign}(\sigma(\mathbf{w}^* \cdot \mathbf{x}) - t)$, which is an instance of learning halfspaces with adversarial noise.¹ In particular, we argue that constant approximate solutions to this halfspace learning problem suffice for our initialization.

1.2. Preliminaries

For $n \in \mathbb{Z}_+$, let $[n] := \{1, \dots, n\}$. We use bold lowercase letters to denote vectors and bold uppercase letters for matrices. For $\mathbf{x} \in \mathbb{R}^d$ and $i \in [d]$, \mathbf{x}_i denotes the i^{th} coordinate of \mathbf{x} , and $\|\mathbf{x}\|_2 := (\sum_{i=1}^d \mathbf{x}_i^2)^{1/2}$ denotes the ℓ_2 -norm of \mathbf{x} . We use $\mathbf{x} \cdot \mathbf{y}$ for the dot product of $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ and $\theta(\mathbf{x}, \mathbf{y})$ for the angle between \mathbf{x}, \mathbf{y} . We slightly abuse notation and denote by \mathbf{e}_i the i -th standard basis vector in \mathbb{R}^d . We use $\mathbb{1}\{A\}$ to denote the characteristic function of the set A . For unit vectors \mathbf{u}, \mathbf{v} , we use $\mathbf{u}^{\perp \mathbf{v}}$ to denote the component of \mathbf{u} that is orthogonal to \mathbf{v} i.e., $\mathbf{u}^{\perp \mathbf{v}} = (\mathbf{I} - \mathbf{v}\mathbf{v}^\top)\mathbf{u}$.

¹Note here that $\text{sign}(\sigma(\mathbf{w}^* \cdot \mathbf{x}) - t)$ being a halfspace crucially relies on σ being monotone.

Finally, we use \mathbb{S}^{d-1} to denote the unit sphere in \mathbb{R}^d and \mathbb{B} to denote the unit ball. For (\mathbf{x}, y) distributed according to \mathcal{D} , we denote by $\mathcal{D}_{\mathbf{x}}$ the marginal distribution of \mathbf{x} . We use the standard $O(\cdot)$, $\Theta(\cdot)$, $\Omega(\cdot)$ asymptotic notation and $\tilde{O}(\cdot)$ to omit polylogarithmic factors in the argument.

Gaussian Space Let $\mathcal{N}(\mathbf{0}, \mathbf{I})$ denote the standard normal distribution. The L_2 norm of a function g with respect to the standard normal is $\|g\|_{L_2} = (\mathbf{E}_{\mathbf{x} \sim \mathcal{N}}[|g(\mathbf{x})|^2])^{1/2}$, while $\|g\|_{L_\infty}$ is the essential supremum of the absolute value of g . We denote by $L_2(\mathcal{N})$ the vector space of all functions $f : \mathbb{R}^d \rightarrow \mathbb{R}$ such that $\|f\|_{L_2} < \infty$. $\text{He}_i(z)$ denotes the *normalized* probabilist’s Hermite polynomial of degree i . For any function $f : \mathbb{R} \rightarrow \mathbb{R}$, $f \in L_2(\mathcal{N})$, we denote by $P_k f(z)$ the degree k partial sum of the Hermite expansion of f , i.e., $P_k f(z) = \sum_{i \leq k} \hat{f}(i) \text{He}_i(z)$, and let $P_{>k} f(z) = \sum_{i > k} \hat{f}(i) \text{He}_i(z)$, where $\hat{f}(i) = \mathbf{E}_{z \sim \mathcal{N}(0,1)}[f(z) \text{He}_i(z)]$. An important tool for our work is the Ornstein–Uhlenbeck semigroup, formally defined below.

Definition 3 (Ornstein–Uhlenbeck Semigroup) Let $\rho \in (0, 1)$. The Ornstein–Uhlenbeck semigroup, denoted by T_ρ , is a linear operator that maps a function $g \in L_2(\mathcal{N})$ to the function $T_\rho g$ defined as: $(T_\rho g)(\mathbf{x}) := \mathbf{E}_{\mathbf{z} \sim \mathcal{N}}[g(\rho \mathbf{x} + \sqrt{1 - \rho^2} \mathbf{z})]$.

2. Data Augmentation and Its Effect on the L_2^2 Loss Landscape

This section describes the basic data augmentation approach and provides some of the key structural properties relating to the data-augmented L_2^2 loss.

2.1. Augmenting the Data: Connection to Ornstein–Uhlenbeck Semigroup

As already discussed in [Section 1.1](#), our algorithm relies on the data augmentation technique, i.e., in each iteration, the algorithm injects Gaussian noise (see [Algorithm 1](#)), which has the effect of improving the regularity properties of the loss landscape, as shown in this section.

Algorithm 1 Augment Dataset with Injected White Noise

- 1: **Input:** Parameters ρ, m ; Sample data $\mathfrak{D} = \{(\mathbf{x}^{(1)}, y^{(1)}), \dots, (\mathbf{x}^{(N)}, y^{(N)})\}$; $S \leftarrow \emptyset$
 - 2: **for** $(\mathbf{x}^{(i)}, y^{(i)}) \in \mathfrak{D}$ **do**
 - 3: **for** $j = 1, \dots, m$ **do**
 - 4: Sample \mathbf{z} from $\mathcal{N}(\mathbf{0}, \mathbf{I})$. Let $\tilde{\mathbf{x}}^{(j)} \leftarrow \rho \mathbf{x}^{(i)} + (1 - \rho^2)^{1/2} \mathbf{z}$ and add to $S \leftarrow S \cup \{(\tilde{\mathbf{x}}^{(j)}, y^{(i)})\}$.
 - 5: **Return:** S
-

The augmentation can be viewed as a transformation of the distribution \mathcal{D} to \mathcal{D}_ρ , where for any $(\tilde{\mathbf{x}}, y) \sim \mathcal{D}_\rho$, we have $\tilde{\mathbf{x}} \sim \rho \mathcal{D}_{\mathbf{x}} + (1 - \rho^2)^{1/2} \mathcal{N}(\mathbf{0}, \mathbf{I})$. The data augmentation introduced in [Algorithm 1](#) in fact simulates the Ornstein–Uhlenbeck semigroup, as stated below.

Lemma 4 Let \mathcal{D} be a distribution of labeled examples (\mathbf{x}, y) such that $\mathcal{D}_{\mathbf{x}} = \mathcal{N}(\mathbf{0}, \mathbf{I})$ and let \mathcal{D}_ρ be the distribution constructed by applying [Algorithm 1](#) to \mathcal{D} . Then, for any $f : \mathbb{R} \rightarrow \mathbb{R}$ and any unit vector $\mathbf{w} \in \mathbb{R}^d$ with $\|\mathbf{E}_{\mathbf{x} \sim \mathcal{D}_{\mathbf{x}}}[f(\mathbf{w} \cdot \mathbf{x})]\| < \infty$, we have $\mathbf{E}_{(\tilde{\mathbf{x}}, y) \sim \mathcal{D}_\rho}[f(\mathbf{w} \cdot \tilde{\mathbf{x}})] = \mathbf{E}_{\mathbf{x} \sim \mathcal{D}_{\mathbf{x}}}[T_\rho f(\mathbf{w} \cdot \mathbf{x})]$.

2.2. Alignment of the Gradients of the Augmented Loss

Our main structural result is to show that the gradients of the square loss applied to the augmented data correlate with a target parameter vector \mathbf{w}^* . We use $\mathcal{L}_\rho(\mathbf{w}) = \mathbf{E}_{(\tilde{\mathbf{x}}, y) \sim \mathcal{D}_\rho}[(\mathbf{w} \cdot \tilde{\mathbf{x}} - y)^2]$ to denote the square loss on the augmented data and refer to it as the “augmented loss.”

Proposition 5 (Main Structural Result) Fix an activation $\sigma : \mathbb{R} \rightarrow \mathbb{R}$. Let \mathcal{D} be a distribution of labeled examples (\mathbf{x}, y) such that $\mathcal{D}_{\mathbf{x}} = \mathcal{N}(\mathbf{0}, \mathbf{I})$ and let \mathcal{D}_{ρ} with $\rho \in (0, 1)$ be the distribution resulting from applying [Algorithm 1](#) to \mathcal{D} . Fix vectors $\mathbf{w}^*, \mathbf{w} \in \mathbb{S}^{d-1}$ such that $\mathcal{L}(\mathbf{w}^*) = \text{OPT}$ and let $\theta = \theta(\mathbf{w}^*, \mathbf{w})$. Let $\mathbf{g}(\mathbf{w}) = (1/(2\rho))(\nabla_{\mathbf{w}} \mathcal{L}_{\rho}(\mathbf{w}))^{\perp \mathbf{w}}$. If $0 < \rho \leq \cos \theta < 1$ and $\sin \theta \geq 3\sqrt{\text{OPT}}/\|\mathbf{T}_{\rho}\sigma'\|_{L_2}$, then, $\mathbf{g}(\mathbf{w}) \cdot \mathbf{w}^* \leq -(2/3)\|\mathbf{T}_{\sqrt{\rho \cos \theta}}\sigma'\|_{L_2}^2 \sin^2 \theta$.

To prove the proposition, we rely on the following auxiliary lemma, which relates $\mathbf{g}(\mathbf{w})$ to the Ornstein–Uhlenbeck semigroup applied to the derivative of the activation.

Lemma 6 Let $\mathbf{g}(\mathbf{w}) = (1/(2\rho))(\nabla_{\mathbf{w}} \mathcal{L}_{\rho}(\mathbf{w}))^{\perp \mathbf{w}}$. Then, $\mathbf{g}(\mathbf{w}) = -\mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}}[y \mathbf{T}_{\rho}\sigma'(\mathbf{w} \cdot \mathbf{x}) \mathbf{x}^{\perp \mathbf{w}}]$.

Proof Sketch of Proposition 5 Assume that $(\mathbf{w}^*)^{\perp \mathbf{w}} \neq \mathbf{0}$; otherwise the statements hold trivially. Let $\mathbf{v} := (\mathbf{w}^*)^{\perp \mathbf{w}} / \|(\mathbf{w}^*)^{\perp \mathbf{w}}\|_2$; then $\mathbf{w}^* = \cos \theta \mathbf{w} + \sin \theta \mathbf{v}$ and $\mathbf{w} \cdot \mathbf{x}, \mathbf{v} \cdot \mathbf{x}$ are independent standard Gaussians. By [Lemma 6](#), $-\mathbf{g}(\mathbf{w}) \cdot \mathbf{w}^* = \mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}}[y \mathbf{T}_{\rho}\sigma'(\mathbf{w} \cdot \mathbf{x}) \mathbf{v} \cdot \mathbf{x}] \sin \theta$. Hence, adding and subtracting $\sigma(\mathbf{w}^* \cdot \mathbf{x})$ to y in the expectation we get $-\mathbf{g}(\mathbf{w}) \cdot \mathbf{w}^* = ((Q_1) + (Q_2)) \sin \theta$, where $(Q_1) := \mathbf{E}_{\mathbf{x} \sim \mathcal{D}_{\mathbf{x}}}[\sigma(\mathbf{w}^* \cdot \mathbf{x}) \mathbf{T}_{\rho}\sigma'(\mathbf{w} \cdot \mathbf{x}) \mathbf{v} \cdot \mathbf{x}]$ and $(Q_2) := \mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}}[(y - \sigma(\mathbf{w}^* \cdot \mathbf{x})) \mathbf{T}_{\rho}\sigma'(\mathbf{w} \cdot \mathbf{x}) \mathbf{v} \cdot \mathbf{x}]$. By Cauchy-Schwarz inequality, $(Q_2) \geq -\sqrt{\text{OPT}} \mathbf{E}_{\mathbf{x} \sim \mathcal{D}_{\mathbf{x}}}[(\mathbf{T}_{\rho}\sigma'(\mathbf{w} \cdot \mathbf{x}))^2] = -\sqrt{\text{OPT}} \|\mathbf{T}_{\rho}\sigma'\|_{L_2}$, where we used the definition of OPT and that $\mathbf{w} \cdot \mathbf{x}$ and $\mathbf{v} \cdot \mathbf{x}$ are independent Gaussians. To bound (Q_1) , applying Stein’s lemma ([Fact 26](#)) as well as the properties of Ornstein–Uhlenbeck semigroup ([Fact 21](#)) we can show that $(Q_1) = \mathbf{E}_{\mathbf{x} \sim \mathcal{D}_{\mathbf{x}}}[\mathbf{T}_{\cos \theta}\sigma'(\mathbf{w} \cdot \mathbf{x}) \mathbf{T}_{\rho}\sigma'(\mathbf{w} \cdot \mathbf{x})] \sin \theta = \|\mathbf{T}_{\sqrt{\rho \cos \theta}}\sigma'\|_{L_2}^2 \sin \theta$. Therefore, we have that $-\mathbf{g}(\mathbf{w}) \cdot \mathbf{w}^* \geq \|\mathbf{T}_{\sqrt{\rho \cos \theta}}\sigma'\|_{L_2}^2 \sin^2 \theta - \sqrt{\text{OPT}} \|\mathbf{T}_{\rho}\sigma'\|_{L_2} \sin \theta$.

To finish the proof, note that $\|\mathbf{T}_{\lambda} f\|_{L_2}$ is non-decreasing in $\lambda \in (0, 1)$ for any function $f \in L_2(\mathcal{N})$ ([Fact 21](#)), therefore $\|\mathbf{T}_{\sqrt{\rho \cos \theta}}\sigma'\|_{L_2} \geq \|\mathbf{T}_{\rho}\sigma'\|_{L_2}$ if $\cos \theta \geq \rho$. Using the assumption that $\sin \theta \geq 3\sqrt{\text{OPT}}/\|\mathbf{T}_{\rho}\sigma'\|_{L_2}$, we obtain $-\mathbf{g}(\mathbf{w}) \cdot \mathbf{w}^* \geq (2/3)\|\mathbf{T}_{\sqrt{\rho \cos \theta}}\sigma'\|_{L_2}^2 \sin^2 \theta$. \blacksquare

2.3. Critical Points and Their Connection to the L_2^2 Loss

[Proposition 5](#) provides sufficient conditions ensuring that the vector $-\mathbf{g}(\mathbf{w})$ guides \mathbf{w} towards the direction of \mathbf{w}^* whenever we are in a region around approximate solutions. Specifically, if the parameter ρ is chosen appropriately and the following alignment condition holds: $\sin \theta \|\mathbf{T}_{\cos \theta}\sigma'\|_{L_2} \geq 3\sqrt{\text{OPT}}$, then $-\mathbf{g}(\mathbf{w})$ has a nontrivial correlation with \mathbf{w}^* . Otherwise, we can guarantee that the angle between \mathbf{w} and \mathbf{w}^* is already sufficiently small. This implies that the region of convergence of an algorithm that relies on $-\mathbf{g}(\mathbf{w})$ depends on the quantity: $\psi_{\sigma}(\theta) := \sin \theta \|\mathbf{T}_{\cos \theta}\sigma'\|_{L_2}$. Motivated by this observation, we define the *Convergence Region*, as follows.

Definition 7 (Critical Point and Convergence Region of σ) Given $\sigma : \mathbb{R} \rightarrow \mathbb{R}$, $\sigma \in L_2(\mathcal{N})$, and $\theta_0 \in [0, \pi/2]$, we define the error alignment function $\psi_{\sigma} : [0, \pi/2] \rightarrow \mathbb{R}_+$ by $\psi_{\sigma}(\theta) := \sin \theta \|\mathbf{T}_{\cos \theta}\sigma'\|_{L_2}$. For any $\epsilon > 0$, we define the Convergence Region $\mathcal{R}_{\sigma, \theta_0}(\epsilon) = \{\theta : \psi_{\sigma}(\theta) \leq \sqrt{\epsilon}\} \cap \{\theta : 0 \leq \theta \leq \theta_0\}$. We say that θ^* is a $(\sigma, \theta_0, \epsilon)$ -Critical Point if $\theta^* = \{\max \theta : \theta \in \mathcal{R}_{\sigma, \theta_0}(\epsilon)\}$.

[Definition 7](#) utilizes an upper bound θ_0 because $\psi_{\sigma}(\theta)$ is not necessarily monotonic. Specifically, it can be shown that $\psi_{\sigma}(\theta)$ is non-decreasing up to some θ' and then non-increasing (see [Figure 1](#) for illustrative examples and [Claim 46](#) in [Appendix D](#) for a more formal statement and proof). Consequently, the region $\mathcal{R}_{\sigma, \theta_0}(\epsilon)$ may consist of two disjoint intervals. The role of (appropriately selected) θ_0 is to ensure that this does not happen. The significance of the above definition comes from the following proposition, which bounds the L_2^2 error within the Convergence Region.

Proposition 8 (Critical Points and L_2^2 Error) *Given $\sigma : \mathbb{R} \rightarrow \mathbb{R}$, $\sigma \in L_2(\mathcal{N})$, and a distribution \mathcal{D} of labeled examples (\mathbf{x}, y) such that $\mathcal{D}_{\mathbf{x}} = \mathcal{N}(\mathbf{0}, \mathbf{I})$, let \mathbf{w}^* be such that $\mathcal{L}(\mathbf{w}^*) = \text{OPT}$. Then, for any unit vector \mathbf{w} with $\theta = \theta(\mathbf{w}, \mathbf{w}^*)$ such that $\theta \leq \theta^*$, where θ^* is the $(\sigma, \theta_0, \text{COPT})$ -Critical Point for some θ_0 and $C > 1$ an absolute constant, $\mathcal{L}(\mathbf{w}) \leq O(\text{OPT}) + 4\|\mathbf{P}_{>(1/\theta^*)^2\sigma}\|_{L_2}^2$.*

To prove [Proposition 8](#), we first prove the following technical lemma, which decomposes the error into $O(\text{OPT})$ and error terms that depend on the properties of the activation σ . A more formal version of [Lemma 9](#) is stated as [Lemma 48](#) in [Appendix D](#), where its proof is also provided.

Lemma 9 (Error Decomposition, Informal) *Under the assumptions of [Proposition 8](#), we have that $\mathcal{L}(\mathbf{w}) \leq 2\text{OPT} + C\theta^2\|\mathbf{T}_{\rho}\sigma'\|_{L_2}^2 + 4\|\mathbf{P}_{>k\sigma}\|_{L_2}^2$, where C is an absolute constant, for the following choices of ρ : (i) if $k \leq 1$, ρ can be any value in $(0, 1)$; (ii) if $k \geq 2$, then $\rho = \sqrt{1 - 1/k}$.*

Proof Sketch of [Proposition 8](#) Since θ^* is the $(\sigma, \theta_0, \text{COPT})$ -Critical point, we have by its definition that $(\theta^*)^2\|\mathbf{T}_{\cos(\theta^*)}\sigma'\|_{L_2}^2 \leq \text{COPT}$. Let $k = \lfloor 1/(\theta^*)^2 \rfloor$. Consider first $\theta^* \leq 1/\sqrt{2}$, which implies that $k \geq 2$. Observe that $(1 - 1/k)^{1/2} \leq \cos \theta^*$, thus as $\|\mathbf{T}_{\rho}\sigma'\|_{L_2}$ is non-decreasing with respect to ρ ([Fact 21](#)), we further have $\|\mathbf{T}_{(1-1/k)^{1/2}}\sigma'\|_{L_2}^2 \leq \|\mathbf{T}_{\cos(\theta^*)}\sigma'\|_{L_2}^2$. Thus, applying [Lemma 9](#), for any $\theta \leq \theta^*$, we get $\mathcal{L}(\mathbf{w}) \leq (2 + 8eC)\text{OPT} + 4\|\mathbf{P}_{>(1/\theta^*)^2\sigma}\|_{L_2}^2$. When $\theta^* > 1/\sqrt{2}$, then $k = 0, 1$. Choose $\rho = \cos(\theta^*) \in (0, 1)$ in [Lemma 9](#), note again that $(\theta^*)^2\|\mathbf{T}_{\cos(\theta^*)}\sigma'\|_{L_2}^2 \leq \text{COPT}$ by the definition of θ^* , thus we have $\mathcal{L}(\mathbf{w}) \leq (2 + C)\text{OPT} + 4\|\mathbf{P}_{>(1/\theta^*)^2\sigma}\|_{L_2}^2$. ■

3. Learning GLMs via Variable Augmentation

In this section, we present our main algorithm ([Algorithm 2](#)) for robustly learning Gaussian GLMs, as stated in [Problem 1](#). Our algorithm applies to the following large class of activations:

Definition 10 ((B, L) -Regular Activations) *Given parameters $B, L > 0$, we define the class of (B, L) -Regular activations, denoted by $\mathcal{H}(B, L)$, as the class containing all functions $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ such that 1) $\|\sigma\|_{L_{\infty}} \leq B$ and 2) $\|\sigma'\|_{L_2} \leq L$. Given $\epsilon > 0$, we define the class of ϵ -Extended (B, L) -Regular activations, denoted by $\mathcal{H}_{\epsilon}(B, L)$, as the class containing all activations $\sigma_1 : \mathbb{R} \rightarrow \mathbb{R}$ for which there exists $\sigma_2 \in \mathcal{H}(B, L)$ such that $\|\sigma_1 - \sigma_2\|_{L_2}^2 \leq \epsilon$.*

Our results hold for any activation that is ϵ -Extended (B, L) -Regular. This class contains all Lipschitz activations and all activations with bounded 4th moment. More examples are in [Appendix C](#).

[Algorithm 2](#) uses the main structural result of [Section 2](#) ([Proposition 5](#)) to update its iterates $\mathbf{w}^{(t)}$. In particular, for $\theta_t = \theta(\mathbf{w}^{(t)}, \mathbf{w}^*)$, we show that after one gradient descent-style update, the angle θ_{t+1} shrinks by a factor $1 - c$, i.e., $\theta_{t+1} \leq (1 - c)\theta_t$, where $0 < c < 1$ is an absolute constant. A crucial feature of [Algorithm 2](#) is that in each iteration it carefully chooses a new value of ρ_t . This variable update of ρ_t ensures the ‘signal’ of the gradient is present until $\mathbf{w}^{(t)}$ reaches a small region centered at \mathbf{w}^* . Within this region, the agnostic noise corrupts the signal of the augmented gradient and convergence to \mathbf{w}^* is no longer guaranteed. However, the region that $\mathbf{w}^{(t)}$ reaches is in fact the Convergence Region $\mathcal{R}_{\sigma, \theta_0}(O(\text{OPT}))$, within which all points are solutions with the target approximation error. We show in [Section 4](#) that for any monotone (B, L) -Regular activation, any $\hat{\mathbf{w}}$ in $\mathcal{R}_{\sigma, \theta_0}(O(\text{OPT}))$ is a solution with error $\text{COPT} + \epsilon$, under suitable initialization. We now present our algorithm and state our main result ([Theorem 11](#)) for general (B, L) -regular activations.

Algorithm 2 SGD – VA: SGD with Variable Augmentation

```

1: Input: Parameters  $\epsilon, T$ ; Sample access to  $\mathcal{D}$ 
2:  $[\mathbf{w}^{(0)}, \bar{\theta}] = \text{Initialization}[\sigma]$  (Section 4.3); set  $\rho_0 = \cos \bar{\theta}$ 
3: for  $t = 0, \dots, T$  do
4:   Draw  $n$  samples  $\widehat{\mathcal{D}}_{\rho_t} = \{(\tilde{\mathbf{x}}^{(i)}, y^{(i)})\}_{i=1}^n$  from  $\mathcal{D}_{\rho_t}$  using Algorithm 1
5:    $\widehat{\mathbf{g}}(\mathbf{w}^{(t)}) = -(1/\rho_t) \mathbf{E}_{(\tilde{\mathbf{x}}, y) \sim \widehat{\mathcal{D}}_{\rho_t}} [y \sigma'(\mathbf{w}^{(t)} \cdot \tilde{\mathbf{x}})(\tilde{\mathbf{x}})^{\perp \mathbf{w}^{(t)}}]$ 
6:    $\eta_t = \sqrt{(1 - \rho_t)/2} / (4 \|\widehat{\mathbf{g}}(\mathbf{w}^{(t)})\|_2)$ 
7:    $\mathbf{w}^{(t+1)} = (\mathbf{w}^{(t)} - \eta_t \widehat{\mathbf{g}}(\mathbf{w}^{(t)})) / \|\mathbf{w}^{(t)} - \eta_t \widehat{\mathbf{g}}(\mathbf{w}^{(t)})\|_2$ 
8:    $\rho_{t+1} = 1 - (1 - 1/256)^2 (1 - \rho_t)$ 
9:    $\widehat{\mathbf{w}} = \text{Test}[\mathbf{w}^{(0)}, \mathbf{w}^{(1)}, \dots, \mathbf{w}^{(T)}]$  (Algorithm 5)
10: Return:  $\widehat{\mathbf{w}}$ 
    
```

Theorem 11 Let $\epsilon > 0$. Let σ be a (B, L) -Regular activation. Algorithm 2, given initialization $\mathbf{w}^{(0)}$ with $\theta(\mathbf{w}^{(0)}, \mathbf{w}^*) \leq \bar{\theta}$, runs at most $T = O(\log(L/\epsilon))$ iterations, draws $\Theta(dB^2 \log(L/\epsilon)/\epsilon + B^4 \log(L/\epsilon)/\epsilon^2)$ samples, and returns a vector $\widehat{\mathbf{w}}$ such that with probability at least $2/3$, $\widehat{\mathbf{w}} \in \mathcal{R}_{\sigma, \theta_0}(O(\text{OPT}))$. Moreover, $\mathcal{L}(\widehat{\mathbf{w}}) \leq O(\text{OPT}) + \epsilon + 4\|\mathbf{P}_{>1/(\theta^*)^2} \sigma\|_{L_2}^2$.

Define $\zeta(\rho) := \sqrt{\text{OPT}} / \|\mathbf{T}_{\rho} \sigma'\|_{L_2}$. Recall that in Proposition 5 we showed when

$$\text{conditions for fast convergence: } \sin \theta_t \geq 3\zeta(\rho_t), \zeta(\rho_t) := \sqrt{\text{OPT}} / \|\mathbf{T}_{\rho_t} \sigma'\|_{L_2}, \rho_t \leq \cos \theta_t \quad (3)$$

hold, $-\mathbf{g}(\mathbf{w}^{(t)})$ aligns well with \mathbf{w}^* , enabling $\theta_{t+1} \leq (1 - c)\theta_t$. However, two critical issues arise: (1) If $\sin \theta_t \lesssim \zeta(\rho_t)$, then conditions in Equation (3) do not hold, and we cannot guarantee that θ_t contracts. Moreover, since $\|\mathbf{T}_{\rho_t} \sigma'\|_{L_2} \leq \|\mathbf{T}_{\cos \theta_t} \sigma'\|_{L_2}$, it is not necessarily the case that $\sin \theta_t \lesssim \zeta(\cos \theta_t)$, thus we also cannot assert that $\mathbf{w}^{(t)}$ has reached the target region $\mathcal{R}_{\sigma, \theta_0}(C^2 \text{OPT})$.

(2) Suppose that the conditions in Equation (3) apply, hence $\theta_{t+1} \leq (1 - c)\theta_t$. Assume that $\mathbf{w}^{(t+1)}$ is still far from \mathbf{w}^* and $\theta_{t+1} \gtrsim \zeta(\cos \theta_{t+1})$. It is possible that $\zeta(\cos \theta_{t+1}) \lesssim \theta_{t+1} \lesssim \zeta(\rho_t)$, because $\|\mathbf{T}_{\cos \theta_{t+1}} \sigma'\|_{L_2} \geq \|\mathbf{T}_{\rho_t} \sigma'\|_{L_2}$, as $\rho_t \leq \cos \theta_t \leq \cos \theta_{t+1}$ and $\|\mathbf{T}_{\rho} \sigma'\|_{L_2}$ is an increasing function of ρ (by Fact 21). This implies that the conditions in Equation (3) might become invalid for ρ_t .

To overcome these issues, we consider the event $\mathcal{E}_t := \{|\cos \theta_t - \rho_t| \leq \sin^2 \theta_t, \sin \theta_t \leq C\zeta(\rho_t)\}$. We first observe that when \mathcal{E}_t is satisfied, then, $|\cos \theta_t - \rho_t| \leq \sin^2 \theta_t$ indicates that ρ_t and $\cos \theta_t$ are sufficiently close and we argue that $\zeta(\rho_t) \approx \zeta(\cos(2\theta_t))$, therefore, we have that $\sin \theta_t \leq C\zeta(\cos(2\theta_t))$. From here we argue that $\mathbf{w}^{(t)} \in \mathcal{R}_{\sigma, \theta_0}(4C^2 \text{OPT})$. This addresses (1). Now suppose \mathcal{E}_t does not hold. We use induction to show that updating ρ_t by Line 8, we have $\rho_{t+1} \leq \cos \theta_{t+1}$. Now if $\sin \theta_{t+1} \geq 3\zeta(\rho_{t+1})$, Equation (3) is satisfied and we decrease θ_{t+1} , whereas if $\sin \theta_{t+1} \leq 3\zeta(\rho_{t+1})$, we know that $\mathbf{w}^{(t+1)}$ is the target vector as discussed above. This addresses issue (2). Figure 4 in the appendix provides a visual illustration of the mechanism of Algorithm 2.

Proof Sketch of Theorem 11 Let $\theta_t = \theta(\mathbf{w}^{(t)}, \mathbf{w}^*)$ and define $\zeta(\rho) := \sqrt{\text{OPT}} / \|\mathbf{T}_{\rho} \sigma'\|_{L_2}$. Assume that $\epsilon \leq \text{OPT}$ (otherwise we can get additive error $O(\epsilon)$). Suppose further that we have access to the population gradients $\mathbf{g}^{(t)}$, so that the statistical error is negligible (we bound it in Appendix E.2).

Define the event $\mathcal{E}_t := \{|\cos \theta_t - \rho_t| \leq \sin^2 \theta_t, \sin \theta_t \lesssim \zeta(\rho_t)\}$. We claim that if \mathcal{E}_t holds at some iteration t , then the algorithm converges to a vector in the region $\mathcal{R}_{\sigma, \theta_0}(O(\text{OPT}))$. In particular, in this case we have that $\rho_t \geq \cos 2\theta_t$, hence $\sin \theta_t \lesssim \zeta(\rho_t) \lesssim \zeta(\cos(2\theta_t))$, i.e., $\psi_{\sigma}(2\theta_t) \lesssim \sqrt{\text{OPT}}$ as $\zeta(\rho)$ is a decreasing function, which implies that $\mathbf{w}^{(t)} \in \mathcal{R}_{\sigma, \theta_0}(O(\text{OPT}))$.

It remains to show that there exists some $t^* \leq T$ for which \mathcal{E}_{t^*} holds. In fact, it suffices to prove that $\rho_t \leq \cos \theta_t$ for all $t \leq t^*$. Since $\rho_t \rightarrow 1$, if no such t^* existed then eventually $\cos \theta_t$ would be arbitrarily close to 1, forcing $\sin \theta_t \lesssim \zeta(1)$ and yielding a contradiction. We prove $\rho_t \leq \cos \theta_t$ for all $t \leq t^*$ by induction. By the assumptions on θ_0 , we have $\rho_0 \leq \cos \theta_0$.

Induction Step. Suppose that for some $0 \leq t < t^*$ we have $\rho_t \leq \cos \theta_t$. We argue that $\rho_{t+1} \leq \cos \theta_{t+1}$. If \mathcal{E}_t already holds for some $t' \leq t$, there is nothing to prove. Otherwise, assume that the condition $\sin \theta_t \lesssim \zeta(\rho_t)$ is violated. Since $\mathbf{g}^{(t)}$ is orthogonal to $\mathbf{w}^{(t)}$, the update is given by $\mathbf{w}^{(t+1)} = \text{proj}_{\mathbb{B}}(\mathbf{w}^{(t)} - \eta_t \mathbf{g}^{(t)})$. Thus, $\|\mathbf{w}^{(t+1)} - \mathbf{w}^*\|_2^2 \leq \|\mathbf{w}^{(t)} - \eta_t \mathbf{g}^{(t)} - \mathbf{w}^*\|_2^2 = \|\mathbf{w}^{(t)} - \mathbf{w}^*\|_2^2 + \eta_t^2 \|\mathbf{g}^{(t)}\|_2^2 + 2\eta_t \mathbf{g}^{(t)} \cdot \mathbf{w}^*$. By Proposition 5, we have $\mathbf{g}^{(t)} \cdot \mathbf{w}^* \lesssim -\|\mathbf{T}_{\rho_t} \sigma'\|_{L_2}^2 \sin^2 \theta_t^2$ and $\|\mathbf{g}^{(t)}\|_2 \lesssim \|\mathbf{T}_{\rho_t} \sigma'\|_{L_2}^2 \sin \theta_t$, hence $\|\mathbf{w}^{(t+1)} - \mathbf{w}^*\|_2^2 \leq \|\mathbf{w}^{(t)} - \mathbf{w}^*\|_2^2 - \eta_t |\mathbf{g}^{(t)} \cdot \mathbf{w}^*|$. Thus, by choosing η_t appropriately, there exists $\xi > 0$ such that $\theta_{t+1} \leq \theta_t - \xi$ and if we choose ρ_{t+1} so that $\cos^{-1} \rho_t - \cos^{-1} \rho_{t+1} < \xi$, we ensure $\rho_{t+1} \leq \cos \theta_{t+1}$. Alternatively, if $\sin \theta_t \lesssim \zeta(\rho_t)$ and $|\cos \theta_t - \rho_t| \geq \sin^2 \theta_t$, then by the triangle inequality we obtain $\|\mathbf{w}^{(t+1)} - \mathbf{w}^*\|_2 \leq \|\mathbf{w}^{(t)} - \mathbf{w}^*\|_2 + \eta_t \|\mathbf{g}^{(t)}\|_2$. In this case, we can choose η_t so that even if $\theta_{t+1} \geq \theta_t$, the increase is bounded by a small $\xi > 0$, i.e., $\theta_{t+1} \leq \theta_t + \xi$. Since $\cos \theta_t \geq \sin^2 \theta_t + \rho_t$, we can adjust ρ_t to ensure that $\cos(\theta_t + \xi) \geq \rho_{t+1}$. This completes the inductive step. \blacksquare

4. SGD – VA Efficiently Learns Monotone GLMs

We have shown in Section 3 that Algorithm 2 converges to a parameter vector \mathbf{w} with an L_2^2 error at most $O(\text{OPT}) + 4\|\mathbf{P}_{>1/(\theta^*)^2 \sigma}\|_{L_2}^2$, where θ^* is a Critical Point. One of the technical difficulties is that in general we cannot bound $\|\mathbf{P}_{>1/(\theta^*)^2 \sigma}\|_{L_2}^2$ by $O(\text{OPT})$. One such example is when $\sigma(t) = \text{He}_{(1/(\theta^*)^2+1)}(t)$; in this case $\|\mathbf{P}_{>1/(\theta^*)^2 \sigma}\|_{L_2}^2 = \|\sigma\|_{L_2}^2$, which can be much larger than OPT . In this section, we show that if the activation is also monotone, then for sufficiently small θ^* , we can bound $\|\mathbf{P}_{>1/(\theta^*)^2 \sigma}\|_{L_2}^2$ by the Ornstein–Uhlenbeck semigroup of σ' . Specifically, we provide an initialization method that along with Algorithm 2 gives an algorithm that guarantees error $O(\text{OPT})$. Formally, our main result is stated in the following theorem.

Theorem 12 (Learning Monotone (B, L) -Regular Activations) *Let $\epsilon > 0$, and let $\sigma \in \mathcal{H}(B, L)$ be a monotone activation. Then, Algorithm 2 draws $N = \tilde{\Theta}(dB^2 \log(L/\epsilon)/\epsilon + d/\epsilon^2)$ samples, runs in $\text{poly}(d, N)$ time, and outputs $\hat{\mathbf{w}}$ such that with probability at least $2/3$, $\hat{\mathbf{w}} \in \mathcal{R}_{\sigma, \theta_0}(O(\text{OPT}) + \epsilon)$ and $\mathcal{L}(\hat{\mathbf{w}}) \leq C\text{OPT} + \epsilon$, where C is an absolute constant independent of ϵ, d, B, L .*

The main result of this section is an initialization routine that allows us to bound the higher coefficients of the spectrum, $\|\mathbf{P}_{>1/(\theta^*)^2 \sigma}\|_{L_2}^2$. In particular, we prove the following.

Proposition 13 (Initialization) *Let $\sigma \in \mathcal{H}(B, L)$ be a monotone activation. There exists an algorithm that draws $N = \tilde{O}(d/\epsilon^2)$ samples, runs in $\text{poly}(N, d)$ time, and with probability at least $2/3$, returns a unit vector $\mathbf{w}^{(0)} \in \mathbb{R}^d$ such that for any unit $\mathbf{w}' \in \mathbb{R}^d$ with $\theta = \theta(\mathbf{w}', \mathbf{w}^*) \leq \theta(\mathbf{w}^{(0)}, \mathbf{w}^*)$, it holds that $\|\mathbf{P}_{>1/\theta^2 \sigma}\|_{L_2}^2 \lesssim \sin^2 \theta \|\mathbf{T}_{\cos \theta} \sigma'\|_{L_2}^2$.*

The proof of Theorem 12 combines Theorem 11 and Proposition 13, and is provided in the appendix.

If σ satisfies $\mathbf{E}_{z \sim \mathcal{N}}[\sigma^{2+\zeta}(z)] \leq B_\sigma$ for $\zeta > 0$, then σ is an ϵ -Extended $((B_\sigma/\epsilon)^{1/\zeta}, (B_\sigma/\epsilon)^{4/\zeta}/\epsilon^2)$ -Regular activation (see Lemma 36). We thus have the following immediate corollary.

Corollary 14 (Learning Monotone Activations With Bounded $(2 + \zeta)$ Moments) *Let $\epsilon > 0$, $\zeta > 0$, and let σ be a monotone activation such that $\mathbf{E}_{z \sim \mathcal{N}}[\sigma^{2+\zeta}(z)] \leq B_\sigma$. Then, [Algorithm 2](#) draws $N = \tilde{\Theta}(d(B_\sigma/\epsilon)^{2/\zeta} \log(B_\sigma/\epsilon)/\epsilon + d/\epsilon^2)$ samples, runs in $\text{poly}(d, N)$ time, and outputs $\hat{\mathbf{w}}$ such that with probability at least $2/3$, $\mathcal{L}(\hat{\mathbf{w}}) \leq \text{COPT} + \epsilon$, where C is an absolute constant.*

To prove [Proposition 13](#), we combine two main technical pieces: (1) proving that there exists a threshold $\bar{\theta}$ such that for any $\theta \leq \bar{\theta}$, $\|\mathbf{P}_{>1/\theta^2}\sigma\|_{L_2}^2 \lesssim \sin^2 \theta \|\mathbf{T}_{\cos \theta} \sigma'\|_{L_2}^2$; and (2) proving that there exists an efficient algorithm that finds a vector $\mathbf{w}^{(0)}$ such that $\theta(\mathbf{w}^{(0)}, \mathbf{w}^*) \leq \bar{\theta}$. [Section 4.1](#) addresses (1), with main technical result stated in [Proposition 16](#). To prove this result, we approximate the considered monotone activations σ by sequences of “monotone staircase” functions.

Definition 15 (Monotone Staircase Functions) *Let $\phi(z; t) := \mathbf{1}\{z \geq t\}$ and let $m \in \mathbb{Z}_+$, $M > 0$. The class of monotone staircase functions (of M -bounded support) are defined as $\mathcal{F}_M := \{\Phi_m : \mathbb{R} \rightarrow \mathbb{R} : \Phi_m(z) = \sum_{i=1}^m A_i \phi(z; t_i) + A_0 : A_0 \in \mathbb{R}; A_i > 0, |t_i| \leq M, \forall i \in [m]; m < \infty\}$.*

If Φ_k converges to σ pointwise, we argue that $\|\mathbf{P}_{>1/\theta^2}\sigma\|_{L_2}^2 \lesssim 2\|\Phi_k - \mathbf{T}_{\cos \theta} \Phi_k\|_{L_2}^2 + \theta^2 \|\mathbf{T}_{\cos \theta} \Phi_k'\|_{L_2}^2$. We further show that $\mathbf{T}_{\cos \theta} \Phi_k' \rightarrow \mathbf{T}_{\cos \theta} \sigma'$, therefore, it remains to show that $\|\Phi_k - \mathbf{T}_{\cos \theta} \Phi_k\|_{L_2}^2 \lesssim \theta^2 \|\mathbf{T}_{\cos \theta} \Phi_k'\|_{L_2}^2$. [Proposition 17](#) in [Section 4.2](#) proves the claim that when ρ is not too small, $\|\Phi - \mathbf{T}_\rho \Phi\|_{L_2}^2 \lesssim \theta^2 \|\mathbf{T}_\rho \Phi'\|_{L_2}^2$, for any $\Phi(z)$ that is a *monotonic staircase function*. These staircase functions constitute a dense subset of the monotone function class and have a simple and easy-to-analyze form, therefore they serve well for our purpose. In [Definition 15](#), M is chosen to be a bound on the support of σ' , which is always finite by [Claim 34](#). In [Section 4.3](#), we prove (2) by providing an initialization algorithm. Finally, combining (1) and (2), we prove [Proposition 13](#).

4.1. Bounding Higher Order Hermite Coefficients of Monotone Activations

The main result of this subsection is the following:

Proposition 16 (From Hermite Tails to Ornstein–Uhlenbeck Semigroup) *Let $\sigma \in L_2(\mathcal{N})$ be a monotone activation, M be the upper bound for the support of $\sigma'(z)$ ². For any $\theta \in [0, \pi]$ such that $1 - C/M^2 < \cos^2 \theta$ with $C > 0$ an absolute constant, it holds $\|\mathbf{P}_{>1/\theta^2}\sigma\|_{L_2}^2 \lesssim \sin^2 \theta \|\mathbf{T}_{\cos \theta} \sigma'\|_{L_2}^2$.*

Proof Sketch of [Proposition 16](#) Let Φ_k be a sequence of monotone staircase functions ([Definition 15](#)) that converges to σ with respect to L_2 ; this is true because piecewise constant functions are dense over compact sets with respect to the L_2 norm (in this case the compact set is $[-M, M]$). For $\rho^2 \geq 1 - C/M^2$, where M is the upper bound on the support of σ' and Φ_k' , by Young’s inequality we have $\|\mathbf{P}_{>1/\theta^2}\sigma\|_{L_2}^2 \leq 2\|\mathbf{P}_{>1/\theta^2}(\sigma - \Phi_k)\|_{L_2}^2 + 4\|\mathbf{P}_{>1/\theta^2}(\Phi_k - \mathbf{T}_\rho \Phi_k)\|_{L_2}^2 + 4\|\mathbf{P}_{>1/\theta^2}\mathbf{T}_\rho \Phi_k\|_{L_2}^2$. Observe that $\|\mathbf{P}_{>m}f\|_{L_2}^2 = \sum_{i>m} \hat{f}(i)^2 \leq \|f\|_{L_2}^2$. Therefore, $\|\mathbf{P}_{>1/\theta^2}(\sigma - \Phi_k)\|_{L_2}^2 \leq \|\sigma - \Phi_k\|_{L_2}^2 \rightarrow 0$. In addition, note that for any $f, f' \in L_2(\mathcal{N})$, it holds $\|\mathbf{P}_{>m}f\|_{L_2}^2 \leq \sum_{i>m} (i/m) \hat{f}(i)^2 \leq (1/m) \|f'\|_{L_2}^2$, thus $\|\mathbf{P}_{>1/\theta^2}\mathbf{T}_\rho \Phi_k\|_{L_2}^2 \leq \theta^2 \|(\mathbf{T}_\rho \Phi_k)'\|_{L_2}^2$. Further, by [Fact 21](#), we have $\|(\mathbf{T}_\rho \Phi_k)'\|_{L_2}^2 \leq \|\mathbf{T}_\rho \Phi_k'\|_{L_2}^2$ since $\rho < 1$, thus, $\|\mathbf{P}_{>1/\theta^2}\sigma\|_{L_2}^2 \leq 4\|\Phi - \mathbf{T}_\rho \Phi\|_{L_2}^2 + 4\theta^2 \|\mathbf{T}_\rho \Phi'\|_{L_2}^2$ when $k \rightarrow \infty$. Next, by [Proposition 17](#), we conclude that $\|\Phi_k - \mathbf{T}_\rho \Phi_k\|_{L_2}^2 \lesssim (1 - \rho^2) \|\mathbf{T}_\rho \Phi_k'\|_{L_2}^2$, and, therefore, we have that $\|\mathbf{P}_{>1/\theta^2}\sigma\|_{L_2}^2 \leq 4((1 - \rho^2) + \theta^2) \|\mathbf{T}_\rho \Phi_k'\|_{L_2}^2$. In [Lemma 63](#), we show that the sequence of smoothed derivatives $\mathbf{T}_\rho \Phi_k'$ also converges to σ' , therefore it holds $\|\mathbf{T}_\rho \Phi_k'\|_{L_2}^2 \rightarrow \|\mathbf{T}_\rho \sigma'\|_{L_2}^2$. Letting $\rho = \cos \theta$ completes the proof. \blacksquare

²In [Claim 34](#), we show that $\forall \sigma \in \mathcal{H}(B, L)$, the support of $\sigma'(z)$ can be truncated at some $M < +\infty$ w.l.o.g.

4.2. Bounding the Augmentation Error

Our main technical result provides an upper bound on the smoothing error of piecewise staircase functions using the $L_2(\mathcal{N})$ norm of the smoothed derivative, as stated below.

Proposition 17 *Let $\Phi \in \mathcal{F}_M$. For any $\rho \in (0, 1)$ such that $\rho^2 \geq 1 - C/M^2$ where $C < M^2$ is an absolute constant, we have $\|\mathsf{T}_\rho \Phi - \Phi\|_{L_2}^2 \lesssim (1 - \rho^2) \|\mathsf{T}_\rho \Phi'\|_{L_2}^2$.*

Proceeding to the proof of [Proposition 17](#), technical difficulties arise when we try to relate $\|\mathsf{T}_\rho \Phi(z) - \Phi(z)\|_{L_2}^2$ with $\|\mathsf{T}_\rho \Phi'(z)\|_{L_2}^2$. The main obstacle is that it is hard to analyze $\mathsf{T}_\rho \phi(z; t) - \phi(z; t)$, since $\mathsf{T}_\rho \phi(z; t) = \mathbf{Pr}_{u \sim \mathcal{N}}[u \geq (t - \rho z)/(1 - \rho^2)^{1/2}]$, and the probability term does not have a closed form. Our workaround is to introduce a new type of ‘centered augmentation (smoothing)’ operator $\mathsf{T}_\rho \Phi(z/\rho)$ that takes a simpler and easy-to-analyze form, and then translate the upper bound on the centered augmentation error back to the upper bound on the standard augmentation error. We show that $\Delta := \|\mathsf{T}_\rho \Phi(z) - \Phi(z)\|_{L_2}^2$ is bounded by the following three terms $\Delta \lesssim \Delta_1 + \Delta_2 + \Delta_3$, where $\Delta_1 := \|\mathsf{T}_\rho \Phi(z) - \mathsf{T}_{\rho_1} \Phi(z)\|_{L_2}^2$, $\Delta_2 := \|\mathsf{T}_{\rho_1} \Phi(z) - \mathsf{T}_{\rho_1} \Phi(z/\rho_1)\|_{L_2}^2$ and $\Delta_3 := \|\mathsf{T}_{\rho_1} \Phi(z/\rho_1) - \Phi(z)\|_{L_2}^2$, with $\rho_1 \in (0, 1)$ being a carefully chosen parameter that is slightly larger than ρ . Taking advantage of the nice analytic form of $\mathsf{T}_\rho \Phi(z/\rho)$, we show that all these three terms can be bounded by $\|\mathsf{T}_\rho \Phi'(z)\|_{L_2}^2$, using the properties of $\mathsf{T}_\rho \Phi(z/\rho)$ provided in [Lemma 18](#).

We define the centered augmentation as $\mathsf{T}_\rho \sigma(z/\rho) = \mathbf{E}_{u \sim \mathcal{N}}[\sigma(z + (\sqrt{1 - \rho^2}/\rho)u)]$. We show that the L_2^2 error between the centered augmentation $\mathsf{T}_\rho \Phi(z/\rho)$ and $\Phi(z)$, $\mathsf{T}_\rho \Phi(z)$ are well controlled, as summarized in the following lemma (see [Appendix F.2](#) for complete statements):

Lemma 18 *Let $\Phi \in \mathcal{F}_M$, $C \in (0, M^2/2]$. For any $\rho^2 \geq 1 - C/M^2$, it holds:*

$$\|\mathsf{T}_\rho \Phi(z/\rho) - \Phi(z)\|_{L_2}^2 \leq 4((1 - \rho^2)/\rho^2) \|\mathsf{T}_\rho \Phi'(z/\rho)\|_{L_2}^2; \quad (4)$$

$$\|\mathsf{T}_\rho \Phi(z) - \mathsf{T}_\rho \Phi(z/\rho)\|_{L_2}^2 \leq C'(1 - \rho^2)(\|\mathsf{T}_\rho \Phi'(z/\rho)\|_{L_2}^2 + \|\mathsf{T}_\rho \Phi'\|_{L_2}^2); \quad (5)$$

$$\|\mathsf{T}_{\rho_1} \Phi'(z/\rho_1)\|_{L_2}^2 \leq 2e^C \|\mathsf{T}_\rho \Phi'(z)\|_{L_2}^2, \text{ where } \rho_1^2 = \rho^2 + C(1 - \rho^2)/M^2. \quad (6)$$

Proof Sketch of Proposition 17 Let $\Delta := \|\mathsf{T}_\rho \Phi(z) - \Phi(z)\|_{L_2}^2$. Observe that by adding and subtracting $\mathsf{T}_{\rho_1} \Phi$, $\mathsf{T}_{\rho_1} \Phi(z/\rho_1)$ in the norm and repeatedly using $(a + b)^2 \leq 2a^2 + 2b^2$, we have $\Delta \leq 4\Delta_1 + 4\Delta_2 + 2\Delta_3$ where $\Delta_1 := \|\mathsf{T}_\rho \Phi(z) - \mathsf{T}_{\rho_1} \Phi(z)\|_{L_2}^2$, $\Delta_2 := \|\mathsf{T}_{\rho_1} \Phi(z) - \mathsf{T}_{\rho_1} \Phi(z/\rho_1)\|_{L_2}^2$ and $\Delta_3 := \|\mathsf{T}_{\rho_1} \Phi(z/\rho_1) - \Phi(z)\|_{L_2}^2$.

For Δ_1 , observe that since $\rho < \rho_1 < 1$, we can use the property that $\mathsf{T}_\rho \Phi(z) = \mathsf{T}_{\rho/\rho_1}(\mathsf{T}_{\rho_1} \Phi(z))$ and $(\mathsf{T}_{\rho_1} \Phi(z))' = \rho_1 \mathsf{T}_{\rho_1} \Phi'(z)$ ([Fact 21](#)). Using [Lemma 24](#) with $f(z) = \mathsf{T}_{\rho_1} \Phi(z)$ and noting that $\|\mathsf{T}_{\rho_1} \Phi'(z)\|_{L_2}^2 \lesssim \|\mathsf{T}_\rho \Phi'(z)\|_{L_2}^2$ for our ρ and ρ_1 ([Claim 74](#)), we have $\Delta_1 \lesssim (1 - \rho^2) \|\mathsf{T}_{\rho_1} \Phi'(z)\|_{L_2}^2 \lesssim (1 - \rho^2) \|\mathsf{T}_\rho \Phi'(z)\|_{L_2}^2$. For Δ_2 , applying [Equation \(5\)](#) with ρ_1 , and noting that $\|\mathsf{T}_{\rho_1} \Phi'(z)\|_{L_2}^2 \lesssim \|\mathsf{T}_\rho \Phi'(z)\|_{L_2}^2$ ([Claim 74](#)), then combining with [Equation \(6\)](#), we obtain: $\Delta_2 \lesssim (1 - \rho^2)^2 \|\mathsf{T}_\rho \Phi'(z)\|_{L_2}^2$. Finally, for Δ_3 , using [Equation \(4\)](#) and [Equation \(6\)](#) from [Lemma 18](#), and plugging in the value of ρ_1 , we get $\Delta_3 \leq 4(1 - \rho_1^2)/\rho_1^2 \|\mathsf{T}_{\rho_1} \Phi'(z/\rho_1)\|_{L_2}^2 \lesssim (1 - \rho^2) \|\mathsf{T}_\rho \Phi'(z)\|_{L_2}^2$. ■

4.3. Initialization Algorithm and Proof of Proposition 13

In this section, we provide an initialization algorithm for σ that is a monotone (B, L) -Regular activation. The algorithm generates a vector $\mathbf{w}^{(0)}$ satisfying $\theta(\mathbf{w}^{(0)}, \mathbf{w}^*) \leq C/M$, where C is an absolute constant and $M \leq \sqrt{\log(B/\epsilon) - \log \log(B/\epsilon)}$. Our key idea is to convert the regression problem to a problem of robustly learning halfspaces via *data transformation*. In particular, we

transform y to $\tilde{y} \in \{0, 1\}$ by truncating the labels y to $\tilde{y} = \mathbb{1}\{y \geq t'\}$, where this t' is a carefully chosen threshold. Then, we show that there exists a halfspace $\phi(\mathbf{w}^* \cdot \mathbf{x}; t) = \mathbb{1}\{\mathbf{w}^* \cdot \mathbf{x} \geq t\}$ such that the transformed labels \tilde{y} can be viewed as the corrupted labels of $\phi(\mathbf{w}^* \cdot \mathbf{x}; t)$. Finally we utilize a previous algorithm from [Diakonikolas et al. \(2022c\)](#) to robustly learn \mathbf{w}^* . In particular, we show:

Proposition 19 *Let σ be a non-decreasing (B, L) -Regular function. Let M be defined as in [Claim 34](#). Then, there exists an algorithm that draws $O(d/\epsilon^2 \log(1/\delta))$ samples, it runs in $\text{poly}(d, N)$ time, and, with probability at least $1 - \delta$, it outputs a vector \mathbf{w} such that $\theta(\mathbf{w}, \mathbf{w}^*) \leq C/M$, where $C > 0$ is a universal constant, independent of any problem parameters.*

We defer the proof of [Proposition 19](#) to [Appendix F.3](#).

Proof of [Proposition 13](#) [Proposition 19](#) implies that there exists an algorithm that uses $O(d/\epsilon^2)$ samples and outputs a vector $\mathbf{w}^{(0)}$ such that $\theta(\mathbf{w}^{(0)}, \mathbf{w}^*) \leq C/M$. Now for any $\theta \leq \theta_0$, it holds $\cos \theta^2 \geq 1 - \theta^2 \geq 1 - C^2/M^2$. Thus, using [Proposition 16](#), we have $\|\mathbf{P}_{>1/\theta^2 \sigma}\|_{L_2}^2 \lesssim \sin^2 \theta \|\mathbf{T}_{\cos \theta \sigma'}\|_{L_2}^2$. ■

5. Conclusions and Open Problems

In this work, we give a constant-factor approximate robust learner for monotone GLMs under the Gaussian distribution, answering a recognized open problem in the field. A number of open questions remain. An immediate goal is to generalize our algorithmic result to Single-Index Models (SIMs), corresponding to the case where the monotone activation is unknown. We believe that progress in this direction is attainable. Another question is whether one can obtain a similarly robust GLM learner (even for the known activation case) for more general marginal distributions, e.g., encompassing all isotropic log-concave distributions. This remains open even for the special case of a single general (i.e., potentially-biased) halfspace, where known constant-factor approximate learners ([Diakonikolas et al., 2018, 2022c](#)) make essential use of the Gaussian assumption.

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Appendix

Organization The appendix is organized as follows. In [Appendix A](#), we provide a detailed comparison with related prior works. In [Appendix B](#), we give additional background on the Ornstein–Uhlenbeck semigroup and introduce useful facts that will repeatedly appear in the technical sections. In [Appendix C](#), we provide detailed discussions on the (Extended)- (B, L) -activation class and our assumptions. In [Appendix D](#), [Appendix E](#), and [Appendix F](#) we provide the full versions of [Section 2](#), [Section 3](#), [Section 4](#), with complete proofs and supplementary lemmas.

Appendix A. Detailed Comparison with Prior Work

In this section, we provide a detailed comparison with related prior works.

	Distribution	Activation	Error Bound
[WZDD23]	Well-Behaved	Monotonic (a, b) -unbounded	$O(\text{poly}(b/a))\text{OPT}$
[WZDD24]	Gaussian	k^* -information exponent	$O(\ \sigma'\ _{L_2})\text{OPT}$
[GV2024]	Gaussian	Biased ReLUs	$C\text{OPT}$
Ours	Gaussian	Monotone + Lipschitz or Bounded $2 + \zeta$ Moment	$C\text{OPT}$

Table 1: Comparison of our approach with prior work on robustly learning GLMs.

[Diakonikolas et al. \(2022b\)](#); [Wang et al. \(2023\)](#); [Zarifis et al. \(2024\)](#) studied agnostic learning of GLMs under ‘well-behaved’ distributions, where σ , possibly not known a priori, is monotone and (a, b) -unbounded, meaning that $|\sigma'(z)| \leq b$ and $\sigma'(z) \geq a$ when $z \geq 0$. They provided an algorithm that finds $\hat{\mathbf{w}} \in \mathbb{B}(W)$ with error $O(\text{poly}(b/a))\text{OPT} + \epsilon$. Note that in these works, rescaling \mathbf{w} to \mathbb{S}^{d-1} is not required; therefore, a, b do not have dependencies on the parameter W . However, the main drawback of these works is that their algorithm cannot be applied to all monotone and Lipschitz functions. In particular, when $a = 0$, the previous works do not provide any useful results at all. Furthermore, if $a = O(\epsilon)$, the algorithms in [Diakonikolas et al. \(2022b\)](#); [Wang et al. \(2023\)](#); [Zarifis et al. \(2024\)](#) only provide an approximate solution with $O(\text{poly}(1/\epsilon))\text{OPT}$ error. In stark comparison, in our work, we can deal with any b -Lipschitz activations and obtain $C\text{OPT} + \epsilon$ error, where the absolute constant C does not depend on b, ϵ , or W , as shown in [Theorem 2](#).

[Wang et al. \(2024\)](#) studied robust learning of GLMs under Gaussian marginals, similar to our setting. They considered a broader class of activations where σ has constant information exponent k^* , defined as the degree of the first non-zero Hermite coefficient: $\sigma(z) \doteq \sum_{k \geq 1} c_k \text{He}_k(z)$, with $k^* = \min\{k \geq 1 : c_k \neq 0\}$. [Wang et al. \(2024\)](#) makes the following assumptions: $\|\mathbf{w}\|_2 = 1$, $\|\sigma\|_{L_2} = 1$, $\|\sigma\|_{L_4} \leq +\infty$, and that c_{k^*} is an absolute constant. Their algorithm requires $O(d^{\lceil k^*/2 \rceil}/c_{k^*} + d/\epsilon)$ samples and outputs $\hat{\mathbf{w}} \in \mathbb{S}^{d-1}$ with error $O(\|\sigma'\|_{L_2})\text{OPT} + \epsilon$.

However, their approach has the following key limitations: (1) It does not generalize to $\mathbf{w}^* \in \mathbb{B}(W)$, as rescaling to \mathbb{S}^{d-1} affects the gradient norm—leading to an error bound of $O(W\|\sigma'\|_{L_2})\text{OPT}$, which depends on W . (2) Rescaling σ to satisfy $\|\sigma\|_{L_2} = 1$ can inadvertently amplify $\|\sigma'\|_{L_2}$, increasing the error. (3) Finally, note that their sample complexity depends on c_1 , therefore their sample complexity can be even larger if c_1 is extremely small.

Our results address these issues: (1) as discussed in the introduction, this work’s error bound in [Theorem 12](#) is independent of all the parameters $\|\sigma'\|_{L_2}$, $\|\sigma\|_{L_\infty}$, d and ϵ , and therefore rescaling the activation will not impact the approximation error; (2) similarly, the quantity $\|\sigma\|_{L_2}$ also does

not impact our approximation error; (3) finally, our sample complexity is independent of c_1 , and will therefore not be impacted if c_1 is very small.

Recent independent work (Guo and Vijayaraghavan, 2024) studied agnostic learning of biased ReLUs under Gaussian \mathbf{x} -marginals, also achieving $C\text{OPT} + \epsilon$ error. We note that their algorithm is tailored to the special case of ReLUs. On the other hand, our framework handles all monotone Lipschitz activations (including all biased ReLUs as a special case), and even all monotone activations with bounded $(2 + \zeta)$ -order moments for $\zeta > 0$; see Lemma 36.

Gollakota et al. (2023b); Hu et al. (2024) studied agnostic learning of GLMs with unknown activation σ . These works focused on general distributions: Gollakota et al. (2023b) only requires the marginal distribution of \mathbf{x} to have its second moment bounded by λ ; and Hu et al. (2024) only requires \mathbf{x} to be supported on a Euclidean ball. However, the error bounds that Gollakota et al. (2023b); Hu et al. (2024) achieve cannot be considered constant factor approximations. Gollakota et al. (2023b) provides $O(W\sqrt{\lambda}\text{OPT})$ error guarantee for 1-Lipschitz activations; their algorithm achieves $O(b/a)\text{OPT} + \epsilon$ error when restricted to (a, b) -bi-Lipschitz activations, i.e., for $0 < a \leq \sigma'(z) \leq b$. Hu et al. (2024) does not provide an L_2^2 -error guarantee but instead focuses on finding an omnipredictor that minimizes a convex surrogate loss.

In Damian et al. (2023), the authors considered GLMs with bounded information exponent and employed a smoothing technique different than ours, with a constant smoothing parameter. Importantly, their algorithm is limited to the realizable setting. As explained in Wang et al. (2024), their algorithm fails in the more challenging robust learning setting, even for monotone functions (with information exponent $k^* = 1$).

Moreover, their smoothing approach differs from ours both conceptually and practically. Conceptually, as discussed in Section 1.1, our method is based on the observation that the gradient of the augmented/Ornstein–Uhlenbeck-semigroup-smoothed L_2^2 loss *maximizes* the signal from \mathbf{w}^* , which is otherwise obscured by agnostic noise. In contrast, Damian et al. (2023) applied a spherical smoothing technique aimed at capturing higher-order information and improving the ratio $\|\mathbf{g}(\mathbf{w})\|_2 / (\mathbf{g}(\mathbf{w}) \cdot \mathbf{w}^*)$, where $\mathbf{g}(\mathbf{w}) = \nabla \mathcal{L}(\mathbf{w})$. This is sufficient for the realizable setting, but not for the more challenging adversarial setting. Practically, our algorithm and techniques diverge significantly from those in Damian et al. (2023). First, whereas they implemented spherical smoothing, we utilize Gaussian noise injection while also reweighting the marginals \mathbf{x} . Second, instead of fixing the smoothing parameter, we employ variable augmentation/smoothing. This variable smoothing is crucial to our algorithm, as it ensures that the signal of the augmented gradient is not obscured by noise in each iteration (see the discussion and analysis in Theorem 11).

Kalai and Sastry (2009); Kakade et al. (2011) studied the problem of learning GLMs in the realizable setting. They considered monotone 1-Lipschitz activations under any distribution \mathcal{D} that is supported on $\mathbb{B} \times [0, 1]$. Their analysis is not applicable to our robust learning setting.

Appendix B. Additional Notation and Preliminaries

Additional Notation Let $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ denote the d -dimensional Gaussian distribution with mean $\boldsymbol{\mu} \in \mathbb{R}^d$ and covariance $\boldsymbol{\Sigma} \in \mathbb{R}^{d \times d}$. In this work we usually consider the standard normal distribution, i.e., $\boldsymbol{\mu} = \mathbf{0}$ and $\boldsymbol{\Sigma} = \mathbf{I}$, and thus denote it simply by \mathcal{N} . The usual inner product for this Gaussian space is $\mathbf{E}_{\mathbf{x} \sim \mathcal{N}}[f(\mathbf{x})g(\mathbf{x})]$. We write $f(z) \doteq g(z)$ to mean that $\mathbf{E}_{z \sim \mathcal{N}(0,1)}[(f(z) - g(z))^2] = 0$. We use *normalized* probabilists’ Hermite polynomial of degree i , defined via $\text{He}_i(x) = \text{he}_i(x)/\sqrt{i!}$,

where by $\text{he}_i(x)$ we denote the probabilist's Hermite polynomial of degree i :

$$\text{he}_k(z) = (-1)^k \exp(z^2/2) \frac{d^k}{dz^k} \exp(-z^2/2).$$

These normalized Hermite polynomials form a complete orthonormal basis for the single-dimensional version of the inner product space defined above. Given a function $f : \mathbb{R} \rightarrow \mathbb{R}$, $f \in L_2(\mathcal{N})$, we compute its Hermite coefficients as $\hat{f}(i) = \mathbf{E}_{z \sim \mathcal{N}(0,1)}[f(z)\text{He}_i(z)]$, and express the function uniquely as $f(z) \doteq \sum_{i \geq 0} \hat{f}(i)\text{He}_i(z)$.

B.1. Ornstein–Uhlenbeck Semigroup

An important tool for our work is the Ornstein–Uhlenbeck semigroup. The Ornstein–Uhlenbeck semigroup and operators are broadly used in stochastic analysis and control theory (see, e.g., [Bogachev \(1998\)](#)). Within learning theory, they have found applications in bounding the sensitivity of a Boolean function ([Klivans et al., 2008](#)). A formal definition of the Ornstein–Uhlenbeck semigroup is provided below.

Definition 20 (Ornstein–Uhlenbeck Semigroup) *Let $\rho \in (0, 1)$. The Ornstein–Uhlenbeck semigroup, denoted by T_ρ , is a linear operator that maps a function $g \in L_2(\mathcal{N})$ to the function $T_\rho g$ defined as:*

$$(T_\rho g)(\mathbf{x}) := \mathbf{E}_{\mathbf{z} \sim \mathcal{N}} \left[g(\rho \mathbf{x} + \sqrt{1 - \rho^2} \mathbf{z}) \right].$$

To simplify the notation, we often write $T_\rho g(\mathbf{x})$ instead of $(T_\rho g)(\mathbf{x})$.

The following fact summarizes useful properties of the Ornstein–Uhlenbeck semigroup.

Fact 21 (see, e.g., [Bogachev \(1998\)](#), [O’Donnell \(2014\)](#)(Chapter 11))

1. For any $f, g \in L_2$ and any $t > 0$, $\mathbf{E}_{\mathbf{x} \sim \mathcal{N}}[(T_t f(\mathbf{x}))g(\mathbf{x})] = \mathbf{E}_{\mathbf{x} \sim \mathcal{N}}[(T_t g(\mathbf{x}))f(\mathbf{x})]$.
2. For any $g : \mathbb{R}^d \rightarrow \mathbb{R}$, $g \in L_2$, all of the following statements hold.
 - (a) For any $t, s > 0$, $T_t T_s g = T_{ts} g$.
 - (b) For any $\rho \in (0, 1)$, $T_\rho g(\mathbf{x})$ is differentiable at every point $\mathbf{x} \in \mathbb{R}^d$.
 - (c) For any $\rho \in (0, 1)$, $T_\rho g(\mathbf{x})$ is $\|g\|_{L_\infty}/(1 - \rho^2)^{1/2}$ -Lipschitz, i.e., $\|\nabla T_\rho g(\mathbf{x})\|_{L_\infty} \leq \|g\|_{L_\infty}/(1 - \rho^2)^{1/2}$, $\forall \mathbf{x} \in \mathbb{R}^d$.
 - (d) For any $\rho \in (0, 1)$, $T_\rho g(\mathbf{x}) \in C^\infty$.
 - (e) For any $p \geq 1$, T_ρ is nonexpansive with respect to the norm $\|\cdot\|_{L_p}$, i.e., $\|T_\rho g\|_{L_p} \leq \|g\|_{L_p}$.
 - (f) $\|T_\rho g(\mathbf{x})\|_{L_2}$ is non-decreasing w.r.t. ρ .
 - (g) If g is, in addition, a differentiable function, then for all $\rho \in (0, 1)$, it holds that: $\nabla_{\mathbf{x}} T_\rho g(\mathbf{x}) = \rho T_\rho \nabla_{\mathbf{x}} g(\mathbf{x})$, for any $\mathbf{x} \in \mathbb{R}^d$.
3. For all $\rho \in (0, 1)$ and $i \in \mathbb{Z}_+$, $T_\rho \text{He}_i(z) = \rho^i \text{He}_i(z)$.

The Ornstein–Uhlenbeck semigroup induces an operator L applying to functions $f \in L_2(\mathcal{N})$, defined below.

Definition 22 (Definition 11.24 in O’Donnell (2014)) *The Ornstein–Uhlenbeck operator is a linear operator applied that applies to functions $f \in L_2(\mathcal{N})$ and is defined by $\mathsf{L}f = \frac{dT_\rho f}{d\rho} \big|_{\rho=1}$, provided that $\mathsf{L}f$ exists.*

Fact 23 *Let $f, g \in L_2(\mathcal{N})$, $\rho \in (0, 1)$. Then:*

1. ((O’Donnell, 2014, Proposition 11.27)) $\frac{dT_\rho f}{d\rho} = \frac{1}{\rho} \mathsf{L}T_\rho f = \frac{1}{\rho} T_\rho \mathsf{L}f$.
2. ((O’Donnell, 2014, Proposition 11.28)) $\mathbf{E}_{\mathbf{x} \sim \mathcal{N}} [f(\mathbf{x}) \mathsf{L}T_\rho g(\mathbf{x})] = \mathbf{E}_{\mathbf{x} \sim \mathcal{N}} [\nabla f(\mathbf{x}) \nabla T_\rho g(\mathbf{x})]$.

We use Fact 23 to prove the following Lemma 24:

Lemma 24 *Let $f \in L_2(\mathcal{N})$ be a continuous and (almost everywhere) differentiable function. Then $\mathbf{E}_{\mathbf{x} \sim \mathcal{N}} [(T_\rho f(\mathbf{x}) - f(\mathbf{x}))^2] \leq 3(1 - \rho) \mathbf{E}_{\mathbf{x} \sim \mathcal{N}} [\|\nabla f(\mathbf{x})\|_2^2]$.*

Proof Observe that $(\mathbf{E}_{\mathbf{x} \sim \mathcal{N}} [(T_\rho f(\mathbf{x}) - f(\mathbf{x}))^2])^{1/2} = \sup_{g \in \mathcal{C}^\infty, \|g\|_{L_2} \leq 1} \mathbf{E}_{\mathbf{x} \sim \mathcal{N}} [g(\mathbf{x}) (T_\rho f(\mathbf{x}) - f(\mathbf{x}))]$. Consider any $g \in \mathcal{C}^\infty$ with $\|g\|_{L_2} \leq 1$. We have that

$$\mathbf{E}_{\mathbf{x} \sim \mathcal{N}} [g(\mathbf{x}) (T_\rho f(\mathbf{x}) - f(\mathbf{x}))] = \mathbf{E}_{\mathbf{x} \sim \mathcal{N}} [f(\mathbf{x}) (T_\rho g(\mathbf{x}) - g(\mathbf{x}))] = \mathbf{E}_{\mathbf{x} \sim \mathcal{N}} \left[f(\mathbf{x}) \int_\rho^1 \frac{dT_t g(\mathbf{x})}{dt} dt \right].$$

As $g \in \mathcal{C}^\infty$, we can use Fact 23 to conclude that

$$\mathbf{E}_{\mathbf{x} \sim \mathcal{N}} \left[f(\mathbf{x}) \int_\rho^1 \frac{dT_t g(\mathbf{x})}{dt} dt \right] = \mathbf{E}_{\mathbf{x} \sim \mathcal{N}} \left[f(\mathbf{x}) \int_\rho^1 (1/t) \mathsf{L}T_t g(\mathbf{x}) dt \right],$$

where L is the Ornstein–Uhlenbeck operator. Using the identity that for f such that $\nabla f \in L_2(\mathcal{N})$ and $g \in \mathcal{C}^\infty$ it holds that $\mathbf{E}_{\mathbf{x} \sim \mathcal{N}} [f(\mathbf{x}) \mathsf{L}T_t g(\mathbf{x})] = \mathbf{E}_{\mathbf{x} \sim \mathcal{N}} [\nabla f(\mathbf{x}) \nabla T_t g(\mathbf{x})]$ (Fact 23), we have that

$$\mathbf{E}_{\mathbf{x} \sim \mathcal{N}} [g(\mathbf{x}) (T_\rho f(\mathbf{x}) - f(\mathbf{x}))] = \int_\rho^1 (1/t) \mathbf{E}_{\mathbf{x} \sim \mathcal{N}} [\nabla f(\mathbf{x}) \nabla T_t g(\mathbf{x})] dt.$$

Note that using Fact 21 (f) and Stein’s lemma Fact 26, we have:

$$\begin{aligned} \nabla T_\rho g(\mathbf{x}) &= \rho T_\rho \nabla g(\mathbf{x}) = \rho \mathbf{E}_{\mathbf{z} \sim \mathcal{N}} [\nabla g(\rho \mathbf{x} + \sqrt{1 - \rho^2} \mathbf{z})] \\ &= (\rho / (\sqrt{1 - \rho^2})) \mathbf{E}_{\mathbf{z} \sim \mathcal{N}} [g(\rho \mathbf{x} + \sqrt{1 - \rho^2} \mathbf{z}) \mathbf{z}]. \end{aligned}$$

Therefore, since \mathbf{z}, \mathbf{x} are independent standard Gaussian random vectors, we have that

$$\begin{aligned} &\mathbf{E}_{\mathbf{x} \sim \mathcal{N}} [g(\mathbf{x}) (T_\rho f(\mathbf{x}) - f(\mathbf{x}))] \\ &= \int_\rho^1 \frac{1}{\sqrt{1 - t^2}} \mathbf{E}_{\mathbf{x}, \mathbf{z} \sim \mathcal{N}} [g(t\mathbf{x} + \sqrt{1 - t^2} \mathbf{z}) \mathbf{z} \cdot \nabla f(\mathbf{x})] dt \\ &\leq \int_\rho^1 \frac{1}{\sqrt{1 - t^2}} \left(\mathbf{E}_{\mathbf{x}, \mathbf{z} \sim \mathcal{N}} [g(t\mathbf{x} + \sqrt{1 - t^2} \mathbf{z})^2] \mathbf{E}_{\mathbf{x}, \mathbf{z} \sim \mathcal{N}} [(\mathbf{z} \cdot \nabla f(\mathbf{x}))^2] \right)^{1/2} dt \\ &= \int_\rho^1 \frac{1}{\sqrt{1 - t^2}} \left(\mathbf{E}_{\mathbf{u} \sim \mathcal{N}} [g(\mathbf{u})^2] \mathbf{E}_{\mathbf{x} \sim \mathcal{N}} [\|\nabla f(\mathbf{x})\|_2^2] \right)^{1/2} dt \\ &\leq \left(\mathbf{E}_{\mathbf{x} \sim \mathcal{N}} [\|\nabla f(\mathbf{x})\|_2^2] \right)^{1/2} \int_\rho^1 \frac{1}{\sqrt{1 - t^2}} dt = \left(\mathbf{E}_{\mathbf{x} \sim \mathcal{N}} [\|\nabla f(\mathbf{x})\|_2^2] \right)^{1/2} \arccos \rho, \end{aligned}$$

where we used the Cauchy-Schwarz inequality and the fact that $\|g\|_{L_2} \leq 1$. Using the inequality $\arccos \rho \leq \sqrt{3} \sqrt{1 - \rho}$, we complete the proof of Lemma 24. \blacksquare

B.2. Gaussian Distribution and Hermite Polynomials

In the following fact, we gather some useful facts about Hermite polynomials that are used throughout the paper.

Fact 25 (See, e.g., [Bogachev \(1998\)](#)) *The following statements hold for Hermite polynomials as defined above.*

1. (Parseval's identity) For any $f \in L_2(\mathcal{N})$, we have $\mathbf{E}_{z \sim \mathcal{N}(0,1)}[(f(z) - \mathbf{P}_k f(z))^2] = \sum_{i=k+1}^{\infty} \hat{f}(i)^2$.
2. (Mehler's Identity) For any real number $|\rho| < 1$ and $x, y \in \mathbb{R}$, it holds

$$\sum_{k \geq 0} \rho^k \text{He}_k(x) \text{He}_k(y) = \frac{1}{\sqrt{1-\rho^2}} \exp\left(-\frac{(\rho x - y)^2}{2(1-\rho^2)} + \frac{y^2}{2}\right). \quad (7)$$

3. (Differentiation) $(\text{He}_i(z))' = \sqrt{i} \text{He}_{i-1}(z)$.

Finally, the following facts about Gaussian distribution are useful to our paper:

Fact 26 (Stein's Lemma ([Stein, 1981](#))) *Suppose that \mathbf{x} is distributed as $\mathcal{N}(\boldsymbol{\mu}, \sigma^2 \mathbf{I})$ for some $\boldsymbol{\mu} \in \mathbb{R}^d$, $\sigma \in \mathbb{R}_+$ and let $g : \mathbb{R}^d \rightarrow \mathbb{R}$ be an almost everywhere differentiable function such that both $\mathbf{E}[g(\mathbf{x})\mathbf{x}]$ and $\mathbf{E}[\nabla g(\mathbf{x})]$ exist. Then, it holds*

$$\mathbf{E}[g(\mathbf{x})(\mathbf{x} - \boldsymbol{\mu})] = \sigma^2 \mathbf{E}[\nabla g(\mathbf{x})].$$

Fact 27 (Komatsu's Inequality) *For any $t \geq 0$ it holds:*

$$C \frac{\exp(-t^2/2)}{t + \sqrt{t^2 + 4}} < \Pr_{x \sim \mathcal{N}(0,1)}[x \geq t] < C \frac{\exp(-t^2/2)}{t + \sqrt{t^2 + 2}},$$

where $C > 0$ is a universal constant.

Appendix C. Discussion on Regular Activations

Let us first recall the definitions of the (Extended-)(B, L)-Regular activations.

Definition 28 ((B, L)-Regular Activations) *Given parameters $B, L > 0$, we define the class of (B, L)-Regular activations, denoted by $\mathcal{H}(B, L)$, as the class containing all functions $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ such that 1) $\|\sigma\|_{L_\infty} \leq B$ and 2) $\|\sigma'\|_{L_2} \leq L$.*

Given $\epsilon > 0$, we define the class of ϵ -Extended (B, L)-Regular activations, denoted by $\mathcal{H}_\epsilon(B, L)$, as the class containing all activations $\sigma_1 : \mathbb{R} \rightarrow \mathbb{R}$ for which there exists $\sigma_2 \in \mathcal{H}(B, L)$ such that $\|\sigma_1 - \sigma_2\|_{L_2}^2 \leq \epsilon$.

We remark that our algorithm can be applied to many non-differentiable activations.

Remark 29 (On Differentiability) In the definition of (Extended-)(B, L)-Regular activations, the differentiability of σ is required. However, this restriction can be relaxed for any activation that is a locally-Lipschitz³ function, since they are differentiable almost-everywhere ([Federer, 1969](#)).

³We say a function σ is locally-Lipschitz if for any $z_0 \in \mathbb{R}$, there exists positive reals b and δ such that for any $z \in [z_0 - \delta, z_0 + \delta]$, it holds $|\sigma(z) - \sigma(z_0)| \leq b|z - z_0|$.

Therefore, since the set of non-differentiable points is measure-zero, we can define the derivative of σ at those non-differentiable points freely (for example, using Clarke Differentials (Clarke, 1990)).

Furthermore, our results can also be applied to functions that are not even locally-Lipschitz. In particular, functions that have finite ‘monotone jumps’ like $\sigma(z) = \text{sign}(z - t)$ are subject to our results (Lemma 39).

In fact, the set of smoothed functions are dense to our functions (i.e., there always exists a $\rho \in (0, 1)$ such that $\|\sigma - T_\rho \sigma\|_{L_2}^2 \leq \epsilon$). Therefore, statistically, there is no difference in using either of the functions.

C.1. Rescaling to the Unit Sphere

Next, we comment on the impact of rescaling the activation σ .

Remark 30 (Rescaling the Parameter) Let σ be a monotone (Extended-) (B, L) -regular activation. In our approach, it is without loss of generality to assume that $\|\mathbf{w}\|_2 = 1$. This is because, for any nonzero vector $\mathbf{w} \in \mathbb{B}(W)$, we can always rescale the activation $\sigma(\mathbf{w} \cdot \mathbf{x})$ to $\sigma(\|\mathbf{w}\|_2(\mathbf{w}/\|\mathbf{w}\|_2) \cdot \mathbf{x}) := \bar{\sigma}((\mathbf{w}/\|\mathbf{w}\|_2) \cdot \mathbf{x}) = \bar{\sigma}(\mathbf{w}' \cdot \mathbf{x})$ where $\|\mathbf{w}'\|_2 = 1$. In other words, we define $\bar{\sigma}(z) = \sigma(\|\mathbf{w}^*\|_2 z)$, where \mathbf{w}^* is one of the target vectors.

After rescaling, the second moment of $\bar{\sigma}'$ increases to $\|\bar{\sigma}'\|_{L_2} \leq \|\mathbf{w}^*\|_2 \|\sigma'(\|\mathbf{w}^*\|_2 z)\|_{L_2}$ (which can be further bounded by using that σ is close to a function $\hat{\sigma}$ with $\|\hat{\sigma}'\|_\infty \leq \|\mathbf{w}^*\|_2 B/\epsilon$). Therefore, the parameter L can potentially scale with W .

For instance, if σ is a b -Lipschitz activation, then the derivative of the rescaled function satisfies $|\bar{\sigma}'(z)| = \|\mathbf{w}^*\|_2 |\sigma'(\|\mathbf{w}^*\|_2 z)| \leq \|\mathbf{w}^*\|_2 b$ meaning that $\bar{\sigma}$ effectively becomes a Wb -Lipschitz activation. However, we emphasize that our approximation error obtained in Theorem 11 does not scale with any of these parameters B, L . These parameters only influence the sample complexity and runtime of our algorithm in a polynomial manner.

In the following lemma, we show that without loss of generality we can assume that we know the norm of the unknown vector \mathbf{w}^* as we can reduce the problem into testing $1/\text{poly}(\epsilon, 1/L, 1/B)$ different values for the norm.

Lemma 31 Fix $\epsilon > 0$ and $\delta \in (0, 1)$. Let $W > 0$ and let σ be an activation such that for all $\lambda \in (0, W)$, $\sigma(\lambda z)$ is a (B, L) -Regular activation. Fix a unit vector \mathbf{w} and assume that for some $0 < \lambda \leq W$, it holds that $\mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}}[(\sigma(\lambda \mathbf{w} \cdot \mathbf{x}) - y)^2] \leq \epsilon$. Then, with $N = \text{poly}(1/\epsilon, W, B, L)$ samples and $\text{poly}(d, N)$ runtime, we can find $\lambda' > 0$ so that $\mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}}[(\sigma(\lambda' \mathbf{w} \cdot \mathbf{x}) - y)^2] \leq 4\epsilon$.

Proof Let $r = \text{poly}(\epsilon, 1/L, 1/B)$ and we fix the following grid $(1+r)^k$, for $k = 1, \dots, O(\log(W)/r)$. From Lemma 32, for some $k \leq O(\log(W)/r)$ it holds that $\mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}}[(\sigma(\lambda \mathbf{w} \cdot \mathbf{x}) - \sigma((1+r)^k \mathbf{w} \cdot \mathbf{x}))^2] \leq 2\epsilon$. Hence, by testing all the possible choices and outputting the one with the minimum error suffices. This testing can be done with $\text{poly}(1/\epsilon, W, B, L)$ samples. ■

Lemma 32 Let σ be a (B, L) -Regular activation. Let $\epsilon > 0$ sufficiently small, then for $r \leq \text{poly}(\epsilon, 1/L, 1/B)$, it holds that $\mathbf{E}_{z \sim \mathcal{N}(0, 1)}[(\sigma(z) - \sigma((1+r)z))^2] \leq \epsilon$.

Proof From Lemma 24, it holds that $\mathbf{E}_{z \sim \mathcal{N}(0, 1)}[(\sigma(z) - T_\delta \sigma(z))^2] \leq 3(1 - \delta)L^2$. Using Markov’s inequality we have that

$$\Pr[|\sigma(z) - T_\delta \sigma(z)| \geq \epsilon] \leq \frac{\mathbf{E}_{z \sim \mathcal{N}(0, 1)}[(\sigma(z) - T_\delta \sigma(z))^2]}{\epsilon^2} \leq \frac{3(1 - \delta)L^2}{\epsilon^2}.$$

Furthermore, note that

$$\mathbf{Pr}_{z \sim \mathcal{N}(0,1)} [|\sigma((1+r)z) - T_\delta \sigma((1+r)z)| \geq \epsilon] = \mathbf{Pr}_{z \sim \mathcal{N}(0,(1+r)^2)} [|\sigma(z) - T_\delta \sigma(z)| \geq \epsilon] .$$

Furthermore, note that the total variation distance between two zero mean Gaussians with variance σ_1 and σ_2 is bound from above by $\sqrt{\log(\sigma_1/\sigma_2) - \sigma_2^2/(2\sigma_1^2) - 1/2}$, for our case this is smaller than $2r$. Therefore, we have that

$$\mathbf{Pr}_{z \sim \mathcal{N}(0,1)} [|\sigma((1+r)z) - T_\delta \sigma((1+r)z)| \geq \epsilon] \leq \mathbf{Pr}_{z \sim \mathcal{N}(0,1)} [|\sigma(z) - T_\delta \sigma(z)| \geq \epsilon] + 2r .$$

Therefore,

$$\mathbf{Pr}_{z \sim \mathcal{N}(0,1)} [|\sigma((1+r)z) - T_\delta \sigma((1+r)z)| \geq \epsilon] \leq \frac{3(1-\delta)L^2}{\epsilon^2} + 2r .$$

Combining, we have that

$$\begin{aligned} \mathbf{E}_{z \sim \mathcal{N}(0,1)} [(\sigma((1+r)z) - T_\delta \sigma((1+r)z))^2] &\leq \epsilon^2 \mathbf{Pr}_{z \sim \mathcal{N}(0,1)} [|\sigma((1+r)z) - T_\delta \sigma((1+r)z)| \leq \epsilon] \\ &\quad + B^2 \mathbf{Pr}_{z \sim \mathcal{N}(0,1)} [|\sigma((1+r)z) - T_\delta \sigma((1+r)z)| \geq \epsilon] \\ &\leq \epsilon^2 + B^2(2r + \frac{3(1-\delta)L^2}{\epsilon^2}) . \end{aligned}$$

Furthermore, from [Fact 21](#) it holds that $T_\delta \sigma(z)$ is $B/(1-\delta^2)^{1/2}$ -Lipschitz. Therefore, we have that

$$\mathbf{E}_{z \sim \mathcal{N}(0,1)} [(T_\delta \sigma(z) - T_\delta \sigma((1+r)z))^2] \leq \frac{B^2 r^2}{(1-\delta^2)} . \quad (8)$$

Choosing δ so that $1-\delta, 1-\delta^2 \leq O(\epsilon^4/(B^2+L^2))$ and $r = \epsilon(1-\delta^2)/(B+1)$, we get that

$$\mathbf{E}_{z \sim \mathcal{N}(0,1)} [(\sigma(z) - \sigma((1+r)z))^2] \leq \epsilon .$$

■

C.2. Truncating the Regular Activations

Next, we observe that for any $\sigma \in \mathcal{H}_\epsilon(B, L)$, one can assume without loss of generality that the labels y are bounded and the support of σ' is also bounded. First, we show that we can truncate the labels y without loss of generality.

Claim 33 *Let σ be a (B, L) -Regular activation. Let $\bar{y} = \text{sign}(y) \min\{|y|, B\}$. Then, $\mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}}[(\bar{y} - \sigma(\mathbf{w}^* \cdot \mathbf{x}))^2] \leq \text{OPT}$. Furthermore, for any $\hat{\mathbf{w}}$ such that $\mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}}[(\bar{y} - \sigma(\hat{\mathbf{w}} \cdot \mathbf{x}))^2] \leq O(\text{OPT})$, we have $\mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}}[(y - \sigma(\hat{\mathbf{w}} \cdot \mathbf{x}))^2] \leq O(\text{OPT})$. Hence it is w.l.o.g. to assume that $|y| \leq B$.*

Proof Let $\Pi(u) = \text{sign}(u) \min\{|u|, B\}$ be the projection operator projecting $u \in \mathbb{R}$ to the interval $[-B, B]$ and let $\bar{y} := \Pi(y)$. Since $|\sigma(z)| \leq B$ almost surely, we have $\Pi(\sigma(z)) = \sigma(z)$. Thus by the property of projection operators, we have $|y - \sigma(\mathbf{w}^* \cdot \mathbf{x})| \geq |\Pi(y) - \Pi(\sigma(\mathbf{w}^* \cdot \mathbf{x}))|$. Therefore, we

have $\mathbf{E}_{(\mathbf{x},y) \sim \mathcal{D}}[(\bar{y} - \sigma(\mathbf{w}^* \cdot \mathbf{x}))^2] = \mathbf{E}_{(\mathbf{x},y) \sim \mathcal{D}}[(\Pi(y) - \Pi(\sigma(\mathbf{w}^* \cdot \mathbf{x})))^2] \leq \mathbf{E}_{(\mathbf{x},y) \sim \mathcal{D}}[(y - \sigma(\mathbf{w}^* \cdot \mathbf{x}))^2] \leq \text{OPT}$. The arguments above shows that \mathbf{w}^* is also an OPT solution when y is truncated.

Now let $\hat{\mathbf{w}}$ be a constant approximate solution with respect to the truncated labels: $\mathbf{E}_{(\mathbf{x},y) \sim \mathcal{D}}[(\bar{y} - \sigma(\hat{\mathbf{w}} \cdot \mathbf{x}))^2] \leq \text{COPT}$. Then, we have

$$\begin{aligned} \mathbf{E}_{(\mathbf{x},y) \sim \mathcal{D}}[(y - \sigma(\hat{\mathbf{w}} \cdot \mathbf{x}))^2] &\leq 2 \mathbf{E}_{(\mathbf{x},y) \sim \mathcal{D}}[(y - \bar{y})^2] + 2 \mathbf{E}_{(\mathbf{x},y) \sim \mathcal{D}}[(\bar{y} - \sigma(\hat{\mathbf{w}} \cdot \mathbf{x}))^2] \\ &\leq 4 \mathbf{E}_{(\mathbf{x},y) \sim \mathcal{D}}[(y - \sigma(\mathbf{w}^* \cdot \mathbf{x}))^2] + 4 \mathbf{E}_{(\mathbf{x},y) \sim \mathcal{D}}[(\bar{y} - \sigma(\mathbf{w}^* \cdot \mathbf{x}))^2] + 2\text{COPT} \\ &\leq (8 + 2C)\text{OPT}. \end{aligned}$$

Therefore, $\hat{\mathbf{w}}$ is also an absolute constant approximate solution with respect to the true labels y , thus, it is without loss of generality to consider the L_2^2 loss with the truncated labels \bar{y} . \blacksquare

Finally, we show that for $\sigma \in \mathcal{H}(B, L)$, σ can be truncated so that the support of σ' can be bounded by $M < \infty$.

Claim 34 *Let σ be a (B, L) -Regular activation. Then, there exists a function $\tilde{\sigma} \in \mathcal{H}(B, L)$ that satisfies $\|\tilde{\sigma} - \sigma\|_{L_2}^2 \leq \epsilon$ and such that the support of $\tilde{\sigma}'$ is bounded by*

$$M \leq \sqrt{2 \log(4B^2/\epsilon) - \log \log(4B^2/\epsilon)}.$$

Moreover, if $\hat{\mathbf{w}}$ satisfies $\mathbf{E}_{(\mathbf{x},y) \sim \mathcal{D}}[(y - \tilde{\sigma}(\hat{\mathbf{w}} \cdot \mathbf{x}))^2] \leq O(\text{OPT}) + \epsilon$, then also $\mathbf{E}_{(\mathbf{x},y) \sim \mathcal{D}}[(y - \sigma(\hat{\mathbf{w}} \cdot \mathbf{x}))^2] \leq O(\text{OPT}) + \epsilon$. Thus, one can replace σ with $\tilde{\sigma}$ and assume without loss of generality that the support of σ' is bounded by M .

Proof Note that by choosing $M = \sqrt{2 \log(4B^2/\epsilon) - \log \log(4B^2/\epsilon)}$, using [Fact 27](#) we have

$$\Pr[|z| \geq M] \leq \frac{2 \exp(-M^2/2)}{M} = \frac{(\epsilon/(4B^2)) \sqrt{\log(4B^2/\epsilon)}}{\sqrt{2 \log(4B^2/\epsilon) - \log \log(4B^2/\epsilon)}} \leq \frac{\epsilon}{4B^2}.$$

Let us define

$$\tilde{\sigma}(z) = \begin{cases} \sigma(z), & \text{when } |z| \leq M \\ \sigma(M), & \text{when } z \geq M \\ \sigma(-M), & \text{when } z \leq -M. \end{cases}$$

Then, since $\|\sigma\|_\infty \leq B$, we have $\|\tilde{\sigma}\|_\infty \leq B$, and it holds

$$\mathbf{E}_{z \sim \mathcal{N}}[(\sigma(z) - \tilde{\sigma}(z))^2] = \mathbf{E}_{z \sim \mathcal{N}}[(\sigma(z) - \tilde{\sigma}(z))^2 \mathbf{1}\{|z| \geq M\}] \leq 4B^2 \Pr[|z| \geq M] \leq \epsilon.$$

In addition, $\|\tilde{\sigma}'(z)\|_{L_2} = \|\sigma'(z) \mathbf{1}\{|z| \leq M\}\|_{L_2} \leq L$. In other words, there exists an activation $\tilde{\sigma} \in \mathcal{H}(B, L)$ such that $\|\tilde{\sigma} - \sigma\|_{L_2}^2 \leq \epsilon$. Furthermore, we have

$$\begin{aligned} \mathbf{E}_{(\mathbf{x},y) \sim \mathcal{D}}[(y - \tilde{\sigma}(\mathbf{w}^* \cdot \mathbf{x}))^2] &\leq 2 \mathbf{E}_{(\mathbf{x},y) \sim \mathcal{D}}[(y - \sigma(\mathbf{w}^* \cdot \mathbf{x}))^2] + 2 \mathbf{E}_{\mathbf{x} \sim \mathcal{D}_{\mathbf{x}}}[(\sigma(\mathbf{w}^* \cdot \mathbf{x}) - \tilde{\sigma}(\mathbf{w}^* \cdot \mathbf{x}))^2] \\ &\leq \text{COPT} + \epsilon, \end{aligned}$$

Now let $\widehat{\mathbf{w}}$ satisfy $\mathbf{E}_{(\mathbf{x},y) \sim \mathcal{D}}[(y - \tilde{\sigma}(\widehat{\mathbf{w}} \cdot \mathbf{x}))^2] \leq \text{COPT} + \epsilon$. We show that $\mathcal{L}(\widehat{\mathbf{w}}) \leq O(\text{OPT}) + \epsilon$. We only need to observe that

$$\begin{aligned} \mathbf{E}_{(\mathbf{x},y) \sim \mathcal{D}}[(y - \sigma(\widehat{\mathbf{w}} \cdot \mathbf{x}))^2] &\leq 2 \mathbf{E}_{(\mathbf{x},y) \sim \mathcal{D}}[(y - \tilde{\sigma}(\widehat{\mathbf{w}} \cdot \mathbf{x}))^2] + 2 \mathbf{E}_{\mathbf{x} \sim \mathcal{D}_{\mathbf{x}}}[(\tilde{\sigma}(\widehat{\mathbf{w}} \cdot \mathbf{x}) - \sigma(\widehat{\mathbf{w}} \cdot \mathbf{x}))^2] \\ &\leq 2\text{COPT} + 4\epsilon. \end{aligned}$$

Hence we can replace σ with $\tilde{\sigma} \in \mathcal{H}(B, L)$ and focus on the L_2^2 loss with respect to $\tilde{\sigma}$. Therefore, we can assume without loss of generality that $\sigma(z)$ is a constant when $|z| \geq M$, in other words, for any $|z| \geq M$, we have $\sigma'(z) = 0$, and the support of σ' is indeed bounded by M . \blacksquare

Remark 35 Since M is an upper bound on the support of σ' , we will assume without loss of generality throughout the rest of the paper that M^2 is larger than any absolute constant C .

C.3. Examples of Regular Activations

We now show that $\mathcal{H}_\epsilon(B, L)$ contains a wide range of activations. First, we show that all monotone functions with bounded $2 + \zeta$ -moment are Extended Regular activations:

Lemma 36 *If σ satisfies $\mathbf{E}_{z \sim \mathcal{N}}[\sigma(z)^{2+\zeta}] \leq B_{\sigma,4}$ for some $\zeta > 0$ and σ is monotone, then $\sigma \in \mathcal{H}_\epsilon(c_1 D, c_2 D^4/\epsilon^2)$ where $D = (B_\sigma/4\epsilon)^{1/\zeta}$ and c_1, c_2 are absolute constants.*

Proof For some $\zeta > 0$, we have $\mathbf{E}_{z \sim \mathcal{N}}[\sigma(z)^{2+\zeta}] \leq B_\sigma$. From Markov's inequality, we have that

$$\Pr[|\sigma(z)| \geq T] \leq \frac{\mathbf{E}_{z \sim \mathcal{N}}[\sigma(z)^{2+\zeta}]}{T^{2+\zeta}} \leq \frac{B_\sigma}{T^{2+\zeta}}.$$

Note that $\mathbf{E}[\sigma^2(z)] = \int_0^\infty \Pr[\sigma^2(z) \geq t] dt = \int_0^\infty 2u \Pr[\sigma^2(z) \geq u^2] du$ (the last part is after change of variables to $u^2 = t$). Therefore, we have that

$$\begin{aligned} \mathbf{E}[\sigma^2(z) \mathbb{1}\{|\sigma(z)| \geq D\}] &= \int_0^\infty 2u \Pr[\sigma^2(z) \mathbb{1}\{|\sigma(z)| \geq D\} \geq u^2] du \\ &= \int_0^\infty 2u \Pr[\sigma(z) \mathbb{1}\{|\sigma(z)| \geq D\} \geq u] du \\ &= \int_D^\infty 2u \Pr[\sigma(z) \geq u] du \leq \int_D^\infty 2u \frac{B_\sigma}{u^{2+\zeta}} du \leq 4 \frac{B_\sigma}{D^\zeta}. \end{aligned}$$

Set $D = (B_\sigma/4\epsilon)^{1/\zeta}$ and let $\bar{\sigma}(z) = \text{sign}(\sigma(z)) \min\{|\sigma(z)|, D\}$. We show that $\mathbf{E}_{z \sim \mathcal{N}}[(\sigma(z) - \bar{\sigma}(z))^2] \leq \epsilon$:

$$\begin{aligned} \mathbf{E}_{z \sim \mathcal{N}}[(\sigma(z) - \bar{\sigma}(z))^2] &= \mathbf{E}_{z \sim \mathcal{N}}[(\sigma(z) - \bar{\sigma}(z))^2 \mathbb{1}\{|\sigma(z)| \geq D\}] \\ &\leq \mathbf{E}_{z \sim \mathcal{N}}[(\sigma(z))^2 \mathbb{1}\{|\sigma(z)| \geq D\}] \leq \epsilon. \end{aligned}$$

Therefore, σ is ϵ -close to a $\bar{\sigma}$ with $\|\bar{\sigma}\|_\infty \leq (B_\sigma/4\epsilon)^{1/\zeta}$.

It remains to show that the activation $\bar{\sigma}$ is also ϵ -close to an activation with bounded $\|\sigma'\|_{L_2}$. Without loss of generality we assume that $\bar{\sigma}(z) \geq 0$ and $\bar{\sigma}(z) \in [0, 2D]$, because we can just add

assume that the function is $\bar{\sigma}(z)' = \bar{\sigma}(z) + D$. Note that it holds that $\bar{\sigma}(z) = \int_0^{2D} \mathbb{1}\{\bar{\sigma}(z) \geq t\} dt$. It suffices to show that there exists a parameter ρ so that $\|\bar{\sigma}(z) - T_\rho \bar{\sigma}(z)\|_{L_2}^2 \leq (1 - \rho^2) \text{poly}(D)$. We have that

$$\begin{aligned} \|\bar{\sigma}(z) - T_\rho \bar{\sigma}(z)\|_{L_2}^2 &= \mathbf{E}_{z \sim \mathcal{N}} \left[\left(\int_0^{2D} \mathbb{1}\{\bar{\sigma}(z) \geq t\} - \mathbb{1}\{T_\rho \bar{\sigma}(z) \geq t\} dt \right)^2 \right] \\ &\leq \mathbf{E}_{z \sim \mathcal{N}} \left[\left(\int_0^{2D} |\mathbb{1}\{\bar{\sigma}(z) \geq t\} - \mathbb{1}\{T_\rho \bar{\sigma}(z) \geq t\}| dt \right)^2 \right] \\ &\leq 2D \mathbf{E}_{z \sim \mathcal{N}} \left[\int_0^{2D} \left(\mathbb{1}\{\bar{\sigma}(z) \geq t\} - \mathbb{1}\{T_\rho \bar{\sigma}(z) \geq t\} \right)^2 dt \right] \\ &= 2D \int_0^{2D} \|\mathbb{1}\{\bar{\sigma}(z) \geq t\} - T_\rho \mathbb{1}\{\bar{\sigma}(z) \geq t\}\|_{L_2}^2 dt, \end{aligned}$$

where we used the Jensen's inequality $((1/(b-a) \int_a^b f(z) dz))^2 \leq (1/(b-a)) \int_a^b f^2(z) dz$ for positive functions f and we exchange the integrals using the Fubini's theorem. Note that because the function $\bar{\sigma}(z)$ is monotone, then there exists a function $q(z)$ so that $\mathbb{1}\{\bar{\sigma}(z) \geq t\} = \mathbb{1}\{z \geq q(t)\}$. Therefore, using this transformation, it suffices to bound the difference

$$\begin{aligned} \|\mathbb{1}\{z \geq q(t)\} - T_\rho \mathbb{1}\{z \geq q(t)\}\|_{L_2}^2 &\leq \mathbf{E}_{x, z \sim \mathcal{N}(0,1)} [(\mathbb{1}\{x \geq q(t)\} - \mathbb{1}\{x\rho + z(1 - \rho^2)^{1/2} \geq q(t)\})^2] \\ &\leq 4(1 - \rho^2)^{1/2}, \end{aligned}$$

where in the first inequality we used Jensen, and in the second one we used that $\mathbf{E}[|\text{sign}(\mathbf{w} \cdot \mathbf{x} + t) - \text{sign}(\mathbf{v} \cdot \mathbf{x} + t)|] \leq \theta(\mathbf{v}, \mathbf{u})$ for any two unit vectors \mathbf{v}, \mathbf{w} (see Fact C.11 of [Diakonikolas et al. \(2022c\)](#)). Hence, we have that

$$\int_0^{2D} \|\mathbb{1}\{\bar{\sigma}(z) \geq t\} - T_\rho \mathbb{1}\{\bar{\sigma}(z) \geq t\}\|_{L_2}^2 dt \leq 8D(1 - \rho^2)^{1/2}.$$

Therefore, the function $\|\bar{\sigma}(z) - T_\rho \bar{\sigma}(z)\|_2^2 \leq \epsilon$ for $\rho = \sqrt{1 - (\epsilon/(16D^2))^2}$. That means that $\|(T_\rho \bar{\sigma}(z))'\|_{L_2}^2 \leq 16^2 D^4 / \epsilon^2$. Thus, we conclude that $\sigma \in \mathcal{H}_\epsilon(2D, 16^2 D^4 / \epsilon^2)$. \blacksquare

Now let us define a special type of activation that has an ‘exponential-tail’ property. We will show in [Lemma 39](#) that all b -Lipschitz functions are such kind of activations.

Definition 37 ((R, r)-Sub-exponential Activations) We say that an activation $\sigma(z)$ is (R, r)-sub-exponential for some positive constants R, r , if for any $p > 0$, we have $(\mathbf{E}_{z \sim \mathcal{N}}[\sigma(z)^p])^{1/p} \leq Rp^r$.

We will make use of the following fact in the proof of [Lemma 39](#).

Fact 38 (([Vershynin, 2018](#)) Theorem 5.2.2) Let $z \sim \mathcal{N}(0, 1)$ and let σ be a b -Lipschitz function. Then, $\sigma(z)$ is a sub-Gaussian random variable with $\|\sigma(z)\|_{\psi_2} \leq cb$, where $\|\cdot\|_{\psi_2}$ is the Orlicz 2-norm and c is an absolute constant.

We show that all the following function classes belong to the Extended Regular activation class:

Lemma 39 All of the following activations are ϵ -Extended (B, L)-Regular.

1. If σ satisfies $\mathbf{E}_{z \sim \mathcal{N}}[\sigma(z)^4] \leq B_{\sigma,4}$ and $\|\sigma'\|_{L_2} \leq L$, then $\sigma \in \mathcal{H}(\sqrt{B_{\sigma,4}/\epsilon}, L)$.
2. If σ is (R, r) -Sub-exponential and $\|\sigma'\|_{L_2} \leq L$, then $\sigma \in \mathcal{H}_\epsilon(cR(r + \log(R/\epsilon))^r, L)$, where c is an absolute constant.
3. If σ is b -Lipschitz, then $\sigma \in \mathcal{H}_\epsilon(cb \log^{1/2}(b/\epsilon), b)$, where c is an absolute constant.
4. If $\sigma = \sigma_1 + \Phi$, where $\sigma_1 \in \mathcal{H}_\epsilon(B, L)$, $|\Phi(z)| \leq A$, $\Phi \in \mathcal{F}_M$ (recall [Definition 15](#)), i.e.,

$$\Phi(z) = \sum_{i=1}^m A_i \phi(z; t_i) + A_0 : A_0 \in \mathbb{R}; A_i > 0, |t_i| \leq M, \forall i \in [m]; m < \infty$$

then $\sigma \in \mathcal{H}_\epsilon(B + A, L + \max\{A^2 L / \sqrt{\epsilon}, A^4 / \epsilon\})$.

Proof We prove each claim in order.

1. Suppose first that $\mathbf{E}_{z \sim \mathcal{N}}[\sigma(z)^4] \leq B_{\sigma,4}$. Let $\bar{\sigma}(z) = \text{sign}(\sigma(z)) \min\{|\sigma(z)|, \sqrt{B_{\sigma,4}/\epsilon}\}$, which is an activation in the (B, L) -Regular class. We show that $\mathbf{E}_{z \sim \mathcal{N}}[(\sigma(z) - \bar{\sigma}(z))^2] \leq \epsilon$:

$$\begin{aligned} \mathbf{E}_{z \sim \mathcal{N}}[(\sigma(z) - \bar{\sigma}(z))^2] &= \mathbf{E}_{z \sim \mathcal{N}}[(\sigma(z) - \bar{\sigma}(z))^2 \mathbf{1}\{|\sigma(z)| \geq \sqrt{B_{\sigma,4}/\epsilon}\}] \\ &\leq \mathbf{E}_{z \sim \mathcal{N}}[(\sigma(z))^2 \mathbf{1}\{|\sigma(z)| \geq \sqrt{B_{\sigma,4}/\epsilon}\}] \\ &\leq \sqrt{\mathbf{E}_{z \sim \mathcal{N}}[\sigma^4(z)] \Pr[|\sigma(z)| \geq \sqrt{B_{\sigma,4}/\epsilon}]}. \end{aligned}$$

By Markov's inequality we have

$$\Pr[|\sigma(z)| \geq \sqrt{B_{\sigma,4}/\epsilon}] = \Pr[\sigma^2(z) \geq B_{\sigma,4}/\epsilon] \leq \frac{\mathbf{E}_{z \sim \mathcal{N}}[\sigma(z)^4] \epsilon^2}{B_{\sigma,4}^2} \leq \frac{\epsilon^2}{B_{\sigma,4}}. \quad (9)$$

Therefore, the L_2^2 difference between σ and $\bar{\sigma}$ is bounded above by

$$\mathbf{E}_{z \sim \mathcal{N}}[(\sigma(z) - \bar{\sigma}(z))^2] \leq \sqrt{B_{\sigma,4} \frac{\epsilon^2}{B_{\sigma,4}}} \leq \epsilon.$$

Therefore, σ is an Extended $(\sqrt{B_{\sigma,4}/\epsilon}, L)$ -Regular activation.

2. Next, assume that σ is (R, r) -Sub-exponential. Similarly, let

$$\bar{\sigma}(z) = \text{sign}(\sigma(z)) \min\{|\sigma(z)|, eR(4r \log(4) + \log(R^4/\epsilon^2))^r\}$$

Denote for simplicity $B_\sigma := eR(4r \log(4) + \log(R^4/\epsilon^2))^r$. Then, $\bar{\sigma}$ is a (B_σ, L) -Regular activation. Using the same arguments as above, we have

$$\begin{aligned} \mathbf{E}_{z \sim \mathcal{N}}[(\sigma(z) - \bar{\sigma}(z))^2] &= \mathbf{E}_{z \sim \mathcal{N}}[(\sigma(z) - \bar{\sigma}(z))^2 \mathbf{1}\{|\sigma(z)| \geq B_\sigma\}] \\ &\leq \sqrt{\mathbf{E}_{z \sim \mathcal{N}}[\sigma^4(z)] \Pr[|\sigma(z)| \geq B_\sigma]}. \end{aligned}$$

Now since σ is a (R, r) -Sub-exponential activation, we have

$$\Pr[|\sigma(z)| \geq t] \leq \frac{\mathbf{E}_{z \sim \mathcal{N}}[\sigma^p(z)]}{t^p} \leq \left(\frac{Rp^r}{t}\right)^p.$$

Choosing $p = (t/(Re))^{1/r}$, we get

$$\Pr[|\sigma(z)| \geq t] \leq \exp\left(- (t/(Re))^{1/r}\right).$$

Let $t = B_\sigma = Re(4r \log(4) + \log(R^4/\epsilon^2))^r$, then it holds

$$\Pr[|\sigma(z)| \geq B_\sigma] \leq \exp\left(-\log(4^{4r}\epsilon^2/R^4)\right) = \frac{\epsilon^2}{R^4 4^{4r}}.$$

Furthermore, since $\mathbf{E}_{z \sim \mathcal{N}}[\sigma^4] \leq R^4 4^{4r}$, we thus obtain

$$\mathbf{E}_{z \sim \mathcal{N}}[(\sigma(z) - \bar{\sigma}(z))^2] \leq \epsilon.$$

3. Next, suppose σ is b -Lipschitz. Then, since $|\sigma'| \leq b$, we have $\|\sigma'\|_{L_2} \leq b$. Next, we show that σ is $(b, 1/2)$ -Sub-exponential. Note that it is without loss of generality to assume that $\mathbf{E}_{z \sim \mathcal{N}}[\sigma(z)] = 0$, since we can always consider shifting the activation σ and the labels y to $\sigma(z) - \mathbf{E}_{z \sim \mathcal{N}}[\sigma(z)]$, $y - \mathbf{E}_{z \sim \mathcal{N}}[\sigma(z)]$ and obtaining the same results. Since $z \sim \mathcal{N}(0, 1)$, we can use [Fact 38](#), which yields that $\sigma(z)$ is sub-Gaussian with sub-Gaussian constant $\|\sigma(z)\|_{\psi_2} = cb$. Because $\sigma(z)$ is sub-Gaussian, we know $\|\sigma(z)\|_{L_p} \leq c\|\sigma(z)\|_{\psi_2} p^{1/2} \leq cbp^{1/2}$, this implies that σ is a $(b, 1/2)$ -Sub-exponential activation. Using the previous result (Part 2), we immediately obtain that σ is an Extended $(cb \log^{1/2}(b/\epsilon), b)$ -Regular activation.

4. Finally, consider $\sigma = \sigma_1 + \Phi$ where $\sigma_1 \in \mathcal{H}_\epsilon(B, L)$, $\Phi \in \mathcal{F}_M$, $|\Phi(z)| \leq A$. Let $\tilde{\sigma} = T_{1-\epsilon_0}\sigma_1 + T_{1-\epsilon_0}\Phi$, where $\epsilon_0 \leq \min(\epsilon/L^2, 1/M^2, (\epsilon/A^2)^2)$, and M is defined as in [Claim 34](#). Then we have

$$\|\tilde{\sigma} - \sigma\|_{L_2}^2 \leq 2\|T_{1-\epsilon_0}\sigma_1 - \sigma_1\|_{L_2}^2 + 2\|T_{1-\epsilon_0}\Phi - \Phi\|_{L_2}^2.$$

By [Lemma 24](#), we have $\|T_{1-\epsilon_0}\sigma_1 - \sigma_1\|_{L_2}^2 \lesssim \epsilon_0 \|\sigma_1'\|_{L_2}^2 \leq \epsilon$. In addition, applying [Proposition 64](#), since $\epsilon_0 \leq 1/M^2$, we have $\|T_{1-\epsilon_0}\Phi - \Phi\|_{L_2}^2 \lesssim \epsilon_0 \|T_{1-\epsilon_0}\Phi'(z)\|_{L_2}^2$. Note that $\max_{z \in \mathbb{R}} \Phi(z) = \sum_{i=1}^m A_i \leq A$. Then, by [Lemma 65](#), we have

$$\|T_{1-\epsilon_0}\Phi'(z)\|_{L_2}^2 \lesssim (1/\sqrt{\epsilon_0}) \sum_{i,j=1}^m A_i A_j = (1/\sqrt{\epsilon_0}) A^2 \leq \max\{A^2 L / \sqrt{\epsilon}, A^4 / \epsilon\}.$$

Therefore, we further obtain $\|T_{1-\epsilon_0}\Phi - \Phi\|_{L_2}^2 \leq \sqrt{\epsilon_0} A^2 \leq \epsilon$. This implies that $\|\tilde{\sigma} - \sigma\|_{L_2}^2 \lesssim \epsilon$. Furthermore, observe that $\|\tilde{\sigma}\|_{L_\infty} \leq \|\sigma_1\|_{L_\infty} + \|\Phi\|_{L_\infty} \leq B + A$, and $\|\tilde{\sigma}'\|_{L_2} \leq \|T_{1-\epsilon_0}\sigma_1'\|_{L_2} + \|T_{1-\epsilon_0}\Phi'\|_{L_2} \leq L + \sqrt{\epsilon}$. Thus, we conclude that $\tilde{\sigma} \in \mathcal{H}_\epsilon(B + A, L + \max\{A^2 L / \sqrt{\epsilon}, A^4 / \epsilon\})$. \blacksquare

C.4. Required Assumptions on Activation

Here we point out that it is information-theoretically impossible to learn all monotone activations, in the realizable setting, if we only assume that $\sigma \in L_2(\mathcal{N})$ —even if we further assume that $\|\sigma\|_{L_2} \leq 1$. The intuition behind this fact is the following: there exists a monotone function σ , which is equal to 0 everywhere except in the tails of a direction \mathbf{v} . That means that in order to see a point where the labels are non-zero, we need to see a label from the tails.

We show that for any choice of the threshold in the tails, i.e., $\Pr_{z \sim \mathcal{N}}[z \geq t]$, there exists a monotone function that has $\|\sigma\|_{L_2} = 1$ and for any unit vectors \mathbf{v}, \mathbf{u} with $\|\mathbf{v} - \mathbf{u}\|_2 = \Omega(1)$, we have $\|\sigma(\mathbf{v} \cdot \mathbf{x}) - \sigma(\mathbf{u} \cdot \mathbf{x})\|_{L_2}^2 = \Omega(1)$. Formally, we show that:

Theorem 40 (Impossibility of Learning All Monotone Functions) *Consider the class \mathcal{F} consisting of all monotone activations $\sigma \in L_2(\mathcal{N})$ satisfying $\|\sigma\|_{L_2} \leq 1$. There is no finite-sample algorithm that realizably learns \mathcal{F} up to error $1/8$.*

Proof Let $\gamma^{-1}(\delta) = \sup_t \{t : \Pr_{z \sim \mathcal{N}(0,1)}[z \geq t] \leq \delta\}$. Let $\delta < 1/16$ and consider the following function $\sigma(t) = (1/\sqrt{\delta})\mathbb{1}\{t \geq \gamma^{-1}(\delta)\}$. Note that this function belongs to the class $\sigma \in L_2(\mathcal{N})$ with $\|\sigma\|_{L_2} \leq 1$, since

$$\mathbf{E}[\sigma^2(z)] = 1/\delta \Pr[t \geq \gamma^{-1}(\delta)] = 1.$$

Consider a set of unit vectors V such that for any $\mathbf{u}, \mathbf{v} \in V$ we have $\|\mathbf{u} - \mathbf{v}\|_2 \geq 1/2$. By standard packing arguments, there exists such a set of size $2^{\Theta(d)}$. Let $\theta := \theta(\mathbf{u}, \mathbf{v})$. Then we have $\|\mathbf{u} - \mathbf{v}\|_2 = 2 \sin(\theta/2) \geq 1/2$, hence $\cos \theta = 1 - 2 \sin^2(\theta/2) \leq 7/8$. Note that for any $\mathbf{u}, \mathbf{v} \in V$, with $\mathbf{u} \neq \mathbf{v}$, it holds that

$$\begin{aligned} \|\sigma(\mathbf{v} \cdot \mathbf{x}) - \sigma(\mathbf{u} \cdot \mathbf{x})\|_{L_2}^2 &= 2(1 - \mathbf{E}[\sigma(\mathbf{v} \cdot \mathbf{x})\sigma(\mathbf{u} \cdot \mathbf{x})]) \\ &= 2 \sum_{k \geq 0} (1 - \cos^k \theta) \hat{\sigma}(i)^2 \\ &\geq 2(1 - \hat{\sigma}(0)^2) - \cos \theta \sum_{k \geq 1} \hat{\sigma}(i)^2 \geq 2(1 - \hat{\sigma}(0)^2) - 2 \cos \theta \|\sigma\|_{L_2}^2, \end{aligned}$$

where $\hat{\sigma}(i)$ are the Hermite coefficients of σ . Furthermore, note that $\hat{\sigma}(0) = \mathbf{E}[\sigma(z)] = \sqrt{\delta}$. Hence, we have that

$$\|\sigma(\mathbf{v} \cdot \mathbf{x}) - \sigma(\mathbf{u} \cdot \mathbf{x})\|_{L_2}^2 \geq 2(1 - \delta - 7/8) \geq 1/4 - 2\delta \geq 1/8.$$

Intuitively, in order to learn up to error $\epsilon < 1/2$, we need to see at least one sample \mathbf{x} such that $\sigma(\mathbf{v} \cdot \mathbf{x}) > 0$, which happens with probability δ . Since δ can be selected to be an arbitrarily small positive number, by taking $\delta \rightarrow 0$, we see that in order to observe one sample, we need $\Omega(1/\delta)$ samples; if we choose \mathbf{v} at random, we succeed with probability at most $\exp(-cd)$.

One way to formalize the argument is to reduce the problem of learning Gaussian halfspaces to the above task. Consider the following transformation: $\forall \mathbf{v} \in V$, let $y'_\mathbf{v} = 1$ when $\sigma(\mathbf{v} \cdot \mathbf{x}) = 1/\sqrt{\delta}$ and $y'_\mathbf{v} = -1$ otherwise. This gives an instance of learning halfspaces under the Gaussian distribution. We have that

$$\|\sigma(\mathbf{v} \cdot \mathbf{x}) - \sigma(\mathbf{u} \cdot \mathbf{x})\|_{L_2}^2 = \Pr[\text{sign}(\mathbf{v} \cdot \mathbf{x} - \gamma^{-1}(\delta)) \neq \text{sign}(\mathbf{u} \cdot \mathbf{x} - \gamma^{-1}(\delta))]/\delta.$$

Therefore, in order to get error of $1/8$ for our monotone GLM learning task, we need to learn halfspaces with accuracy better than $\delta/8$. This task is known to have a sample complexity lower

bound of $\Omega(d/\delta)$. Therefore, since an algorithm that learns $\sigma(\mathbf{v} \cdot \mathbf{x})$ with error better than $1/8$ will also learn halfspaces, it follows that achieving error better than $1/8$ requires $\gtrsim d/\delta$ samples. As $\delta \rightarrow 0$, the number of samples becomes unbounded. If we output a function at random, the probability of success is lower bounded by the number of elements in $|V|$, which gives the result. ■

Appendix D. Full Version of Section 2

D.1. Augmenting the Data: Connection to Ornstein–Uhlenbeck Semigroup

As already discussed in Section 1.1, our algorithm relies on the data augmentation technique, i.e., in each iteration, the algorithm injects Gaussian noise (see Algorithm 3), which has the effect of improving the regularity properties of the loss landscape, as shown in this section.

Algorithm 3 Augment Dataset with Injected White Noise

- 1: **Input:** Parameters ρ, m ; Sample data $\mathfrak{D} = \{(\mathbf{x}^{(1)}, y^{(1)}), \dots, (\mathbf{x}^{(N)}, y^{(N)})\}$; $S \leftarrow \emptyset$.
 - 2: **for** $(\mathbf{x}^{(i)}, y^{(i)}) \in \mathfrak{D}$ **do**
 - 3: **for** $j = 1, \dots, m$ **do**
 - 4: Sample \mathbf{z} from $\mathcal{N}(\mathbf{0}, \mathbf{I})$. Let $\tilde{\mathbf{x}}^{(j)} \leftarrow \rho \mathbf{x}^{(i)} + (1 - \rho^2)^{1/2} \mathbf{z}$ and add to $S \leftarrow S \cup \{(\tilde{\mathbf{x}}^{(j)}, y^{(i)})\}$.
 - 5: **Return:** S .
-

Remark 41 Note that Algorithm 3 does not require new samples from the distribution \mathcal{D} . As we will see in Lemma 44, the purpose of Algorithm 3 is to estimate $T_\rho \sigma'(\mathbf{w} \cdot \mathbf{x}^{(i)})$ for each sample $\mathbf{x}^{(i)}$ we received. By standard concentration bounds using at most $\tilde{O}(L/\epsilon^2)$ independent (unlabeled) Gaussian samples suffices for all $\{\mathbf{x}^{(i)}, y^{(i)}\} \in \mathfrak{D}$. Since this affects only the runtime in polynomially, for simplicity of analysis we assume Algorithm 3 can be executed efficiently and have access to the population one.

The augmentation can be viewed as a transformation of the distribution \mathcal{D} to \mathcal{D}_ρ , where for any $(\tilde{\mathbf{x}}, y) \sim \mathcal{D}_\rho$, it holds $\tilde{\mathbf{x}} \sim \rho \mathcal{D}_\mathbf{x} + (1 - \rho^2)^{1/2} \mathcal{N}(\mathbf{0}, \mathbf{I})$. The data augmentation introduced in Algorithm 3 in fact simulates the Ornstein–Uhlenbeck semigroup, which we formalize in the following lemma.

Lemma 42 *Let \mathcal{D} be a distribution of labeled examples $(\mathbf{x}, y) \in \mathbb{R}^d \times \mathbb{R}$ such with \mathbf{x} -marginal $\mathcal{D}_\mathbf{x} = \mathcal{N}(\mathbf{0}, \mathbf{I})$. Furthermore, let \mathcal{D}_ρ be the distribution constructed by applying Algorithm 1 to samples from \mathcal{D} . Then for any $f : \mathbb{R} \rightarrow \mathbb{R}$ and any unit vector $\mathbf{w} \in \mathbb{R}^d$ with $|\mathbf{E}_{\mathbf{x} \sim \mathcal{D}_\mathbf{x}}[f(\mathbf{w} \cdot \mathbf{x})]| < \infty$, we have $\mathbf{E}_{\tilde{\mathbf{x}} \sim (\mathcal{D}_\rho)_\tilde{\mathbf{x}}} [f(\mathbf{w} \cdot \tilde{\mathbf{x}})] = \mathbf{E}_{\mathbf{x} \sim \mathcal{D}_\mathbf{x}} [T_\rho f(\mathbf{w} \cdot \mathbf{x})]$.*

Proof Using the definition of \mathcal{D}_ρ , we have that

$$\begin{aligned} \mathbf{E}_{\tilde{\mathbf{x}} \sim (\mathcal{D}_\rho)_\tilde{\mathbf{x}}} [f(\mathbf{w} \cdot \tilde{\mathbf{x}})] &= \mathbf{E}_{\mathbf{x} \sim \mathcal{D}_\mathbf{x}} [\mathbf{E}_{\mathbf{z} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})} [f(\rho \mathbf{w} \cdot \mathbf{x} + \sqrt{1 - \rho^2} \mathbf{w} \cdot \mathbf{z})]] \\ &= \mathbf{E}_{\mathbf{x} \sim \mathcal{D}_\mathbf{x}} [\mathbf{E}_{\zeta \sim \mathcal{N}(0, 1)} [f(\rho \mathbf{w} \cdot \mathbf{x} + \sqrt{1 - \rho^2} \zeta)]] = \mathbf{E}_{\mathbf{x} \sim \mathcal{D}_\mathbf{x}} [T_\rho f(\mathbf{w} \cdot \mathbf{x})], \end{aligned}$$

where we have used that $\mathbf{w} \cdot \mathbf{z}$ is distributed according to the standard normal distribution. ■

[Lemma 42](#) shows that our data augmentation technique is equivalent to applying the Ornstein–Uhlenbeck semigroup to our dataset. This application of the Ornstein–Uhlenbeck semigroup to the dataset has the effect of smoothing the landscape of the square loss, which in turn allows us to prove that the gradient of the smoothed/augmented loss carries information about the direction of the target vector. This is the main structural result obtained in the next subsection.

D.2. Alignment of the Gradients of the Augmented Loss

In this section, we provide the main structural result of this work, showing that the gradients of the square loss applied to the augmented data correlate with a target parameter vector \mathbf{w}^* . For notational convenience, we use

$$\mathcal{L}_\rho(\mathbf{w}) = \mathbf{E}_{(\tilde{\mathbf{x}}, y) \sim \mathcal{D}_\rho} [(\sigma(\mathbf{w} \cdot \tilde{\mathbf{x}}) - y)^2] \quad (10)$$

to denote the square loss on the augmented data and refer to it as the “augmented loss.”

Proposition 43 (Main Structural Result) *Fix an activation $\sigma : \mathbb{R} \rightarrow \mathbb{R}$. Let \mathcal{D} be a distribution of labeled examples $(\mathbf{x}, y) \in \mathbb{R}^d \times \mathbb{R}$ such that its \mathbf{x} -marginal $\mathcal{D}_\mathbf{x}$ is $\mathcal{N}(\mathbf{0}, \mathbf{I})$. Moreover, let \mathcal{D}_ρ be the distribution constructed by applying [Algorithm 1](#) with parameter $\rho \in (0, 1)$ to the distribution \mathcal{D} . Fix unit vectors $\mathbf{w}^*, \mathbf{w} \in \mathbb{R}^d$ such that $\mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}} [(\sigma(\mathbf{w}^* \cdot \mathbf{x}) - y)^2] = \text{OPT}$ and let $\theta = \theta(\mathbf{w}^*, \mathbf{w})$. Let $\mathbf{g}(\mathbf{w})$ be the gradient of the loss $\mathcal{L}_\rho(\mathbf{w}) = \mathbf{E}_{(\tilde{\mathbf{x}}, y) \sim \mathcal{D}_\rho} [(\sigma(\mathbf{w} \cdot \tilde{\mathbf{x}}) - y)^2]$ projected on the subspace \mathbf{w}^\perp and scaled by $1/(2\rho)$, i.e., $\mathbf{g}(\mathbf{w}) = (1/(2\rho))(\nabla_{\mathbf{w}} \mathcal{L}_\rho(\mathbf{w}))^\perp_{\mathbf{w}}$. Then,*

$$\mathbf{g}(\mathbf{w}) \cdot \mathbf{w}^* \leq -\|\mathbf{T}_{\sqrt{\rho \cos \theta}} \sigma'\|_{L_2}^2 \sin^2 \theta + \sqrt{\text{OPT}} \|\mathbf{T}_\rho \sigma'\|_{L_2} \sin \theta.$$

In particular, if $0 < \rho \leq \cos \theta < 1$ and $\sin \theta \geq 3\sqrt{\text{OPT}}/\|\mathbf{T}_\rho \sigma'\|_{L_2}$, then

$$\mathbf{g}(\mathbf{w}) \cdot \mathbf{w}^* \leq -(2/3)\|\mathbf{T}_{\sqrt{\rho \cos \theta}} \sigma'\|_{L_2}^2 \sin^2 \theta.$$

Before proving the proposition, we first prove the following auxiliary lemma, which establishes a connection between the Riemannian gradient of [Equation \(10\)](#) and the Ornstein–Uhlenbeck semigroup applied to the derivative of the activation.

Lemma 44 *Let $\mathbf{g}(\mathbf{w}) = (1/(2\rho))(\nabla_{\mathbf{w}} \mathcal{L}_\rho(\mathbf{w}))^\perp_{\mathbf{w}}$. Then, $\mathbf{g}(\mathbf{w}) = -\mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}} [y \mathbf{T}_\rho \sigma'(\mathbf{w} \cdot \mathbf{x}) \mathbf{x}^\perp_{\mathbf{w}}]$.*

Proof By definition, the projected gradient vector $\mathbf{g}(\mathbf{w})$ equals

$$\mathbf{g}(\mathbf{w}) = \frac{1}{\rho} \left(\mathbf{E}_{(\tilde{\mathbf{x}}, y) \sim \mathcal{D}_\rho} [\sigma(\mathbf{w} \cdot \tilde{\mathbf{x}}) \sigma'(\mathbf{w} \cdot \tilde{\mathbf{x}}) \tilde{\mathbf{x}}^\perp_{\mathbf{w}}] - \mathbf{E}_{(\tilde{\mathbf{x}}, y) \sim \mathcal{D}_\rho} [y \sigma'(\mathbf{w} \cdot \tilde{\mathbf{x}}) \tilde{\mathbf{x}}^\perp_{\mathbf{w}}] \right).$$

Note that since $(\mathcal{D}_\rho)_{\tilde{\mathbf{x}}}$ is also a standard Gaussian distribution, we have $\mathbf{E}_{\tilde{\mathbf{x}} \sim (\mathcal{D}_\rho)_{\tilde{\mathbf{x}}}} [\tilde{\mathbf{x}}^\perp_{\mathbf{w}}] = \mathbf{0}$, hence $\mathbf{E}_{(\tilde{\mathbf{x}}, y) \sim \mathcal{D}_\rho} [\sigma(\mathbf{w} \cdot \tilde{\mathbf{x}}) \sigma'(\mathbf{w} \cdot \tilde{\mathbf{x}}) \tilde{\mathbf{x}}^\perp_{\mathbf{w}}] = \mathbf{0}$, as $\mathbf{w} \cdot \tilde{\mathbf{x}}$ and $\tilde{\mathbf{x}}^\perp_{\mathbf{w}}$ are independent. Thus, $\rho \mathbf{g}(\mathbf{w}) = -\mathbf{E}_{(\tilde{\mathbf{x}}, y) \sim \mathcal{D}_\rho} [y \sigma'(\mathbf{w} \cdot \tilde{\mathbf{x}}) \tilde{\mathbf{x}}^\perp_{\mathbf{w}}]$. Now, since $(\mathcal{D}_\rho)_{\tilde{\mathbf{x}}} = \rho \mathcal{D}_\mathbf{x} + (1 - \rho^2)^{1/2} \mathcal{N}(\mathbf{0}, \mathbf{I})$, we have

$$\mathbf{g}(\mathbf{w}) = -\frac{1}{\rho} \mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}, \mathbf{z} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})} [y \sigma'(\mathbf{w} \cdot (\rho \mathbf{x} + \sqrt{1 - \rho^2} \mathbf{z})) (\rho \mathbf{x} + \sqrt{1 - \rho^2} \mathbf{z})^\perp_{\mathbf{w}}].$$

As \mathbf{z} is independent of \mathbf{x}, y and follows the standard Gaussian distribution, it must be $\mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}, \mathbf{z} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})} [y \sigma'(\mathbf{w} \cdot \tilde{\mathbf{x}}) \mathbf{z}^{\perp \mathbf{w}}] = 0$, and thus we further have

$$\begin{aligned} \mathbf{g}(\mathbf{w}) &= -\frac{1}{\rho} \mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}, \mathbf{z} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})} [y \sigma'(\mathbf{w} \cdot (\rho \mathbf{x} + \sqrt{1 - \rho^2} \mathbf{z})) \rho \mathbf{x}^{\perp \mathbf{w}}] \\ &= -\mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}} [y \mathbf{E}_{\mathbf{z} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})} [\sigma'(\mathbf{w} \cdot (\rho \mathbf{x} + \sqrt{1 - \rho^2} \mathbf{z}))] \mathbf{x}^{\perp \mathbf{w}}] \\ &= -\mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}} [y T_{\rho} \sigma'(\mathbf{w} \cdot \mathbf{x}) \mathbf{x}^{\perp \mathbf{w}}], \end{aligned}$$

completing the proof. \blacksquare

We are now ready to prove our main structural result.

Proof of Proposition 43 When \mathbf{w}^* is parallel to \mathbf{w} , then $\mathbf{g}(\mathbf{w}) \cdot \mathbf{w}^* = 0$ since $\mathbf{g}(\mathbf{w})$ is orthogonal to \mathbf{w} , and $\sin \theta = 0$, hence the statements hold trivially. Thus in the rest of the proof we assume that $(\mathbf{w}^*)^{\perp \mathbf{w}} \neq \mathbf{0}$. Denote $\mathbf{v} := (\mathbf{w}^*)^{\perp \mathbf{w}} / \|(\mathbf{w}^*)^{\perp \mathbf{w}}\|_2$. Then, $\mathbf{w}^* = \mathbf{w} \cos \theta + \mathbf{v} \sin \theta$, where $\theta := \theta(\mathbf{w}, \mathbf{w}^*)$. Using Lemma 6, the inner product between \mathbf{w}^* and $-\mathbf{g}(\mathbf{w})$ equals

$$-\mathbf{g}(\mathbf{w}) \cdot \mathbf{w}^* = \mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}} [y T_{\rho} \sigma'(\mathbf{w} \cdot \mathbf{x}) \mathbf{v} \cdot \mathbf{x}] \sin \theta.$$

By adding and subtracting $\sigma(\mathbf{w}^* \cdot \mathbf{x})$ on the right-hand side, we get that

$$\begin{aligned} -\mathbf{g}(\mathbf{w}) \cdot \mathbf{w}^* &= \mathbf{E}_{\mathbf{x} \sim \mathcal{D}_{\mathbf{x}}} [\sigma(\mathbf{w}^* \cdot \mathbf{x}) T_{\rho} \sigma'(\mathbf{w} \cdot \mathbf{x}) \mathbf{v} \cdot \mathbf{x}] \sin \theta \\ &\quad + \mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}} [(y - \sigma(\mathbf{w}^* \cdot \mathbf{x})) T_{\rho} \sigma'(\mathbf{w} \cdot \mathbf{x}) \mathbf{v} \cdot \mathbf{x}] \sin \theta. \end{aligned} \quad (11)$$

Observe that since $\mathbf{w}^* = \cos \theta \mathbf{w} + \sin \theta \mathbf{v}$ and \mathbf{x} is a standard Gaussian random vector, we have $\mathbf{w} \cdot \mathbf{x}$ and $\mathbf{v} \cdot \mathbf{x}$ are independent standard Gaussian random variables. By applying Cauchy-Schwarz inequality to the expectation in the last term, we obtain

$$\begin{aligned} &\mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}} [(y - \sigma(\mathbf{w}^* \cdot \mathbf{x})) T_{\rho} \sigma'(\mathbf{w} \cdot \mathbf{x}) \mathbf{v} \cdot \mathbf{x}] \\ &\geq -\left(\mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}} [(y - \sigma(\mathbf{w}^* \cdot \mathbf{x}))^2] \mathbf{E}_{\mathbf{x} \sim \mathcal{D}_{\mathbf{x}}} [(T_{\rho} \sigma'(\mathbf{w} \cdot \mathbf{x}) \mathbf{v} \cdot \mathbf{x})^2] \right)^{1/2} \\ &= -\sqrt{\text{OPT}} \mathbf{E}_{\mathbf{x} \sim \mathcal{D}_{\mathbf{x}}} [(T_{\rho} \sigma'(\mathbf{w} \cdot \mathbf{x}))^2] = -\sqrt{\text{OPT}} \|T_{\rho} \sigma'\|_{L_2}, \end{aligned}$$

where in the first equality we used the definition of OPT and that $\mathbf{w} \cdot \mathbf{x}$ and $\mathbf{v} \cdot \mathbf{x}$ are independent standard Gaussian random variables noted above.

To bound the first term on the right-hand side of (11), we again use that $\mathbf{w} \cdot \mathbf{x}$ and $\mathbf{v} \cdot \mathbf{x}$ are independent standard Gaussian random variables and apply Stein's lemma (Fact 26) to obtain

$$\begin{aligned} \mathbf{E}_{\mathbf{x} \sim \mathcal{D}_{\mathbf{x}}} [\sigma(\mathbf{w}^* \cdot \mathbf{x}) T_{\rho} \sigma'(\mathbf{w} \cdot \mathbf{x}) \mathbf{v} \cdot \mathbf{x}] &= \mathbf{E}_{\mathbf{x} \sim \mathcal{D}_{\mathbf{x}}} [\sigma(\cos \theta \mathbf{w} \cdot \mathbf{x} + \sin \theta \mathbf{v} \cdot \mathbf{x}) T_{\rho} \sigma'(\mathbf{w} \cdot \mathbf{x}) \mathbf{v} \cdot \mathbf{x}] \\ &= \mathbf{E}_{\mathbf{x} \sim \mathcal{D}_{\mathbf{x}}} [\sigma'(\cos \theta \mathbf{w} \cdot \mathbf{x} + \sin \theta \mathbf{v} \cdot \mathbf{x}) \sin \theta T_{\rho} \sigma'(\mathbf{w} \cdot \mathbf{x})] \\ &= \mathbf{E}_{\mathbf{x} \sim \mathcal{D}_{\mathbf{x}}} [T_{\cos \theta} \sigma'(\mathbf{w} \cdot \mathbf{x}) T_{\rho} \sigma'(\mathbf{w} \cdot \mathbf{x})] \sin \theta = \|T_{\sqrt{\rho \cos \theta}} \sigma'\|_{L_2}^2 \sin \theta, \end{aligned}$$

where in the last equality we used the identity $T_a T_b = T_{ab} = T_{\sqrt{ab}} T_{\sqrt{ab}}$. Therefore, we have that

$$-\mathbf{g}(\mathbf{w}) \cdot \mathbf{w}^* \geq \|T_{\sqrt{\rho \cos \theta}} \sigma'\|_{L_2}^2 \sin^2 \theta - \sqrt{\text{OPT}} \|T_{\rho} \sigma'\|_{L_2} \sin \theta.$$

To argue the last part of [Proposition 43](#), recall (by [Fact 21](#), Part 2(e)) that the function $g(\lambda) := \|T_{\lambda} f\|_{L_2}$ is non-decreasing in $\lambda \in (0, 1)$ for any function $f \in L_2(\mathcal{N})$, therefore $\|T_{\sqrt{\rho \cos \theta}} \sigma'\|_{L_2} \geq \|T_{\rho} \sigma'\|_{L_2}$ if $\cos \theta \geq \rho$. By using the assumption that $\sin \theta \geq 3\sqrt{\text{OPT}}/\|T_{\rho} \sigma'\|_{L_2}$, we obtain that

$$-\mathbf{g}(\mathbf{w}) \cdot \mathbf{w}^* \geq (2/3) \|T_{\sqrt{\rho \cos \theta}} \sigma'\|_{L_2}^2 \sin^2 \theta.$$

This completes the proof of [Proposition 43](#). ■

D.3. Critical Points and Their Connection to the L_2^2 Loss

[Proposition 43](#) provides sufficient conditions ensuring that the vector $-\mathbf{g}(\mathbf{w})$ directs \mathbf{w} towards the direction of \mathbf{w}^* whenever we are in a region around approximate solutions. Specifically, if the parameter ρ is chosen appropriately and the following alignment condition holds: $\sin \theta \|T_{\cos \theta} \sigma'\|_{L_2} \geq 3\sqrt{\text{OPT}}$, then $-\mathbf{g}(\mathbf{w})$ has a nontrivial correlation with \mathbf{w}^* . Otherwise, we can guarantee that the angle between \mathbf{w} and \mathbf{w}^* is already sufficiently small. This implies that the region of convergence of an algorithm that relies on $-\mathbf{g}(\mathbf{w})$ depends on the quantity:

$$\psi_{\sigma}(\theta) := \sin \theta \|T_{\cos \theta} \sigma'\|_{L_2}.$$

Motivated by this observation, we define the *Convergence Region*, which characterizes the region of θ (and, equivalently, the region of \mathbf{w}) for which the algorithm makes progress towards \mathbf{w}^* .

Definition 45 (Critical Point and Convergence Region of σ) *Given $\sigma : \mathbb{R} \rightarrow \mathbb{R}$, $\sigma \in L_2(\mathcal{N})$, and $\theta_0 \in [0, \pi/2]$, we define the error alignment function $\psi_{\sigma} : [0, \pi/2] \rightarrow \mathbb{R}_+$ with respect to σ as follows: $\psi_{\sigma}(\theta) := \sin \theta \|T_{\cos \theta} \sigma'\|_{L_2}$. For any $\epsilon > 0$, we define the Convergence Region $\mathcal{R}_{\sigma, \theta_0}(\epsilon) = \{\theta : \psi_{\sigma}(\theta) \leq \sqrt{\epsilon}\} \cap \{\theta : 0 \leq \theta \leq \theta_0\}$. We say that θ^* is $(\sigma, \theta_0, \epsilon)$ -Critical Point if θ^* is the maximum θ in $\mathcal{R}_{\sigma, \theta_0}(\epsilon)$.*

[Definition 45](#) defines the Convergence Region using an upper bound θ_0 . This upper bound is necessary because $\psi_{\sigma}(\theta)$ is not necessarily monotonic. Specifically, it can be shown that $\psi_{\sigma}(\theta)$ is non-decreasing up to a critical point θ' and then non-increasing (see [Figure 1](#) for illustrative examples). Consequently, the region $\mathcal{R}_{\sigma, \theta_0}(\epsilon)$ may consist of two disjoint intervals. The role of (an appropriately selected) θ_0 is to ensure that this does not happen.

Claim 46 *Let $\sigma \in L_2(\mathcal{N})$. Then there exists a real number $\bar{\theta} \in (0, \pi/2)$, such that for any $\theta \leq \bar{\theta}$, $\psi_{\sigma}(\theta)$ is non-decreasing. If $\|\sigma''\|_{L_2} \leq L'$, then $\bar{\theta} \geq \min(\pi/3, \|T_{1/2} \sigma'\|_{L_2}^2 / (L')^2)$.*

Proof Since $\psi_{\sigma}(\theta) \geq 0$, to show that $\psi_{\sigma}(\theta)$ is non-decreasing is equivalent to show that $\psi_{\sigma}^2(\theta)$ is non-decreasing. Let us calculate the derivative of $\psi_{\sigma}^2(\theta)$:

$$\begin{aligned} (\psi_{\sigma}^2(\theta))' &= \frac{d}{d\theta} (\sin^2 \theta \mathbf{E}_{z \sim \mathcal{N}}[(T_{\cos \theta} \sigma')^2]) \\ &= 2 \sin \theta \cos \theta \|T_{\cos \theta} \sigma'\|_{L_2}^2 + 2 \sin^2 \theta \mathbf{E}_{z \sim \mathcal{N}} \left[T_{\cos \theta} \sigma' \frac{d}{d\theta} T_{\cos \theta} \sigma' \right] \end{aligned}$$

Using [Fact 23](#), since $\frac{d}{d\rho} T_\rho f = (LT_\rho f)/\rho$, we further have

$$\begin{aligned} (\psi_\sigma^2(\theta))' &= 2 \sin \theta \cos \theta \|T_{\cos \theta} \sigma'\|_{L_2}^2 + 2 \sin^3 \theta \mathbf{E}_{z \sim \mathcal{N}} \left[T_{\cos \theta} \sigma' \frac{1}{\cos \theta} LT_{\cos \theta} \sigma' (-\sin \theta) \right] \\ &= 2 \sin \theta \cos \theta \|T_{\cos \theta} \sigma'\|_{L_2}^2 - 2 \frac{\sin^3 \theta}{\cos \theta} \mathbf{E}_{z \sim \mathcal{N}} \left[\frac{d}{dz} T_{\cos \theta} \sigma' \frac{d}{dz} T_{\cos \theta} \sigma' \right] \\ &= 2 \sin \theta \cos \theta \left(\|T_{\cos \theta} \sigma'\|_{L_2}^2 - \tan^2 \theta \| (T_{\cos \theta} \sigma')' \|_{L_2}^2 \right), \end{aligned}$$

where in the second equality we used [Fact 23](#) that $\mathbf{E}_{z \sim \mathcal{N}} [f(z) LT_\rho g(z)] = \mathbf{E}_{z \sim \mathcal{N}} [f'(z) (T_\rho g(z))']$. Therefore, we only need to prove that $h(\theta) := \|T_{\cos \theta} \sigma'\|_{L_2}^2 - \tan^2 \theta \| (T_{\cos \theta} \sigma')' \|_{L_2}^2 \geq 0$ in a region $(0, \bar{\theta})$. Note that $h(0) = \|\sigma'\|_{L_2}^2 > 0$. Furthermore, since $\|T_{\cos \theta} \sigma'\|_{L_2}^2$ and $\| (T_{\cos \theta} \sigma')' \|_{L_2}^2$ are continuous functions of θ (as we can see by Hermite expansion), we know that there exists a threshold $\bar{\theta}$ such that for any $\theta \leq \bar{\theta}$, it holds $\psi'_\sigma(z) \geq 0$. Furthermore, if σ'' is in $L_2(\mathcal{N})$ and $\|\sigma''\|_{L_2} \leq L'$, then since $(T_\rho f(z))' = \rho T_\rho f'(z)$ and T_ρ is a non-expansive operator ([Fact 21](#)), we have

$$h(\theta) = \|T_{\cos \theta} \sigma'\|_{L_2}^2 - \sin^2 \theta \|T_{\cos \theta} \sigma''\|_{L_2}^2 \geq \|T_{\cos \theta} \sigma'\|_{L_2}^2 - \sin^2 \theta (L')^2.$$

Assuming $\theta \leq \pi/3$, we have $h(\theta) \geq 0$ as long as $\theta \leq \bar{\theta} = \min(\pi/3, \|T_{1/2} \sigma'\|_{L_2}^2 / (L')^2)$. \blacksquare

The significance of the Critical Point and the Convergence Region comes from the following proposition, which bounds the L_2^2 error for points within the Convergence Region.

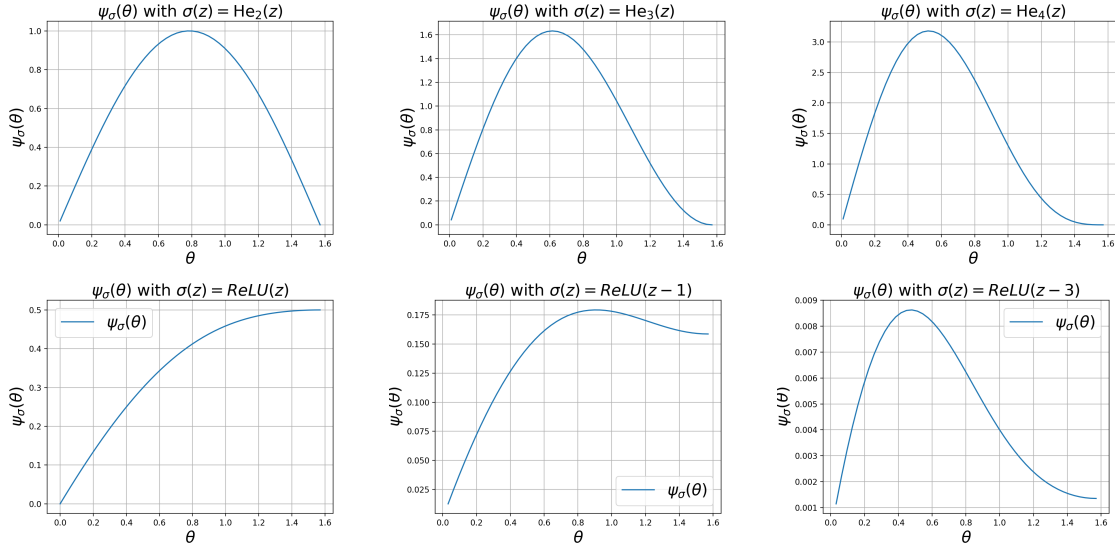


Figure 1: (Up): The plot of $\psi_\sigma(\theta)$, where $\sigma(z) = \text{He}_i(z)$, $i = 2, 3, 4$. (Down): The plot of $\psi_\sigma(\theta)$, where $\sigma(z) = \text{ReLU}(z - t)$, $t = 0, 1, 3$.

Proposition 47 (Critical Points and L_2^2 Error) Given $\sigma : \mathbb{R} \rightarrow \mathbb{R}$, $\sigma \in L_2(\mathcal{N})$, and a distribution \mathcal{D} of labeled examples $(\mathbf{x}, y) \in \mathbb{R}^d \times \mathbb{R}$ such that $\mathcal{D}_\mathbf{x} = \mathcal{N}(\mathbf{0}, \mathbf{I})$, let $\mathbf{w}^* \in \mathbb{R}^d$ be such that

$\mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}}[(\sigma(\mathbf{w}^* \cdot \mathbf{x}) - y)^2] = \text{OPT}$. Then, for any unit vector $\mathbf{w} \in \mathbb{R}^d$ with $\theta = \theta(\mathbf{w}, \mathbf{w}^*)$ such that $\theta \leq \theta^*$, where θ^* is the $(\sigma, \theta_0, C\text{OPT})$ -Critical Point for some θ_0 and $C > 1$ an absolute constant, it holds that $\mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}}[(\sigma(\mathbf{w} \cdot \mathbf{x}) - y)^2] \leq O(\text{OPT}) + 4\|P_{>(1/\theta^*)^2\sigma}\|_{L_2}^2$.

[Proposition 47](#) provides a sufficient condition for proving that our algorithm converges to a region with the target approximation error. In particular, if we argue that the iterates of the algorithm we use land in the Critical Region, then [Proposition 47](#) gives us the target L_2^2 error.

To prove [Proposition 47](#), we first prove the following technical lemma, which decomposes the error into $O(\text{OPT})$ and error terms that depend on the properties of the activation σ .

Lemma 48 (Error Decomposition) *Given $\sigma : \mathbb{R} \rightarrow \mathbb{R}$, $\sigma \in L_2(\mathcal{N})$, and a distribution \mathcal{D} of labeled examples $(\mathbf{x}, y) \in \mathbb{R}^d \times \mathbb{R}$ such that $\mathcal{D}_{\mathbf{x}} = \mathcal{N}(\mathbf{0}, \mathbf{I})$, let $\mathbf{w}^* \in \mathbb{R}^d$ be such that $\mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}}[(\sigma(\mathbf{w}^* \cdot \mathbf{x}) - y)^2] = \text{OPT}$. Then, for any unit vector $\mathbf{w} \in \mathbb{R}^d$ with $\theta = \theta(\mathbf{w}, \mathbf{w}^*)$ and any $k \in \mathbb{Z}_+$, it holds that*

$$\mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}}[(\sigma(\mathbf{w} \cdot \mathbf{x}) - y)^2] \leq 2\text{OPT} + 4\theta^2\|P_k\sigma'\|_{L_2}^2 + 4\|P_{>k}\sigma\|_{L_2}^2. \quad (12)$$

Furthermore, if $k \geq 2$, then for any $c \in [1, (k/2)^{1/4}]$,

$$\mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}}[(\sigma(\mathbf{w} \cdot \mathbf{x}) - y)^2] \leq 2\text{OPT} + 8e^{c^2}\theta^2\|T_{\sqrt{1-c^2/k}}\sigma'\|_{L_2}^2 + 4\|P_{>k}\sigma\|_{L_2}^2. \quad (13)$$

Finally, if $k = 0, 1$, then for any $\rho \in (0, 1)$ it holds

$$\mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}}[(\sigma(\mathbf{w} \cdot \mathbf{x}) - y)^2] \leq 2\text{OPT} + \theta^2\|T_\rho\sigma'\|_{L_2}^2 + 4\|P_{>k}\sigma\|_{L_2}^2.$$

Proof First, we decompose the error into the minimum error (the one achieved by \mathbf{w}^*) and the alignment error (the one from the misalignment of \mathbf{w} and \mathbf{w}^*). By Young's inequality, we have that

$$\begin{aligned} \mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}}[(\sigma(\mathbf{w} \cdot \mathbf{x}) - y)^2] &\leq 2 \mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}}[(\sigma(\mathbf{w}^* \cdot \mathbf{x}) - y)^2] + 2 \mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}}[(\sigma(\mathbf{w}^* \cdot \mathbf{x}) - \sigma(\mathbf{w} \cdot \mathbf{x}))^2] \\ &= 2\text{OPT} + 2 \underbrace{\mathbf{E}_{\mathbf{x} \sim \mathcal{D}_{\mathbf{x}}}[(\sigma(\mathbf{w}^* \cdot \mathbf{x}) - \sigma(\mathbf{w} \cdot \mathbf{x}))^2]}_{\text{Alignment Error}}. \end{aligned} \quad (14)$$

In the rest of the proof, we bound above the alignment error.

Claim 49 (Angle and Alignment Error) *Let $\mathbf{w}, \mathbf{w}^* \in \mathbb{R}^d$ be unit vectors and let $\theta := \theta(\mathbf{w}, \mathbf{w}^*)$. Then, for any $k \in \mathbb{Z}_+$,*

$$\mathbf{E}_{\mathbf{x} \sim \mathcal{D}_{\mathbf{x}}}[(\sigma(\mathbf{w}^* \cdot \mathbf{x}) - \sigma(\mathbf{w} \cdot \mathbf{x}))^2] \leq 2\theta^2\|P_k\sigma'\|_{L_2}^2 + 2 \mathbf{E}_{t \sim \mathcal{N}}[(P_{>k}\sigma(t))^2].$$

Proof By expanding the square, since \mathbf{w}, \mathbf{w}^* are fixed vectors independent of \mathbf{x} , we have that

$$\begin{aligned} \mathbf{E}_{\mathbf{x} \sim \mathcal{D}_{\mathbf{x}}}[(\sigma(\mathbf{w}^* \cdot \mathbf{x}) - \sigma(\mathbf{w} \cdot \mathbf{x}))^2] &= 2 \left(\mathbf{E}_{t \sim \mathcal{N}}[(\sigma(t))^2] - \mathbf{E}_{\mathbf{x} \sim \mathcal{D}_{\mathbf{x}}}[\sigma(\mathbf{w}^* \cdot \mathbf{x})\sigma(\mathbf{w} \cdot \mathbf{x})] \right) \\ &= 2 \left(\mathbf{E}_{t \sim \mathcal{N}}[(\sigma(t))^2] - \mathbf{E}_{t \sim \mathcal{N}}[\sigma(t)T_{\cos\theta}\sigma(t)] \right), \end{aligned} \quad (15)$$

where in the second inequality we used that $t = \mathbf{w}^* \cdot \mathbf{x} \sim \mathcal{N}(0, 1)$, decomposed \mathbf{w} into components parallel and orthogonal to \mathbf{w}^* , and applied the definition of $T_{\cos \theta}$.

Let $\sigma(t) \doteq \sum_{i=0}^{\infty} a_i \text{He}_i(t)$ with $a_i = \mathbf{E}_{z \sim \mathcal{N}}[\sigma(z) \text{He}_i(z)]$ be the Hermite expansion of σ . Using the property that $T_\rho \text{He}_i(t) = \rho^i \text{He}_i(t)$, for any integer $k \geq 1$ (see [Fact 21](#), Part 3), we have that

$$\begin{aligned} \mathbf{E}_{t \sim \mathcal{N}}[(\sigma(t))^2] - \mathbf{E}_{t \sim \mathcal{N}}[\sigma(t) T_{\cos \theta} \sigma(t)] &= \sum_{i=1}^{+\infty} a_i^2 (1 - \cos^i \theta) \\ &= a_1^2 (1 - \cos \theta) + \sum_{i=2}^{+\infty} a_i^2 (1 - (1 - \sin^2 \theta)^{i/2}) \\ &\leq (1/2) a_1^2 \theta^2 + \theta^2 \sum_{i=2}^k (i/2) a_i^2 + \sum_{i=k+1}^{+\infty} a_i^2, \end{aligned} \quad (16)$$

where for the first term we used that $(1 - \cos \theta) = 2 \sin^2(\theta/2) \leq \theta^2/2$, and for the terms from $i = 2, \dots, k$, we used the Bernoulli inequality and that $\sin \theta \leq \theta$ for $\theta \geq 0$.

Furthermore, note that if $\sigma(t) \doteq \sum_{i=0}^{\infty} a_i \text{He}_i(t)$ is the Hermite expansion of σ , then $\sigma'(t) \doteq \sum_{i=1}^{\infty} a_i \sqrt{i} \text{He}_{i-1}(t)$ is the Hermite expansion of σ' . Therefore, we have that

$$\mathbf{E}_{t \sim \mathcal{N}}[(\sigma(t))^2] - \mathbf{E}_{t \sim \mathcal{N}}[\sigma(t) T_{\cos \theta} \sigma(t)] \leq \theta^2 \|P_k \sigma'\|_{L_2}^2 + \|P_{>k} \sigma\|_{L_2}^2,$$

which, combined with [Equation \(15\)](#), completes the proof. \blacksquare

To complete the proof, it remains to bound $\|P_k \sigma'\|_{L_2}^2$ above, which is done in the following claim.

Claim 50 *When $k \geq 2$, for any $c \in [1, (k/2)^{1/4}]$,*

$$\|P_k \sigma'(t)\|_{L_2}^2 \leq 2e^{c^2} \|T_{\sqrt{1-c^2/k}} \sigma'(t)\|_{L_2}^2.$$

Proof Note that the $T_{\sqrt{1-c^2/k}} \sigma'(t) \doteq \sum_{i=1}^{\infty} (1 - c^2/k)^{(i-1)/2} \sqrt{i} a_i \text{He}_{i-1}(t)$. Therefore, we have that

$$\|T_{\sqrt{1-c^2/k}} \sigma'\|_{L_2}^2 = \sum_{i=1}^{+\infty} i (1 - c^2/k)^{i-1} a_i^2 \geq \sum_{i=1}^k i (1 - c^2/k)^i a_i^2.$$

By assumption, we have $c^4/k \leq 1/2$. Therefore, for any $i \leq k$, using the inequality $(1 - c^2/k)^i \geq (1 - c^2/k)^k \geq e^{-c^2} (1 - c^4/k) \geq e^{-c^2}/2$, we have that

$$\|T_{\sqrt{1-c^2/k}} \sigma'\|_{L_2}^2 \geq \sum_{i=1}^k i e^{-c^2} (1/2) a_i^2 = e^{-c^2} (1/2) \|P_k \sigma'\|_{L_2}^2.$$

This completes the proof. \blacksquare

Combining [Claim 50](#) with [Claim 49](#) and bring back the bounds to [Equation \(14\)](#), we have that when $k \geq 2$ and for any $c \in [1, (k/2)^{1/4}]$, it holds

$$\mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}}[(\sigma(\mathbf{w} \cdot \mathbf{x}) - y)^2] \leq 2\text{OPT} + 8e^{c^2} \theta^2 \|T_{\sqrt{1-c^2/k}} \sigma'\|_{L_2}^2 + 4\|P_{>k} \sigma\|_{L_2}^2.$$

When $k = 0$, then according to Equation (16), for any $\rho \in (0, 1)$, we have:

$$\mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}} [(\sigma(\mathbf{w}^* \cdot \mathbf{x}) - \sigma(\mathbf{w} \cdot \mathbf{x}))^2] \leq 2\|P_{>0}\sigma\|_{L_2}^2 \leq \frac{1}{2}\theta^2\|T_\rho\sigma'\|_{L_2}^2 + 2\|P_{>0}\sigma\|_{L_2}^2,$$

since $\frac{1}{2}\theta^2\|T_\rho\sigma'\|_{L_2}^2 \geq 0$ for any $\rho \in (0, 1)$. When $k = 1$, similarly according to Equation (16), we have

$$\mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}} [(\sigma(\mathbf{w}^* \cdot \mathbf{x}) - \sigma(\mathbf{w} \cdot \mathbf{x}))^2] \leq \frac{1}{2}\theta^2 a_1^2 + 2\|P_{>1}\sigma\|_{L_2}^2.$$

Observe that for any $\rho \in (0, 1)$, we have $T_\rho\sigma'(z) \doteq \sum_{i \geq 1} \rho^{i-1} \sqrt{i} a_i \text{He}_{i-1}(z)$, therefore $\|T_\rho\sigma'\|_{L_2}^2 = \sum_{i \geq 1} \rho^{2(i-1)} i a_i^2 \geq a_1^2$. Thus, we further obtain

$$\mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}} [(\sigma(\mathbf{w}^* \cdot \mathbf{x}) - \sigma(\mathbf{w} \cdot \mathbf{x}))^2] \leq \frac{1}{2}\theta^2\|T_\rho\sigma'\|_{L_2}^2 + 2\|P_{>1}\sigma\|_{L_2}^2.$$

Plugging the above bounds for the cases $k = 0, 1$ back into Equation (14) completes the proof. \blacksquare

Having proved Lemma 48, it is not hard to see that Proposition 47 follows as a direct corollary:

Proof of Proposition 47 Since θ^* is the $(\sigma, \theta_0, \text{COPT})$ -Critical point, we have $(\theta^*)^2\|T_{\cos(c\theta^*)}\sigma'\|_{L_2}^2 \leq \text{COPT}$. We apply Lemma 48 with $k = \lfloor 1/(\theta^*)^2 \rfloor$. Consider first $\theta^* \leq 1/\sqrt{2}$, which implies that $k \geq 2$. Then for any $c \in [1, (k/2)^{1/4}]$, since $c\theta^* \geq \sin(c\theta^*)$ and $c \geq 1$, it holds

$$\sqrt{1 - \frac{c^2}{k}} \leq \sqrt{1 - (c\theta^*)^2} \leq \sqrt{1 - \sin^2(c\theta^*)} = \cos(c\theta^*) \leq \cos\theta^*.$$

Thus as $\|T_\rho\sigma'\|_{L_2}$ is non-decreasing with respect to ρ (Fact 21), we further have $\|T_{\sqrt{1-c^2/k}}\sigma'\|_{L_2}^2 \leq \|T_{\cos(\theta^*)}\sigma'\|_{L_2}^2$. Therefore, applying Lemma 48, for any $\theta \leq \theta^*$, we obtain

$$\begin{aligned} \mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}} [(\sigma(\mathbf{w} \cdot \mathbf{x}) - y)^2] &\leq 2\text{OPT} + 8e^2\theta^2\|T_{\sqrt{1-c^2/k}}\sigma'\|_{L_2}^2 + 4\|P_{>k}\sigma\|_{L_2}^2 \\ &\leq 2\text{OPT} + 8e^2(\theta^*)^2\|T_{\cos(\theta^*)}\sigma'\|_{L_2}^2 + 4\|P_{>(1/\theta^*)^2}\sigma\|_{L_2}^2 \\ &\leq C'\text{OPT} + 4\|P_{>(1/\theta^*)^2}\sigma\|_{L_2}^2, \end{aligned}$$

where C' is an absolute constant. In particular, when $c = 1$, we have $C' = 2 + 8eC$. When $\theta^* > 1/\sqrt{2}$, then $k = 0, 1$. Choose $\rho = \cos(\theta^*) \in (0, 1)$ in Lemma 48, we have

$$\mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}} [(\sigma(\mathbf{w} \cdot \mathbf{x}) - y)^2] \leq 2\text{OPT} + \theta^2\|T_{\cos(\theta^*)}\sigma'\|_{L_2}^2 + 4\|P_{>k}\sigma\|_{L_2}^2 \leq (2 + C)\text{OPT} + 4\|P_{>(1/\theta^*)^2}\sigma\|_{L_2}^2$$

In summary, for all $\theta^* \in (0, \pi/2)$, we have $\mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}} [(\sigma(\mathbf{w} \cdot \mathbf{x}) - y)^2] \leq O(\text{OPT}) + 4\|P_{>(1/\theta^*)^2}\sigma\|_{L_2}^2$. \blacksquare

Appendix E. Full Version of Section 3

In this section, we present our main algorithm (Algorithm 2) for learning GLMs under Gaussian marginals with adversarial corruptions, as stated in Problem 1. Algorithm 2 uses the main structural result of Section 2 (Proposition 5) to update its iterates $\mathbf{w}^{(t)}$. In particular, for $\theta_t = \theta(\mathbf{w}^{(t)}, \mathbf{w}^*)$, we show that after one gradient descent-style update, the angle θ_{t+1} shrinks by a factor $1 - c$, i.e., $\theta_{t+1} \leq (1 - c)\theta_t$, where $0 < c < 1$ is an absolute constant. A crucial feature of Algorithm 2 is that in each iteration it carefully chooses a new value of ρ_t . This *variable* update of ρ_t ensures the ‘signal’ of the gradient is present until $\mathbf{w}^{(t)}$ reaches a small region centered at \mathbf{w}^* . Within this region, the agnostic noise corrupts the signal of the augmented gradient and convergence to \mathbf{w}^* is no longer guaranteed. However, the region that $\mathbf{w}^{(t)}$ reaches is in fact the Convergence Region $\mathcal{R}_{\sigma, \theta_0}(O(\text{OPT}))$, within which all points are solutions with the target approximation error. We will show in Section 4 that for any *monotone* (B, L) -Regular activations, any point $\hat{\mathbf{w}}$ in $\mathcal{R}_{\sigma, \theta_0}(O(\text{OPT}))$ is a solution with error $C\text{OPT} + \epsilon$, provided that the initialized angle $\theta_0 = \theta(\mathbf{w}^{(0)}, \mathbf{w}^*)$ is suitably small.

We now present our main algorithm.

Algorithm 4 SGD – VA: SGD with Variable Augmentation

- 1: **Input:** Parameters ϵ, T ; Sample access to \mathcal{D} .
 - 2: $[\mathbf{w}^{(0)}, \bar{\theta}] = \text{Initialization}[\sigma]$ (Appendix F.3); set $\rho_0 = \cos \bar{\theta}$.
 - 3: **for** $t = 0, \dots, T$ **do**
 - 4: Draw n samples $\{(\tilde{\mathbf{x}}^{(i)}, y^{(i)})\}_{i=1}^n$ from \mathcal{D}_{ρ_t} using Algorithm 3 and construct the empirical distribution $\hat{\mathcal{D}}_{\rho_t}$.
 - 5: $\hat{\mathbf{g}}(\mathbf{w}^{(t)}) = -(1/\rho_t) \mathbf{E}_{(\tilde{\mathbf{x}}, y) \sim \hat{\mathcal{D}}_{\rho_t}} [y \sigma'(\mathbf{w}^{(t)} \cdot \tilde{\mathbf{x}})(\tilde{\mathbf{x}})^\perp \mathbf{w}^{(t)}]$
 - 6: $\eta_t = \sqrt{(1 - \rho_t)/2} / (4 \|\hat{\mathbf{g}}(\mathbf{w}^{(t)})\|_2)$.
 - 7: $\mathbf{w}^{(t+1)} = (\mathbf{w}^{(t)} - \eta_t \hat{\mathbf{g}}(\mathbf{w}^{(t)})) / \|\mathbf{w}^{(t)} - \eta_t \hat{\mathbf{g}}(\mathbf{w}^{(t)})\|_2$.
 - 8: $\rho_{t+1} = 1 - (1 - 1/256)^2 (1 - \rho_t)$
 - 9: $\hat{\mathbf{w}} = \text{Test}[\mathbf{w}^{(0)}, \mathbf{w}^{(1)}, \dots, \mathbf{w}^{(T)}]$. (Algorithm 5)
 - 10: **Return:** $\hat{\mathbf{w}}$
-

Our main result concerning the general setting of (B, L) -Regular activations is summarized in the following theorem.

Theorem 51 *Let $\epsilon > 0$. Let σ be a (B, L) -Regular activation. Algorithm 4 given initialization $\mathbf{w}^{(0)}$ with $\theta(\mathbf{w}^{(0)}, \mathbf{w}^*) \leq \bar{\theta}$, runs at most $T = O(\log(L/\epsilon))$ iterations, draws $\tilde{\Theta}(dB^2 \log(L/\epsilon)/\epsilon + B^4 \log(L/\epsilon)/\epsilon^2)$ samples, and returns a vector $\hat{\mathbf{w}}$ such that with probability at least $2/3$, $\hat{\mathbf{w}}$ lies in the target region $\mathcal{R}_{\sigma, \theta_0}(O(\text{OPT}))$. Moreover, $\mathcal{L}(\hat{\mathbf{w}}) = O(\text{OPT}) + \epsilon + 4\|\mathbf{P}_{>1/(\theta^*)^2} \sigma\|_{L_2}^2$.*

Let us provide a roadmap of the proof. Suppose for simplicity that we have taken enough samples so that we have access to the population gradient $\mathbf{g}(\mathbf{w}^{(t)})$. Furthermore, for the convenience of notation, let us use $\zeta(\rho)$ to denote the value $\sqrt{\text{OPT}}/\|\mathbf{T}_\rho \sigma'\|_{L_2}$. Our main tool is the structural result in Proposition 43, which shows that when

$$\text{conditions for fast convergence: } \sin \theta_t \geq 3\zeta(\rho_t), \zeta(\rho_t) := \sqrt{\text{OPT}}/\|\mathbf{T}_{\rho_t} \sigma'\|_{L_2}, \rho_t \leq \cos \theta_t \quad (17)$$

are satisfied, the gradient $\mathbf{g}(\mathbf{w}^{(t)})$ correlates strongly with the target vector \mathbf{w}^* , hence providing sufficient information about the direction of \mathbf{w}^* . This structural result enables us to decrease the angle θ_{t+1} efficiently so that $\theta_{t+1} \leq (1 - c)\theta_t$. However, two critical problems arise:

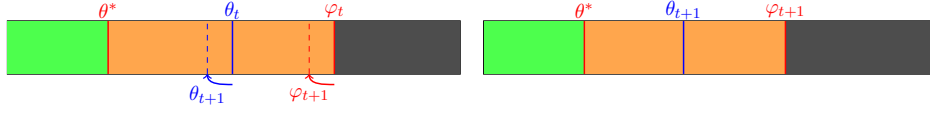


Figure 2: Successfull Update

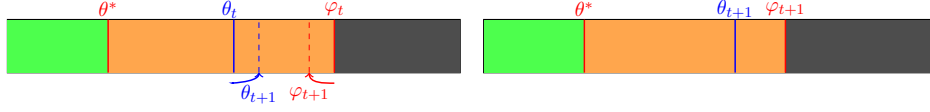


Figure 3: Wrong Update

Figure 4: Illustration of θ^* , θ_t , and φ_t at different stages. The green region represents the Convergence Region, while the black region denotes the area that θ_t will never enter. Notably, the black region consistently expands, irrespective of whether the update is successful. The parameter θ_t is always guaranteed to never reach the black region.

1. If $\sin \theta_t \lesssim \zeta(\rho_t)$, then since the conditions in Equation (17) are not valid, we cannot guarantee that the angle θ_t contracts. On the other hand, since $\|\mathbf{T}_{\rho_t} \sigma'\|_{L_2} \leq \|\mathbf{T}_{\cos \theta_t} \sigma'\|_{L_2}$, it is not necessarily the case that $\sin \theta_t \lesssim \zeta(\cos \theta_t)$, therefore we also cannot assert that $\mathbf{w}^{(t)}$ has reached the target region $\mathcal{R}_{\sigma, \theta_0}(C^2\text{OPT})$.
2. Suppose that the conditions in Equation (17) are satisfied, and we have contraction of angle $\theta_{t+1} \leq (1 - c)\theta_t$. Assume that $\mathbf{w}^{(t+1)}$ is still far away from \mathbf{w}^* and $\theta_{t+1} \gtrsim \zeta(\cos \theta_{t+1})$, meaning that we still need to further decrease the angle between $\mathbf{w}^{(t+1)}$ and \mathbf{w}^* . However, it is possible that $\zeta(\cos \theta_{t+1}) \lesssim \theta_{t+1} \lesssim \zeta(\rho_t)$, because $\|\mathbf{T}_{\cos \theta_{t+1}} \sigma'\|_{L_2} \geq \|\mathbf{T}_{\rho_t} \sigma'\|_{L_2}$, as we have $\rho_t \leq \cos \theta_t \leq \cos \theta_{t+1}$ and $\|\mathbf{T}_{\rho} \sigma'\|_{L_2}$ is an increasing function of ρ (see Fact 21). This implies that the fast convergence conditions (Equation (17)) might be invalid if we continue using ρ_t . Thus, we need to carefully increase ρ_t to ρ_{t+1} so that $\sin \theta_{t+1} \gtrsim \zeta(\rho_{t+1})$, while maintaining the other condition $\rho_{t+1} \leq \cos \theta_{t+1}$. This seems impossible since we do not have any lower bound on θ_{t+1} .

To overcome these hurdles, let us study the event $\mathcal{E}_t := \{|\cos \theta_t - \rho_t| \leq \sin^2 \theta_t, \sin \theta_t \leq C\zeta(\rho_t)\}$. We first observe that when \mathcal{E}_t is satisfied, then, since $\sin \theta_t \leq C\zeta(\rho_t)$ the algorithm may not be converging anymore, as discussed in Case 1 above. However, since \mathcal{E}_t also satisfies $|\cos \theta_t - \rho_t| \leq \sin^2 \theta_t$, one can show that $\zeta(\rho_t) \approx \zeta(\cos \theta_t)$, therefore, we have that $\sin \theta_t \leq C\zeta(\cos \theta_t)$. In other words, we can certify that $\mathbf{w}^{(t)}$ lies in the target region $\mathcal{R}_{\sigma, \theta_0}(C^2\text{OPT})$. This solves the first problem above.

Now suppose \mathcal{E}_t is not satisfied. We use induction to show that updating ρ_t by Line 8, it always holds $\rho_{t+1} \leq \cos \theta_{t+1}$. To see this, suppose $\rho_t \leq \cos \theta_t$ holds at iteration t and \mathcal{E}_t is not satisfied. Then if we have $\sin \theta_t \geq C\zeta(\rho_t)$, the conditions in Equation (17) are satisfied hence we have control of θ_{t+1} . We can then show that $\rho_{t+1} \leq \cos \theta_{t+1}$ and $\sin \theta_{t+1} \gtrsim \zeta(\rho_{t+1})$ with ρ_{t+1} defined by Line 8. On the other hand, if $|\cos \theta_t - \rho_t| \geq \sin^2 \theta_t$, then since $\rho_t \leq \cos \theta_t$ we know that ρ_t is much smaller compared to $\cos \theta_t$. Thus, since we are taking small gradient steps and making very small increments to ρ_t , we have that $\rho_{t+1} \leq \cos \theta_{t+1}$ and $\sin \theta_{t+1} \gtrsim \zeta(\rho_{t+1})$ continue to hold. This resolves the second problem. See Figure 4 for a visual illustration of the mechanism of Algorithm 4.

We can now proceed to the proof of [Theorem 51](#).

Proof of Theorem 51 In the proof, we denote the angle between $\mathbf{w}^{(t)}$ and \mathbf{w}^* by $\theta_t = \theta(\mathbf{w}^{(t)}, \mathbf{w}^*)$ and denote $\widehat{\mathbf{g}}(\mathbf{w}^{(t)})$ by $\widehat{\mathbf{g}}^{(t)}$. After initialization, it holds $\theta_0 = \theta(\mathbf{w}^{(0)}, \mathbf{w}^*) \leq \bar{\theta}$. Furthermore, the algorithm uses the following parameters: $\varphi_t = (1/\sqrt{2})(1 - \beta)^t \sin \bar{\theta}$, $\beta = 1/256$; $\rho_t := 1 - 2\varphi_t^2$, $\rho_0 = \cos \bar{\theta}$; $\eta_t = \varphi_t/(4\|\widehat{\mathbf{g}}^{(t)}\|_2)$. Note that if $\epsilon \geq C\text{OPT}$, then we can run the algorithm with $\epsilon' = \epsilon/(2C)$ and assume that we have more noise of order $\text{OPT}' = 2\epsilon'$. In this case, the final error bound will be $C\text{OPT}' \leq \epsilon/2 \leq \text{OPT} + \epsilon$. So, without loss of generality, we can assume that $\epsilon \leq \text{OPT}$. The goal of the algorithm is to converge to a vector in the region $\mathcal{R}_{\sigma, \theta_0}(C\text{OPT})$ where $C > 0$ is an absolute constant. For this reason, we consider the following event

$$\mathcal{E}_t := \left\{ |\cos \theta_t - \rho_t| \leq \sin^2 \theta_t, \sin \theta_t \leq \frac{C\sqrt{\text{OPT}}}{\|\mathbf{T}_{\rho_t} \sigma'\|_{L_2}} \right\},$$

where $C > 0$ is an absolute constant.

We argue that if \mathcal{E}_t is satisfied at some iteration t , then the algorithm converges to a vector that lies in the $\mathcal{R}_{\sigma, \theta_0}(C\text{OPT})$ region. We consider two cases, the first case is if $\rho_t \geq \cos \theta_t$ and the second one if $\rho_t \leq \cos \theta_t$. Assume first that $1 > \rho_t \geq \cos \theta_t$. Then, since $\|\mathbf{T}_{\rho} \sigma'\|_{L_2}$ is an increasing function with respect to the variable ρ (see [Fact 21](#)), and \mathcal{E}_t implies $\sin \theta_t \leq C\sqrt{\text{OPT}}/\|\mathbf{T}_{\rho_t} \sigma'\|_{L_2}$, we also have that $\sin \theta_t \leq C\sqrt{\text{OPT}}/\|\mathbf{T}_{\cos \theta_t} \sigma'\|_{L_2}$ and therefore that means that $\mathbf{w}^{(t)}$ is inside the region $\mathcal{R}_{\sigma, \theta_0}(C^2\text{OPT})$. Next, we consider the case where $\rho_t \leq \cos \theta_t$. Since \mathcal{E}_t implies that $|\cos \theta_t - \rho_t| \leq \sin^2 \theta_t$, we further have

$$\rho_t \geq \cos \theta_t - \sin^2 \theta_t \geq \cos^2 \theta_t - \sin^2 \theta_t = \cos(2\theta_t).$$

Therefore, we have $\sin \theta_t \leq C\sqrt{\text{OPT}}/\|\mathbf{T}_{\rho_t} \sigma'\|_{L_2} \leq C\sqrt{\text{OPT}}/\|\mathbf{T}_{\cos(2\theta_t)} \sigma'\|_{L_2}$, i.e., $\psi_{\sigma}(2\theta_t) \leq 2C\sqrt{\text{OPT}}$. Let θ^* be the $(\sigma, \theta_0, 4C^2\text{OPT})$ -Critical Point, we thus have $2\theta_t \leq \theta^*$ and $\mathbf{w}^{(t)}$ is inside the region $\mathcal{R}_{\sigma, \theta_0}(4C^2\text{OPT})$. Now since $\theta_t \leq \theta^*$, applying [Proposition 47](#) yields

$$\mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}} [(\sigma(\mathbf{w}^{(t)} \cdot \mathbf{x}) - y)^2] \leq O(\text{OPT}) + 4\|\mathbf{P}_{>1/(\theta^*)^2} \sigma\|_{L_2}^2.$$

indicating that $\mathbf{w}^{(t)}$ is a solution that achieves the target error.

We proceed to show that the algorithm is guaranteed to generate a solution $\mathbf{w}^{(t^*)}$ that satisfies the event \mathcal{E}_{t^*} at some iteration $t^* \leq T = O(\log(\|\sigma'\|_{L_2}^2/\text{OPT}))$. Our strategy is to prove that in every iteration $t \leq t^*$, it holds that $\rho_t \leq \cos \theta_t$, due to the careful design of the algorithm. Furthermore, we guarantee that ρ_t can grow geometrically; therefore, we obtain an exponentially growing lower bound on $\cos \theta_t$, which implies that $\sin \theta_t$ shrinks at a linear rate and hence the event \mathcal{E}_t will eventually be satisfied at some iteration t^* .

Claim 52 *Let t' be the maximum $t \in [0, T]$ such that for all $t = 0, \dots, t'$, \mathcal{E}_t is not satisfied. Then, for all $t \leq t'$, it holds that $\rho_t \leq \cos \theta_t$.*

Proof of Claim 52 We use induction to show the claim that $\rho_t \leq \cos \theta_t$ for all the iterations $t = 0, \dots, t'$ where the event \mathcal{E}_t is not satisfied.

Base Case $t = 0$. Recall that:

$$\varphi_t = (1/\sqrt{2})(1 - \beta)^t \sin \bar{\theta}, \beta = 1/256; \rho_t := 1 - 2\varphi_t^2, \rho_0 = \cos \bar{\theta}; \eta_t = \varphi_t/(4\|\widehat{\mathbf{g}}^{(t)}\|_2).$$

Therefore, since $\theta_0 \leq \bar{\theta}$, $\rho_0 = \cos(\bar{\theta}) \leq \cos \theta_0$ is satisfied in the base case.

Induction Step. For the induction step, suppose that \mathcal{E}_t is not satisfied, in other words, we have either $|\cos \theta_t - \rho_t| \geq \sin^2 \theta_t$ or $\sin \theta_t \geq C\sqrt{\text{OPT}}/\|\mathbf{T}_{\rho_t}\sigma'\|_{L_2}$. Assume that $\rho_t \leq \cos \theta_t$ for the iterations $0, \dots, t$. We argue that $\rho_{t+1} \leq \cos \theta_{t+1}$ continues to hold after one iteration.

Case I. Consider first the case where $\sin \theta_t \geq C\sqrt{\text{OPT}}/\|\mathbf{T}_{\rho_t}\sigma'\|_{L_2}$. We study the distance between $\mathbf{w}^{(t)}$ and \mathbf{w}^* after one iteration from t to $t+1$. Since $\hat{\mathbf{g}}^{(t)}$ is orthogonal to $\mathbf{w}^{(t)}$, it must be $\|\mathbf{w}^{(t)} - \eta_t \hat{\mathbf{g}}^{(t)}\|_2 \geq 1$, therefore, $\mathbf{w}^{(t+1)} = \text{proj}_{\mathbb{B}}(\mathbf{w}^{(t)} - \eta_t \hat{\mathbf{g}}^{(t)})$. By the non-expansiveness of the projection operator, we have

$$\begin{aligned} \|\mathbf{w}^{(t+1)} - \mathbf{w}^*\|_2^2 &= \|\text{proj}_{\mathbb{B}}(\mathbf{w}^{(t)} - \eta_t \hat{\mathbf{g}}^{(t)}) - \mathbf{w}^*\|_2^2 \leq \|\mathbf{w}^{(t)} - \eta_t \hat{\mathbf{g}}^{(t)} - \mathbf{w}^*\|_2^2 \\ &= \|\mathbf{w}^{(t)} - \mathbf{w}^*\|_2^2 + \eta_t^2 \|\hat{\mathbf{g}}^{(t)}\|_2^2 - 2\eta_t \hat{\mathbf{g}}^{(t)}(\mathbf{w}^{(t)} - \mathbf{w}^*) \\ &= \|\mathbf{w}^{(t)} - \mathbf{w}^*\|_2^2 + \eta_t^2 \|\hat{\mathbf{g}}^{(t)}\|_2^2 + 2\eta_t \hat{\mathbf{g}}^{(t)} \cdot \mathbf{w}^*. \end{aligned} \quad (18)$$

Next, we use the following lemma about the concentration of $\hat{\mathbf{g}}$.

Lemma 53 *Suppose σ is a (B, L) -Regular activation. If $0 < \rho \leq \cos \theta < 1$ and $\sin \theta \geq 4(\sqrt{\text{OPT}})/\|\mathbf{T}_{\rho}\sigma'\|_{L_2}$, then using*

$$n = \Theta\left(\frac{dB^2}{\sin^2 \theta \|\mathbf{T}_{\rho}\sigma'\|_{L_2}^2 \delta}\right)$$

samples, with probability at least $1 - \delta$, we have

$$\begin{aligned} \|\hat{\mathbf{g}}(\mathbf{w})\|_2 &\leq (3/2) \mathbf{E}_{\mathbf{x} \sim \mathcal{D}_{\mathbf{x}}} [\mathbf{T}_{\cos \theta} \sigma'(\mathbf{w} \cdot \mathbf{x}) \mathbf{T}_{\rho} \sigma'(\mathbf{w} \cdot \mathbf{x})] \sin \theta, \\ \hat{\mathbf{g}}(\mathbf{w}) \cdot \mathbf{w}^* &\leq -\frac{1}{2} \mathbf{E}_{\mathbf{x} \sim \mathcal{D}_{\mathbf{x}}} [\mathbf{T}_{\cos \theta} \sigma'(\mathbf{w} \cdot \mathbf{x}) \mathbf{T}_{\rho} \sigma'(\mathbf{w} \cdot \mathbf{x})] \sin^2 \theta. \end{aligned}$$

We defer the proof of [Lemma 53](#) to [Appendix E.2](#). Using [Lemma 53](#), we know that with a batch size of

$$n = \Theta\left(\frac{dB^2}{\sin^2 \theta \|\mathbf{T}_{\rho_t}\sigma'\|_{L_2}^2 \delta}\right) \leq \Theta\left(\frac{dB^2}{\epsilon \delta}\right)$$

and if the following conditions are satisfied

$$\rho_t \leq \cos \theta_t, \quad (19)$$

$$\text{and } \sin \theta_t \geq C\sqrt{\text{OPT}}/\|\mathbf{T}_{\rho_t}\sigma'\|_{L_2}, \quad (20)$$

then, with probability at least $1 - \delta$, we have that

$$\hat{\mathbf{g}}^{(t)} \cdot \mathbf{w}^* \leq -(1/2) \mathbf{E}_{\mathbf{x} \sim \mathcal{D}_{\mathbf{x}}} [\mathbf{T}_{\cos \theta_t} \sigma'(\mathbf{w}^{(t)} \cdot \mathbf{x}) \mathbf{T}_{\rho_t} \sigma'(\mathbf{w}^{(t)} \cdot \mathbf{x})] \sin^2 \theta_t, \quad (21)$$

$$\|\hat{\mathbf{g}}^{(t)}\|_2 \leq 2 \mathbf{E}_{\mathbf{x} \sim \mathcal{D}_{\mathbf{x}}} [\mathbf{T}_{\cos \theta_t} \sigma'(\mathbf{w}^{(t)} \cdot \mathbf{x}) \mathbf{T}_{\rho_t} \sigma'(\mathbf{w}^{(t)} \cdot \mathbf{x})] \sin \theta_t. \quad (22)$$

Combining [Equation \(21\)](#) with [Equation \(22\)](#) we get $\hat{\mathbf{g}}^{(t)} \cdot \mathbf{w}^* \geq (1/4)\|\hat{\mathbf{g}}^{(t)}\|_2 \sin \theta_t$. Therefore, bringing in our choice of stepsize $\eta_t = \varphi_t/(4\|\hat{\mathbf{g}}^{(t)}\|_2)$, and noticing $\sin \theta_t \geq \sin(\theta_t/2)$ we obtain:

$$\begin{aligned} 4 \sin^2(\theta_{t+1}/2) &= \|\mathbf{w}^{(t+1)} - \mathbf{w}^*\|_2^2 \leq \|\mathbf{w}^{(t)} - \mathbf{w}^*\|_2^2 + \frac{\varphi_t^2}{16} - \frac{\varphi_t \sin \theta_t}{8} \\ &\leq 4 \sin^2(\theta_t/2) + \frac{\varphi_t}{16}(\varphi_t - 2 \sin(\theta_t/2)). \end{aligned} \quad (23)$$

Since, by assumption, we have $\rho_t \leq \cos \theta_t$, it holds $2 \sin^2(\theta_t/2) = 1 - \cos \theta_t \leq 1 - \rho_t = 2\varphi_t^2$, in other words, $\sin(\theta_t/2) \leq \varphi_t$. Consider first the case that $\varphi_t \geq \sin(\theta_t/2) \geq (3/4)\varphi_t$. Then, according to Equation (23) we get

$$4 \sin^2(\theta_{t+1}/2) \leq 4 \sin^2(\theta_t/2) - \frac{1}{32} \varphi_t^2 \leq 4(1 - 1/128) \varphi_t^2.$$

Hence, since $1 - 1/128 \leq (1 - 1/256)^2$, we get $\sin(\theta_{t+1}/2) \leq (1 - 1/256) \varphi_t = \varphi_{t+1}$.

On the other hand, if $\sin(\theta_t/2) \leq (3/4)\varphi_t$, then by the triangle inequality and the non-expansiveness of the projection operator, we have

$$\begin{aligned} 2 \sin(\theta_{t+1}/2) &= \|\mathbf{w}^{(t+1)} - \mathbf{w}^*\|_2 = \|\text{proj}_{\mathbb{B}}(\mathbf{w}^{(t)} - \eta_t \hat{\mathbf{g}}(\mathbf{w}^{(t)})) - \mathbf{w}^*\|_2 \\ &\leq \|\mathbf{w}^{(t)} - \mathbf{w}^* - \eta_t \hat{\mathbf{g}}^{(t)}\|_2 \\ &\leq \|\mathbf{w}^{(t)} - \mathbf{w}^*\|_2 + (1/4)\varphi_t = 2 \sin(\theta_t/2) + (1/4)\varphi_t \\ &\leq (3/2)(\varphi_t/2) + (1/4)\varphi_t \leq (7/4)\varphi_t. \end{aligned}$$

Therefore, it holds that $\sin(\theta_{t+1}/2) \leq (7/8)\varphi_t \leq \varphi_{t+1}$.

To conclude, we proved that if $\sin \theta_t \geq C\sqrt{\text{OPT}}/\|\mathbf{T}_{\rho_t} \sigma'\|_{L_2}$ and $\rho_t \leq \cos \theta_t$, we have $\sin(\theta_{t+1}/2) \leq \varphi_{t+1}$ after one step of the algorithm, which immediately implies that $\cos \theta_{t+1} = 1 - 2 \sin^2(\theta_{t+1}/2) \geq 1 - 2\varphi_{t+1}^2 = \rho_{t+1}$. Note that when $\sin \theta_t \geq C\sqrt{\text{OPT}}/\|\mathbf{T}_{\rho_t} \sigma'\|_{L_2}$, our argument indicates that $\cos \theta_{t+1} \geq \rho_{t+1}$ holds regardless of whether $|\cos \theta_t - \rho_t| \leq \sin^2 \theta_t$ or not.

Case II. It remains to consider the case where $|\cos \theta_t - \rho_t| \geq \sin^2 \theta_t$. In fact, we consider the setting where $\cos \theta_t - \rho_t \geq \sin^2 \theta_t$, because from the induction argument we have $\rho_t \leq \cos \theta_t$. Observe that this case only requires discussing the setting where $\sin \theta_t \leq C\sqrt{\text{OPT}}/\|\mathbf{T}_{\rho_t} \sigma'\|_{L_2}$, because if $\sin \theta_t \geq C\sqrt{\text{OPT}}/\|\mathbf{T}_{\rho_t} \sigma'\|_{L_2}$ then our previous argument already implies that $\rho_{t+1} \leq \cos \theta_{t+1}$ after one iteration. Therefore, assuming that $\sin \theta_t \leq C\sqrt{\text{OPT}}/\|\mathbf{T}_{\rho_t} \sigma'\|_{L_2}$, applying triangle inequality and the non-expansiveness of projection operator, it holds

$$\begin{aligned} 2 \sin(\theta_{t+1}/2) &= \|\mathbf{w}^{(t+1)} - \mathbf{w}^*\|_2 = \|\text{proj}_{\mathbb{B}}(\mathbf{w}^{(t)} - \eta_t \hat{\mathbf{g}}(\mathbf{w}^{(t)})) - \mathbf{w}^*\|_2 \leq \|\mathbf{w}^{(t)} - \eta_t \hat{\mathbf{g}}^{(t)} - \mathbf{w}^*\|_2 \\ &\leq \|\mathbf{w}^{(t)} - \mathbf{w}^*\|_2 + \eta_t \|\hat{\mathbf{g}}^{(t)}\|_2 = 2 \sin(\theta_t/2) + \varphi_t/4. \end{aligned}$$

Using the assumption $\cos \theta_t - \rho_t \geq \sin^2 \theta_t$, we observe that

$$1 - \sin^2(\theta_t/2) - (1 - 2\varphi_t^2) \geq \sin^2 \theta_t \geq 2 \sin^2(\theta_t/2),$$

in other words, we have $\sin(\theta_t/2) \leq \sqrt{2/3}\varphi_t$. Hence, it holds

$$\sin(\theta_{t+1}/2) \leq \sin(\theta_t/2) + \varphi_t/8 \leq (\sqrt{2/3} + 1/8)\varphi_t \leq (1 - 1/256)\varphi_t = \varphi_{t+1}.$$

Since $\sin(\theta_{t+1}/2) \leq \varphi_{t+1}$, using similar argument we have $\cos \theta_{t+1} \geq 1 - 2\varphi_{t+1}^2 = \rho_{t+1}$, therefore the induction argument continues to hold at step $t + 1$. \blacksquare

In conclusion, from Claim 52, we have $\cos \theta_t \geq \rho_t$ holds for all the iterations $t = 0, \dots, t^* - 1$. It remains to show that the event \mathcal{E}_{t^*} is satisfied at some iteration t^* .

Claim 54 *If $T = c \log(\|\sigma'\|_{L_2}^2/\text{OPT})$, where $c > 0$ is a sufficiently large absolute constant, then there exists $t^* \leq T$, so that the event \mathcal{E}_{t^*} is satisfied.*

Proof of Claim 54 Since $\sin \theta_t \leq \varphi_t \leq (1 - \beta)^t$ for all the iterations $0, 1, \dots, t$ where the event \mathcal{E}_t is not satisfied, we have $\theta_t \rightarrow 0$. After at most $T = (1/\beta) \log(\|\sigma'\|_{L_2}/\sqrt{\text{OPT}}) = O(\log(\|\sigma'\|_{L_2}^2/\text{OPT}))$ iterations, it must hold that $\sin \theta_t \leq \sqrt{\text{OPT}}/\|\sigma'\|_{L_2}$. Note that $\sqrt{\text{OPT}}/\|\sigma'\|_{L_2} \leq \sqrt{\text{OPT}}/\|T_\rho \sigma'\|_{L_2}$ for any $\rho \in (0, 1)$ therefore there exists an iteration t^* for which \mathcal{E}_{t^*} is satisfied. ■

Therefore, we guarantee that \mathcal{E}_t will be satisfied in at most $T = O(\log(\|\sigma'\|_{L_2}^2/\text{OPT})) = O(\log(L/\epsilon))$ iterations. Setting $\delta = 2/(3T)$ and using a union bound, we have that at most

$$N_1 = nT = \Theta\left(\frac{dB^2T^2}{\epsilon}\right) = \tilde{\Theta}\left(\frac{dB^2 \log(L/\epsilon)}{\epsilon}\right)$$

samples suffices to guarantee that the algorithm generates a target solution with probability at least $2/3$. To pick out the target vector $\hat{\mathbf{w}}$ from the list, we can apply a testing procedure with $N_2 = \tilde{\Theta}(B^4/\epsilon^2)$ samples (see Algorithm 5 and Lemma 55 in Appendix E.1). Thus, in summary, the sample complexity is $N_1 + N_2 = \tilde{\Theta}(dB^2 \log(L/\epsilon)/\epsilon + B^4 \log(L/\epsilon)/\epsilon^2)$. ■

E.1. Finding the Best Parameter

As we showed in Theorem 51, Algorithm 4 returns a list of vectors that contains at least one vector $\hat{\mathbf{w}}$ in the target region $\mathcal{R}_{\sigma, \theta_0}(O(\text{OPT}) + \epsilon)$. We now present a simple testing algorithm to identify one of such target vectors.

Algorithm 5 Testing

- 1: **Input:** Vectors $\{\mathbf{w}^{(0)}, \dots, \mathbf{w}^{(T)}\}$; Number of Samples m
 - 2: Sample $\{(\mathbf{x}^{(1)}, y^{(1)}), \dots, (\mathbf{x}^{(m)}, y^{(m)})\}$ from \mathcal{D} and construct the empirical distribution $\hat{\mathcal{D}}$.
 - 3: For $t = 0, \dots, T$, let $\hat{\mathcal{L}}(\mathbf{w}^{(t)}) \leftarrow \mathbf{E}_{(\mathbf{x}, y) \sim \hat{\mathcal{D}}}[(\sigma(\mathbf{w}^{(t)} \cdot \mathbf{x}) - y)^2]$
 - 4: $\hat{\mathbf{w}} = \text{argmin}\{\mathbf{w}^{(t)}, t \in [T] : \hat{\mathcal{L}}(\mathbf{w}^{(t)})\}$.
 - 5: **Output:** $\hat{\mathbf{w}}$.
-

Lemma 55 (Testing) *Let σ be a (B, L) -Regular activation. Let $\{\mathbf{w}^{(t)}\}_{t \in [T]}$ be the list of vectors generated by Algorithm 4 with $T = O(\log(L/\epsilon))$. Let $t^* \in [T]$ be the index such that $\mathcal{L}(\mathbf{w}^{(t^*)}) \in \text{argmin}_{t \in [T]} \mathcal{L}(\mathbf{w}^{(t)})$. We have*

1. *If $\|\mathbf{P}_{>1/(\theta_{t^*})^2} \sigma\|_{L_2}^2 \leq (\theta_{t^*})^2 \|\mathbf{T}_{\cos(\theta_{t^*})} \sigma'\|_{L_2}^2$ for an absolute constant C , then using*

$$m \leq \Theta\left(\frac{B^2 \log(L/\epsilon)}{\epsilon}\right)$$

samples, Algorithm 5 finds a vector $\hat{\mathbf{w}} \in \{\mathbf{w}^{(t)}\}_{t \in [T]}$ such that $\hat{\mathbf{w}} \in \mathcal{R}_{\sigma, \theta_0}(O(\text{OPT}) + \epsilon)$ and $\mathcal{L}(\hat{\mathbf{w}}) \leq O(\text{OPT}) + \epsilon$.

2. *Otherwise, using*

$$m \leq \Theta\left(\frac{B^4 \log \log(L/\epsilon)}{\epsilon^2}\right)$$

samples, Algorithm 5 outputs a vector $\hat{\mathbf{w}} \in \{\mathbf{w}^{(t)}\}_{t \in [T]}$ such that $\hat{\mathbf{w}} \in \mathcal{R}_{\sigma, \theta_0}(O(\text{OPT}) + \epsilon)$ and $\mathcal{L}(\hat{\mathbf{w}}) \leq O(\text{OPT}) + \epsilon + 4\|\mathbf{P}_{>1/(\theta_{t^})^2} \sigma\|_{L_2}^2$.*

Proof Let $t^* \in \operatorname{argmin}_{t \in [T]} \mathcal{L}(\mathbf{w}^{(t)})$. Let $\ell(\mathbf{w}; \mathbf{x}, y) := (\sigma(\mathbf{w} \cdot \mathbf{x}) - y)^2$, and $\Delta(\mathbf{w}^1, \mathbf{w}^2) := \ell(\mathbf{w}^1; \mathbf{x}, y) - \ell(\mathbf{w}^2; \mathbf{x}, y)$. Given a data set $\{(\mathbf{x}^{(i)}, y^{(i)})\}_{i=1}^m$, we denote by $\widehat{\mathcal{L}}(\mathbf{w})$ the empirical version of $\mathcal{L}(\mathbf{w})$, i.e., $\widehat{\mathcal{L}}(\mathbf{w}) = (1/m) \sum_{i=1}^m \ell(\mathbf{w}; \mathbf{x}^{(i)}, y^{(i)})$.

Consider first the case when $\|\mathbf{P}_{>1/(\theta_{t^*})^2} \sigma\|_{L_2}^2 \leq C(\theta_{t^*})^2 \|\mathbf{T}_{\cos(\theta_{t^*})} \sigma'\|_{L_2}^2$ for an absolute constant C . Our goal is to show that $\widehat{\mathcal{L}}(\mathbf{w}^{(t^*)})$ can be separated from all $\widehat{\mathcal{L}}(\mathbf{w}^{(t)})$ for all $\mathbf{w}^{(t)}$ with large L_2^2 error. As shown in [Claim 33](#), when σ is a (B, L) -Regular activation, labels y can be assumed to be bounded above by B without loss of generality. Therefore, the variance of $\Delta(\mathbf{w}^1, \mathbf{w}^2)$ is bounded by

$$\begin{aligned} \mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}} [\Delta^2(\mathbf{w}^1, \mathbf{w}^2)] &= \mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}} [(\sigma(\mathbf{w}^1 \cdot \mathbf{x}) - \sigma(\mathbf{w}^2 \cdot \mathbf{x}))^2 (\sigma(\mathbf{w}^1 \cdot \mathbf{x}) + \sigma(\mathbf{w}^2 \cdot \mathbf{x}) - 2y)^2] \\ &\leq 16B^2 \mathbf{E}_{\mathbf{x} \sim \mathcal{D}_{\mathbf{x}}} [(\sigma(\mathbf{w}^1 \cdot \mathbf{x}) - \sigma(\mathbf{w}^2 \cdot \mathbf{x}))^2]. \end{aligned}$$

On the other hand, suppose without loss of generality that $\mathcal{L}(\mathbf{w}^1) \geq \mathcal{L}(\mathbf{w}^2)$; then, the expectation of $\Delta(\mathbf{w}^1, \mathbf{w}^2)$ can be bounded below by

$$\begin{aligned} &\mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}} [\Delta(\mathbf{w}^1, \mathbf{w}^2)] \\ &= \mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}} [(\sigma(\mathbf{w}^1 \cdot \mathbf{x}) - \sigma(\mathbf{w}^2 \cdot \mathbf{x}))(\sigma(\mathbf{w}^1 \cdot \mathbf{x}) - \sigma(\mathbf{w}^2 \cdot \mathbf{x}) + 2\sigma(\mathbf{w}^2 \cdot \mathbf{x}) - 2y)] \\ &= \mathbf{E}_{\mathbf{x} \sim \mathcal{D}_{\mathbf{x}}} [(\sigma(\mathbf{w}^1 \cdot \mathbf{x}) - \sigma(\mathbf{w}^2 \cdot \mathbf{x}))^2] + 2 \mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}} [(\sigma(\mathbf{w}^1 \cdot \mathbf{x}) - \sigma(\mathbf{w}^2 \cdot \mathbf{x}))(\sigma(\mathbf{w}^2 \cdot \mathbf{x}) - y)] \\ &\geq \mathbf{E}_{\mathbf{x} \sim \mathcal{D}_{\mathbf{x}}} [(\sigma(\mathbf{w}^1 \cdot \mathbf{x}) - \sigma(\mathbf{w}^2 \cdot \mathbf{x}))^2] - \frac{1}{2} \mathbf{E}_{\mathbf{x} \sim \mathcal{D}_{\mathbf{x}}} [(\sigma(\mathbf{w}^1 \cdot \mathbf{x}) - \sigma(\mathbf{w}^2 \cdot \mathbf{x}))^2] - 2\mathcal{L}(\mathbf{w}^2) \\ &= \frac{1}{2} \mathbf{E}_{\mathbf{x} \sim \mathcal{D}_{\mathbf{x}}} [(\sigma(\mathbf{w}^1 \cdot \mathbf{x}) - \sigma(\mathbf{w}^2 \cdot \mathbf{x}))^2] - 2\mathcal{L}(\mathbf{w}^2). \end{aligned}$$

where in the last inequality we used Young's inequality $ab \geq -((1/4)a^2 + b^2)$.

Now consider first $\mathcal{L}(\mathbf{w}^2) \leq (1/8) \mathbf{E}_{\mathbf{x} \sim \mathcal{D}_{\mathbf{x}}} [(\sigma(\mathbf{w}^1 \cdot \mathbf{x}) - \sigma(\mathbf{w}^2 \cdot \mathbf{x}))^2]$. Then, we have

$$\mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}} [\Delta(\mathbf{w}^1, \mathbf{w}^2)] \geq \frac{1}{4} \mathbf{E}_{\mathbf{x} \sim \mathcal{D}_{\mathbf{x}}} [(\sigma(\mathbf{w}^1 \cdot \mathbf{x}) - \sigma(\mathbf{w}^2 \cdot \mathbf{x}))^2].$$

Therefore, using Markov's inequality, we have

$$\begin{aligned} \Pr \left[\left| \frac{1}{m} \sum_{i=1}^m \left(\ell(\mathbf{w}^1; \mathbf{x}^{(i)}, y^{(i)}) - \ell(\mathbf{w}^2; \mathbf{x}^{(i)}, y^{(i)}) \right) - \mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}} [\Delta(\mathbf{w}^1, \mathbf{w}^2)] \right| \geq \frac{1}{3} \mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}} [\Delta(\mathbf{w}^1, \mathbf{w}^2)] \right] \\ \leq \frac{\mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}} [\Delta^2(\mathbf{w}^1, \mathbf{w}^2)]}{m((1/3) \mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}} [\Delta(\mathbf{w}^1, \mathbf{w}^2)])^2} \leq \frac{cB^2}{m \mathbf{E}_{\mathbf{x} \sim \mathcal{D}_{\mathbf{x}}} [(\sigma(\mathbf{w}^1 \cdot \mathbf{x}) - \sigma(\mathbf{w}^2 \cdot \mathbf{x}))^2]} \leq \frac{cB^2}{m \mathcal{L}(\mathbf{w}^2)}. \end{aligned}$$

Let $\mathbf{w}^2 = \mathbf{w}^{(t^*)}$. With $m \geq (cB^2/(\mathcal{L}(\mathbf{w}^{(t^*)})\delta))$, the inequality above implies that when $\mathbf{w}^1 = \mathbf{w}^{(t)}$, $t \in [T]$, such that $\mathcal{L}(\mathbf{w}^{(t)}) > \mathcal{L}(\mathbf{w}^{(t^*)})$ and $\mathcal{L}(\mathbf{w}^{(t^*)}) \leq (1/8) \mathbf{E}_{\mathbf{x} \sim \mathcal{D}_{\mathbf{x}}} [(\sigma(\mathbf{w}^{(t)} \cdot \mathbf{x}) - \sigma(\mathbf{w}^{(t^*)} \cdot \mathbf{x}))^2]$, it holds with probability at least $1 - \delta$:

$$|\widehat{\mathcal{L}}(\mathbf{w}^{(t)}) - \widehat{\mathcal{L}}(\mathbf{w}^{(t^*)}) - \mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}} [\Delta(\mathbf{w}^{(t)}, \mathbf{w}^{(t^*)})]| \leq \frac{1}{3} \mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}} [\Delta(\mathbf{w}^{(t)}, \mathbf{w}^{(t^*)})],$$

in other words, we have

$$\widehat{\mathcal{L}}(\mathbf{w}^{(t^*)}) \leq \widehat{\mathcal{L}}(\mathbf{w}^{(t)}) - \frac{2}{3} \mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}} [\Delta(\mathbf{w}^{(t)}, \mathbf{w}^{(t^*)})].$$

Let $\delta = 1/(3T)$. Applying a union bound to all $\mathbf{w}^{(t)}$, $t \in [T]$, we have that with probability at least $2/3$, using $m \geq (cB^2T/\mathcal{L}(\mathbf{w}^{(t^*)}))$ samples suffices to distinguish $\mathbf{w}^{(t^*)}$ from other vectors $\mathbf{w}^{(t)}$ that have large L_2^2 error.

But what if $\mathcal{L}(\mathbf{w}^{(t^*)}) \geq (1/8) \mathbf{E}_{\mathbf{x} \sim \mathcal{D}_{\mathbf{x}}} [(\sigma(\mathbf{w}^{(t)} \cdot \mathbf{x}) - \sigma(\mathbf{w}^{(t^*)} \cdot \mathbf{x}))^2]$? In this case, note that

$$\begin{aligned} \mathcal{L}(\mathbf{w}^{(t^*)}) &\geq \frac{1}{8} \mathbf{E}_{\mathbf{x} \sim \mathcal{D}_{\mathbf{x}}} [(\sigma(\mathbf{w}^{(t)} \cdot \mathbf{x}) - \sigma(\mathbf{w}^{(t^*)} \cdot \mathbf{x}))^2] \\ &= \frac{1}{8} \left(\mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}} [(\sigma(\mathbf{w}^{(t)} \cdot \mathbf{x}) - y)^2] + \mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}} [(\sigma(\mathbf{w}^{(t^*)} \cdot \mathbf{x}) - y)^2] \right. \\ &\quad \left. - 2 \mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}} [(\sigma(\mathbf{w}^{(t)} \cdot \mathbf{x}) - y)(\sigma(\mathbf{w}^{(t^*)} \cdot \mathbf{x}) - y)] \right) \\ &\geq \frac{1}{8} \left(\mathcal{L}(\mathbf{w}^{(t)}) + \mathcal{L}(\mathbf{w}^{(t^*)}) - \frac{1}{2} \mathcal{L}(\mathbf{w}^{(t)}) - 2\mathcal{L}(\mathbf{w}^{(t^*)}) \right), \end{aligned}$$

where in the last inequality we used Young's inequality $ab \leq (1/4)a^2 + b^2$. The inequality above indicates that $\mathcal{L}(\mathbf{w}^{(t)}) \leq 18\mathcal{L}(\mathbf{w}^{(t^*)})$. Note that according to [Theorem 51](#), $\mathbf{w}^{(t^*)}$ is guaranteed to reside in the $\mathcal{R}_{\sigma, \theta_0}(COPT + \epsilon)$ region, therefore, by definition of the region $\mathcal{R}_{\sigma, \theta_0}(COPT + \epsilon)$,

$$(\theta_{t^*})^2 \|\mathbf{T}_{\cos(\theta_{t^*})} \sigma'\|_{L_2}^2 \leq COPT + \epsilon.$$

When $\|\mathbf{P}_{>1/(\theta_{t^*})^2} \sigma\|_{L_2}^2 \leq C(\theta_{t^*})^2 \|\mathbf{T}_{\cos(\theta_{t^*})} \sigma'\|_{L_2}^2$, according to the error bound displayed in [Proposition 47](#), we have $\mathcal{L}(\mathbf{w}^{(t^*)}) \leq O(OPT) + 4C(\theta_{t^*})^2 \|\mathbf{T}_{\cos(\theta_{t^*})} \sigma'\|_{L_2}^2 \leq O(OPT) + 4C\epsilon$. Therefore, any vector $\mathbf{w}^{(t)}$ that satisfies $\mathcal{L}(\mathbf{w}^{(t^*)}) \geq (1/8) \mathbf{E}_{\mathbf{x} \sim \mathcal{D}_{\mathbf{x}}} [(\sigma(\mathbf{w}^{(t)} \cdot \mathbf{x}) - \sigma(\mathbf{w}^{(t^*)} \cdot \mathbf{x}))^2]$ is in the $\mathcal{R}_{\sigma, \theta_0}(C_1 OPT + \epsilon)$ region and can be output as a constant factor solution.

In summary, when $\|\mathbf{P}_{>1/(\theta_{t^*})^2} \sigma\|_{L_2}^2 \lesssim (\theta_{t^*})^2 \|\mathbf{T}_{\cos(\theta_{t^*})} \sigma'\|_{L_2}^2$, using

$$m \leq \Theta\left(\frac{B^2 \log(L/\epsilon)}{\epsilon}\right)$$

samples suffices for the testing algorithm.

Now consider the general case where $\|\mathbf{P}_{>1/(\theta_{t^*})^2} \sigma\|_{L_2}^2 \gtrsim (\theta_{t^*})^2 \|\mathbf{T}_{\cos(\theta_{t^*})} \sigma'\|_{L_2}^2$. We still have $\mathcal{L}(\mathbf{w}^{(t^*)}) \leq O(OPT) + 4\|\mathbf{P}_{>1/(\theta_{t^*})^2} \sigma\|_{L_2}^2$ but it is no longer acceptable to output a vector $\widehat{\mathbf{w}}$ such that $\mathcal{L}(\widehat{\mathbf{w}}) = C\mathcal{L}(\mathbf{w}^{(t^*)})$ for constant $C > 1$. This is because in the worst case the L_2^2 error of $\widehat{\mathbf{w}}$ can be as large as $\mathcal{L}(\widehat{\mathbf{w}}) = O(OPT) + 4C\|\mathbf{P}_{>1/(\theta_{t^*})^2} \sigma\|_{L_2}^2$. When $\|\mathbf{P}_{>1/(\theta_{t^*})^2} \sigma\|_{L_2}^2$ is large, this error bound does not imply that $\widehat{\mathbf{w}}$ lies in the target region $\mathcal{R}_{\sigma, \theta_0}(O(OPT))$. Therefore, we need a different analysis.

To find a vector $\widehat{\mathbf{w}}$ from the set $\{\mathbf{w}^{(t)}\}_{t \in [T]}$ such that $\widehat{\mathbf{w}} \in \mathcal{R}_{\sigma, \theta_0}(O(OPT) + \epsilon)$, we need to approximate $\mathcal{L}(\mathbf{w}^{(t)})$ to error at most $O(OPT) + \epsilon$ for each $\mathbf{w}^{(t)}$, $t \in [T]$. Since $|\ell(\mathbf{w}; \mathbf{x}, y)| \leq 4B^2$, we know that $\ell(\mathbf{w}; \mathbf{x}, y)$ is a sub-Gaussian random variable with $\|\ell(\mathbf{w}; \mathbf{x}, y)\|_{\psi_2} \leq cB^2$ for some absolute constant c . Then using Hoeffding's inequality, we have

$$\Pr \left[\left| \sum_{i=1}^m \frac{1}{m} \ell(\mathbf{w}; \mathbf{x}^{(i)}, y^{(i)}) - \mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}} [\ell(\mathbf{w}; \mathbf{x}, y)] \right| \geq (OPT + \epsilon) \right] \leq \exp \left(- \frac{c'(OPT + \epsilon)^2}{mB^4} \right).$$

Therefore, using $m \leq B^4 \log(1/\delta)/\epsilon^2$ samples suffices to approximate $\mathcal{L}(\mathbf{w})$ to error $\text{OPT} + \epsilon$. Using a union bound on all $\mathbf{w}^{(t)}$, $t \in [T]$, and set $\delta = 1/(3T)$, we obtain that with probability at least $2/3$, using

$$m \leq \Theta\left(\frac{B^4 \log \log(L/\epsilon)}{\epsilon^2}\right)$$

samples, it holds

$$|\widehat{\mathcal{L}}(\mathbf{w}^{(t)}) - \mathcal{L}(\mathbf{w}^{(t)})| \leq \text{OPT} + \epsilon$$

for all $\mathbf{w}^{(t)}$, $t \in [T]$. Therefore, since $\mathcal{L}(\mathbf{w}^{(t^*)}) \leq O(\text{OPT}) + \epsilon + 4\|\mathbf{P}_{>1/(\theta_{t^*})^2}\sigma\|_{L_2}^2$, by outputting $\widehat{\mathbf{w}} = \min\{\mathbf{w}^{(t)}, t \in [T] : \widehat{\mathcal{L}}(\mathbf{w}^{(t)})\}$ we guarantee that $\mathcal{L}(\widehat{\mathbf{w}}) \leq O(\text{OPT}) + \epsilon + 4\|\mathbf{P}_{>1/(\theta_{t^*})^2}\sigma\|_{L_2}^2$, hence $\widehat{\mathbf{w}} \in \mathcal{R}_{\sigma, \theta_0}(O(\text{OPT}) + \epsilon)$. \blacksquare

E.2. Proof of Lemma 53

In this subsection, we provide technical lemmas that determine the number of samples required for each iteration. We start with Lemma 56 that bounds the population gradient $\|\mathbf{g}(\mathbf{w})\|_2$. Then in Lemma 53 we provide the sufficient batch size of samples per iteration, utilizing the bounds on $\|\mathbf{g}(\mathbf{w})\|_2$ and the truncated upper bounds on the activation $\sigma(z)$ and labels y .

Lemma 56 *Let $\mathbf{g}(\mathbf{w}) := \mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}}[y \mathbf{T}_\rho \sigma'(\mathbf{w} \cdot \mathbf{x}) \mathbf{x}^{\perp \mathbf{w}}]$ and $\theta = \theta(\mathbf{w}, \mathbf{w}^*)$. Then, we have*

$$\|\mathbf{g}(\mathbf{w})\|_2 \leq \sqrt{\text{OPT}} \|\mathbf{T}_\rho \sigma'\|_{L_2} + \|\mathbf{T}_{\sqrt{\rho \cos \theta}} \sigma'\|_{L_2}^2 \sin \theta.$$

If, in addition, $0 < \rho \leq \cos \theta < 1$ and $\sin \theta \geq 4\sqrt{\text{OPT}}/\|\mathbf{T}_\rho \sigma'\|_{L_2}$, then

$$\|\mathbf{g}(\mathbf{w})\|_2 \leq (5/4) \|\mathbf{T}_{\sqrt{\rho \cos \theta}} \sigma'\|_{L_2}^2 \sin \theta$$

Proof By the variational definition of vector norms, we have

$$\begin{aligned} \|\mathbf{g}(\mathbf{w})\|_2 &= \max_{\|\mathbf{u}\|_2=1} \mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}} [y \mathbf{T}_\rho \sigma'(\mathbf{w} \cdot \mathbf{x}) \mathbf{x}^{\perp \mathbf{w}} \cdot \mathbf{u}] \\ &= \max_{\|\mathbf{u}\|_2=1} \mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}} [(y - \sigma(\mathbf{w}^* \cdot \mathbf{x})) \mathbf{T}_\rho \sigma'(\mathbf{w} \cdot \mathbf{x}) \mathbf{x}^{\perp \mathbf{w}} \cdot \mathbf{u}] + \mathbf{E}_{\mathbf{x} \sim \mathcal{D}_\mathbf{x}} [\sigma(\mathbf{w}^* \cdot \mathbf{x}) \mathbf{T}_\rho \sigma'(\mathbf{w} \cdot \mathbf{x}) \mathbf{x}^{\perp \mathbf{w}} \cdot \mathbf{u}]. \end{aligned} \quad (24)$$

Observe here that the maximizing \mathbf{u} depends on the expectation defining $\mathbf{g}(\mathbf{w})$ and is thus deterministic. To bound the right-hand side in Equation (24), we fix an arbitrary unit vector \mathbf{u} and bound the two summands. Using the Cauchy-Schwarz inequality, the first term in Equation (24) above can be bounded by:

$$\begin{aligned} &\mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}} [(y - \sigma(\mathbf{w}^* \cdot \mathbf{x})) \mathbf{T}_\rho \sigma'(\mathbf{w} \cdot \mathbf{x}) \mathbf{x}^{\perp \mathbf{w}} \cdot \mathbf{u}] \\ &\leq \sqrt{\mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}} [(y - \sigma(\mathbf{w}^* \cdot \mathbf{x}))^2]} \sqrt{\mathbf{E}_{\mathbf{x} \sim \mathcal{D}_\mathbf{x}} [(\mathbf{T}_\rho \sigma'(\mathbf{w} \cdot \mathbf{x}))^2 (\mathbf{x}^{\perp \mathbf{w}} \cdot \mathbf{u})^2]} \\ &\leq \sqrt{\text{OPT}} \sqrt{\mathbf{E}_{\mathbf{x} \sim \mathcal{D}_\mathbf{x}} [(\mathbf{T}_\rho \sigma'(\mathbf{w} \cdot \mathbf{x}))^2]} \sqrt{\mathbf{E}_{\mathbf{x} \sim \mathcal{D}_\mathbf{x}} [(\mathbf{x}^{\perp \mathbf{w}} \cdot \mathbf{u})^2]} \\ &= \sqrt{\text{OPT}} \sqrt{\mathbf{E}_{\mathbf{x} \sim \mathcal{D}_\mathbf{x}} [(\mathbf{T}_\rho \sigma'(\mathbf{w} \cdot \mathbf{x}))^2]}, \end{aligned}$$

where in the second inequality we used the fact that $\mathbf{w} \cdot \mathbf{x}$ is independent of $\mathbf{x}^{\perp \mathbf{w}} \cdot \mathbf{u}$, due to the Gaussianity. The last equality uses $\mathbf{x}^{\perp \mathbf{w}} \cdot \mathbf{u} \sim \mathcal{N}(0, 1)$, as \mathbf{u} is independent of \mathbf{x} and $\mathbf{x} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$.

For the second term in Equation (24), observe that if $\mathbf{u} \perp \mathbf{w}, \mathbf{w}^*$, then the expectation takes value zero due to the independence between Gaussian random variables $\mathbf{x}^{\perp \mathbf{w}} \cdot \mathbf{u}$ and $\mathbf{w} \cdot \mathbf{x}, \mathbf{w}^* \cdot \mathbf{x}$. Therefore, we only need to consider \mathbf{u} in the span of \mathbf{w}, \mathbf{w}^* , which can be expressed as $\mathbf{u} = \cos \alpha \mathbf{w} + \sin \alpha (\mathbf{w}^*)^{\perp \mathbf{w}} / \|(\mathbf{w}^*)^{\perp \mathbf{w}}\|_2$, for some $\alpha \in [0, 2\pi]$. Thus, plugging this \mathbf{u} back into the second term in Equation (24), and setting $z_1 = \mathbf{w} \cdot \mathbf{x}$, $z_2 = (\mathbf{w}^*)^{\perp \mathbf{w}} / \|(\mathbf{w}^*)^{\perp \mathbf{w}}\|_2 \cdot \mathbf{x}$ which are independent Gaussian random variables, we get

$$\begin{aligned} \mathbf{E}_{\mathbf{x} \sim \mathcal{D}_{\mathbf{x}}} [\sigma(\mathbf{w}^* \cdot \mathbf{x}) T_{\rho} \sigma'(\mathbf{w} \cdot \mathbf{x}) \mathbf{x}^{\perp \mathbf{w}} \cdot \mathbf{u}] &= \mathbf{E}_{z_1, z_2 \sim \mathcal{N}} [\sigma(\cos(\theta) z_1 + \sin(\theta) z_2) T_{\rho} \sigma'(z_1) \sin(\alpha) z_2] \\ &= \mathbf{E}_{z_1} \left[\mathbf{E}_{z_2} [\sigma(\cos(\theta) z_1 + \sin(\theta) z_2) z_2 \mid z_1] T_{\rho} \sigma'(z_1) \sin(\alpha) \right] \\ &= \mathbf{E}_{z_1, z_2 \sim \mathcal{N}} [\sigma'(\cos(\theta) z_1 + \sin(\theta) z_2) T_{\rho} \sigma'(z_1) \sin(\alpha) \sin(\theta)], \end{aligned}$$

where in the last inequality we applied Fact 26. Moreover, recalling the definition of the Ornstein–Uhlenbeck semigroup, we further have

$$\begin{aligned} \mathbf{E}_{\mathbf{x} \sim \mathcal{D}_{\mathbf{x}}} [\sigma(\mathbf{w}^* \cdot \mathbf{x}) T_{\rho} \sigma'(\mathbf{w} \cdot \mathbf{x}) \mathbf{x}^{\perp \mathbf{w}} \cdot \mathbf{u}] &= \mathbf{E}_{z_1} \left[\mathbf{E}_{z_2} [\sigma'(\cos \theta z_1 + \sin \theta z_2) \mid z_1] T_{\rho} \sigma'(z_1) \sin \alpha \sin \theta \right] \\ &\leq \mathbf{E}_{z_1 \sim \mathcal{N}} [T_{\cos \theta} \sigma'(z_1) T_{\rho} \sigma'(z_1)] \sin \theta, \end{aligned}$$

where the last inequality holds since $\mathbf{E}_{z_1 \sim \mathcal{N}} [T_{\cos \theta} \sigma'(z_1) T_{\rho} \sigma'(z_1)] = \|T_{\sqrt{\rho \cos \theta}} \sigma'\|_{L_2}^2 \geq 0$. Plugging in these bounds on the first and second terms of Equation (24), we get

$$\|\mathbf{g}(\mathbf{w})\|_2 \leq \sqrt{\text{OPT}} \sqrt{\mathbf{E}_{\mathbf{x} \sim \mathcal{D}_{\mathbf{x}}} [(T_{\rho} \sigma'(\mathbf{w} \cdot \mathbf{x}))^2]} + \mathbf{E}_{\mathbf{x} \sim \mathcal{D}_{\mathbf{x}}} [T_{\cos \theta} \sigma'(\mathbf{w} \cdot \mathbf{x}) T_{\rho} \sigma'(\mathbf{w} \cdot \mathbf{x})] \sin \theta.$$

As we have argued in the proof of Proposition 43, if $\rho \leq \cos \theta$, then $\mathbf{E}_{\mathbf{x} \sim \mathcal{D}_{\mathbf{x}}} [T_{\cos \theta} \sigma'(\mathbf{w} \cdot \mathbf{x}) T_{\rho} \sigma'(\mathbf{w} \cdot \mathbf{x})] \geq \mathbf{E}_{\mathbf{x} \sim \mathcal{D}_{\mathbf{x}}} [(T_{\rho} \sigma'(\mathbf{w} \cdot \mathbf{x}))^2]$, hence we further get that if in addition it holds

$$\sin \theta \geq 4\sqrt{\text{OPT}} / \sqrt{\mathbf{E}_{\mathbf{x} \sim \mathcal{D}_{\mathbf{x}}} [(T_{\rho} \sigma'(\mathbf{w} \cdot \mathbf{x}))^2]},$$

then we obtain

$$\begin{aligned} \|\mathbf{g}(\mathbf{w})\|_2 &\leq (\mathbf{E}_{\mathbf{x} \sim \mathcal{D}_{\mathbf{x}}} [T_{\cos \theta} \sigma'(\mathbf{w} \cdot \mathbf{x}) T_{\rho} \sigma'(\mathbf{w} \cdot \mathbf{x})] + (1/4) \mathbf{E}_{\mathbf{x} \sim \mathcal{D}_{\mathbf{x}}} [(T_{\rho} \sigma'(\mathbf{w} \cdot \mathbf{x}))^2]) \sin \theta \\ &\leq (5/4) \mathbf{E}_{\mathbf{x} \sim \mathcal{D}_{\mathbf{x}}} [T_{\cos \theta} \sigma'(\mathbf{w} \cdot \mathbf{x}) T_{\rho} \sigma'(\mathbf{w} \cdot \mathbf{x})] \sin \theta, \end{aligned}$$

completing the proof. ■

We now proceed to determine the sample complexities required to estimate the gradient. Lemma 53 provides the sample complexity to approximate the norm of the population gradient $\|\mathbf{g}(\mathbf{w})\|_2$ and the inner product between the population gradient and \mathbf{w}^* . We restate and prove Lemma 53:

Lemma 53 *Suppose σ is a (B, L) -Regular activation. If $0 < \rho \leq \cos \theta < 1$ and $\sin \theta \geq 4(\sqrt{\text{OPT}}) / \|T_{\rho} \sigma'\|_{L_2}$, then using*

$$n = \Theta \left(\frac{dB^2}{\sin^2 \theta \|T_{\rho} \sigma'\|_{L_2}^2 \delta} \right)$$

samples, with probability at least $1 - \delta$, we have

$$\begin{aligned}\|\widehat{\mathbf{g}}(\mathbf{w})\|_2 &\leq (3/2) \mathbf{E}_{\mathbf{x} \sim \mathcal{D}_{\mathbf{x}}} [\mathrm{T}_{\cos \theta} \sigma'(\mathbf{w} \cdot \mathbf{x}) \mathrm{T}_{\rho} \sigma'(\mathbf{w} \cdot \mathbf{x})] \sin \theta, \\ \widehat{\mathbf{g}}(\mathbf{w}) \cdot \mathbf{w}^* &\leq -\frac{1}{2} \mathbf{E}_{\mathbf{x} \sim \mathcal{D}_{\mathbf{x}}} [\mathrm{T}_{\cos \theta} \sigma'(\mathbf{w} \cdot \mathbf{x}) \mathrm{T}_{\rho} \sigma'(\mathbf{w} \cdot \mathbf{x})] \sin^2 \theta.\end{aligned}$$

Proof Observe first that, by Chebyshev inequality,

$$\Pr[\|\widehat{\mathbf{g}}(\mathbf{w}) - \mathbf{g}(\mathbf{w})\|_2 \geq t] \leq \frac{\mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}} [\|\mathbf{g}(\mathbf{w}; \mathbf{x}, y) - \mathbf{g}(\mathbf{w})\|_2^2]}{nt^2}. \quad (25)$$

Now we proceed to bound the variance $\mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}} [\|\mathbf{g}(\mathbf{w}; \mathbf{x}, y) - \mathbf{g}(\mathbf{w})\|_2^2]$. Let $\mathbf{e}_1, \dots, \mathbf{e}_d$ be the standard basis of \mathbb{R}^d ; we have

$$\begin{aligned}\mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}} [\|\mathbf{g}(\mathbf{w}; \mathbf{x}, y) - \mathbf{g}(\mathbf{w})\|_2^2] &\leq \mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}} [\|\mathbf{g}(\mathbf{w}; \mathbf{x}, y)\|_2^2] = \mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}} \left[\sum_{j=1}^d (\mathbf{g}(\mathbf{w}; \mathbf{x}, y) \cdot \mathbf{e}_j)^2 \right] \\ &= \sum_{j=1}^d \mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}} [(y \mathrm{T}_{\rho} \sigma'(\mathbf{w} \cdot \mathbf{x}) \mathbf{x} \cdot (\mathbf{e}_j)^{\perp \mathbf{w}})^2] \\ &\leq dB^2 \|\mathrm{T}_{\rho} \sigma'\|_{L_2}^2 \mathbf{E}_{\mathbf{x} \sim \mathcal{D}_{\mathbf{x}}} [(\mathbf{x} \cdot (\mathbf{e}_1)^{\perp \mathbf{w}})^2] \leq dB^2 \|\mathrm{T}_{\rho} \sigma'\|_{L_2}^2.\end{aligned}$$

In the second inequality above, we used $|y| \leq B$, which is w.l.o.g., as shown in [Claim 33](#). Therefore, plugging the upper bound on the variance back into [Equation \(25\)](#), we get

$$\Pr[\|\widehat{\mathbf{g}}(\mathbf{w}) - \mathbf{g}(\mathbf{w})\|_2 \geq t] \leq \frac{dB^2 \|\mathrm{T}_{\rho} \sigma'\|_{L_2}^2}{nt^2}.$$

Now choosing $t = \frac{1}{6} \|\mathrm{T}_{\sqrt{\rho \cos \theta}} \sigma'\|_{L_2}^2 \sin \theta$ and setting

$$n = \Theta \left(\frac{dB^2 \|\mathrm{T}_{\rho} \sigma'\|_{L_2}^2}{\sin^2 \theta \|\mathrm{T}_{\sqrt{\rho \cos \theta}} \sigma'\|_{L_2}^4 \delta} \right),$$

we obtain that with probability at least $1 - \delta$, it holds

$$\|\widehat{\mathbf{g}}(\mathbf{w}) - \mathbf{g}(\mathbf{w})\|_2 \leq \frac{1}{6} \|\mathrm{T}_{\sqrt{\rho \cos \theta}} \sigma'\|_{L_2}^2 \sin \theta, \quad (26)$$

and hence

$$\|\widehat{\mathbf{g}}(\mathbf{w})\|_2 \leq \|\mathbf{g}(\mathbf{w})\|_2 + (1/6) \|\mathrm{T}_{\sqrt{\rho \cos \theta}} \sigma'\|_{L_2}^2 \sin \theta.$$

Applying the upper bound on $\|\mathbf{g}(\mathbf{w})\|_2$ we have provided in [Lemma 56](#), we obtain

$$\|\widehat{\mathbf{g}}(\mathbf{w})\|_2 \leq \sqrt{\mathrm{OPT}} \|\mathrm{T}_{\rho} \sigma'\|_{L_2} + (7/6) \|\mathrm{T}_{\sqrt{\rho \cos \theta}} \sigma'\|_{L_2}^2 \sin \theta.$$

In particular, if $0 < \rho \leq \cos \theta < 1$ and $\sin \theta \geq 4\sqrt{\mathrm{OPT}}/\|\mathrm{T}_{\rho} \sigma'\|_{L_2}$, then we have

$$\|\widehat{\mathbf{g}}(\mathbf{w})\|_2 \leq (3/2) \|\mathrm{T}_{\sqrt{\rho \cos \theta}} \sigma'\|_{L_2}^2 \sin \theta.$$

Since $0 < \rho \leq \cos \theta < 1$, it must be $\|\mathbf{T}_\rho \sigma'\|_{L_2} \leq \|\mathbf{T}_{\sqrt{\rho \cos \theta}} \sigma'\|_{L_2}$, and thus using

$$n = \Theta\left(\frac{dB^2}{\sin^2 \theta \|\mathbf{T}_\rho \sigma'\|_{L_2}^2 \delta}\right) \quad (27)$$

samples suffices to guarantee that $\|\widehat{\mathbf{g}}(\mathbf{w})\|_2 \leq (3/2)\|\mathbf{T}_{\sqrt{\rho \cos \theta}} \sigma'\|_{L_2}^2 \sin \theta$.

For the inner product between $\widehat{\mathbf{g}}(\mathbf{w})$ and \mathbf{w}^* , let us denote $\mathbf{v} = (\mathbf{w}^*)^\perp / \|(\mathbf{w}^*)^\perp\|_2$, and $\mathbf{w}^* = \sin \theta \mathbf{v} + \cos \theta \mathbf{w}$. Then, since $\widehat{\mathbf{g}}(\mathbf{w})$ is orthogonal to \mathbf{w} , we have $\widehat{\mathbf{g}}(\mathbf{w}) \cdot \mathbf{w}^* = \widehat{\mathbf{g}}(\mathbf{w}) \cdot \mathbf{v} \sin \theta$. Therefore, using Equation (26), we obtain that when the batch size n satisfies Equation (27), with probability at least $1 - \delta$, we have

$$\begin{aligned} \widehat{\mathbf{g}}(\mathbf{w}) \cdot \mathbf{w}^* &= (\widehat{\mathbf{g}}(\mathbf{w}) - \mathbf{g}(\mathbf{w})) \cdot \mathbf{v} \sin \theta + \mathbf{g}(\mathbf{w}) \cdot \mathbf{v} \sin \theta \\ &\leq \|\widehat{\mathbf{g}}(\mathbf{w}) - \mathbf{g}(\mathbf{w})\|_2 \sin \theta + \mathbf{g}(\mathbf{w}) \cdot \mathbf{w}^* \\ &\leq (1/6)\|\mathbf{T}_{\sqrt{\rho \cos \theta}} \sigma'\|_{L_2}^2 \sin^2 \theta + \mathbf{g}(\mathbf{w}) \cdot \mathbf{w}^*. \end{aligned} \quad (28)$$

Now applying Proposition 43 we get

$$\widehat{\mathbf{g}}(\mathbf{w}) \cdot \mathbf{w}^* \leq -\frac{5}{6}\|\mathbf{T}_{\sqrt{\rho \cos \theta}} \sigma'\|_{L_2}^2 \sin^2 \theta + \sqrt{\text{OPT}}\|\mathbf{T}_\rho \sigma'\|_{L_2} \sin \theta.$$

In particular, when $\sin \theta \geq 3\sqrt{\text{OPT}}/\|\mathbf{T}_\rho \sigma'\|_{L_2}$, in Proposition 43 we showed that

$$\mathbf{g}(\mathbf{w}) \cdot \mathbf{w}^* \leq -(2/3)\|\mathbf{T}_{\sqrt{\rho \cos \theta}} \sigma'\|_{L_2}^2 \sin^2 \theta.$$

Thus, when $\sin \theta \geq 3\sqrt{\text{OPT}}/\|\mathbf{T}_\rho \sigma'\|_{L_2}$, using Equation (28) we have that with probability at least $1 - \delta$,

$$\widehat{\mathbf{g}}(\mathbf{w}) \cdot \mathbf{w}^* \leq -\frac{1}{2}\|\mathbf{T}_{\sqrt{\rho \cos \theta}} \sigma'\|_{L_2}^2 \sin^2 \theta,$$

completing the proof. ■

Appendix F. Full Version of Section 4

We have shown in Appendix E that Algorithm 4 converges to a parameter vector \mathbf{w} with an L_2^2 error bounded above by $O(\text{OPT}) + \|\mathbf{P}_{>1/(\theta^*)^2} \sigma\|_{L_2}^2$, where θ^* is a Critical Point. One of the technical difficulties is that in general we cannot bound $\|\mathbf{P}_{>1/(\theta^*)^2} \sigma\|_{L_2}^2$ by OPT . One such example is when $\sigma(t) = \text{He}_{(1/(\theta^*)^2+1)}(t)$; in this case $\|\mathbf{P}_{>1/(\theta^*)^2} \sigma\|_{L_2}^2 = \|\sigma\|_{L_2}^2$, which can be $\omega(\text{OPT})$. In this section, we show that if the activation is also monotone, then given that θ^* is sufficiently small, we can bound $\|\mathbf{P}_{>1/(\theta^*)^2} \sigma\|_{L_2}^2$ by the Ornstein–Uhlenbeck semigroup of σ' . Specifically, we provide an initialization method that along with Algorithm 4 gives an algorithm that guarantees error $O(\text{OPT})$. Formally, we show the following.

Theorem 57 (Learning Monotone (B, L) -Regular Activations) *Let $\epsilon > 0$, and let σ be a monotone (B, L) -Regular activation. Then, Algorithm 4 draws $N = \tilde{\Theta}(dB^2 \log(L/\epsilon)/\epsilon + d/\epsilon^2)$ samples, runs in $\text{poly}(d, N)$ time and returns a vector $\widehat{\mathbf{w}}$ such that with probability at least $2/3$, $\widehat{\mathbf{w}} \in \mathcal{R}_{\sigma, \theta_0}(O(\text{OPT}) + \epsilon)$, and it holds that $\mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}}[(\sigma(\widehat{\mathbf{w}} \cdot \mathbf{x}) - y)^2] \leq C\text{OPT} + \epsilon$, where C is an absolute constant independent of ϵ, d, B, L .*

The main result of this section is an initialization routine that allows us to bound the higher coefficients of the spectrum, $\|P_{>1/(\theta^*)^2}\sigma\|_{L_2}^2$. In particular, we prove the following.

Proposition 58 (Initialization) *Let $\sigma : \mathbb{R} \rightarrow \mathbb{R}$, $\sigma \in L_2(\mathcal{N})$, be a monotone (B, L) -Regular activation. Let \mathcal{D} be a distribution of labeled examples $(\mathbf{x}, y) \in \mathbb{R}^d \times \mathbb{R}$ such that $\mathcal{D}_{\mathbf{x}} = \mathcal{N}(\mathbf{0}, \mathbf{I})$. Fix a unit vector $\mathbf{w}^* \in \mathbb{R}^d$ such that $\mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}}[(\sigma(\mathbf{w}^* \cdot \mathbf{x}) - y)^2] = \text{OPT}$. There exists an algorithm that draws $N = \tilde{O}(d/\epsilon^2)$ samples, runs in $\text{poly}(N, d)$ time, and with probability at least $2/3$, returns a unit vector $\mathbf{w}^{(0)} \in \mathbb{R}^d$ such that for any unit $\mathbf{w}' \in \mathbb{R}^d$ with $\theta = \theta(\mathbf{w}', \mathbf{w}^*) \leq \theta(\mathbf{w}^{(0)}, \mathbf{w}^*)$, it holds that*

$$\|P_{>1/\theta^2}\sigma\|_{L_2}^2 \lesssim \sin^2 \theta \|T_{\cos \theta} \sigma'\|_{L_2}^2.$$

Combining Theorem 51 with Proposition 58, we can prove Theorem 57.

Proof of Theorem 57 Theorem 51 implies that Algorithm 4 generates a vector $\hat{\mathbf{w}} \in \mathcal{R}_{\sigma, \theta_0}(C\text{OPT} + \epsilon)$ where C is an absolute constant. This implies that $\theta^2 \|T_{\cos(\theta)} \sigma'\|_{L_2}^2 \leq C\text{OPT} + \epsilon$. Since $\theta(\hat{\mathbf{w}}, \mathbf{w}^*) \leq \theta_0$, combining with Proposition 58, i.e., $\|P_{>1/\theta^2}\sigma\|_{L_2}^2 \lesssim \sin^2 \theta \|T_{\cos \theta} \sigma'\|_{L_2}^2$, we further have $\|P_{>1/\theta^2}\sigma\|_{L_2}^2 \lesssim \text{OPT} + \epsilon$. Finally, using the error bound on $\mathcal{L}(\hat{\mathbf{w}})$ developed in Proposition 47, we get $\mathcal{L}(\hat{\mathbf{w}}) \leq C\text{OPT} + \epsilon$.

As displayed in Theorem 51, the main algorithm uses $N_1 = \tilde{\Theta}(dB^2/\epsilon + B^2/\epsilon)$ samples (since according to Lemma 55, when $\|P_{>1/\theta^2}\sigma\|_{L_2}^2 \lesssim \sin^2 \theta \|T_{\cos \theta} \sigma'\|_{L_2}^2$, using $m = \tilde{\Theta}(B^2/\epsilon)$ samples suffices for testing Algorithm 5), and in Proposition 58 we showed that the initialization procedure requires $\tilde{\Theta}(d/\epsilon^2)$ samples. Thus, in summary, for monotone (B, L) -Regular activations, Algorithm 4 uses $N = \tilde{\Theta}(dB^2/\epsilon + d/\epsilon^2)$ samples and runs in $\text{poly}(d, N)$ times. ■

For monotone b -Lipschitz activations σ , we know from Lemma 39 that σ is an ϵ -Extended $(b \log^{1/2}(b/\epsilon), b)$ -Regular activation, meaning that there exists a truncated activation $\bar{\sigma}$ such that $\mathbf{E}_{z \sim \mathcal{N}}[(\bar{\sigma}(z) - \sigma(z))^2] \leq \epsilon$ and $\bar{\sigma}$ is $(b \log^{1/2}(b/\epsilon), b)$ -Regular. Hence applying Theorem 57 to $\bar{\sigma}$, we obtain the following corollary:

Corollary 59 (Learning Monotone & Lipschitz Activations) *Let $\epsilon, b > 0$, and let σ be a monotone b -Lipschitz activation. Then, Algorithm 4 draws $N = \tilde{\Theta}(db^2/\epsilon + d/\epsilon^2)$ samples, runs in $\text{poly}(d, N)$ time, and returns a vector $\hat{\mathbf{w}}$ such that with probability at least $2/3$, it holds that $\mathcal{L}(\hat{\mathbf{w}}) \leq C\text{OPT} + \epsilon$, where C is an absolute constant independent of ϵ, d, b .*

Similarly, if σ has bounded $2 + \zeta$ moment $\mathbf{E}_{z \sim \mathcal{N}}[\sigma^{2+\zeta}(z)] \leq B_\sigma$, then according to Lemma 36 we know that σ is an ϵ -Extended $((B_\sigma/\epsilon)^{1/\zeta}, (B_\sigma/\epsilon)^{4/\zeta}/\epsilon^2)$ -Regular activation. Therefore, replacing B with $(B_\sigma/\epsilon)^{1/\zeta}$ and replace L with $(B_\sigma/\epsilon)^{4/\zeta}/\epsilon^2$ in Theorem 57, we obtain:

Corollary 60 (Learning Monotone Activations With Bounded $2 + \zeta$ Moments) *Let $\epsilon > 0$, and let σ be a monotone activation that satisfies $\mathbf{E}_{z \sim \mathcal{N}}[\sigma^{2+\zeta}(z)] \leq B_\sigma$. Then, Algorithm 4 draws $N = \tilde{\Theta}(d(B_\sigma/\epsilon)^{2/\zeta} \log(B_\sigma/\epsilon)/\epsilon + d/\epsilon^2)$ samples, runs in $\text{poly}(d, N)$ time, and returns a vector $\hat{\mathbf{w}}$ such that with probability at least $2/3$, it holds that $\mathcal{L}(\hat{\mathbf{w}}) \leq C\text{OPT} + \epsilon$, where C is an absolute constant independent of ϵ, d, B_σ, L .*

The main contents of this section are the following: To prove Proposition 58, we need to combine two main technical pieces: (1) proving that there exists a threshold θ_0 such that for any $\theta \leq \theta_0$, $\|P_{>1/\theta^2}\sigma\|_{L_2}^2 \lesssim \sin^2 \theta \|T_{\cos \theta} \sigma'\|_{L_2}^2$; (2) proving that there exists an efficient algorithm that finds a parameter $\mathbf{w}^{(0)}$ such that $\theta(\mathbf{w}^{(0)}, \mathbf{w}^*) \leq \theta_0$.

Appendix F.1 is devoted to the proof of (1), i.e., that there exists θ_0 such that for $\theta \leq \theta_0$, $\|P_{>1/\theta^2}\sigma\|_{L_2}^2 \lesssim \sin^2 \theta \|T_{\cos \theta} \sigma'\|_{L_2}^2$ (Proposition 62). Unfortunately, it was technically hard to prove this claim directly for all monotone functions due to the versatility of such functions. Hence, the natural idea is that if we can prove (1) for a sequence of simple and ‘nice’ functions Φ_k that can converge to σ , then by the convergence theorems the desired claim will also hold true for σ . In particular, let Φ_k be a sequence of functions; then, one can show that the higher order coefficients can be bounded by

$$\|P_{>1/\theta^2}\sigma\|_{L_2}^2 \leq 2\|\sigma - \Phi_k\|_{L_2}^2 + 4\|\Phi_k - T_\rho \Phi_k\|_{L_2}^2 + 4\theta^2 \|T_\rho \Phi_k'\|_{L_2}^2.$$

If Φ_k converges to σ pointwise, one can show that the first term above goes to 0 and the second term converges to $\theta^2 \|T_{\cos \theta} \sigma'\|_{L_2}^2$, which is the bound we are looking for. Thus, it remains to show that $\|\Phi_k - T_\rho \Phi_k\|_{L_2}^2 \lesssim \theta^2 \|T_\rho \Phi_k'\|_{L_2}^2$.

Appendix F.2 proves the claim that $\|\Phi - T_\rho \Phi\|_{L_2}^2 \lesssim \theta^2 \|T_\rho \Phi'\|_{L_2}^2$ (Proposition 64), for any $\Phi(z)$ that is a *monotonic staircase function*:

Definition 61 (Monotonic Staircase Functions) *For simplicity, denote the indicator function $\mathbb{1}\{z \geq t\}$ by $\phi(z; t)$. Let m be a positive integer and let $M > 0$. The monotonic staircase functions (of M -bounded support) are defined by*

$$\mathcal{F}_M := \left\{ \sum_{i=1}^m A_i \phi(z; t_i) + A_0 : A_0 \in \mathbb{R}; A_i > 0, |t_i| \leq M, \forall i \in [m]; m < \infty \right\}.$$

These staircase functions constitute a dense subset of the monotone function class and have a simple and easy-to-analyze form, therefore they serve well for our purpose. However, though the staircase function Φ already takes a concise and simple expression, many technical difficulties arise when analyzing $T_\rho \Phi(z) - \Phi(z)$, mainly due to the complicated form of $T_\rho \Phi(z)$. Our workaround is to introduce a new type of smoothing/augmentation method, which we call centered augmentation, defined by $T_\rho(\Phi(z/\rho))$. This recentered augmentation takes a much simpler form compared to $T_\rho \Phi(z)$. In particular, we show that when the smoothing parameter ρ is not too small, namely, when $1 - \rho^2 \leq O(1/\log(1/\epsilon))$, then: (i) the L_2^2 distance between $T_\rho \Phi(z/\rho)$ and $\Phi(z)$ can be bounded above by $(1 - \rho^2) \|T_\rho \Phi'(z/\rho)\|_{L_2}^2$ (Lemma 68); (ii) the L_2^2 distance between $T_\rho \Phi(z/\rho)$ and $T_\rho \Phi(z)$ can be bounded above by $(1 - \rho^2) (\|T_\rho \Phi'(z/\rho)\|_{L_2}^2 + \|T_\rho \Phi'(z)\|_{L_2}^2)$ (Lemma 69); (iii) finally, choosing the smoothing strength ρ_1 slightly larger than ρ , we have $\|T_{\rho_1} \Phi'(z/\rho_1)\|_{L_2}^2 \lesssim \|T_\rho \Phi'(z)\|_{L_2}^2$ (Lemma 71). Combining these 3 results on the relations between $T_\rho \Phi(z/\rho)$ and $T_\rho \Phi(z)$, we prove Proposition 64 in Appendix F.2.2, completing the last piece of the puzzle in the proof of Proposition 62.

Finally, in Appendix F.3, we prove (2) by providing an *SQ* initialization algorithm. The main idea is to transform the labels y to $\tilde{y} = \mathcal{T}(y) := \mathbb{1}\{y \geq t'\}$ for a carefully chosen threshold t' . Then, we show that there exists a halfspace $\phi(\mathbf{w}^* \cdot \mathbf{x}; t) = \mathbb{1}\{\mathbf{w}^* \cdot \mathbf{x} \geq t\}$ such that the transformed labels \tilde{y} can be viewed as the corrupted labels of $\phi(\mathbf{w}^* \cdot \mathbf{x}; t)$. Then, utilizing the algorithm for learning halfspaces Diakonikolas et al. (2022c), we can obtain an initial vector $\mathbf{w}^{(0)}$ such that $\theta(\mathbf{w}^{(0)}, \mathbf{w}^*) \leq \theta_0$.

Finally, combining (1) and (2), we prove Proposition 58.

F.1. Bounding Higher Order Hermite Coefficients of Monotone Activations

The main result of this section is the following:

Proposition 62 (From Hermite Tails to Ornstein–Uhlenbeck Semigroup) *Let $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ be a monotone activation and $\sigma \in L_2(\mathcal{N})$. Let M be the upper bound for the support of $\sigma'(z)$,⁴ i.e., $\forall z \in \mathbb{R}$ such that $|z| \geq M$, we have $\sigma'(z) = 0$. For any $\theta \in [0, \pi]$ such that $1 - C/M^2 < \cos^2 \theta$ with $C > 0$ an absolute constant, it holds that $\|P_{>1/\theta^2} \sigma\|_{L_2}^2 \lesssim \sin^2 \theta \|T_{\cos \theta} \sigma'\|_{L_2}^2$.*

Proof Instead of proving Proposition 62 directly for the activation σ , we chose another function Φ that works as a surrogate for σ and satisfies certain regularity properties. Let Φ be any function in $L_2(\mathcal{N})$, then by Young’s inequality we have that

$$\begin{aligned} \|P_{>1/\theta^2} \sigma\|_{L_2}^2 &\leq 2\|P_{>1/\theta^2}(\sigma - \Phi)\|_{L_2}^2 + 2\|P_{>1/\theta^2} \Phi\|_{L_2}^2 \\ &\leq 2\|P_{>1/\theta^2}(\sigma - \Phi)\|_{L_2}^2 + 4\|P_{>1/\theta^2}(\Phi - T_\rho \Phi)\|_{L_2}^2 + 4\|P_{>1/\theta^2} T_\rho \Phi\|_{L_2}^2. \end{aligned}$$

Observe that $P_{>m}$ is a non-expansive operator since for any $f \in L_2(\mathcal{N})$, $f(z) \doteq \sum_{i \geq 0} a_i \text{He}_i(z)$ it holds

$$\|P_{>m} f\|_{L_2}^2 = \sum_{i > m} a_i^2 \leq \sum_{i \geq 0} a_i^2 = \|f\|_{L_2}^2.$$

Therefore, $\|P_{>1/\theta^2}(\sigma - \Phi)\|_{L_2}^2 \leq \|\sigma - \Phi\|_{L_2}^2$. In addition, note that we have the following inequality for any $f, f' \in L_2(\mathcal{N})$:

$$\|P_{>m} f\|_{L_2}^2 = \sum_{i > m} a_i^2 \leq \sum_{i > m} (i/m) a_i^2 \leq \sum_{i=1}^m (i/m) a_i^2 + \sum_{i > m} (i/m) a_i^2 = (1/m) \|f'\|_{L_2}^2,$$

therefore $\|P_{>1/\theta^2} T_\rho \Phi\|_{L_2}^2 \leq \theta^2 \|(T_\rho \Phi)'\|_{L_2}^2$. Finally by Fact 21 we have $\|(T_\rho \Phi)'\|_{L_2}^2 = \|\rho T_\rho \Phi'\|_{L_2}^2 \leq \|T_\rho \Phi'\|_{L_2}^2$ since $\rho < 1$, thus, it holds

$$\|P_{>1/\theta^2} \sigma\|_{L_2}^2 \leq 2\|\sigma - \Phi\|_{L_2}^2 + 4\|\Phi - T_\rho \Phi\|_{L_2}^2 + 4\theta^2 \|T_\rho \Phi'\|_{L_2}^2. \quad (29)$$

Let Φ_k be any sequence of functions such that $\lim_{k \rightarrow \infty} \|\Phi_k - \sigma\|_{L_2} = 0$. For this sequence we have that Equation (29) becomes

$$\|P_{>1/\theta^2} \sigma\|_{L_2}^2 \leq 2\|\sigma - \Phi_k\|_{L_2}^2 + 4\|\Phi_k - T_\rho \Phi_k\|_{L_2}^2 + 4\theta^2 \|T_\rho \Phi_k'\|_{L_2}^2. \quad (30)$$

In particular, let Φ_k be a sequence of staircase monotonic functions (see Definition 61) that converges to σ uniformly; then, for $\rho^2 \geq 1 - C/M^2$ where M is the upper bound on the support of σ' (which is also the upper bound on the support of all Φ_k' ’s) and C is an absolute constant, from Proposition 64, we conclude that $\|\Phi_k - T_\rho \Phi_k\|_{L_2}^2 \lesssim (1 - \rho^2) \|T_\rho \Phi_k'\|_{L_2}^2$ and therefore we have that

$$\|P_{>1/\theta^2} \sigma\|_{L_2}^2 \leq 2\|\sigma - \Phi_k\|_{L_2}^2 + 4((1 - \rho^2) + \theta^2) \|T_\rho \Phi_k'\|_{L_2}^2. \quad (31)$$

Our next goal is to show that the sequence of smoothed derivatives $T_\rho \Phi_k'$ also converge to σ' , as stated in the following lemma.

Lemma 63 (Convergence of Derivatives) *Let $\sigma : \mathbb{R} \rightarrow \mathbb{R}$, $\sigma \in L_2(\mathcal{N})$, and let $\Phi_k : \mathbb{R} \rightarrow \mathbb{R}$ be a sequence of functions such that $\|\sigma - \Phi_k\|_{L_2} \rightarrow 0$ as $k \rightarrow \infty$. Then, for any $\rho \in (0, 1)$, it holds that*

$$\|T_\rho \Phi_k' - T_\rho \sigma'\|_{L_2} \rightarrow 0, \text{ as } k \rightarrow \infty.$$

⁴In Claim 34, we show that for any $\sigma \in \mathcal{H}(B, L)$, the support of σ' can always be bounded by $M \lesssim \sqrt{\log(B/\epsilon) - \log \log(B/\epsilon)}$.

Proof For any function $f \in L_2(\mathcal{N})$, we have that (by the definition of Ornstein–Uhlenbeck semi-group and Stein’s lemma, stated in [Fact 26](#))

$$\mathbf{T}_\rho f'(z) = \frac{1}{\sqrt{1-\rho^2}} \mathbf{E}_{t \sim \mathcal{N}}[f(\rho z + \sqrt{(1-\rho^2)}t)] .$$

Therefore, we have that

$$\begin{aligned} \|\mathbf{T}_\rho \Phi'_k - \mathbf{T}_\rho \sigma'\|_{L_2}^2 &= \frac{1}{1-\rho^2} \mathbf{E}_{z \sim \mathcal{N}} \left[\left(\mathbf{E}_{t \sim \mathcal{N}} \left[\left(\Phi_k(\rho z + \sqrt{(1-\rho^2)}t) - \sigma(\rho z + \sqrt{(1-\rho^2)}t) \right) t \right] \right)^2 \right] \\ &\leq \frac{1}{1-\rho^2} \mathbf{E}_{z \sim \mathcal{N}} \left[\mathbf{E}_{t \sim \mathcal{N}} \left[\left(\Phi_k(\rho z + \sqrt{(1-\rho^2)}t) - \sigma(\rho z + \sqrt{(1-\rho^2)}t) \right)^2 \right] \mathbf{E}_{t \sim \mathcal{N}}[t^2] \right] \\ &= \frac{1}{1-\rho^2} \mathbf{E}_{z \sim \mathcal{N}} \left[\mathbf{E}_{t \sim \mathcal{N}} \left[\left(\Phi_k(\rho z + \sqrt{(1-\rho^2)}t) - \sigma(\rho z + \sqrt{(1-\rho^2)}t) \right)^2 \right] \right] \\ &= \frac{1}{1-\rho^2} \mathbf{E}_{z \sim \mathcal{N}} \left[(\Phi_k(z) - \sigma(z))^2 \right] , \end{aligned}$$

where the inequality is by the Cauchy-Schwarz inequality and the last inequality is by $\rho z + \sqrt{(1-\rho^2)}t \sim \mathcal{N}(0, 1)$ for independent $z \sim \mathcal{N}(0, 1)$, $t \sim \mathcal{N}(0, 1)$. It remains to take the limit with $k \rightarrow \infty$. \blacksquare

Combining [Lemma 63](#) with [Equation \(31\)](#), and letting $\rho = \cos \theta$ now completes the proof of [Proposition 62](#). \blacksquare

We recall that the assumption that the support of $\sigma'(z)$ is bounded by $M < +\infty$ is without loss of generality, as we have proved in [Claim 34](#).

F.2. Bounding the Augmentation Error

In this subsection, we prove the main technical result, which provides an upper bound on the smoothing error of piecewise staircase functions using the $L_2(\mathcal{N})$ norm of the smoothed derivative. We recall the class of the piecewise staircase functions \mathcal{F}_M below:

$$\mathcal{F}_M := \left\{ \sum_{i=1}^m A_i \phi(z; t_i) + A_0 : A_0 \in \mathbb{R}; A_i > 0, |t_i| \leq M, \forall i \in [m]; m < \infty \right\} .$$

Our result is the following proposition:

Proposition 64 *Let $\Phi \in \mathcal{F}_M$ be any staircase function that consists of m indicator functions with thresholds t_i , $i \in [m]$, and suppose $|t_i| \leq M$ for all $i \in [m]$, where $1 < M < +\infty$. For any $\rho \in (0, 1)$ such that $\rho^2 \geq 1 - C/M^2$ where $C < M^2/4$ is an absolute constant, we have*

$$\mathbf{E}_{z \sim \mathcal{N}}[(\mathbf{T}_\rho \Phi(z) - \Phi(z))^2] \lesssim (1 - \rho^2) \mathbf{E}_{z \sim \mathcal{N}}[(\mathbf{T}_\rho \Phi'(z))^2] .$$

As we have remarked in the comment after the proof of [Lemma 39](#), since M is an upper bound on the support of σ' , we will assume without loss of generality throughout the rest of the paper that M^2 is larger than constant $4C$.

Some remarks about the staircase functions are in order. Observe first that according to [Claim 34](#), when σ is (B, L) -Regular, we can always bound M by $\sqrt{2\log(4B^2/\epsilon) - \log\log(4B^2/\epsilon)}$. Next, for any function $\Phi \in \mathcal{F}_M$, its derivative can be written as:

$$\Phi'(z) = \sum_{i=1}^m A_i \phi'(z; t_i) = \sum_{i=1}^m A_i \delta(z - t_i),$$

where $\delta(z - t_i)$ is the Dirac delta function. Certainly, when $|z| \geq M$ we have $\Phi'(z) = 0$. Also note that for any non-decreasing function σ with the support of its derivative $\sigma'(z)$ bounded by M , there exists a sequence of staircase functions $\Phi_k \in \mathcal{F}_M$ such that Φ_k converges to σ uniformly. To prove this claim, we note that since for any $|z| \geq M$, $\sigma'(z) = 0$, therefore $\sigma(z) = \sigma(M)$ when $z \geq M$ and $\sigma(z) = \sigma(-M)$ for all $z \leq -M$. Hence, let

$$\Phi_k(z) = \sum_{i=1}^m \frac{1}{k} \phi(z; t_i) + \sigma(-M),$$

where $\begin{cases} m = \lceil \sigma(M) - \sigma(-M)/k \rceil + 1, \\ t_i = \min_{t \in [-M, M]} \{\sigma(t) \geq (i-1)(1/k) + \sigma(-M)\}, i = 1, \dots, m-1; t_m = M. \end{cases}$

By construction, we have $|\Phi_k(z) - \sigma(z)| \leq 1/k$ for all $z \in \mathbb{R}$, therefore Φ_k converges to σ uniformly.

To prove [Proposition 64](#), we decompose $\mathbf{E}_{z \sim \mathcal{N}}[(T_\rho \Phi(z) - \Phi(z))^2]$ into the following terms and provide upper bounds on each term respectively:

$$\begin{aligned} & \mathbf{E}_{z \sim \mathcal{N}}[(T_\rho \Phi(z) - \Phi(z))^2] \\ & \leq 2 \mathbf{E}_{z \sim \mathcal{N}}[(T_\rho \Phi(z) - T_{\rho_1} \Phi(z/\rho_1))^2] + 2 \mathbf{E}_{z \sim \mathcal{N}}[(T_{\rho_1} \Phi(z/\rho_1) - \Phi(z))^2] \\ & \lesssim \mathbf{E}_{z \sim \mathcal{N}}[(T_\rho \Phi(z) - T_{\rho_1} \Phi(z))^2] + \mathbf{E}_{z \sim \mathcal{N}}[(T_{\rho_1} \Phi(z) - T_{\rho_1} \Phi(z/\rho_1))^2] + \mathbf{E}_{z \sim \mathcal{N}}[(T_{\rho_1} \Phi(z/\rho_1) - \Phi(z))^2], \end{aligned} \tag{32}$$

where we repeatedly used the inequality $(a+b)^2 \leq 2a^2 + 2b^2$. As we have discussed at the beginning of [Appendix F](#), we introduced this ‘recentered smoothing’ operator $T_\rho \Phi(z/\rho)$ to overcome the difficulty of analyzing $T_\rho \Phi(z) - \Phi(z)$, since $T_\rho \Phi(z/\rho)$ takes a more simple and easy-to-analyze form. Here, $\rho_1 \in (0, 1)$ is a carefully chosen smoothing parameter that is slightly larger than ρ , so that we can bound $\|T_{\rho_1} \Phi'(z/\rho_1)\|_{L_2}^2$ above using $\|T_\rho \Phi'(z)\|_{L_2}^2$ ([Lemma 71](#)).

Coming back to [Equation \(32\)](#), we show that: (1) the first term $\mathbf{E}_{z \sim \mathcal{N}}[(T_\rho \Phi(z) - T_{\rho_1} \Phi(z))^2]$ can be bounded above by $(1-\rho)\|T_{\rho_1} \Phi'(z)\|_{L_2}^2$, using [Lemma 24](#); (2) the second term $\mathbf{E}_{z \sim \mathcal{N}}[(T_{\rho_1} \Phi(z) - T_{\rho_1} \Phi(z/\rho_1))^2]$ is bounded above by $(1-\rho)(\|T_{\rho_1} \Phi'(z/\rho_1)\|_{L_2}^2 + \|T_{\rho_1} \Phi'(z)\|_{L_2}^2)$, using [Lemma 69](#); and (3) the third term $\mathbf{E}_{z \sim \mathcal{N}}[(T_{\rho_1} \Phi(z/\rho_1) - \Phi(z))^2]$ is bounded above by $(1-\rho)\|T_{\rho_1} \Phi'(z/\rho_1)\|_{L_2}^2$, using [Lemma 68](#).

Thus, in summary, we have $\mathbf{E}_{z \sim \mathcal{N}}[(T_\rho \Phi(z) - \Phi(z))^2] \lesssim (1-\rho)(\|T_{\rho_1} \Phi'(z/\rho_1)\|_{L_2}^2 + \|T_{\rho_1} \Phi'(z)\|_{L_2}^2)$. Since ρ_1 is chosen so that $\|T_{\rho_1} \Phi'(z/\rho_1)\|_{L_2}^2 \lesssim \|T_\rho \Phi'(z)\|_{L_2}^2$ (see [Lemma 71](#)), and furthermore, since it holds that $\|T_{\rho_1} \Phi'(z)\|_{L_2}^2 \lesssim \|T_\rho \Phi'(z)\|_{L_2}^2$, combining these results we prove that $\mathbf{E}_{z \sim \mathcal{N}}[(T_\rho \Phi(z) - \Phi(z))^2] \lesssim (1-\rho)\|T_\rho \Phi'(z)\|_{L_2}^2$.

We first derive an explicit expression for $\mathbf{E}_{z \sim \mathcal{N}}[(T_\rho \Phi'(z))^2]$, for any $\Phi \in \mathcal{F}_M$.

Lemma 65 For any $\Phi \in \mathcal{F}_M$, it holds that

$$\mathbf{E}_{z \sim \mathcal{N}}[(T_\rho \Phi'(z))^2] = \sum_{i,j=1}^m \frac{A_i A_j}{2\pi \sqrt{1-\rho^4}} \exp\left(-\frac{t_i^2 + t_j^2}{2(1-\rho^4)} + \frac{\rho^2 t_i t_j}{1-\rho^4}\right).$$

Proof By the linearity of the Ornstein–Uhlenbeck semigroup, we have $T_\rho \Phi'(z) = \sum_{i=1}^m A_i T_\rho \phi'(z; t_i)$. In fact, each summand in this summation has an explicit expression, which we derive in the following:

$$\begin{aligned} T_\rho \phi'(z; t_i) &= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \phi'(\rho z + \sqrt{1-\rho^2}u; t_i) \exp(-u^2/2) du \\ &= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \delta(\rho z + \sqrt{1-\rho^2}u - t_i) \exp(-u^2/2) du \\ &= \frac{1}{\sqrt{2\pi(1-\rho^2)}} \exp\left(-\frac{(\rho z - t_i)^2}{2(1-\rho^2)}\right), \end{aligned} \tag{33}$$

where we have used that δ is the Dirac delta function, and so $\delta(u)$ satisfies $\delta(au) = \delta(u)/a$ for any real positive number a . Therefore, we get that

$$\begin{aligned} &\mathbf{E}_{z \sim \mathcal{N}}[(T_\rho \Phi'(z))^2] \\ &= \mathbf{E}_{z \sim \mathcal{N}}\left[\sum_{i,j=1}^m \frac{A_i A_j}{2\pi(1-\rho^2)} \exp\left(-\frac{(\rho z - t_i)^2 + (\rho z - t_j)^2}{2(1-\rho^2)}\right)\right] \\ &= \sum_{i,j=1}^m \frac{A_i A_j}{2\pi(1-\rho^2)} \exp\left(-\frac{t_i^2 + t_j^2}{2(1-\rho^2)}\right) \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\rho^2}{1-\rho^2}z^2 + \frac{(t_i + t_j)\rho}{1-\rho^2}z - \frac{z^2}{2}\right) dz \\ &\stackrel{(i)}{=} \sum_{i,j=1}^m \frac{A_i A_j}{2\pi(1-\rho^2)} \exp\left(-\frac{t_i^2 + t_j^2}{2(1-\rho^2)}\right) \sqrt{\frac{1-\rho^2}{1+\rho^2}} \exp\left(\frac{(t_i + t_j)^2 \rho^2}{2(1-\rho^4)}\right) \\ &= \sum_{i,j=1}^m \frac{A_i A_j}{2\pi \sqrt{1-\rho^4}} \exp\left(-\frac{t_i^2 + t_j^2}{2(1-\rho^4)} + \frac{\rho^2 t_i t_j}{1-\rho^4}\right), \end{aligned}$$

where in (i) we used the fact that $\int \exp(-az^2 + bz) dz = \sqrt{\pi/a} \exp(b^2/4a)$. ■

A byproduct of the above proof is that:

Claim 66 For any $\Phi \in \mathcal{F}_M$, it holds

$$T_\rho \Phi'(z) = \sum_{k \geq 0} \rho^k \alpha_k \text{He}_k(z), \text{ where } \alpha_k = \sum_{i=1}^m \frac{A_i}{\sqrt{2\pi}} \exp(-t_i^2/2) \text{He}_k(t_i).$$

Furthermore, the function $\zeta(\rho) := \mathbf{E}_{z \sim \mathcal{N}}[(T_\rho \Phi'(z))^2]$ is a non-decreasing function of $\rho \in (0, 1)$.

Proof It is easy to see that $T_\rho \phi'(z; t_i)$ is square-integrable under the Gaussian measure, therefore the Hermite expansion of $T_\rho \phi'(z; t_i)$ exists. In particular, using Mehler's formula (Fact 25), we can derive the Hermite expansion of $T_\rho \phi'(z; t_i)$ immediately:

$$T_\rho \phi'(z; t_i) = \frac{1}{\sqrt{2\pi}} \sum_{k \geq 0} \rho^k \exp(-t_i^2/2) \text{He}_k(t_i) \text{He}_k(z),$$

which then implies that it holds

$$\begin{aligned} T_\rho \Phi'(z) &= \sum_{i=1}^m \frac{A_i}{\sqrt{2\pi}} \sum_{k \geq 0} \rho^k \exp(-t_i^2/2) \text{He}_k(t_i) \text{He}_k(z) \\ &= \sum_{k \geq 0} \rho^k \left(\sum_{i=1}^m \frac{A_i}{\sqrt{2\pi}} \exp(-t_i^2/2) \text{He}_k(t_i) \right) \text{He}_k(z). \end{aligned} \quad (34)$$

For the monotonicity of $\zeta(\rho)$, observe that by the Hermite expansion of $T_\rho \Phi'(z)$, we have

$$\zeta(\rho) = \mathbf{E}_{z \sim \mathcal{N}}[(T_\rho \Phi'(z))^2] = \sum_{k \geq 0} \rho^{2k} \alpha_k^2,$$

which is an increasing function of $\rho \in (0, 1)$. ■

Claim 66 implies that though $\Phi'(z)$ is not in $L_2(\mathcal{N})$ (since the square of the Dirac delta function $\delta^2(z)$ is not integrable), $T_\rho \Phi'(z)$ is well-defined and is continuous and smooth. Consequently, all the facts presented in **Fact 21** apply to $T_\rho \Phi'(z)$ as well.

Proceeding to the analysis of $\mathbf{E}_{z \sim \mathcal{N}}[(T_\rho \Phi(z) - \Phi(z))^2]$, however, technical difficulties arise when we try to relate $\mathbf{E}_{z \sim \mathcal{N}}[(T_\rho \Phi(z) - \Phi(z))^2]$ with $\mathbf{E}_{z \sim \mathcal{N}}[(T_\rho \Phi'(z))^2]$. The main obstacle is that it is hard to analyze $T_\rho \phi(z; t) - \phi(z; t)$, since

$$T_\rho \phi(z; t) - \phi(z; t) = \mathbf{Pr}_{u \sim \mathcal{N}}[u \geq (t - \rho z)/\sqrt{1 - \rho^2}] - \mathbb{1}\{z \geq t\},$$

and the probability term does not have a close form. The workaround is to study the centered augmentation (centered smoothing), and then translate the upper bound on the centered augmentation error back to the upper bound on the standard augmentation error.

F.2.1. CENTERED AUGMENTATION

We define the centered augmentation as the following:

$$T_\rho \sigma(z/\rho) = \mathbf{E}_{u \sim \mathcal{N}}[\sigma(z + (\sqrt{1 - \rho^2}/\rho)u)].$$

Note that for the staircase functions $\Phi \in \mathcal{F}_M$, it holds

$$\begin{aligned} T_\rho \Phi(z/\rho) &= \sum_{i=1}^m A_i \mathbf{E}_{u \sim \mathcal{N}}[\mathbb{1}\{z + (\sqrt{1 - \rho^2}/\rho)u \geq t\}] \\ &= \sum_{i=1}^m A_i \mathbf{E}_{u \sim \mathcal{N}}[\mathbb{1}\{\rho z + \sqrt{1 - \rho^2}u \geq \rho t_i\}] = \sum_{i=1}^m A_i T_\rho \phi(z; \rho t_i). \end{aligned}$$

We first provide explicit expressions for $T_\rho \Phi'(z/\rho)$ and $\mathbf{E}_{z \sim \mathcal{N}}[(T_\rho \Phi'(z/\rho))^2]$.

Lemma 67 *For any $\Phi(z) = \sum_{i=1}^m A_i \phi(z; t_i) + A_0 \in \mathcal{F}_M$, we have*

$$\begin{aligned} T_\rho \Phi'(z/\rho) &= \sum_{i=1}^m \frac{\rho A_i}{\sqrt{2\pi(1 - \rho^2)}} \exp\left(-\frac{\rho^2(z - t_i)^2}{2(1 - \rho^2)}\right), \text{ and} \\ \mathbf{E}_{z \sim \mathcal{N}}[(T_\rho \Phi'(z/\rho))^2] &= \sum_{i,j=1}^m \frac{\rho^2 A_i A_j}{2\pi\sqrt{1 - \rho^4}} \exp\left(-\frac{\rho^2(t_i^2 + t_j^2)}{2(1 - \rho^4)} + \frac{\rho^4 t_i t_j}{1 - \rho^4}\right). \end{aligned}$$

Proof The proof follows similar steps as the proof of [Lemma 65](#). Observe first that by the definition of $\Phi(z)$, the derivative of Φ equals

$$\Phi'(z) = \sum_{i=1}^m A_i \phi'(z; t_i) = \sum_{i=1}^m A_i \delta(z - t_i),$$

where δ is the Dirac delta function. As $\delta(u)$ satisfies $\delta(au) = \delta(u)/a$ for any real positive number a ,

$$\Phi'(z/\rho) = \sum_{i=1}^m A_i \delta((z - \rho t_i)/\rho) = \sum_{i=1}^m \rho A_i \delta(z - \rho t_i) = \rho \sum_{i=1}^m A_i \phi'(z; \rho t_i).$$

This implies that

$$\mathbf{T}_\rho \Phi'(z/\rho) = \rho \sum_{i=1}^m A_i \mathbf{T}_\rho \phi'(z; \rho t_i),$$

which leads to the first claim in the statement after combining with [Equation \(33\)](#). The second claim now follows from [Lemma 65](#), by replacing t_i, t_j with ρt_i and ρt_j . \blacksquare

We now show that the centered augmentation error can be bounded above by $\mathbf{E}_{z \sim \mathcal{N}}[(\mathbf{T}_\rho \Phi'(z/\rho))^2]$.

Lemma 68 *Let $\Phi \in \mathcal{F}_M$. Then, for any $\rho \in (0, 1)$,*

$$\mathbf{E}_{z \sim \mathcal{N}}[(\mathbf{T}_\rho \Phi(z/\rho) - \Phi(z))^2] \leq 4((1 - \rho^2)/\rho^2) \mathbf{E}_{z \sim \mathcal{N}}[(\mathbf{T}_\rho \Phi'(z/\rho))^2].$$

Proof Observe that after augmentation, the indicator function $\mathbf{1}\{z \geq t\} = \phi(z; t)$ becomes $\mathbf{T}_\rho \phi(z/\rho; t) = \mathbf{T}_\rho \phi(z; \rho t) = \mathbf{Pr}_{u \sim \mathcal{N}}[u \geq \rho(t - z)/\sqrt{1 - \rho^2}]$. Therefore, $\mathbf{T}_\rho \phi(z/\rho; t) - \phi(z; t)$ can be expressed as:

$$\mathbf{T}_\rho \phi(z/\rho; t) - \phi(z; t) = \begin{cases} \mathbf{Pr}_{u \sim \mathcal{N}}[u \geq \rho(t - z)/\sqrt{1 - \rho^2}] & z < t, \\ -\mathbf{Pr}_{u \sim \mathcal{N}}[u \leq \rho(t - z)/\sqrt{1 - \rho^2}] & z \geq t. \end{cases}$$

Hence, $\mathbf{E}_{z \sim \mathcal{N}}[(\mathbf{T}_\rho \Phi(z/\rho) - \Phi(z))^2]$ equals:

$$\begin{aligned} & \mathbf{E}_{z \sim \mathcal{N}}[(\mathbf{T}_\rho \Phi(z/\rho) - \Phi(z))^2] \\ &= \mathbf{E}_{z \sim \mathcal{N}} \left[\sum_{i,j=1}^m A_i A_j (\mathbf{T}_\rho \phi(z/\rho; t_i) - \phi(z; t_i)) (\mathbf{T}_\rho \phi(z/\rho; t_j) - \phi(z; t_j)) \right] \\ &= \sum_{i,j=1}^m \frac{A_i A_j}{\sqrt{2\pi}} \int_{-\infty}^{\min\{t_i, t_j\}} \mathbf{Pr}_{u \sim \mathcal{N}} \left[u \geq \frac{\rho(t_i - z)}{\sqrt{1 - \rho^2}} \right] \mathbf{Pr}_{u \sim \mathcal{N}} \left[u \geq \frac{\rho(t_j - z)}{\sqrt{1 - \rho^2}} \right] e^{-z^2/2} dz \\ & \quad - \sum_{i,j=1}^m \frac{A_i A_j}{\sqrt{2\pi}} \int_{\min\{t_i, t_j\}}^{\max\{t_i, t_j\}} \mathbf{Pr}_{u \sim \mathcal{N}} \left[u \geq \frac{\rho(\max\{t_i, t_j\} - z)}{\sqrt{1 - \rho^2}} \right] \mathbf{Pr}_{u \sim \mathcal{N}} \left[u \leq \frac{\rho(\min\{t_i, t_j\} - z)}{\sqrt{1 - \rho^2}} \right] e^{-z^2/2} dz \\ & \quad + \sum_{i,j=1}^m \frac{A_i A_j}{\sqrt{2\pi}} \int_{\max\{t_i, t_j\}}^{+\infty} \mathbf{Pr}_{u \sim \mathcal{N}} \left[u \leq \frac{\rho(t_i - z)}{\sqrt{1 - \rho^2}} \right] \mathbf{Pr}_{u \sim \mathcal{N}} \left[u \leq \frac{\rho(t_j - z)}{\sqrt{1 - \rho^2}} \right] e^{-z^2/2} dz. \end{aligned}$$

When $z \leq \min\{t_i, t_j\}$, since both $\rho(t_i - z)$ and $\rho(t_j - z)$ are positive, by standard Gaussian concentration,

$$\Pr_{u \sim \mathcal{N}} \left[u \geq \frac{\rho(t_i - z)}{\sqrt{1 - \rho^2}} \right] \leq \frac{1}{2} \exp \left(-\frac{\rho^2(t_i - z)^2}{2(1 - \rho^2)} \right), \quad \Pr_{u \sim \mathcal{N}} \left[u \geq \frac{\rho(t_j - z)}{\sqrt{1 - \rho^2}} \right] \leq \frac{1}{2} \exp \left(-\frac{\rho^2(t_j - z)^2}{2(1 - \rho^2)} \right).$$

The same inequalities hold for $\Pr[u \leq \rho(t_i - z)/\sqrt{1 - \rho^2}]$ and $\Pr[u \leq \rho(t_j - z)/\sqrt{1 - \rho^2}]$ when $z \geq \max\{t_i, t_j\}$. Thus, we can further upper bound $\mathbf{E}_{z \sim \mathcal{N}}[(T_\rho \Phi(z/\rho) - \Phi(z))^2]$ by

$$\begin{aligned} & \mathbf{E}_{z \sim \mathcal{N}}[(T_\rho \Phi(z/\rho) - \Phi(z))^2] \\ & \leq \sum_{i,j=1}^m \frac{A_i A_j}{\sqrt{2\pi}} \int_{-\infty}^{\min\{t_i, t_j\}} \Pr_{u \sim \mathcal{N}} \left[u \geq \frac{\rho(t_i - z)}{\sqrt{1 - \rho^2}} \right] \Pr_{u \sim \mathcal{N}} \left[u \geq \frac{\rho(t_j - z)}{\sqrt{1 - \rho^2}} \right] e^{-z^2/2} dz \\ & \quad + \sum_{i,j=1}^m \frac{A_i A_j}{\sqrt{2\pi}} \int_{\max\{t_i, t_j\}}^{+\infty} \Pr_{u \sim \mathcal{N}} \left[u \leq \frac{\rho(t_i - z)}{\sqrt{1 - \rho^2}} \right] \Pr_{u \sim \mathcal{N}} \left[u \leq \frac{\rho(t_j - z)}{\sqrt{1 - \rho^2}} \right] e^{-z^2/2} dz \\ & \leq \sum_{i,j=1}^m \frac{1}{2} \frac{A_i A_j}{\sqrt{2\pi}} \left(\int_{-\infty}^{\min\{t_i, t_j\}} + \int_{\max\{t_i, t_j\}}^{+\infty} \right) \exp \left(-\frac{\rho^2((t_i - z)^2 + (t_j - z)^2)}{2(1 - \rho^2)} - \frac{z^2}{2} \right) dz \\ & \leq \sum_{i,j=1}^m \frac{1}{2} \frac{A_i A_j}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp \left(-\frac{\rho^2((t_i - z)^2 + (t_j - z)^2)}{2(1 - \rho^2)} - \frac{z^2}{2} \right) dz \\ & = \sum_{i,j=1}^m \frac{1}{2} A_i A_j \sqrt{\frac{1 - \rho^2}{1 + \rho^2}} \exp \left(-\frac{\rho^2(t_i^2 + t_j^2)}{2(1 - \rho^4)} + \frac{\rho^4 t_i t_j}{1 - \rho^4} \right), \end{aligned}$$

where in the last inequality we used the definition of Gaussian pdf with variance $\frac{1 - \rho^2}{1 + \rho^2}$ and the fact that its integral over the real line is equal to one. Comparing with the expression for $\mathbf{E}_{z \sim \mathcal{N}}[(T_\rho \Phi'(z/\rho))^2]$ from Lemma 67, we immediately get the claimed bound on $\mathbf{E}_{z \sim \mathcal{N}}[(T_\rho \Phi'(z/\rho))^2]$. \blacksquare

Our next result shows that when ρ is close to 1, the centered augmentation $T_\rho \Phi(z/\rho)$ does not differ much from the uncentered augmentation $T_\rho \Phi(z)$, as stated below.

Lemma 69 *Let $\Phi \in \mathcal{F}_M$. Suppose $1 > \rho^2 \geq 1 - C/M^2$ for an absolute constant $C \in (0, M^2/2]$. Then:*

$$\mathbf{E}_{z \sim \mathcal{N}}[(T_\rho \Phi(z) - T_\rho \Phi(z/\rho))^2] \leq C'(1 - \rho^2)(\|T_\rho \Phi'(z/\rho)\|_{L_2}^2 + \|T_\rho \Phi'(z)\|_{L_2}^2),$$

where C' is an absolute constant.

Proof We first observe that since T_ρ is a linear operator on functionals, we have $T_\rho \Phi(z) - T_\rho \Phi(z/\rho) = T_\rho(\Phi(z) - \Phi(z/\rho))$. Given a staircase function $\Phi \in \mathcal{F}_M$, $\Phi(z) = \sum_{i=1}^m A_i \phi(z; t_i) + A_0$, let $I_+ = \{i : t_i > 0\}$ and $I_- = \{i : t_i < 0\}$. Expressing $T_\rho(\Phi(z) - \Phi(z/\rho))$ in terms of the sum of

indicator functions we get

$$\begin{aligned}
 |\mathbb{T}_\rho(\Phi(z) - \Phi(z/\rho))| &= \left| \mathbb{T}_\rho \left(\sum_{i=1}^m A_i(\phi(z; t_i) - \phi(z/\rho; t_i)) \right) \right| \\
 &= \left| \mathbb{T}_\rho \left(\sum_{i=1}^m A_i(-\mathbb{1}\{\rho t_i \leq z \leq t_i, t_i \geq 0\} + \mathbb{1}\{t_i \leq z \leq \rho t_i, t_i \leq 0\}) \right) \right| \\
 &\leq \sum_{i=1}^m A_i \left| \mathbb{T}_\rho \left(-\mathbb{1}\{\rho t_i \leq z \leq t_i, t_i \geq 0\} + \mathbb{1}\{t_i \leq z \leq \rho t_i, t_i \leq 0\} \right) \right| \\
 &= \sum_{i \in I_+} A_i \mathbb{T}_\rho(\mathbb{1}\{\rho t_i \leq z \leq t_i\}) + \sum_{i \in I_-} A_i \mathbb{T}_\rho(\mathbb{1}\{t_i \leq z \leq \rho t_i\})
 \end{aligned}$$

Suppose first $t_i \geq 0$. Then by the definition of Ornstein–Uhlenbeck semigroup, we have

$$g_i(z) := \mathbb{T}_\rho(\mathbb{1}\{\rho t_i \leq z \leq t_i\}) = \mathbf{E}_{u \sim \mathcal{N}}[\mathbb{1}\{\rho t_i \leq \rho z + \sqrt{1-\rho^2}u \leq t_i\}] = \int_{\rho(t_i-z)/\sqrt{1-\rho^2}}^{(t_i-\rho z)/\sqrt{1-\rho^2}} \frac{e^{-u^2/2}}{\sqrt{2\pi}} du.$$

When $z \leq t_i$ or $z \geq t_i/\rho$, $t_i - \rho z$ and $\rho(t_i - z)$ are both positive or negative, therefore, when $z \in (-\infty, t_i] \cup [t_i/\rho, +\infty)$, the function $g_i(z)$ can be bounded by

$$\begin{aligned}
 g_i(z) &= \int_{\rho(t_i-z)/\sqrt{1-\rho^2}}^{(t_i-\rho z)/\sqrt{1-\rho^2}} \frac{e^{-u^2/2}}{\sqrt{2\pi}} du \\
 &\leq \frac{1}{\sqrt{2\pi}} \left(\frac{t_i - \rho z}{\sqrt{1-\rho^2}} - \frac{\rho(t_i - \rho z)}{\sqrt{1-\rho^2}} \right) \exp \left(-\frac{1}{2} \min \left\{ \frac{(t_i - \rho z)^2}{1-\rho^2}, \frac{\rho^2(t_i - z)^2}{1-\rho^2} \right\} \right) \\
 &\leq \frac{(1-\rho)t_i}{\sqrt{2\pi}(1-\rho^2)} \left(\exp \left(-\frac{(t_i - \rho z)^2}{2(1-\rho^2)} \right) + \exp \left(-\frac{\rho^2(t_i - z)^2}{2(1-\rho^2)} \right) \right).
 \end{aligned}$$

Comparing the right-hand side of the inequality above with the expressions for $\mathbb{T}_\rho \phi'(z; t_i)$ and $\mathbb{T}_\rho \phi'(z; \rho t_i)$ displayed in [Equation \(33\)](#) and [Lemma 67](#),

$$\begin{aligned}
 \mathbb{T}_\rho \phi'(z; t_i) &= \frac{1}{\sqrt{2\pi}(1-\rho^2)} \exp \left(-\frac{(\rho z - t_i)^2}{2(1-\rho^2)} \right) \\
 \mathbb{T}_\rho \phi'(z; \rho t_i) &= \frac{1}{\sqrt{2\pi}(1-\rho^2)} \exp \left(-\frac{\rho^2(z - t_i)^2}{2(1-\rho^2)} \right)
 \end{aligned}$$

we obtain that

$$g_i(z) \mathbb{1}\{z \leq t_i \text{ or } z \geq t_i/\rho\} \leq (1-\rho)t_i(\mathbb{T}_\rho \phi'(z; t_i) + \mathbb{T}_\rho \phi'(z; \rho t_i)).$$

On the other hand, when $z \in [t_i, t_i/\rho]$, since $0 \in [\rho(t_i - z)/\sqrt{1-\rho^2}, (t_i - \rho z)/\sqrt{1-\rho^2}]$ we can bound $g_i(z)$ above by

$$g(z) \leq \int_{\rho(t_i-z)/\sqrt{1-\rho^2}}^{(t_i-\rho z)/\sqrt{1-\rho^2}} \frac{1}{\sqrt{2\pi}} du \leq \frac{(1-\rho)t_i}{\sqrt{2\pi}(1-\rho^2)}.$$

Thus, in summary, $g_i(z)$ is bounded above by

$$\begin{aligned} g_i(z) &= g_i(z) \mathbb{1}\{z \leq t_i \text{ or } z \geq t_i/\rho\} + g_i(z) \mathbb{1}\{t_i \leq z \leq t_i/\rho\} \\ &\leq (1-\rho)t_i(\mathsf{T}_\rho\phi'(z; t_i) + \mathsf{T}_\rho\phi'(z; \rho t_i)) + \frac{(1-\rho)t_i}{\sqrt{2\pi(1-\rho^2)}} \mathbb{1}\{t_i \leq z \leq t_i/\rho\}. \end{aligned}$$

Similarly, for $i \in I_-$, with the same arguments we obtain that

$$g_i(z) \leq (1-\rho)|t_i|(\mathsf{T}_\rho\phi'(z; t_i) + \mathsf{T}_\rho\phi'(z; \rho t_i)) + \frac{(1-\rho)|t_i|}{\sqrt{2\pi(1-\rho^2)}} \mathbb{1}\{t_i/\rho \leq z \leq t_i\}.$$

Therefore, the L_2^2 difference between $\mathsf{T}_\rho\Phi(z)$ and $\mathsf{T}_\rho\Phi(z/\rho)$ can be bounded by (note that $A_i, g_i(z) > 0$ for all $i \in [m]$)

$$\begin{aligned} \mathbf{E}_{z \sim \mathcal{N}}[(\mathsf{T}_\rho\Phi(z) - \mathsf{T}_\rho\Phi(z/\rho))^2] &= \mathbf{E}_{z \sim \mathcal{N}} \left[\left(\sum_{i=1}^m A_i g_i(z) \right)^2 \right] \\ &\leq \mathbf{E}_{z \sim \mathcal{N}} \left[\left(\sum_{i=1}^m A_i (1-\rho) |t_i| (\mathsf{T}_\rho\phi'(z; t_i) + \mathsf{T}_\rho\phi'(z; \rho t_i)) \right. \right. \\ &\quad \left. \left. + \frac{A_i (1-\rho) |t_i|}{\sqrt{2\pi(1-\rho^2)}} \sum_{i=1}^m (\mathbb{1}\{z \in [t_i, t_i/\rho]\} + \mathbb{1}\{z \in [t_i/\rho, t_i]\}) \right)^2 \right] \\ &\leq 2 \underbrace{\mathbf{E}_{z \sim \mathcal{N}} \left[\left(\sum_{i=1}^m A_i (1-\rho) |t_i| (\mathsf{T}_\rho\phi'(z; t_i) + \mathsf{T}_\rho\phi'(z; \rho t_i)) \right)^2 \right]}_{(Q_1)} \\ &\quad + 2 \underbrace{\mathbf{E}_{z \sim \mathcal{N}} \left[\left(\frac{(1-\rho)}{\sqrt{2\pi(1-\rho^2)}} \sum_{i=1}^m A_i |t_i| (\mathbb{1}\{z \in [t_i, t_i/\rho]\} + \mathbb{1}\{z \in [t_i/\rho, t_i]\}) \right)^2 \right]}_{(Q_2)}. \quad (35) \end{aligned}$$

Note that in (Q_2) above, we used the convention that if $a > b$ then $[a, b] = \emptyset$ and $\mathbb{1}\{z \in \emptyset\} = 0$. For (Q_1) , using Young's inequality again yields:

$$\begin{aligned} (Q_1) &\leq 2 \mathbf{E}_{z \sim \mathcal{N}} \left[\left(\sum_{i=1}^m A_i (1-\rho) |t_i| \mathsf{T}_\rho\phi'(z; t_i) \right)^2 \right] + 2 \mathbf{E}_{z \sim \mathcal{N}} \left[\left(\sum_{i=1}^m A_i (1-\rho) |t_i| \mathsf{T}_\rho\phi'(z; \rho t_i) \right)^2 \right] \\ &\leq 2(1-\rho)^2 \left(\max_{i \in [m]} \{|t_i|\} \right)^2 \left(\mathbf{E}_{z \sim \mathcal{N}} \left[\left(\sum_{i=1}^m A_i \mathsf{T}_\rho\phi'(z; t_i) \right)^2 \right] + \mathbf{E}_{z \sim \mathcal{N}} \left[\left(\sum_{i=1}^m A_i \mathsf{T}_\rho\phi'(z; \rho t_i) \right)^2 \right] \right) \\ &\leq 2(1-\rho)^2 M^2 (\|\mathsf{T}_\rho\Phi'(z)\|_{L_2}^2 + \|\mathsf{T}_\rho\Phi'(z/\rho)\|_{L_2}^2), \end{aligned}$$

where in the last inequality, we use the fact that since $\Phi \in \mathcal{F}_M$, we have $|t_i| \leq M$ for all $i \in [m]$. Now, by our assumption, $\rho^2 \geq 1 - C/M^2$, therefore, $(1-\rho)(1+\rho) \leq C/M^2$ and hence $1-\rho \leq C/M^2$, which implies

$$(Q_1) \leq 2C(1-\rho)(\|\mathsf{T}_\rho\Phi'(z)\|_{L_2}^2 + \|\mathsf{T}_\rho\Phi'(z/\rho)\|_{L_2}^2).$$

For (Q_2) , since $|t_i| \leq M$ for all $i \in [m]$ and $1 - \rho \leq C/M^2$, expanding the square yields

$$\begin{aligned} (Q_2) &\leq \frac{M^2(1-\rho)}{2\pi(1+\rho)} \mathbf{E}_{z \sim \mathcal{N}} \left[\left(\sum_{i=1}^m A_i (\mathbb{1}\{z \in [t_i, t_i/\rho]\} + \mathbb{1}\{z \in [t_i/\rho, t_i]\}) \right)^2 \right] \\ &\leq \frac{C}{4\pi} \mathbf{E}_{z \sim \mathcal{N}} \left[\sum_{i,j \in I_+} A_i A_j \mathbb{1}\{z \in [t_i, t_i/\rho] \cap [t_j, t_j/\rho]\} + \sum_{i,j \in I_-} A_i A_j \mathbb{1}\{z \in [t_i/\rho, t_i] \cap [t_j/\rho, t_j]\} \right] \end{aligned}$$

By the symmetry of Gaussian distribution, we have

$$\mathbf{Pr}[z \in [t_i/\rho, t_i] \cap [t_j/\rho, t_j], t_i, t_j \leq 0] = \mathbf{Pr}[z \in [|t_i|, |t_i|/\rho] \cap [|t_j|, |t_j|/\rho]],$$

therefore, it suffices to discuss only the case where $t_i, t_j \in I_+$. Suppose without loss of generality that $0 < t_i \leq t_j$. Observe that $\mathbf{E}_{z \sim \mathcal{N}}[\mathbb{1}\{z \in [t_i, t_i/\rho] \cap [t_j, t_j/\rho]\}] \neq 0$ if and only if $0 < t_i \leq t_j < t_i/\rho \leq t_j\rho$, therefore, the expectation of the indicator is bounded by:

$$\begin{aligned} \mathbf{E}_{z \sim \mathcal{N}}[\mathbb{1}\{z \in [t_i, t_i/\rho] \cap [t_j, t_j/\rho]\}] &= \mathbf{Pr}[z \in [t_j, t_i/\rho]] = \int_{t_j}^{t_i/\rho} \frac{e^{-u^2/2}}{\sqrt{2\pi}} du \\ &\leq \frac{\exp(-t_j^2/2)}{\sqrt{2\pi}} (t_i/\rho - t_j) \leq \frac{(1-\rho)t_i}{\rho\sqrt{2\pi}} \exp\left(-\frac{t_j^2}{2}\right). \end{aligned} \quad (36)$$

Recall that in [Lemma 67](#), we proved

$$\mathbf{E}_{z \sim \mathcal{N}}[(T_\rho \Phi'(z/\rho))^2] = \sum_{i,j=1}^m \frac{\rho^2 A_i A_j}{2\pi\sqrt{1-\rho^4}} \exp\left(-\frac{\rho^2(t_i^2 + t_j^2)}{2(1-\rho^4)} + \frac{\rho^4 t_i t_j}{1-\rho^4}\right),$$

hence our strategy is to show that:

$$\exp\left(-\frac{t_j^2}{2}\right) \leq \exp\left(-\frac{\rho^2(t_i^2 + t_j^2)}{2(1-\rho^4)} + \frac{\rho^4 t_i t_j}{1-\rho^4}\right), \text{ for } 0 < t_i \leq t_j < t_i/\rho \leq t_j/\rho. \quad (37)$$

We show:

Claim 70 *Let $t_i, t_j > 0$ satisfy $t_i \leq t_j \leq t_i/\rho$. Then, for any $\rho \in (0, 1)$, it holds*

$$-\frac{t_j^2}{2} \leq -\frac{\rho^2(t_i^2 + t_j^2)}{2(1-\rho^4)} + \frac{\rho^4 t_i t_j}{1-\rho^4}.$$

The proof of [Claim 70](#) is deferred to [Appendix F.2.3](#). Therefore, for each $t_i, t_j, i, j \in [m]$, the expectation in [Equation \(36\)](#) is bounded above by

$$\begin{aligned} &\mathbf{E}_{z \sim \mathcal{N}}[\mathbb{1}\{z \in [t_i, t_i/\rho] \cap [t_j, t_j/\rho], t_i t_j > 0\}] \\ &\leq \frac{(1-\rho)\sqrt{2\pi(1-\rho^4)}|t_i|}{\rho^3} \frac{\rho^2}{2\pi\sqrt{1-\rho^4}} \exp\left(-\frac{\rho^2(t_i^2 + t_j^2)}{2(1-\rho^4)} + \frac{\rho^4 t_i t_j}{1-\rho^4}\right), \end{aligned}$$

which, combining with the fact that $\sqrt{1-\rho} \leq \sqrt{C}/M$ and $|t_i| \leq M$, yields

$$\begin{aligned}
 (Q_2) &\leq \frac{C}{4\pi} \left(\sum_{i,j \in I_+} A_i A_j \mathbf{E}_{z \sim \mathcal{N}} [\mathbb{1}\{z \in [t_i, t_i/\rho] \cap [t_j, t_j/\rho]\}] \right. \\
 &\quad \left. + \sum_{i,j \in I_-} A_i A_j \mathbf{E}_{z \sim \mathcal{N}} [\mathbb{1}\{z \in [t_i/\rho, t_i] \cap [t_j\rho, t_j]\}] \right) \\
 &\leq \sum_{i,j=1}^m \frac{C(1-\rho)\sqrt{2\pi(1-\rho^4)}|t_i|}{4\pi\rho^3} \frac{\rho^2}{2\pi\sqrt{1-\rho^4}} \exp\left(-\frac{\rho^2(t_i^2+t_j^2)}{2(1-\rho^4)} + \frac{\rho^4 t_i t_j}{1-\rho^4}\right) \\
 &\leq C'(1-\rho) \|\mathbf{T}_\rho \Phi'(z/\rho)\|_{L_2}^2.
 \end{aligned}$$

Plugging the bounds on (Q_1) , (Q_2) back to [Equation \(35\)](#), we finally obtain:

$$\mathbf{E}_{z \sim \mathcal{N}} [(\mathbf{T}_\rho \Phi(z) - \mathbf{T}_\rho \Phi(z/\rho))^2] \leq C''(1-\rho)(\|\mathbf{T}_\rho \Phi'(z/\rho)\|_{L_2}^2 + \|\mathbf{T}_\rho \sigma'(z)\|_{L_2}^2).$$

Since $1-\rho \leq 1-\rho^2$, we complete the proof of [Lemma 69](#). ■

Our last step is to show that $\mathbf{E}_{z \sim \mathcal{N}} [(\mathbf{T}_\rho \Phi'(z/\rho))^2]$ is not much larger than $\mathbf{E}_{z \sim \mathcal{N}} [(\mathbf{T}_\rho \Phi'(z))^2]$ when ρ is close to 1.

Lemma 71 *Let $\Phi \in \mathcal{F}_M$ be any staircase function that is constructed from m indicator functions with thresholds t_i , $i \in [m]$, and suppose that $|t_i| \leq M$, for all $i \in [m]$, where $1 < M < +\infty$. For any $\rho \in (0, 1)$ such that $\rho^2 \geq 1 - C/M^2$ where $C < M^2$ is an absolute constant, let $\rho_1 = \sqrt{\rho^2 + C(1-\rho^2)/M^2}$. Then,*

$$\mathbf{E}_{z \sim \mathcal{N}} [(\mathbf{T}_{\rho_1} \Phi'(z/\rho_1))^2] \leq 2e^C \mathbf{E}_{z \sim \mathcal{N}} [(\mathbf{T}_\rho \Phi'(z))^2].$$

Proof Observe first that $1-\rho_1^2 = (1-\rho^2)(1-C/M^2) \in (0, 1)$, hence $\rho_1 \in (0, 1)$ and $\mathbf{T}_{\rho_1} \Phi'(z/\rho_1)$ is well-defined. To proceed, we compare each term of $\mathbf{E}_{z \sim \mathcal{N}} [(\mathbf{T}_{\rho_1} \Phi'(z/\rho_1))^2]$ and $\mathbf{E}_{z \sim \mathcal{N}} [(\mathbf{T}_\rho \Phi'(z))^2]$ that are given in [Lemma 67](#) and [Lemma 65](#) separately.

Since $\rho_1^2/(1-\rho_1^4)$ appears in the exponential terms of $\mathbf{E}_{z \sim \mathcal{N}} [(\mathbf{T}_{\rho_1} \Phi'(z/\rho_1))^2]$ while the coefficient in the exponential terms of $\mathbf{E}_{z \sim \mathcal{N}} [(\mathbf{T}_\rho \Phi'(z))^2]$ is $1/(1-\rho^4)$, we first need to compare these two factors. The proof of [Claim 72](#) is deferred to [Appendix F.2.3](#).

Claim 72 *Let $\rho_1^2 = \rho^2 + C(1-\rho^2)/M^2$. If $1 > \rho^2 \geq 1 - C/M^2$, then $\rho_1^2/(1-\rho_1^4) \geq 1/(1-\rho^4)$.*

Observe that for any $t_i, t_j \in \mathbb{R}$ and $\rho \in (0, 1)$, we have $t_i^2 + t_j^2 - 2\rho_1^2 t_i t_j \geq (1 - \rho^2)(t_i^2 + t_j^2) \geq 0$, and recalling the expression for $\mathbf{E}_{z \sim \mathcal{N}}[(T_{\rho_1} \Phi'(z/\rho_1))^2]$ given in [Lemma 67](#), we thus obtain

$$\begin{aligned}
 \mathbf{E}_{z \sim \mathcal{N}}[(T_{\rho_1} \Phi'(z/\rho_1))^2] &= \sum_{i,j=1}^m \frac{A_i A_j}{2\pi} \sqrt{\frac{\rho_1^4}{1 - \rho_1^4}} \exp\left(-\frac{\rho_1^2(t_i^2 + t_j^2 - 2\rho_1^2 t_i t_j)}{2(1 - \rho_1^4)}\right) \\
 &\stackrel{(i)}{\leq} \sum_{i,j=1}^m \frac{A_i A_j}{2\pi} \sqrt{\frac{\rho_1^4}{1 - \rho_1^4}} \exp\left(-\frac{t_i^2 + t_j^2}{2(1 - \rho^4)} + \frac{\rho_1^2 t_i t_j}{1 - \rho^4}\right) \\
 &\stackrel{(ii)}{=} \sum_{i,j=1}^m \frac{A_i A_j}{2\pi} \sqrt{\frac{\rho_1^4}{1 - \rho_1^4}} \exp\left(-\frac{t_i^2 + t_j^2}{2(1 - \rho^4)} + \frac{\rho^2 t_i t_j}{1 - \rho^4} + \frac{C(1 - \rho^2)t_i t_j}{(1 - \rho^4)M^2}\right) \\
 &= \sum_{i,j=1}^m \frac{A_i A_j}{2\pi} \sqrt{\frac{\rho_1^4}{1 - \rho_1^4}} \exp\left(-\frac{t_i^2 + t_j^2}{2(1 - \rho^4)} + \frac{\rho^2 t_i t_j}{1 - \rho^4}\right) \exp\left(\frac{C t_i t_j}{(1 + \rho^2)M^2}\right),
 \end{aligned} \tag{38}$$

where in (i) we plugged in [Claim 72](#) and in (ii) we used the definition that $\rho_1^2 = \rho^2 + C(1 - \rho^2)/M^2$.

Since M is an upper bound on $|t_i|, i \in [m]$, we can assume without loss of generality that $M^2 \geq 2C$. We next observe that when $M^2 \geq 2C$, we have the following inequality, whose proof is relocated to [Appendix F.2.3](#).

Claim 73 *If $M^2 \geq 2C$ then $\rho_1^4/(1 - \rho_1^4) \leq 4/(1 - \rho^4)$.*

Thus, plugging in [Claim 73](#) into [Equation \(38\)](#), and recalling that it is assumed $|t_i|^2 \leq M^2$ for any $i \in [m]$, we further get

$$\begin{aligned}
 \mathbf{E}_{z \sim \mathcal{N}}[(T_{\rho_1} \Phi'(z/\rho_1))^2] &\leq \sum_{i,j=1}^m \frac{A_i A_j}{\pi} \frac{1}{\sqrt{1 - \rho^4}} \exp\left(-\frac{t_i^2 + t_j^2}{2(1 - \rho^4)} + \frac{\rho^2 t_i t_j}{1 - \rho^4}\right) \exp\left(\frac{C t_i t_j}{(1 + \rho^2)M^2}\right) \\
 &\leq 2e^C \sum_{i,j=1}^m \frac{A_i A_j}{2\pi \sqrt{1 - \rho^4}} \exp\left(-\frac{t_i^2 + t_j^2}{2(1 - \rho^4)} + \frac{\rho^2 t_i t_j}{1 - \rho^4}\right) \\
 &= 2e^C \mathbf{E}_{z \sim \mathcal{N}}[(T_{\rho} \Phi'(z))^2],
 \end{aligned}$$

which completes the proof. ■

F.2.2. PROOF OF [PROPOSITION 64](#)

We can now restate [Proposition 64](#) and present its proof.

Proposition 64 *Let $\Phi \in \mathcal{F}_M$ be any staircase function that consists of m indicator functions with thresholds $t_i, i \in [m]$, and suppose $|t_i| \leq M$ for all $i \in [m]$, where $1 < M < +\infty$. For any $\rho \in (0, 1)$ such that $\rho^2 \geq 1 - C/M^2$ where $C < M^2/4$ is an absolute constant, we have*

$$\mathbf{E}_{z \sim \mathcal{N}}[(T_{\rho} \Phi(z) - \Phi(z))^2] \lesssim (1 - \rho^2) \mathbf{E}_{z \sim \mathcal{N}}[(T_{\rho} \Phi'(z))^2].$$

Proof of Proposition 64 Observe that $\mathbf{E}_{z \sim \mathcal{N}}[(T_\rho \Phi(z) - \Phi(z))^2]$ can be bounded as

$$\mathbf{E}_{z \sim \mathcal{N}}[(T_\rho \Phi(z) - \Phi(z))^2] \leq 2 \mathbf{E}_{z \sim \mathcal{N}}[(T_\rho \Phi(z) - T_{\rho_1} \Phi(z/\rho_1))^2] + 2 \mathbf{E}_{z \sim \mathcal{N}}[(T_{\rho_1} \Phi(z/\rho_1) - \Phi(z))^2]. \quad (39)$$

We first bound the second term in Equation (39). Since we have assumed that $\rho^2 \geq 1 - C/M^2$, where C is an absolute constant, using Lemma 68 and Lemma 71 and plugging in $\rho_1 = \sqrt{\rho^2 + C(1 - \rho^2)/M^2}$, we get

$$\begin{aligned} \mathbf{E}_{z \sim \mathcal{N}}[(T_{\rho_1} \Phi(z/\rho_1) - \Phi(z))^2] &\leq \frac{4(1 - \rho_1^2)}{\rho_1^2} \mathbf{E}_{z \sim \mathcal{N}}[(T_{\rho_1} \Phi'(z/\rho_1))^2] \\ &\leq \frac{8e^C(1 - \rho^2 - C(1 - \rho^2)/M^2)}{(\rho^2 + C(1 - \rho^2)/M^2)} \mathbf{E}_{z \sim \mathcal{N}}[(T_\rho \Phi'(z))^2] \\ &\lesssim e^C(1 - \rho^2) \mathbf{E}_{z \sim \mathcal{N}}[(T_\rho \Phi'(z))^2]. \end{aligned} \quad (40)$$

Now we turn to bounding the first term in Equation (39). First, we add and subtract $T_{\rho_1} \Phi'(z)$ in the squared parentheses to obtain:

$$\mathbf{E}_{z \sim \mathcal{N}}[(T_\rho \Phi(z) - T_{\rho_1} \Phi(z/\rho_1))^2] \leq 2 \underbrace{\mathbf{E}_{z \sim \mathcal{N}}[(T_\rho \Phi(z) - T_{\rho_1} \Phi(z))^2]}_{Q_1} + 2 \underbrace{\mathbf{E}_{z \sim \mathcal{N}}[(T_{\rho_1} \Phi(z) - T_{\rho_1} \Phi(z/\rho_1))^2]}_{Q_2}. \quad (41)$$

For Q_1 , observe that since $\rho < \rho_1 < 1$, using the property of Ornstein–Uhlenbeck semigroup presented in Fact 21, we have $T_\rho \Phi(z) = T_{\rho_1(\rho/\rho_1)} \Phi(z) = T_{\rho/\rho_1}(T_{\rho_1} \Phi(z))$. Therefore, using Lemma 24 with $f(z) = T_{\rho_1} \Phi(z)$ we have

$$\begin{aligned} \mathbf{E}_{z \sim \mathcal{N}}[(T_{\rho/\rho_1}(T_{\rho_1} \Phi(z)) - T_{\rho_1} \Phi(z))^2] &\leq 3(1 - \rho/\rho_1) \mathbf{E}_{z \sim \mathcal{N}} \left[\left(\frac{d}{dz} T_{\rho_1} \Phi(z) \right)^2 \right] \\ &\stackrel{(i)}{=} \frac{3(\rho_1 - \rho)}{\rho_1} \rho_1^2 \mathbf{E}_{z \sim \mathcal{N}}[(T_{\rho_1} \Phi'(z))^2] \\ &= \frac{3(\rho_1^2 - \rho^2)\rho_1}{\rho_1 + \rho} \mathbf{E}_{z \sim \mathcal{N}}[(T_{\rho_1} \Phi'(z))^2] \\ &\stackrel{(ii)}{=} \frac{3C(1 - \rho^2)\rho_1}{M^2(\rho_1 + \rho)} \mathbf{E}_{z \sim \mathcal{N}}[(T_{\rho_1} \Phi'(z))^2], \end{aligned} \quad (42)$$

note that in (i) we applied Fact 21, Part 2(g), and in (ii) we brought in the definition of $\rho_1^2 = \rho^2 + C(1 - \rho^2)/M^2$. It remains to bound $\mathbf{E}_{z \sim \mathcal{N}}[(T_{\rho_1} \Phi'(z))^2]$ above by $\mathbf{E}_{z \sim \mathcal{N}}[(T_\rho \Phi'(z))^2]$. The proof of Claim 74 is deferred to Appendix F.2.3.

Claim 74 Let $\rho^2 \geq 1 - C/M^2$ and $\rho_1^2 = \rho^2 + C(1 - \rho^2)/M^2$. Then, $\|T_{\rho_1} \Phi'(z)\|_{L_2}^2 \leq e^C \|T_\rho \Phi'(z)\|_{L_2}^2$.

Plugging the upper bound on $\mathbf{E}_{z \sim \mathcal{N}}[(T_{\rho_1} \Phi'(z))^2]$ from Claim 74 back into Equation (42) yields

$$Q_1 = 2 \mathbf{E}_{z \sim \mathcal{N}}[(T_{\rho/\rho_1}(T_{\rho_1} \Phi(z)) - T_{\rho_1} \Phi(z))^2] \leq \frac{C(1 - \rho^2)\rho_1}{M^2(\rho_1 + \rho)} 4e^C \mathbf{E}_{z \sim \mathcal{N}}[(T_\rho \Phi'(z))^2].$$

Since $C/M^2 \leq 1/4$ we have $1 > \rho_1 > \rho \geq 1/2$, thus we finally get

$$Q_1 \leq 2e^C(1 - \rho^2)\|\mathsf{T}_\rho\Phi'\|_{L_2}^2.$$

We now turn to bounding the term Q_2 in Equation (40). Applying Lemma 69 with ρ_1 , we obtain

$$Q_2 \leq C''(1 - \rho^2)(\|\mathsf{T}_{\rho_1}\Phi'(z)\|_{L_2}^2 + \|\mathsf{T}_{\rho_1}\Phi'(z/\rho_1)\|_{L_2}^2).$$

Applying Claim 74 and Lemma 71 again, we obtain

$$Q_2 \leq 4C''(1 - \rho^2)e^C\|\mathsf{T}_\rho\Phi'\|_{L_2}^2.$$

Plugging the upper bounds on Q_1 and Q_2 back into Equation (41) yields $\mathbf{E}_{z \sim \mathcal{N}}[(\mathsf{T}_\rho\Phi(z) - \mathsf{T}_{\rho_1}\Phi(z/\rho_1))^2] \lesssim e^C(1 - \rho^2)\|\mathsf{T}_\rho\Phi'(z)\|_{L_2}^2$. Finally, combining with Equation (40), we obtain

$$\mathbf{E}_{z \sim \mathcal{N}}[(\mathsf{T}_\rho\Phi(z) - \Phi(z))^2] \lesssim e^C(1 - \rho^2)\|\mathsf{T}_\rho\Phi'\|_{L_2}^2.$$

Since e^C is an absolute constant, this completes the proof of Proposition 64. \blacksquare

F.2.3. PROOF OF SUPPLEMENTARY CLAIMS

Below, we provide proofs for the supplementary claims appeared in Appendix F.

Claim 70 *Let $t_i, t_j > 0$ satisfy $t_i \leq t_j \leq t_i/\rho$. Then, for any $\rho \in (0, 1)$, it holds*

$$-\frac{t_j^2}{2} \leq -\frac{\rho^2(t_i^2 + t_j^2)}{2(1 - \rho^4)} + \frac{\rho^4 t_i t_j}{1 - \rho^4}.$$

Proof of Claim 70 Simple algebraic calculation yields

$$\begin{aligned} -\frac{t_j^2}{2} + \frac{\rho^2(t_i^2 + t_j^2)}{2(1 - \rho^4)} - \frac{\rho^4 t_i t_j}{1 - \rho^4} &= \frac{1}{2(1 - \rho^4)}(-t_j^2 + \rho^2 t_i^2 + \rho^2 t_j^2 + \rho^4 t_j^2 - 2\rho^4 t_i t_j) \\ &= \frac{1}{2(1 - \rho^4)}(-(1 - \rho^2)t_j^2 + \rho^2 t_i^2 + \rho^4(t_j - t_i)^2 - \rho^4 t_i^2) \\ &= \frac{1}{2(1 - \rho^4)}(-(1 - \rho^2)t_j^2 + \rho^2(1 - \rho^2)t_i^2 + \rho^4(t_j - t_i)^2). \end{aligned}$$

Now since $0 < t_j - t_i < t_i/\rho - t_i = (1 - \rho)t_i/\rho$, we further have

$$\begin{aligned} &-\frac{t_j^2}{2} + \frac{\rho^2(t_i^2 + t_j^2)}{2(1 - \rho^4)} - \frac{\rho^4 t_i t_j}{1 - \rho^4} \\ &\leq \frac{1}{2(1 - \rho^4)}(-(1 - \rho^2)t_j^2 + \rho^2(1 - \rho^2)t_i^2 + \rho^2(1 - \rho)^2 t_i^2) \\ &\leq \frac{1}{2(1 - \rho^4)}(-(1 - \rho^2) + \rho^2(1 - \rho^2) + \rho^2(1 - \rho)^2)t_j^2 \\ &= \frac{t_j^2}{2(1 + \rho)(1 + \rho^2)}(-(1 + \rho) + \rho^2(1 + \rho) + \rho^2(1 - \rho)) = \frac{-t_j^2(1 - \rho)(1 + 2\rho)}{2(1 + \rho)(1 + \rho^2)} < 0. \end{aligned}$$

Thus, indeed we have $-t_j^2/2 \leq -\rho^2(t_i^2 + t_j^2 - 2\rho^2 t_i t_j)/(2(1 - \rho^4))$. \blacksquare

Claim 72 Let $\rho_1^2 = \rho^2 + C(1 - \rho^2)/M^2$. If $1 > \rho^2 \geq 1 - C/M^2$, then $\rho_1^2/(1 - \rho_1^4) \geq 1/(1 - \rho^4)$.

Proof of Claim 72 Since $\rho_1, \rho < 1$, we only need to show that $\rho_1^2(1 - \rho^4) \geq 1 - \rho_1^4$. Plugging in the value of ρ_1 , we have

$$\begin{aligned} \rho_1^2(1 - \rho^4) &= (\rho^2 + C(1 - \rho^2)/M^2)(1 - \rho^4) = (\rho^2 + C(1 - \rho^2)/M^2)(1 - \rho^2)(1 + \rho^2); \\ 1 - \rho_1^4 &= 1 - (\rho^2 + C(1 - \rho^2)/M^2)^2 = (1 + \rho^2 + C(1 - \rho^2)/M^2)(1 - \rho^2)(1 - C/M^2). \end{aligned}$$

Therefore, our goal is to prove that

$$(\rho^2 + C(1 - \rho^2)/M^2)(1 - \rho^2)(1 + \rho^2) \geq (1 + \rho^2 + C(1 - \rho^2)/M^2)(1 - \rho^2)(1 - C/M^2).$$

Dividing both sides of the inequality above by $1 - \rho^2 > 0$ yields that it is sufficient to show the following inequality:

$$\rho^2 + C(1 - \rho^2)/M^2 + (\rho^2 + C(1 - \rho^2)/M^2)\rho^2 \geq (1 - C/M^2) + (\rho^2 + C(1 - \rho^2)/M^2)(1 - C/M^2).$$

Since $\rho^2 \geq 1 - C/M^2$, we have $\rho^2 + C(1 - \rho^2)/M^2 \geq 1 - C/M^2$ and

$$(\rho^2 + C(1 - \rho^2)/M^2)\rho^2 \geq (\rho^2 + C(1 - \rho^2)/M^2)(1 - C/M^2).$$

Thus, it holds that $\rho_1^2(1 - \rho^2) \geq 1 - \rho_1^4$. ■

Claim 73 If $M^2 \geq 2C$ then $\rho_1^4/(1 - \rho_1^4) \leq 4/(1 - \rho^4)$.

Proof of Claim 73 For any fixed $\rho \in (0, 1)$, let us define

$$h(M) = \frac{(\rho^2 + C(1 - \rho^2)/M^2)(1 - \rho^4)}{1 - (\rho^2 + C(1 - \rho^2)/M^2)^2}.$$

It is easy to see that $h(M)$ is a decreasing function with respect to $M > 0$, therefore, for any fixed $\rho \in (0, 1)$ and any $M^2 \geq 2C$, we have

$$h(M) = \frac{(\rho^2 + C(1 - \rho^2)/M^2)(1 - \rho^4)}{1 - (\rho^2 + C(1 - \rho^2)/M^2)^2} \leq h(\sqrt{2C}) = \frac{((1 + \rho^2)/2)^2(1 - \rho^4)}{1 - ((1 + \rho^2)/2)^2} \leq 4.$$

Therefore, for any $\rho \in (0, 1)$, it holds $\rho_1^4/(1 - \rho_1^4) \leq 4/(1 - \rho^4)$. ■

Claim 74 Let $\rho^2 \geq 1 - C/M^2$ and $\rho_1^2 = \rho^2 + C(1 - \rho^2)/M^2$. Then, $\|\mathbb{T}_{\rho_1}\Phi'(z)\|_{L_2}^2 \leq e^C \|\mathbb{T}_\rho\Phi'(z)\|_{L_2}^2$.

Proof of Claim 74 To prove this claim, we recall the explicit expression of $\mathbf{E}_{z \sim \mathcal{N}}[(T_{\rho_1} \Phi'(z))^2]$ using the formula displayed in Lemma 65:

$$\begin{aligned}
 \mathbf{E}_{z \sim \mathcal{N}}[(T_{\rho_1} \Phi'(z))^2] &= \sum_{i,j=1}^m \frac{A_i A_j}{2\pi \sqrt{1 - \rho_1^4}} \exp\left(-\frac{t_i^2 + t_j^2}{2(1 - \rho_1^4)} + \frac{\rho_1^2 t_i t_j}{1 - \rho_1^4}\right) \\
 &= \sum_{i,j=1}^m \frac{A_i A_j}{2\pi \sqrt{1 - \rho_1^4}} \exp\left(-\frac{(t_i^2 + t_j^2 - 2\rho_1^2 t_i t_j)}{2(1 - \rho_1^4)}\right) \\
 &\stackrel{(i)}{\leq} \sum_{i,j=1}^m \frac{A_i A_j}{2\pi \sqrt{(1 - \rho_1^2)(1 + \rho_1^2)}} \exp\left(-\frac{t_i^2 + t_j^2}{2(1 - \rho^4)} + \frac{\rho_1^2 t_i t_j}{1 - \rho^4}\right) \\
 &\stackrel{(ii)}{\leq} \sum_{i,j=1}^m \frac{A_i A_j}{2\pi \sqrt{(1 - \rho^2)(1 - C/M^2)(1 + \rho_1^2)}} \exp\left(-\frac{t_i^2 + t_j^2}{2(1 - \rho^4)} + \frac{\rho^2 t_i t_j}{1 - \rho^4} + \frac{(\rho_1^2 - \rho^2) t_i t_j}{1 - \rho^4}\right) \\
 &\stackrel{(iii)}{\leq} 2 \sum_{i,j=1}^m \frac{A_i A_j}{2\pi \sqrt{1 - \rho^4}} \exp\left(-\frac{t_i^2 + t_j^2}{2(1 - \rho^4)} + \frac{\rho_1^2 t_i t_j}{1 - \rho^4}\right) \exp\left(\frac{(\rho_1^2 - \rho^2) t_i t_j}{1 - \rho^4}\right) \\
 &\leq 2 \mathbf{E}_{\mathbf{x} \sim \mathcal{D}_{\mathbf{x}}}[(T_{\rho} \Phi'(z))^2] \exp\left(\frac{(\rho_1^2 - \rho^2) M^2}{1 - \rho^4}\right).
 \end{aligned}$$

Inequality (i) is due to the facts that $(t_i^2 + t_j^2 - 2\rho_1^2 t_i t_j) \geq 0$ for any $t_i, t_j \in \mathbb{R}$ and that $1/(1 - \rho_1^4) \geq 1/(1 - \rho^2)$ since $\rho < \rho_1 < 1$; in (ii) we plugged in the definition ρ_1 ; (iii) comes from the assumption that $C/M^2 \leq 1/4$ and the fact that $1 + \rho_1^2 \geq 1 + \rho^2$ since $\rho_1 > \rho$. Finally, for the term $\exp((\rho_1^2 - \rho^2) t_i t_j / (1 - \rho^4))$, bringing in the definition of ρ_1^2 and the fact that $|t_i| \leq M, i \in [m]$ we get

$$\exp\left(\frac{(\rho_1^2 - \rho^2) M^2}{1 - \rho^4}\right) = \exp\left(\frac{(1 - \rho^2)(C/M^2) M^2}{(1 - \rho^2)(1 + \rho^2)}\right) \leq e^C.$$

Thus, in summary, we have $\mathbf{E}_{z \sim \mathcal{N}}[(T_{\rho_1} \Phi'(z))^2] \leq e^C \mathbf{E}_{z \sim \mathcal{N}}[(T_{\rho} \Phi'(z))^2]$. ■

F.3. Initialization Algorithm for Monotone Activations

In this section, we provide an initialization algorithm for σ that is an ϵ -Extended monotone (B, L) -Regular activation. The algorithm generates a vector $\mathbf{w}^{(0)}$ satisfying $\theta(\mathbf{w}^{(0)}, \mathbf{w}^*) \leq C/M$, where C is an absolute constant and M is at most $\sqrt{\log(B/\epsilon) - \log \log(B/\epsilon)}$. Our key idea is to convert the regression problem to a problem of robustly learning halfspaces via *data transformation*. In particular, we transform y to $\tilde{y} \in \{0, 1\}$ by truncating the labels y to $\tilde{y} = \mathbb{1}\{y \geq t'\}$, where this t' is a carefully chosen threshold. Then we utilize a previous algorithm from Diakonikolas et al. (2022c) to robustly learn \mathbf{w}^* .

Proposition 75 *Let σ be a non-decreasing (B, L) -Regular function. Let M be defined as in Claim 34. Then, there exists an algorithm that draws $O(d/\epsilon^2 \log(1/\delta))$ samples, it runs in $\text{poly}(d, N)$ time, and, with probability at least $1 - \delta$, it outputs a vector \mathbf{w} such that $\theta(\mathbf{w}, \mathbf{w}^*) \leq \min\{\pi/6, C/M\}$, where $C > 0$ is a universal constant, independent of any problem parameters.*

The proof of [Proposition 58](#) follows from [Proposition 75](#) and [Proposition 62](#).

Proof of [Proposition 58](#) [Proposition 75](#) implies that there exists an algorithm using $O(d/\epsilon^2)$ samples and outputs a vector $\mathbf{w}^{(0)}$ such that $\theta(\mathbf{w}^{(0)}, \mathbf{w}^*) \leq C/M$. Now for any $\theta \leq \theta_0$, it holds $\cos \theta^2 \leq 1 - \theta^2 \geq 1 - C^2/M^2$. Thus, using [Proposition 62](#) we know that $\|\mathbf{P}_{>1/\theta^2} \sigma\|_{L_2}^2 \lesssim \sin^2 \theta \|\mathbf{T}_{\cos \theta} \sigma'\|_{L_2}^2$. This finishes the proof of [Proposition 58](#). \blacksquare

Since we are only looking for a constant factor approximate solution, it is sufficient to truncate the activation σ to $\tilde{\sigma}$ so that $\|\sigma - \tilde{\sigma}\|_{L_2}^2 \leq C_1 \text{OPT}$ for some absolute constant C_1 in [Claim 34](#). Hence, given an activation $\sigma \in \mathcal{H}(B, L)$, the parameter M is defined as following: fix an absolute constant $C_1 \geq 1$; there exists a function $\tilde{\sigma} \in \mathcal{H}(B, L)$ satisfying $\|\tilde{\sigma} - \sigma\|_{L_2}^2 \leq 2C_1\epsilon$ such that $\sigma'(z) = 0$ for all $|z| \geq M$. In fact, in the proof of [Claim 34](#), we chose $\tilde{\sigma}(z) = \sigma(z)\mathbb{1}\{-M_- \leq z \leq M_+\} + \sigma(M_+)\mathbb{1}\{z \geq M_+\} + \sigma(-M_-)\mathbb{1}\{z \leq -M_-\}$, such that $\mathbf{E}_{z \sim \mathcal{N}}[(\sigma(z) - \sigma(M_+))^2 \mathbb{1}\{z \geq M_+\}] = C_1\epsilon$ and $\mathbf{E}_{z \sim \mathcal{N}}[(\sigma(z) - \sigma(-M_-))^2 \mathbb{1}\{z \leq -M_-\}] = C_1\epsilon$, then $M = \max\{M_+, M_-\} \leq \sqrt{\log(B/\epsilon) - \log \log(B/\epsilon)}$. In the following, let us assume without loss of generality that $M = M_+ \geq M_-$ since if $M_+ \leq M_-$, we can instead consider $-\sigma(-z)$.

Our goal is to show that, there exists $M^* \geq M$ such that it holds: $\Pr[\tilde{y} \neq \mathbb{1}\{\mathbf{w}^* \cdot \mathbf{x} \geq M^*\}] \leq (4/\sqrt{C_1}) \Pr[\mathbf{w}^* \cdot \mathbf{x} \geq M^*]$, where $\tilde{y} = \mathcal{T}(y) = \mathbb{1}\{y \geq \sigma(M^*)\}$. Then, we will use the following fact from [Diakonikolas et al. \(2022c\)](#), which states:

Fact 76 ([Diakonikolas et al. \(2022c\)](#), Corollary of Lemma C.3 and Theorem C.1) *There is an algorithm that given a halfspace $\phi(\mathbf{w}^* \cdot \mathbf{x}; t)$ and a distribution $(\mathbf{x}, \tilde{y}) \sim \mathcal{D}$ of labeled examples with OPT' -adversarial noise—meaning that $\Pr[\phi(\mathbf{w}^* \cdot \mathbf{x}; t) \neq \tilde{y}] \leq \text{OPT}'$, if $\exp(-t^2/2)/t \geq C_2 \text{OPT}'$, where $C_2 > 1$ is a large universal constant—returns \mathbf{w} such that $\theta(\mathbf{w}, \mathbf{w}^*) \exp(-t^2/2) \leq C_3 \text{OPT}'$ and $\theta(\mathbf{w}, \mathbf{w}^*) \leq \pi/6$, where C_3 is also a universal constant, with probability at least $1 - \delta$, using $O(d/\epsilon^2 \log(1/\delta))$ samples.*

With the error $\text{OPT}' \leq \Pr[z \geq M^*]$ and $t = M^*$ in [Fact 76](#), we obtain a vector \mathbf{w} that satisfies $\theta(\mathbf{w}, \mathbf{w}^*) \leq C_3 \exp((M^*)^2/2) \Pr[z \geq M^*] \leq C_3/M^* \leq C_3/M$, where we used the fact that $\Pr[z \geq M^*] \approx \exp(-(M^*)^2/2)/M^*$. This will complete our initialization argument.

To proceed, we prove the following key lemma:

Lemma 77 *Fix $C > 1$. Let f be a monotone function such that $f \geq 0$ and $\|f - y\|_{L_2}^2 \leq \epsilon$. Assume that for all $q > 0$ it holds that $\mathbf{E}[\mathbb{1}\{f(z) \geq q\} - \mathbb{1}\{y \geq q\}] \geq \Pr[f(z) \geq q]/C$. Then, it holds that $\|f\|_{L_2}^2 \leq 5C^2\epsilon$.*

Proof Let $T(q) = \Pr[f(z) \geq q]$ and $\Delta(q) = \mathbf{E}[\mathbb{1}\{f(z) \geq q\} - \mathbb{1}\{y \geq q\}]$. From the assumption we have that $T(q) \leq C\Delta(q)$. Therefore, we have that

$$\begin{aligned} \mathbf{E}[f^2] &= \int_0^\infty 2qT(q) dq \leq \int_0^\infty 2q(C\Delta(q)) dq \\ &= 2C \int_0^\infty q(\Pr[f \geq q, y < q] + \Pr[f < q, y \geq q]) dq \\ &= 2C \left(\mathbf{E} \left[\frac{f^2 - y^2}{2} \mathbb{1}\{f > y\} \right] + \mathbf{E} \left[\frac{y^2 - f^2}{2} \mathbb{1}\{y > f\} \right] \right) \\ &= 2C \mathbf{E} \left[\frac{|f^2 - y^2|}{2} \right] = C \mathbf{E}[|f - y||f + y|] \\ &\leq C \sqrt{\mathbf{E}[(f - y)^2] \mathbf{E}[(f + y)^2]} \leq C \sqrt{\epsilon \mathbf{E}[(f + y)^2]}. \end{aligned}$$

Note that $\mathbf{E}[(f + y)^2] \leq 4 \mathbf{E}[f^2] + 4\epsilon$, therefore we have that

$$\mathbf{E}[f^2] \leq C \sqrt{4\epsilon(\mathbf{E}[f^2] + \epsilon)}.$$

Let $\tau = \mathbf{E}[f^2]$, the above becomes

$$\tau^2 \leq 4C^2\epsilon\tau + 4C^2\epsilon^2.$$

Maximizing over τ , we have that $\tau \leq 5C^2\epsilon$ provided that $C > 1$. Therefore $\mathbf{E}[f^2] \leq 5C^2\epsilon$. \blacksquare

We can now prove [Proposition 75](#).

Proof of Proposition 75 Let $\sigma \in \mathcal{H}(B, L)$. Throughout the proof, we make the following assumptions that are without loss of generality:

1. There exists $\bar{M} < \infty$ such that $\sigma(z) = \sigma(\bar{M})$ when $z \geq \bar{M}$ and $\sigma(z) = \sigma(-\bar{M})$ when $z \leq -\bar{M}$. This is without loss of generality according to [Claim 34](#).
2. $B = \max\{|\sigma(\bar{M})|, |\sigma(-\bar{M})|\} = \sigma(\bar{M})$, since we can always shift $\sigma(z)$ to $\sigma(z) + |\sigma(-\bar{M})|$ without affecting any of the results.
3. By [Claim 33](#), it holds $|y| \leq B = \sigma(\bar{M})$ without loss of generality.
4. $\mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}}[(y - \sigma(\mathbf{w}^* \cdot \mathbf{x}))^2] \leq \epsilon$, and $h(z) = \sigma(0)$ is not an approximate solution, i.e., for any absolute constant C we have $\mathbf{E}_{(\mathbf{x}, y) \sim \mathcal{D}}[(y - h(\mathbf{w}^* \cdot \mathbf{x}))^2] \geq C\epsilon$.
5. There exists $M \in [0, \bar{M}]$ such that $\mathbf{E}_{z \sim \mathcal{N}}[(\sigma(z) - \sigma(M))^2 \mathbf{1}\{z \geq M\}] \geq C_1\epsilon$, where $C_1 > 1$ is a large absolute constant. In the following proof, we will denote M as the smallest value in $[0, \bar{M}]$ such that $\mathbf{E}_{z \sim \mathcal{N}}[(\sigma(z) - \sigma(M))^2 \mathbf{1}\{z \geq M\}] \geq C_1\epsilon$. Such M exists because if $\mathbf{E}_{z \sim \mathcal{N}}[(\sigma(z) - \sigma(0))^2 \mathbf{1}\{z \geq 0\}] < C_1\epsilon$, then it must hold that $\mathbf{E}_{z \sim \mathcal{N}}[(\sigma(z) - \sigma(0))^2 \mathbf{1}\{z \leq 0\}] \geq C_1\epsilon$, otherwise we have $\mathbf{E}_{z \sim \mathcal{N}}[(\sigma(z) - \sigma(0))^2] \leq 2C_1\epsilon$ which further implies $h(z) = \sigma(0)$ is a constant approximate solution as $\mathbf{E}_{z \sim \mathcal{N}}[(y - h(z))^2] \leq 2 \mathbf{E}_{z \sim \mathcal{N}}[(y - \sigma(z))^2] + 2 \mathbf{E}_{z \sim \mathcal{N}}[(\sigma(z) - h(z))^2] \leq (2C_1 + 2)\epsilon$, contradicting the assumption above. Since $\mathbf{E}_{z \sim \mathcal{N}}[(\sigma(z) - \sigma(0))^2 \mathbf{1}\{z \leq 0\}] \geq C_1\epsilon$, we can consider $\tilde{\sigma}(z) = -\sigma(-z)$ as the activation for the initialization algorithm, and we have $\mathbf{E}_{z \sim \mathcal{N}}[(\sigma(z) - \sigma(0))^2 \mathbf{1}\{z \geq 0\}] \geq C_1\epsilon$. Since $\mathbf{E}_{z \sim \mathcal{N}}[(\sigma(z) - \sigma(t))^2 \mathbf{1}\{z \geq t\}]$ is a decreasing function from $t = 0$ to $t = +\infty$ and $\mathbf{E}_{z \sim \mathcal{N}}[(\sigma(z) - \sigma(\bar{M}))^2 \mathbf{1}\{z \geq \bar{M}\}] = 0$, we know such M must exist.
6. In fact, using the same argument in the bullet above, we can assume without loss of generality that $\mathbf{E}_{z \sim \mathcal{N}}[(\sigma(z) - \sigma(-M))^2 \mathbf{1}\{z \leq -M\}] \leq \mathbf{E}_{z \sim \mathcal{N}}[(\sigma(z) - \sigma(M))^2 \mathbf{1}\{z \geq M\}]$, because if $\mathbf{E}_{z \sim \mathcal{N}}[(\sigma(z) - \sigma(-M))^2 \mathbf{1}\{z \leq -M\}] \geq \mathbf{E}_{z \sim \mathcal{N}}[(\sigma(z) - \sigma(M))^2 \mathbf{1}\{z \geq M\}]$ we can consider $\tilde{\sigma} = -\sigma(-z)$ and choose $M' \geq M$ that is the smallest value such that $\mathbf{E}_{z \sim \mathcal{N}}[(\tilde{\sigma}(z) - \tilde{\sigma}(M))^2 \mathbf{1}\{z \geq M\}] \geq C_1\epsilon$. Then $\mathbf{E}_{z \sim \mathcal{N}}[(\tilde{\sigma}(z) - \tilde{\sigma}(-M'))^2 \mathbf{1}\{z \leq -M'\}] \leq \mathbf{E}_{z \sim \mathcal{N}}[(\tilde{\sigma}(z) - \tilde{\sigma}(M'))^2 \mathbf{1}\{z \geq M'\}]$. Thus, in summary, M satisfies the condition of [Claim 34](#) and hence it is enough to initialize to \mathbf{w}^0 such that $\theta(\mathbf{w}^0, \mathbf{w}^*) \leq C/M$.

Given the assumptions above, we claim that there must exist $q' \in [0, B - \sigma(M)]$ such that

$$\Pr[\mathbf{1}\{y \geq \sigma(M) + q'\} \neq \mathbf{1}\{\sigma(z) \geq \sigma(M) + q'\}] \leq 4/\sqrt{C_1} \Pr[\sigma(z) \geq \sigma(M) + q']. \quad (43)$$

Suppose for the sake of contradiction that for any $q \in [0, B - \sigma(M)]$ it holds $\Pr[\mathbb{1}\{y \geq \sigma(M) + q\} \neq \mathbb{1}\{\sigma(z) \geq \sigma(M) + q\}] \geq 4/\sqrt{C_1} \Pr[\sigma(z) \geq \sigma(M) + q]$. Note that for $q \geq B - \sigma(M)$, since $\sigma(z) \leq B$ and $y \leq B$, we have $\mathbb{1}\{y \geq \sigma(M) + q\} = \mathbb{1}\{\sigma(z) \geq \sigma(M) + q\} = 0$. Thus in summary, we have $\Pr[\mathbb{1}\{y \geq \sigma(M) + q\} \neq \mathbb{1}\{\sigma(z) \geq \sigma(M) + q\}] \geq 4/\sqrt{C_1} \Pr[\sigma(z) \geq \sigma(M) + q]$ for all $q \geq 0$.

Now let $f(z) = (\sigma(z) - \sigma(M))\mathbb{1}\{z \geq M\}$, $y' = (y - \sigma(M))\mathbb{1}\{y \geq \sigma(M)\}$. Then, for any $q \geq 0$,

$$\begin{aligned} 4/\sqrt{C_1} \Pr[\sigma(z) \geq \sigma(M) + q] &= 4/\sqrt{C_1} \Pr[f(z) \geq q] \\ &\leq \Pr[\mathbb{1}\{y \geq \sigma(M) + q\} \neq \mathbb{1}\{\sigma(z) \geq \sigma(M) + q\}] \\ &= \Pr[\mathbb{1}\{y' \geq q\} \neq \mathbb{1}\{f(z) \geq q\}]. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} \mathbf{E}_{z \sim \mathcal{N}}[(f(z) - y')^2] &= \mathbf{E}_{z \sim \mathcal{N}}[(\sigma(z) - \sigma(M))\mathbb{1}\{z \geq M\} - (y - \sigma(M))\mathbb{1}\{y \geq \sigma(M)\}]^2 \\ &\leq 2 \mathbf{E}_{z \sim \mathcal{N}}[(\sigma(z) - y)^2 \mathbb{1}\{z \geq M\}] + 2 \mathbf{E}_{z \sim \mathcal{N}}[(y - \sigma(M))^2 (\mathbb{1}\{y \geq \sigma(M)\} - \mathbb{1}\{z \geq M\})^2] \\ &\leq 2\epsilon + 2 \mathbf{E}_{z \sim \mathcal{N}}[(y - \sigma(M))^2 (\mathbb{1}\{y \geq \sigma(M)\} - \mathbb{1}\{z \geq M\})^2]. \end{aligned}$$

Note that it holds $0 \leq (y - \sigma(M))\mathbb{1}\{y \geq \sigma(M), z < M\} \leq (y - \sigma(z))\mathbb{1}\{y \geq \sigma(M), z < M\}$ and $0 \leq (\sigma(M) - y)\mathbb{1}\{y < \sigma(M), z \geq M\} \leq (\sigma(z) - y)\mathbb{1}\{y < \sigma(M), z \geq M\}$. Therefore,

$$\begin{aligned} &\mathbf{E}_{z \sim \mathcal{N}}[(y - \sigma(M))^2 (\mathbb{1}\{y \geq \sigma(M)\} - \mathbb{1}\{z \geq M\})^2] \\ &= \mathbf{E}_{z \sim \mathcal{N}}[(y - \sigma(z))^2 \mathbb{1}\{y \geq \sigma(M), z < M\}] + \mathbf{E}_{z \sim \mathcal{N}}[(y - \sigma(z))^2 \mathbb{1}\{y < \sigma(M), z \geq M\}] \leq 2\epsilon. \end{aligned}$$

Combining with the upper bound on $\mathbf{E}_{z \sim \mathcal{N}}[(f(z) - y')^2]$ yields $\mathbf{E}_{z \sim \mathcal{N}}[(f(z) - y')^2] \leq 6\epsilon$. Hence the conditions of [Lemma 77](#) are satisfied, and applying [Lemma 77](#) we obtain $\mathbf{E}_{z \sim \mathcal{N}}[f^2] = \mathbf{E}_{z \sim \mathcal{N}}[(\sigma(z) - \sigma(M))^2 \mathbb{1}\{z \geq M\}] \leq 2(C_1/16)(6\epsilon) \leq (3/4)C_1\epsilon$. However, recall that M is chosen such that $\mathbf{E}_{z \sim \mathcal{N}}[(\sigma(z) - \sigma(M))^2 \mathbb{1}\{z \geq M\}] \geq C_1\epsilon$, therefore we have reached a contradiction.

Now let $M^* = \operatorname{argmin}\{0 \leq M' \leq \bar{M} : \sigma(M') = \sigma(M) + q'\}$, q' satisfying [Equation \(43\)](#), we have $M^* \in [M, \bar{M}]$. Note that this q' can be found via a grid search on the interval $[0, B - \sigma(M)]$, as we can discretize the label y and activation σ using a $\sqrt{\epsilon}$ grid, therefore, there will only be $\text{poly}(1/\sqrt{\epsilon}, B)$ number of possible choices of q' . The procedure is standard and we omit it here. The argument above implies that it holds $\text{OPT}' = \Pr[\mathbb{1}\{y \geq \sigma(M) + q'\} \neq \mathbb{1}\{\sigma(z) \geq \sigma(M) + q'\}] \leq 4/\sqrt{C_1} \Pr[z \geq M^*] = (1/C_2) \exp(-(M^*)^2/2)/M^*$ for C_2 being a large absolute constant. Hence the conditions of [Fact 76](#) are satisfied, which then implies that there exists an algorithm that given labels $\tilde{y} = \mathbb{1}\{y \geq \sigma(M^*)\}$ and a target halfspace $\phi(\mathbf{w}^* \cdot \mathbf{x}; M^*)$, returns a vector \mathbf{w} such that $\theta(\mathbf{w}, \mathbf{w}^*) \leq \min\{\pi/6, C_3 \exp((M^*)^2/2) \text{OPT}'\} = \min\{\pi/6, C_3/M^*\} \leq \min\{\pi/6, C_3/M\}$. This completes the proof of [Proposition 75](#). \blacksquare