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Optimal Online Bookmaking for Any Number of Outcomes

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Abstract

We study the *Online Bookmaking* problem, where a bookmaker dynamically updates betting odds on the possible outcomes of an event. In each betting round, the bookmaker can adjust the odds based on the cumulative betting behavior of gamblers, aiming to maximize profit while mitigating potential loss. We show that for any event and any number of betting rounds, in a worst-case setting over all possible gamblers and outcome realizations, the bookmaker's optimal loss is the largest root of a simple polynomial. Our solution shows that bookmakers can be as fair as desired while avoiding financial risk, and the explicit characterization reveals an intriguing relation between the bookmaker's regret and Hermite polynomials. We develop an efficient algorithm that computes the optimal bookmaking strategy: when facing an optimal gambler, the algorithm achieves the optimal loss, and in rounds where the gambler is suboptimal, it reduces the achieved loss to the *optimal opportunistic* loss, a notion that is related to subgame perfect Nash equilibrium. The key technical contribution to achieve these results is an explicit characterization of the *Bellman-Pareto frontier*, which unifies the dynamic programming updates for Bellman's value function with the multi-criteria optimization framework of the Pareto frontier in the context of vector repeated games.
Keywords: Online decision making, regret analysis, game theory

1. Introduction

The online betting industry, particularly in the realm of sports betting, has experienced remarkable growth in recent years (Zion Market Research, 2024). Central to this market is the role of bookmakers, who set odds for uncertain events and accept wagers, while managing their exposure to risk. In online bookmaking, odds are dynamically adjusted based on incoming bets and evolving information, making it an intricate algorithmic task. The challenge is to dynamically calibrate the odds such that the bookmaker guarantees a profit regardless of the event's outcome. In real-world settings, betting markets exhibit a wide range in the number of possible outcomes K. For instance, in a National Football League (NFL) game, $K = 2$, as gamblers can bet on either of the two competing teams, while the Eurovision Song Contest presents about $K = 40$ possible outcomes, each indicating a possible winner of the contest. As K increases, the bookmaker faces greater uncertainty and must balance risk across a broader set of outcomes.

We consider *online bookmaking* as a sequential game between a bookmaker and a gambler (Bhatt et al., 2025), noting that the behavior of multiple gamblers can be unified as a single gambler from the bookmaker's perspective. The game unfolds over a horizon T and involves an event with K possible outcomes, where exactly one outcome $k \in [K]$ materializes. At time t, the bookmaker assigns the odds $\gamma_t(k)$ for each possible outcome k, and the gambler distributes a betting unit as a probability vector q_t, where $q_t(k)$ is the bet placed on outcome k. Following T rounds of this procedure, the event's outcome is disclosed. If k is indeed the outcome of the event, the gambler receives a payout of $\gamma_t(k) \cdot q_t(k)$, and otherwise the gambler loses their placed bet $q_t(k)$.

The bookmaker's fairness is often measured using the *overround* parameter, also known as the bookmaker margin or *vigorish* (Cortis, 2015),

$$\Gamma = \sum_{k\in[K]} \frac{1}{\gamma_t(k)}, \tag{1}$$

assumed here to be time-independent. The bookmaker's odds are considered *fair* when $\Gamma = 1$, but in practice, $\Gamma > 1$. For instance, in UK football betting, the overround typically[1] remains below 1.1 for $K = 2$ markets. As the number of possible outcomes K increases, the overround also tends to rise: reaching $\Gamma \approx 1.15$ for $K = 3$ markets (e.g., football match winner), and potentially up to 1.6 for $K = 24$ markets (e.g., final score predictions). The overround also serves to define the bookmaker's offered odds as a probability vector $r_t(k) = \frac{1}{\Gamma\gamma_t(k)}$.

In each round, the bookmaker collects a betting unit from the gambler, resulting in a total of T units collected over the game. The return to the gambler upon collecting the bets $q_1, \ldots, q_T$ is

$$\frac{1}{\Gamma}\sum_{k=1}^{K} \mathbb{1}\{I = k\} \sum_{t=1}^{T} \frac{q_t(k)}{r_t(k)},$$

where $\mathbb{1}\{I = k\}$ is the indicator function of the materialized outcome. In this paper, we consider the bookmaker's worst-case scenario: (i) the gambler aims to maximize the bookmaker's loss, and (ii) the event's outcome results in the largest possible return. The optimal bookmaking loss is then defined as

$$L^\star_{T,K} = \inf_{r_1,\ldots r_T} \max_{q_1,\ldots,q_T} \max_{k\in[K]} \sum_{t=1}^{T} \frac{q_t(k)}{r_t(k)}, \tag{2}$$

where r_t, q_t depend on past offered bets and wagers (formally defined in Section 2.2). Note that the overround is omitted from (2), but the optimal loss still directly characterizes the fundamental limits of the overround. In particular, after collecting T money units and paying the worst-case loss to the gambler, the bookmaker is left with $T - \frac{1}{\Gamma}L^\star_{T,K}$ money units. Thus, to guarantee a positive gain, the bookmaker must choose the overround to be $\Gamma > \frac{L^\star_{T,K}}{T}$. We will show the convergence of $\frac{L^\star_{T,K}}{T}$ to 1 as T grows for all K. This implies that, even when facing events with many possible outcomes, bookmakers can choose the overround to be arbitrarily close to 1, given a sufficiently large T.

The online bookmaking problem is a *repeated zero-sum game* falling under the broader framework of game theory pioneered by Von Neumann's Minimax Theorem (1928). Blackwell's approachability theorem (1956) and Hannan's no-regret algorithm for competing with the best fixed action (1957) gave rise to online learning algorithms with provable regret bounds. Unlike the classical experts problem (Cover, 1965; Vovk, 1990; Littlestone and Warmuth, 1994) where the optimal adversarial strategy for more than 4 experts is unknown (Gravin et al., 2016), our setting benefits from a fully characterized optimal adversarial strategy (Bhatt et al., 2025, detailed below). Sequential prediction and adversarial decision-making are central to online learning, and incorporate concepts such as potential-based strategies (Cesa-Bianchi and Lugosi, 2003) and approachability (Abernethy et al., 2011). These settings have also been examined from an information-theoretic perspective (Merhav and Feder, 2002; Shtar'kov, 1987), where the minimax log-loss plays a key

1. According to Football-Data.co.uk

role, corresponding to the optimal code length in lossless compression (Cesa-Bianchi and Lugosi, 2006, Ch. 9).

The optimal bookmaking loss in (2) is difficult to compute for several reasons. First, the bookmaker's strategy is a sequence of mappings that depend on the entire history of placed bets, resulting in a domain that grows exponentially with time. Second, the event's outcome becomes apparent only after the final betting round. In particular, the maximization over k implies that the bookmaker does not know the *true* objective function during the game, turning it into a vector-valued game in which the bookmaker must balance the losses across outcomes throughout the betting process. Lastly, each possible objective function is unbounded and non-Lipschitz. While many online learning problems can be cast as instances of online convex optimization (OCO) (see Hazan, 2016), the regret analyses in OCO hinge on losses being bounded and Lipschitz. In our bookmaking setting, however, neither condition holds. In particular, an optimal bookmaking strategy may approach the simplex boundary. Likewise, the loss is not convex–concave, ruling out minimax theorems such as Sion (1958). Constant-factor considerations further complicate the analysis: while a loss of $\$1\mathrm{M}$ or $\$100\mathrm{M}$ might be considered equivalent in an asymptotic sense, their practical implications differ significantly. To address the latter, one might formulate the problem as an *extensive-form* game with a discrete action space (Kroer and Sandholm, 2015); however, the exponential growth of the game tree makes backward induction computationally infeasible.

Most sports betting studies adopt the gambler's perspective, emphasizing outcome prediction and bankroll management. *Kelly betting* (Kelly, 1956; Rotando and Thorp, 1992), which optimizes long-term wealth through bet sizing, is commonly used to model gambler behavior in bookmaking algorithms (e.g., Zhu et al., 2024). Various techniques have been developed to adjust the bookmaker's *implied probabilities*, such as additive, multiplicative, *Shin*, and power methods (see Clarke et al., 2017). Other works examine dynamic pricing strategies (e.g., Divos et al., 2018; Lorig et al., 2021), but do not consider adversarial betting behavior. Most closely related to our work is Bhatt et al. (2025), which introduced the problem of online bookmaking and solved it in terms of an explicit loss and an optimal algorithm for binary games, i.e., $K = 2$. In contrast, we solve the online bookmaking problem for all K and extend it to the opportunistic setting when the gambler is suboptimal (see Section 2.3).

Contributions We present an analysis of the online bookmaking problem.

- We derive an exact and computable expression for the optimal bookmaking loss $L^\star_{T,K}$ for any number of outcomes K and betting rounds T.
- We identify the regret $R_{T,K} = L^\star_{T,K} - T$, measured with respect to a bookmaker that foresees the aggregated wagers but does not know the event's outcome. We prove that the regret scales as $\sqrt{T}$ and, for any K, its scaling factor converges to the largest root of the K-th Hermite polynomial. This establishes the fact that bookmakers can be as fair as desired.
- We introduce the notion of opportunistic bookmaking, which generalizes the setting above by accounting for the possibility that the gambler's past behavior was not optimal. We characterize the *optimal opportunistic bookmaking loss* and present an efficient algorithm that achieves this loss at any time.
- A key technique to achieve our explicit results is the characterization of the *Bellman-Pareto Frontier*, corresponding to the union of all optimal future payouts in vector games.

Paper Structure Section 2 includes the preliminaries and the formulation of the online bookmaking problem, along with its generalization to opportunistic bookmaking strategies. Section 3 presents our main results. Section 4 provides the main ideas and proofs. The paper is concluded in Section 5.

2. Problem Setting

2.1. Preliminaries

Notation Constants are denoted by uppercase letters (e.g. T). For $K \in \mathbb{N}_+$, we define $[K] = \{1, \ldots, K\}$. The standard basis of $\mathbb{R}^K$ is denoted $\mathcal{E}_K = \{\mathbf{e}_1, \ldots, \mathbf{e}_K\}$. Vectors are denoted with lowercase letters, e.g., r. Vector sequences are expressed as $r^T = (r_1, \ldots, r_T)$, where subscripts indicate time indices, i.e., r_t is the t-th element of r^T. For vectors $r, r_t \in \mathbb{R}^K$, their k-th elements are $r(k)$ and $r_t(k)$, respectively. The notation $r^{\setminus k}$ denotes r without its k-th element. The simplex in $\mathbb{R}^K$ is denoted by $\Delta \equiv \Delta^{K-1}$ when the dimension is clear from context. For vectors $x, y \in \mathbb{R}^K$, the notation $x \oslash y$ represents element-wise division. The *weak coordinate-wise partial order* on $\mathbb{R}^K$ is denoted by $\succeq$ and defined as $x \succeq y$ if $x(k) \geq y(k)$ for all $k \in [K]$, while *strict coordinate-wise partial order* is denoted by $\succ$ and holds if $x \succeq y$ and there exists some $k \in [K]$ such that $x(k) > y(k)$. For $x \in \mathbb{R}$ and $m \in \mathbb{N}$, the *falling factorial* is defined as $x^{\underline{m}} := \prod_{i=0}^{m-1}(x - i)$, and the *rising factorial* is defined as $x^{\overline{m}} := \prod_{i=0}^{m-1}(x + i)$.

Definition 1 (Elementary Symmetric Polynomial) *For $s \in \mathbb{R}^K$ and $m \in \mathbb{N}$, the* elementary symmetric polynomial (ESP) *of degree m over the elements of s is*

$$\sigma_m(s) := \sum_{\mathfrak{I} \in \binom{[K]}{m}} \prod_{k \in \mathfrak{I}} s(k), \tag{3}$$

where $\binom{[K]}{m}$ denotes the set of all subsets of $[K]$ with cardinality m.

Example 1 *The non-zero elementary symmetric polynomials of $s \in \mathbb{R}^3$ are:*

$$\sigma_0(s) = 1, \qquad \sigma_1(s) = s(1) + s(2) + s(3),$$
$$\sigma_2(s) = s(1)s(2) + s(1)s(3) + s(2)s(3), \qquad \sigma_3(s) = s(1)s(2)s(3).$$

2.2. Problem Formulation

The *online bookmaking problem* is modeled as a repeated game of T rounds with a fixed number of outcomes K and a fixed overround parameter $\Gamma \geq 1$. In each round t, the bookmaker selects a probability distribution $r_t \in \Delta^{K-1}$, and publishes betting odds $\gamma_t(k) = \frac{1}{\Gamma r_t(k)}$ for every outcome $k \in [K]$. The gambler then chooses $q_t \in \Delta^{K-1}$, which represents the distribution of its bets across the different outcomes. The bookmaker and the gambler repeat this procedure T times, where at each round they can use the previous odds r^{t-1} and placed bets q^{t-1}. Formally, the bookmaker's strategy Ψ^T is a sequence of mappings, defined by

$$\Psi_t : \Delta^{t-1} \times \Delta^{t-1} \to \Delta, \quad t \in [T], \tag{4}$$

that determines the odds r_t as $r_t = \Psi_t\,(r^{t-1},\, q^{t-1})$. Similarly, the gambler's strategy Λ^T is a sequence of mappings, defined by

$$\Lambda_t : \Delta^t \times \Delta^{t-1} \to \Delta, \quad t \in [T], \tag{5}$$

which produces the bet vector q_t as $q_t = \Lambda_t\,(r^t,\, q^{t-1})$.

We consider the worst-case scenario, which defines the *optimal bookmaking loss* as

$$L^\star_{T,K} = \inf_{\Psi^T} \max_{q^T \in \Delta^T} \max_{k \in [K]} \sum_{t=1}^{T} \frac{q_t(k)}{r_t(k)}, \tag{6}$$

where r^T is determined by Ψ^T. Note that the maximization over Λ^T is replaced with $q^T \in \Delta^T$ without loss of optimality, as the gambler acts as an adversary that responds to the bookmaker's strategy. Also, since the bookmaker's strategy is deterministic, we consider, without loss of optimality, a bookmaker's strategy Ψ_t that depends on q^{t-1}. Our main objective is to characterize the optimal loss as a simple computable expression for all T and K, and to find an optimal bookmaking strategy Ψ^T that achieves the infimum in (6).

By Bhatt et al. (2025, Theorem 2), the maximization over $q^T \in \Delta^T$ in (6) can be restricted to the vertices of the simplex, i.e., it suffices to maximize over $q^T \in \mathcal{E}_K^T$. This restriction is related to the notion of decisive gamblers. If the gambler adheres to this optimal behavior, we derive a strategy Ψ^T that achieves the optimal bookmaking loss simultaneously for all decisive gamblers. In real-world settings, however, the gambler may deviate from optimality, reflecting the aggregate behavior of multiple gamblers. This motivates the design of a bookmaker's strategy capable of handling—and even benefiting from—such non-optimal behavior. In the next section, we refine the notion of online bookmaking loss to capture the idea of *opportunistic strategies*.

2.3. Opportunistic Bookmaking Strategy

The optimal loss $L^\star_{T,K}$ is characterized as the solution to a repeated min-max game. In this game, the first player, a bookmaker, selects a strategy, while the second player, the gambler, responds to it. The second player can be viewed as an adversarial environment that consistently chooses the worst-case scenario for the bookmaker at each step. However, the gambler may not always play optimally; that is, the environment is not as adversarial as expected. We then ask

Can the bookmaker reduce its optimal loss when facing a suboptimal gambler, and if so, how?

This question is structured as follows: we provide the bookmaker with a state vector $s \in \mathbb{R}^K$ corresponding to the payouts that are already committed to the possible event outcomes. The state vector, for example, may arise from past betting rounds as $s = \sum_{i=1}^{t-1} q_i \oslash r_i$. The dependence of the state on t is omitted since only the committed payouts matter, and not the number of rounds that led to the state vector. The bookmaker's objective is then to design a strategy for H future betting rounds. An opportunistic bookmaking strategy $\hat{\Psi}^H$ is a sequence of mappings, defined by

$$\hat{\Psi}_h : \mathbb{R}^K \times \Delta^{h-1} \times \Delta^{h-1} \to \Delta, \quad h \in [H],$$

such that, for each of the remaining rounds $h \in [H]$, the odds are given by $r_h = \hat{\Psi}_h(s,\, r^{h-1},\, q^{h-1})$.

The optimal opportunistic bookmaking loss (OOBL) given a state s is defined as

$$L^\star_{H,K}(s) = \inf_{\hat{\Psi}^H} \max_{q^H \in \Delta^H} \max_{k\in[K]} s(k) + \sum_{h=1}^{H} \frac{q_h(k)}{r_h(k)}. \tag{7}$$

This general formulation allows us to define the loss with respect to an arbitrary state, rather than a state resulting from optimal adversarial dynamics. The expression $s(k)+\sum_{h=1}^{H}\frac{q_h(k)}{r_h(k)}$ quantifies the total payout for outcome k, combining the prior commitments encoded in $s(k)$ and the additional payout incurred over the remaining H rounds. Although the gambler may have acted suboptimally, we retain the worst-case maximization over q^H, since the bookmaker cannot predict future behavior. This conservative formulation avoids making optimistic assumptions about how the gambler will act going forward. Clearly, (7) specializes to the online bookmaking loss $L^\star_{T,K}$ in (6) by choosing $s = \mathbf{0}_K$ and $H = T$. Even though the latter is a special case, we present the results for this setting first, as they are cleaner and provide stronger insights.

3. Main Results

This section presents our main results. We first present Theorems A and B, which characterize the optimal bookmaking loss and its regret growth rate. In Section 3.1, we establish the optimal opportunistic bookmaking loss, a strategy to achieve it (Theorem C), and present Algorithm 1, which provides an opportunistically optimal strategy at all times.

Our first result characterizes the optimal bookmaking loss.

Theorem A (The Optimal Bookmaking Loss) *The optimal bookmaking loss $L^\star_{T,K}$ is equal to the largest root of the polynomial*

$$\mathcal{P}_{T,K}(x) = \sum_{m=0}^{K} \binom{K}{m} (-T)^{\overline{K-m}} x^m, \tag{8}$$

where $\alpha^{\overline{\beta}}$ denotes the rising factorial (defined above).

Theorem A provides an explicit and computable expression for any horizon T and any number of outcomes K. In particular, to compute the loss for any pair of parameters, all that is required is to find the maximal root of a polynomial of degree K. That is, the computational complexity does not scale with T. We illustrate the benefits of the explicit loss expression via special cases.

- For $K = 2$, we obtain $\mathcal{P}_{T,2}(x) = x^2 - 2Tx + T(T-1)$ whose largest root is

  $$L^\star_{T,2} = T + \sqrt{T}. \tag{9}$$

  This reveals the optimal online bookmaking loss for binary games in Bhatt et al. (2025).

- For $K = 3$, we obtain $\mathcal{P}_{T,3}(x) = x^3 - 3Tx^2 + 3T(T-1)x - T(T-1)(T-2)$. It is a cubic equation, and by *Cardano's method* the largest root is

  $$L^\star_{T,3} = T + 2\sqrt{T}\cos\left(\tfrac{1}{3}\arccos\left(T^{-\frac{1}{2}}\right)\right). \tag{10}$$

For $K \geq 4$, explicit computation of $L^\star_{T,K}$ is difficult, but we can identify a structure for the optimal loss based on $K = 2, 3$:

$$L^\star_{T,K} = T + R_{T,K}, \tag{11}$$

where $R_{T,K}$ denotes the *regret*. The notion of regret is not arbitrary: the linear T term corresponds to a lower bound on the optimal loss, achieved by a bookmaker that knows the sequence of bets in hindsight but does not know the event's outcome, so its optimal action is $r_t = q_t$. Recall that if $R_{T,K}/T$ converges to zero, the overround can be arbitrarily close to 1 (fair regime). We establish this convergence by characterizing the growth rate of the regret.

Observe that for $K = 2, 3$, the regret scales as $O(\sqrt{T})$; as it holds that $R_{T,2} = \sqrt{T}$, and $\lim_{T\to\infty} R_{T,3}/\sqrt{T} = \sqrt{3}$. This suggests studying the asymptotic growth of regret by defining the *asymptotic regret factor*

$$\beta_K \coloneqq \lim_{T\to\infty} \frac{R_{T,K}}{\sqrt{T}}, \tag{12}$$

provided that the limit exists. Here, K is treated as a fixed constant, independent of T. Analysis shows the limit indeed exists and, in some cases, can even be expressed by radicals, e.g., $\beta_4 = \sqrt{3+\sqrt{6}}$ and $\beta_5 = \sqrt{5+\sqrt{10}}$. We provide a precise characterization of β_K for all K.

Theorem B (The Asymptotic Regret Factor) *The asymptotic regret factor β_K in (12) is equal to the largest root of the K-th Hermite polynomial*

$$\mathrm{He}_K(x) = (-1)^K \frac{1}{\varphi(x)} \frac{\mathrm{d}^K}{\mathrm{d}x^K} \varphi(x),$$

where $\varphi(x)$ is the probability density function of the standard normal distribution. Consequently,

$$\beta_K = 2\sqrt{K} + o(\sqrt{K}),$$

and explicit bounds on β_K are given below in (13).

Theorem B establishes the $\sqrt{T}$-scaling of the regret for all K, and provides a simple method to compute the optimal regret factor via Hermite polynomials. The roots of Hermite polynomials are well-studied: all roots are real, and the largest root satisfies $A_K \leq \beta_K \leq B_K$ where the bounds are given by

$$A_K = 2\sqrt{K} - 9 \cdot 2^{-5/3} \cdot K^{-1/6}, \; B_K = \sqrt{4K+2} - \sqrt{2} \cdot (6^{-1/3} i_1) \cdot (2K+1)^{-1/6}, \tag{13}$$

and i_1 denotes the smallest positive root of the *Airy function* ($6^{-1/3} i_1 \approx 1.85574$) (Krasikov, 2004).

Remark 2 *For any fixed $T \in \mathbb{N}_+$,*

$$\lim_{K\to\infty} \frac{R_{T,K}}{\sqrt{T}} = \infty. \tag{14}$$

Thus, the optimal bookmaking loss grows to infinity as the number of outcomes K grows. The proof is given in Appendix D.2.

3.1. The Opportunistically Optimal Bookmaking Strategy

We illustrate in Figure 1 the main idea of the optimal algorithm presented in this section. The gray bars correspond to the state vector s, representing the payouts committed by the bookmaker prior to designing an opportunistic strategy with horizon H. Given a state vector and H, the bookmaker computes $L^\star_{H,K}(s)$, which corresponds to the total loss—comprising the current state and the forthcoming worst-case payouts. In Theorem 9, we establish a Nash equilibrium of the game: for all state vectors, if the gambler bets decisively in each of the remaining betting rounds (a strategy we refer to as a *decisive gambler*), the bookmaker is forced to return $L^\star_{H,K}(s)$—that is, the maximum among the gray bars will reach the water level. We emphasize that a decisive gambler may choose a different outcome in each betting round. Conversely, regardless of the initial state vector, the optimal bookmaker is able to guarantee the same loss $L^\star_{H,K}(s)$ for any decisive gambler's betting sequence. This equilibrium resembles a *water-filling* solution: the bookmaker balances the losses across outcomes such that the maximum loss reaches the water level $L^\star_{H,K}(s)$. The idea behind the opportunistic strategy is now apparent: when a decisive bet is placed by the gambler, the corresponding gray bar increases, while all other coordinates of the state remain unchanged. However, if the gambler distributes its bet among multiple outcomes, the gambler's strategy is suboptimal, and the bookmaker exploits this opportunity by lowering the water level accordingly. The difference between the water level and the current state vector is called the *residual loss* vector.

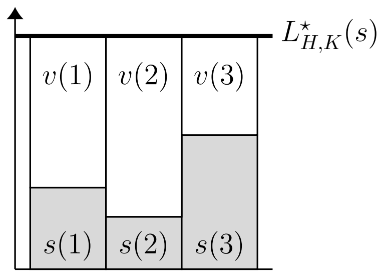

Figure 1: Illustration of a single step in a game with $K = 3$. The state $s(k)$ corresponds to the committed payouts to each possible outcome so far, and the residual losses $v(k)$ correspond to the anticipated future losses. Theorem 9 establishes the Nash equilibrium that these vectors sum up to the optimal loss $L^\star_{H,K}(s)$.

The optimal opportunistic bookmaking loss will be defined using the polynomial

$$\mathcal{F}_{H,s}(x) = \sum_{m=0}^{K} c_{H,m}(s)\, x^{K-m}, \quad \text{where} \quad c_{H,m}(s) = (-1)^m \sum_{n=0}^{m} H^{\overline{m-n}} \binom{K-n}{m-n} \sigma_n(s)\,, \tag{15}$$

where the dependence on K is omitted from the notation $\mathcal{F}_{H,s}$ for brevity. The optimal strategy will be calculated from the residual loss vector using the polynomial

$$\mathcal{D}_{H,K}(v) \coloneqq \sum_{m=0}^{K} (-H)^{\overline{K-m}} \sigma_m(v)\,. \tag{16}$$

Algorithm 1 provides a simple procedure to sequentially compute the odds based on the bets placed so far. The main idea is to compute the odds based on the current residual loss vector. In each round, if the gambler is not decisive, the loss can be decreased to the OOBL, and if the gambler is optimal, the bookmaker sticks to the strategy computed using the previous loss. We formalize the performance of Algorithm 1 in Theorem C.

Theorem C (Optimal Opportunistic Bookmaking: Loss and Algorithm) *For any number of outcomes K, horizon H, and state $s \in \mathbb{R}^K$, the optimal opportunistic bookmaking loss $L^\star_{H,K}(s)$ is equal to the largest real root of the polynomial $\mathcal{F}_{H,s}$ in (15). The optimal odds for the k-th outcome are*

$$r(k) = \frac{\mathcal{D}_{H-1,K-1}\left(v^{\setminus k}\right)}{\mathcal{D}_{H-1,K}\left(v\right)}, \tag{17}$$

where $v = L^\star_{H,K}\left(s\right) \cdot \mathbf{1}_K - s$ is the residual loss vector. Consequently, Algorithm 1 is opportunistically optimal at any time.

We note that (17) constitutes a bookmaking strategy: since $r(k)$ depends only on the residual loss vector v and the horizon H, recomputing v after each bet and reusing this formula yields the bookmaker's action at every step. This is the main procedure in Algorithm 1.

Remark 3 *The polynomials $\mathcal{P}_{T,K}\left(x\right)$ and $\mathcal{F}_{H,s}\left(x\right)$ that are used to compute the optimal bookmaking loss and the optimal opportunistic bookmaking loss are related to $\mathcal{D}_{H,K}\left(v\right)$ via the relation*

$$\mathcal{F}_{H,s}\left(x\right) \coloneqq \mathcal{D}_{H,K}\left(x \cdot \mathbf{1}_K - s\right). \tag{18}$$

Consequently, we also have $\mathcal{P}_{T,K}\left(x\right) \coloneqq \mathcal{D}_{T,K}(x \cdot \mathbf{1}_K)$. The particular vector argument of $\mathcal{D}_{H,K}\left(\cdot\right)$ in (18) corresponds to a residual vector when the water level is x. That is, the OOBL is the minimal water level such that the difference to the state constitutes a valid residual vector. The proof of the identity in (18) appears in Appendix D.3.

Remark 4 (Computational Complexity) *Algorithm 1 is implemented in a forward manner, so that only T sequential operations are required. This is noteworthy since it is uncommon for an exactly optimal algorithm to be both forward-running and to exhibit a per-round computational complexity that is independent of the horizon T. The overall complexity of the algorithm is $O(TK^2)$ with a per-round complexity of $O(K^2)$. The complexity analysis can be found in Appendix D.4.*

Remark 5 *Bhatt et al. (2025) consider the binary case ($K = 2$) and design two algorithms for the bookmaker, depending on the gambler's behavior. Against a decisive gambler, they propose the* Optimal Strategy For Decisive Gamblers *algorithm (`ODG`), which coincides with Algorithm 1. Against a non-decisive gambler, they construct a mixture of `ODG` strategies, based on the observed betting sequence. This approach is shown to be suboptimal in Appendix D.5. They also propose a Monte Carlo approximation of the mixture, which incurs a computational cost of $\omega(T^3)$. In contrast, Algorithm 1 applies to both gambler types and achieves optimality with significantly lower complexity.*

Remark 6 (The Effect of ε-Approximate Root-Finding) *Algorithm 1 computes the optimal opportunistic bookmaking loss by evaluating the largest real root of a polynomial (Line 6). In Appendix D.6, we present a modified version of the algorithm that approximates this root with precision ε. In this case, for every non-decisive gambler action (Line 5), an additional loss of at most 2ε is incurred. Therefore, the additional loss accumulated over the course of the game is at most $2T\varepsilon$, which is $o(1)$ when $\varepsilon = O(T^{-1})$. Utilizing the standard Newton–Raphson method— which converges in $O(K \log \varepsilon^{-1})$ iterations—for root-finding, the resulting algorithm incurs only an additional constant loss, with overall $O(TK^2 + KT \log T)$ complexity.*

Algorithm 1: Opportunistically Optimal Bookmaking Algorithm

Input: K, T, q^{T-1} (bets obtained sequentially)

Output: $r_1, \dots, r_T$ (outputs r_t after observing q^{t-1})

Initialization: $s \leftarrow \mathbf{0}_K,\ L \leftarrow L^\star_{T,K}$

output $r_1 \leftarrow \frac{1}{K} \cdot \mathbf{1}_K$

for $t = 2 : T$ **do**

  $s \leftarrow s + q_{t-1} \oslash r_{t-1}$ // update the state vector

  if $q_{t-1} \notin \mathcal{E}_K$ **then** // check if the gambler is not optimal (decisive)

    $L \leftarrow \arg\max \text{Roots}\left(\mathcal{F}_{T-t+1,s}\right)$ // reduce loss to the optimal opportunistic loss

  end

  $v \leftarrow (L \cdot \mathbf{1}_K - s)$ // calculate the residual (future) loss vector

  for $k = 1 : K$ **do**

    $r(k) \leftarrow \mathcal{D}_{T-t,K-1}\left(v^{\setminus k}\right)$ // $r(k) \propto$ the optimal betting odds in (17)

  end

  output $r_t \leftarrow r/\|r\|_1$

end

4. Proofs and Key Ideas

This section presents the proofs of the main results. In Section 4.1, we establish a Nash Equilibrium of the game (Theorem 9). Section 4.2 introduces and characterizes the *Bellman-Pareto*. The proofs of the main results in Section 3 are provided in Section 4.3.

4.1. Nash Equilibrium of the Game

We present the optimal opportunistic bookmaking loss (OOBL), defined in (7), as the result of a sequential optimization process, where at each step, we optimize a function that depends on the previous function in a backward manner. This formulation allows optimization only over the instantaneous bookmaker and gambler actions.

Definition 7 (Value Function) *The value function quantifies the OOBL given a specified horizon H and state vector s. It is defined recursively for $H > 0$*

$$\mathcal{V}_H(s) \coloneqq \inf_{r \in \Delta} \max_{q \in \Delta} \mathcal{V}_{H-1}(s + q \oslash r) \tag{19}$$

with the initial condition $\mathcal{V}_0(s) \coloneqq \max_{k \in [K]} s(k)$.

Note that the value function is akin to that in Markov Decision Processes using the dynamic programming principle. Indeed, it corresponds to the Bellman operator, but the maximization over $k \in [K]$ at the last round implies that the immediate reward is zero at all times, and the overall reward is measured at the final time. It is straightforward to see that $\mathcal{V}_H(s) = L^\star_{H,K}(s)$. In the special case $s = \mathbf{0}_K$, we obtain $\mathcal{V}_H(\mathbf{0}_K) = L^\star_{H,K}$. Before presenting a Nash equilibrium, we define the notion of a *decisive gambler*.

Definition 8 (Decisive Gambler) *A gambler is decisive if it bets on a single outcome, i.e., $q \in \mathcal{E}_K$ in* (19).

We now present a result that characterizes a Nash equilibrium and establishes the existence and uniqueness of an optimal opportunistic bookmaking strategy.

Theorem 9 (Nash Equilibrium of the Game) *For any horizon $H \geq 1$ and state s, the optimization of the value function in (19) satisfies the following properties: (i) for any $r \in \Delta$, there exists an optimal gambler who is decisive, meaning it is sufficient to maximize over $q \in \mathcal{E}_K$, and (ii) the optimal $r^\star \in \Delta$ is unique and satisfies*

$$\mathcal{V}_H(s) = \mathcal{V}_{H-1}(s + q \oslash r^\star) \quad \forall q \in \mathcal{E}_K. \tag{20}$$

Theorem 9 characterizes optimal strategies for both the bookmaker and the gambler: a decisive gambler maximizes the bookmaker's return, and the optimal bookmaker equalizes the loss associated with each decisive gambler's bet $q \in \mathcal{E}_K$. This balancing mechanism resembles the *water-filling algorithm* (Boyd and Vandenberghe, 2004, Ex. 5.2), but only the loss of a single outcome will reach the water level as we illustrate in Figure 1. We remark that the optimality of decisive gamblers was established in Bhatt et al. (2025) for $s = \mathbf{0}_K$, while Theorem 9 extends it to general s and proves the existence of a balancing $r^\star$. The proof of Theorem 9 is provided in Appendix B.

Although the value function enables us to characterize the players' optimal strategies, its recursive structure makes it computationally intractable for large H. A further challenge lies in the state space, as (19) must be solved for any state $s \in \mathbb{R}^K$. Leveraging the implication of Theorem 9, we restrict our attention to decisive gamblers without loss of optimality.

4.2. Bellman-Pareto Frontier

Theorem 9 implies that the optimal opportunistic bookmaking strategy $\hat{\Psi}^H$ and any decisive gambler's betting sequence $q^H \in (\mathcal{E}_K)^H$ satisfy

$$\mathcal{V}_H(s) = \max_{k \in [K]} \left(s(k) + \sum_{h=1}^{H} \frac{q_h(k)}{r_h(k)} \right). \tag{21}$$

The next lemma is a simple observation that the remaining optimization over $k \in [K]$ in (21) is determined by the gambler's last bet q_H.

Lemma 10 *The index $k \in [K]$ that attains the maximum in (21) is determined by the gambler's bet in the final round as $q_H = \mathbf{e}_k$.*

The proof of Lemma 10 is provided in Appendix E.3. Combining (21) and Lemma 10 allows us to omit the maximization over k, thereby obtaining the simpler problem of finding the unique bookmaking strategy that equates

$$\mathcal{V}_H(s) = \left(s(k) + \sum_{h=1}^{H} \frac{q_h(k)}{r_h(k)} \right), \tag{22}$$

for all $q^H \in (\mathcal{E}_K)^H$, where k is determined by the last bet as $q_H = \mathbf{e}_k$. Recall that the sum in (22) is the *residual loss* vector (RLV), illustrated in Figure 1, as the difference between the current state and the water level. Our solution approach is not to solve (22) directly, but to look at a more general problem. Our proposed idea is to find the set of all possible residual vectors (possible sums in (22)).

That is, we ignore the state vector and the optimal strategy looked for, and aim to characterize the set of all possible residual vectors when the gambler is decisive. If we are able to characterize the set of all residual vectors, then it is straightforward to find the residual vector that solves (22) since the sum of the state and the residual vectors should be a constant vector. We proceed to formalize the set of possible residual vectors.

Definition 11 (Bellman-Pareto Frontier) *A strategy Ψ^H achieves a residual vector $v \in \mathbb{R}^K$ if for any sequence $q^H \in (\mathcal{E}_K)^H$ it holds that*

$$\sum_{h=1}^{H} \frac{q_h(k)}{r_h(k)} = v(k), \tag{23}$$

*where $k \in [K]$ corresponds to the gambler's bet at the final betting round $q_H = \mathbf{e}_k$. A vector v is H-*achievable *if there exists a strategy Ψ^H that achieves it. The set of all H-achievable vectors, denoted by $\mathfrak{P}_{H,K}$, is the* Bellman-Pareto (BP) frontier *of the online bookmaking game.*

The BP frontier captures the fundamental trade-offs in the bookmaker's decision-making. Each residual vector in the frontier represents a possible configuration of future payouts that cannot be improved for one outcome without worsening another.

The following theorem characterizes the Bellman-Pareto frontier.

Theorem 12 (The Bellman-Pareto Frontier) *The Bellman-Pareto frontier of the online bookmaking game is given by*

$$\mathfrak{P}_{H,K} = \left\{ L^\star_{H,K}(s) \cdot \mathbf{1}_K - s \;\middle|\; s \in \mathbb{R}^K,\ \min_{k \in [K]} s(k) = 0 \right\}, \tag{24}$$

where $L^\star_{H,K}(s)$ is equal to the largest real root of $\mathcal{F}_{H,s}$ in (15).

Theorem 12 identifies the BP frontier through its connection to the OOBL. The fact that $L^\star_{H,K}(s)$ is efficiently computable for any state s allows the bookmaker to determine all achievable RLVs, and thus, all valid future payouts. To prove Theorem 12, we first establish several intermediate results.

Lemma 13 (Recurrence Relation for $\mathcal{D}_{H,K}$) *For every $H \geq 1$ and $k \in [K]$,*

$$\mathcal{D}_{H,K}\left(v\right) = v(k) \cdot \mathcal{D}_{H,K-1}(v^{\setminus k}) - H \cdot \mathcal{D}_{H-1,K-1}(v^{\setminus k}).$$

Lemma 13, proved in Appendix E.3, is established primarily by leveraging the recurrence relation for ESPs (Lemma 24) and the identity $H \cdot (H-1)^{\underline{m}} = H^{\underline{m+1}}$ for all $H, m \in \mathbb{N}$.

Lemma 14 (Necessary Constraints for H-Achievable Vectors) *If $v \in \mathfrak{P}_{H,K}$ then*

1. *The first action of a bookmaker that achieves v is given by (17).*
2. *For every $k \in [K]$, the value $v(k)$ is determined by the remaining $K-1$ entries $v^{\setminus k}$ via*

$$v(k) = \frac{H \cdot \mathcal{D}_{H-1,K-1}\left(v^{\setminus k}\right)}{\mathcal{D}_{H,K-1}\left(v^{\setminus k}\right)}. \tag{25}$$

Consequently, by Lemma 13, $\mathcal{D}_{H,K}\left(v\right) = 0$.

3. *For all* $u \in \mathbb{R}^K$, $u \succ v \implies \mathcal{D}_{H,K}(u) > 0$.

Lemma 14 establishes structural constraints on any vector v that lies on the Bellman-Pareto frontier. It highlights the built-in trade-off among the coordinates: reducing one component necessarily increases another. The proof of Lemma 14 is deferred to Appendix C.

Proof [Proof of Theorem 12] We prove that u is an H-achievable vector if and only if there exists a state s such that u is the RLV for s.

$\Leftarrow$ Let u be the RLV for s, that is, $s + u = L^{\star}_{H,K}(s) \cdot \mathbf{1}_K$. By Theorem 9, there exists a bookmaking strategy $\hat{\Psi}^H$ whose loss is $L^{\star}_{H,K}(s)$; combined with Lemma 10, this implies that for any betting sequence $q^H = (q_1, \dots, q_{H-1}, \mathbf{e}_k) \in (\mathcal{E}_K)^H$,

$$\sum_{h=1}^{H} \frac{q_h(k)}{r_h(k)} = u(k).$$

$\Rightarrow$ Suppose that u is an H-achievable vector. We prove that u is the RLV for the state $\hat{s} = -u$. Let $x = \mathcal{V}_H(\hat{s})$ denote the optimal loss given the state $\hat{s}$. By Theorem 9, an optimal bookmaking strategy exists whose loss equals x. As shown in (22) and the subsequent discussion, this strategy achieves the RLV

$$v = x \cdot \mathbf{1}_K - \hat{s}.$$

The vector u satisfies $u = 0 \cdot \mathbf{1}_K - \hat{s}$, i.e., it follows the structure of v with $x = 0$. Since u is H-achievable, by Lemma 14.2, $\mathcal{D}_{H,K}(u) = 0$. Thus, by Lemma 14.3, for any $x < 0$, the vector $x \cdot \mathbf{1}_K - \hat{s}$ is not H-achievable. As a result, the RLV for $\hat{s}$ is u.

We conclude that

$$\begin{aligned}
\mathfrak{P}_{H,K} &= \left\{ \mathcal{V}_H(s) \cdot \mathbf{1}_K - s \;\middle|\; s \in \mathbb{R}^K \right\} \\
&= \left\{ \mathcal{V}_H\left(s - \min_{k \in [K]} s(k) \cdot \mathbf{1}_K \right) \cdot \mathbf{1}_K - \left(s - \min_{k \in [K]} s(k) \cdot \mathbf{1}_K \right) \;\middle|\; s \in \mathbb{R}^K \right\},
\end{aligned}$$

where the second equality follows from the uniform translation property of the value function, stated in Lemma 32. Substituting $\mathcal{V}_H(s) = L^{\star}_{H,K}(s)$ yields the characterization in (24).

We next show that $L^{\star}_{H,K}(s)$ equals the largest real root of $\mathcal{F}_{H,s}(x) \coloneqq \mathcal{D}_{H,K}(x \cdot \mathbf{1}_K - s)$, given explicitly in (15), following Remark 3. Let $v = L^{\star}_{H,K}(s) \cdot \mathbf{1}_K - s$ be the RLV for a state $s \in \mathbb{R}^K$. Given that $v \in \mathfrak{P}_{H,K}$, Lemma 14.2 implies $\mathcal{D}_{H,K}(L^{\star}_{H,K}(s) \cdot \mathbf{1}_K - s) = 0$. Let $\rho = \{\rho_1, \dots, \rho_Z\}$ denote the real roots of $\mathcal{F}_{H,s}$, ordered increasingly. Since $L^{\star}_{H,K}(s) \in \rho$, this set contains a unique $z \in [Z]$ such that $L^{\star}_{H,K}(s) = \rho_z$. If $z < Z$, it holds that $\rho_z \cdot \mathbf{1}_K - s \prec \rho_Z \cdot \mathbf{1}_K - s$, which, by Lemma 14.3, implies that $\rho_z \cdot \mathbf{1}_K - s$ is not H-achievable. Thus, $L^{\star}_{H,K}(s)$ must equal ρ_Z. ■

4.3. Proofs of Main Results

This section includes the proofs for Theorems A and C. The proof of Theorem B is proved in Appendix D.1.

Proof [Proof of Theorem C] Theorem 12 shows that $L^{\star}_{H,K}(s)$ equals the largest real root of $\mathcal{F}_{H,s}$, for any $H, K \in \mathbb{N}_+$ and $s \in \mathbb{R}^K$. Once $L^{\star}_{H,K}(s)$ is computed, the RLV $v = L^{\star}_{H,K}(s) \cdot \mathbf{1}_K - s$ determines the optimal bookmaker's first action r via (17), as established in Lemma 14.1.

The correctness of Algorithm 1 for a game with T rounds then follows. Before the first betting round, the state is $s = \mathbf{0}_K$, and the loss is initialized at $L = L^\star_{T,K}$. Since s is a constant vector, so is the RLV v; by the symmetry in (17), $r_1 = \frac{1}{K} \cdot \mathbf{1}_K$ is the optimal action. Let $t \in [2, T]$ be a round in the game, with state s and horizon $T - t + 1$. In Line 4, the state is updated based on the gambler's previous action q_{t-1}. In Lines 9-12, the odds r_t are computed from the vector v according to (17). It remains to show that Line 8 correctly computes the RLV v. Suppose that upon entering round t, it holds that L equals the optimal loss prior to observing q_{t-1}, and that the previous odds r_{t-1} were computed optimally. This holds for $t = 2$ by initialization and is preserved inductively. If $q_{t-1} \in \mathcal{E}_K$, i.e., the gambler is decisive, then by Theorem 9 the optimal loss cannot decrease. As r_{t-1} is chosen to achieve the RLV from the previous round, the updated v remains consistent with the definition of the RLV. If $q_{t-1} \notin \mathcal{E}_K$, then L is recomputed using the updated state s and horizon $T - t + 1$ (Line 10), and Line 8 produces the RLV. ■

Proof [Proof of Theorem A] The optimal bookmaking loss is a special case of the optimal opportunistic bookmaking loss. Thus, we can utilize Theorem C to conclude that $L^\star_{T,K}$ is the largest root of the polynomial $\mathcal{F}_{T,\mathbf{0}_K}(x)$ as follows

$$\begin{aligned}\mathcal{F}_{T,\mathbf{0}_K}(x) &\overset{(a)}{=} \mathcal{D}_{T,K}(x \cdot \mathbf{1}_K)\\ &\overset{(b)}{=} \sum_{m=0}^{K} \binom{K}{m} (-T)^{\overline{K-m}} x^m,\end{aligned}$$

where (a) follows from $\mathcal{F}_{H,s}(x) := \mathcal{D}_{H,K}(x \cdot \mathbf{1}_K - s)$ in (18) and (b) follows from $\mathcal{D}_{H,K}(v) := \sum_{m=0}^{K} (-H)^{\overline{K-m}} \sigma_m(v)$ in (16), together with Definition 1, noting that $\sigma_m(x \cdot \mathbf{1}_K) = \binom{K}{m} x^m$. This concludes the proof that $\mathcal{F}_{T,\mathbf{0}_K}(x) = \mathcal{P}_{T,K}(x)$. ■

5. Conclusions

We considered the online bookmaking problem, and derived the optimal bookmaking loss for any number of possible outcomes K and number of betting rounds T. Our solution, expressed as the largest root of a polynomial, provides a tractable and exact characterization of the optimal loss and strategy, a form of exact characterization that is uncommon in repeated vector games. This solution enabled us to show that the regret scales as $\sqrt{T}$ and, for any K, its scaling factor converges to the largest root of the K-th Hermite polynomial. We introduced the concept of opportunistic strategies and characterized the optimal opportunistic bookmaking loss, as well as an efficient algorithm that achieves it. This can be viewed as a water-filling solution in which the water level decreases whenever the stronger player (in our case, the gambler) fails to play optimally. The key to our analysis was the characterization of the Bellman-Pareto frontier, which may have broader applications to vector-valued problems in online learning and game theory. Future work may consider extending our approach to alternative loss functions or constrained settings, building on the structure offered by the Bellman-Pareto framework.

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Appendix Structure Appendix A presents preliminaries for the analysis. Appendix B establishes the proof of Theorem 9. Appendix C provides the proof of Lemma 14. Appendix D contains the omitted proofs from Section 3. Appendix E provides technical proofs that support the overall analysis.

Appendix A. Preliminaries

This appendix presents preliminary results used in the main analysis. Appendix A.1 defines notation used in the appendix, Appendix A.2 reviews formal definitions and properties of correspondences. Appendix A.3 provides key identities involving falling and rising factorials and Stirling numbers. Appendix A.4 provides several properties of elementary symmetric polynomials.

A.1. Additional Notation

We denote $\mathbb{R}_{++} \coloneqq \{x \in \mathbb{R} \mid x > 0\}$. For vectors $x, y \in \mathbb{R}^K$, the notation $x \odot y$ represents element-wise multiplication. For a set $\mathfrak{I} \subseteq [K]$, we write $x^{\setminus \mathfrak{I}}$ for x excluding indices in $\mathfrak{I}$. The interior of the simplex in $\mathbb{R}^K$ is denoted by $\operatorname{Int}\left(\Delta^{K-1}\right)$.

A.2. Correspondences

Definition 15 (Correspondence) *Let Θ and X be topological spaces. A* correspondence *from Θ to X, denoted $\mathcal{Z} : \Theta \rightrightarrows X$, is a mapping that assigns to each $\theta \in \Theta$ a subset of X, that is, $\mathcal{Z}(\theta) \subseteq X$, for every $\theta \in \Theta$.*

Definition 16 (Compact-valued Correspondence) *Let X and Θ be topological spaces. A correspondence $\mathcal{Z} : \Theta \rightrightarrows X$ is* compact-valued *if for every $\theta \in \Theta$, the set $\mathcal{Z}(\theta)$ is compact in X.*

Definition 17 (Continuity of a Correspondence) *Let X and Θ be topological spaces, and let $\mathcal{Z} : \Theta \rightrightarrows X$ be a correspondence.*

- *$\mathcal{Z}$ is* **upper hemicontinuous** *at $\theta \in \Theta$ if for every open set V with $\mathcal{Z}(\theta) \subset V$, there exists a neighborhood U of θ such that*

$$\mathcal{Z}(\theta') \subset V, \quad \forall \theta' \in U.$$

- *$\mathcal{Z}$ is* **lower hemicontinuous** *at $\theta \in \Theta$ if for every open set V with $V \cap \mathcal{Z}(\theta) \neq \emptyset$, there exists a neighborhood U of θ such that*

$$\mathcal{Z}(\theta') \cap V \neq \emptyset, \quad \forall \theta' \in U.$$

- *$\mathcal{Z}$ is* **continuous** *if it is both upper and lower hemicontinuous at every $\theta \in \Theta$.*

Lemma 18 (Berge's Maximum Theorem) *Let X and Θ be topological spaces, and let $f : X \times \Theta \to \mathbb{R}$ be a continuous function. Suppose that $\mathcal{Z} : \Theta \rightrightarrows X$ is a compact-valued correspondence and that $\mathcal{Z}(\theta) \neq \emptyset$ for all $\theta \in \Theta$. Define the function*

$$f^{\star}(\theta) = \sup_{x \in \mathcal{Z}(\theta)} f(x, \theta)$$

and the solution correspondence

$$\mathcal{Z}^{\star}(\theta) = \arg \max_{x \in \mathcal{Z}(\theta)} f(x, \theta).$$

If $\mathcal{Z}$ is continuous, then the function $f^{\star} : \Theta \to \mathbb{R}$ is continuous and the solution correspondence $\mathcal{Z}^{\star} : \Theta \rightrightarrows X$ is compact-valued and upper hemicontinuous.

For more information see Beavis and Dobbs (1990).

A.3. Falling and Rising Factorials and Stirling Numbers

For clarity, we restate the factorial operators given in Section 2.1.

Definition 19 (Falling and Rising Factorials) *For $x \in \mathbb{R}$ and $m \in \mathbb{N}$, the* falling factorial *is defined as*

$$x^{\underline{m}} := \prod_{i=0}^{m-1} (x - i) = x(x-1) \cdots (x - m + 1),$$

and the rising factorial *is defined as*

$$x^{\overline{m}} := \prod_{i=0}^{m-1} (x + i) = x(x+1) \cdots (x + m - 1).$$

In both cases, the empty product is equal to one so that $x^{\underline{0}} = x^{\overline{0}} = 1$.

Lemma 20 *For all $T, m \in \mathbb{N}$,*

1. $T^{\underline{m+1}} = T \cdot (T-1)^{\underline{m}}$
2. $T^{\underline{m}} + m \cdot T^{\underline{m-1}} = (T+1)^{\underline{m}}$
3. $T^{\underline{m}} = (-1)^m (-T)^{\overline{m}}$

Definition 21 (The Stirling Numbers of the First Kind) *The* Stirling numbers of the first kind $\begin{bmatrix} n \\ k \end{bmatrix}$ *count the number of permutations of n elements consisting of exactly k disjoint cycles, where $1 \leq k \leq n$. They satisfy the recurrence relation:*

$$\begin{bmatrix} n+1 \\ k \end{bmatrix} = n \begin{bmatrix} n \\ k \end{bmatrix} + \begin{bmatrix} n \\ k-1 \end{bmatrix}, \quad \textit{for} \;\; n \geq 1, \; k \geq 1,$$

with base cases:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 1, \qquad \begin{bmatrix} n \\ 0 \end{bmatrix} = 0 \;\; \textit{for} \;\; n > 0, \quad \begin{bmatrix} 0 \\ k \end{bmatrix} = 0 \;\; \textit{for} \;\; k > 0.$$

The signed Stirling numbers of the first kind $\mathrm{s}(n,k)$ *are defined by:*

$$\mathrm{s}(n,k) = (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}.$$

Lemma 22 (Expansion of Falling Factorials) *The rising and falling factorial can be expressed in terms of the Stirling numbers of the first kind as:*

$$x^{\overline{n}} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k, \qquad x^{\underline{n}} = \sum_{k=0}^{n} \mathrm{s}(n,k)\, x^k.$$

Graham et al. (1994) provides a thorough overview of rising and falling factorials, along with Stirling numbers and their combinatorial properties.

A.4. Properties of Elementary Symmetric Polynomials

Remark 23 (Generating Function for ESPs) *The ESPs of a vector $x \in \mathbb{R}^K$ can be expressed through their generating function by*

$$\sum_{m=0}^{K} \sigma_m(x) t^m = \prod_{k=1}^{K} (1 + x(k) \cdot t)$$

Lemma 24 (Recurrence Relation for ESPs) *Let $m, K \in \mathbb{N}_+$ and $x \in \mathbb{R}^K$. It holds that*

$$\sigma_m(x) = x(k) \cdot \sigma_{m-1}\left(x^{\setminus k}\right) + \sigma_m\left(x^{\setminus k}\right),$$

for every $k \in [K]$.

See Macdonald (1998) for more information about recurrence relations in symmetric polynomials.

Lemma 25 (Summation of reduced ESPs) *For all $K \in \mathbb{N}_+$ and $x \in \mathbb{R}^K$,*

$$\sum_{i=1}^{K} \sigma_m\left(x^{\setminus i}\right) = (K-m) \cdot \sigma_m(x), \quad 0 \le m \le K.$$

Lemma 26 (Shifted ESP Expansion) *Let $x \in \mathbb{R}^K$, $t \in \mathbb{R}$, and $n \in \mathbb{N}$. Then,*

$$\sigma_n(t \cdot \mathbf{1}_K - x) = \sum_{i=0}^{n} (-1)^i \sigma_i(x) \binom{K-i}{n-i} t^{n-i}.$$

The proofs of Lemmas 25 and 26 are provided in Appendix E.1.

Appendix B. Proof of Theorem 9

In this section, we prove Theorem 9, which characterizes a Nash equilibrium of the game.

Lemma 27 (Compact Action Space and Continuity) *For every $H \geq 1$, there exists a* continuous, non-empty and compact-valued correspondence $\mathcal{Z}_H : \mathbb{R}^K \rightrightarrows \mathrm{Int}\left(\Delta^{K-1}\right)$ *(see Appendix A.2 for definitions), such that*

$$\mathcal{V}_H\left(s\right) = \min_{r \in \mathcal{Z}_H(s)} \max_{q \in \Delta} \mathcal{V}_{H-1}\left(s + q \oslash r\right). \tag{B.1}$$

Consequently, $\mathcal{V}_H$ is continuous for all $H \in \mathbb{N}$.

While no fixed compact subset of the simplex interior contains the optimal bookmaker's actions for all H and s, Lemma 27 establishes that for each specific pair (H, s), such a compact set, $\mathcal{Z}_H(s)$, exists. The continuity of the correspondence $\mathcal{Z}_H$ in s ensures that, for all H, the value function $\mathcal{V}_H$ is continuous—a key property supporting our analysis and results. Lemma 27 is built on bounding the value function by the loss of a naïve bookmaker acting with the uniform distribution $r = \frac{1}{K} \cdot \mathbf{1}_K$ in all rounds. See Appendix B.1 for the full proof.

Lemma 28 *Let $H \geq 1$ and suppose that the value function $\mathcal{V}_{H-1}$ is convex. Then, for every state s, there exists an optimal gambler who is decisive; that is,*

$$\mathcal{V}_H\left(s\right) = \min_{r \in \mathcal{Z}_H(s)} \max_{k \in [K]} \mathcal{V}_{H-1}\left(s + \mathbf{e}_k \oslash r\right). \tag{B.2}$$

Proof [Proof of Lemma 28] Let $H \geq 1$, $s \in \mathbb{R}^K$ and suppose that $\mathcal{V}_{H-1}$ is a convex function. The gambler's action is a mixture of K decisive actions; that is, any vector $q \in \Delta$ can be written as a convex combination of the standard basis vectors $\mathcal{E}_K$. Let $X \in \mathcal{E}_K$ be a one-hot random variable distributed according to q, i.e.,

$$\mathbb{P}(X = \mathbf{e}_k) = q(k) \quad \text{for all } k \in [K], \tag{B.3}$$

and express (B.1) as

$$\mathcal{V}_H\left(s\right) = \min_{r \in \mathcal{Z}_H(s)} \max_{q \in \Delta} \mathcal{V}_{H-1}\left(s + \mathbb{E}_q\left[X\right] \oslash r\right).$$

For any $r \in \mathcal{Z}_H(s)$ it holds that

$$\begin{aligned}
\max_{q \in \Delta} \mathcal{V}_{H-1}\left(s + \mathbb{E}_q\left[X\right] \oslash r\right) &\overset{(a)}{=} \max_{q \in \Delta} \mathcal{V}_{H-1}\left(\mathbb{E}_q\left[s + X \oslash r\right]\right) \\
&\overset{(b)}{\leq} \max_{q \in \Delta} \mathbb{E}_q\left[\mathcal{V}_{H-1}\left(s + X \oslash r\right)\right] \\
&\overset{(c)}{=} \max_{q \in \Delta} \sum_{k \in [K]} q(k) \cdot \mathcal{V}_{H-1}\left(s + \mathbf{e}_k \oslash r\right) \\
&\overset{(d)}{=} \max_{q \in \mathcal{E}_K} \sum_{k \in [K]} q(k) \cdot \mathcal{V}_{H-1}\left(s + \mathbf{e}_k \oslash r\right) \\
&= \max_{k \in [K]} \mathcal{V}_{H-1}\left(s + \mathbf{e}_k \oslash r\right),
\end{aligned}$$

where: (a) follows by the linearity of expectation; (b) follows by Jensen's inequality and the assumption that $\mathcal{V}_{H-1}$ is a convex function; (c) follows by (B.3); and (d) holds since the sum is linear in q, and a maximum of a linear function over the simplex is attained at a vertex. ∎

As the state vector $s \in \mathbb{R}^K$ captures the payouts already committed to the possible outcomes, it is natural to expect that if $\hat{s} \succeq s$, then the bookmaker's optimal loss given $\hat{s}$ cannot be smaller than given s. Interestingly, even a single coordinate in s being smaller than the corresponding coordinate in $\hat{s}$ results in a strictly smaller optimal loss.

Lemma 29 (Coordinate-wise Monotonicity of the Value Function) *Let $\hat{s}, s \in \mathbb{R}^K$.*

1. **(Weak)** *For every $H \in \mathbb{N}$, $\hat{s} \succeq s \implies \mathcal{V}_H(\hat{s}) \geq \mathcal{V}_H(s)$.*
2. **(Strict)** *For every $H \in \mathbb{N}_+$, $\hat{s} \succ s \implies \mathcal{V}_H(\hat{s}) > \mathcal{V}_H(s)$.*

The proof of Lemma 29 is presented in Appendix B.2.

Lemma 30 *Let $H \geq 1$ and suppose that the value function $\mathcal{V}_{H-1}$ is convex. Then, for every state s, the optimal bookmaker's action $r^\star$ is unique and satisfies*

$$\mathcal{V}_H(s) = \mathcal{V}_{H-1}(s + q \oslash r^\star) \quad \forall q \in \mathcal{E}_K. \tag{20}$$

The proof of Lemma 30, given in Appendix B.3, proceeds by contradiction: any action r that fails to satisfy (20) enables a redistribution of probability mass that reduces the loss, contradicting optimality. Uniqueness follows from the coordinate-wise strict monotonicity of the value function (Lemma 29.2), which ensures no two distinct actions yield the same loss.

Lemma 31 *$\mathcal{V}_H$ is a convex function for every $H \in \mathbb{N}$.*

Proof [Proof of Lemma 31] We prove by induction on H. In the base case, $\mathcal{V}_0$ is a convex function as it is the maximum of finite convex functions. Let $H \in \mathbb{N}_+$ and assume that $\mathcal{V}_{H-1}$ is a convex function. Let $a, b \in \mathbb{R}^K$, $\lambda \in [0,1]$, and let $r_a, r_b, r \in \Delta^{K-1}$ be the optimal bookmaker actions for states a, b and $\lambda a + (1-\lambda)b$, respectively. It holds that

$$\begin{aligned}
\mathcal{V}_H(\lambda a + (1-\lambda)b) &\overset{(a)}{=} \max_{k\in[K]} \mathcal{V}_{H-1}(\lambda a + (1-\lambda)b + \mathbf{e}_k \oslash (\lambda r + (1-\lambda)r)) \\
&\overset{(b)}{\leq} \max_{k\in[K]} \mathcal{V}_{H-1}(\lambda a + (1-\lambda)b + \mathbf{e}_k \oslash (\lambda r_a + (1-\lambda)r_b)) \\
&\overset{(c)}{\leq} \max_{k\in[K]} \mathcal{V}_{H-1}(\lambda(a + \mathbf{e}_k \oslash r_a) + (1-\lambda)(b + \mathbf{e}_k \oslash r_b)) \\
&\overset{(d)}{\leq} \max_{k\in[K]} (\lambda \mathcal{V}_{H-1}(a + \mathbf{e}_k \oslash r_a) + (1-\lambda)\mathcal{V}_{H-1}(b + \mathbf{e}_k \oslash r_b)) \\
&\overset{(e)}{\leq} \lambda \max_{k\in[K]} \mathcal{V}_{H-1}(a + \mathbf{e}_k \oslash r_a) + (1-\lambda) \max_{k\in[K]} \mathcal{V}_{H-1}(b + \mathbf{e}_k \oslash r_b) \\
&\overset{(f)}{=} \lambda \mathcal{V}_H(a) + (1-\lambda)\mathcal{V}_{H+1}(b),
\end{aligned}$$

where the steps are justified as follows:

(a) Follows from Lemma 30.

(b) Follows from the optimality of r, which implies that the action $\lambda r_a + (1-\lambda) r_b$ induces a larger objective.

(c) Follows from the convexity of the function $x \mapsto \frac{1}{x}$ in $\mathbb{R}_+$ and Lemma 29.1. In particular, define the vectors

$$s := \lambda a + (1-\lambda) b + \mathbf{e}_k \oslash (\lambda r_a + (1-\lambda)\, r_b),$$
$$\hat{s} := \lambda\, (a + \mathbf{e}_k \oslash r_a) + (1-\lambda)\, (b + \mathbf{e}_k \oslash r_b)\,.$$

As $r_a(k), r_b(k) > 0$, it holds that $\lambda r_a(k) + (1-\lambda)\, r_b(k) > 0$. By weak coordinate-wise monotonicity of the value function, $\mathcal{V}_{H-1}\,(s) \leq \mathcal{V}_{H-1}\,(\hat{s})$.

(d) By the induction hypothesis, $\mathcal{V}_{H-1}$ is a convex function.

(e) Holds as maximizing each function separately can only increase the overall value.

(f) Follows from our choice of r_a, r_b as the optimal actions for states a and b, respectively.

■

Proof [Proof of Theorem 9] Fix $H \in \mathbb{N}_+$ and $s \in \mathbb{R}^K$. By Lemma 31, the value function $\mathcal{V}_{H-1}$ is convex. This implies, by Lemma 28, the existence of an optimal gambler who is decisive. Moreover, Lemma 30 guarantees uniqueness of the optimal bookmaker action $r^\star \in \Delta$, which satisfies

$$\mathcal{V}_H\,(s) = \mathcal{V}_{H-1}\,(s + q \oslash r^\star) \quad \forall q \in \mathcal{E}_K. \tag{20}$$

■

B.1. Proof of Lemma 27

This section proves Lemma 27, showing that for all $H \geq 1$, the minimization over r in the value function (Definition 7) can be restricted to a continuous, compact-valued correspondence

$$\mathcal{Z}_H : \mathbb{R}^K \rightrightarrows \mathrm{Int}\left(\Delta^{K-1}\right).$$

Building on this property, we establish the continuity of the value function $\mathcal{V}_H$ with respect to s.

We begin with the following lemma:

Lemma 32 (Uniform Translation) *For any $H \in \mathbb{N}$, $s \in \mathbb{R}^K$ and $c \in \mathbb{R}$,*

$$\mathcal{V}_H\,(s + c \cdot \mathbf{1}_K) = \mathcal{V}_H\,(s) + c.$$

The proof of Lemma 32, provided in Appendix E.2, proceeds by induction on H using the recursive definition of the value function in Definition 7.

Corollary 33 *Let $H \in \mathbb{N}$, $s \in \mathbb{R}^K$ and $c \in \mathbb{R}$. For the vector $\hat{s} = s + c \cdot \mathbf{1}_K$, it holds that*

$$\arg\inf_{r\in\Delta} \left\{\max_{q\in\Delta} \mathcal{V}_H\,(s + q \oslash r)\right\} = \arg\inf_{r\in\Delta} \left\{\max_{q\in\Delta} \mathcal{V}_H\,(\hat{s} + q \oslash r)\right\}.$$

The key idea in the proof of Lemma 27 is that the value function must be bounded above by the maximal loss incurred under any alternative bookmaking strategy. In particular, the value function is upper bounded by the performance of a naïve bookmaker acting with the uniform distribution $r = \frac{1}{K} \cdot \mathbf{1}_K$ in all rounds. For $H \in \mathbb{N}_+$ define the function $\omega_H : \mathbb{R}^K \to \mathbb{R}_{++}$ as

$$\omega_H(s) \ := \ \frac{1}{\max_{i\in[K]} s(i) - \min_{j\in[K]} s(j) + HK}, \tag{B.4}$$

and define the correspondence $\mathcal{Z}_H : \mathbb{R}^K \rightrightarrows \mathrm{Int}\left(\Delta^{K-1}\right)$ as

$$\mathcal{Z}_H(s) \ := \ \{r \in \Delta^{K-1} \mid r \succeq \omega_H(s) \cdot \mathbf{1}_K\}. \tag{B.5}$$

Lemma 34 *The correspondence $\mathcal{Z}_H$ is continuous, non-empty and compact-valued (see Appendix A.2 for definitions) for all $H \in \mathbb{N}_+$.*

The proof of Lemma 34 is provided in Appendix E.2.
Proof [Proof of Lemma 27] Let $H \in \mathbb{N}_+$, $s \in \mathbb{R}^K$ and define

$$\hat{s} := s - \min_{j\in[K]} s(j) \cdot \mathbf{1}_K. \tag{B.6}$$

Consider a game with H rounds and initial state vector $\hat{s}$. An optimal loss is bounded above by the loss of any other bookmaker, hence

$$\mathcal{V}_H\left(\hat{s}\right) \le \max_{i\in[K]} \hat{s}(i) + HK, \tag{B.7}$$

where the RHS of (B.7) is the maximal loss of the naïve bookmaker acting with the uniform distribution $r = \frac{1}{K} \cdot \mathbf{1}_K$ in all rounds. We claim that

$$\arg\inf_{r\in\Delta} \left\{\max_{q\in\Delta} \mathcal{V}_{H-1}\left(\hat{s} + q \oslash r\right)\right\} \subseteq \mathcal{Z}_H(s). \tag{B.8}$$

Assume in contradiction there exists r in the LHS of (B.8) which is not in the RHS. Then there exist $k \in [K]$ and $\kappa > 0$ such that

$$r(k) = \frac{1}{\max_{i\in[K]} s(i) - \min_{j\in[K]} s(j) + HK + \kappa}.$$

Since the q term in the LHS of (B.8) can be $\mathbf{e}_k$, we obtain

$$\begin{aligned}
\mathcal{V}_H\left(\hat{s}\right) &\ge \max_{i\in[K]} s(i) - \min_{j\in[K]} s(j) + HK + \kappa \\
&\overset{(a)}{=} \max_{i\in[K]} \hat{s}(i) + HK + \kappa \\
&> \max_{i\in[K]} \hat{s}(i) + HK \\
&\overset{(b)}{\ge} \mathcal{V}_H\left(\hat{s}\right),
\end{aligned}$$

which yields a contradiction. Here, (a) follows from the construction of $\hat{s}$ in (B.6), and (b) follows from the upper bound on $\mathcal{V}_H\left(\hat{s}\right)$ in (B.7). Combining (B.8) with Corollary 33, we conclude that

$$\arg\inf_{r\in\Delta} \left\{\max_{q\in\Delta} \mathcal{V}_{H-1}\left(s + q \oslash r\right)\right\} \subseteq \mathcal{Z}_H(s). \tag{B.9}$$

Thus, the value function $\mathcal{V}_H$ is given by

$$\mathcal{V}_H\left(s\right)=\min_{r\in\mathcal{Z}_H(s)}\max_{q\in\Delta}\mathcal{V}_{H-1}\left(s+q\oslash r\right),$$

where by Lemma 34, $\mathcal{Z}_H(s)$ is a continuous, non-empty and compact-valued correspondence.

We prove by induction on H that $\mathcal{V}_H$ is continuous.

- *Base case* $(H=0)$: In the base case, $\mathcal{V}_0$ is continuous as it is the maximum of finite continuous functions.
- *Inductive step* $(H-1\to H)$: Let $H\in\mathbb{N}_+$ and define the function $\mathcal{U}:\mathbb{R}^K\times\mathbb{R}_+^K\to\mathbb{R}$ as

$$\mathcal{U}(s,r)\ :=\ \max_{q\in\Delta}\ \mathcal{V}_{H-1}\left(\,s+q\oslash r\,\right). \tag{B.10}$$

  We claim that $\mathcal{U}(s,r)$ is continuous in (s,r): for any fixed $q\in\Delta$, the mapping

$$(s,r)\mapsto s+q\oslash r$$

  is continuous in (s,r) on $\mathbb{R}^K\times\mathbb{R}_+^K$ as division by a positive number and addition of vectors are continuous. By the induction hypothesis, $\mathcal{V}_{H-1}$ is continuous, and thus the composition $(s,r)\mapsto\mathcal{V}_{H-1}\left(s+q\oslash r\right)$ is continuous. Since Δ^{K-1} is a compact set, the function $\mathcal{U}$ is the pointwise maximum of continuous functions over a compact set, and hence continuous in (s,r). We express the value function as:

$$\begin{aligned}\mathcal{V}_H\left(s\right)&=\inf_{r\in\mathcal{Z}_H(s)}\ \max_{q\in\Delta}\ \mathcal{V}_{H-1}\left(\,s+q\oslash r\,\right)\\&=\inf_{r\in\mathcal{Z}_H(s)}\ \mathcal{U}(s,r),\end{aligned}$$

  where the first equality follows from (B.9), and the second equality follows from the definition of $\mathcal{U}(s,r)$ in (B.10). The function $\mathcal{U}(s,r)$ is continuous, as established above, and the correspondence $\mathcal{Z}_H:\mathbb{R}^K\rightrightarrows\Delta^{K-1}$ is compact-valued, continuous, and non-empty-valued by Lemma 34. Hence, the conditions of Berge's Maximum Theorem (Lemma 18) are satisfied, and $\mathcal{V}_H$ is continuous.

■

B.2. Proof of Lemma 29

This section provides the proof of Lemma 29, which establishes coordinate-wise monotonicity of the value function. We first prove the strict case (Lemma 29.2) and then derive the weak case (Lemma 29.1) as a direct adaptation.

B.2.1. PROOF OF LEMMA 29.2

The degenerate case $K=1$ follows directly from Definition 7: Let $H\in\mathbb{N}_+$ and $\hat{s},s$ be 1-dimensional vectors such that $\hat{s}\succ s$; that is, $\hat{s}(1)>s(1)$. It holds that

$$\begin{aligned}\mathcal{V}_H\left(\hat{s}\right)&=\hat{s}(1)+H\\&>s(1)+H\\&=\mathcal{V}_H\left(s\right),\end{aligned}$$

where the equalities follow from the fact that $q_t(1) = r_t(1) = 1$ for all $t \in [H]$.

To prove coordinate-wise strict monotonicity in the general case, we analyze the game at the last round, as its structure differs from other rounds. The following lemma establishes a Nash equilibrium of the game when $H = 1$.

Lemma 35 *In case $H = 1$, for every $s \in \mathbb{R}^K$ the optimal bookmaking action $r^\star \in \mathcal{Z}_1(s)$ is unique and satisfies*

$$\mathcal{V}_1(s) = \mathcal{V}_0(s + q \oslash r^\star) \quad \forall q \in \mathcal{E}_K. \tag{B.11}$$

The proof of Lemma 35 is presented in Appendix E.2.

Proof [Proof of Lemma 29.2] Let $\hat{s}, s \in \mathbb{R}^K$ be such that $\hat{s} \succ s$. We prove by induction on H.

- *Base case* ($H = 1$): By Lemma 35, it holds that

$$\mathcal{V}_1(s) = s(k) + \frac{1}{r(k)} \quad \forall k \in [K]$$

for some $r \in \Delta$. Assume in contradiction that $\mathcal{V}_1(\hat{s}) \le \mathcal{V}_1(s)$. Then there exists $\hat{r} \in \Delta$ such that

$$\hat{s}(k) + \frac{1}{\hat{r}(k)} \le s(k) + \frac{1}{r(k)} \quad \forall k \in [K]. \tag{B.12}$$

Since $\hat{s} \succ s$, there exists $i \in [K]$ such that $\hat{s}(i) > s(i)$. By (B.12), it must hold that $\hat{r}(i) > r(i)$. Then, there must exist $j \in [K] \setminus \{i\}$ such that $\hat{r}(j) < r(j)$. It follows that

$$\hat{s}(j) + \frac{1}{\hat{r}(j)} \overset{(a)}{\ge} s(j) + \frac{1}{\hat{r}(j)} \overset{(b)}{>} s(j) + \frac{1}{r(j)},$$

where (a) holds as $\hat{s} \succeq s$, and (b) holds since $\hat{r}(j) < r(j)$. This contradicts (B.12).

- *Inductive step* ($H \to H + 1$): Define

$$\Delta_\omega = \{r \in \Delta \mid r \succeq \min\{\omega_{H+1}(s), \omega_{H+1}(\hat{s})\} \cdot \mathbf{1}_K\}.$$

Δ_ω is a compact set for which $\mathcal{Z}_H(s), \mathcal{Z}_H(\hat{s}) \subseteq \Delta_\omega$. Hence, by Lemma 27,

$$\mathcal{V}_{H+1}(s) = \min_{r \in \Delta_\omega} \max_{q \in \Delta} \mathcal{V}_H(s + q \oslash r) \tag{B.13}$$

$$\mathcal{V}_{H+1}(\hat{s}) = \min_{r \in \Delta_\omega} \max_{q \in \Delta} \mathcal{V}_H(\hat{s} + q \oslash r). \tag{B.14}$$

By (B.14), there exists $\hat{r} \in \Delta_\omega$ such that

$$\mathcal{V}_{H+1}(\hat{s}) = \max_{q \in \Delta} \mathcal{V}_H(\hat{s} + q \oslash \hat{r}). \tag{B.15}$$

Fix

$$\bar{q} \in \arg\max_{q \in \Delta} \mathcal{V}_H(s + q \oslash \hat{r}). \tag{B.16}$$

It holds that

$$\begin{aligned}
\mathcal{V}_{H+1}(s) &\overset{(a)}{=} \min_{r\in\Delta_\omega}\max_{q\in\Delta}\mathcal{V}_H(s+q\oslash r)\\
&\overset{(b)}{\leq} \max_{q\in\Delta}\mathcal{V}_H(s+q\oslash\hat{r})\\
&\overset{(c)}{=} \mathcal{V}_H(s+\bar{q}\oslash\hat{r})\\
&\overset{(d)}{<} \mathcal{V}_H(\hat{s}+\bar{q}\oslash\hat{r})\\
&\overset{(e)}{\leq} \max_{q\in\Delta}\mathcal{V}_H(\hat{s}+q\oslash\hat{r})\\
&\overset{(f)}{=} \mathcal{V}_{H+1}(\hat{s}),
\end{aligned}$$

where the steps are justified as follows:

(a) Follows from (B.13).

(b) Holds since the minimizing r yields an objective no larger than that induced by any $\hat{r}$.

(c) Follows from the choice of $\bar{q}$ in (B.16).

(d) Follows from the induction hypothesis on H. In particular, since $s \prec \hat{s}$, it holds that

$$s+\bar{q}\oslash\hat{r} \prec \hat{s}+\bar{q}\oslash\hat{r},$$

and thus, by the induction hypothesis,

$$\mathcal{V}_H(s+\bar{q}\oslash\hat{r}) < \mathcal{V}_H(\hat{s}+\bar{q}\oslash\hat{r}).$$

(e) Holds since taking the maximum over q, with $\hat{s}$ and $\hat{r}$ fixed, can only increase the value.

(f) Follows from the choice of $\hat{r}$ as one that satisfies (B.15).

This establishes that $\mathcal{V}_{H+1}(s) < \mathcal{V}_{H+1}(\hat{s})$, completing the proof.

■

B.2.2. PROOF OF LEMMA 29.1

Proof [Proof of Lemma 29.1] Let $\hat{s}, s \in \mathbb{R}^K$ be such that $\hat{s} \succeq s$. For the case $H = 0$, it holds that

$$\mathcal{V}_0(\hat{s}) := \max_{k\in[K]}\hat{s}(k) \geq \max_{k\in[K]}s(k) = \mathcal{V}_0(s).$$

Assume $H \geq 1$. If $\hat{s} = s$ the statement is trivial. Otherwise, since $\hat{s} \succ s$, the result follows from Lemma 29.2. ■

B.3. Proof of Lemma 30

Proof [Proof of Lemma 30] Let $H \geq 1$, $K \geq 2$ and $s \in \mathbb{R}^K$. The case $H = 1$ is treated in Lemma 35, and for $K = 1$ the result follows immediately. By Lemma 28, there exists $r \in \mathcal{Z}_H(s)$ for which

$$\mathcal{V}_H(s) = \max_{k \in [K]} \mathcal{V}_{H-1}(s + \mathbf{e}_k \oslash r).$$

By coordinate-wise strict monotonicity (Lemma 29.2), there exists vector $u \in \mathbb{R}^K_{++}$ such that

$$\mathcal{V}_{H-1}(s + \mathbf{e}_k \oslash r) = \mathcal{V}_{H-1}(s) + u(k) \quad \forall k \in [K]. \tag{B.17}$$

Assume, towards a contradiction, that r does not satisfy (20); that is, there exists $i \in [K]$ and $\varepsilon > 0$ such that

$$\max_{k \in [K]} u(k) - u(i) = \varepsilon. \tag{B.18}$$

Relying on the continuity of the value function (Lemma 27) and the fact that

$$\lim_{r(i) \to 0} \mathcal{V}_{H-1}\left(s + \frac{\mathbf{e}_i}{r(i)}\right) = \infty,$$

the intermediate value theorem guarantees the existence of a scalar $0 < \tilde{r} < r(i)$ for which

$$\mathcal{V}_{H-1}\left(s + \frac{\mathbf{e}_i}{\tilde{r}}\right) = \mathcal{V}_{H-1}(s) + u(i) + \frac{\varepsilon}{2}. \tag{B.19}$$

We construct a new action $\hat{r}$ as follows:

$$\hat{r}(k) = \begin{cases} \tilde{r} & \text{if } k = i, \\ r(k) + \dfrac{r(i) - \tilde{r}}{K-1} & \text{if } k \neq i. \end{cases}$$

It is easy to verify that $\hat{r} \in \Delta^{K-1}$. By acting with $\hat{r}$ the bookmaker's loss decreases:

- *For* $k = i$:

$$\begin{aligned} \mathcal{V}_{H-1}\left(s + \frac{\mathbf{e}_i}{\hat{r}(i)}\right) &\overset{(a)}{=} \mathcal{V}_{H-1}(s) + u(i) + \frac{\varepsilon}{2} \\ &\overset{(b)}{<} \mathcal{V}_{H-1}(s) + \max_{k \in [K]} u(k), \end{aligned}$$

  where (a) follows from our choice of $\hat{r}(i) = \tilde{r}$ to satisfy (B.19), and (b) follows from (B.18), verifying a gap of $\frac{\varepsilon}{2} > 0$.

- *For* $k \in [K] \setminus \{i\}$:

$$\begin{aligned} \mathcal{V}_{H-1}\left(s + \frac{\mathbf{e}_k}{\hat{r}(k)}\right) &\overset{(a)}{<} \mathcal{V}_{H-1}\left(s + \frac{\mathbf{e}_k}{r(k)}\right) \\ &\overset{(b)}{\leq} \mathcal{V}_{H-1}(s) + \max_{k \in [K]} u(k), \end{aligned}$$

  where (a) follows from coordinate-wise strict monotonicity (Lemma 29.2) combined with the fact that $\hat{r}(k) > r(k)$, and (b) follows from (B.17).

It follows that r is suboptimal, contradicting the optimality assumption.

It remains to show that the optimal bookmaking action is unique. Let $r^\star, \hat{r} \in \mathcal{Z}_H(s)$ be two vectors that satisfy (20). I.e.,

$$\mathcal{V}_{H-1}\left(s + \frac{\mathbf{e}_k}{r^\star(k)}\right) = \mathcal{V}_{H-1}\left(s + \frac{\mathbf{e}_k}{r(k)}\right) \quad \forall k \in [K].$$

Suppose, for the sake of contradiction, that there exists $k \in [K]$ such that $r^\star(k) \neq r(k)$, and without loss of generality, assume $r(k) > r^\star(k)$. By coordinate-wise strict monotonicity (Lemma 29.2), it follows that

$$\mathcal{V}_{H-1}\left(s + \frac{\mathbf{e}_k}{r^\star(k)}\right) < \mathcal{V}_{H-1}\left(s + \frac{\mathbf{e}_k}{r(k)}\right),$$

contradicting the optimality of r. ■

Appendix C. Proof of Lemma 14

In this section, we prove Lemma 14, which establishes necessary constraints for H-achievable vectors. We begin by examining the partial derivatives of the polynomial $\mathcal{D}_{H,K}$, defined in (16), as presented in the following lemma.

Lemma 36 (Partial Derivatives of $\mathcal{D}_{H,K}$) *For every $H, K \in \mathbb{N}_+$,*

1. *$\mathcal{D}_{H,K}$ is an infinitely differentiable function.*
2. *For every $m \in [K]$ and $\mathfrak{I} \in \binom{[K]}{m}$*

$$\frac{\partial^m \mathcal{D}_{H,K}}{\partial^m v_{\mathfrak{I}}}(v) = \mathcal{D}_{H,K-m}\left(v^{\setminus \mathfrak{I}}\right),$$

   where for $\mathfrak{I} = \{i_1, \ldots, i_m\}$ the expression $\partial^m v_{\mathfrak{I}}$ stands for $\partial v(i_1) \ldots \partial v(i_m)$.

3. *For every $k \in [K]$ and $m > 1$,*

$$\frac{\partial^m \mathcal{D}_{H,K}}{\partial v(k)^m}(v) = 0.$$

The proof of Lemma 36 is provided in Appendix E.3 and builds on the recurrence relation established in Lemma 13:

$$\mathcal{D}_{H,K}(v) = v(k) \cdot \mathcal{D}_{H,K-1}(v^{\setminus k}) - H \cdot \mathcal{D}_{H-1,K-1}(v^{\setminus k}),$$

for all $H, K \in \mathbb{N}_+$.

For the analysis, we use an equivalent form of the polynomial $\mathcal{D}_{H,K}$, given by

$$\mathcal{D}_{H,K}(v) = \sum_{m=0}^{K} (-1)^m H^{\underline{m}} \sigma_{K-m}(v) \tag{C.1}$$

for all $H, K \in \mathbb{N}$, which follows from Lemma 20.3. For convenience, we introduce an alternative notation for the polynomial $\mathcal{D}_{H,K}$. For $H \in \mathbb{N}_+$ and $K \in \mathbb{N}$, define the polynomial

$$\mathcal{N}_{H,K}(v) \;\coloneqq\; H \cdot \mathcal{D}_{H-1,K}(v). \tag{C.2}$$

This allows us to express Lemma 13 in the following equivalent form:

$$\mathcal{D}_{H,K}(v) = v(k) \cdot \mathcal{D}_{H,K-1}(v^{\setminus k}) - \mathcal{N}_{H,K-1}(v^{\setminus k}), \tag{C.3}$$

for all $H, K \in \mathbb{N}_+$.

We prove Lemma 14.2 using the equivalent from of (25), expressed with the notation $\mathcal{N}_{H,K}(v)$:

$$v(k) = \frac{\mathcal{N}_{H,K-1}\left(v^{\setminus k}\right)}{\mathcal{D}_{H,K-1}\left(v^{\setminus k}\right)}.$$

To prove Lemma 14.3, we will show that

$$\forall\, m \in [K],\ \forall\, \mathfrak{I} \in \binom{[K]}{m}, \quad \mathcal{D}_{H,K-m}\left(v^{\setminus \mathfrak{I}}\right) > 0. \tag{C.4}$$

The following lemma establishes that this condition implies the same result:

Lemma 37 *Let $H, K \in \mathbb{N}_+$ and $v \in \mathbb{R}^K$. If v satisfies the constraint in Lemma 14.2, then the positivity condition (C.4) implies that v also satisfies the constraint in Lemma 14.3.*

Proof [Proof of Lemma 37] For a fixed $k \in [K]$, it holds that

$$\frac{\partial \mathcal{D}_{H,K}}{\partial v(k)}(v) \overset{(a)}{=} \mathcal{D}_{H,K-1}\left(v^{\setminus k}\right) \overset{(b)}{>} 0\,, \tag{C.5}$$

where (a) follows from Lemma 36.2 and (b) follows by (C.4). Thus,

- By Lemma 14.2, $\mathcal{D}_{H,K}(v) = 0$.
- By (C.5), $\nabla \mathcal{D}_{H,K}(v)$ is a strictly positive vector.
- By Lemma 36.3, for every $k \in [K]$ and $m > 1$,

  $$\frac{\partial^m \mathcal{D}_{H,K}}{\partial v(k)^m}(v) = 0,$$

  indicating $\mathcal{D}_{H,K}$ has no concavity along any axis.

We conclude that for all $u \in \mathbb{R}^K,\ \ u \succ v \implies \mathcal{D}_{H,K}(u) > 0$, as Lemma 14.3 states. ■

The following lemma provides a lower bound on any H-achievable vector.

Lemma 38 (Lower Bound on H-Achievable Vectors) *For all $H, K \in \mathbb{N}_+$, if $v \in \mathfrak{P}_{H,K}$ then $v \succeq H \cdot \mathbf{1}_K$. Moreover, when $K > 1$, the inequality is strict: $v \succ H \cdot \mathbf{1}_K$.*

This lower bound follows from the fact that the gambler may commit to a single outcome in each of the remaining H rounds. According to the definition of the ESP (Definition 1), it follows that for any H-achievable vector v,

$$\forall\, m \in [K],\ \forall\, \mathfrak{I} \in \binom{[K]}{m}, \quad \sigma_{K-m}\left(v^{\setminus \mathfrak{I}}\right) > 0. \tag{C.6}$$

Consequently, throughout the proof of Lemma 14, we assume that no division by zero occurs when dividing by sums or products of elements of v. A formal proof of Lemma 38 is provided in Appendix E.3.

Proof [Proof of Lemma 14] The case $K = 1$ is immediate from Definition 11; for completeness, a formal proof is provided in Appendix E.3. We then consider $K \geq 2$, and prove by induction on H.

Base case ($H = 1$). If $v \in \mathfrak{P}_{1,K}$ then, by Definition 11, there exists $r \in \mathbb{R}^K$ such that

1.i. $\sum_{k=1}^{K} r(k) = 1$.

1.ii. $\forall k \in [K]\,,\ v(k) = \frac{1}{r(k)}$.

The following hold for every $k \in [K]$:

$$\mathcal{D}_{0,K}\left(v\right) = \sigma_K\left(v\right) \tag{C.7}$$

$$\mathcal{D}_{0,K-1}\left(v^{\setminus k}\right) = \sigma_{K-1}\left(v^{\setminus k}\right) \tag{C.8}$$

$$\mathcal{D}_{1,K-1}\left(v^{\setminus k}\right) = \prod_{i\in[K]\setminus\{k\}} v(i)\left(1 - \sum_{j\in[K]\setminus\{k\}} \frac{1}{v(j)}\right) \tag{C.9}$$

$$\mathcal{N}_{1,K-1}\left(v^{\setminus k}\right) = \sigma_{K-1}\left(v^{\setminus k}\right) \tag{C.10}$$

Condition 1.ii is satisfied if and only if

$$\forall k \in [K] \quad r(k) = \frac{1}{v(k)}. \tag{C.11}$$

For every $k \in [K]$, it holds that

$$\begin{aligned}\frac{1}{v(k)} &\overset{(a)}{=} \frac{\sigma_{K-1}\left(v^{\setminus k}\right)}{\sigma_K\left(v\right)} \\ &\overset{(b)}{=} \frac{\mathcal{D}_{0,K-1}\left(v^{\setminus k}\right)}{\mathcal{D}_{0,K}\left(v\right)},\end{aligned}$$

where (a) follows from Definition 1 of the ESP, and (b) follows from (C.8) and (C.7). Therefore, Lemma 14.1 holds for the base case.

The vector r, which is defined as in (C.11), should satisfy Condition 1.i; i.e.,

$$\sum_{k\in[K]} \frac{1}{v(k)} = 1. \tag{C.12}$$

For every $k \in [K]$, it holds that

$$\begin{aligned}\frac{1}{v(k)} &= 1 - \sum_{j\in[K]\setminus\{k\}} \frac{1}{v(j)} \\ &= \frac{\prod_{i\in[K]\setminus\{k\}} v(i)}{\prod_{i\in[K]\setminus\{k\}} v(i)}\left(1 - \sum_{j\in[K]\setminus\{k\}} \frac{1}{v(j)}\right) \\ &= \frac{\mathcal{D}_{1,K-1}\left(v^{\setminus k}\right)}{\mathcal{N}_{1,K-1}\left(v^{\setminus k}\right)},\end{aligned}$$

where the last equality follows from (C.9) and (C.10). Therefore,

$$v(k) = \frac{\mathcal{N}_{1,K-1}\left(v^{\setminus k}\right)}{\mathcal{D}_{1,K-1}\left(v^{\setminus k}\right)}, \tag{C.13}$$

and Lemma 14.2 holds for the base case.

We prove that (C.4) holds by induction on m. Combined with Lemma 37, this completes the proof of Lemma 14.3 for the base case.

- *Base case* $(m = 1)$: For every $k \in [K]$, it holds that

$$\begin{aligned} 0 &\stackrel{(a)}{=} \mathcal{D}_{1,K}(v) \\ &\stackrel{(b)}{=} v(k) \cdot \mathcal{D}_{1,K-1}\left(v^{\setminus k}\right) - \mathcal{N}_{1,K-1}\left(v^{\setminus k}\right) \\ &\stackrel{(c)}{=} v(k) \cdot \mathcal{D}_{1,K-1}\left(v^{\setminus k}\right) - \sigma_{K-1}\left(v^{\setminus k}\right) \end{aligned}$$

  where (a) follows from Lemma 14.2; (b) follows from the identity in (C.3); and (c) follows from (C.10). By (C.6), both $v(k)$ and $\sigma_{K-1}\left(v^{\setminus k}\right)$ are > 0, and therefore, $\mathcal{D}_{1,K-1}\left(v^{\setminus k}\right)$ must be > 0.

- *Inductive step* $(m \to m+1)$: Let $m \in [K-1]$, $\mathfrak{I} \in \binom{[K]}{m}$ and $k \in [K] \setminus \mathfrak{I}$ be fixed. By (C.3), it holds that

$$\mathcal{D}_{1,K-m}\left(v^{\setminus \mathfrak{I}}\right) = v(k) \cdot \mathcal{D}_{1,K-(m+1)}\left(v^{\setminus \mathfrak{I}\cup\{k\}}\right) - \sigma_{K-(m+1)}\left(v^{\setminus \mathfrak{I}\cup\{k\}}\right).$$

  From the induction hypothesis (on m) The LHS of the equation is positive. By (C.6), the term $\mathcal{D}_{1,K-(m+1)}\left(v^{\setminus \mathfrak{I}\cup\{k\}}\right)$ must be > 0.

Inductive step ($H \to H+1$). If $v \in \mathfrak{P}_{H+1,K}$ then there exists an action $r \in \mathbb{R}^K$ such that

2.i. $\sum_{k=1}^{K} r(k) = 1$.

2.ii. For every $k \in [K]$, the vector ${}^{k}v$, defined as

$$ {}^{k}v(i) := \begin{cases} v(i) - \frac{1}{r(k)} & \text{if } k = i, \\ v(i) & \text{otherwise.} \end{cases} \tag{C.14}$$

is H-achievable.

Note that

$$ {}^{k}v^{\setminus k} = v^{\setminus k}, \quad \forall k \in [K]. \tag{C.15}$$

For every $k \in [K]$, it holds that

$$\begin{aligned} 0 &\stackrel{(a)}{=} \mathcal{D}_{H,K}\left({}^{k}v\right) &\text{(C.16)} \\ &\stackrel{(b)}{=} {}^{k}v(k) \cdot \mathcal{D}_{H,K-1}\left({}^{k}v^{\setminus k}\right) - \mathcal{N}_{H,K-1}\left({}^{k}v^{\setminus k}\right) \\ &\stackrel{(c)}{=} \left(v(k) - \frac{1}{r(k)}\right) \cdot \mathcal{D}_{H,K-1}\left(v^{\setminus k}\right) - \mathcal{N}_{H,K-1}\left(v^{\setminus k}\right) \\ &= v(k) \cdot \mathcal{D}_{H,K-1}\left(v^{\setminus k}\right) - \mathcal{N}_{H,K-1}\left(v^{\setminus k}\right) - \frac{\mathcal{D}_{H,K-1}\left(v^{\setminus k}\right)}{r(k)} \\ &\stackrel{(d)}{=} \mathcal{D}_{H,K}(v) - \frac{\mathcal{D}_{H,K-1}\left(v^{\setminus k}\right)}{r(k)}, &\text{(C.17)} \end{aligned}$$

where the steps are justified as follows:

(a) Follows from Condition 2.ii and the induction hypothesis on Lemma 14.2.

(b) Follow from the identity in (C.3).

(c) Follow from the definition of the vector ${}^k v$ in (C.14) and (C.15).

(d) Follow from the identity in (C.3).

Condition 2.ii, (C.15) and the induction hypothesis on Lemma 14.3, imply

$$\mathcal{D}_{H,K-1}\left(v^{\setminus k}\right) > 0 \quad \forall k \in [K]. \tag{C.18}$$

By (C.17), (C.18) and the fact that $r(k)$ must be > 0, we conclude that

$$\mathcal{D}_{H,K}\left(v\right) \neq 0, \tag{C.19}$$

and we can divide by this term. We obtain that r is given by

$$r(k) = \frac{\mathcal{D}_{H,K-1}\left(v^{\setminus k}\right)}{\mathcal{D}_{H,K}\left(v\right)} \qquad \forall k \in [K]\,. \tag{C.20}$$

This completes the inductive step for Lemma 14.1.

Let $k \in [K]$ be fixed and let r be a vector that is generated as in (C.20). r satisfies Condition 2.ii if and only if

$$r(k) = 1 - \sum_{i \in [K] \setminus \{k\}} r(i). \tag{C.21}$$

By (C.19), (C.21) holds if and only if

$$\mathcal{D}_{H,K}\left(v\right) \cdot r(k) = \mathcal{D}_{H,K}\left(v\right) - \mathcal{D}_{H,K}\left(v\right) \sum_{i \in [K] \setminus \{k\}} r(i). \tag{C.22}$$

By (C.20), the LHS of (C.22) is $\mathcal{D}_{H,K-1}\left(v^{\setminus k}\right)$, and for every $i \in [K] \setminus \{k\}$,

$$\begin{aligned}\mathcal{D}_{H,K}\left(v\right) \cdot r(i) &= \mathcal{D}_{H,K-1}\left(v^{\setminus i}\right) \\ &= v(k) \cdot \mathcal{D}_{H,K-2}\left(v^{\setminus\{i,k\}}\right) - \mathcal{N}_{H,K-2}\left(v^{\setminus\{i,k\}}\right),\end{aligned}$$

where the second equality follows from the identity in (C.3). Thus,

$$\mathcal{D}_{H,K}\left(v\right) \sum_{i \in [K] \setminus \{k\}} r(i) = v(k) \sum_{i \in [K] \setminus \{k\}} \mathcal{D}_{H,K-2}\left(v^{\setminus\{i,k\}}\right) - \sum_{i \in [K] \setminus \{k\}} \mathcal{N}_{H,K-2}\left(v^{\setminus\{i,k\}}\right),$$

and (C.22) holds if and only if

$$\mathcal{D}_{H,K-1}\left(v^{\setminus k}\right) = \mathcal{D}_{H,K}\left(v\right) - v(k) \sum_{i \in [K] \setminus \{k\}} \mathcal{D}_{H,K-2}\left(v^{\setminus\{i,k\}}\right) + \sum_{i \in [K] \setminus \{k\}} \mathcal{N}_{H,K-2}\left(v^{\setminus\{i,k\}}\right).$$

Further expanding $\mathcal{D}_{H,K}\left(v\right)$ using (C.3) and rearrange, we get that

$$\mathtt{A} = v(k) \cdot \mathtt{B}, \tag{C.23}$$

with

$$\mathtt{A} = \mathcal{D}_{H,K-1}\left(v^{\setminus k}\right) + \mathcal{N}_{H,K-1}\left(v^{\setminus k}\right) - \sum_{i\in[K]\setminus\{k\}} \mathcal{N}_{H,K-2}\left(v^{\setminus\{i,k\}}\right), \tag{C.24}$$

$$\mathtt{B} = \mathcal{D}_{H,K-1}\left(v^{\setminus k}\right) - \sum_{i\in[K]\setminus\{k\}} \mathcal{D}_{H,K-2}\left(v^{\setminus\{i,k\}}\right). \tag{C.25}$$

A simplification of Equations (C.24) and (C.25) is presented in Appendix E.3.1, yielding:

$$\mathtt{A} = \mathcal{N}_{H+1,K-1}\left(v^{\setminus k}\right) \quad \mathtt{B} = \mathcal{D}_{H+1,K-1}\left(v^{\setminus k}\right). \tag{C.26}$$

Thus, (C.23) holds if and only if

$$\mathcal{N}_{H+1,K-1}\left(v^{\setminus k}\right) = v(k)\cdot\mathcal{D}_{H+1,K-1}\left(v^{\setminus k}\right). \tag{C.27}$$

The LHS of (C.27) $\neq 0$; otherwise, by (C.2), it implies that $\mathcal{D}_{H,K-1}\left(v^{\setminus k}\right) = 0$, which contradicts (C.18). By Lemma 38, $v(k) > 0$. It follows that $\mathcal{D}_{H+1,K-1}\left(v^{\setminus k}\right) \neq 0$, allowing us to divide both sides of (C.27), yielding

$$v(k) = \frac{\mathcal{N}_{H+1,K-1}\left(v^{\setminus k}\right)}{\mathcal{D}_{H+1,K-1}\left(v^{\setminus k}\right)}.$$

This completes the inductive step for Lemma 14.2.

We prove that (C.4) holds by induction on m. Combined with Lemma 37, this completes the proof of Lemma 14.3.

- *Base case* $(m = 1)$: For every $k \in [K]$, it holds that

$$\begin{aligned} 0 &\overset{(a)}{=} \mathcal{D}_{H+1,K}(v) \\ &\overset{(b)}{=} v(k)\cdot\mathcal{D}_{H+1,K-1}\left(v^{\setminus k}\right) - (H+1)\cdot\mathcal{D}_{H,K-1}\left(v^{\setminus k}\right) \end{aligned}$$

  where (a) follows from Lemma 14.2 and (b) follows from Lemma 13. By (C.18) and Lemma 38, $\mathcal{D}_{H+1,K-1}\left(v^{\setminus k}\right)$ must be > 0.

- *Inductive step* $(m \to m+1)$: Let $m \in [K-1]$, $\mathfrak{I} \in \binom{[K]}{m}$ and $k \in [K]\setminus\mathfrak{I}$ be fixed. By the recurrence relation in Lemma 13,

$$\mathcal{D}_{H+1,K-m}\left(v^{\setminus\mathfrak{I}}\right) = v(k)\cdot\mathcal{D}_{H+1,K-(m+1)}\left(v^{\setminus\mathfrak{I}\cup\{k\}}\right) - (H+1)\cdot\mathcal{D}_{H,K-(m+1)}\left(v^{\setminus\mathfrak{I}\cup\{k\}}\right)$$

  By the induction hypothesis on m, the LHS of the equation is > 0. By Lemma 38, $v(k) > 0$. Similarly to (C.15), it holds that $v^{\setminus\mathfrak{I}\cup\{k\}} = {}^{k}v^{\setminus\mathfrak{I}\cup\{k\}}$. By Condition 2.ii and the induction hypothesis (on H) on Lemma 14.3, $\mathcal{D}_{H,K-(m+1)}\left(v^{\setminus\mathfrak{I}\cup\{k\}}\right) > 0$. Therefore, the term $\mathcal{D}_{H,K-(m+1)}\left(v^{\setminus\mathfrak{I}\cup\{k\}}\right)$ must be > 0.

■

Remark 39 *For every $v \in \mathfrak{P}_{H,K}$, the vector r defined in (17) is a valid probability distribution with non-zero entries. This follows from the structure of $\mathfrak{P}_{H,K}$: for each such v, r is the only choice that satisfies the necessary constraints. For completeness, an explicit proof is provided in Appendix E.3.2.*

Appendix D. Omitted Proofs for Section 3 (Main Results)

This appendix contains the proofs of Theorem B and the remarks from Section 3. Appendix D.1 proves Theorem B, while Appendices D.2, D.3, D.4, D.5 and D.6 prove Remarks 2, 3, 4, 5 and 6, respectively.

D.1. Proof of Theorem B (The Asymptotic Regret Factor)

To characterize the regret in (11), we begin by defining the polynomial

$$\widehat{\mathcal{P}}_{T,K}\left(x\right) \coloneqq \mathcal{P}_{T,K}\left(x+T\right). \tag{D.1}$$

By Theorem A, it holds that

$$L^{\star}_{T,K} = T + \arg\max \mathrm{Roots}\left(\widehat{\mathcal{P}}_{T,K}\right). \tag{D.2}$$

Hence, the regret can be expressed as

$$R_{T,K} = \arg\max \mathrm{Roots}\left(\widehat{\mathcal{P}}_{T,K}\right). \tag{D.3}$$

We show in Appendix E.4.1 that $\widehat{\mathcal{P}}_{T,K}$ can be written as

$$\widehat{\mathcal{P}}_{T,K}\left(x\right) = \sum_{m=0}^{K} x^{K-m}\binom{K}{m}\left(\sum_{d=0}^{m}\sum_{i=0}^{d}(-1)^{d}\binom{m}{d}\mathrm{s}\left(d,i\right)\,T^{m-(d-i)}\right), \tag{D.4}$$

where $\mathrm{s}\left(\cdot,\cdot\right)$ stands for the signed Stirling numbers of the first kind (see Definition 21). We define the polynomial $\widetilde{\mathcal{P}}_{T,K}$ as

$$\widetilde{\mathcal{P}}_{T,K}\left(x\right) \coloneqq \widehat{\mathcal{P}}_{T,K}\left(\sqrt{T}x\right).$$

Following Equations (D.2) and (D.3),

$$L^{\star}_{T,K} = T + \sqrt{T}\cdot\arg\max \mathrm{Roots}\left(\widetilde{\mathcal{P}}_{T,K}\right),$$

and

$$\beta_{T,K} \coloneqq \frac{R_{T,K}}{\sqrt{T}} = \arg\max \mathrm{Roots}\left(\widetilde{\mathcal{P}}_{T,K}\right). \tag{D.5}$$

In Appendix E.4.2 we show that

$$\widetilde{\mathcal{P}}_{T,K}\left(x\right) = \sum_{m=0}^{K} x^{K-m}\binom{K}{m}\cdot T^{\frac{K}{2}}\cdot\widetilde{c}_{T,m}, \tag{D.6}$$

$$\text{where}\qquad \widetilde{c}_{T,m} = \sum_{n=0}^{m} T^{\frac{m}{2}-n}\sum_{d=0}^{m}(-1)^{d}\binom{m}{d}\mathrm{s}\left(d,d-n\right). \tag{D.7}$$

Lemma 40 *For all $m\in\mathbb{N}$, the maximal power of T in $\widetilde{c}_{T,m}$ is ≤ 0. As a result, the maximal power of T in $\widetilde{\mathcal{P}}_{T,K}$ is at most $\frac{K}{2}$.*

The proof of Lemma 40 is deferred to Appendix D.1.1.

As defined in (12), $\beta_K = \lim_{T\to\infty} \beta_{T,K}$. By (D.5), $\beta_K = \arg\max \text{Roots}\left(\lim_{T\to\infty} \widetilde{\mathcal{P}}_{T,K}\right)$. With the established limit, we proceed to establish Theorem B.

Proof [Proof of Theorem B] We prove

$$\lim_{T\to\infty} \widetilde{\mathcal{P}}_{T,K}\left(x\right) = \mathrm{He}_K(x). \tag{D.8}$$

By Lemma 40, the maximal power of T in $\widetilde{\mathcal{P}}_{T,K}$ is at most $\frac{K}{2}$. The coefficient of $T^{\frac{K}{2}}$ in (D.6) contains the summand of $\widetilde{c}_{T,m}$ with $m = 2n$. Assuming that the coefficient of $T^{\frac{K}{2}}$ does not vanish, we have

$$\lim_{T\to\infty} \widetilde{\mathcal{P}}_{T,K}\left(x\right) = \sum_{n=0}^{\lfloor K/2\rfloor} x^{K-2n}\binom{K}{2n}\sum_{d=0}^{2n}(-1)^d\binom{2n}{d}\,\mathrm{s}\left(d, d-n\right).$$

As for $d < n$ it holds that $\mathrm{s}\left(d, d-n\right) = 0$, we denote $d = n+m$, and obtain

$$\begin{aligned}\lim_{T\to\infty} \widetilde{\mathcal{P}}_{T,K}\left(x\right) &= \sum_{n=0}^{\lfloor K/2\rfloor} x^{K-2n}\binom{K}{2n}\sum_{m=0}^{n}(-1)^{n+m}\binom{2n}{n+m}\,\mathrm{s}\left(n+m, m\right)\\ &= K!\sum_{n=0}^{\lfloor K/2\rfloor}\frac{x^{K-2n}(-1)^n}{(K-2n)!(2n)!}\sum_{m=0}^{n}(-1)^m\binom{2n}{n+m}\,\mathrm{s}\left(n+m, m\right).\end{aligned}$$

By Gould et al. (2015, Theorem 1),

$$\sum_{m=0}^{n}(-1)^m\binom{2n}{n+m}\,\mathrm{s}\left(n+m, m\right) = \frac{(2n)!}{n!2^n}.$$

Therefore,

$$\lim_{T\to\infty} \widetilde{\mathcal{P}}_{T,K}\left(x\right) = K!\sum_{n=0}^{\lfloor K/2\rfloor}\frac{(-1)^n}{n!(K-2n)!}\frac{x^{K-2n}}{2^n}. \tag{D.9}$$

The RHS of (D.9) is precisely the K-th probabilist's Hermite polynomial $\mathrm{He}_K(x)$ (e.g. Patarroyo, 2019, Eq. 3). ■

D.1.1. PROOF OF LEMMA 40

We make use of the following two auxiliary results.

Lemma 41 (Alternating Binomial Sum of Polynomials) *Let $P(x)$ be a polynomial of degree less than n. Then, the alternating sum of binomial coefficients weighted by $P(j)$ satisfies the identity:*

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}P(j) = 0. \tag{D.10}$$

Lemma 41 follows from the theory of finite differences (see Graham et al., 1994, Ch. 2).

Lemma 42 (Polynomiality of Stirling Numbers of the First Kind) *For a fixed $n \in \mathbb{N}$, there exists a polynomial $G_n(d)$ of degree $\leq 2n$ such that*

$$\begin{bmatrix} d \\ d-n \end{bmatrix} = G_n(d).$$

Lemma 42 follows from the combinatorial interpretation of Stirling numbers and their polynomial nature as established by Gessel and Stanley (1978).

Proof [Proof of Lemma 40] Let $0 \leq m \leq K$. It holds that

$$\begin{aligned}
\widetilde{c}_{T,m} &\overset{(a)}{\coloneqq} \sum_{n=0}^{m} T^{\frac{m}{2}-n} \sum_{d=0}^{m} (-1)^d \binom{m}{d} \mathrm{s}\,(d, d-n) \\
&\overset{(b)}{=} \sum_{n=0}^{m} (-1)^n T^{\frac{m}{2}-n} \sum_{d=0}^{m} (-1)^d \binom{m}{d} \begin{bmatrix} d \\ d-n \end{bmatrix} \\
&\overset{(c)}{=} \sum_{n=0}^{m} (-1)^n T^{\frac{m}{2}-n} \sum_{d=0}^{m} (-1)^d \binom{m}{d} G_n(d),
\end{aligned}$$

where the steps are justified as follows:

(a) Follows from the definition of the term $\widetilde{c}_{T,m}$ in (D.7).

(b) Follows from Definition 21, where the signed Stirling numbers of the first kind are defined by

$$\mathrm{s}\,(n, k) = (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}.$$

(c) Follows from Lemma 42, which states that each term $\begin{bmatrix} d \\ d-n \end{bmatrix}$ can be written as a polynomial $G_n(d)$ of degree at most $2n$.

Thus, if $\frac{m}{2} > n$, then $\deg\left(G_n(d)\right) < m$, and by Lemma 41 we obtain

$$\sum_{d=0}^{m} (-1)^d \binom{m}{d} G_n(d) = 0.$$

■

D.2. Proof of Remark 2

Lemma 43 *For any $T, K, m \in \mathbb{N}_+$ it holds that $L^\star_{mT,K} \leq m \cdot L^\star_{T,K}$.*

Proof [Proof of Lemma 43] Fix $T, K, m \in \mathbb{N}_+$, and let Ψ^T be an optimal bookmaker whose loss is $L^\star_{T,K}$. Construct a bookmaker $\tilde{\Psi}$ for horizon mT by applying Ψ^T independently to each of the m disjoint blocks of T rounds. That is, partition the gambler's sequence $q_1, \ldots, q_{mT}$ into m segments and apply Ψ^T to each, starting from a zero state. The total loss of $\tilde{\Psi}$ is at most $m \cdot L^\star_{T,K}$, and thus $L^\star_{mT,K} \leq m \cdot L^\star_{T,K}$, by the optimality of the loss $L^\star_{mT,K}$. ■

Proof [Proof of Remark 2] Assume, towards a contradiction, there exists $\widehat{T} \in \mathbb{N}_+$ and $C \in \mathbb{R}$ such that

$$\lim_{K \to \infty} \frac{R_{\widehat{T},K}}{\sqrt{\widehat{T}}} = C.$$

Since adding an outcome to the game can only increase the bookmaker's optimal loss,

$$\forall K \in \mathbb{N}_+ \quad L^\star_{\widehat{T},K} \leq \widehat{T} + C\sqrt{\widehat{T}} \tag{D.11}$$

Theorem B states that

$$\forall K \in \mathbb{N}_+ \quad \lim_{T \to \infty} \beta_{T,K} = \arg\max \text{Roots}\left(\text{He}_K\right). \tag{D.12}$$

Lower bound on the RHS of (D.12) (e.g., Krasikov, 2004) implies there exist $m \in \mathbb{N}_+$ for which

$$\forall K \in \mathbb{N}_+ \quad \beta_{m\widehat{T},K} \geq \sqrt{K} \tag{D.13}$$

Define

$$\widehat{K} \coloneqq \left\lceil mC^2 \right\rceil + 1, \tag{D.14}$$

and consider the optimal bookmaking loss in a game with $\widehat{K}$ outcomes and $m\widehat{T}$ rounds:

$$\begin{aligned}
L^\star_{m\widehat{T},\widehat{K}} &\overset{(a)}{\leq} m \cdot L^\star_{\widehat{T},\widehat{K}} \\
&\overset{(b)}{\leq} m \cdot \left(\widehat{T} + C\sqrt{\widehat{T}}\right) \\
&= m\widehat{T} + \sqrt{mC^2}\sqrt{m\widehat{T}} \\
&\overset{(c)}{<} m\widehat{T} + \sqrt{\widehat{K}}\sqrt{m\widehat{T}} \\
&\overset{(d)}{\leq} m\widehat{T} + \beta_{m\widehat{T},\widehat{K}}\sqrt{m\widehat{T}} \\
&= L^\star_{m\widehat{T},\widehat{K}},
\end{aligned}$$

where (a) follows from Lemma 43, stated above; (b) follows from the upper bound on $L^\star_{\widehat{T},\widehat{K}}$ in (D.11); (c) follows from our definition of $\widehat{K}$ in (D.14); and, (d) follows from the lower bound on the regret factor $\beta_{m\widehat{T},\widehat{K}}$ in (D.13). This yields a contradiction, completing the argument. ∎

D.3. Proof of Remark 3

Proof [Proof of Remark 3] We prove that evaluating $\mathcal{D}_{H,K}$ at $x \cdot \mathbf{1}_K - s$, as defined in (18), yields the polynomial expression in (15).

$$
\begin{aligned}
\mathcal{D}_{H,K}\left(x \cdot \mathbf{1}_K - s\right) &\overset{(a)}{=} \sum_{j=0}^{K} (-1)^{K-j} H^{\underline{K-j}} \sigma_j\left(x \cdot \mathbf{1}_K - s\right) \\
&\overset{(b)}{=} \sum_{j=0}^{K} (-1)^{K-j} H^{\underline{K-j}} \sum_{i=0}^{j} (-1)^i \sigma_i(s) \binom{K-i}{j-i} x^{j-i} \\
&= \sum_{j=0}^{K} \sum_{i=0}^{j} (-1)^{K-(j-i)} H^{\underline{K-j}} \sigma_i(s) \binom{K-i}{j-i} x^{j-i} \\
&\overset{(c)}{=} \sum_{j=0}^{K} \sum_{n=0}^{j} (-1)^{K-n} H^{\underline{K-j}} \sigma_{j-n}(s) \binom{K-j+n}{n} x^{n} \\
&= \sum_{n=0}^{K} (-1)^{K-n} \left(\sum_{j=n}^{K} H^{\underline{K-j}} \sigma_{j-n}(s) \binom{K-(j-n)}{n} \right) x^{n} \\
&\overset{(d)}{=} \sum_{m=0}^{K} (-1)^{m} \left(\sum_{j=K-m}^{K} H^{\underline{K-j}} \sigma_{j-(K-m)}(s) \binom{K-j+K-m}{K-m} \right) x^{K-m} \\
&\overset{(e)}{=} \sum_{m=0}^{K} \left((-1)^{m} \sum_{n=0}^{m} H^{\underline{m-n}} \binom{K-n}{K-m} \sigma_n(s) \right) x^{K-m},
\end{aligned}
$$

where: (a) uses the definition of $\mathcal{D}_{H,K}$ in (C.1); (b) follows from Lemma 26; (c) substitutes $n = j - i$; (d) by the substitution $m = K - n$; and (e) via the change of variable $n = j - K + m$. Accordingly, we obtain the expression stated in (15). ∎

D.4. Proof of Remark 4 (Algorithm 1 Computational Complexity)

In this section, we analyze the computational complexity of Algorithm 1.

Remark 44 *All the elementary symmetric polynomials of a vector in $\mathbb{R}^K$ can be computed in $O(K \log K)$ time using FFT-based polynomial multiplication techniques (Harvey and van der Hoeven, 2021; Ben-Sasson et al., 2023, Thm. 7.1). We refer to this method as the* `FFT-ESP` *algorithm.*

We first address the efficient computation of partial elementary symmetric polynomials, handled by Algorithm 2.

Lemma 45 *Algorithm 2, hereafter referred to as* `PESP`*, runs in $O(K^2)$ time and, given a vector $v \in \mathbb{R}^K$, returns a matrix $A \in \mathbb{R}^{K \times K}$ such that*

$$
(A)_{k,m} = \sigma_{m-1}\left(v^{\setminus k}\right)
$$

for all $k, m \in [K]$.

Algorithm 2: Partial Elementary Symmetric Polynomials

Input: $v \in \mathbb{R}^K$
Output: $A = [a_1 \cdots a_K] \in \mathbb{R}^{K \times K}$ where $A_{k,m} = \sigma_{m-1}\left(v^{\setminus k}\right)$
Initialization: $A \leftarrow \mathbf{0}^{K \times K}$, $s \leftarrow \mathbf{0}_{K+1}$
$\sigma_0(v), \ldots, \sigma_K(v) \leftarrow$ `FFT-ESP`(v)
$a_1 \leftarrow \mathbf{1}_K$
for $m = 1 : K-1$ **do**
  $a_{m+1} \leftarrow \sigma_m(v) \cdot \mathbf{1}_K - v \odot a_m$
end
output A

Proof [Proof of Lemma 45] Let $K \in \mathbb{N}_+$, $k \in [K]$ and $v \in \mathbb{R}^K$. By Lemma 24, for every $m \in [K]$

$$\sigma_m(v) = v(k) \cdot \sigma_{m-1}\left(v^{\setminus k}\right) + \sigma_m\left(v^{\setminus k}\right).$$

Hence, for every $m \in [K-1]$,

$$\sigma_m\left(v^{\setminus k}\right) = \sigma_m(v) - v(k) \cdot \sigma_{m-1}\left(v^{\setminus k}\right), \tag{D.15}$$

with the base case $\sigma_0\left(v^{\setminus k}\right) = 1$. Thus, for every $k \in [K]$,

$$A_{k,1} = a_1(k) = 1 = \sigma_0\left(v^{\setminus k}\right),$$

and by induction on $m-1$, for every $m \in [2:K]$,

$$A_{k,m} = \sigma_{m-1}(v) - v(k) \cdot a_{m-1}(k) = \sigma_{m-1}(v) - v(k) \cdot \sigma_{m-2}\left(v^{\setminus k}\right) = \sigma_{m-1}\left(v^{\setminus k}\right)$$

In terms of complexity, `FFT-ESP` executes once in $O(K \log K)$ time (Remark 44), followed by K iterations where each of the K entries performs $O(1)$ operations. This results in a total complexity of $O(K^2)$. ■

Proof [Proof of Remark 4] To analyze the per-time-step complexity, consider an arbitrary round $t \in [2, T]$ with associated state vector $s \in \mathbb{R}^K$, and let $H = T - t + 1$ denote the remaining horizon. The operations executed at round t are as follows, along with their per-step time complexity.

1. *State and Residual loss Vectors Update* (Lines 4 and 8): element-wise vector operation, runs in $O(K)$ time.

2. *Opportunistic Loss update* (Line 6):

   (a) *Computing the polynomial $\mathcal{F}_{H,K}$* (15): This process consists of the following steps:

   - Computing the binomial coefficients

     $$\binom{K-n}{m-n} \quad \text{for all} \quad 0 \le m, n \le K.$$

     These coefficients can be precomputed before the algorithm begins. Using Pascal's triangle, they can be computed in $O(K^2)$ time.

- Computing the elementary symmetric polynomials using `FFT-ESP` in $O(K \log K)$ (Remark 44).
- Computing the falling factorials

$$H^{\underline{0}}, \ldots, H^{\underline{K}}.$$

These are computable in $O(K)$ time directly via Definition 19.

Once all components are precomputed in $O(K^2)$, the polynomial is obtained by iterating over all $0 \le n \le m \le K$, requiring an additional $O(K^2)$ operations.

(b) *Root Finding*: Since $\mathcal{F}_{H,s}$ is a degree-K polynomial, its largest root can be computed numerically to precision ε in $O(K \log(1/\varepsilon))$ time using standard root-finding methods, such as Newton's Method or Laguerre's method. By applying the upper bound from Theorem B, we can efficiently compute a good initial point in constant time, where the number of required iterations is only $O(\log\log(1/\varepsilon))$.

3. *Optimal Odds Update* (Line 10): Using `PESP`, we compute all the necessary ESPs for the polynomials

$$\mathcal{D}_{H-1,K-1}(v^{\setminus 1}),\ \ldots, \mathcal{D}_{H-1,K-1}(v^{\setminus K})$$

in $O(K^2)$. The falling factorials required for these computations are obtained in $O(K)$. Each polynomial, for $k \in [K]$, involves $K+1$ iterations, each requiring $O(1)$ operations, resulting in an overall complexity of $O(K^2)$. Finally, computing the quotient for each term in the sum incurs an additional $O(K)$ cost.

Thus, the overall complexity per time step is $O(K^2)$, leading to a total complexity of $O(TK^2)$ over T time steps. ■

D.5. Proof of Remark 5

Bhatt et al. (2025) consider the binary case ($K = 2$) and design two algorithms for the bookmaker, depending on the gambler's behavior. Against a decisive gambler, they propose the *Optimal Strategy For Decisive Gamblers* algorithm (`ODG`), which coincides with Algorithm 1. Against a non-decisive gambler, who may place continuous bets $q_t \in [0, 1]$, they construct a mixture of `ODG` strategies, based on the observed betting sequence. In particular, let $r_t^{\texttt{ODG}}(x^{t-1})$ denote the odds produced by `ODG` algorithm for some input $x^{t-1} \in \{0, 1\}^t$. Their idea is to view $q_t \in [0, 1]$ as the expected value of a binary random variable $X_t \sim \text{Ber}(q_t)$. The strategy for continuous bets (Bhatt et al., 2025, Eq. 25) is

$$\bar{r}_t(q^{t-1}) = \mathbb{E}\left[r_t^{\texttt{ODG}}(X^{t-1})\right], \tag{D.16}$$

where the expected value is taken with respect to the sequence of independent random variables $X_i \sim \text{Ber}(q_i)$.

As computing the odds according to (D.16) can take exponential time, they propose to evaluate it via a Monte Carlo simulation and provide *Monte Carlo Based Efficient Strategy* Algorithm (referred to as `MC`). At each round t, the algorithm runs N parallel simulations of `ODG`, and the strategy is computed as the average over these N independent runs. The computational complexity of `MC` is $\mathcal{O}(NT)$, where N denotes the number of samples per round and T is the total number of betting rounds. The computational efficiency of MC hinges on the number of samples required to achieve a desired approximation accuracy. Specifically, to ensure an additive error tolerance of ε with probability at least $1-\delta$, the required number of samples must satisfy:

$$N \geq \frac{T}{2\varepsilon^2} \log\left(\frac{2T}{\delta}\right).$$

Substituting this bound on N yields an overall complexity of $O\left(T^{3+2\alpha} \log\left(\frac{T}{\delta}\right)\right)$.

Proof [Proof of Remark 5] We show there exist $T \in \mathbb{N}_+$, $t \in [T]$, and $s \in \mathbb{R}^2$ for which (D.16) does not output the same odds as Algorithm 1 (OOPT). Let $T = 4$. At the first round,

$$r_1^{\texttt{OOPT}} = \bar{r}_1 = (0.5, 0.5).$$

Hence, after any gambler's action $q_1 = (q_1(1),\, q_1(2))$, both algorithms have the same state vector

$$s = (2q_1(1),\, 2q_1(2)).$$

It holds that

$$\begin{aligned}
\bar{r}_2\left((q_1(1))\right) &= \mathbb{P}\left(\ (1)\ \mid\ (q_1(1))\ \right) \cdot r_2^{\texttt{ODG}}\left((1)\right) + \mathbb{P}\left(\ (2)\ \mid\ (q_1(1))\ \right) \cdot r_2^{\texttt{ODG}}\left((2)\right) \\
&= q_1(1) \cdot \left(\frac{2}{3}, \frac{1}{3}\right) + q_1(2) \cdot \left(\frac{1}{3}, \frac{2}{3}\right) \\
&= \frac{1}{3}\left(2 \cdot q_1(1) + q_1(2),\ q_1(1) + 2 \cdot q_1(2)\right).
\end{aligned}$$

At $t = 2$, by Theorem C, $L^\star_{3,2}(s) = \arg\max \mathrm{Roots}\left(\mathcal{F}_{3,s}\right)$ where

$$\mathcal{F}_{3,s}(x) = x^2 - \left(6 + 2q_1(1) + 2q_1(2)\right)x + \left(6 + 6q_1(1) + 6q_1(2) + 4q_1(1)q_1(2)\right).$$

Choosing $q_1 = (0.4, 0.6)$ gives $\mathcal{F}_{3,s}(x) = x^2 - 8x + 12.96$, and the optimal loss is 5.7435. The residual loss is then

$$v \approx (5.7435 - 0.8, 5.7435 - 1.2) = (4.9435, 4.5435).$$

Using (17), we have that

$$r_2^{\texttt{OOPT}} \approx (0.4635, 0.5364) \neq (0.4666, 0.5333) \approx \bar{r}_2\left((q_1(1))\right).$$

■

D.6. Proof of Remark 6 (The Effect of ε-Approximate Root-Finding)

In this section, we analyze a variant of Algorithm 1 that replaces exact root-finding with an ε-approximate oracle. The underlying principle of the algorithm is to obtain an upper approximation to the optimal opportunistic loss such that the resulting "residual loss vector" v is still achievable.

Let $\varepsilon > 0$, and assume access to an oracle $\mathbf{O}_\varepsilon$ that, given a univariate polynomial $\mathcal{P}$:

Algorithm 3: Opportunistic Bookmaking Algorithm with Maximal Real Root Oracle

∗ **Input:** K, T, ε (precision), $\mathbf{O}_\varepsilon$, q^{T-1} (bets obtained sequentially)
Output: $r_1, \ldots, r_T$ (outputs r_t after observing q^{t-1})
∗ **Initialization:** $s \leftarrow \varepsilon \cdot \mathbf{1}_K, \quad L \leftarrow \mathbf{O}_\varepsilon(\mathcal{F}_{T,s})$
output $r_1 \leftarrow \frac{1}{K} \cdot \mathbf{1}_K$
for $t = 2 : T$ **do**
  $s \leftarrow s + q_{t-1} \oslash r_{t-1}$ // update the state vector with additional $\varepsilon \cdot \mathbf{1}_K$
  if $q_{t-1} \notin \mathcal{E}_K$ **then**
∗ $s \leftarrow s + \varepsilon \cdot \mathbf{1}_K$
∗ $L \leftarrow \mathbf{O}_\varepsilon(\mathcal{F}_{T-t+1,s})$ // call the oracle instead of calculating $\arg\max \text{Roots}\,(\mathcal{F}_{T-t+1,s})$
  end
  $v \leftarrow (L \cdot \mathbf{1}_K - s)$
  for $k = 1 : K$ **do**
    $r(k) \leftarrow \mathcal{D}_{T-t,K-1}\left(v^{\setminus k}\right)$
  end
  output $r_t \leftarrow r / \|r\|_1$
end

- *If $\mathcal{P}$ has real roots:* returns a value $\rho^\varepsilon \in \mathbb{R}$, such that

$$|\rho^\varepsilon - \rho| < \varepsilon, \tag{D.17}$$

  where ρ is $\mathcal{P}$'s maximal real root.

- *Otherwise:* returns ∞.

We present Algorithm 3, where modified lines of Algorithm 1 are marked with $*$.
Proof [Proof of Remark 6] We show that for every non-decisive gambler action, the bookmaker's loss exceeds the optimal opportunistic bookmaking loss by at most 2ε.

Let $H \in \mathbb{N}_+$ and $s \in \mathbb{R}^K$. Denote by $L^\varepsilon_{H,K}(s)$ a fixed response of $\mathbf{O}_\varepsilon(\mathcal{F}_{H,s})$. By Theorem C, $L^\star_{H,K}(s)$ is equal to the largest real root of the polynomial $\mathcal{F}_{H,s}$; thus, from (D.17), it follows that

$$L^\star_{H,K}(s) - \varepsilon < L^\varepsilon_{H,K}(s) < L^\star_{H,K}(s) + \varepsilon. \tag{D.18}$$

By Definition 7 of the value function,

$$\mathcal{V}_H(s) - \varepsilon < L^\varepsilon_{H,K}(s) < \mathcal{V}_H(s) + \varepsilon. \tag{D.19}$$

Observe a fixed value of $L^\varepsilon_{H,K}(s + \varepsilon \cdot \mathbf{1}_K)$: By (D.19),

$$\mathcal{V}_H(s + \varepsilon \cdot \mathbf{1}_K) - \varepsilon < L^\varepsilon_{H,K}(s + \varepsilon \cdot \mathbf{1}_K) < \mathcal{V}_H(s + \varepsilon \cdot \mathbf{1}_K) + \varepsilon.$$

By uniform translation property of the value function (Lemma 32),

$$(\mathcal{V}_H(s) + \varepsilon) - \varepsilon < L^\varepsilon_{H,K}(s + \varepsilon \cdot \mathbf{1}_K) < (\mathcal{V}_H(s) + \varepsilon) + \varepsilon,$$

i.e.,

$$\mathcal{V}_H(s) < L^\varepsilon_{H,K}(s + \varepsilon \cdot \mathbf{1}_K) < \mathcal{V}_H(s) + 2\varepsilon.$$

By continuity and coordinate-wise strict monotonicity of the value functions (Lemmas 27 and 29.2), and since the value of $L^{\varepsilon}_{H,K}(s + \varepsilon \cdot \mathbf{1}_K)$ is bounded, there exists a vector $b \in \mathbb{R}^K$ such that $0 \prec b \prec 2\varepsilon \cdot \mathbf{1}_K$ and

$$L^{\varepsilon}_{H,K}(s + \varepsilon \cdot \mathbf{1}_K) = L^{\star}_{H,K}(s + b). \tag{D.20}$$

Denote by $\mathbf{O}_0$ the oracle who returns the precise value of the maximal real root (if it exists). By (D.20), an equivalent interpretation of Algorithm 3 is as follows:

- Before round $t \in [T]$ with some initial state s_t, we sample a vector

$$b_t \in \{b \in \mathbb{R}^K_+ \ : \ b \prec 2\varepsilon \cdot \mathbf{1}_K\} \tag{D.21}$$

  according to some unknown probability distribution. We interpret $b_t(k)$ as a given random tax that we are forced to pay, if outcome k materializes.

- In general, the state s_t accounts for some given loss when choosing a bookmaking action (see the motivation for the OOBL in Section 2.3). Thus, our effective state to consider is $s_t + b_t$.

- After observing b_t, we have an access to the oracle $\mathbf{O}_0$. By Theorem C, the optimal strategy is to query $\mathbf{O}_0$ with the polynomial $\mathcal{F}_{T-t+1,s_t+b_t}$.

A direct result of coordinate-wise strict monotonicity property (Lemma 29.2), is that the worst random tax b_t, as in (D.21) is $2\varepsilon\mathbf{1}_K$ (formally, infinitesimally close to $2\varepsilon\mathbf{1}_K$). By uniform translation property, we obtain that the bookmaker's loss exceeds the optimal opportunistic bookmaking loss by at most 2ε.

■

Appendix E. Supplementary Technical Details

This appendix collects additional technical proofs. Appendix E.1 supports the preliminary material in Appendix A, Appendix E.2 corresponds to Appendix B, Appendix E.3 provides supporting proofs for both Section 4.2 and Appendix C, and Appendix E.4 presents technical arguments for Appendix D .

E.1. Proofs for Appendix A

Proof [Proof of Lemma 25] For $m = 0$, the lemma holds trivially due to the fact that $\sigma_0(\cdot) = 1$. For $0 < m \leq K$, by Definition 1, it holds that

$$\sum_{i=1}^{K} \sigma_m\left(x^{\setminus i}\right) = \sum_{i=1}^{K} \sum_{\mathfrak{J} \in \binom{[K]\setminus\{i\}}{m}} \prod_{k \in \mathfrak{J}} x(k).$$

Every subset of indices $\mathfrak{J} \in \binom{[K]}{m}$ appears in $K - m$ different configurations of i within the summation. This results from choosing any of the $K - m$ indices that do not belong to $\mathfrak{J}$ for the value of i. ■

Proof [Proof of Lemma 26] Substituting the vector $t \cdot \mathbf{1}_K - x$ into the generating function for ESPs (see Remark 23), we obtain:

$$\begin{aligned}\sum_{n=0}^{K} \sigma_n \left(t \cdot \mathbf{1}_K - x\right) y^n &= \prod_{k=1}^{K} \left(1 + \left(t - x(k)\right) y\right) \\ &= \prod_{k=1}^{K} \left((1+ty) - x(k)y\right)\end{aligned} \tag{E.1}$$

Expanding this product involves selecting for each k either the $1 + ty$ term or the $-x(k)y$ term. Specifically, all the terms in the product which involves i of the $-x(k)y$ terms will contribute to the generating function the term

$$(-1)^i \sigma_i(x) y^i (1+ty)^{K-i}. \tag{E.2}$$

Thus,

$$\begin{aligned}\prod_{k=1}^{K} \left(1 + \left(t - x(k)\right) y\right) &\overset{(a)}{=} \sum_{i=0}^{K} (-1)^i \sigma_i(x) y^i \cdot (1+ty)^{K-i} \\ &\overset{(b)}{=} \sum_{i=0}^{K} (-1)^i \sigma_i(x) y^i \sum_{j=0}^{K-i} \binom{K-i}{j} t^j y^j \\ &= \sum_{i=0}^{K} \sum_{j=0}^{K-i} (-1)^i \sigma_i(x) \binom{K-i}{j} t^j y^{i+j} \\ &\overset{(c)}{=} \sum_{n=0}^{K} \left(\sum_{i=0}^{n} (-1)^i \sigma_i(x) \binom{K-i}{n-i} t^{n-i} \right) y^n,\end{aligned} \tag{E.3}$$

where (a) follows by (E.2), (b) follows from the expansion of $(1+ty)^{K-i}$ using the binomial theorem, and (c) uses the variable substitution $n = i + j$. By equating the coefficients of y^n on the LHS of (E.1) with those in (E.3), we obtain:

$$\sigma_n(t \cdot \mathbf{1}_K - x) = \sum_{i=0}^{n} (-1)^i \sigma_i(x) \binom{K-i}{n-i} t^{n-i}$$

■

E.2. Proofs for Appendix B

Proof [Proof of Lemma 32] Let $s \in \mathbb{R}^K$ and $c \in \mathbb{R}$. We prove by induction on H.

- *Base Case* ($H = 0$): It holds that

$$\begin{aligned}\mathcal{V}_0 \left(s + c \cdot \mathbf{1}_K\right) &\coloneqq \max_{k \in [K]} \left(s + c \cdot \mathbf{1}_K\right)(k) \\ &= \max_{k \in [K]} s(k) + c \\ &= \mathcal{V}_0 \left(s\right) + c.\end{aligned}$$

- *Inductive step* ($H \to H+1$):

$$\begin{aligned}\mathcal{V}_{H+1}\left(s+c\cdot\mathbf{1}_K\right) &\coloneqq \inf_{r\in\Delta^{K-1}}\ \max_{q\in\Delta^{K-1}} \mathcal{V}_H\left(s+c\cdot\mathbf{1}_K+q\oslash r\right)\\ &\overset{(*)}{=} \inf_{r\in\Delta^{K-1}}\ \max_{q\in\Delta^{K-1}} \mathcal{V}_H\left(s+q\oslash r\right)+c\\ &= \mathcal{V}_{H+1}\left(s\right)+c,\end{aligned}$$

where $(*)$ follows from the induction hypothesis.

■

Proof [Proof of Lemma 34] The correspondence $\mathcal{Z}_H$ is compact-valued by its definition in (B.5), and is non-empty as for every $s\in\mathbb{R}^K$, $\frac{1}{K}\in\mathcal{Z}_H(s)$.

A correspondence is continuous if it is both upper and lower hemicontinuous (see Definition 17). From the continuity of the maximum and minimum functions on $\mathbb{R}^K$, the function ω_H, defined in (B.4), is continuous. Hence for any $\bar{s}\in\mathbb{R}^K$ and $\delta>0$, there exists an open neighborhood U of $\bar{s}$ such that for all $s\in U$, $|\omega_H(s)-\omega_H(\bar{s})|<\delta$.

Let $\bar{s}\in\mathbb{R}^K$ and let $V\subset\Delta^{K-1}$ be an open set with $\mathcal{Z}_H(\bar{s})\subset V$. Since every $r\in\mathcal{Z}_H(\bar{s})$ is an interior point of V, for each there exists $\varepsilon_r>0$ such that $B(r,\varepsilon_r)\subset V$. By compactness of $\mathcal{Z}_H(\bar{s})$, a finite subcover yields a uniform margin $\delta>0$ ensuring that if $|\omega_H(s)-\omega_H(\bar{s})|<\delta$, then for every $r\in\Delta^{K-1}$ and $k\in[K]$, if $r(k)\geq\omega_H(s)$ then $r\in V$. By continuity of ω_H, there exists an open neighborhood U of $\bar{s}$, such that for every $s\in U$, $\mathcal{Z}_H(s)\subset V$, and thus $\mathcal{Z}_H$ is upper hemicontinuous.

Let $\bar{s}\in\mathbb{R}^K$ and suppose $V\subset\Delta^{K-1}$ is open with $\mathcal{Z}_H(\bar{s})\cap V\neq\emptyset$. Choose $\bar{r}\in\mathcal{Z}_H(\bar{s})\cap V$. Since V is open, there exists $\varepsilon>0$ with $B(\bar{r},\varepsilon)\subset V$.

- *If $\bar{r}$ is an interior point of $\mathcal{Z}_H(\bar{s})$:* For all $k\in[K]$, $\bar{r}(k)>\omega_H(\bar{s})$. By the continuity of ω_H, there exists a neighborhood U of $\bar{s}$ such that for all $s\in U$ and $k\in[K]$, $\omega_H(s)<\bar{r}(k)$. Hence, $\bar{r}\in\mathcal{Z}_H(s)\cap V$.

- *If $\bar{r}$ lies on the boundary of $\mathcal{Z}_H(\bar{s})$:* Then for some coordinate, $\bar{r}(k)=\omega_H(\bar{s})$. Since $\bar{r}$ is in V, we can select $r'\in B(\bar{r},\varepsilon)$ with $r'(k)>\omega_H(\bar{s})$ for all k. Again, by continuity of ω_H, for s in a neighborhood U of $\bar{s}$ we have $\omega_H(s)<r'(k)$ for every $k\in[K]$, so that $r'\in\mathcal{Z}_H(s)\cap V$.

In either case, there exists an open neighborhood U of $\bar{s}$ such that $\mathcal{Z}_H(s)\cap V\neq\emptyset$ for all $s\in U$, and thus $\mathcal{Z}_H$ is lower hemicontinuous.

■

Proof [Proof of Lemma 35] Let $s\in\mathbb{R}^K$. By Definition 7, $\mathcal{V}_0\left(s\right)\coloneqq\max_{k\in[K]}s(k)$ and hence a convex function. By Lemma 28,

$$\mathcal{V}_1\left(s\right)=\min_{r\in\mathcal{Z}_1(s)}\max_{k\in[K]}s(k)+\frac{1}{r(k)}.$$

Assume, towards a contradiction, that there exists an optimal bookmaker action $r\in\mathcal{Z}_1(s)$ that does not satisfy (B.11); that is, r achieves the optimal loss $\ell^\star$, yet there exists an index $i\in[K]$ such that

$$s(i)+\frac{1}{r(i)}=\ell^\star-\kappa \quad \text{for some } \kappa>0. \tag{E.4}$$

Assume $K \geq 2$ (for the case $K = 1$, the statement is trivially false). It holds that

$$\frac{K}{K-1} r(i) \; , \; \frac{K\kappa(r(i)^2)}{(K-1)(1+\kappa r(i))} > 0.$$

Therefore, there exists $\varepsilon \in \mathbb{R}_{++}$ such that

$$0 < \varepsilon < \min\left\{ \frac{K}{K-1} r(i) \; , \; \frac{K\kappa(r(i)^2)}{(K-1)(1+\kappa r(i))} \right\}.$$

We construct a new distribution $\hat{r} \in \Delta^{K-1}$ as follows:

$$\hat{r}(k) = \begin{cases} r(k) - \frac{K-1}{K}\varepsilon & \text{if } k = i, \\ r(k) + \frac{1}{K}\varepsilon & \text{if } k \neq i. \end{cases}$$

$\hat{r}$ is indeed in Δ^{K-1} as it satisfies:

- *Sum Constraint*:

$$\sum_{k=1}^{K} \hat{r}(k) = r(i) - \frac{K-1}{K}\varepsilon + \sum_{k \neq i} \left(r(k) + \frac{1}{K}\varepsilon \right) = 1.$$

- *Positivity Constraint*: For $k = i$, since $\varepsilon < \frac{K}{K-1} r(i)$ we obtain:

$$\hat{r}(i) = r(i) - \frac{K-1}{K}\varepsilon > r(i) - \frac{K-1}{K}\frac{K}{K-1} r(i) = 0.$$

  For $k \in [K] \setminus \{i\}$:

$$\hat{r}(k) = r(k) + \frac{1}{K}\varepsilon > r(k) > 0.$$

We show that acting with $\hat{r}$ produces a loss that is $< \ell^\star$.

- *For $k = i$*:

$$\frac{1}{\hat{r}(i)} - \frac{1}{r(i)} = \frac{r(i) - \hat{r}(i)}{r(i)\hat{r}(i)} = \frac{\frac{K-1}{K}\varepsilon}{r(i)(r(i) - \frac{K-1}{K}\varepsilon)}. \tag{E.5}$$

  From the definition of ε, we have:

$$\varepsilon < \frac{\kappa r(i)^2 K}{(K-1)(1+\kappa r(i))} \quad \implies \frac{\frac{K-1}{K}\varepsilon}{r(i)\left(r(i) - \frac{K-1}{K}\varepsilon\right)} < \kappa. \tag{E.6}$$

  Thus, by Equations (E.4), (E.5) and (E.6)

$$s(i) + \frac{1}{\hat{r}(i)} = \ell^\star - \kappa + \frac{1}{\hat{r}(i)} - \frac{1}{r(i)} < \ell^\star.$$

- *For* $k \in [K] \setminus \{i\}$:

$$\frac{1}{r(k)} > \frac{1}{r(k) + \frac{1}{K}\varepsilon} \implies s(k) + \frac{1}{r(k)} > s(k) + \frac{1}{\hat{r}(k)}$$

Hence,

$$\ell^\star = \max_{k \in [K]\setminus\{i\}} s(k) + \frac{1}{r(k)} > \max_{k \in [K]\setminus\{i\}} s(k) + \frac{1}{\hat{r}(k)}.$$

Combining both cases, we obtain:

$$\max_{k \in [K]} \left(s(k) + \frac{1}{\hat{r}(k)} \right) < \ell^\star.$$

It follows that r is suboptimal, contradicting the optimality assumption.

It remains to show that the optimal bookmaking action is unique. Let $r^\star, \hat{r} \in \mathcal{Z}_1(s)$ be two vectors that satisfy (B.11), i.e.,

$$s(k) + \frac{1}{r^\star(k)} = s(k) + \frac{1}{\hat{r}(k)} = \mathcal{V}_1(s) \quad \forall k \in [K].$$

As the function $x \mapsto \frac{1}{x}$ is strictly decreasing in $\mathbb{R}_+$, for every $k \in [K]$, $\hat{r}(k) = r^\star(k)$, and thus the optimal bookmaking action $r^\star$ is unique. ■

E.3. Proofs and Technical Details for Section 4.2 and Appendix C

Proof [Proof of Lemma 10] Let $k \in [K]$ be the index for which $q_H = \mathbf{e}_k$. When $K = 1$, the claim is immediate. Suppose $K \geq 2$, and assume, towards a contradiction, that some $j \in [K] \setminus \{k\}$ maximizes the RHS of (21). By Lemma 27 (see Appendix B), any optimal bookmaker's action r_H assigns a positive probability mass $r_H(i) > 0$ for every $i \in [K]$. Thus, for the betting sequence $(q_1, \ldots, q_{H-1}, \mathbf{e}_j) \in (\mathcal{E}_K)^H$, the resulting loss would exceed $\mathcal{V}_H(s)$, a contradiction to the bookmaker's optimality assumption. ■

Proof [Proof of Lemma 13] By Lemma 24, we have for $m \geq 1$

$$v(k)\, \sigma_{m-1}\left(v^{\setminus k}\right) + \sigma_m\left(v^{\setminus k}\right) = \sigma_m(v).$$

In particular, with m replaced by $K - m$ (noting that $K - m \geq 1$ for $m \leq K - 1$) we obtain

$$v(k)\, \sigma_{K-1-m}\left(v^{\setminus k}\right) = \sigma_{K-m}(v) - \sigma_{K-m}\left(v^{\setminus k}\right). \tag{E.7}$$

By (E.7), we have

$$v(k) \cdot \mathcal{D}_{H,K-1}\left(v^{\setminus k}\right) = \sum_{m=0}^{K-1} (-1)^m\, H^{\underline{m}}\Big(\sigma_{K-m}(v) - \sigma_{K-m}\left(v^{\setminus k}\right)\Big). \tag{E.8}$$

We combine the second summand of (E.8) and $H \cdot \mathcal{D}_{H-1,K-1}\left(v^{\setminus k}\right)$ as follows:

$$\begin{aligned}
&H \cdot \mathcal{D}_{H-1,K-1}\left(v^{\setminus k}\right) + \sum_{m=0}^{K-1} (-1)^m H^{\underline{m}} \sigma_{K-m}\left(v^{\setminus k}\right) \\
&\overset{(a)}{=} \sum_{m=0}^{K-1} (-1)^m H^{\underline{m+1}} \sigma_{K-1-m}\left(v^{\setminus k}\right) + \sum_{m=0}^{K-1} (-1)^m H^{\underline{m}} \sigma_{K-m}\left(v^{\setminus k}\right) \\
&\overset{(b)}{=} \sum_{m=1}^{K} (-1)^{m-1} H^{\underline{m}} \sigma_{K-m}\left(v^{\setminus k}\right) + \sum_{m=0}^{K-1} (-1)^m H^{\underline{m}} \sigma_{K-m}\left(v^{\setminus k}\right) \\
&\overset{(c)}{=} (-1)^{K-1} H^{\underline{K}} \sigma_0\left(v^{\setminus k}\right) + \sigma_K\left(v^{\setminus k}\right) \\
&= (-1)^{K-1} H^{\underline{K}} \sigma_0\left(v^{\setminus k}\right),
\end{aligned} \tag{E.9}$$

where (a) follows from the identity $H(H-1)^{\underline{m}} = H^{\underline{m+1}}$, (b) follows by re-indexing with $m+1$, and (c) follows by noting that the two sums are telescoping. We can now combine (E.8) and (E.9) to complete the proof

$$\begin{aligned}
&v(k) \cdot \mathcal{D}_{H,K-1}\left(v^{\setminus k}\right) - H \cdot \mathcal{D}_{H-1,K-1}\left(v^{\setminus k}\right) \\
&= \sum_{m=0}^{K-1} (-1)^m H^{\underline{m}} \sigma_{K-m}(v) + (-1)^K H^{\underline{K}} \sigma_0\left(v^{\setminus k}\right) \\
&= \sum_{m=0}^{K} (-1)^m H^{\underline{m}} \sigma_{K-m}(v) \\
&= \mathcal{D}_{H,K}(v),
\end{aligned}$$

where we use the fact that $\sigma_0\left(v^{\setminus k}\right) = \sigma_0(v) = 1$. ■

Proof [Proof of Lemma 14, case $K=1$] When $K=1$, the only possible state is $v = (v(1))$, and, by Definition 11, $v \in \mathfrak{P}_{H,1}$ if and only if $v(1) = H$. Moreover,

$$\mathcal{D}_{H,1}(v) = v(1) - H = 0, \qquad \mathcal{D}_{H-1,0}(\cdot) = 1.$$

Thus the unique action $r(1) = 1$ satisfies

$$\frac{\mathcal{D}_{H-1,0}\left(v^{\setminus 1}\right)}{\mathcal{D}_{H-1,1}(v)} = \frac{1}{H-(H-1)} = 1 = r(1),$$

verifying Lemma 14.1, and

$$v(1) = H = \frac{H \cdot \mathcal{D}_{H-1,0}(\cdot)}{\mathcal{D}_{H,0}(\cdot)} = \frac{\mathcal{N}_{H,0}\left(v^{\setminus 1}\right)}{\mathcal{D}_{H,0}\left(v^{\setminus 1}\right)},$$

verifying Lemma 14.2. Since $\mathcal{D}_{H,0}\left(v^{\setminus 1}\right) = 1 > 0$, Lemma 14.3 holds. ■

Proof [Proof of Lemma 36] Let $H, K \in \mathbb{N}_+$.

1. By (16), $\mathcal{D}_{H,K}(v)$ is a finite sum of elementary symmetric polynomials (each of which is a polynomial in v). Hence $\mathcal{D}_{H,K}$ is itself a multivariate polynomial, and therefore C^∞ on $\mathbb{R}^K$.

2. We prove by induction on m.

   - *Base case* ($m = 1$): Let $k \in [K]$ be some index. By Lemma 13,

$$\begin{aligned}\frac{\partial}{\partial v(k)}\mathcal{D}_{H,K}(v) &= \frac{\partial}{\partial v(k)}\left(v(k)\cdot\mathcal{D}_{H,K-1}\left(v^{\setminus k}\right) - H\cdot\mathcal{D}_{H-1,K-1}\left(v^{\setminus k}\right)\right)\\ &= \mathcal{D}_{H,K-1}\left(v^{\setminus k}\right)\end{aligned}$$

   - *Inductive step* ($m \to m+1$): Let $m \in [K-1]$. Fix any subset of indices $\mathfrak{I} = \{i_1, \ldots, i_m\} \subset [K]$ of size m, and $k \in [K] \setminus \mathfrak{I}$. It holds that

$$\begin{aligned}\frac{\partial^m \mathcal{D}_{H,K}}{\partial^m v_{\mathfrak{I}\cup\{k\}}}(v) &= \frac{\partial}{\partial v(k)}\frac{\partial^m \mathcal{D}_{H,K}}{\partial^m v_{\mathfrak{I}}}(v)\\ &\overset{(a)}{=} \frac{\partial}{\partial v(k)}\mathcal{D}_{H,K-m}\left(v^{\setminus \mathfrak{I}}\right)\\ &\overset{(b)}{=} \frac{\partial}{\partial v(k)}\left(v(k)\cdot\mathcal{D}_{H,K-(m+1)}\left(v^{\setminus \mathfrak{I}\cup\{k\}}\right) - H\cdot\mathcal{D}_{H-1,K-(m+1)}\left(v^{\setminus \mathfrak{I}\cup\{k\}}\right)\right)\\ &= \mathcal{D}_{H,K-(m+1)}\left(v^{\setminus \mathfrak{I}\cup\{k\}}\right)\end{aligned}$$

   where (a) is implied by the induction hypothesis, and (b) follows from Lemma 13.

3. Let $k \in [K]$ and $m > 1$. By Lemma 36.2 it holds that

$$\frac{\partial^m}{\partial v(k)^m}\mathcal{D}_{H,K}(v) = \frac{\partial^{m-1}}{\partial v(k)^{m-1}}\mathcal{D}_{H,K-1}\left(v^{\setminus k}\right) = 0.$$

■

Proof [Proof of Lemma 38] Let r^H be any sequence of actions chosen by the bookmaker Ψ^H. Then, if the gambler bets on a single outcome $k \in [K]$ in each of the remaining H rounds, the total payout for outcome k will be

$$\sum_{h=1}^{H}\frac{1}{r_h(k)} \geq \sum_{h=1}^{H} 1 = H.$$

In case $K = 1$, we have $r_h(1) = 1$ for all h, and the inequality becomes an equality. Otherwise, setting $r_h(k) = 1$ at any h would cause the remaining components of the vector to become infinite.
■

E.3.1. SIMPLIFICATION OF EQUATIONS (C.24) AND (C.25)

For any $H \geq 1$ and $K \geq 2$, it holds that

$$\begin{aligned}
\sum_{i\in[K]\setminus\{k\}} \mathcal{D}_{H,K-2}\left(v^{\setminus\{i,k\}}\right) &\overset{(a)}{=} \sum_{i\in[K]\setminus\{k\}} \sum_{m=0}^{K-2} (-1)^m \cdot H^{\underline{m}} \cdot \sigma_{K-2-m}\left(v^{\setminus\{i,k\}}\right) \\
&= \sum_{i\in[K]\setminus\{k\}} \sum_{m=0}^{K-2} (-1)^{(m+1)-1} \cdot H^{\underline{(m+1)-1}} \cdot \sigma_{(K-1)-(m+1)}\left(v^{\setminus\{i,k\}}\right) \\
&\overset{(b)}{=} \sum_{i\in[K]\setminus\{k\}} \sum_{n=1}^{K-1} (-1)^{n-1} \cdot H^{\underline{n-1}} \cdot \sigma_{(K-1)-n}\left(v^{\setminus\{i,k\}}\right) \\
&= \sum_{n=1}^{K-1} (-1)^{n-1} \cdot H^{\underline{n-1}} \cdot \sum_{i\in[K]\setminus\{k\}} \sigma_{(K-1)-n}\left(v^{\setminus\{i,k\}}\right) \\
&\overset{(c)}{=} \sum_{n=1}^{K-1} (-1)^{n-1} \cdot n \cdot H^{\underline{n-1}} \cdot \sigma_{(K-1)-n}\left(v^{\setminus k}\right), \qquad \text{(E.10)}
\end{aligned}$$

where: (a) follows from the expression of $\mathcal{D}_{H,K}(v)$ in (C.1); (b) follows by reindexing the summation via $n = m + 1$; and (c) follows from Lemma 25. We obtain

$$\begin{aligned}
\mathtt{B} &\overset{(a)}{:=} \mathcal{D}_{H,K-1}\left(v^{\setminus k}\right) - \sum_{i\in[K]\setminus\{k\}} \mathcal{D}_{H,K-2}\left(v^{\setminus\{i,k\}}\right) \\
&\overset{(b)}{=} \sum_{m=0}^{K-1} (-1)^m \cdot H^{\underline{m}} \cdot \sigma_{(K-1)-m}\left(v^{\setminus k}\right) + \sum_{m=0}^{K-1} (-1)^m \cdot m \cdot H^{\underline{m-1}} \cdot \sigma_{(K-1)-m}\left(v^{\setminus k}\right) \\
&= \sum_{m=0}^{K-1} (-1)^m \cdot \sigma_{(K-1)-m}\left(v^{\setminus k}\right) \cdot \left(H^{\underline{m}} + m \cdot H^{\underline{m-1}}\right) \\
&\overset{(c)}{=} \sum_{m=0}^{K-1} (-1)^m \cdot (H+1)^{\underline{m}} \cdot \sigma_{(K-1)-m}\left(v^{\setminus k}\right) \\
&= \mathcal{D}_{H+1,K-1}\left(v^{\setminus k}\right), \qquad \text{(E.11)}
\end{aligned}$$

where: (a) follows from our definition of $\mathtt{B}$ in (C.25); (b) follows from the expression of $\mathcal{D}_{H,K}(v)$ in (C.1) and a substitution of (E.10); and (c) follows from the identity $T^{\underline{m}} + m \cdot T^{\underline{m-1}} = (T+1)^{\underline{m}}$ in Lemma 20.2.

Similarly to the derivation of (E.10), by the definition of $\mathcal{N}_{H,K}(v)$ in (C.2), it holds that

$$\sum_{i\in[K]\setminus\{k\}} \mathcal{N}_{H,K-2}\left(v^{\setminus\{i,k\}}\right) = \sum_{n=0}^{K-1} (-1)^{n-1} \cdot n \cdot H^{\underline{n}} \cdot \sigma_{(K-1)-n}\left(v^{\setminus k}\right). \qquad \text{(E.12)}$$

It follows that

$$\begin{aligned}
\mathtt{A} &\overset{(a)}{\coloneqq} \mathcal{D}_{H,K-1}\left(v^{\setminus k}\right) + \mathcal{N}_{H,K-1}\left(v^{\setminus k}\right) - \sum_{i\in[K]\setminus\{k\}} \mathcal{N}_{H,K-2}\left(v^{\setminus\{i,k\}}\right) \\
&\overset{(b)}{=} \sum_{m=0}^{K-1}(-1)^m \cdot H^{\underline{m}} \cdot \sigma_{(K-1)-m}\left(v^{\setminus k}\right) + H \cdot \sum_{m=0}^{K-1}(-1)^m \cdot (H-1)^{\underline{m}} \cdot \sigma_{(K-1)-m}\left(v^{\setminus k}\right) \\
&\quad + \sum_{m=0}^{K-1}(-1)^m \cdot m \cdot H^{\underline{m}} \cdot \sigma_{(K-1)-m}\left(v^{\setminus k}\right) \\
&\overset{(c)}{=} \sum_{m=0}^{K-1}(-1)^m \cdot \sigma_{(K-1)-m}\left(v^{\setminus k}\right)\left((m+1)\cdot H^{\underline{m}} + H^{\underline{m+1}}\right) \\
&\overset{(d)}{=} \sum_{m=0}^{K-1}(-1)^m \cdot (H+1)^{\underline{m+1}} \cdot \sigma_{(K-1)-m}\left(v^{\setminus k}\right) \\
&= \mathcal{N}_{H+1,K-1}\left(v^{\setminus k}\right),
\end{aligned}$$

where the steps are justified as follows:

(a) Follows from our definition of $\mathtt{A}$ in (C.24).

(b) Follows from the expression of $\mathcal{D}_{H,K}(v)$ in (C.1), the definition of $\mathcal{N}_{H,K}(v)$ in (C.2), and a substitution of (E.12).

(c) Follows from the identity $H \cdot (H-1)^{\underline{m}} = H^{\underline{m+1}}$.

(d) Follows from the identity $T^{\underline{m}} + m \cdot T^{\underline{m-1}} = (T+1)^{\underline{m}}$ in Lemma 20.2.

E.3.2. PROOF OF REMARK 39

Lemma 46 *Let $H, K \in \mathbb{N}_+$. If $v \in \mathfrak{P}_{H,K}$ then*

$$\sum_{k=1}^{K} \mathcal{D}_{H-1,K-1}\left(v^{\setminus k}\right) = \mathcal{D}_{H-1,K}(v)$$

The lemma must hold by construction: for $H = 1$, v was defined to satisfy condition 1.i; for $H > 1$, it satisfy condition 2.i. For the sake of completeness, we provide an explicit proof.

Proof [Proof of Lemma 46] Let $H, K \in \mathbb{N}_+$ and $v \in \mathfrak{P}_{H,K}$. It holds that

$$\begin{aligned}
\sum_{k=1}^{K} \mathcal{D}_{H-1,K-1}\left(v^{\setminus k}\right) &= \sum_{k=1}^{K}\sum_{m=0}^{K-1}(-1)^m (H-1)^{\underline{m}} \sigma_{(K-1)-m}\left(v^{\setminus k}\right) \\
&= \sum_{m=0}^{K-1}(-1)^m (H-1)^{\underline{m}} \sum_{k=1}^{K} \sigma_{K-(m+1)}\left(v^{\setminus k}\right) \\
&\overset{(a)}{=} \sum_{m=0}^{K-1}(-1)^m (H-1)^{\underline{m}} \cdot (m+1) \cdot \sigma_{K-(m+1)}(v) \\
&\overset{(b)}{=} \sum_{n=1}^{K}(-1)^{n-1} (H-1)^{\underline{n-1}} \cdot (n) \cdot \sigma_{K-n}(v) \\
&\overset{(c)}{=} \sum_{n=1}^{K}(-1)^{n-1} \left(H^{\underline{n}} - (H-1)^{\underline{n}}\right) \cdot \sigma_{K-n}(v) \\
&= -\left(\sum_{n=1}^{K}(-1)^n H^{\underline{n}} \cdot \sigma_{K-n}(v)\right) + \left(\sum_{n=1}^{K}(-1)^n (H-1)^{\underline{n}} \sigma_{K-n}(v)\right) \\
&= -\left(\mathcal{D}_{H,K}(v) - \sigma_K(v)\right) + \left(\mathcal{D}_{H-1,K}(v) - \sigma_K(v)\right) \\
&= -\mathcal{D}_{H,K}(v) + \mathcal{D}_{H-1,K}(v) \\
&\overset{(d)}{=} \mathcal{D}_{H-1,K}(v)
\end{aligned}$$

where the steps are justified as follows:

(a) Follows from Lemma 25.

(b) Follows from the substitution $n = m + 1$.

(c) Follows from the identity $n \cdot (H-1)^{\underline{n-1}} + (H-1)^{\underline{n}} = H^{\underline{n}}$ stated in Lemma 20.2.

(d) By Lemma 14.2, any $v \in \mathfrak{P}_{H,K}$ satisfies $\mathcal{D}_{H,K}(v) = 0$.

■

Proof [Proof of Remark 39] Let $H, K \in \mathbb{N}_+$ and $v \in \mathfrak{P}_{H,K}$. We prove the vector r, given by

$$r(k) = \frac{\mathcal{D}_{H-1,K-1}\left(v^{\setminus k}\right)}{\mathcal{D}_{H-1,K}(v)} \tag{17}$$

for every $k \in [K]$, is a probability distribution with non-zero elements.

- *Sum Constraint:* Holds by Lemma 46.
- *Positivity Constraint:* We distinguish two cases based on the value of H.
  - *Case $H = 1$:* For every $k \in [K]$

$$r(k) = \frac{1}{v(k)}. \tag{C.11}$$

By Lemma 38, $v \succeq \mathbf{1}_K$; thus, r satisfies the positivity constraint.

- *Case* $H > 1$*:* For every $k \in [K]$ the vector ${}^k v := v - \mathbf{e}_k / r(k)$ must be in $\mathfrak{P}_{H-1,K}$. By the proof of Lemma 14.3 (see Lemma 37) for every $m \in [K]$ and $\mathfrak{I} \in \binom{[K]}{m}$, $\mathcal{D}_{H-1,K-m}\left({}^k v^{\setminus \mathfrak{I}}\right) > 0$. Choosing $\mathfrak{I} = \{k\}$ gives

$$\mathcal{D}_{H-1,K-1}\left(v^{\setminus k}\right) = \mathcal{D}_{H-1,K-1}\left({}^k v^{\setminus k}\right) > 0.$$

Thus, for every $k \in [K]$, the numerator in (17) is positive. By Lemma 46, the denominator is a sum of positive numbers and hence positive. Thus, r satisfies the positivity constraint.

■

E.4. Technical Details for Appendix D

E.4.1. DERIVATION OF THE EXPLICIT FORM OF $\widehat{\mathcal{P}}_{T,K}$ IN (D.4)

$$\begin{aligned}
\widehat{\mathcal{P}}_{T,K}(x) &\overset{(a)}{:=} \mathcal{P}_{T,K}(x+T) \\
&= \sum_{m=0}^{K} (-1)^m \binom{K}{K-m} T^{\underline{m}} (x+T)^{K-m} \\
&\overset{(b)}{=} \sum_{n=0}^{K} (-1)^{K-n} \binom{K}{n} T^{\underline{K-n}} (x+T)^n . \\
&\overset{(c)}{=} \sum_{n=0}^{K} (-1)^{K-n} \binom{K}{n} T^{\underline{K-n}} \sum_{m=0}^{n} \binom{n}{m} x^m T^{n-m} \\
&= \sum_{n=0}^{K} \sum_{m=0}^{n} x^m (-1)^{K-n} \binom{K}{n} \binom{n}{m} T^{\underline{K-n}} T^{n-m} \\
&\overset{(d)}{=} \sum_{n=0}^{K} \sum_{m=0}^{n} x^m \binom{K}{m} (-1)^{K-n} \binom{K-m}{K-n} T^{\underline{K-n}} T^{n-m} \\
&= \sum_{m=0}^{K} x^m \binom{K}{m} \left(\sum_{n=m}^{K} (-1)^{K-n} \binom{K-m}{K-n} T^{n-m} T^{\underline{K-n}} \right) \\
&\overset{(e)}{=} \sum_{m=0}^{K} x^m \binom{K}{m} \left(\sum_{d=0}^{K-m} (-1)^d \binom{K-m}{d} T^{K-m-d} T^{\underline{d}} \right) \\
&\overset{(f)}{=} \sum_{m=0}^{K} x^m \binom{K}{m} \left(\sum_{d=0}^{K-m} \sum_{i=0}^{d} (-1)^d \binom{K-m}{d} \mathrm{s}(d,i)\, T^{K-(m+d-i)} \right) \\
&\overset{(g)}{=} \sum_{n=0}^{K} x^{K-n} \binom{K}{n} \left(\sum_{d=0}^{n} \sum_{i=0}^{d} (-1)^d \binom{m}{d} \mathrm{s}(d,i)\, T^{n-(d-i)} \right),
\end{aligned}$$

where the steps are justified as follows:

(a) By definition of $\widehat{\mathcal{P}}_{T,K}$ in (D.4).

(b) Substitution $n = K - m$.

(c) From the binomial expansion $(x+T)^n = \sum_{m=0}^{n} \binom{n}{m} x^m T^{n-m}$.

(d) By the identity $\binom{K}{n}\binom{n}{m} = \binom{K}{m}\binom{K-m}{n-m}$.

(e) Substituting $d = K - n$.

(f) By Lemma 22, which expresses the falling factorial in terms of Stirling numbers of the first kind.

(g) Substituting $n = K - m$.

E.4.2. DERIVATION OF THE EXPLICIT FORM OF $\widetilde{\mathcal{P}}_{T,K}$ IN (D.6)

$$\begin{aligned}
\widetilde{\mathcal{P}}_{T,K}(x) &\overset{(a)}{=} \widehat{\mathcal{P}}_{T,K}\left(\sqrt{T}x\right) \\
&\overset{(b)}{=} \sum_{m=0}^{K} T^{\frac{K-m}{2}} x^{K-m} \binom{K}{m} \left(\sum_{d=0}^{m} \sum_{i=0}^{d} (-1)^d \binom{m}{d} \mathrm{s}(d,i)\, T^{m-(d-i)} \right) \\
&= \sum_{m=0}^{K} x^{K-m} \binom{K}{m} T^{\frac{K}{2}} \left(\sum_{d=0}^{m} \sum_{i=0}^{d} (-1)^d \binom{m}{d} \mathrm{s}(d,i)\, T^{\frac{m}{2}-(d-i)} \right),
\end{aligned}$$

where (a) follows from our definition of $\widetilde{\mathcal{P}}_{T,K}$ in (D.6), and in (b) we substitute (D.4). Denote

$$\widetilde{c}_{T,m} \coloneqq \sum_{d=0}^{m} \sum_{i=0}^{d} (-1)^d \binom{m}{d} \mathrm{s}(d,i)\, T^{\frac{m}{2}-(d-i)}.$$

It holds that

$$\begin{aligned}
\widetilde{c}_{T,m} &= \sum_{d=0}^{m} \sum_{n=0}^{d} (-1)^d \binom{m}{d} \mathrm{s}(d,d-n)\, T^{\frac{m}{2}-n} \\
&= \sum_{n=0}^{m} \sum_{d=n}^{m} (-1)^d \binom{m}{d} \mathrm{s}(d,d-n)\, T^{\frac{m}{2}-n} \\
&= \sum_{n=0}^{m} T^{\frac{m}{2}-n} \sum_{d=0}^{m} (-1)^d \binom{m}{d} \mathrm{s}(d,d-n),
\end{aligned}$$

where the first equality follows by re-indexing with $n = d - i$.