Stochastic block models with many communities and the Kesten–Stigum bound

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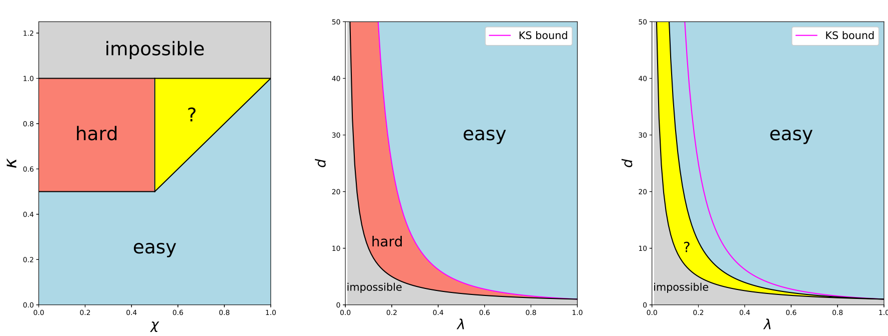
We study the inference of communities in stochastic block models with a growing number of communities. For block models with n vertices and a fixed number of communities q, it was predicted in Decelle et al. (2011) that there are computationally efficient algorithms for recovering the communities above the Kesten–Stigum (KS) bound and that efficient recovery is impossible below the KS bound. This conjecture has since stimulated a lot of interest, with the achievability side proven in a line of research culminating in work of Abbe and Sandon (2018). Conversely, the hardness side of the conjecture has been supported by recent progress based on the low-degree paradigm.

In this paper we investigate community recovery in the regime $q \to \infty$ where no such predictions exist. We show that efficient inference of communities remains possible above the KS bound. Furthermore, we show that recovery of block models is low-degree-hard below the KS bound when the number of communities $q \ll \sqrt{n}$. Perhaps surprisingly, we find that when $q \gg \sqrt{n}$, there is an efficient algorithm based on non-backtracking walks for recovery even below the KS bound. We identify a new threshold which we conjecture is the threshold for weak recovery in this regime. Finally, we show that detection is easy and identify (up to a constant) the information-theoretic threshold for community recovery as q diverges. Our low-degree hardness results also naturally have consequences for graphon estimation, improving results of Luo and Gao (2023).

In the figure below, the blue regions represent computationally efficient regimes, the red regions represent low-degree hardness, the gray regions represent information-theoretic impossibility, and the yellow regions are still unknown. The left plot shows κ vs χ where $q \asymp n^{\chi}$ and $\lambda \asymp d^{-\kappa}$. The middle and right plots show d vs λ for $q = n^{1/3}$ and $q = n^{2/3}$ respectively.[1]



1. Extended abstract. Full version can be found at [arXiv:2503.03047, v2].

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