

Boosting with the Logistic Loss is Consistent

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Abstract

This manuscript provides optimization guarantees, generalization bounds, and statistical consistency results for AdaBoost variants which replace the exponential loss with the logistic and similar losses (specifically, twice differentiable convex losses which are Lipschitz and tend to zero on one side).

The heart of the analysis is to show that, in lieu of explicit regularization and constraints, the structure of the problem is fairly rigidly controlled by the source distribution itself. The first control of this type is in the separable case, where a distribution-dependent relaxed weak learning rate induces speedy convergence with high probability over any sample. Otherwise, in the nonseparable case, the convex surrogate risk itself exhibits distribution-dependent levels of curvature, and consequently the algorithm’s output has small norm with high probability.

Keywords: Boosting, additive logistic regression, coordinate descent, convex analysis.

1. Introduction

Boosting algorithms form accurate predictors by combining many simple ones. These methods are practically effective (Caruana and Niculescu-Mizil, 2006), theoretically alluring (Schapire, 1990), and continue to be the topic of extensive research (Schapire and Freund, 2012).

The most popular scheme, AdaBoost (Freund and Schapire, 1997), was eventually revealed to be a coordinate descent method applied to a convex empirical risk minimization problem (Breiman, 1999). Due to the lack of regularization and constraints, this optimization problem eschews the typical structure which leads to a fast-converging, well-conditioned optimization problem: it typically fails to have minimizers (let alone possessing compact level sets or strong convexity), and the simple predictors (weak learners) can be linearly dependent, meaning the Hessian is singular. Consequently, fairly customized convergence analyses must be developed (Freund and Schapire, 1997; Collins et al., 2002; Mukherjee et al., 2011; Telgarsky, 2012), with data-dependent quantities dictating behavior. This, however, can be a great boon: one such quantity, the *weak learning rate*—a measure of the compatibility of the weak learners to the target function—allows for linear convergence in settings far removed from the strong convexity typical of fast convergence in convex optimization.

Difficulties also arise on the statistical side: each round typically selects a new hypothesis from some VC class, and the method is consequently building hypotheses in the linear span, which generally has infinite VC dimension, and is thus statistically unstable (Devroye et al., 1996, Theorem 14.3). It was therefore the topic of great research to establish consistency

of AdaBoost (Zhang and Yu, 2005; Jiang, 2000), a question finally closed by Bartlett and Traskin (2007).

AdaBoost originally used the exponential loss, however much practical and theoretical research has been devoted to the logistic loss (Friedman et al., 2000; Lafferty, 1999; Collins et al., 2002), both due to intuitive appeal (e.g., less attention to outliers), and statistical connections (e.g., consistency of maximum likelihood (Ferguson, 1996, Section 17)). Even so, this choice has not been subjected to the same intensive consistency study as the exponential loss, and as discussed by Bartlett and Traskin (2007, Section 4), current analyses for the exponential loss do not carry over.

1.1. Outline

The primary goal of this manuscript is to close the gap with the exponential loss; namely, boosting with losses similar to the logistic loss is consistent under the same assumptions as those assumed for the exponential loss (Bartlett and Traskin, 2007, Corollary 9), moreover with comparable rates (Schapire and Freund, 2012, Theorem 12.2).

The algorithm and related notation are detailed in Section 2. To fit practical regimes, both the selection of simple predictors (also termed *weak learners* and coordinates) and step size may be approximate; crucially, however, the analysis covers the case of unconstrained step sizes. The usual early stopping threshold is employed: m^a iterations are performed, where m is the sample size and $a \in (0, 1)$ is a scalar parameter to the algorithm. Lastly, rather than simply outputting the final predictor, the method returns the iterate which achieved the smallest classification error. While perhaps unnecessary, this choice leads to a pleasantly simple convergence analysis in the separable case.

The general consistency result is presented in Section 3, along with a sketch of the analysis. As usual, the Borel-Cantelli Lemma is used to convert finite sample guarantees into a consistency result; the finite sample guarantees themselves are split into two cases: a separable case in Section 4, and a nonseparable case in Section 5. In either case, when using the logistic loss, the classification risk will decay roughly as m^{-c} for some $c < 1$.

Proofs are only outlined in the body, with details deferred to the appendices.

1.2. Related Work

On the general topic of AdaBoost, both the original papers (Schapire, 1990; Freund, 1995; Freund and Schapire, 1997) as well as the textbook by the original authors (Schapire and Freund, 2012) are indispensable.

Additive logistic regression was introduced by Friedman et al. (2000), with extensive additional discussion appearing shortly thereafter (Friedman, 2000; Lafferty, 1999; Mason et al., 2000). The particular method studied in this manuscript, which is essentially AdaBoost but with the exponential loss replaced by losses similar to the logistic loss, was shown to produce a sequence of empirical risks converging to the infimum by Collins et al. (2002), with (optimization) rates in the general case coming later (Telgarsky, 2012), and (optimization) rates in the margin case coming earlier (Duffy and Helmbold, 2000).

The consistency of AdaBoost was first analyzed under various regularization strategies. Most notably, the work of Blanchard et al. (2003) and Lugosi and Vayatis (2004) studied the solutions of penalized estimators; the former work in particular achieving excellent

finite sample guarantees, with convex risk decaying roughly as $\mathcal{O}(m^{-1/2})$ (where m is the sample size), with improvements under various noise conditions. This work, however, did not demonstrate tractable algorithms to produce these estimators, which was a goal of the work by [Zhang and Yu \(2005\)](#); namely, there it is shown that merely constraining the step size taken by an AdaBoost-style scheme (with a variety of losses) suffices to achieve a convex risk rate of roughly $\mathcal{O}(m^{-1/4})$ (in the case of the logistic loss), which includes the effect of approximate solutions produced by the algorithm. As will be discussed later, the present work, in the nonseparable case, fits well with the development by [Zhang and Yu \(2005\)](#).

Two works give a consistency analysis of AdaBoost without any algorithmic modifications, under the condition that the algorithm is stopped after m^a iterations (with arbitrary $a \in (0, 1)$). The first such analysis, due to [Bartlett and Traskin \(2007\)](#), was focused on establishing consistency, and established the convex risk decays roughly as $\mathcal{O}(1/\sqrt{\ln(m)})$; the analysis depends on a curvature lower bound, which follows from a lower bound on the convex risk since the exponential loss is equal to its derivative. This derivative structure is of course not present with the logistic loss, and the present analysis must find another way. A streamlined consistency analysis of AdaBoost appears in the textbook of [Schapire and Freund \(2012, Theorem 12.2\)](#), with a rate of roughly $\mathcal{O}(m^{-1/9})$ (by choosing $a := 5/9$); the analysis is short and clean, but it is not clear how to decouple the exponential loss.

In the separable case, the analysis here relies upon ideas from weak learnability, just as with the original analysis of AdaBoost (under margin assumptions) ([Freund and Schapire, 1997](#)). The relaxed notion of margin here is very close to the quantity AvgMin_k as developed by [Shalev-Shwartz and Singer \(2008, Section 4.1\)](#); the main contrasting point is that the present manuscript is concerned with statistical properties, and in particular how these relaxed margin properties behave under sampling. The optimization analysis in the separable case here shares ideas both with the original AdaBoost analysis ([Freund and Schapire, 1997](#)), but also with the literature on hard cores ([Impagliazzo, 1995](#); [Barak et al., 2009](#)); one distinction is that the latter methods take the target weak learning rate as input, whereas here (and in general with *adaptive boosting*), it must be found by the algorithm. Interestingly, the loss function implicit in the boosting algorithm due to [Impagliazzo \(1995, Proof of Lemma 1\)](#) achieves superior constants to the logistic loss in [Theorem 7](#); nearly the same loss was presented and praised by [Zhang \(2004, see the definition at the end of Section 4.6\)](#).

As stated previously, the nonseparable case fits well with the scheme laid down by [Zhang and Yu \(2005\)](#), where the algorithm is modified to constrain step sizes. Indeed, the analysis here first establishes that the iterates are well-behaved with exactly the sorts of norm bounds needed by the analysis of [Zhang and Yu \(2005\)](#) (compare for instance the summability conditions ([Zhang and Yu, 2005, Equation \(4\)](#)) with [Lemma 17](#)). In order to produce these results, the present work uses a dual optimum as a witness to the difficulty of the convex risk problem over the source distribution; this technique follows structural properties of boosting laid in the finite-dimensional case by [Telgarsky \(2012\)](#). That convergence analysis appears statistically unstable, and the subsequent analysis here follows a similar path to the one by [Zhang and Yu \(2005\)](#), with additional help from [Bartlett and Traskin \(2007\)](#). One interesting distinction between the present work and those by [Bartlett and Traskin \(2007\)](#) and [Zhang and Yu \(2005\)](#) is that the latter two require a more strenuous algorithm: the weak learner and step size selection must be performed simultaneously. Decoupling these

does not appear to impact the rates, however, this distinction prevents those results from being directly invoked here, meaning they must instead be reworked.

Lastly, note that the translation between convex and classification risks follows standard results on classification calibration as first developed by [Zhang \(2004\)](#), and later extended by [Bartlett et al. \(2006\)](#).

2. Notation and Algorithm

Let \mathcal{H} be the collection of weak learners, where each $h \in \mathcal{H}$ is a function of the form $h : \mathcal{X} \rightarrow [-1, +1]$, with \mathcal{X} being an abstract instance space, and the crucial property of the output space $[-1, +1]$ is that it is bounded. Given any weighting λ of \mathcal{H} satisfying $\sum_{h \in \mathcal{H}} |\lambda(h)| < \infty$, define the function

$$(H\lambda)_x := (H\lambda)(x) := \sum_{h \in \mathcal{H}} h(x)\lambda(h).$$

Since $\sum_h \lambda(h)$ is absolutely convergent and $\sup_{x,h} |h(x)| \leq 1$, then $H\lambda$ is well-defined.

Let Λ denote the space of all absolutely convergent weightings over \mathcal{H} ; formally, Λ is the Banach space $L^1(\rho)$, where ρ is the counting measure over \mathcal{H} . In this way, H can be viewed as a function from Λ to the vector space of bounded functions over \mathcal{X} . The algorithm itself only considers finite sets of hypotheses over a finite sample, and thus H can be viewed as a matrix, but the Banach space generalization will be useful when considering the abstract problem over the distribution.

For additional convenience, define a second function

$$(A\lambda)_{x,y} := (A\lambda)(x,y) := -y(H\lambda)_x := -y \sum_{h \in \mathcal{H}} h(x)\lambda(h),$$

which is again well-defined. Let $\mathbf{e}_h \in \Lambda$ denote the weighting placing unit weight on a fixed $h \in \mathcal{H}$, and zero weight elsewhere. For more properties of these Banach spaces, as well as the linear operators H and A , please see [Appendix A](#).

The basic measure of the complexity of \mathcal{H} is its VC dimension.

Definition 1 *Let \mathcal{F}_{vc} contain all classes \mathcal{H} of finite VC dimension, denoted $\mathcal{V}(\mathcal{H}) < \infty$.*

The source distribution over $\mathcal{X} \times \{-1, +1\}$ will always be denoted by μ , with a factorization (disintegration) into a marginal $\mu^{\mathcal{X}}$ over \mathcal{X} and conditional $\Pr(Y = 1|X = x)$, the latter considered as a function over \mathcal{X} . When a sample $\{(x_i, y_i)\}_{i=1}^m$ is available, $\hat{\mu}$ will denote the corresponding empirical measure. Many results hold for arbitrary probability measures over $\mathcal{X} \times \{-1, +1\}$, in which case the variable ν will be adopted; the σ -algebra over \mathcal{X} is always the Borel σ -algebra (and it is tacitly supposed \mathcal{X} is a topological space). With the measures defined, a second notion of class complexity is as follows.

Definition 2 *Let $\mathcal{F}_{\text{ds}}(\nu)$ contain every class \mathcal{H} whose linear span $\text{span}(\mathcal{H})$ is dense (in the $L^1(\nu)$ topology) in the collection of all bounded measurable functions over \mathcal{X} .*

Conditions similar to those defining $\mathcal{F}_{\text{ds}}(\nu)$ are usually called *dense class assumptions* ([Bartlett and Traskin, 2007](#), Condition 1, Denseness), or *completeness assumptions* ([Breiman,](#)

2000, Definition 1); for a more extensive discussion of these conditions, please see Appendix B; for the time being, the important point is that reasonable elements of $\mathcal{F}_{\text{vc}} \cap \mathcal{F}_{\text{ds}}(\nu)$ exist; in particular, the following result provides that if $\mathcal{X} = \mathbb{R}^d$, then decision lists and decision trees with d axis-aligned splits suffice.

Proposition 3 *Suppose $\mathcal{X} = \mathbb{R}^d$, and let ν be a Borel probability measure over \mathcal{X} . If $\text{span}(\mathcal{H})$ contains all indicators of products of half-open intervals of the form $\times_{i=1}^d [a_i, b_i)$, where $a_i < b_i$, then $\mathcal{H} \in \mathcal{F}_{\text{ds}}(\nu)$.*

Given a loss function $\ell : \mathbb{R} \rightarrow \mathbb{R}_+$ (where \mathbb{R}_+ denotes nonnegative reals, and later \mathbb{R}_{++} will denote positive reals), a first version of the relevant optimization problem over the source distribution is

$$\bar{\mathcal{L}} := \inf \left\{ \int \ell(-y(H\lambda)_x) d\mu(x, y) : \lambda \in \Lambda \right\} = \inf \left\{ \int \ell(A\lambda) d\mu : \lambda \in \Lambda \right\},$$

where $\bar{\mathcal{L}}$ denotes the optimal value, and the final expression both exhibits the practice of dropping integration variables, and the convenience of A . For further simplification, define the simplified surrogate risk functions

$$\mathcal{L}(A\lambda) := \int \ell(A\lambda) d\mu \quad \text{and} \quad \hat{\mathcal{L}}(A\lambda) := \int \ell(A\lambda) d\hat{\mu} = \frac{1}{m} \sum_{i=1}^m \ell((A\lambda)_{x_i, y_i}),$$

meaning $\hat{\mathcal{L}}$ denotes the usual empirical risk. The classes of loss functions considered here are as follows.

Definition 4 *Let \mathbb{L}_{2d} denote twice continuously differentiable convex losses. Additionally, let \mathbb{L}_{lg} contain all differentiable convex Lipschitz losses $\ell : \mathbb{R} \rightarrow \mathbb{R}_+$ with tightest Lipschitz constant $\beta_2 := \sup_{x \neq y} |\ell(x) - \ell(y)| / |x - y|$ as follows. First, every ℓ has $\ell' \in [0, \beta_2]$ everywhere and $\ell' \in [\beta_1, \beta_2]$ over \mathbb{R}_+ for some $0 < \beta_1 \leq \beta_2$. Second, every ℓ has Lipschitz gradients with (tightest) parameter B_2 , meaning $B_2 := \sup_{x, y \in \mathbb{R}} (\ell'(y) - \ell'(x)) / (y - x) < \infty$.*

Although the most general guarantees require $\ell \in \mathbb{L}_{\text{lg}} \cap \mathbb{L}_{2d}$, the separable case needs only $\ell \in \mathbb{L}_{\text{lg}}$, which allows consideration of an interesting piecewise quadratic loss $\ell_{\text{russ}}(x) := 0.5(x+1)^2 \mathbb{1}[-1 < x < 0] + (x+0.5) \mathbb{1}[0 \leq x]$, which was used by Impagliazzo (1995, Proof of Lemma 1) in the study of hard cores. Both ℓ_{russ} and the logistic loss $\ell_{\text{log}}(x) := \ln(1 + \exp(x))$ are in \mathbb{L}_{lg} , whereas ℓ_{log} and the exponential loss are within \mathbb{L}_{2d} (cf. Lemma 25).

Let \mathcal{R} denote the classification risk, meaning

$$\mathcal{R}(H\lambda) := \int (\mathbb{1}[y = +1 \wedge (H\lambda)_x < 0] + \mathbb{1}[y = -1 \wedge (H\lambda)_x \geq 0]) d\mu(x, y).$$

Analogously to \mathcal{L} , let $\hat{\mathcal{R}}$ denote empirical classification risk, and $\bar{\mathcal{R}}$ denote optimal classification risk over $\text{span}(\mathcal{H})$. Notice that these definitions embed the fact that boosting procedures provide a real-valued function $H\lambda$, which is then thresholded to produce a binary classifier.

Borrowing from the probability literature, brackets denote a shorthand for sets; for instance $[y(H\lambda)_x > 0] = \{(x, y) \in \mathcal{X} \times \{-1, +1\} : y(H\lambda)_x > 0\}$ is the subset of $\mathcal{X} \times \{-1, +1\}$ where λ achieves strictly positive margins. When the variables are clear, they will be suppressed; e.g., $[p > 0] = [p(x, y) > 0] = \{(x, y) \in \mathcal{X} \times \{-1, +1\} : p(x, y) > 0\}$.

Algorithm 1: BOOST.

Input: loss ℓ and empirical measure $\hat{\mu}$ (granting $m, \hat{\mathcal{L}}, \nabla \hat{\mathcal{L}}$), hypothesis class \mathcal{H} (granting A, A^\top), stopping and coordinate search parameters $a \in (0, 1)$ and $\rho \in (0, 1)$.

Output: Coefficient vector $\hat{\lambda}$.

Initialize $\lambda_0 := 0$.

for $t = 1, 2, \dots, \lceil m^a \rceil$: **do**

 Choose approximate best coordinate (weak learner) $h_t \in \mathcal{H}$ satisfying

$$|\nabla \hat{\mathcal{L}}(A\lambda_{t-1})^\top A e_{h_t}| \geq \rho \sup_{h \in \mathcal{H}} |\nabla \hat{\mathcal{L}}(A\lambda_{t-1})^\top A e_h| = \rho \|\nabla \hat{\mathcal{L}}(A\lambda_{t-1})^\top A\|_\infty.$$

 Set descent direction $v_t \in \{\pm e_{h_t}\}$, whereby

$$-\rho \|A^\top \nabla \hat{\mathcal{L}}(A\lambda_{t-1})\|_\infty \geq \nabla \hat{\mathcal{L}}(A\lambda_{t-1})^\top A v_t \geq -\|A^\top \nabla \hat{\mathcal{L}}(A\lambda_{t-1})\|_\infty.$$

 Set $\bar{\alpha}_t := \arg \min_{\alpha > 0} \hat{\mathcal{L}}(A(\lambda_{t-1} + \alpha v_t))$, and choose a step α_t as follows:

option 1: If $\bar{\alpha}_t < \infty$, set $\alpha_t = \bar{\alpha}_t$ (i.e., make an optimal unconstrained step).

option 2: If $\ell \in \mathbb{L}_{\text{lg}}$, choose any $\alpha_t \in [-\nabla \hat{\mathcal{L}}(A\lambda_{t-1})^\top A v_t / B_2, \bar{\alpha}_t]$.

option 3: Choose any α_t satisfying the Wolfe conditions (please see Remark 35.)

 Update $\lambda_t := \lambda_{t-1} + \alpha_t v_t$.

end

return $\hat{\lambda}$ achieving the best classification error (i.e., $\hat{\mathcal{R}}(H\hat{\lambda}) = \min_{i \in \lceil m^a \rceil} \hat{\mathcal{R}}(H\lambda_i)$).

2.1. Algorithm

The algorithm itself is spelled out in Algorithm 1. As the method is coordinate descent applied to $\hat{\mathcal{L}} \circ A$, the relevant gradient term is $A^\top \nabla \hat{\mathcal{L}}(A\lambda) = \nabla(\hat{\mathcal{L}} \circ A)(\lambda)$ (which can be computed from the sample; see Appendix F for details). The scalar $\rho \in (0, 1)$ allows for approximate weak learner selection, and furthermore the step size also has some flexibility, though as stated previously, the unconstrained case is the tricky one. Lastly, note that the iterate achieving the best classification error is returned.

3. Consistency Statement and Analysis Sketch

The analysis considers two cases: either $\bar{\mathcal{L}} = 0$ (*separable*) or $\bar{\mathcal{L}} > 0$ (*nonseparable*). (By Proposition 29, $\bar{\mathcal{L}} = 0$ implies finite samples have a separating choice $\lambda \in \Lambda$ almost surely.)

When the instance is separable, the improvement in objective value $\mathcal{L}(A\lambda)$ in early iterations may be lower bounded by a margin-based quantity related to the classical weak learning rate; while this quantity is a random variable, with high probability it can be lower bounded by the analogous quantity over the distribution (which will be shown positive iff the instance is separable). The bulk of the analysis is in constructing and controlling this quantity; the optimization and generalization analysis thereafter is straightforward, yielding a rate of roughly $\mathcal{O}(1/m^{1/3})$ when $a = 2/3$.

When the instance is not separable, every weak learner makes a fair number of mistakes, and thus the algorithm makes more hesitant progress. Concretely, with high probability,

the norms of the iterates are bounded, and moreover the quantity $m^{-1} \sum_i \ell''((A\lambda)_{x_i, y_i})$, which is roughly the Hessian in axis-aligned directions (and relevant to coordinate descent), is also lower bounded. This in turn allows adaptation of the optimization analysis due to [Zhang and Yu \(2005\)](#). While the rate in this case is still roughly $\mathcal{O}(1/m^{1/c})$, unfortunately the exponent $c > 1$ depends both on μ and on \mathcal{H} (but is of course finite).

As a final point of interest, each case, in order to respectively establish either fast decrease or the norm constraints, considers the behavior of the reweighted average margins

$$\int (A\lambda)p d\mu = \int (A\lambda)_{x,y} p(x,y) d\mu(x,y), \quad (1)$$

where $\lambda \in \Lambda$ and $p \in L^\infty(\nu)$. In the separable case, this quantity is studied for a single good λ as p varies, whereas the nonseparable case studies a single bad p as λ varies.

Combining these finite sample results with the Borel-Cantelli Lemma gives the following.

Theorem 5 *Let loss $\ell \in \mathbb{L}_{\text{lg}} \cap \mathbb{L}_{2\text{d}}$, probability measure μ over $\mathcal{X} \times \{-1, +1\}$, binary class $\mathcal{H} \in \mathcal{F}_{\text{vc}} \cap \mathcal{F}_{\text{ds}}(\mu^{\mathcal{X}})$, and any stopping parameter $a \in (0, 1)$ be given. Let $\hat{\lambda}_m$ denote the output of Algorithm 1 when run on m examples, and let \mathcal{R}_\star to denote the Bayes error rate. Then $\mathcal{R}(A\hat{\lambda}_m) \rightarrow \mathcal{R}_\star$ almost surely as $m \rightarrow \infty$.*

4. The Separable Case ($\bar{\mathcal{L}} = 0$)

The rates in the case $\bar{\mathcal{L}} = 0$ will depend on the following quantity $\gamma_\epsilon(\nu)$, which directly embeds the reweighted margin expression in eq. (1).

Definition 6 *Let ν be any probability measure over $\mathcal{X} \times \{-1, +1\}$ (relevant choices are μ and $\hat{\mu}$), and let $\epsilon \in [0, 1]$ be given. Define a permissible set of densities (with respect to ν)*

$$\mathcal{D}_\epsilon(\nu) := \{p \in L^1(\nu) : p \geq 0 \text{ } \nu\text{-a.e.}, \|p\|_1 = 1, \|p\|_\infty \leq 1/\epsilon\},$$

with the convention $1/0 = \infty$ in the case $\epsilon = 0$. Additionally define

$$\gamma_\epsilon(\nu) := \inf_{p \in \mathcal{D}_\epsilon(\nu)} \sup \left\{ - \int (A\lambda)p d\nu : \lambda \in \Lambda, \|\lambda\|_1 \leq 1 \right\}.$$

(When ν is a discrete measure, $\gamma_\epsilon(\nu)$ is almost equivalent to AvgMin_k as developed by [Shalev-Shwartz and Singer \(2008, Section 4.1\)](#).)

This quantity will play a role analogous to the *weak learning rate* in AdaBoost, which guarantees the algorithm makes speedy progress in certain separable cases. The correspondence between these two quantities will occupy much of this section; but first, note primary guarantee in the separable case.

Theorem 7 *Let $\ell \in \mathbb{L}_{\text{lg}}$ (with parameters β_1, β_2, B_2) and any $\mathcal{H} \in \mathcal{F}_{\text{vc}}$ be given, and suppose $\bar{\mathcal{L}} = 0$. Let any error tolerance $\epsilon \in (0, 1]$ and any confidence parameter $\delta \in (0, 1]$ be given, and for convenience set $\epsilon' := \epsilon\beta_1/\beta_2$; by these choices, $\gamma_{\epsilon'}(\mu) > 0$. Suppose Algorithm 1 is run with stopping parameter $a \in (0, 1)$, and the sample size m satisfies*

$$m \geq \max \left\{ \frac{2}{(\epsilon\gamma_{\epsilon'}(\mu))^2} \ln \left(\frac{4}{\delta} \right), \left(\frac{24B_2\ell(0)}{(\rho\beta_1\epsilon\gamma_{\epsilon'}(\mu))^2} \right)^{1/a} \right\}.$$

Then, with probability at least $1 - \delta$, the algorithm's output $\hat{\lambda}$ satisfies

$$\mathcal{R}(H\hat{\lambda}) \leq \epsilon + 2\sqrt{\epsilon \frac{(\mathcal{V}(\mathcal{H}) + 1) \ln(2em) + \ln(8/\delta)}{m^{1-a}}} + 4 \frac{(\mathcal{V}(\mathcal{H}) + 1) \ln(2em) + \ln(8/\delta)}{m^{1-a}}.$$

To simplify this bound, first note that, for the logistic loss, $B_2 = 1/4$, $\beta_1 = 1/2$, and $\beta_2 = 1$ (cf. Lemma 25). Ignoring these terms, as well as $\ell(0) = 1$, ρ (which can be set to $1/2$), and $\gamma_{\epsilon'}(\mu)$; the choices $a = 1/2$ and $\epsilon := \mathcal{O}(m^{-a/2}) = \mathcal{O}(m^{-1/4})$ grant that $\mathcal{O}(1/\epsilon^2)$ iterations suffice to achieve classification risk $\mathcal{O}(1/\sqrt{m})$, whereas the choices $a = 2/3$ and $\epsilon := \mathcal{O}(m^{-a/2}) = \mathcal{O}(m^{-1/3})$ provide that $\mathcal{O}(1/\epsilon^3)$ suffice to achieve error $\mathcal{O}(m^{-1/3})$.

4.1. The Quantity $\gamma_{\epsilon}(\nu)$

To develop the meaning and necessity of $\gamma_{\epsilon}(\nu)$, first recall the classical definitions associated with *weak learnability* (adjusted here so that “binary” means $\{-1, +1\}$ and not $\{0, 1\}$).

Definition 8 (*Schapire and Freund (2012, Chapter 2).*) A class \mathcal{H} is weakly PAC-learnable with rate γ if for any measure ν over $\mathcal{X} \times \{-1, +1\}$, there exists $h \in \mathcal{H}$ with $\int y h(x) d\nu(x, y) \geq \gamma$. Additionally, class \mathcal{H} and empirical measure ν are empirically weakly learnable with rate $\gamma(\nu)$ if there exists $h \in \mathcal{H}$ so that $\int y h(x) p(x, y) d\nu(x, y) \geq \gamma(\nu)$ for every reweighting p of measure ν .

The definitions of γ and $\gamma(\nu)$ are close to the definition of $\gamma_{\epsilon}(\nu)$: the latter replaces the quantifiers and inequalities with explicit infima and suprema, which grants the following correspondence.

Proposition 9 Let class \mathcal{H} , probability measure μ (over $\mathcal{X} \times \{-1, +1\}$), and empirical counterpart $\hat{\mu}$ be given. Then the weak PAC-learning rate γ satisfies $\gamma \leq \gamma_0(\mu)$, and the empirical weak learning rate $\gamma(\hat{\mu})$ satisfies $\gamma(\hat{\mu}) \leq \gamma_0(\hat{\mu})$.

The following example highlights why $\gamma_0(\nu)$ can be problematic, even when $\bar{\mathcal{L}} = 0$.

Example 1 (Nightmare scenario #1) Suppose $\mathcal{X} = (0, 1]$, and

$$\Pr[y = +1|x] = \begin{cases} 1 & \text{when } x \in (1/(i+1), 1/i] \text{ for some odd integer } i, \\ 0 & \text{when } x \in (1/(i+1), 1/i] \text{ for some even (positive) integer } i. \end{cases}$$

Let \mathcal{H} consist of threshold functions (decision stumps). Given any integer k , a combination of thresholds may be constructed which is correct on k intervals, and thus $\bar{\mathcal{L}} = 0$ by considering $k \rightarrow \infty$. Unfortunately, the norm of these solutions also grows unboundedly, suggesting γ and $\gamma(\mu)$ are tiny. Indeed, consider a distribution over \mathcal{X} which is uniform on k of the intervals, and zero elsewhere. Any threshold is incorrect on nearly half of these intervals, and by considering $k \rightarrow \infty$, it follows that $\gamma \leq \gamma_0(\mu) = 0$.

In precise terms, this nightmare, and suggested sequence of distributions, provide the following property.

Proposition 10 *There exist choices for \mathcal{H} and μ so that $\bar{\mathcal{L}} = 0$, but $\gamma \leq \gamma_0(\mu) = 0$ and, with any probability $1 - \delta$ and sample size m large enough that $k := \lfloor m^{1/4} / (3\sqrt{2\ln(4/\delta)}) \rfloor$ satisfies $k \geq 2$, then $\gamma(\hat{\mu}) \leq \gamma_0(\hat{\mu}) \leq \gamma_\epsilon(\hat{\mu}) \leq \mathcal{O}((\ln(4/\delta) + \ln(m))^{1/2}/m^{1/4})$, where $\epsilon = \mathcal{O}(1/k) = \mathcal{O}(m^{-1/4}/\ln(4/\delta))$ and the $\mathcal{O}(\cdot)$ only suppresses terms independent of m and δ .*

But something is wrong here — Example 1 seems quite easy! The reason γ indicates otherwise is that it simply tries too hard: Example 1 is easy if giving up on an ϵ -fraction of the data is acceptable. This reasoning leads to the relaxation $\gamma_\epsilon(\nu)$, which, in contrast to Proposition 10, carries the following guarantee.

Proposition 11 *Let probability μ over $\mathcal{X} \times \{-1, +1\}$ and class \mathcal{H} be given.*

1. *Let loss $\ell \in \mathbb{L}_{\text{lg}}$ be given. Then $\bar{\mathcal{L}} = 0$ iff $\gamma_\epsilon(\mu) > 0$ for all $\epsilon > 0$.*
2. *Let any $\epsilon > 0$, confidence parameter $\delta \in (0, 1]$, and empirical measure $\hat{\mu}$ be given. Then with probability at least $1 - \delta$,*

$$\gamma_\epsilon(\hat{\mu}) \geq \gamma_\epsilon(\mu) - \frac{1}{\epsilon} \sqrt{\frac{1}{2m} \ln \left(\frac{2}{\delta} \right)}.$$

In order to prove this result, and also a few other components in the proof of Theorem 7, the following dual representation of $\gamma_\epsilon(\nu)$ is used. A similar result was proved by Shalev-Shwartz and Singer (2008, see the quantity AvgMin_k) in the case of measures with finite support and finite cardinality hypothesis classes; the proof here invokes Sion's Minimax Theorem (Komiya, 1988), which operates in fairly general topological vector spaces.

Lemma 12 *Let probability measure ν over $\mathcal{X} \times \{-1, +1\}$, any \mathcal{H} , and any $\epsilon \in (0, 1]$ be given. Then*

$$\begin{aligned} \gamma_\epsilon(\nu) &= \min_{p \in \mathcal{D}_\epsilon(\nu)} \sup \left\{ \int (A\lambda) p d\nu : \lambda \in \Lambda, \|\lambda\|_1 \leq 1 \right\} \\ &= \sup \left\{ \min_{p \in \mathcal{D}_\epsilon(\nu)} \int (A\lambda) p d\nu : \lambda \in \Lambda, \|\lambda\|_1 \leq 1 \right\} \\ &= \min_{p \in \mathcal{D}_\epsilon(\nu)} \|A^\top p\|_\infty, \end{aligned}$$

where A^\top is the unique adjoint operator to A (cf. Lemma 22), and

$$\|A^\top p\|_\infty = \sup \left\{ \left| \int y h(x) p(x, y) d\nu(x, y) \right| : h \in \mathcal{H} \right\}.$$

In order to use this to prove the first part of Theorem 7, first note that whenever $\gamma_\epsilon(\nu) = 0$, there exists a dual element $p \in \mathcal{D}_\epsilon(\nu)$ certifying this property, which in turn can be related to the duality structure of \mathcal{L} (presented later in Proposition 14), and gives the result. For the second part of Theorem 7, similarly the infimum in the definition of $\gamma_\epsilon(\nu)$ can be removed by considering a single bad certificate $p \in \mathcal{D}_\epsilon(\mu)$, and the supremum can be removed by considering a single good $\lambda \in \Lambda$. The certificate p can be shown to have a simple structure (it emphasizes margin violations for the fixed good λ), and in turn the deviations are easy to control.

4.2. Proof Sketch of Theorem 7

The pieces are in place to establish the finite sample guarantees in Theorem 7. First, note the following empirical risk guarantee.

Lemma 13 *Let any $\ell \in \mathbb{L}_{\text{lg}}$, empirical measure $\hat{\mu}$, and \mathcal{H} be given. Suppose Algorithm 1 is run with any of the three step size choices for T iterations, let $\epsilon_t = \widehat{\mathcal{R}}(H\lambda_t)$ denote the classification error of $H\lambda_t$, and set $\epsilon'_t := \epsilon_t\beta_1/\beta_2$ for convenience. Then*

$$\widehat{\mathcal{L}}(A\lambda_T) \leq \ell(0) - \sum_{t=1}^T \frac{(\rho\beta_1\epsilon_{t-1}\gamma_{\epsilon'_{t-1}}(\hat{\mu}))^2}{6B_2}.$$

Notice that this result indicates that the convex risk decreases quickly in the presence of classification errors. The proof, sketched as follows, is fairly straightforward. First, standard properties of the line search choices show that $\widehat{\mathcal{L}}$ drops in round t proportionally to $\|A^\top \nabla \widehat{\mathcal{L}}(A\lambda_{t-1})\|_\infty$. Considering $\nabla \widehat{\mathcal{L}}(A\lambda_{t-1})$ as a reweighting of $\hat{\mu}$, this expression appears in the dual form of dual form of $\gamma_\epsilon(\hat{\mu})$ as presented in Lemma 12. In order to make the correspondence precise, $\nabla \widehat{\mathcal{L}}(A\lambda_{t-1})$ must be rescaled to unit norm; but, by the Lipschitz property, the rescaling is by at most $\beta_1\epsilon_{t-1}$! After some algebra, and summing across all iterations, the result follows.

From here, there is little to do. By Lemma 13, until some iteration has low error, progress is quick. The selection rule (returning $\hat{\lambda}$ with minimal classification risk) ensures there are no problems if the classification risk happens to go back up, and Proposition 11 allows $\gamma_\epsilon(\mu)$ to replace $\gamma_\epsilon(\hat{\mu})$. As this reasoning provides a direct guarantee on the empirical classification risk, standard uniform convergence techniques give the result.

5. The Nonseparable Case ($\bar{\mathcal{L}} > 0$)

When $\bar{\mathcal{L}} > 0$, the essential object will be an optimum to the convex dual of the central optimization problem $\inf_\lambda \mathcal{L}(A\lambda)$, specified as follows.

Proposition 14 *Let loss $\ell \in \mathbb{L}_{\text{lg}}$ (with tightest Lipschitz parameter β_2), class \mathcal{H} , and probability measure ν over $\mathcal{X} \times \{-1, +1\}$ be given. Then*

$$\inf \left\{ \int \ell(A\lambda) d\nu : \lambda \in \Lambda \right\} = \max \left\{ - \int \ell^*(p) : p \in L^\infty(\nu), p \in [0, \beta_2] \text{ } \nu\text{-a.e.}, \|A^\top p\|_\infty = 0 \right\},$$

where ℓ^* is the Fenchel conjugate to ℓ , and the adjoint A^\top is as in Lemma 12 and Lemma 22. Additionally, the dual optimum \bar{p} satisfies $\mu([\bar{p} \geq \tau]) \geq \tau$, where $\tau > 0$ whenever the optimal value $\bar{\mathcal{L}}_\nu$ is positive, and moreover τ has the explicit form $\tau := \ell^{-*}(-\bar{\mathcal{L}}_\nu)/2$, where ℓ^{-*} is the (well-defined) inverse of ℓ^* along $[0, \ell'(0)]$.

The strategy in the nonseparable case is to exhibit curvature in the objective function (i.e., a lower bound on the second-order expression $m^{-1} \sum_i \ell((A\lambda)_{x_i, y_i})$), and the dual optimum \bar{p} will be provide the mechanism. Making these statement precise is the topic of this section, however, for the time being, note that the dual problem resembles a maximum entropy problem, where the constraint $\|A^\top p\|_\infty = 0$ requires reweightings (including \bar{p}) to

decorrelate all predictors from the target, and the objective function $-\int \ell^*$ prefers weightings which are large and close to uniform (cf. Lemma 28; in the case of the logistic loss ℓ_{\log} , these statements are fairly concrete: $\ell_{\log}^*(\phi) = \phi \ln(\phi) + (1 - \phi) \ln(1 - \phi)$, the Fermi-Dirac Entropy).

Theorem 15 *Let loss $\ell \in \mathbb{L}_{\lg} \cap \mathbb{L}_{2d}$, binary class $\mathcal{H} \in \mathcal{F}_{\text{vc}}$, probability measure μ over $\mathcal{X} \times \{-1, +1\}$ with empirical counterpart $\hat{\mu}$ corresponding to a sample of size m , time horizon $t \leq m^a$ with $a \in (0, 1)$, and any confidence $\delta \in (0, 1]$ be given. Suppose $\bar{\mathcal{L}} > 0$, and let $\bar{p} \in L^\infty(\mu)$ denote the dual optimum as in Proposition 14, with corresponding real number τ so that $\mu([p \geq \tau]) \geq \tau$. Define the quantities*

$$c := \frac{16\ell(0)}{\tau\ell'(0)} \max \left\{ 1, \frac{1}{\tau} \right\} \max \{1, \|p\|_\infty\}, \quad B_1 := \frac{\tau}{8} \inf_{z \in [-c, +c]} \ell''(z),$$

$$R_i := \sqrt{i} \sqrt{\frac{\ell(0) \max\{5, 2B_1/B_2\}}{\rho^2 B_1}},$$

and suppose the sample size is large enough to satisfy $m \geq (2/\tau^2) \ln(4/\delta)$ and

$$\frac{2(R_t + 2c)\|\bar{p}\|_\infty}{m^{1/2}} \left(2\sqrt{2\mathcal{V}(\mathcal{H}) \ln(m+1)} + \sqrt{2 \ln(4/\delta)} \right) \leq \frac{c\bar{\tau}^2}{8}$$

(which happens for all large m since $\lim_{m \rightarrow \infty} R_t \sqrt{\ln(m)/m} = \lim_{m \rightarrow \infty} \sqrt{\ln(m)/m^{1-a}} = 0$). Then it holds that the above values τ , c , B_1 , and R_i (for $0 < i \leq t$) are all positive, and moreover the following statements hold simultaneously with probability at least $1 - \delta$.

1. The final coefficient vector $\lambda_t \in \Lambda$ satisfies

$$\begin{aligned} \mathcal{L}(A\lambda_t) &\leq \inf_{\|\lambda\|_1 \leq R_{t-1}} \mathcal{L}(A\bar{\lambda}) + m^{-a/4} + R_{t-1} \sqrt{\frac{2}{m} \ln \left(\frac{6}{\delta} \right)} \\ &\quad + \frac{2\beta_2 R_t}{m^{1/2}} \left(2\sqrt{2\mathcal{V}(\mathcal{H}) \ln(m+1)} + \ell(R_t) \sqrt{2 \ln(6/\delta)} \right) \\ &\quad + \ell(0) \left(\frac{\|\bar{\lambda}\|_1}{\|\bar{\lambda}\|_1 + \rho m^{a/4}/(4B_2 R_1)} \right)^{9B_2/(B_1 \rho^3)}. \end{aligned}$$

2. If $\mathcal{H} \in \mathcal{F}_{\text{ds}}(\mu^{\mathcal{X}})$ (where $\mu^{\mathcal{X}}$ is the marginal of μ over \mathcal{X}), and letting \mathcal{R}_\star denote the Bayes error rate, there exists $\psi : \mathbb{R} \rightarrow \mathbb{R}$ satisfying $\psi(\mathcal{R}(A\lambda_t) - \mathcal{R}_\star) \leq \mathcal{L}(A\lambda_t) - \bar{\mathcal{L}}$ and $\psi(z) \rightarrow 0$ as $z \rightarrow 0$. (For instance, when $\ell = \ell_{\log}$, then $\psi(z) = z^2/2$.)

3. The returned coefficients $\hat{\lambda}$ satisfy

$$\begin{aligned} \mathcal{R}(H\hat{\lambda}) &\leq \mathcal{R}(H\lambda_t) + 4\sqrt{\widehat{\mathcal{R}}(H\lambda_t) \frac{(\mathcal{V}(\mathcal{H}) + 1) \ln(2em) + \ln(24/\delta)}{m^{1-a}}} \\ &\quad + 8 \frac{(\mathcal{V}(\mathcal{H}) + 1) \ln(2em) + \ln(24/\delta)}{m^{1-a}}. \end{aligned}$$

This bound is inferior to the guarantee in the separable case; while it is still of the form $1/m^{1/c}$, the exponent $1/c$ is distribution-dependent. The source of weakness is the optimization guarantee (cf. Lemma 18), which is brute-forced and should be improvable.

5.1. Curvature

Recall that the dual optimum \bar{p} satisfies $\|A^\top \bar{p}\|_\infty = 0$, which implies $\int (A\lambda) \bar{p} d\mu = 0$ for every $\lambda \in \Lambda$ (cf. Lemma 22). To see how this helps locate bad examples and produce curvature, note the rearrangement

$$-\int_{[A\lambda < 0]} (A\lambda) \bar{p} d\nu = \int_{[A\lambda \geq 0]} (A\lambda) \bar{p} d\nu,$$

meaning ν has been reweighted by \bar{p} so that negative and positive margins are equal (in a sense, \bar{p} renders every $\lambda \in \Lambda$ equivalent to random guessing). Since \bar{p} is fairly well-behaved (it is within $[0, \beta_2]$ μ -a.e. (where β_2 is the Lipschitz constant for ℓ), and is fairly flat since $\mu([\bar{p} \geq \tau]) \geq \tau$), then some algebra allows the removal of \bar{p} from the above display, which yields the statement: if $A\lambda$ has many good margins, it also has many bad margins. This constrains the norms of solutions found by the algorithm, and generates curvature in the sense that progress in any direction quickly leads to \mathcal{L} increasing.

Of course, \bar{p} could have instead been directly constructed from the presence of noise, but then the results would not be applicable to cases where ν itself is noiseless, but \mathcal{H} is simply very weak. The following example emphasizes this role of noise, but also shows that the above development overlooked the effect of sampling.

Example 2 (Nightmare scenario #2) *Pick any \mathcal{X} , (marginal) distribution $\mu^\mathcal{X}$ over \mathcal{X} , hypothesis class $\mathcal{H} \in \mathcal{F}_{\text{ds}}(\nu^\mathcal{X})$, and any $\bar{\lambda} \in \Lambda$. Define the conditional density $\Pr[y = +1|x]$ to be 0.9 when $(H\bar{\lambda})_x \geq 0$, and 0.1 otherwise when $(H\bar{\lambda})_x < 0$. By this construction, $H\bar{\lambda}$ attains the Bayes error rate (which is 0.1), and every other $H\lambda$ does at best this well. Any weighting λ with favorable convex risk $\mathcal{L}(A\lambda)$ will necessarily have a small norm in consequence of the guaranteed 10% classification error.*

Unfortunately, finite samples look slightly different. Suppose $\mathcal{X} = \mathbb{R}^d$ and $\mu^\mathcal{X}$ is absolutely continuous with respect to Lebesgue measure. With probability 1, a random sample of any size will contain no noise, and $\text{span}(\mathcal{H})$ has a perfect predictor $H\tilde{\lambda}$ (over the sample); in particular, nothing inhibits the norms of solutions over $\tilde{\mathcal{L}}$.

In this example, the good predictor $H\tilde{\lambda}$ is potentially very complex, as it is fitting noise. The solution here will be to only control those predictors with small norms; note that this deviation inequality embeds the reweighted average margin expression from eq. (1).

Lemma 16 *Let probability measure μ over $\mathcal{X} \times \{-1, +1\}$ with empirical counterpart $\hat{\mu}$, any hypothesis class $\mathcal{H} \in \mathcal{F}_{\text{vc}}$, reweighting $p \in L^\infty(\mu)$ with $\|A^\top p\|_\infty = 0$, and norm bound C be given. Then, with probability at least $1 - \delta$, $p \in [0, \|p\|_\infty]$ $\hat{\mu}$ -a.e., and*

$$\sup_{\substack{\lambda \in \Lambda \\ \|\lambda\|_1 \leq C}} \left| \int (A\lambda) p d\hat{\mu} \right| \leq \frac{2C\|p\|_\infty}{m^{1/2}} \left(2\sqrt{2\mathcal{V}(\mathcal{H}) \ln(m+1)} + \sqrt{2\ln(1/\delta)} \right).$$

Armed with these tools, the structure of the nonseparable problem is as follows. Note that the term B_1 is the aforementioned curvature lower bound, and furthermore the facts $\sum_i \alpha_i = \infty$ and $\sum_i \alpha_i^2 < \infty$ mean that the step sizes exactly fit the constrained step size regime studied by Zhang and Yu (2005, Equation (4)).

Lemma 17 *Suppose the setting and quantities in the preamble of Theorem 15; the following statements hold simultaneously with probability at least $1 - \delta/2$.*

1. *Every $\lambda \in \Lambda$ with $\|\lambda\|_1 \leq R_t + 4c$ and $\widehat{\mathcal{L}}(A\lambda) < 2\ell(0)$ has $m^{-1} \sum_{i=1}^m \ell''((A\lambda)_{x_i, y_i}) \geq B_1$.*
2. *For every choice of step size, $\|\lambda_i\|_1 \leq R_i$ and*

$$\alpha_i^2 \leq \min \left\{ \frac{9\|A^\top \nabla \widehat{\mathcal{L}}(A\lambda_{i-1})\|_\infty^2}{4\rho^2 B_1^2}, \frac{\max\{5, 2B_2/B_1\}(\widehat{\mathcal{L}}(A\lambda_{i-1}) - \widehat{\mathcal{L}}(A\lambda_i))}{\rho^2 B_1} \right\}.$$

3. *Let $\bar{\lambda} \in \Lambda$ with $\|\bar{\lambda}\|_1 \leq R_1\sqrt{t-1} = R_{t-1}$ and $\epsilon := \min_{i \in [t-1]} \widehat{\mathcal{L}}(A\lambda_i) - \widehat{\mathcal{L}}(A\bar{\lambda}) \geq 0$ be arbitrary. For every choice of step size,*

$$\alpha_i \geq \frac{\rho(\widehat{\mathcal{L}}(A\lambda_{i-1}) - \widehat{\mathcal{L}}(A\bar{\lambda}))}{2B_2(\|\bar{\lambda}\|_1 + R_1\sqrt{i-1})} \quad \text{and} \quad \sum_{i=1}^t \alpha_i \geq \frac{\rho\epsilon\sqrt{t-1}}{4B_2R_1}.$$

5.2. Proof of Theorem 15

The convergence analysis due to [Zhang and Yu \(2005\)](#) can be adjusted to the present setting (where step and coordinate selection are decoupled), yielding the following guarantee. Note that Lemma 17 also allows the application of the analysis due to [Bartlett and Traskin \(2007\)](#) (again with decoupling modifications), however this leads to a rate of roughly $\mathcal{O}(1/\sqrt{\ln(m)})$.

Lemma 18 *Let $\ell \in \mathbb{L}_{\text{lg}} \cap \mathbb{L}_{2\text{d}}$ with Lipschitz gradient parameter B_2 , binary class \mathcal{H} , time horizon t , and empirical probability measure $\hat{\mu}$ be given. Let $\bar{\lambda} \in \Lambda$ be arbitrary, and suppose there exists $c_3 > 0$ with $c_3\alpha_i \leq \|A^\top \nabla \widehat{\mathcal{L}}(A\lambda_{i-1})\|_\infty$ for all $0 \leq i \leq t$. Then*

$$\widehat{\mathcal{L}}(A\lambda_t) - \widehat{\mathcal{L}}(A\bar{\lambda}) \leq \left(\widehat{\mathcal{L}}(A\lambda_0) - \widehat{\mathcal{L}}(A\bar{\lambda}) \right) \left(\frac{\|\bar{\lambda}\|_1}{\|\bar{\lambda}\|_1 + \sum_{i \leq t} \alpha_i} \right)^{6B_2/(c_3\rho^2)}.$$

From here, there is little to do: the conditions for this rate are met with high probability thanks to Lemma 17, and the rest is standard uniform convergence.

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Appendix A. Spaces and Linear Operators

As stated in Section 2, H and A are mappings which produce bounded functions; the bulk of the analysis, however, considers them as producing functions over $L^1(\nu^{\mathcal{X}})$ and $L^1(\nu)$ as follows (where ν is a probability distribution over $\mathcal{X} \times \{-1, +1\}$).

Lemma 19 *Let ν be a probability measure over $\mathcal{X} \times \{-1, +1\}$, and let $\nu^{\mathcal{X}}$ denote the marginal distribution over \mathcal{X} .*

1. *The definition of H and A is valid for arbitrary weightings $\lambda \in \Lambda$; in particular, $\text{supp}(\lambda)$ is countable, and*

$$(H\lambda)_x = \int h(x)\lambda(h)d\rho(h) = \sum_{h \in \text{supp}(\lambda)} (H\lambda)_x \lambda(h),$$

$$(A\lambda)_{x,y} = -y \int (H\lambda)_x \lambda(h)d\rho(h) = -y \sum_{h \in \text{supp}(\lambda)} (H\lambda)_x \lambda(h),$$

2. *H and A are linear operators.*

3. *$H : \Lambda \rightarrow L^1(\nu^{\mathcal{X}})$ and $A : \Lambda \rightarrow L^1(\nu)$ are continuous linear operators (with unit norm).*

Proof If $\lambda \in \Lambda$, then $\int |\lambda(h)|d\rho(h) < \infty$, and since ρ is a counting measure, it follows that $\text{supp}(\lambda)$ is countable. Furthermore, for any x, y, h , $|h(x)| \leq |-yh(x)| \leq 1$, and thus the rescalings $h(x)\lambda$ and $-yh(x)\lambda$ are both in Λ , and in particular

$$(H\lambda)_x = \int h(x)\lambda(h)d\rho(h) = \sum_{h \in \text{supp}(\lambda)} (H\lambda)_x \lambda(h),$$

and similarly for A .

It follows by definition (and another check for integrability) that $(H(a\lambda_1 + b\lambda_2))_x = a(H\lambda_1)_x + b(H\lambda_2)_x$, and thus H is a linear operator; the proof for A is the same.

Lastly, H is continuous with unit norm, since boundedness of each h combined with $\nu^{\mathcal{X}}$ being a probability measure gives

$$\begin{aligned} \sup \{ \|H\lambda\| : \lambda \in \Lambda, \|\lambda\|_1 \leq 1 \} &= \sup \left\{ \int (H\lambda)d\nu^{\mathcal{X}} : \lambda \in \Lambda, \|\lambda\|_1 \leq 1 \right\} \\ &\leq \sup \left\{ \int \sum_{h \in \text{supp}(\lambda)} h(x)\lambda(h)d\nu^{\mathcal{X}}(x) : \lambda \in \Lambda, \|\lambda\|_1 \leq 1 \right\} \\ &\leq \sup \left\{ \sup_{x,h} |h(x)| \left| \sum_{h \in \text{supp}(\lambda)} \lambda(h) \right| : \lambda \in \Lambda, \|\lambda\|_1 \leq 1 \right\} \\ &= 1. \end{aligned}$$

(The proof for A is the same, since $y \in \{-1, +1\}$ implies $|yh(x)| = |h(x)|$.) ■

Note, H and A may also be defined as Bochner (or similar) integrals.

Next, to develop the adjoint of A^\top , relevant dual spaces need to be established (the adjoint of H^\top does not appear, but is similar).

Lemma 20 *If ν is a probability measure over $\mathcal{X} \times \{-1, +1\}$, then $L^1(\nu)^*$ is isometrically isomorphic to $L^\infty(\nu)$, and in particular for every $Q \in L^1(\nu)^*$ there exists $q \in L^\infty(\nu)$ so that $Q(f) = \int qf d\nu$ for every $f \in L^1(\nu)$. Similarly, recalling $\Lambda = L^1(\rho)$ where ρ is counting measure over some class \mathcal{H} , the dual $L^1(\rho)^*$ is isometrically isomorphic to $L^\infty(\rho)$, and once again elements of $L^1(\rho)^*$ can be written as integrals over ρ with an element of $L^\infty(\rho)$.*

Proof The first relationship follows since ν is a probability measure and thus σ -finite (Folland, 1999, Theorem 6.15), and the second is a general property of counting measures (even though the cardinality of \mathcal{H} may preclude ρ from being σ -finite) (Folland, 1999, Exercises 3.15 and 6.25). \blacksquare

Remark 21 *This manuscript always identifies the above dual spaces by the provided isometric isomorphism, a fact which will be crucial in the convex duality theory of \mathcal{L} (cf. Lemma 34).*

Lastly, the adjoint A^\top has the following structure.

Lemma 22 *Let probability measure ν over $\mathcal{X} \times \{-1, +1\}$ and any \mathcal{H} be given.*

1. *Considering A as a linear operator from Λ to $L^1(\nu)$, its adjoint $A^\top : \Lambda^* \rightarrow L^1(\nu)^*$ is the unique continuous linear operator satisfying $(A^\top p)(\lambda) = \int (A\lambda)p$, where $p \in L^\infty(\nu)$ and $\lambda \in \Lambda$ (and dual spaces have been identified via isomorphism as in Lemma 20).*
2. *Again identifying $A^\top p$ for $p \in L^\infty(\nu)$ with an element of $L^\infty(\rho)$,*

$$\begin{aligned} \|A^\top p\|_\infty &= \sup \left\{ \left| \int y h(x) p(x, y) d\nu(x, y) \right| : h \in \mathcal{H} \right\} \\ &= \sup \left\{ \int (A\lambda)_{x,y} p(x, y) d\nu(x, y) : \lambda \in \Lambda, \|\lambda\|_1 \leq 1 \right\}. \end{aligned}$$

3. *The map $p \mapsto \|A^\top p\|_\infty$ is a convex function over $L^\infty(\nu)$, and is lower semi-continuous in the weak* topology (i.e., the weak topology induced on $L^\infty(\nu)$ by $L^1(\nu)$).*

Proof

1. Recall by Lemma 19 that A is a continuous linear operator; the basic properties of A^\top follow by properties of adjoints of continuous linear operators (Rudin, 1973, Theorem 4.10) combined with the isometric isomorphism of the relevant dual spaces as provided by Lemma 20.

2. Let $p \in L^\infty(\nu)$ be given. Since the isometric isomorphism provided by Lemma 20 allows $A^\top p$ to be identified with an element of $L^\infty(\rho)$, the operator norm of $A^\top p$ is simply the $L^\infty(\rho)$ norm of the element it has been identified with by the isomorphism. Since ρ is a counting measure, letting $\mathbf{e}_h \in \Lambda$ be an indicator function for a single $h \in \mathcal{H}$ (it is 1 on h and 0 elsewhere), using the above adjoint relation $(A^\top p)(\mathbf{e}_h) = \int (A\mathbf{e}_h)p$, and using the definition of norms on $L^\infty(\rho)$,

$$\begin{aligned} \|A^\top p\|_\infty &= \inf \left\{ a \geq 0 : \rho(\{h \in \mathcal{H} : |(A^\top p)(\mathbf{e}_h)| > a\}) = 0 \right\} \\ &= \inf \left\{ a \geq 0 : \rho(\{h \in \mathcal{H} : \left| \int -yh(x)p(x,y)d\nu(x,y) \right| > a\}) = 0 \right\} \\ &= \inf \left\{ a \geq 0 : \forall h \in \mathcal{H}, \left| \int -yh(x)p(x,y)d\nu(x,y) \right| \leq a \right\} \\ &= \sup_{h \in \mathcal{H}} \left| \int -yh(x)p(x,y)d\nu(x,y) \right|, \end{aligned}$$

where the last equality can be established by noting the domain of the infimum includes all $a \geq 0$ satisfying $a \geq \sup_{h \in \mathcal{H}} \left| \int -yh(x)p(x,y)d\nu(x,y) \right|$, but no values satisfying $a < \sup_{h \in \mathcal{H}} \left| \int -yh(x)p(x,y)d\nu(x,y) \right|$.

Next, to show

$$\begin{aligned} &\sup \left\{ \left| \int yh(x)p(x,y)d\nu(x,y) \right| : h \in \mathcal{H} \right\} \\ &= \sup \left\{ \int (A\lambda)_{x,y}p(x,y)d\nu(x,y) : \lambda \in \Lambda, \|\lambda\|_1 \leq 1 \right\}, \end{aligned}$$

one direction is immediate, since positive and negative copies $\pm \mathbf{e}_h$ of the indicator elements satisfy $\pm \mathbf{e}_h \in \Lambda$ and $\|\pm \mathbf{e}_h\|_1 = 1$. For the other direction, let $\tau > 0$ be arbitrary, and choose any $\lambda_\tau \in \Lambda$ which is within τ of the supremum on the right side of the display. Then, since $|\text{supp}(\lambda)|$ is countable (via Lemma 19), and since $\|\lambda_\tau\|_1 \leq 1$ implies $\|A\lambda_\tau\|_1 \leq 1$, the dominated convergence theorem (Folland, 1999, Theorem 2.25 (summation form)) may be applied (with dominating function 1), and

$$\begin{aligned} &\sup \left\{ \int (A\lambda)_{x,y}p(x,y)d\nu(x,y) : \lambda \in \Lambda, \|\lambda\|_1 \leq 1 \right\} \\ &\leq \tau + \int (A\lambda_\tau)_{x,y}p(x,y)d\nu(x,y) \\ &= \tau + \int \sum_{h \in \text{supp}(\lambda_\tau)} -yh(x)\lambda_\tau(h)p(x,y)d\nu(x,y) \\ &= \tau + \sum_{h \in \text{supp}(\lambda_\tau)} \lambda_\tau(h) \int -yh(x)p(x,y)d\nu(x,y) \\ &\leq \tau + \|\lambda_\tau\|_1 \sup \left\{ \left| \int yh(x)p(x,y)d\nu(x,y) \right| : h \in \mathcal{H} \right\}; \end{aligned}$$

since $\|\lambda_\tau\|_1 \leq 1$, and since $\tau > 0$ was arbitrary, the result follows.

3. For the last part, define a convex indicator over $L^1(\nu)$ as

$$\iota(f) = \begin{cases} 0 & \text{when } f \in \{A\lambda : \lambda \in \Lambda, \|\lambda\|_1 \leq 1\}, \\ \infty & \text{otherwise.} \end{cases}$$

(Note that ι is not necessarily lower semi-continuous over $L^1(\nu)$, since as discussed shortly in Lemma 23, the subspace $A\Lambda$ might not be closed.) The conjugate of ι is, for any $p \in L^\infty(\nu)$,

$$\begin{aligned} \iota^*(p) &= \sup \left\{ \int fp : \exists \lambda \in \Lambda, \|\lambda\|_1 \leq 1 \text{ s.t. } f = A\lambda \right\} \\ &= \sup \left\{ \int (A\lambda)p : \lambda \in \Lambda, \|\lambda\|_1 \leq 1 \right\} \\ &= \|A^\top p\|_\infty, \end{aligned}$$

where the last step used the earlier equalities for A^\top . Since $p \mapsto \|A^\top p\|_\infty$ is the conjugate of a convex function, it is lower semi-continuous in the weak* topology (Zălinescu, 2002, Theorem 2.3.1(i)).

■

Lastly, note the following properties of the sets $H\Lambda$ and $A\Lambda$.

Lemma 23 *Let any \mathcal{H} and any probability measure ν over $\mathcal{X} \times \{-1, +1\}$ with marginal $\nu^\mathcal{X}$ over \mathcal{X} be given. Then $H\Lambda$ and $A\Lambda$ are subspaces, but it is possible that neither is closed in its respective $L^1(\nu^\mathcal{X})$ and $L^1(\nu)$ topology (indeed, Example 1 provides the counterexample).*

Proof Since Λ is a Banach space and H and A are linear operators, it follows that $H\Lambda$ and $A\Lambda$ are subspaces.

For the lack of closure, consider the setting of Example 1, and in particular building a sequence of functions $\{f_k\}_{k=1}^\infty$ which are a combination of k thresholds, and predict correctly on the last k intervals. This sequence has a limit point in $L^1(\mu^\mathcal{X})$ (in particular, it is a countable sum of indicators over intervals), but no such function is in $H\Lambda$, which is therefore not closed in $L^1(\nu^\mathcal{X})$. To obtain a similar result for $A\Lambda$, define $g_k(x, y) := f_k(x)$. ■

Appendix B. The Family of Dense Classes $\mathcal{F}_{\text{ds}}(\nu)$

As the goal of a consistency analysis is to show that the Bayes predictor is approximated arbitrarily finely, necessarily the function class considered by a purportedly consistent algorithm must be very large.

As discussed in Section 2, one choice is the class $\mathcal{F}_{\text{ds}}(\nu)$ of functions dense according to $L^1(\nu)$ in the family of bounded measurable functions. A partial survey of density assumptions in other work is as follows.

- Breiman (2000, Definition 1) works with a similar definition: the relevant metric is $L^2(\nu)$, and the closure must contain $L^2(\nu)$, where ν is constrained to be continuous with respect to Lebesgue measure. By contrast, the metric for $\mathcal{F}_{\text{ds}}(\nu)$ is $L^1(\nu)$, where ν is an arbitrary measure over the Borel σ -algebra, and the closure of the class must contain bounded measurable functions, which are a subspace of $L^\infty(\nu)$, which is contained within $L^2(\nu)$.

Proposition 3, which will be proved shortly, states that it suffices for $\text{span}(\mathcal{H})$ to contain boxes formed by half-open intervals. This result was stated by Breiman (1999, Proposition 1) with an abbreviated proof for his setting of Lebesgue-continuous measures, thus the present result can be taken as merely proving that result with slightly more generality and verbosity.

- The closest assumption and family of results to those here were provided by Zhang (2004, Section 4); while an analog to Proposition 3 is not shown there, the proofs rely on a form of Lusin's Theorem, which is used in Proposition 3 as well; indeed, the proofs here owe their existence to those earlier ones by Zhang (2004, Section 4).
- Another approach, suggested by Lugosi and Vayatis (2004, Theorem 1 and subsequent remarks), and later used by Bartlett and Traskin (2007, Condition 1) and Schapire and Freund (2012, eq. (12.11)), is to require the weaker condition that

$$\inf\{\mathcal{L}(A\lambda) : \lambda \in \Lambda\} = \inf\left\{\int \ell(-yf(x))d\nu(x, y) : f \text{ measurable from } \mathcal{X} \text{ to } \mathbb{R}\right\};$$

for a verification that this property is indeed weaker, see Lemma 24. Lugosi and Vayatis (2004, Lemma 1) show that this assumption is satisfied by classes whose convex hull contains indicators of all Borel sets, and thus Lemma 24 can be considered a simplification which suffices to grant consistency with more computationally tractable classes (like decision lists and trees).

As discussed above, the essential property of $\mathcal{F}_{\text{ds}}(\nu)$ is that it implies the weaker condition used by Lugosi and Vayatis (2004, Theorem 1 and subsequent remarks), which in turn is directly needed for the classification calibration methods in the consistency proof (cf. Theorem 5). The Lipschitz condition here is not crucial, and for instance can be removed by adjusting $\mathcal{F}_{\text{ds}}(\nu)$ to require approximants to a function to carry nearly the same uniform bound.

Lemma 24 *Let distribution ν over $\mathcal{X} \times \{-1, +1\}$, class $\mathcal{H} \in \mathcal{F}_{\text{ds}}(\nu^{\mathcal{X}})$, and nonnegative Lipschitz convex loss ℓ be given (with Lipschitz constant β_2). Then*

$$\inf\{\mathcal{L}(A\lambda) : \lambda \in \Lambda\} = \inf\left\{\int \ell(-yf(x))d\nu(x, y) : f \text{ measurable from } \mathcal{X} \text{ to } \mathbb{R}\right\}.$$

Proof One direction is immediate, since $H\lambda$ defines a family of measurable functions.

Going the other direction, first define, for any measurable f , a clamping

$$[f]_r(z) := \begin{cases} f(z) & \text{when } f(z) \leq r, \\ r & \text{otherwise} \end{cases}.$$

For any $\epsilon > 0$, based on four cases for the structure of ℓ , a clamping value r_ϵ is defined as follows in order to satisfy, for any f and z , $\ell([f]_{r_\epsilon}(z)) \leq \ell(f(z)) + \epsilon$.

- If $\lim_{z \rightarrow -\infty} \ell(z) = \lim_{z \rightarrow +\infty} \ell(z) < \infty$, then ℓ is a constant function, and $r_\epsilon = 0$ suffices.
- If $\lim_{z \rightarrow -\infty} \ell(z) = \lim_{z \rightarrow +\infty} \ell(z) = \infty$, then ℓ has compact level sets, and in particular an r_ϵ exists so that

$$\{z : \ell(z) \leq \inf_q \ell(q) + \epsilon\} \subseteq \{z : |z| \leq r_\epsilon\}.$$

It follows that $\ell([f]_{r_\epsilon}(z)) \leq \ell(f(z))$.

- If $\lim_{z \rightarrow -\infty} \ell(z) < \infty$ and $\lim_{z \rightarrow +\infty} \ell(z) = \infty$, then set

$$r_\epsilon := \inf\{|z| : \ell(z) \leq \inf_q \ell(q) + \epsilon\}.$$

Unlike the preceding two cases, clamping here can increase the value, but not by more than ϵ .

- If $\lim_{z \rightarrow -\infty} \ell(z) = \infty$ and $\lim_{z \rightarrow +\infty} \ell(z) < \infty$, then this case is handled by the preceding one by considering the reflection $z \mapsto \ell(-z)$.

Consequently, let $\{f_i\}_{i=1}^\infty$ be a minimizing sequence for the target infimum above so that

$$\int \ell(-yf_i(x))d\nu(x, y) \leq 2^{-i} + \inf\left\{\int \ell(-yf(x))d\nu(x, y) : f \text{ measurable from } \mathcal{X} \text{ to } \mathbb{R}\right\}.$$

Each f_i might not be bounded, so define $g_i := [f_i]_{r_{\epsilon_i}}$ where $\epsilon_i := 2^{-i}$; by this choice,

$$\begin{aligned} \int \ell(-yg_i(x))d\nu(x, y) &= \int \ell(-y[f_i]_{r_{\epsilon_i}}(x))d\nu(x, y) \\ &\leq 2^{-i} + \int \ell(-yf_i(x))d\nu(x, y) \\ &\leq 2^{-i+1} + \inf\left\{\int \ell(-yf(x))d\nu(x, y) : f \text{ measurable from } \mathcal{X} \text{ to } \mathbb{R}\right\}. \end{aligned}$$

Lastly, since $\text{span}(\mathcal{H})$ is dense in the $L^1(\nu^\mathcal{X})$ metric, let $h_i \in \text{span}(\mathcal{H})$ satisfy $\|h_i - g_i\|_1 \leq 2^{-i}$; since ℓ is Lipschitz with constant β_2 , then

$$\begin{aligned} \int \ell(-yh_i(x))d\nu(x, y) &\leq \int \ell(-yh_i(x))d\nu(x, y) + \int (\ell(-yg_i(x)) - \ell(-yh_i(x)))d\nu(x, y) \\ &\leq \int \ell(-yh_i(x))d\nu(x, y) + \int \beta_2 |-yg_i(x) + yh_i(x)|d\nu(x, y) \\ &\leq \int \ell(-yh_i(x))d\nu(x, y) + \beta_2 \|g_i - h_i\|_1 \\ &\leq (2 + \beta_2)2^{-i} + \inf\left\{\int \ell(-yf(x))d\nu(x, y) : f \text{ measurable from } \mathcal{X} \text{ to } \mathbb{R}\right\}, \end{aligned}$$

and the result follows. ■

To close, the proof of Proposition 3, which avoids strong structural assumptions on the measure (for instance, a relationship to Lebesgue measure) via an invocation of Lusin's Theorem.

Proof (of Proposition 3) Let $\epsilon > 0$ and bounded measurable g with $\|g\|_u := \sup_x |g(x)| \in (0, \infty)$ (when $\|g\|_u = 0$, then $g \in \text{span}(\mathcal{H})$ and the proof is complete). By Lusin's Theorem, there exists compactly-support continuous $h \in L^1(\nu)$ which satisfies $\nu([h \neq g]) \leq \epsilon/(4\|g\|_u)$, and $\|h\|_u \leq \|g\|_u$ (Folland, 1999, Theorem 7.10). Let C denote the compact support of h ; continuity over a compact subset of \mathbb{R}^d means uniform continuity, and therefore let $\tau > 0$ be sufficiently small that the bounding box of C may be partitioned into finitely many cubes of side length τ (products of half-open intervals of length τ) so that, for any x_1 and x_2 within a single cube, $|h(x_1) - h(x_2)| \leq \epsilon/2$. Now let f be a sum of indicators of these cubes, where each indicator is weighted by $h(x)$ with x being an arbitrary point in the corresponding cube. By construction and since $\text{span}(\mathcal{H})$ contains such cubes, $f \in \text{span}(\mathcal{H})$, and moreover $\|f - h\|_1 \leq \epsilon/2$ since ν is a probability measure, which provides

$$\|f - g\|_1 \leq \|f - h\|_1 + \|h - g\|_1 \leq \epsilon/2 + \int_{[h \neq g]} |h - g| d\nu \leq \epsilon/2 + 2\|g\|_u \nu([h \neq g]) \leq \epsilon.$$
■

Appendix C. Loss Function Classes \mathbb{L}_{lg} and $\mathbb{L}_{2\text{d}}$

First, note that \mathbb{L}_{lg} and $\mathbb{L}_{2\text{d}}$ contain a few useful things.

Lemma 25 $\ell_{\text{log}} \in \mathbb{L}_{\text{lg}}$ with parameters $B_2 = 1/4$, $\beta_1 = 1/2$, $\beta_2 = 1$. $\ell_{\text{russ}} \in \mathbb{L}_{\text{lg}}$ with parameters $B_2 = \beta_1 = \beta_2 = 1$. Lastly, $\exp \in \mathbb{L}_{2\text{d}}$ and $\ell_{\text{log}} \in \mathbb{L}_{2\text{d}}$.

Proof For the logistic loss ℓ_{log} , note $0 \leq \sup_x \ell''_{\text{log}}(x) \leq 1/4$, thus the mean value theorem grants Lipschitz gradients with parameter $B_2 \leq 1/4$. ℓ_{log} 's Lipschitz parameters are $\beta_1 = 1/2$ and $\beta_2 = 1$.

Since ℓ_{russ} is not twice differentiable, gradient slopes must be checked manually. To start, note

$$\ell'_{\text{russ}}(x) = \begin{cases} 0 & \text{when } x \leq -1, \\ x + 1 & \text{when } x \in (-1, 0), \\ 1 & \text{when } x \geq 0, \end{cases}$$

whereby $\beta_1 = \beta_2 = 1$. Within each line segment, the gradient slopes are 0, 1, and 0. By manually checking pairs $x < y$ in the first and second, first and third, and second and third intervals, the tightest Lipschitz constant on the gradients is 1.

The containments within $\mathbb{L}_{2\text{d}}$ are direct. ■

The next two results establish the value of Lipschitz gradients: the standard Taylor expansion inequality used in conjunction with twice differentiability is still valid.

Lemma 26 *Let $\ell \in \mathbb{L}_{\text{lg}}$ with Lipschitz gradient parameter B_2 be given. Then, for any $x, y \in \mathbb{R}$,*

$$\ell(y) \leq \ell(x) + \ell'(x)(y - x) + \frac{B_2}{2}(x - y)^2.$$

Proof Suppose $x \leq y$; by the mean value theorem and the definition of B_2 ,

$$\begin{aligned} \ell(y) &= \ell(x) + \int_x^y \ell'(t) dt \\ &= \ell(x) + \int_x^y \left(\ell'(x) + \frac{\ell'(t) - \ell'(x)}{t - x}(t - x) \right) dt \\ &\leq \ell(x) + \ell'(x)(y - x) + B_2 \left(\frac{t^2}{2} - xt \right) \Big|_x^y \\ &\leq \ell(x) + \ell'(x)(y - x) + \frac{B_2}{2}(x - y)^2. \end{aligned}$$

Almost identically, when $x > y$,

$$\begin{aligned} \ell(y) &= \ell(x) + \int_x^y \ell'(t) dt \\ &= \ell(x) + \int_x^y \ell'(x) dt + \int_y^x \frac{\ell'(t) - \ell'(x)}{t - x}(x - t) dt \\ &\leq \ell(x) + \ell'(x)(y - x) + \frac{B_2}{2}(x - y)^2. \end{aligned}$$

■

Corollary 27 *Let $\ell \in \mathbb{L}_{\text{lg}}$ with Lipschitz gradient parameter B_2 be given. Then, for any $x, y \in \mathbb{R}^m$,*

$$\frac{1}{m} \sum_i \ell(y_i) \leq \frac{1}{m} \sum_i \ell(x_i) + \frac{1}{m} \sum_i \ell'(x_i)(y_i - x_i) + \frac{B_2}{2m} \sum_i (x_i - y_i)^2.$$

Proof It suffices to apply Lemma 26 m times. ■

Lastly, the following convexity properties of losses will be useful. Note that the non-negativity of ℓ^* is the reason losses were chosen to be increasing functions (much of the literature uses decreasing functions); this makes the dual space more readily interpretable as a space of reweightings.

Lemma 28 *Suppose $\ell : \mathbb{R} \rightarrow \mathbb{R}_+$ is convex with $\lim_{z \rightarrow -\infty} \ell(z) = 0$.*

1. ℓ is lower semi-continuous, whereby ℓ^* is convex lower semi-continuous, and $\ell = \ell^{**}$.
2. $\ell^*(\phi) = \infty$ for $\phi < 0$, and $\ell^*(0) = 0$.

3. Let $\beta := \sup_{x \neq y} |\ell(x) - \ell(y)|/|x - y|$ denote the tightest Lipschitz constant for ℓ . If $\beta < \infty$, then $\ell^*(\phi) = \infty$ when $\phi > \beta$, and $\ell^*(\phi) < \infty$ when $\phi \in [0, \beta]$.

4. If $g \in \partial\ell(0)$ is any subgradient of ℓ at the origin and $\ell(0) > 0$, then $\ell^*(\phi) < 0$ for $\phi \in (0, g)$, and ℓ^* attains its minimum value at g .

Proof Since ℓ is finite everywhere, it is continuous (thus lower semi-continuous), and thus $\ell = \ell^{**}$ and ℓ^* is convex lower semi-continuous (Rockafellar, 1970, Theorem 12.2).

For any $x \in \mathbb{R}$ and subgradient $g_x \in \partial\ell(x)$, $\ell(0) \geq \ell(x) + g_x(0 - x)$. Since $\lim_{z \rightarrow -\infty} \ell(z) = 0$ and ℓ is convex, it follows that ℓ is nondecreasing, meaning $g \geq 0$, and thus, for any $\phi < 0$,

$$\ell^*(\phi) = \sup_{x \in \mathbb{R}} \phi x - \ell(x) \geq \sup_{x \in \mathbb{R}} \phi x - \ell(0) - g_x x \geq \sup_{x < 0} (\phi - g_x)x - \ell(0) = \infty.$$

Additionally, since $\inf_x \ell(x) = 0$,

$$\phi^*(0) = \sup_x 0 \cdot x - \ell(x) = -\inf_x \ell(x) = 0.$$

Next, suppose ℓ has tightest Lipschitz parameter β , whereby the any subgradient g_x at a point x satisfies $|g_x| \leq \beta$. Consequently, proceeding just as in the study of the case $\phi < 0$, for any $\phi > \beta$,

$$\ell^*(\phi) \geq \sup_{x \in \mathbb{R}} \phi x - \ell(0) - g_x x \geq \sup_{x > 0} (\phi - \beta)x - \ell(0) = \infty.$$

On the other hand, let $\beta' \in (0, \beta)$ be arbitrary, whereby there must exist $x > y$ with

$$r < \frac{\ell(x) - \ell(y)}{x - y}$$

(where the absolute values were dropped since $x > y$ and ℓ is nondecreasing). Taking any $h \in \partial\ell(x)$, note

$$r < \frac{\ell(x) - \ell(y)}{x - y} \leq \frac{\ell(x) - (\ell(x) + h(y - x))}{x - y} = h.$$

Consequently, by the Fenchel-Young inequality,

$$\ell^*(h) = hx - \ell(x) < \infty.$$

Since ℓ^* is convex, it is finite over a convex set. Since r was arbitrary, it follows that ℓ^* is finite over $[0, \beta)$. Since ℓ^* is lower semi-continuous, it must also hold that $\ell^*(\beta) < \infty$.

For the final property, let $g \in \partial\ell(0)$ be given; by the Fenchel-Young inequality and $\ell(0) > 0$,

$$\ell^*(g) = 0 \cdot g - \ell(0) < 0.$$

Since $\ell^*(0) = 0$ and ℓ^* is closed and convex, the first part follows. For the second part, since ℓ is closed and convex, $g \in \partial\ell(0)$ implies $0 \in \partial\ell^*(g)$ (Rockafellar, 1970, Theorem 23.5), which is precisely the first order optimality condition (Borwein and Lewis, 2000, Proposition 3.1.5). ■

As a final basic result about ℓ , note that the terminology “separable” is at least somewhat justified.

Proposition 29 *Suppose $\ell : \mathbb{R} \rightarrow \mathbb{R}_{++}$ is convex with $\lim_{z \rightarrow -\infty} \ell(z) = 0$, and let any \mathcal{H} and any probability measure ν over $\mathcal{X} \times \{-1, +1\}$ be given. Suppose $\inf_{\lambda} \int \ell(A\lambda) d\nu = 0$.*

1. *For any $\epsilon > 0$, there exists $\lambda_{\epsilon} \in \Lambda$ so that $\nu([A\lambda_{\epsilon} \leq -1]) \geq 1 - \epsilon$.*
2. *With probability 1 over the draw of a sample $\{(x_i, y_i)\}_{i=1}^m$ (for any $m < \infty$), there exists $\lambda \in \Lambda$ so that $(A\lambda)_{x_i, y_i} \leq -1$ for every i .*
3. *In general, there does not exist $\lambda \in \Lambda$ so that $\nu([A\lambda \leq 0]) = 1$ (indeed, Example 1 provides a counterexample).*

Proof Let $\epsilon > 0$ be given, and choose $\{\lambda_i\}_{i=1}^{\infty}$ so that $\int \ell(A\lambda) d\nu \leq 1/i$. Since $\ell(A\lambda_i) \rightarrow 0$ ν -a.e., by Egoroff's theorem there exists S with $\nu(S) \geq 1 - \epsilon$ so that $\ell(A\lambda_i) \rightarrow 0$ uniformly on S (Folland, 1999, Theorem 2.33). But since $\ell > 0$ everywhere and $\lim_{z \rightarrow -\infty} \ell(z) = 0$ and ℓ is convex, it must be the case that $(A\lambda_i) \rightarrow -\infty$ uniformly on S , and so there exists i with $A\lambda_i \leq -1$ on S , which gives the first result.

For the second result, take any $\epsilon > 0$, and choose λ_{ϵ} as granted by the first part. Let $\hat{\nu}$ denote the empirical measure over the provided sample; then

$$\Pr[\forall i \cdot (A\lambda_{\epsilon})_{x_i, y_i} \leq -1] = \nu([A\lambda_{\epsilon} \leq -1])^m \geq (1 - \epsilon)^m.$$

Since $\epsilon > 0$ was arbitrary, the second result follows.

For the third result, recall that Example 1 (whose properties are provided in Proposition 10) gave an instance where every element of $\text{span}(\mathcal{H})$ makes some mistakes. ■

Appendix D. Duality Properties of γ_{ϵ}

In order to develop $\gamma_{\epsilon}(\nu)$, the set $\mathcal{D}_{\epsilon}(\nu)$ must first be studied.

Proposition 30 (Basic properties of $\mathcal{D}_{\epsilon}(\nu)$) *Let ν be an arbitrary probability measure over $\mathcal{X} \times \{-1, +1\}$, and let $\epsilon \in [0, 1]$ be arbitrary. The set $\mathcal{D}_{\epsilon}(\nu)$ has the following properties.*

1. $\mathcal{D}_{\epsilon}(\nu)$ is convex.
2. $\mathcal{D}_{\epsilon}(\nu)$ is closed in the $L^1(\nu)$ topology.
3. If $\epsilon > 0$, then $\mathcal{D}_{\epsilon}(\nu)$ is closed in the $L^{\infty}(\nu)$ topology, and also closed in the weak* topology (i.e., the weak topology induced upon $L^{\infty}(\nu)$ by $L^1(\nu)$).
4. $\mathcal{D}_{\epsilon}(\nu)$ is compact in the weak* topology on $L^{\infty}(\nu)$ (as discussed in the preceding point).
5. $\mathcal{D}_{\epsilon}(\nu)$ is not guaranteed to be compact in the $L^1(\nu)$ or $L^{\infty}(\nu)$ topologies; indeed, it is not compact when $\epsilon = 1/2$, $\mathcal{X} = [0, 1]$, the marginal distribution $\nu^{\mathcal{X}}$ is uniform on $[0, 1]$, and the conditional distribution $\Pr(Y = 1|X = x)$ is arbitrary.

Proof

1. For convexity, let any $\alpha \in (0, 1)$ and $p_1, p_2 \in \mathcal{D}_\epsilon(\nu)$ be given, and define sets $N_j := p_j^{-1}((-\infty, 0))$ for $j \in \{1, 2\}$, where necessarily $\nu(N_j) = 0$. The goal is to show $p := \alpha p_1 + (1 - \alpha)p_2 \in \mathcal{D}_\epsilon(\nu)$.

Define $N := N_1 \cup N_2$ (where again $\nu(N) = 0$). First, for any $(x, y) \in N^c$,

$$p(x, y) = \alpha p_1(x, y) + (1 - \alpha)p_2(x, y) \geq \alpha \cdot 0 + (1 - \alpha) \cdot 0 = 0,$$

whereby it follows that $p \geq 0$ ν -a.e.. Second,

$$\|p\|_\infty \leq \alpha \|p_1\|_\infty + (1 - \alpha) \|p_2\|_\infty \leq 1/\epsilon,$$

again using the convention $1/\infty = 0$, whereby $\|p\|_\infty \leq 1/\epsilon$ as desired. Lastly,

$$\|p\|_1 = \int |\alpha p_1 + (1 - \alpha)p_2| = \int_{N^c} (\alpha p_1 + (1 - \alpha)p_2) = \alpha \|p_1\|_1 + (1 - \alpha) \|p_2\|_1 = 1,$$

meaning all conditions are met, and $p \in \mathcal{D}_\epsilon(\nu)$. Since α, p_1 , and p_2 were arbitrary, it follows that $\mathcal{D}_\epsilon(\nu)$ is convex.

2. For closure within $L^1(\nu)$, since $L^1(\nu)$ is a metric space, it is first countable, and thus it suffices to check that any sequence $\{p_j\}_{j=1}^\infty$ with $p_j \in \mathcal{D}_\epsilon(\nu)$ and $p_j \rightarrow p \in L^1(\nu)$ satisfies $p \in \mathcal{D}_\epsilon(\nu)$ (Folland, 1999, Proposition 4.6). Given any such sequence $\{p_j\}_{j=1}^\infty$, choose a subsequence $\{q_i\}_{i=1}^\infty$ so that $q_i \rightarrow p$ ν -a.e. (Folland, 1999, Corollary 2.32).

Let N_p be the (null) set of points for which convergence fails, and additionally, for each i , define $N_i := q_i^{-1}((-\infty, 0))$; lastly, set $N := N_p \cup (\cup_i N_i)$, where again $\nu(N) = 0$. Thus for any $(x, y) \in N^c$,

$$\begin{aligned} p(x, y) &= q_i(x, y) + (p(x, y) - q_i(x, y)) \\ &\geq q_i(x, y) - |p(x, y) - q_i(x, y)| \\ &\geq \liminf_{i \rightarrow \infty} q_i(x, y) - |p(x, y) - q_i(x, y)| \\ &\geq 0, \end{aligned}$$

thus $p \geq 0$ ν -a.e.. Additionally,

$$\|p\|_1 = \int |p| = \int_N p = \int_N p_i + \int_N (p_i - p) = \|p_i\|_1 + \int (p_i - p),$$

whereby

$$|\|p\|_1 - \|p_i\|_1| = \left| \int (p_i - p) \right| \leq \|p_i - p\|_1 \rightarrow 0,$$

and $\|p\|_1 = 1$ as desired.

For the last property, if $\epsilon = 0$, there is nothing to show, thus suppose $\epsilon \in (0, 1]$, set $P_i := q_i^{-1}((1/\epsilon, \infty])$, and $Z := N_p \cup (\cup_i P_i)$, whereby it follows that

$$\nu(Z) = 0, \quad q_i \rightarrow p \text{ over } Z^c, \quad q_i \leq 1/\epsilon \text{ over } Z^c.$$

Then, for any $(x, y) \in Z^c$,

$$p(x, y) = q_i(x, y) + (p(x, y) - q_i(x, y)) \leq \limsup_{i \rightarrow \infty} q_i(x, y) + |p(x, y) - q_i(x, y)| \leq 1/\epsilon,$$

which establishes $\|p\|_\infty \leq 1/\epsilon$, and thus $p \in \mathcal{D}_\epsilon(\nu)$.

3. Note firstly that if $\epsilon = 0$, then $\mathcal{D}_\epsilon(\nu)$ can contain members which are not elements of $L^\infty(\nu)$, and thus discussing this set in the $L^\infty(\nu)$ topology does not make sense. For the remainder of this case, suppose $\epsilon > 0$.

Just as in the case of $L^1(\nu)$, for $L^\infty(\nu)$ it suffices to let a sequence $\{p_i\}_{i=1}^\infty \subseteq \mathcal{D}_\epsilon(\nu)$ be given with $p_i \rightarrow p$ in the $L^\infty(\nu)$ topology, and to show that $p \in \mathcal{D}_\epsilon(\nu)$. Notice however, since ν is a probability measure, that

$$\|p_i - p\|_1 = \int |p_i - p| \leq \int \|p_i - p\|_\infty = \|p_i - p\|_\infty,$$

meaning $p_i \rightarrow p$ in $L^1(\nu)$ as well, which by the preceding case provides that $p \in \mathcal{D}_\epsilon(\nu)$ as desired.

Lastly, since $\mathcal{D}_\epsilon(\nu)$ is convex and additionally closed according to $L^\infty(\nu)$, then it is also weak* closed (Rudin, 1973, Theorem 3.12).

4. Again suppose $\epsilon > 0$, and define

$$B_\epsilon := \{p \in L^\infty(\nu) : \|p\|_\infty \leq 1/\epsilon\}.$$

By Alaoglu's Theorem (Folland, 1999, Theorem 5.18), B_1 is compact in the weak* topology, thus $B_\epsilon = \epsilon^{-1}B_1$ is weak*-compact as well. The result follows since $\mathcal{D}_\epsilon(\nu)$ is a weak*-closed subset of B_ϵ , and closed subsets of compact sets are compact (Folland, 1999, Theorem 4.22).

5. Noncompactness can be understood from the fact that norm balls are in general not compact, but an explicit construction is provided for completeness. Since both $L^1(\nu)$ and $L^\infty(\nu)$ are metric spaces, to prove non-compactness, it suffices to prove $\mathcal{D}_{1/2}(\nu)$ is not totally bounded. In particular, a countably infinite subset of $\mathcal{C} \subset \mathcal{D}_{1/2}(\nu)$ will be constructed satisfying the property $(f, g) \in \mathcal{C} \times \mathcal{C}$ with $f \neq g$ implies $\|f - g\|_1 = 1/2$ and $\|f - g\|_\infty = 2$, which suffices to show that \mathcal{C} (and thus $\mathcal{D}_{1/2}(\nu)$) is not totally bounded (in either metric) for the following reason. Let S be any finite subset of $L^1(\nu)$ or $L^\infty(\nu)$. Since \mathcal{C} and S have respectively infinite and finite cardinalities, there must exist $h \in S$ which is a closest element in S to two distinct functions $f \neq g$ in \mathcal{C} . Let $\|\cdot\|$ denote either norm under consideration, and note that

$$1/2 \leq \|f - g\| \leq \|h - f\| + \|h - g\| \leq 2 \max\{\|h - f\|, \|h - g\|\},$$

which means that one of these two distances is at least $1/4$. Since S was an arbitrary finite set, it follows that there is no finite set of balls of radius $1/8$ which covers \mathcal{C} , and thus \mathcal{C} and $\mathcal{D}_{1/2}(\nu)$ are not totally bounded according to either norm.

The construction is as follows. For every positive integer $i \in \mathbb{Z}_{++}$, define the function

$$f_i(x, y) := 2 \sum_{j=0}^{2^i-1} \mathbf{1} [x \in [(2j)2^{-i-1}, (2j+1)2^{-i-1})].$$

Define $\mathcal{C} := \{f_i : i \in \mathbb{Z}_{++}\}$. By construction, $\mathcal{C} \subset \mathcal{D}_{1/2}(\nu)$ (i.e., $\|f_i\|_1 = 1$ and $\|f_i\|_\infty = 2$), and moreover $i \neq j$ implies f_i and f_j disagree on exactly half of their support, which yields $\|f_i - f_j\|_1 = 1/2$. and $\|f_i - f_j\|_\infty = \|f_i\|_\infty = 2$.

■

With the structure of $\mathcal{D}_\epsilon(\nu)$ established, the basic duality structure of $\gamma_\epsilon(\nu)$ follows. Note that the value of establishing the weak*-compactness of $\mathcal{D}_\epsilon(\nu)$ is to grant an application of Sion's minimax Theorem without making any topological assumptions on \mathcal{H} (or rather, on the subspace $H\Lambda$). Additionally, Lemma 12 in Section 4 is a combination of this result and part of Lemma 22.

Lemma 31 *Let probability measure ν over $\mathcal{X} \times \{-1, +1\}$, any \mathcal{H} , and any $\epsilon \in [0, 1]$ be given. Then*

$$\begin{aligned} \gamma_\epsilon(\nu) &= \min_{p \in \mathcal{D}_\epsilon(\nu)} \sup \left\{ \int (A\lambda) p d\nu : \lambda \in \Lambda \right\} \\ &= \sup \left\{ \min_{p \in \mathcal{D}_\epsilon(\nu)} \int (A\lambda) p d\nu : \lambda \in \Lambda \right\} \\ &= \min_{p \in \mathcal{D}_\epsilon(\nu)} \|A^\top p\|_\infty, \end{aligned}$$

where $\|A^\top p\|_\infty$ is discussed in Lemma 22.

Proof (of Lemma 31) Before applying the duality result, it must be established that the various infima are attained. To start, consider the final expression $\inf_{p \in \mathcal{D}_\epsilon(\nu)} \|A^\top p\|_\infty$, and let $\{p_i\}_{i=1}^\infty$ with $p_i \in \mathcal{D}_\epsilon(\nu)$ be a minimizing sequence to the infimum. Since Proposition 30 establishes that $\mathcal{D}_\epsilon(\nu)$ is weak*-compact, there is a subsequence $\{q_i\}_{i=1}^\infty$ which weak*-converges to some $q \in \mathcal{D}_\epsilon$ (Folland, 1999, Theorem 4.29). But Lemma 22 established that $p \mapsto \|A^\top p\|_\infty$ is weak* lower semi-continuous, and since it is finite over $L^\infty(\nu)$, it is therefore weak* continuous, and therefore the limit point q attains the infimum. Furthermore, Lemma 22 provides that $\|A^\top p\|_\infty$ is the same as the first infimand, whereby both expressions attain their minimizers and are equal.

The middle expression is the easiest; once again constructing a weak*-convergent sequence $q_i \rightarrow q$ with $q_i \in \mathcal{D}_\epsilon(\nu)$, the definition of weak*-convergence explicitly grants $\int q_i f \rightarrow \int q f$ for every $f \in L^1(\nu)$, and since $A\lambda \in L^1(\nu)$ is held fixed within this inner expression, it follows that q attains the infimum.

What remains is to swap minimization and maximization. This in turn follows by Sion's minimax theorem (Komiyu, 1988); to verify this application, note that $(p, \lambda) \mapsto \int p\lambda$ is linear and continuous in both parameters (indeed, this is by construction, since $L^\infty(\nu)$ is isometrically isomorphic to the topological dual to $L^1(\nu)$, and the weak* topology over $L^\infty(\nu)$ ensures that this integral relation is continuous for every $\lambda \in \Lambda$), also that Λ is a topological vector space, and lastly that $\mathcal{D}_\epsilon(\nu)$ is a convex compact subset of a topological vector space (namely, the weak* topology, and not the $L^\infty(\nu)$ topology, where $\mathcal{D}_\epsilon(\nu)$ is not necessarily compact as per Proposition 30). ■

Appendix E. Duality Properties of \mathcal{L}

Throughout this section, the identification of $L^1(\nu)^*$ with $L^\infty(\nu)$ and $L^1(\rho)^*$ with $L^\infty(\rho)$ via isometric isomorphism as provided by Lemma 20 will be central to obtaining meaningful expressions for the various conjugates.

To start, note the convexity structure of $\int \ell$.

Lemma 32 *Let $\ell : \mathbb{R} \rightarrow \mathbb{R}_+$ be convex with $\lim_{z \rightarrow -\infty} \ell(z) = 0$ and finite tightest Lipschitz constant $\beta := \sup_{x \neq y} |\ell(x) - \ell(y)|/|x - y| < \infty$, and let ν be a probability measure over $\mathcal{Z} := \mathcal{X} \times \{-1, +1\}$.*

1. *If $f \in L^1(\nu)$, then $\int \ell(f(z)) d\nu(z)$ is well-defined and finite.*
2. *$\int \ell$ is convex lower semi-continuous over $L^1(\nu)$.*
3. *Its conjugate $(\int \ell)^*$ is also convex lower semi-continuous as a function over $L^\infty(\nu)$.*
4. *If $p \in L^\infty(\nu)$, then $(\int \ell)^*(p) = \int \ell^*(p)$, which is finite $(\int \ell)^*(p)$ iff $p \in [0, \beta]$ ν -a.e..*

Proof Let $f \in L^1(\nu)$ be arbitrary. Since ℓ is convex and finite, it is continuous, so $\ell \circ f$ is measurable, and moreover it is nonnegative thus $\int \ell(f)$ is well-defined. Additionally,

$$\begin{aligned} \int \ell(f(z)) d\nu(z) &= \int \ell(0) d\nu(z) + \int (\ell(f(z)) - \ell(0)) d\nu(z) \\ &\leq \ell(0) \nu(\mathcal{Z}) + \int \beta |f(z) - 0| d\nu(z) = \ell(0) \nu(\mathcal{Z}) + \beta \|f\|_1 < \infty. \end{aligned}$$

Next, for any $f_1, f_2 \in L^1(\nu)$ and $\alpha \in [0, 1]$,

$$\begin{aligned} \int \ell(\alpha f_1(z) + (1 - \alpha) f_2(z)) d\nu(z) &\leq \int (\alpha \ell(f_1(z)) + (1 - \alpha) \ell(f_2(z))) d\nu(z) \\ &\leq \alpha \int \ell(f_1(z)) d\nu(z) + (1 - \alpha) \int \ell(f_2(z)) d\nu(z), \end{aligned}$$

whereby $\int \ell$ is convex. Since it is finite over $L^1(\nu)$ (as above), it is necessarily lower semi-continuous.

Since $\int \ell$ is convex lower semi-continuous, so is its conjugate (Zălinescu, 2002, Theorem 2.3.3), where the dual space $L^1(\nu)^*$ is identified with $L^\infty(\nu)$ as per the isomorphism statements in Lemma 20.

The remainder of this proof will reason about the conjugate to $\int \ell$. First let $p \in L^\infty(\nu)$ be given with $\nu(p^{-1}([0, \beta]^c)) > 0$; it will follow that $(\int \ell)^*(p) = \infty$. Define the sets

$$S_- := p^{-1}((-\infty, 0)) \quad S_0 := p^{-1}([0, \beta]), \quad S_+ := p^{-1}((\beta, \infty)),$$

as well as, for every $c \in \mathbb{R}$, the reals

$$g_- \in \partial \ell(-c), \quad g_0 \in \partial \ell(0), \quad g_+ \in \partial \ell(+c),$$

and lastly the simple functions

$$\begin{aligned} f_c(z) &:= -c\mathbf{1}[z \in S_-] + 0\mathbf{1}[z \in S_0] + c\mathbf{1}[z \in S_+], \\ g_c(z) &:= g_-\mathbf{1}[z \in S_-] + g_0\mathbf{1}[z \in S_0] + g_+\mathbf{1}[z \in S_+]. \end{aligned}$$

By these choices, f_c and g_c are measurable and within $L^1(\nu)$, and moreover $g_c \in \partial\ell(f_c)$ everywhere. As such,

$$\begin{aligned} \left(\int \ell\right)^*(p) &= \sup \left\{ \int (fp - \ell(f))d\nu : f \in L^1(\nu) \right\} \\ &\geq \sup \left\{ \int (f_c p - \ell(f_c))d\nu : c \in \mathbb{R} \right\} \\ &\geq \sup \left\{ \int (f_c p - \ell(0) + g_c(0 - f_c))d\nu(z) : c \in \mathbb{R} \right\} \\ &\geq \sup \left\{ c \int_{S_-} (g_c - p)d\nu + c \int_{S_+} (p - g_c)d\nu : c \in \mathbb{R} \right\} - \ell(0)\nu(\mathcal{Z}) \\ &= \infty, \end{aligned}$$

the last step following since $g_c \in [0, \beta]$ everywhere and $\nu(S_- \cup S_+) > 0$. As such, $(\int \ell)^*(p) = 0$, and since $\int \ell^*(p) = \infty$ by properties of ℓ^* (cf. Lemma 28), it follows that $\int \ell^*(p) = (\int \ell)^*(p) = \infty$.

In the remainder of the proof, suppose $p \in [0, \beta]$ ν -a.e..

Now consider the case that p is a simple function with $p \in (0, \beta)$ everywhere. Since ℓ^* is finite over $[0, \beta]$ (cf. Lemma 28), p is within the relative interior of the domain of ℓ^* everywhere, and thus $\partial\ell(p(z))$ is a nonempty set for every $z \in \mathcal{Z}$ (Rockafellar, 1970, Theorem 23.4). Consequently, construct q so that $q(z) \in \partial\ell^*(p(z))$ everywhere, and moreover q is also a simple function (i.e., pick the same subgradient along each of the finitely many regions composing p); these choices will ensure that there are no measurability issues with q (otherwise, the arguments pass through for arbitrary $p \in (0, \beta)$); additionally, $q \in L^1(\nu)$ since ν is a finite measure. Since ℓ is lower semi-continuous, $q(z) \in \partial\ell^*(p(z))$ implies $p(z) \in \partial\ell^{**}(q(z)) = \partial\ell(q(z))$, and the Fenchel-Young inequality implies

$$\ell^*(p(z)) = p(z)q(z) - \ell^{**}(q(z)) = p(z)q(z) - \ell(q(z)).$$

As such,

$$\begin{aligned} \left(\int \ell\right)^*(p) &= \sup \left\{ \int (fp - \ell(f))d\nu : f \in L^1(\nu) \right\} \\ &\geq \int (qp - \ell(q))d\nu \\ &= \int \ell^*(p)d\nu \end{aligned}$$

Now using the fact that $p(z) \in \partial \ell(q(z))$,

$$\begin{aligned} \left(\int \ell \right)^* (p) &= \sup \left\{ \int (fp - \ell(f)) d\nu : f \in L^1(\nu) \right\} \\ &\leq \sup \left\{ \int (fp - \ell(q) - p(f - q)) d\nu : f \in L^1(\nu) \right\} \\ &= \sup \left\{ \int (pq - \ell(q)) d\nu : q \in L^1(\nu) \right\} \\ &= \int \ell^*(p); \end{aligned}$$

combining these two inequalities, $(\int \ell)^*(p) = \int \ell^*(p)$.

Now consider the case that $p \in (0, \beta)$ is just measurable. Since the simple functions are dense in $L^\infty(\nu)$ (Folland, 1999, Theorem 6.8), there exists a simple function $\phi_i \in L^\infty(\nu)$ with $\|p - \phi_i\|_\infty \leq 1/i$, and moreover ϕ_i may be clamped to the range $[1/i, \beta - 1/i]$ (with i sufficiently large to make this interval nonempty), whereby this clamped simple function ψ_i satisfies $\|p - \psi_i\|_\infty \leq 2/i$. Since $(\int \ell)^*$ is lower semi-continuous,

$$\left(\int \ell \right)^* (p) = \lim_i \left(\int \ell \right)^* (\psi_i) = \lim_i \int \ell^*(\psi_i) = \int \ell^*(p),$$

where the last step used the dominated convergence theorem applied with dominating constant map $z \mapsto \sup_{q \in [0, \beta]} |\ell^*(q)|$, which is finite since ℓ^* is continuous over the compact set $[0, \beta]$ (cf. Lemma 28).

Next consider the case that measurable $p \in (0, \beta)$ ν -a.e.; then $\tilde{p}(z) := p(z)\mathbb{1}[p(z) \in (0, \beta)] + (\beta/2)\mathbb{1}[p(z) \notin (0, \beta)]$ satisfies $(\int \ell)^*(p) = (\int \ell)^*(\tilde{p})$ by definition of the conjugate (the integrals ignore measure zero sets), whereby $(\int \ell)^*(p) = (\int \ell)^*(\tilde{p}) = \int \ell^*(\tilde{p}) = \int \ell^*(p)$.

Lastly, suppose measurable $p \in [0, \beta]$ ν -a.e.. For each i , define $p_i = (1 - 1/i)p + \beta/(2i)$. Then $p_i \in (0, \beta)$ ν -a.e., and $\|p_i - p\|_\infty \rightarrow 0$, whereby the lower semi-continuity of $(\int \ell)^*$ and dominated convergence theorem cover this case in the same way as the move away from simple functions.

Note lastly that these last choices provide a finite integral, since $\sup_{z \in [0, \beta]} |\ell^*(z)| < \infty$ as above, and ν is a finite measure. \blacksquare

While the above proof (properties of $\int \ell$) may have seemed like a technical exercise, note that these structural properties can not be taken for granted; in particular, the following result establishes that the $L^1(\nu)$ topology is not the correct way to study the exponential loss.

Proposition 33 *Let ν denote the standard Gaussian measure over \mathbb{R} , and define $f(x) := x^2$ and $f_i(x) := x^2 \mathbb{1}[|x| \leq i]$. Then $f_i \in L^1(\nu)$, $f \in L^1(\nu)$, and $\|f_i - f\|_1 \rightarrow 0$, but*

$$\int \exp(f_i(x)) d\nu(x) < \infty \quad \text{and} \quad \int \exp(f(x)) d\nu(x) = \infty.$$

In particular, $\int \exp$ is not lower semi-continuous over $L^1(\nu)$.

Proof To start, $\int f d\nu = 1$ (variance of a standard Gaussian), and thus $\int f_i \rightarrow f$ by the monotone convergence theorem (and so $\|f_i - f\|_1 \rightarrow 0$). But

$$\begin{aligned} \int \exp(f_i(x)) d\nu(x) &\leq e^{i^2} \int d\nu(x) < \infty, \\ \int \exp(f(x)) d\nu(x) &= \frac{1}{\sqrt{2\pi}} \int e^{x^2/2} dx = \infty. \end{aligned}$$

It follows that there are convergent sequences within $L^1(\nu)$ for which the values of $\int \exp$ do not converge, and consequently $\int \exp$ is not lower semi-continuous over $L^1(\nu)$. \blacksquare

Returning to Lipschitz losses, the desired duality relation follows.

Lemma 34 *Let $\ell : \mathbb{R} \rightarrow \mathbb{R}_+$ be convex with $\lim_{z \rightarrow -\infty} \ell(z) = 0$ and finite tightest Lipschitz constant $\beta := \sup_{x \neq y} |\ell(x) - \ell(y)|/|x - y| < \infty$. Additionally, let ν be a probability measure over $\mathcal{X} \times \{-1, +1\}$, and \mathcal{H} be arbitrary. Then*

$$\inf \left\{ \int \ell(A\lambda) d\nu : \lambda \in \Lambda \right\} = \max \left\{ - \int \ell^*(p) : p \in L^\infty(\nu), p \in [0, \beta] \text{ } \nu\text{-a.e.}, \|A^\top p\|_\infty = 0 \right\}.$$

Proof Consider the following two Fenchel problems:

$$\begin{aligned} V_p &:= \inf \left\{ \int \ell(A\lambda) d\nu + \int 0 \cdot \lambda d\rho : \lambda \in \Lambda \right\}, \\ V_d &:= \sup \left\{ - \int \ell^*(p) - \iota_{\{0\}}(A^\top p) : p \in L^\infty(\nu) \right\}, \end{aligned}$$

where $\iota_{\{0\}}$ is the indicator for the set $\{0\}$,

$$\iota_{\{0\}}(\lambda) = \begin{cases} 0 & \text{when } \lambda = 0, \\ \infty & \text{otherwise,} \end{cases}$$

and is the conjugate to $\int 0 \cdot \lambda$. In order to show $V_p = V_d$ and attainment occurs in the dual, an appropriate Fenchel duality rule will be applied (Zălinescu, 2002, Corollary 2.8.5 using condition (vii)), which requires the verification of the following properties.

- First note that $\int \ell$ and $\int \ell^*$ are both convex lower semi-continuous, and moreover mutually conjugate (cf. Lemma 32). The function $\lambda \mapsto 0 = \int 0\lambda$ is immediately convex lower semi-continuous (over Λ), and thus its conjugate $\iota_{\{0\}}$ is similarly convex lower semi-continuous, and the two are mutually conjugate (Zălinescu, 2002, Theorem 2.3.3).
- Both $L^1(\nu)$ and $\Lambda = L^1(\rho)$ are Banach and therefore Fréchet spaces. (The present proof is one of the reasons Λ was taken to be a Banach space and not merely, say, weightings with finite support as used by the algorithm).

- Let $\text{dom}(f) = \{x : f(x) < \infty\}$ denote the effective domain of a convex function, meaning those values where it is finite. As provided by Lemma 32, $\text{dom}(\int \ell) = L^1(\nu)$, and thus, since $A : \Lambda \rightarrow L^1(\nu)$,

$$A(\text{dom}(\lambda \mapsto 0)) = \text{dom}\left(\int \ell\right) = A\Lambda + L^1(\nu) = L^1(\nu),$$

which settles the constraint qualification. (Recall that $A\Lambda$ is not necessarily a closed subspace (cf. Lemma 23); thus further problems would occur here if this proof were attempted for $\ell = \exp$, as $\text{dom}(\int \ell)$ would not swallow the closure issues of $A\Lambda$.)

This completes the conditions necessary for the Fenchel duality result. To adjust the proof into the desired form, Lemma 32 provided that $\int \ell^*$ is finite iff its input lies within $[0, \beta]$ ν -a.e. (thus other values may safely be discarded from the optimization problem, which always has feasible point $0 \in L^\infty(\nu)$), and secondly $\iota_0(A^\top p) < \infty$ iff $\|A^\top p\|_\infty = 0$ (recall the form of $\|A^\top p\|_\infty$ in Lemma 22). \blacksquare

Appendix F. Line Search Guarantees

Before proceeding with the various properties of the line searches, it is a good time to discuss expressions involving $\nabla \widehat{\mathcal{L}}$, upon which these line searches depend. In the context of the algorithm, the sample size is finite and $|\text{supp}(\lambda)| < \infty$, thus

$$\widehat{\mathcal{L}}(A\lambda) = \frac{1}{m} \sum_{i=1}^m \left(\sum_{h \in \text{supp}(h)} -y_i h(x_i) \lambda(h) \right)$$

always involves only finitely many computations. In this way, A may be simply viewed as a matrix with m rows and at most $\sup_{t \leq \lceil m^a \rceil} |\text{supp}(\lambda_t)| \leq \lceil m^a \rceil$ columns; furthermore, if \mathcal{H} is binary, 2^m columns suffice and are known a priori (and the Sauer-Shelah Lemma can further reduce the dimensions). As such, when working with gradient computations, this manuscript adopts the familiar notation of the form

$$\begin{aligned} \nabla \widehat{\mathcal{L}}(A\lambda)^\top A\lambda' &= \left\langle A^\top \nabla f(A\lambda), \lambda' \right\rangle \\ &= \left\langle \nabla f(A\lambda), A\lambda' \right\rangle \\ &= \frac{1}{m} \sum_{i=1}^m \ell'((A\lambda)_{x_i, y_i}) (A\lambda')_{x_i, y_i}, \end{aligned}$$

and moreover the matrix rule $\nabla(\widehat{\mathcal{L}} \circ A)(\lambda) = A^\top \nabla \widehat{\mathcal{L}}(A\lambda)$ makes sense.

This manuscript never considers gradients of \mathcal{L} (e.g., in the sense of Gâteaux or Fréchet). However, to connect the above expressions to the development of the spaces (e.g., $L^1(\mu)$ and Λ) and linear operators (e.g., A and A^\top) from Appendix A, note firstly that $\partial \mathcal{L}$ is a subset of $L^1(\mu) \rightarrow \mathbb{R}$ (identified with $L^\infty(\mu)$ via Lemma 20), meaning never a singleton since it contains μ -a.e. equivalent copies of functions. Modulo these details, $(A^\top g)$, for some $g \in \partial \mathcal{L}(A\lambda)$, can be identified with an element of $L^\infty(\rho)$ as in Lemma 22, and thus

$(A^\top g)(\lambda')$ makes sense (and indeed, by properties of the adjoint A^\top and the dual space identification from Lemma 20, $(A^\top g)(\lambda') = \int (A\lambda') g d\mu$). Of course, these expressions are nonsense from a computational standpoint.

The remainder of this section gives basic guarantees for various line searches.

Remark 35 (Wolfe line search) *The Wolfe line search chooses any α_t which satisfies the following conditions (where this manuscript makes the simple choice $c_1 = 1/3$ and $c_2 = 1/2$):*

$$\begin{aligned} \widehat{\mathcal{L}}(A(\lambda_{t-1} + \alpha v_t)) &\leq \widehat{\mathcal{L}}(A\lambda_{t-1}) + \alpha c_1 \left\langle A^\top \nabla \widehat{\mathcal{L}}(A\lambda_{t-1}), v_t \right\rangle \\ &\leq \widehat{\mathcal{L}}(A\lambda_{t-1}) - \frac{\alpha \rho}{3} \|A^\top \nabla \widehat{\mathcal{L}}(A\lambda_{t-1})\|_\infty, \end{aligned} \quad (2)$$

$$\begin{aligned} \nabla \widehat{\mathcal{L}}(A(\lambda_{t-1} + \alpha v_t))^\top A v_t &\geq c_2 \left\langle A^\top \nabla \widehat{\mathcal{L}}(A\lambda_{t-1}), v_t \right\rangle \\ &\geq -\frac{1}{2} \|A^\top \nabla \widehat{\mathcal{L}}(A\lambda_{t-1})\|_\infty. \end{aligned} \quad (3)$$

The method itself may be implemented (in the convex case) similarly to binary search (Telgarsky, 2012, Section D.1).

Lemma 36 *Let $\ell \in \mathbb{L}_{\text{lg}}$ with Lipschitz gradient parameter B_2 , and iteration t be given, and suppose α_t is chosen according to one of the first two step choices in Algorithm 1, meaning either $\alpha_t = \bar{\alpha}_t$ or $\alpha_t \in [-\nabla \widehat{\mathcal{L}}(A\lambda_{t-1})^\top A v_t / B_2, \bar{\alpha}_t]$. Then*

$$\begin{aligned} \alpha_t &\geq \frac{\rho \|A^\top \nabla \widehat{\mathcal{L}}(A\lambda_{t-1})\|_\infty}{B_2}, \\ \widehat{\mathcal{L}}(A\lambda_t) &\leq \widehat{\mathcal{L}}(A\lambda_{t-1}) - \frac{\rho^2 \|A^\top \nabla \widehat{\mathcal{L}}(A\lambda_{t-1})\|_\infty^2}{2B_2}. \end{aligned}$$

Proof By Corollary 27, for every $\alpha > 0$, since A has entries within $[-1, +1]$,

$$\begin{aligned} \widehat{\mathcal{L}}(A(\lambda_{t-1} + \alpha v_t)) &\leq \widehat{\mathcal{L}}(A\lambda_{t-1}) + \alpha \left\langle A^\top \nabla \widehat{\mathcal{L}}(A\lambda_{t-1}), v_t \right\rangle + \frac{B_2}{2m} \sum_i (\alpha A v_t)^2 \\ &\leq \widehat{\mathcal{L}}(A\lambda_{t-1}) + \alpha \left\langle A^\top \nabla \widehat{\mathcal{L}}(A\lambda_{t-1}), v_t \right\rangle + \frac{B_2}{2m} \sum_i (\alpha A v_t)_i^2 \\ &\leq \widehat{\mathcal{L}}(A\lambda_{t-1}) + \alpha \left\langle A^\top \nabla \widehat{\mathcal{L}}(A\lambda_{t-1}), v_t \right\rangle + \frac{B_2 \alpha^2}{2}. \end{aligned}$$

This final expression defines a univariate quadratic with minimum $\bar{\alpha} := -\left\langle A^\top \nabla \widehat{\mathcal{L}}(A\lambda_{t-1}), v_t \right\rangle / B_2$.

This function has slopes everywhere exceeding $\widehat{\mathcal{L}} \circ A$ along $[\lambda_{t-1}, \lambda_t]$ (for either choice of step size), and so $\bar{\alpha} \leq \alpha_t \leq \bar{\alpha}_t$. (Indeed, these bounds give a derivation for the second step size choices.) To get the second guarantee, note that plugging $\bar{\alpha}$ into the above quadratic and simplifying via

$$\left\langle A^\top \nabla \widehat{\mathcal{L}}(A\lambda_{t-1}), v_t \right\rangle^2 \geq \rho^2 \|A^\top \nabla \widehat{\mathcal{L}}(A\lambda_{t-1})\|_\infty^2$$

gives the desired minimum quadratic upper bound. \blacksquare

Lemma 37 *Let $\ell \in \mathbb{L}_{\text{lg}}$ with Lipschitz gradient parameter B_2 , and iteration t be given, and suppose α_t satisfies the Wolfe conditions for some $0 < c_1 < c_2 < 1$. Then*

$$\begin{aligned}\alpha_t &\geq \frac{\rho(1-c_2)\|A^\top \nabla \widehat{\mathcal{L}}(A^\top \lambda_{t-1})\|_\infty}{B_2}, \\ \widehat{\mathcal{L}}(A\lambda_t) &\leq \widehat{\mathcal{L}}(A\lambda_{t-1}) - \frac{\rho^2 c_1(1-c_2)\|A^\top \nabla \widehat{\mathcal{L}}(A\lambda_{t-1})\|_\infty^2}{B_2}.\end{aligned}$$

Proof By the definition of B_2 and since A has entries in $[-1, +1]$,

$$\begin{aligned}\left\langle A^\top \nabla \widehat{\mathcal{L}}(A^\top \lambda_t) - A^\top \nabla \widehat{\mathcal{L}}(A^\top \lambda_{t-1}), v_t \right\rangle &= \frac{1}{m} \sum_i (\ell'((A^\top \lambda_t)_i) - \ell'((A^\top \lambda_{t-1})_i))(Av_t)_i \\ &= \sum_i \frac{(\ell'((A^\top \lambda_t)_i) - \ell'((A^\top \lambda_{t-1})_i))}{m(\alpha_t Av_t)_i} \alpha_t (Av_t)_i^2 \\ &\leq \alpha_t B_2.\end{aligned}$$

The rest of the proof is just as for standard Wolfe search guarantees (cf. [Nocedal and Wright \(2006, Theorem 3.2\)](#) or [Telgarsky \(2012, Proposition D.6\)](#)), and direct from the Wolfe conditions. First, subtracting $\left\langle A^\top \nabla \widehat{\mathcal{L}}(A\lambda_{t-1}), v_t \right\rangle$ from both sides of eq. (3) gives

$$\left\langle A^\top \nabla \widehat{\mathcal{L}}(A^\top \lambda_t) - A^\top \nabla \widehat{\mathcal{L}}(A^\top \lambda_{t-1}), v_t \right\rangle \geq (c_2 - 1) \left\langle A^\top \nabla \widehat{\mathcal{L}}(A^\top \lambda_{t-1}), v_t \right\rangle,$$

which can be combined with the above derivation to yield

$$\alpha_t \geq \frac{(c_2 - 1) \left\langle A^\top \nabla \widehat{\mathcal{L}}(A^\top \lambda_{t-1}), v_t \right\rangle}{B_2} \geq \frac{\rho(1-c_2)\|A^\top \nabla \widehat{\mathcal{L}}(A^\top \lambda_{t-1})\|_\infty}{B_2}$$

Plugging this into eq. (2) gives

$$\widehat{\mathcal{L}}(A\lambda_t) \leq \widehat{\mathcal{L}}(A\lambda_{t-1}) - \frac{\rho^2 c_1(1-c_2)\|A^\top \nabla \widehat{\mathcal{L}}(A\lambda_{t-1})\|_\infty^2}{B_2}.$$

\blacksquare

Appendix G. Reweighted Margin Deviations (with p Fixed)

Lemma 38 *Let probability measure μ over $\mathcal{X} \times \{-1, +1\}$ with empirical counterpart $\widehat{\mu}$, any hypothesis class $\mathcal{H} \in \mathcal{F}_{\text{vc}}$, reweighting $p \in L^\infty(\mu)$, and norm bound C be given. Then, with probability at least $1 - \delta$, $p \in [0, \|p\|_\infty]$ $\widehat{\mu}$ -a.e., and*

$$\sup_{\substack{\lambda \in \Lambda \\ \|\lambda\|_1 \leq C}} \left| \int (A\lambda) p d\widehat{\mu} - \int (A\lambda) p d\mu \right| \leq \frac{2C\|p\|_\infty}{m^{1/2}} \left(2\sqrt{2\mathcal{V}(\mathcal{H}) \ln(m+1)} + \sqrt{2\ln(1/\delta)} \right).$$

Proof (of Lemma 38) First, define a simplified reweighting $p'(x, y) := p(x, y)\mathbf{1}[|p(x, y)| \leq \|p\|_\infty]$; by the definition of $\|\cdot\|_\infty$, then $p' = p$ μ -a.e., and thus, with probability 1, any finite sample of any size has p' and p agreeing. The proof will work with p' , which satisfies $\sup_{x,y} |p'(x, y)| \leq \|p\|_\infty$, and then close by discarding a measure zero set and thus relating to p .

The main part of the proof is an almost standard application of Rademacher complexity techniques for voted classifiers (Boucheron et al., 2005, Theorem 4.1 and its proof, which controls for a surrogate loss and not just the classification loss); the only modification will be to work with a loss function which is sensitive to each example in the sample $S = \{(x_i, y_i)\}_{i=1}^m$, which will require a slightly refined Lipschitz contraction principle for Rademacher complexities (Shalev-Shwartz, 2009, Section 22.2, Lemma 15).

Specifically, define the loss

$$\phi((A\lambda)_{x,y}) := \begin{cases} -Cp'(x, y) & \text{when } (A\lambda)_{x,y} \leq -C, \\ (A\lambda)_{x,y}p'(x, y) & \text{when } |(A\lambda)_{x,y}| \leq C, \\ +Cp'(x, y) & \text{when } (A\lambda)_{x,y} \geq C. \end{cases}$$

Since $\sup_{h,x} |h(x)| \leq 1$ and $\|\lambda\|_1 \leq C$, it follows that $|(A\lambda)_{x,y}| \leq C$, and thus the extremal cases are never encountered, meaning

$$\phi((A\lambda)_{x,y}) = (A\lambda)_{x,y}p'(x, y),$$

and by construction ϕ is Lipschitz with parameter $\|p\|_\infty$ (as a function of $(A\lambda)$) and $\phi \circ (A\lambda)$ has uniform bound $C\|p\|_\infty$.

As such, letting R denote Rademacher complexity, by the Lipschitz contraction principle for per-coordinate losses (Shalev-Shwartz, 2009, Section 22.2, Lemma 15), behavior of Rademacher complexity on convex hulls (Boucheron et al., 2005, Theorem 3.3), and relationship between Rademacher complexity and VC dimension (Boucheron et al., 2005, See the display after eq. (7)),

$$R(\phi \circ (A\lambda)) \leq \|p\|_\infty R((A\lambda)) \leq \|p\|_\infty CR(\mathcal{H}) \leq \|p\|_\infty C \sqrt{\frac{2\mathcal{V}(\mathcal{H}) \ln(m+1)}{m}}.$$

This handling of a per-coordinate Lipschitz loss may be inserted into a standard deviation bound for uniformly bounded Lipschitz losses (Boucheron et al., 2005, Theorem 4.1 and its proof) — albeit with an extra factor two to control deviations in both directions — and it follows, with probability at least $1 - \delta$, that

$$\sup_{\substack{\lambda \in \Lambda \\ \|\lambda\|_1 \leq C}} \left| \int (A\lambda)p' d\hat{\mu} - \int (A\lambda)p' d\mu \right| \leq \frac{2C\|p\|_\infty}{m^{1/2}} \left(2\sqrt{2\mathcal{V}(\mathcal{H}) \ln(m+1)} + \sqrt{2\ln(1/\delta)} \right).$$

To complete the proof, recall that $p' = p$ μ -a.e., and a measure zero event was discarded, whereby $p' = p$ $\hat{\mu}$ -a.e. as well. \blacksquare

Appendix H. Deferred Material from Section 4

H.1. Deviations of $\gamma_\epsilon(\nu)$

This subsection establishes the following one-sided deviation bound on $\gamma_\epsilon(\nu)$.

Lemma 39 *Let any \mathcal{H} , any $\epsilon \in (0, 1]$, any confidence parameter $\delta \in (0, 1]$, and any probability measure μ with empirical counterpart $\hat{\mu}$ be given. Then with probability at least $1 - \delta$,*

$$\gamma_\epsilon(\hat{\mu}) \geq \gamma_\epsilon(\mu) - \frac{1}{\epsilon} \sqrt{\frac{1}{2m} \ln \left(\frac{2}{\delta} \right)}.$$

The difficulty in the analysis is that the definition of $\gamma_\epsilon(\nu)$ involves an infimum over $p \in \mathcal{D}_\epsilon(\nu)$ and a supremum over $\lambda \in \Lambda$ with $\|\lambda\|_1 \leq 1$. The proof strategy employed here is to consider a single good choice for λ , and to consider the effect on deviations as p varies. These deviations do not appear to be amenable to the usual approach, as $\mathcal{D}_\epsilon(\nu)$ is massive: it is in general not compact in the relevant metric topologies (cf. Proposition 30), and does not obviously possess other structure granting a uniform convergence result. The approach here is to instead identify that the dual optimum has very simple structure, and moreover this structure is robust to sampling.

Considering again the definition of $\gamma_\epsilon(\nu)$, while it is true that $p \in \mathcal{D}_\epsilon(\nu)$ is defined over a potentially massive space, when placed in the expression $\int (A\lambda) p d\nu$, all that matters is the behavior of p for each value of $A\lambda$, which ranges over $[-1, +1]$. That is to say, p is really reweighting the univariate margin distribution of $A\lambda$, and the best it can do is emphasize bad margins. In particular, the following lemma proves basic properties of an idealized univariate distillation of this scenario.

Lemma 40 *Let a probability measure ξ supported on $[-1, +1]$ and some $\epsilon \in (0, 1]$ be given. Correspondingly define*

$$\begin{aligned} S_\epsilon &:= \{c \in [-1, +1] : \xi((-\infty, c)) \leq \epsilon\}, \\ c_\epsilon &:= \sup S_\epsilon, \\ I_\epsilon &:= (-\infty, c_\epsilon), \\ p_\epsilon(r) &:= \frac{1}{\epsilon} \left(\mathbb{1}[r \in I_\epsilon] + \frac{(\epsilon - \xi(I_\epsilon))}{\xi(\{c_\epsilon\})} \mathbb{1}[r = c_\epsilon] \right), \end{aligned}$$

with the convention $0/0 = 0$ in the definition of p_ϵ . These objects have the following properties.

1. S_ϵ is the closed interval $[-1, c_\epsilon]$.
2. $\xi(I_\epsilon) \leq \epsilon$, and $\xi(I_\epsilon \cup \{c_\epsilon\}) \geq \epsilon$.
3. $\|p_\epsilon\|_1 = 1$ and $\|p_\epsilon\|_\infty \leq 1/\epsilon$.
4. The optimization problem

$$\inf \left\{ \int r p(r) d\xi(r) : p \in L^\infty(\xi), \|p\|_1 = 1, p \in [0, 1/\epsilon] \text{ } \xi\text{-a.e.} \right\}$$

is minimized at p_ϵ .

Proof First note that $-1 \in S_\epsilon$, since ξ is supported on $[-1, +1]$ and thus $\xi((-\infty, -1)) = 0$.

Next, S_ϵ is an interval, since if $-1 \leq c_1 \leq c_2$ and $c_2 \in S_\epsilon$, then $\xi((-\infty, c_1)) \leq \xi((-\infty, c_2))$ and thus $c_1 \in S_\epsilon$.

To show that S_ϵ is indeed a closed interval, consider any increasing sequence $\{c_i\}_{i=1}^\infty$ with $c_i \in S_\epsilon$, thus $c_i \rightarrow c$ for some $c \in [-1, +1]$ since $[-1, +1]$ is compact and the sequence is increasing. Then

$$(-\infty, c) = \cup_{i=1}^\infty (-\infty, c_i),$$

and thus, by continuity of measures [Folland \(1999, Theorem 1.8\)](#),

$$\xi((-\infty, c)) = \xi(\cup_{i=1}^\infty (-\infty, c_i)) = \lim_{i \rightarrow \infty} \xi((-\infty, c_i)) \leq \limsup_{i \rightarrow \infty} \xi((-\infty, c_i)) \leq \epsilon,$$

meaning $c \in S_\epsilon$ and S_ϵ is closed.

Since S_ϵ is a closed interval, then $c_\epsilon = \sup S_\epsilon \in S_\epsilon$, and it follows by the preceding properties that $S_\epsilon = [-1, c_\epsilon]$.

By definition, for every $c \in S_\epsilon$, it holds that $\xi((-\infty, c)) \leq \epsilon$, thus $c_\epsilon \in S_\epsilon$ implies that $\xi(I_\epsilon) \leq \epsilon$.

Next, for every positive integer $i \in \mathbb{Z}_{++}$, it holds by definition of c_ϵ that $c_\epsilon + 1/i \notin S_\epsilon$, and thus, again by continuity of measures [Folland \(1999, Theorem 1.8\)](#),

$$\xi([-1, c_\epsilon]) = \xi(\cap_{i=1}^\infty [-1, c_\epsilon + 1/i]) \geq \liminf_{i \rightarrow \infty} \xi([-1, c_\epsilon + 1/i]) \geq \epsilon.$$

For the norms of p_ϵ (which is a simple function over the Borel σ -algebra), notice that

$$\|p_\epsilon\|_1 = \frac{1}{\epsilon} \left(\xi(I_\epsilon) + \frac{(\epsilon - \xi(I_\epsilon))}{\xi(\{c_\epsilon\})} \xi(\{c_\epsilon\}) \right) = 1.$$

Moreover, $p_\epsilon = 1/\epsilon$ on $(-\infty, c_\epsilon)$, and $p_\epsilon = 0$ on (c_ϵ, ∞) ; to show $\|p\|_\infty \leq 1$, the behavior of p_ϵ on c_ϵ is all that needs to be checked. Since $\xi((-\infty, c_\epsilon]) \geq \epsilon$, then

$$\epsilon - \xi(I_\epsilon) = \epsilon - \xi((-\infty, c_\epsilon]) + \xi(\{c_\epsilon\}) \leq \xi(\{c_\epsilon\}),$$

so $p_\epsilon(c_\epsilon) \leq 1/\epsilon$. Additionally $\xi(I_\epsilon) \leq \epsilon$ implies $p_\epsilon(c_\epsilon) \geq 0$, and thus $\|p\|_\infty \leq 1/\epsilon$ as desired.

Lastly, for the minimization problem, consider any feasible p (meaning $\|p\|_1 = 1$ and $\|p\|_\infty \leq 1/\epsilon$) with $\|p_\epsilon - p\|_1 > 0$. But since p_ϵ is as large as possible along I_ϵ , it follows that $p < p_\epsilon$ for a positive measure subset of I_ϵ , and $p > p_\epsilon$ for a positive measure subset of $[c_\epsilon, \infty)$. Consequently $\int r p_\epsilon(r) d\xi(r) < \int r p'(r) d\xi(r)$. Since p' was arbitrary, it follows that p_ϵ is a minimal choice. \blacksquare

The task now is to map the optimization over $\mathcal{D}_\epsilon(\nu)$ down to this idealized univariate search problem. Temporarily adopting notation from probability theory, a first step in this direction would be to write

$$\int (A\lambda) p d\nu = \mathbb{E}_\nu((A\lambda)p) = \mathbb{E}_\nu(\mathbb{E}((A\lambda)p | (A\lambda) = r)),$$

where the latter notation signifies a conditional expectation with respect the σ -algebra generated by events such that $(A\lambda)$ falls in some Borel subset of \mathbb{R} (recall that all σ -algebras here are Borel). In some circumstances, the function $\mathbb{E}((A\lambda)p | (A\lambda) = r)$ can

be converted into integration over a function that takes r as input, which would directly allow conversion to the above univariate idealization; these techniques generally require assumptions on $\mathcal{X} \times \{-1, +1\}$ which would rather be avoided here (Durrett, 2010, Section 5.1.3, regular conditional probabilities). As such, the following result exhibits the desired correspondence manually, albeit keeping the above idea in mind.

Lemma 41 *Let any $\epsilon \in (0, 1]$, any probability measure ν over $\mathcal{X} \times \{-1, +1\}$, any \mathcal{H} , and any $\lambda \in \Lambda$ with $\|\lambda\|_1 \leq 1$ be given. Define a probability measure ξ over \mathbb{R} as the pushforward of ν through $A\lambda$, meaning, for any Borel subset S of \mathbb{R} ,*

$$\xi(S) := \nu((A\lambda)^{-1}(S)).$$

Then ξ is supported on $[-1, +1]$, and moreover the function $p_\epsilon^\nu(x, y) := p_\epsilon((A\lambda)_{x,y})$, where p_ϵ is as defined in Lemma 40, is a (feasible) minimizer to the optimization problem

$$\inf \left\{ \int (A\lambda) p d\nu : p \in \mathcal{D}_\epsilon(\nu) \right\}.$$

Proof Since $\|\lambda\|_1 \leq 1$ and A is a continuous linear operator with unit norm (cf. Lemma 19, or recall the definition of A and the property $\sup_{x,h} |h(x)| \leq 1$), then $|(A\lambda)_{x,y}| \leq 1$, and thus $(x, y) \mapsto (A\lambda)_{x,y}$ maps $\mathcal{X} \times \{-1, +1\}$ to $[-1, +1]$, and so the corresponding pushforward measure ξ is supported on $[-1, +1]$. Therefore Lemma 40 provides the structure of $p_\epsilon \in L^\infty(\xi)$ attaining the minimum in

$$\inf \left\{ \int r p(r) d\xi(r) : p \in L^\infty(\xi), \|p\|_1 = 1, p \in [0, 1/\epsilon] \text{ } \xi\text{-a.e.} \right\}.$$

Setting $p_\epsilon^\nu := p_\epsilon \circ (A\lambda)$ as in the statement, by the above optimality guarantee and by properties of pushforward measures (Resnick, 1999, Theorem 5.5.1),

$$\begin{aligned} & \inf \left\{ \int r p(r) d\xi(r) : p \in L^\infty(\xi), \|p\|_1 = 1, p \in [0, 1/\epsilon] \text{ } \xi\text{-a.e.} \right\} \\ &= \int r p_\epsilon(r) d\xi(r) \\ &= \int (A\lambda) p_\epsilon^\nu d\mu \\ &\geq \inf \left\{ \int (A\lambda) p d\nu : p \in \mathcal{D}_\epsilon(\nu) \right\}. \end{aligned} \tag{4}$$

Now let $\sigma > 0$ and $p \in \mathcal{D}_\epsilon(\nu)$ be arbitrary. A corresponding element q with $\|q\|_1 = \|p\|_1$ and $\|q\|_\infty \leq \|p\|_\infty$ will be constructed as follows in order to satisfy

$$\left| \int (A\lambda) p d\mu - \int r q(r) d\xi(r) \right| \leq \sigma.$$

Cover $[-1, +1]$ with at most $1 + \lceil 1/\sigma \rceil$ disjoint half-open intervals $\{I_i\}_{i=1}^k$ of the form $[-1 + i\sigma', -1 + (i+1)\sigma')$ where i is a nonnegative integer and $\sigma' := \sigma/(1 + \lceil 1/\sigma \rceil)$. Define

$$q(r) := \sum_{i=1}^k \xi(I_i)^{-1} \mathbf{1}[r \in I_i] \int_{[A\lambda \in I_i]} p d\mu,$$

with the convention $0/0 = 0$ (i.e., $q(I_i) = 0$ when $\xi(I_i) = 0$). By this choice, $\|q\|_\infty \leq \|p\|_\infty$, and

$$\|q\|_1 = \int q(r) d\xi(r) = \sum_{i=1}^k \int_{[A\lambda \in I_i]} p d\mu = \|p\|_1.$$

More importantly, using Fubini's Theorem to interchange the integrals over ξ and μ ,

$$\begin{aligned} \left| \int (A\lambda) p d\mu - \int r q(r) d\xi(r) \right| &\leq \sum_{i=1}^k \left| \int_{[A\lambda \in I_i]} (A\lambda) p d\mu - \int_{I_i} r \xi(I_i)^{-1} \left(\int_{[A\lambda \in I_i]} p d\mu \right) d\xi(r) \right| \\ &\leq \sum_{i=1}^k \int_{[A\lambda \in I_i]} \left| (A\lambda) p - p \int_{I_i} r \xi(I_i)^{-1} d\xi(r) \right| d\mu \\ &\leq \sum_{i=1}^k \int_{[A\lambda \in I_i]} |p| |\sigma'| d\mu \leq \sigma. \end{aligned}$$

Since σ and p were arbitrary,

$$\begin{aligned} &\inf \left\{ \int (A\lambda) p d\nu : p \in \mathcal{D}_\epsilon(\nu) \right\} \\ &\geq \inf \left\{ \int r p(r) d\xi(r) : p \in L^\infty(\xi), \|p\|_1 = 1, p \in [0, 1/\epsilon] \text{ } \xi\text{-a.e.} \right\}, \end{aligned}$$

which combined with the inequalities starting with eq. (4) provides that p'_ϵ is indeed a minimizer. \blacksquare

With these tools in place, the proof of Lemma 39 follows.

Proof (of Lemma 39) Consider the form of $\gamma_\epsilon(\nu)$ provided by Lemma 31, whereby the supremum over $\lambda \in \Lambda$ is on the outside. Let $\sigma > 0$ be arbitrary, choose $\lambda \in \Lambda$ which is within $\sigma > 0$ of achieving the supremum, and let p_ϵ^μ be an optimal dual element as provided by Lemma 41, together meaning

$$\gamma_\epsilon(\mu) \leq \sigma + \inf_{p \in \mathcal{D}_\epsilon(\mu)} \int (A\lambda) p d\mu = \sigma + \int (A\lambda) p_\epsilon^\mu d\mu. \quad (5)$$

Now consider the behavior of p_ϵ^μ over $\hat{\mu}$. By construction, $\|p_\epsilon^\mu\|_{L^\infty(\hat{\mu})} \leq 1/\epsilon$, however $\|p_\epsilon^\mu\|_{L^1(\hat{\mu})}$ is a random variable; but by Hoeffding's inequality, with probability at least $1 - \delta$,

$$\left| \|p_\epsilon^\mu\|_{L^1(\hat{\mu})} - \|p_\epsilon^\mu\|_{L^1(\mu)} \right| = \left| \|p_\epsilon^\mu\|_{L^1(\hat{\mu})} - 1 \right| \leq \frac{1}{\epsilon} \sqrt{\frac{1}{2m} \ln \left(\frac{2}{\delta} \right)};$$

henceforth discard this failure event.

Next instantiate another dual optimum $p_\epsilon^{\hat{\mu}}$ via Lemma 41, but now over the empirical measure $\hat{\mu}$; since $\lambda \in \Lambda$ is primal feasible in the definition of $\gamma_\epsilon(\hat{\mu})$, and again using the form from Lemma 31 with the supremum on the outside, it follows that

$$\gamma_\epsilon(\hat{\mu}) \geq \int (A\lambda) p_\epsilon^{\hat{\mu}} d\hat{\mu}. \quad (6)$$

Now recall the exact form of p_ϵ^μ and $p_\epsilon^{\hat{\mu}}$ as provided by Lemma 41 (and more specifically Lemma 40), which are both exactly $1/\epsilon$ up to some point, within $[0, 1/\epsilon]$ at that point (potentially distinct for $p_\epsilon^{\hat{\mu}}$ and p_ϵ^μ), and zero thereafter; if $\|p_\epsilon^\mu\|_{L^1(\hat{\mu})} \geq 1$, then

$$\int |p_\epsilon^\mu - p_\epsilon^{\hat{\mu}}| d\hat{\mu} = \int p_\epsilon^\mu d\hat{\mu} - \int p_\epsilon^{\hat{\mu}} d\hat{\mu} = \int p_\epsilon^\mu d\hat{\mu} - 1,$$

whereas $\|p_\epsilon^\mu\|_{L^1(\hat{\mu})} \leq 1$ implies

$$\int |p_\epsilon^\mu - p_\epsilon^{\hat{\mu}}| d\hat{\mu} = \int p_\epsilon^{\hat{\mu}} d\hat{\mu} - \int p_\epsilon^\mu d\hat{\mu} = 1 - \int p_\epsilon^\mu d\hat{\mu}.$$

In either case, using as usual the fact $\sup_{x,y} |(A\lambda)_{x,y}| \leq \|\lambda\|_1 \leq 1$, and additionally the controls on $\|p_\epsilon^{\hat{\mu}}\|_{L^1(\hat{\mu})}$ from above,

$$\begin{aligned} \left| \int (A\lambda) p_\epsilon^{\hat{\mu}} d\hat{\mu} - \int (A\lambda) p_\epsilon^\mu d\hat{\mu} \right| &\leq \left| \int p_\epsilon^{\hat{\mu}} d\hat{\mu} - \int p_\epsilon^\mu d\hat{\mu} \right| \\ &\leq \int |p_\epsilon^{\hat{\mu}} - p_\epsilon^\mu| d\hat{\mu} \\ &\leq \frac{1}{\epsilon} \sqrt{\frac{1}{2m} \ln \left(\frac{2}{\delta} \right)}. \end{aligned}$$

Combining this with eqs. (5) and (6),

$$\begin{aligned} \gamma_\epsilon(\hat{\mu}) &\geq \int (A\lambda) p_\epsilon^{\hat{\mu}} d\hat{\mu} \\ &\geq \int (A\lambda) p_\epsilon^\mu d\hat{\mu} - \frac{1}{\epsilon} \sqrt{\frac{1}{2m} \ln \left(\frac{2}{\delta} \right)} \\ &\geq \gamma_\epsilon(\mu) - \sigma - \frac{1}{\epsilon} \sqrt{\frac{1}{2m} \ln \left(\frac{2}{\delta} \right)}. \end{aligned}$$

Since $\sigma > 0$ was arbitrary, the result follows. ■

H.2. Other Results

Lemma 42 *Let ν be a probability measure on $\mathcal{X} \times \{-1, +1\}$. If $0 \leq \epsilon_1 \leq \epsilon_2 \leq 1$, then $0 \leq \gamma_{\epsilon_1}(\nu) \leq \gamma_{\epsilon_2}(\nu) \leq 1$.*

Proof Let $0 \leq \epsilon_1 \leq \epsilon_2 \leq 1$ be given; then $\mathcal{D}_{\epsilon_1}(\nu) \supseteq \mathcal{D}_{\epsilon_2}(\nu)$ by definition, and thus $\gamma_{\epsilon_1}(\nu) \leq \gamma_{\epsilon_2}(\nu)$. Next, $\gamma_{\epsilon_1}(\nu) \geq 0$ follows by Lemma 31 since $\|A^\top p\|_\infty \geq 0$, or by considering the effect of the primal player choosing $\lambda = 0 \in \Lambda$. For the upper bound, since $\sup_{h,x} |h(x)| \leq 1$, then $\gamma_{\epsilon_2}(\nu) \leq \sup \{\|\lambda\|_1 : \lambda \in \Lambda, \|\lambda\|_1 \leq 1\} \leq 1$. ■

Proof (of Proposition 9) Since every $p \in L^1(\nu)$ with $\|p\|_1 = 1$ and $p \geq 0$ ν -a.e. defines a probability measure $p d\nu$ (ignoring a ν -null set which does not affect that value of integration with respect to ν), and since $\|\pm \mathbf{e}_h\|_1 = 1$ for every $h \in \mathcal{H}$,

$$\begin{aligned} \gamma &\leq \inf \left\{ \sup_{h \in \mathcal{H}} \left| \int y h(x) d\xi(x, y) \right| : \xi \text{ is a Borel probability measure over } \mathcal{X} \times \{-1, +1\} \right\} \\ &\leq \inf \left\{ \sup_{\|\lambda\|_1 \leq 1} \int (A\lambda)_{x,y} p(x, y) d\nu(x, y) : p \in L^1(\nu), \|p\|_1 = 1, p \geq 0 \text{ } \nu\text{-a.e.} \right\} \\ &= \gamma_0(\mu). \end{aligned}$$

For $\gamma(\nu) \leq \gamma_0(\nu)$ with ν a discrete measure over a finite set, the proof is as above (indeed with a tiny refinement, since in this case both $\gamma(\nu)$ and $\gamma_0(\nu)$ consider the same set of weightings over ν). \blacksquare

Proof (of Proposition 10) For convenience, define $I_i := (1/(i+1), i]$. This proof will proceed by establishing, for every k , a bounded weighting p_k , which will establish an upper bound on $\gamma_\epsilon(\mu)$ for some $\epsilon > 0$ which is a function of k . The result will then follow for $\gamma_0(\mu)$ by the monotonicity of $\gamma_\epsilon(\mu)$ as a function of μ (cf. Lemma 42), and result for $\gamma_0(\hat{\mu})$ will use deviation bounds on $\gamma_\epsilon(\mu)$ and again the monotonicity property.

Define p_k to be positive over intervals I_i with $1 \leq i \leq 2k$, and zero elsewhere as follows. For any $x \in I_i$, $p_k(x, y) = i(i+1)/(2k)$. By this choice,

$$\int_{I_i} p(x, y) d\mu(x, y) = \frac{i(i+1)}{2k} \left(\frac{1}{i} - \frac{1}{i+1} \right) = \frac{1}{2k};$$

It follows that $\|p\|_1 = 1$, $\|p\|_\infty = 2k+1$, and p_k makes μ look like the uniform distribution over k consecutive intervals.

Now consider any hypothesis $h \in \mathcal{H}$, with some threshold r . If r lies outside this set of intervals, then h is equally correct and incorrect, thus $\int y h(x) p(x, y) d\mu(x, y) = 0$. Otherwise, suppose there are a intervals before the threshold, and b intervals after it; h must be incorrect on at least $a/2 - 1$ of the left intervals, and $b/2 - 1$ of the right intervals; since $2k - 1 \leq a + b \leq 2k$ (this proof is charitable), thus at least $k - 2$ intervals are predicted completely incorrectly. Consequently,

$$\int y h(x) p(x, y) d\mu(x, y) \leq \frac{k+2}{2k} - \frac{k-2}{2k} = \frac{2}{k}.$$

Thus the form of $\gamma_\epsilon(\nu)$ from Lemma 31 provides $\gamma_{1/2k}(\mu) \leq 2/k$, and so $\gamma_0(\mu) = 0$ by monotonicity (Lemma 42), and $\gamma \leq 0$ by Proposition 9.

The remainder of the proof considers finite sample effects. Let $\delta > 0$, and a sample of size $m \geq 2$ be given. Choose integer $k := \lfloor m^{1/4} / (3\sqrt{2 \ln(4/\delta)}) \rfloor$ (where the lower bound on m provides $k \geq 1$), and consider the behavior of density p_k , defined as above. Note firstly that $\|p_k\|_\infty \leq 2k+1 \leq 3k \leq m^{1/4} / \sqrt{2 \ln(4/\delta)}$. Next, with probability at least $1 - \delta/2$, Hoeffding's bound grants

$$\left| \int p d\hat{\mu} - \int p d\mu \right| \leq \|p\|_\infty \sqrt{\frac{1}{2m} \ln \left(\frac{4}{\delta} \right)} \leq \frac{1}{2m^{1/4}} \leq \frac{1}{2}.$$

Now define $p'_k := p_k / (\int p_k d\hat{\mu})$, which means $\int p'_k d\hat{\mu} = 1$, and furthermore $\|p'_k\|_\infty \leq 2\|p_k\|_\infty$. Since \mathcal{H} has VC dimension $\mathcal{V}(\mathcal{H}) = 2$, Lemma 38 grants, with probability at least $1 - \delta/2$,

$$\begin{aligned} \sup_{h \in \mathcal{H}} \left| \int y h(x) p'_k(x, y) d\hat{\mu}(x, y) - \int y h(x) p'_k(x, y) d\mu(x, y) \right| \\ \leq \frac{2\|p'_k\|_\infty}{\sqrt{m}} \left(4\sqrt{\ln(m+1)} + \sqrt{2\ln(2/\delta)} \right). \end{aligned}$$

Now set $\epsilon := 1/\|p'_k\|_\infty \geq \sqrt{2\ln(4/\delta)}/(2m^{1/4})$. Then $p'_k \in \mathcal{D}_\epsilon(\hat{\mu})$, and the above computations provide

$$\gamma_\epsilon(\hat{\mu}) \leq \frac{16(\sqrt{\ln(m+1)} + 1)}{m^{1/4}} + \frac{2}{k} \leq \frac{16(\sqrt{\ln(m+1)} + 1)}{m^{1/4}} + \frac{2}{m^{1/4}/(3\sqrt{2\ln(4/\delta)}) - 1},$$

and lastly Lemma 42 and Proposition 9 grant $\gamma(\hat{\mu}) \leq \gamma_0(\hat{\mu}) \leq \gamma_\epsilon(\hat{\mu})$. \blacksquare

Proof (of Proposition 11)

1. This proof will proceed by establishing the contrapositive twice, and then using the fact that $\bar{\mathcal{L}} \geq 0$ and $\gamma_\epsilon(\mu) \geq 0$.

If $\bar{\mathcal{L}} > 0$, then there must exist a nonzero dual feasible point to the dual of \mathcal{L} in Lemma 34, since Lemma 28 grants that $\ell^*(0) = 0$. This nonzero dual feasible point p satisfies $p \in L^\infty(\mu)$ by the form of the duality problem, and thus $\tilde{p} := p/\|p\|_1$ also has $\tilde{p} \in L^\infty(\mu)$. The dual constraint provides $\|A^\top p\|_\infty = 0$, thus $\|A^\top \tilde{p}\|_\infty = 0$, and so Lemma 22 grants $\gamma_\epsilon(\mu) = 0$ with the choice $\epsilon = 1/\|\tilde{p}\|_\infty$ (and $\epsilon \leq 1$ since $\|\tilde{p}\|_1 = 1$ and μ a probability measure means $\|\tilde{p}\|_\infty \geq 1$).

On the other hand, if there exists ϵ so that $\gamma_\epsilon(\mu) = 0$, then attainment in the duality formula in Lemma 31 provides the existence of $p \in L^\infty(\mu)$ with $\|p\|_1 = 1$ and $\|A^\top p\|_\infty = 0$. By Lemma 28, there exists $c > 0$ so that ℓ^* is strictly negative along $(0, c)$. Consequently, $\tilde{p} := cp/\|p\|_\infty$ satisfies $\|\tilde{p}\|_1 > 0$, and $\tilde{p} \in (0, c)$ μ -a.e., thus $\ell^*(\tilde{p}) < 0$ μ -a.e., and also $\|A^\top \tilde{p}\|_\infty = 0$; together, it follows that $\bar{\mathcal{L}} \geq -\int \ell^*(\tilde{p}) > 0$ as desired.

2. This result is the same as Lemma 39. \blacksquare

Proof (of Lemma 12) This result is the combination of Lemma 31 and Lemma 22. \blacksquare

H.3. Optimization Guarantees

Note that the following proof does not overtly use convexity; convexity however is used both algorithmically by the line searches (otherwise they are not efficient), and for their guarantees (cf. Lemmas 36 and 37).

Proof (of Lemma 13) Consider any $0 \leq t \leq T - 1$. Since $\ell \in \mathbb{L}_{\text{lg}}$,

$$\|\nabla \widehat{\mathcal{L}}(A\lambda_t)\|_1 \geq \frac{1}{m} \sum_{\substack{i \in [m] \\ (A\lambda_t)_{x_i, y_i} \geq 0}} \ell'((A\lambda_t)_{x_i, y_i}) \geq \epsilon_t \beta_1.$$

Combining this with the fact that $\ell' \in [0, \beta_2]$, the vector $p_t := \nabla \widehat{\mathcal{L}}(A\lambda_t) / \|\nabla \widehat{\mathcal{L}}(A\lambda_t)\|_1$ satisfies $\|p_t\|_1 = 1$ and

$$\|p_t\|_\infty \leq \frac{\beta_2}{\|\nabla \widehat{\mathcal{L}}(A\lambda_t)\|_1} \leq \frac{\beta_2}{\epsilon_t \beta_1} = \frac{1}{\epsilon'_t},$$

where ϵ'_t is as provided in the statement (and $\epsilon'_t \leq 1$ since $\epsilon_t \leq 1$ and $\beta_1 \leq \beta_2$). Recalling the dual form $\gamma_{\epsilon'_t}(\widehat{\mu}) = \min\{\|A^\top p\|_\infty : p \in \mathcal{D}_{\epsilon'_t}(\widehat{\mu})\}$ from Lemma 31, and noting that $p_t \in \mathcal{D}_{\epsilon'_t}(\widehat{\mu})$,

$$\|A^\top \nabla \widehat{\mathcal{L}}(A\lambda_t)\|_\infty = \|\nabla \widehat{\mathcal{L}}(A\lambda_t)\|_1 \|A^\top p_t\|_\infty \geq \epsilon_t \beta_1 \gamma_{\epsilon'_t}(\widehat{\mu}).$$

Plugging this into the single-step guarantees from the three line search choices (cf. Lemmas 36 and 37),

$$\begin{aligned} \widehat{\mathcal{L}}(A\lambda_t) &\leq \widehat{\mathcal{L}}(A\lambda_{t-1}) - \frac{\rho^2 \|A^\top \nabla \widehat{\mathcal{L}}(A\lambda_{t-1})\|_\infty^2}{6B_2} \\ &\leq \widehat{\mathcal{L}}(A\lambda_{t-1}) - \frac{(\rho \beta_1 \epsilon_{t-1} \gamma_{\epsilon'_{t-1}}(\widehat{\mu}))^2}{6B_2}. \end{aligned}$$

The desired result comes by summing across all iterations and noting $\widehat{\mathcal{L}}(A\lambda_0) = \widehat{\mathcal{L}}(0) = \ell(0)$. \blacksquare

H.4. Statistical Guarantees

Proof (of Theorem 7) The first step of the proof is to show $\widehat{\mathcal{R}}(H\hat{\lambda}) \leq \epsilon$. Thus consider the case that every iteration has $\widehat{\mathcal{R}}(H\lambda_t) > \epsilon$; by the monotonicity of $\gamma_\epsilon(\mu)$ (cf. Lemma 42), positivity of $\gamma_\epsilon(\mu)$ (cf. Proposition 11), together with the bound on m (and the deviations on $\gamma_\epsilon(\mu)$ in Proposition 11), with probability at least $1 - \delta/2$,

$$\gamma_{\epsilon'_t}(\widehat{\mu}) \geq \gamma_{\epsilon'}(\widehat{\mu}) \geq \gamma_{\epsilon'}(\mu) - \frac{1}{\epsilon} \sqrt{\frac{1}{2m} \ln \left(\frac{4}{\delta} \right)} \geq \frac{\gamma_{\epsilon'}(\mu)}{2} > 0,$$

where the last equality is also by Proposition 11. Thus, by Lemma 13, and the monotonicity of $\gamma_\epsilon(\mu)$ in ϵ , and the second lower bound on m (and thus on m^a),

$$\begin{aligned} \widehat{\mathcal{L}}(A\lambda_T) &\leq \ell(0) - \sum_{t=1}^T \frac{(\rho \beta_1 \epsilon_{t-1} \gamma_{\epsilon'_{t-1}}(\widehat{\mu}))^2}{6B_2} \\ &\leq \ell(0) - \frac{m^a (\rho \beta_1 \epsilon \gamma_{\epsilon'}(\mu))^2}{24B_2} \\ &\leq 0, \end{aligned}$$

a contradiction since ℓ is nonnegative, and moreover positive on regions where it makes mistakes, therefore the above indicates $\widehat{\mathcal{R}}(A\lambda_T) = 0 < \epsilon$. As such, thanks to the final step of Algorithm 1 picking out the iterate with lowest classification error, $\widehat{\mathcal{R}}(H\hat{\lambda}) \leq \epsilon$.

What remains is to establish a deviation inequality. Let \mathcal{H}_t denote the hypothesis class used by predictor λ_t (i.e., $\mathcal{H}_t = \{\sum_{i=1}^t c_i h_i : c_i \in \mathbb{R}, h_i \in \mathcal{H}\}$), and let $\mathcal{S}_{\mathcal{H}_t}(m)$ denote the corresponding shatter coefficient when \mathcal{H}_t is applied to the sample of size m (Boucheron et al., 2005, Section 3). It follows (Schapire and Freund, 2012, Lemma 4.5) that

$$\mathcal{S}_{\mathcal{H}_t}(m) \leq \left(\frac{em}{t}\right)^t \left(\frac{em}{\mathcal{V}(\mathcal{H})}\right)^{t\mathcal{V}(\mathcal{H})}.$$

Plugging this and $t \leq m^a$ into an appropriate VC theorem and simplifying (Boucheron et al., 2005, Theorem 5.1 and subsequent discussion), with probability at least $1 - \delta/2$,

$$\begin{aligned} \mathcal{R}(A\hat{\lambda}) &\leq \epsilon + 2\sqrt{\epsilon \frac{\ln(\mathcal{S}_{\mathcal{H}_t}(2m)) + \ln(8/\delta)}{m}} + 4\frac{\ln(\mathcal{S}_{\mathcal{H}_t}(2m)) + \ln(8/\delta)}{m} \\ &\leq \epsilon + 2\sqrt{\epsilon \frac{(\mathcal{V}(\mathcal{H}) + 1) \ln(2em) + \ln(8/\delta)}{m^{1-a}}} + 4\frac{(\mathcal{V}(\mathcal{H}) + 1) \ln(2em) + \ln(8/\delta)}{m^{1-a}}. \end{aligned}$$

■

Appendix I. Deferred Material from Section 5

I.1. Proof of Proposition 14

Lemma 43 *Let loss $\ell \in \mathbb{L}_{\text{lg}}$ and any $g \in \partial\ell(0)$ be given.*

1. *The restriction of ℓ^* to $[0, g]$, denoted $\ell^*_{[0, g]}$, is a (decreasing) bijection between $[0, g]$ and $[-\ell(0), 0]$.*
2. *Let $f(z) := (\ell^*_{[0, g]})^{-1}(z)$ denote the inverse of $\ell^*_{[0, g]}$. If ν is a probability measure, and $p \in L^\infty(\nu)$, and $r := -\int \ell^*(p) \in [0, \ell(0)]$, then $\nu([p \geq f(-r)]) \geq f(-r)$, with $f(-r) > 0$ iff $r > 0$.*

Proof Choose any $g \in \partial\ell(0)$; by Lemma 28, ℓ^* is 0 at 0, negative along $(0, g)$, and attains its minimum at g . Since ℓ^* is finite for every $z \in (0, g)$, then $\partial\ell^*(z)$ exists (Rockafellar, 1970, Theorem 23.4), and every $x \in \partial\ell^*(z)$ satisfies $z \in \nabla\ell(x)$ (Rockafellar, 1970, Theorem 23.5), and so ℓ^* is strictly convex along (Hiriart-Urruty and Lemaréchal, 2001, Theorem E.4.1.2), meaning ℓ^* is injective along $[0, g]$. Since $\ell^*(0) = 0$, and $-\ell^*(g) = \ell(0)$ (by the Fenchel-Young inequality), and since ℓ^* is lower semi-continuous (Rockafellar, 1970, Theorem 12.2), then ℓ^* is also surjective from $[0, g]$ to $[-\ell(0), 0]$.

Now let f denote the inverse map (from $[-\ell(0), 0]$ to $[0, g]$), and let probability measure ν , function $p \in L^\infty(\nu)$, and scalar $r := -\int \ell^*(p) \in [0, \ell(0)]$ be given (where the containment provides $f(-r)$ is valid). By Jensen's inequality,

$$-r = \int \ell^*(p) \geq \ell^*\left(\int p\right);$$

since f is a decreasing map, this implies $\int p \geq f(-r)$. Furthermore,

$$f(-r) \leq \int p \leq \int_{[p < f(-r)/2]} \frac{f(-r)}{2} + \int_{[p \geq f(-r)/2]} 1 \leq \frac{f(-r)}{2} + \nu([p \geq f(-r)/2]),$$

meaning $\nu([p \geq f(-r)/2]) \geq f(-r)/2$ as desired.

Lastly, the statement $f(-r) > 0$ iff $r > 0$ follows from the bijectivity of $\ell_{[0,g]}^*$. \blacksquare

Proof (of Proposition 14) The basic duality relation is provided by Lemma 34. Since the optimal value satisfies $\bar{\mathcal{L}}_\nu \in [0, \ell(0)]$ (since ν is a probability measure, $0 \in \Lambda$ is primal feasible, and $\ell \geq 0$), then Lemma 43 may be applied with parameter p being the dual optimum and parameter r being the corresponding objective value $\bar{\mathcal{L}}_\nu$. \blacksquare

I.2. Proof of Lemma 17

The first step is to use \bar{p} to show that if λ has low error and norm, then $H\lambda$ will have very small margins over some positive measure set.

Lemma 44 *Let convex differentiable $\ell : \mathbb{R} \rightarrow \mathbb{R}_+$ with $\ell'(0) > 0$, any class \mathcal{H} , and any probability measure ν over $\mathcal{X} \times \{-1, +1\}$ with empirical counterpart $\hat{\mu}$ be given. Suppose the following quantities and constants exist.*

1. *Suppose there exists $p \in L^\infty(\mu)$ with $p \in [0, \|p\|_\infty]$ $\hat{\mu}$ -a.e., and that there exists $\tau > 0$ with $\hat{\mu}([p \geq \tau]) \geq \tau/2$.*
2. *Set*

$$c := \frac{16\ell(0)}{\tau\ell'(0)} \max \left\{ 1, \frac{1}{\tau} \right\} \max \{1, \|p\|_\infty\},$$

and suppose there exist $C > 0$ and $D \leq c\|p\|_\infty\tau^2/8$ so that every $\lambda \in \Lambda$ with $\|\lambda\|_1 \leq C$ satisfies $|\int (A\lambda)p d\hat{\mu}| \leq D$.

If $\lambda \in \Lambda$ satisfies $\|\lambda\|_1 \leq C$ and $\hat{\mu}(|H\lambda| \geq c) \geq 1 - \tau/8$, then $\widehat{\mathcal{L}}(A\lambda) \geq 2\widehat{\mathcal{L}}(A\lambda_0) = 2\ell(0)$.

Proof (of Lemma 44) First consider the case that $\hat{\mu}([y(H\lambda)_x \leq -c]) = \hat{\mu}([A\lambda \geq c]) \geq \tau/8$. By subgradient rules for convex functions, since $\ell \geq 0$ and $\ell'(0) > 0$, and using the definition of c ,

$$\begin{aligned} \int \ell(A\lambda) d\hat{\mu} &\geq \int_{[y(H\lambda)_x \leq -c]} \ell(A\lambda) d\hat{\mu} \\ &\geq \int_{[y(H\lambda)_x \leq -c]} (\ell(0) + \ell'(0)(A\lambda)) d\hat{\mu} \\ &\geq \frac{\ell'(0)c\tau}{8} \\ &\geq 2\ell(0), \end{aligned}$$

meaning $\widehat{\mathcal{L}}(A\lambda) \geq 2\ell(0) = 2\widehat{\mathcal{L}}(0) = 2\mathcal{L}(A\lambda_0)$.

Now consider the remaining possibility that $\hat{\mu}([y(H\lambda)_x \leq -c]) < \tau/8$. Since by assumption $\hat{\mu}([|y(H\lambda)_x| \geq c]) \geq 1 - \tau/8$, it follows that $\hat{\mu}([y(H\lambda)_x \geq c]) \geq 1 - \tau/4$. In turn, it also holds that

$$\hat{\mu}([y(H\lambda)_x \geq c] \cap [p \geq \tau]) \geq \tau/4.$$

Next, the definition of D provides that

$$\begin{aligned} D &\geq \left| \int A\lambda p d\hat{\mu} \right| \\ &\geq \int_{[y(H\lambda)_x \leq 0]} y(H\lambda)_x p(x, y) d\hat{\mu}(x, y) + \int_{[y(H\lambda)_x > 0]} y(H\lambda)_x p(x, y) d\hat{\mu}(x, y), \end{aligned}$$

meaning

$$\int_{[y(H\lambda)_x \leq 0]} -y(H\lambda)_x p(x, y) d\hat{\mu}(x, y) \geq \int_{[y(H\lambda)_x > 0]} y(H\lambda)_x p(x, y) d\hat{\mu}(x, y) - D.$$

As such, since $p \in [0, \|p\|_\infty]$ over the sample,

$$\begin{aligned} -\|p\|_\infty \int_{[y(H\lambda)_x \leq 0]} y(H\lambda)_x d\hat{\mu}(x, y) &\geq \int_{[y(H\lambda)_x \leq 0]} -y(H\lambda)_x p(x, y) d\hat{\mu}(x, y) \\ &\geq \int_{[y(H\lambda)_x > 0]} y(H\lambda)_x p(x, y) d\hat{\mu}(x, y) - D \\ &\geq \int_{[y(H\lambda)_x \geq c] \cap [p \geq \tau]} y(H\lambda)_x p(x, y) d\hat{\mu}(x, y) - D \\ &\geq \frac{c\tau^2}{4} - D. \end{aligned}$$

Turning back to $\hat{\mathcal{L}}$ and proceeding similarly to the earlier case,

$$\begin{aligned} \int \ell(A\lambda) d\hat{\mu} &\geq \int_{[y(H\lambda)_x \leq 0]} \ell(A\lambda) d\hat{\mu} \\ &\geq \int_{[y(H\lambda)_x \leq 0]} (\ell(0) + \ell'(0)(A\lambda)) d\hat{\mu} \\ &\geq \frac{\ell'(0)}{\|p\|_\infty} \left(\frac{c\tau^2}{4} - D \right) \\ &\geq 2\ell(0), \end{aligned}$$

which again yields $\hat{\mathcal{L}}(A\lambda) \geq 2\ell(0) = 2\hat{\mathcal{L}}(0) = 2\hat{\mathcal{L}}(A\lambda_0)$. ■

Next, these small margins in turn cause the line search to not look too far, meaning the next iterate will also have some small margins. Note that \mathcal{H} is assumed binary; this is in order to changes in $H\lambda$ to changes in λ .

Lemma 45 *Let convex $\ell : \mathbb{R} \rightarrow \mathbb{R}_+$, binary \mathcal{H} , and probability measure μ with empirical counterpart $\hat{\mu}$ be given. Let positive reals C, c, τ be given so that $\lambda \in \Lambda$ with $\|\lambda_1\|_1 \leq C + 2c$*

and $\hat{\mu}(|H\lambda| \geq c) \geq 1 - \tau/8$ implies $\hat{\mathcal{L}}(A\lambda) \geq 2\ell(0)$. Then for any $\lambda \in \Lambda$ with $\|\lambda\|_1 \leq C$ and $\mathcal{L}(A\lambda) \leq \ell(0)$, the set of line search candidates

$$S_\lambda := \left\{ \lambda' \in \Lambda : \hat{\mathcal{L}}(A\lambda') < 2\ell(0), \exists \alpha \in \mathbb{R}, h \in \mathcal{H} \cdot \lambda' := \lambda + \alpha \mathbf{e}_h \right\}$$

satisfies $\hat{\mu}(|H\lambda'| \geq c) < 1 - \tau/8$ for every $\lambda' \in S_\lambda$.

Proof Let λ with $\|\lambda\|_1 \leq C$ be given, and consider any λ' of the form $\lambda' := \lambda + \alpha' \mathbf{e}_h$ for some $\alpha' \in \mathbb{R}$ and $h \in \mathcal{H}$. The desired statement will be shown by contrapositive; namely, $\hat{\mu}(|H\lambda'| \geq c) \geq 1 - \tau/8$ implies $\lambda' \notin S_\lambda$.

For any example (x, y) , since H has binary predictors, the map $\alpha \mapsto \mathbb{1}[y(H(\lambda + \alpha \mathbf{e}_h))_x \geq c]$ is constant for $\alpha > 2c$ and for $\alpha < -2c$; consequently, since $\hat{\mu}$ is a discrete measure over a finite set, the map $\alpha \mapsto \hat{\mu}(|y((H(\lambda + \alpha \mathbf{e}_h)))_x| \geq c)$ is also constant for $\alpha > 2c$ and $\alpha < -2c$. As such, the existence of λ' as above implies the existence of $\lambda'' := \lambda + \alpha'' \mathbf{e}_h$ where $h \in \mathcal{H}$ is as before, α'' and α' have the same sign, and $|\alpha''| \leq \min\{2c, |\alpha'|\}$; in other words, λ'' is along the path from λ to λ' , but moreover satisfies $\|\lambda - \lambda''\|_1 \leq 2c$. But this means $\|\lambda''\| \leq \|\lambda\|_1 + \|\lambda - \lambda''\|_1 \leq C + 2c$, whereby the stated assumptions combined with $\hat{\mu}(|H\lambda'| \geq c) \geq 1 - \tau/8$ provide $\hat{\mathcal{L}}(A\lambda'') \geq 2\ell(0)$, thus $\lambda'' \notin S_\lambda$. Furthermore, since λ'' is along the path from λ to λ'' , and $\hat{\mathcal{L}}(A\lambda) \leq \ell(0)$ and $\hat{\mathcal{L}}(A\lambda'') \geq 2\ell(0) \geq \hat{\mathcal{L}}(A\lambda)$, it follows by convexity that $\hat{\mathcal{L}}(A\lambda') \geq 2\ell(0)$, and thus $\lambda' \notin S_\lambda$ as well. \blacksquare

These small margin controls directly give a bound on step sizes.

Lemma 46 (see also [Bartlett and Traskin \(2007, eq. \(28\)\)](#)) *Let $\ell \in \mathbb{L}_{\text{lg}} \cap \mathbb{L}_{2\text{d}}$ be given with Lipschitz gradient parameter B_2 , binary class \mathcal{H} , time horizon t , and empirical probability measure $\hat{\mu}$ corresponding to a sample of size m be given. Let positive real $B_1 > 0$ be given so that for any $\lambda \in \Lambda$ with*

$$\|\lambda\|_1 \leq \sqrt{\frac{t \max\{5, 2B_2/B_1\}(\hat{\mathcal{L}}(A\lambda_0) - \hat{\mathcal{L}}(A\lambda_t))}{\rho^2 B_1}},$$

and any line search candidate $\lambda' := \lambda + \alpha \mathbf{e}_h$ for $h \in \mathcal{H}$, $\alpha \in \mathbb{R}$, and satisfying $\hat{\mathcal{L}}(A\lambda') \leq \hat{\mathcal{L}}(A\lambda)$, then $B_1 \leq \frac{1}{m} \sum_{i=1} \ell''((A\lambda')_i)$. The following properties hold.

1. *For every integer $0 \leq i < t$, an optimal step $\bar{\alpha}_i$ exists, and every step sizes choice satisfies*

$$\alpha_i^2 \leq \min \left\{ \frac{9\|A^\top \nabla \hat{\mathcal{L}}(A\lambda_{i-1})\|_\infty^2}{4\rho^2 B_1^2}, \frac{\max\{5, 2B_2/B_1\}(\hat{\mathcal{L}}(A\lambda_{i-1}) - \hat{\mathcal{L}}(A\lambda_i))}{\rho^2 B_1} \right\}.$$

2. *For every integer $0 \leq i < t$ and any sequence of step size choices,*

$$\begin{aligned} \|\lambda_i\|_1 &\leq \sqrt{i} \sqrt{\sum_{j=1}^i \alpha_j^2} \leq \sqrt{i} \sqrt{\frac{\max\{5, 2B_2/B_1\}(\hat{\mathcal{L}}(A\lambda_0) - \hat{\mathcal{L}}(A\lambda_i))}{\rho^2 B_1}} \\ &\leq \sqrt{i} \sqrt{\frac{\ell(0) \max\{5, 2B_2/B_1\}}{\rho^2 B_1}}. \end{aligned}$$

Proof This proof establishes both properties simultaneously by induction on i . In the base case $\lambda_i = \lambda_0 = 0$ and there is nothing to show, thus suppose $i \geq 1$.

Define the interval $I_i := \{\alpha \geq 0 : \widehat{\mathcal{L}}(A(\lambda_{i-1} + \alpha v_i)) \leq \widehat{\mathcal{L}}(A\lambda_{i-1})\}$. Combining the inductive hypothesis (controlling $\|\lambda_{i-1}\|_1$) with the assumptions on line search candidates means the second-order lower bound B_1 is active along I_i . Now consider a Taylor expansion of $\widehat{\mathcal{L}}$, but in the direction reverse to Lemma 36, and using the fact that H is binary; then for any $\alpha \in I_i$ and some $z \in [\lambda_{i-1}, \lambda_{i-1} + \alpha v_i]$,

$$\begin{aligned} \widehat{\mathcal{L}}(A(\lambda_{i-1} + \alpha v_i)) &= \widehat{\mathcal{L}}(A\lambda_{i-1}) + \alpha \left\langle A^\top \widehat{\mathcal{L}}(A\lambda_{i-1}), v_i \right\rangle + \frac{\alpha^2}{2m} \sum_i \ell''((Az)_i) (Av_i)^2 \\ &\geq \widehat{\mathcal{L}}(A\lambda_{i-1}) - \alpha \|A^\top \widehat{\mathcal{L}}(A\lambda_{i-1})\|_\infty + \frac{\alpha^2}{2} B_1. \end{aligned}$$

This last expression defines a univariate quadratic which lies below $\widehat{\mathcal{L}}(A(\lambda_{i-1} + \alpha v_i))$ along I_i (outside of I_i , the constraints granting the lower bound B_1 may be violated). Consequently, I_i is bounded, and the optimal step $\bar{\alpha}_i$ must exist, and moreover satisfies

$$\bar{\alpha}_i \leq \alpha_* := \frac{\|A^\top \widehat{\mathcal{L}}(A\lambda_{i-1})\|_\infty}{B_1}, \quad (7)$$

where α_* is the minimizer to the above quadratic. Plugging α_* back into the quadratic, for any $\alpha \in I_i$,

$$\widehat{\mathcal{L}}(A(\lambda_{i-1} + \alpha v_i)) \geq \widehat{\mathcal{L}}(A\lambda_{i-1}) - \frac{\|A^\top \widehat{\mathcal{L}}(A\lambda_{i-1})\|_\infty^2}{2B_1}. \quad (8)$$

Now consider the first two step size options in Algorithm 1; combining eq. (8) with Lemma 36,

$$\alpha_i^2 \leq \bar{\alpha}_i^2 \leq \frac{\|A^\top \widehat{\mathcal{L}}(A\lambda_{i-1})\|_\infty^2}{B_1^2} \leq \frac{2B_2(\widehat{\mathcal{L}}(A\lambda_{i-1}) - \widehat{\mathcal{L}}(A\lambda_i))}{\rho^2 B_1^2}$$

as desired.

For option 3 (the Wolfe search), combining eq. (2) with eq. (8) grants

$$\alpha_i^2 \leq \frac{9(\widehat{\mathcal{L}}(A\lambda_{i-1}) - \widehat{\mathcal{L}}(A\lambda_i))^2}{\rho^2 \|A^\top \nabla \widehat{\mathcal{L}}(A\lambda_{i-1})\|_\infty^2} \leq \frac{9(\widehat{\mathcal{L}}(A\lambda_{i-1}) - \widehat{\mathcal{L}}(A\lambda_i))}{2\rho^2 B_1} \leq \frac{9\|A^\top \nabla \widehat{\mathcal{L}}(A\lambda_{i-1})\|_\infty^2}{4\rho^2 B_1^2},$$

which establishes the first inductive property for all step sizes.

The second statement is just Cauchy-Schwarz combined with the bound on α_i^2 :

$$\begin{aligned}
\|\lambda_i\|_1 &\leq \sum_{j=1}^i \alpha_j \|v_j\|_1 \\
&\leq \sqrt{i} \sqrt{\sum_{j=1}^i \alpha_j^2} \\
&\leq \sqrt{i} \sqrt{\frac{\max\{5, 2B_2/B_1\} \sum_{j=1}^i (\widehat{\mathcal{L}}(A\lambda_{j-1}) - \widehat{\mathcal{L}}(A\lambda_j))}{\rho^2 B_1}} \\
&= \sqrt{i} \sqrt{\frac{\max\{5, 2B_2/B_1\} (\widehat{\mathcal{L}}(A\lambda_0) - \widehat{\mathcal{L}}(A\lambda_i))}{\rho^2 B_1}},
\end{aligned}$$

and nonnegativity of ℓ and the fact that all steps perform descent grants $\widehat{\mathcal{L}}(A\lambda_0) - \widehat{\mathcal{L}}(A\lambda_i) \leq \ell(0)$. \blacksquare

After some algebra, the upper bound also grants a lower bound; due to this indirection, it should be possible to improve this bound. Note that the beginning of this derivation, when initially lower bounding α_i , uses derivations similar to those used by [Zhang and Yu \(2005\)](#) and [Bartlett and Traskin \(2007\)](#).

Lemma 47 *Let $\ell \in \mathbb{L}_{\text{lg}} \cap \mathbb{L}_{\text{2d}}$ be given with Lipschitz gradient parameter B_2 , binary class \mathcal{H} , time horizon t , and empirical probability measure $\widehat{\mu}$ corresponding to a sample of size m be given. Suppose there exists $c_2 > 0$ with $\|\lambda_i\|_1 \leq c_2 \sqrt{i}$ for all $0 \leq i \leq t$. Additionally, let $C_2 > 0$ and $\bar{\lambda} \in \Lambda$ be given with $\|\bar{\lambda}\|_1 \leq C_2$ and $\epsilon := \min_{i \in [t-1]} \widehat{\mathcal{L}}(A\lambda_i) - \widehat{\mathcal{L}}(A\bar{\lambda}) \geq 0$. Then*

$$\alpha_i \geq \frac{\rho(\widehat{\mathcal{L}}(A\lambda_{i-1}) - \widehat{\mathcal{L}}(A\bar{\lambda}))}{2B_2(\|\bar{\lambda}\|_1 + c_2 \sqrt{i-1})}$$

and

$$\sum_{i=1}^t \alpha_i \geq \frac{\rho\epsilon}{2B_2 c_2} \left(2\sqrt{t-1} + \frac{c_2}{C_2} + \frac{2C_2}{c_2} \ln \left(\frac{C_2}{C_2 + c_2 \sqrt{t-1}} \right) \right),$$

or more simply $\sum_{i=1}^t \alpha_i \geq (\rho\epsilon\sqrt{t-1})/(4B_2 c_2)$ if $C_2 \leq c_2 \sqrt{t-1}$.

Proof For any $1 \leq i \leq t$, note by Lemma 22 that

$$\begin{aligned}
\|A^\top \nabla \widehat{\mathcal{L}}(A\lambda_{i-1})\|_\infty &= \sup_{\|\lambda\|_1 \leq 1} \left\langle A^\top \nabla \widehat{\mathcal{L}}(A\lambda_{i-1}), \lambda \right\rangle \\
&\geq \frac{1}{\|\lambda_{i-1} - \bar{\lambda}\|_1} \left\langle \nabla \widehat{\mathcal{L}}(A\lambda_{i-1}), A(\lambda_{i-1} - \bar{\lambda}) \right\rangle \\
&\geq \frac{1}{\|\lambda_{i-1} - \bar{\lambda}\|_1} (\widehat{\mathcal{L}}(A\lambda_{i-1}) - \widehat{\mathcal{L}}(A\bar{\lambda})).
\end{aligned}$$

Thus, by the lower bounds in Lemmas 36 and 37 for every step size,

$$\alpha_i \geq \frac{\rho \|A^\top \nabla \hat{\mathcal{L}}(A\lambda_{i-1})\|_\infty}{2B_2} \geq \frac{\rho(\hat{\mathcal{L}}(A\lambda_{i-1}) - \hat{\mathcal{L}}(A\bar{\lambda}))}{2B_2 \|\lambda_{i-1} - \bar{\lambda}\|_1} \geq \frac{\rho\epsilon}{2B_2 \|\lambda_{i-1} - \bar{\lambda}\|_1}.$$

Combining this with the provided upper bound on $\|\lambda_i\|_1$,

$$\alpha_i \geq \frac{\rho\epsilon}{2B_2(\|\bar{\lambda}\|_1 + \|\lambda_{i-1}\|_1)} \geq \frac{\rho\epsilon}{2B_2(\|\bar{\lambda}\|_1 + c_2\sqrt{i-1})}$$

As a consequence of this, and recalling the simplification $\|\bar{\lambda}\|_1 \leq C_2$,

$$\begin{aligned} \sum_{i=1}^t \alpha_i &\geq \frac{\rho\epsilon}{2B_2c_2} \left(\frac{c_2}{C_2} + \sum_{i=1}^{t-1} \frac{c_2}{C_2 + c_2\sqrt{i}} \right) \\ &\geq \frac{\rho\epsilon}{2B_2c_2} \left(\frac{c_2}{C_2} + \int_0^{t-1} \frac{c_2 dx}{C_2 + c_2\sqrt{x}} \right) \\ &= \frac{\rho\epsilon}{2B_2c_2} \left(\frac{c_2}{C_2} + \left(2\sqrt{x} - \frac{2C_2 \ln(\sqrt{x} + C_2/c_2)}{c_2} \right) \Big|_0^{t-1} \right) \\ &= \frac{\rho\epsilon}{2B_2c_2} \left(\frac{c_2}{C_2} + 2\sqrt{t-1} + \frac{2C_2}{c_2} \ln \left(\frac{C_2}{C_2 + c_2\sqrt{t-1}} \right) \right). \end{aligned}$$

When $C_2 \leq c_2\sqrt{t-1}$, it suffices to instantiate the above bound with $C'_2 := c_2\sqrt{t-1}$ (whereby it still holds that $\|\bar{\lambda}\|_1 \leq C_2 \leq C'_2$), and then rearrange, noting $1 - \ln(2) \geq 1/4$ and deleting the nonnegative standalone term c_2/C_2 . \blacksquare

By combining the above chain of results, the proof of Lemma 17 follows.

Proof (of Lemma 17) Recalling the structure from Proposition 14, let dual optimum \bar{p} and real $\tau > 0$ be given so that $\mu([\bar{p} \geq \tau]) \geq \tau$. By Hoeffding's inequality and the first lower bound on m , with probability at least $1 - \delta/4$

$$\hat{\mu}([\bar{p} \geq \tau]) \geq \mu([\bar{p} \geq \tau]) - \sqrt{\frac{1}{2m} \ln \left(\frac{4}{\delta} \right)} \geq \frac{\tau}{2}.$$

Henceforth disregard the corresponding failure event.

Next define

$$D := \frac{2(R_t + 2c)\|\bar{p}\|_\infty}{m^{1/2}} \left(2\sqrt{2\mathcal{V}(\mathcal{H}) \ln(m+1)} + \sqrt{2 \ln(4/\delta)} \right).$$

The second condition on m grants $D \leq c\tau^2/8$, whereby Lemma 16 grants, with probability at least $1 - \delta/4$, that every $\lambda \in \Lambda$ with $\|\lambda\|_1 \leq R_t + 4c$ satisfies

$$\left| \int (A\lambda) \bar{p} d\hat{\mu} \right| \leq D \leq \frac{c\tilde{\tau}^2}{8}.$$

Discard the corresponding failure event as well; unioning this with the earlier failure event, the remaining steps hold with probability at least $1 - \delta/2$.

It follows from Lemma 44 that every such $\lambda \in \Lambda$ with $\|\lambda\|_1 \leq R_t + 2c$ either satisfies $\widehat{\mu}(|H\lambda| \geq c) < 1 - \bar{\tau}/8$, or else $\widehat{\mathcal{L}}(A\lambda) \geq 2\ell(0)$. This in turn means the preconditions to Lemma 45 are met (with $C := R_t$), and in particular, for any $\lambda \in \Lambda$ with $\|\lambda\|_1 \leq R_t$ and $\widehat{\mathcal{L}}(A\lambda) \leq \widehat{\mathcal{L}}(A\lambda_0)$, the set of line search candidates

$$S_\lambda := \left\{ \lambda' \in \Lambda : \widehat{\mathcal{L}}(A\lambda') < 2\ell(0), \exists \alpha \in \mathbb{R}, h \in \mathcal{H} \bullet \lambda' := \lambda + \alpha \mathbf{e}_h \right\}$$

satisfies $\widehat{\mu}(|H\lambda'| \geq c) < 1 - \tau/8$ for every $\lambda' \in S_\lambda$. But then, for every $\lambda' \in S_\lambda$ (and note $\lambda \in S_\lambda$),

$$\frac{1}{m} \sum_{i \in [m]} \ell''((A\lambda)_{x_i, y_i}) \geq \frac{1}{m} \sum_{\substack{i \in [m] \\ |(A\lambda)_{x_i, y_i}| \leq c}} \ell''((A\lambda)_{x_i, y_i}) \geq \widehat{\mu}(|H\lambda| \leq c) \inf_{z \in [-c, +c]} \ell''(z) \geq B_1.$$

This establishes the first desired statement.

For the second statement (upper bounds on α_i and $\|\lambda_i\|_1$), note that the above properties satisfy the preconditions to Lemma 46, whereby the desired upper bounds follow.

Similarly, for the third statement (lower bounds on α_i and $\|\lambda_i\|_1$), the preconditions for Lemma 47 are now met. \blacksquare

I.3. Optimization Guarantees

As stated previously, the following proof is a reworking of a proof due to Zhang and Yu (2005), albeit with the present decoupling of line search and coordinate selection.

Proof (of Lemma 18) Let $1 \leq i \leq t$ be arbitrary. The first step of this proof is to develop two lower bounds on $\|A^\top \nabla \widehat{\mathcal{L}}(A\lambda_{i-1})\|_\infty$. First, just as in the proof of Lemma 47,

$$\begin{aligned} \|A^\top \nabla \widehat{\mathcal{L}}(A\lambda_{i-1})\|_\infty &= \sup_{\|\lambda\|_1 \leq 1} \left\langle A^\top \nabla \widehat{\mathcal{L}}(A\lambda_{i-1}), \lambda \right\rangle \\ &\geq \frac{1}{\|\lambda_{i-1} - \bar{\lambda}\|_1} \left\langle \nabla \widehat{\mathcal{L}}(A\lambda_{i-1}), A(\lambda_{i-1} - \bar{\lambda}) \right\rangle \\ &\geq \frac{1}{\|\lambda_{i-1} - \bar{\lambda}\|_1} (\widehat{\mathcal{L}}(A\lambda_{i-1}) - \widehat{\mathcal{L}}(A\bar{\lambda})). \end{aligned}$$

The second lower bound is provided by assumption, and thus, by the guarantee on any line search as in Lemmas 36 and 37,

$$\begin{aligned} \widehat{\mathcal{L}}(A\lambda_i) - \widehat{\mathcal{L}}(A\bar{\lambda}) &\leq \widehat{\mathcal{L}}(A\lambda_{i-1}) - \widehat{\mathcal{L}}(A\bar{\lambda}) - \frac{\rho^2 \|A^\top \nabla \widehat{\mathcal{L}}(A\lambda_{i-1})\|_\infty^2}{6B_2} \\ &\leq \widehat{\mathcal{L}}(A\lambda_{i-1}) - \widehat{\mathcal{L}}(A\bar{\lambda}) - \frac{c_3 \rho^2 \alpha_i (\widehat{\mathcal{L}}(A\lambda_{i-1}) - \widehat{\mathcal{L}}(A\bar{\lambda}))}{6B_2 \|\bar{\lambda} - \lambda_{i-1}\|_1} \\ &= \left(\widehat{\mathcal{L}}(A\lambda_{i-1}) - \widehat{\mathcal{L}}(A\bar{\lambda}) \right) \left(1 - \frac{c_3 \rho^2 \alpha_i}{6B_2 \|\bar{\lambda} - \lambda_{i-1}\|_1} \right) \\ &\leq \left(\widehat{\mathcal{L}}(A\lambda_{i-1}) - \widehat{\mathcal{L}}(A\bar{\lambda}) \right) \exp \left(- \frac{c_4 \alpha_i}{\|\bar{\lambda} - \lambda_{i-1}\|_1} \right). \end{aligned}$$

where the last step took $c_4 := c_3 \rho^2 / (6B_2)$ for convenience. Iterating this bound,

$$\widehat{\mathcal{L}}(A\lambda_t) - \widehat{\mathcal{L}}(A\bar{\lambda}) \leq \left(\widehat{\mathcal{L}}(A\lambda_0) - \widehat{\mathcal{L}}(A\bar{\lambda}) \right) \exp \left(-c_4 \sum_{i=1}^t \frac{\alpha_i}{\|\bar{\lambda} - \lambda_{i-1}\|_1} \right).$$

Focusing on the summation, define $S_i := \sum_{j \leq i} \alpha_j$ with $S_0 := 0$, whereby $\|\lambda_i\|_1 \leq S_i$. Using this (see also the similar derivation by [Zhang and Yu \(2005, Proof of Lemma 4.2\)](#)),

$$\begin{aligned} \sum_{i=1}^t \frac{\alpha_i}{\|\bar{\lambda}\|_1 + \|\lambda_{i-1}\|_1} &\geq \sum_{i=1}^t \frac{\alpha_i}{\|\bar{\lambda}\|_1 + S_{i-1}} \\ &= \sum_{i=1}^t \left(\frac{\|\bar{\lambda}\|_1 + S_i}{\|\bar{\lambda}\|_1 + S_{i-1}} - \frac{\|\bar{\lambda}\|_1 + S_{i-1}}{\|\bar{\lambda}\|_1 + S_{i-1}} \right) \\ &\geq \sum_{i=1}^t \ln \left(\frac{\|\bar{\lambda}\|_1 + S_i}{\|\bar{\lambda}\|_1 + S_{i-1}} \right) = \ln \left(\frac{\|\bar{\lambda}\|_1 + S_t}{\|\bar{\lambda}\|_1} \right). \end{aligned}$$

Plugging this into the preceding display and collecting terms, the result follows. \blacksquare

Note that the substitution $\|\lambda_i\|_1 \leq S_i$ at the end of the proof of Lemma 18 works around the fact that $\sum_i \alpha_i$ could be much larger than $\|\lambda_i\|_1$; this issue is frequently avoided in the literature by assuming that \mathcal{H} is closed under negation, whereby $v_i = \mathbf{e}_{h_i}$ in each round (i.e., rather than $v_i \in \{\pm \mathbf{e}_{h_i}\}$).

I.4. Statistical Guarantees

Proof (of Theorem 15) Let $\sigma > 0$ be arbitrary, and choose $\bar{\lambda} \in \Lambda$ with $\|\bar{\lambda}\|_1 \leq R_{t-1}$ so that

$$\mathcal{L}(A\bar{\lambda}) \leq \sigma + \inf_{\|\lambda\|_1 \leq R_{t-1}} \mathcal{L}(A\lambda).$$

By McDiarmid's inequality and the fact that $\sup_{x,y} |(A\bar{\lambda})_{x,y}| \leq R_{t-1}$, with probability at least $1 - \delta/6$,

$$\widehat{\mathcal{L}}(A\bar{\lambda}) \leq R_{t-1} \sqrt{\frac{2}{m} \ln \left(\frac{6}{\delta} \right)} + \mathcal{L}(A\bar{\lambda}) \leq \sigma + R_{t-1} \sqrt{\frac{2}{m} \ln \left(\frac{6}{\delta} \right)} + \inf_{\|\lambda\|_1 \leq R_{t-1}} \mathcal{L}(A\lambda).$$

Now let $b \in (0, 1/2)$ be arbitrary, and consider two cases for the difference $\epsilon := \min_{i \in [t-1]} \widehat{\mathcal{L}}(A\lambda_i) - \widehat{\mathcal{L}}(A\bar{\lambda})$.

- If $\epsilon \leq (t-1)^{-b}$, then

$$\widehat{\mathcal{L}}(A\lambda_t) \leq \widehat{\mathcal{L}}(A\lambda_{t-1}) \leq (t-1)^{-b} + \widehat{\mathcal{L}}(A\bar{\lambda}).$$

- Otherwise $\epsilon > (t-1)^{-b}$. Thus Lemma 17 grants, with probability at least $1 - \delta/2$,

$$\alpha_i^2 \leq \frac{9 \|A^\top \nabla \widehat{\mathcal{L}}(A\lambda_{i-1})\|_\infty^2}{4\rho^2 B_1^2} \quad \text{and} \quad \sum_{i=1}^t \alpha_i \geq \frac{\rho \epsilon \sqrt{t-1}}{4B_2 R_1},$$

which can then be plugged into Lemma 18 (with $c_3 := 2\rho B_1/3$), together with the lower bound on ϵ , to yield

$$\begin{aligned}\widehat{\mathcal{L}}(A\lambda_t) - \widehat{\mathcal{L}}(A\bar{\lambda}) &\leq \left(\widehat{\mathcal{L}}(A\lambda_0) - \widehat{\mathcal{L}}(A\bar{\lambda})\right) \left(\frac{\|\bar{\lambda}\|_1}{\|\bar{\lambda}\|_1 + \sum_{i \leq t} \alpha_i}\right)^{6B_2/(c_3\rho^2)} \\ &\leq \left(\widehat{\mathcal{L}}(A\lambda_0) - \widehat{\mathcal{L}}(A\bar{\lambda})\right) \left(\frac{\|\bar{\lambda}\|_1}{\|\bar{\lambda}\|_1 + \rho(t-1)^{1/2-b}/(4B_2R_1)}\right)^{9B_2/(B_1\rho^3)}.\end{aligned}$$

Summing these two bounds gives a relation which holds in general; consequently, with probability at least $1 - \delta/2$ (due to the invocation of Lemma 17),

$$\widehat{\mathcal{L}}(A\lambda_t) - \widehat{\mathcal{L}}(A\bar{\lambda}) \leq (t-1)^{-b} + \ell(0) \left(\frac{\|\bar{\lambda}\|_1}{\|\bar{\lambda}\|_1 + \rho(t-1)^{1/2-b}/(4B_2R_1)}\right)^{9B_2/(B_1\rho^3)}.$$

The first desired claim follows (with probability $1 - 5\delta/6$) by recalling the earlier application of McDiarmid's inequality to $\bar{\lambda}$, combined with $\sigma \downarrow 0$, the choice $b := 1/4$ (which is not optimal, but neither is the exponent $9B_2/(B_1\rho^3)$), and standard Rademacher bounds for voting classifiers applied to Lipschitz losses (Boucheron et al., 2005, Theorem 4.1, eq. (8), and their proofs, which control for \mathcal{L}), which makes use of the bound $\|\lambda_t\|_1 \leq R_t$ (granted by the earlier instantiation of Lemma 17), and simplifying via $t-1 \leq t$,

$$\begin{aligned}\mathcal{L}(A\lambda_t) &\leq \inf_{\|\lambda\|_1 \leq R_{t-1}} \mathcal{L}(A\bar{\lambda}) + t^{-1/4} + R_{t-1} \sqrt{\frac{2}{m} \ln \left(\frac{6}{\delta}\right)} \\ &\quad + \frac{2\beta_2 R_t}{m^{1/2}} \left(2\sqrt{2\mathcal{V}(\mathcal{H}) \ln(m+1)} + \ell(R_t) \sqrt{2 \ln(6/\delta)}\right) \\ &\quad + \ell(0) \left(\frac{\|\bar{\lambda}\|_1}{\|\bar{\lambda}\|_1 + \rho t^{1/4}/(4B_2R_1)}\right)^{9B_2/(B_1\rho^3)}.\end{aligned}$$

Plugging in $t = m^a$ gives the first result.

For the second guarantee, since $\mathcal{H} \in \mathcal{F}_{\text{ds}}(\mu^{\mathcal{X}})$, then Lemma 24 grants

$$\inf\{\mathcal{L}(A\lambda) : \lambda \in \Lambda\} = \inf\left\{\int \ell(-yf(x))d\nu(x, y) : f \in \mathcal{F}_b\right\},$$

where \mathcal{F}_b is the family of Borel measurable functions from \mathcal{X} to \mathbb{R} . From here, since ℓ is *classification calibrated* (Bartlett et al., 2006, Theorem 2, noting that the present manuscript instead takes losses to be nondecreasing rather than nonincreasing), there exists a function $\psi : \mathbb{R} \rightarrow \mathbb{R}$ satisfying

$$\psi(\mathcal{R}(A\lambda_t) - \bar{\mathcal{R}}) \leq \mathcal{L}(A\lambda_t) - \inf_{f \in \mathcal{F}_b} \int \ell(-yf(x))d\mu(x, y) = \mathcal{L}(A\lambda_t) - \inf_{\lambda \in \Lambda} \mathcal{L}(A\lambda)$$

(where the last step used the previous display), and moreover $\psi(z) \rightarrow 0$ as $z \rightarrow 0$ (Bartlett et al., 2006, Theorem 1). The specialization for ℓ_{\log} is due to Zhang (2004, Subsection 3.5 and Corollary 3.1).

The final guarantee follows by applying a version of the VC theorem to the predictors, and follows the exact strategy as in Theorem 7 (but using failure probability $\delta/6$), and making use of the equality

$$\mathcal{R}(H\hat{\lambda}) - \mathcal{R}(H\lambda_t) = (\mathcal{R}(H\hat{\lambda}) - \widehat{\mathcal{R}}(H\hat{\lambda})) + (\widehat{\mathcal{R}}(H\hat{\lambda}) - \widehat{\mathcal{R}}(H\lambda_t)) + (\widehat{\mathcal{R}}(H\lambda_t) - \mathcal{R}(H\lambda_t)),$$

and the fact that $\widehat{\mathcal{R}}(H\hat{\lambda}) \leq \widehat{\mathcal{R}}(H\lambda_t)$. \blacksquare

Appendix J. Proof of Consistency

Proof (of Theorem 5) This proof is a standard application of the Borel-Cantelli Lemma; for an exposition on such applications, please see the proof of consistency of AdaBoost due to [Schapire and Freund \(2012, Proof of Corollary 12.3\)](#). In particular, let any $\epsilon > 0$ be given, and let $E_{m,\epsilon}$ be the event that the output $\hat{\lambda}_m$, trained on m examples, has classification risk $\mathcal{R}(H\hat{\lambda}_m)$ exceeding the Bayes risk \mathcal{R}_\star by more than ϵ ; to prove consistency, it suffices (thanks to Borel-Cantelli) to show that $\sum_m \Pr(E_{m,\epsilon}) < \infty$.

There are two cases to consider: either $\bar{\mathcal{L}} = 0$, or $\bar{\mathcal{L}} > 0$. In the case that $\bar{\mathcal{L}} = 0$, instantiate the finite sample guarantee in Theorem 7 for each $m \geq 1$ with $\epsilon/2$ and $\delta := 1/m^2$; there is a real $M < \infty$ where $m > M$ provides the preconditions on the bound are met and the bound is at most ϵ , and thus

$$\sum_{m=1}^{\infty} \Pr(E_{m,\epsilon}) \leq \sum_{m=1}^M 1 + \sum_{m=M+1}^{\infty} \frac{1}{m^2} \leq M + \frac{\pi^2}{6} < \infty.$$

When $\bar{\mathcal{L}} > 0$, once again instantiate a relevant finite sample guarantee, this time from Theorem 15, with $\delta := 1/m^2$. It will be necessary to use all three guarantees; first, let M_1 be sufficiently large so that the third guarantee provides $\mathcal{R}(H\hat{\lambda}_m) \leq \mathcal{R}(H\lambda'_m) + \epsilon/2$ with failure probability m^{-2} for all $m > M_1$, where λ'_m is the last iterate considered by the algorithm when run on m examples (thus λ'_m is basically λ_m^a , modulo rounding issues). Next, by the second guarantee, there exists ϵ' small enough so that

$$\epsilon' \geq \psi(\mathcal{R}(A\lambda'_m) - \mathcal{R}_\star) \implies \epsilon \geq \mathcal{R}(A\lambda'_m) - \mathcal{R}_\star.$$

As such, now consider the first guarantee, where the goal will be to establish that $\mathcal{L}(A\lambda'_m) - \bar{\mathcal{L}} \leq \epsilon'$ for all large m . But note firstly that the quantity $R_{t-1} \rightarrow \infty$ as $m \rightarrow \infty$, which combined with

$$\inf \{ \mathcal{L}(A\lambda) : \lambda \in \Lambda \} = \inf_{C>0} \inf \{ \mathcal{L}(A\lambda) : \lambda \in \Lambda, \|\lambda\|_1 \leq C \},$$

grants that, if $m > M_2$ (for some M_2), then $\inf_{\|\lambda\| \leq R_t} \mathcal{L}(A\lambda) \leq \bar{\mathcal{L}} + \epsilon'/2$. Finally, the rest of the terms in the first guarantee are at most $\epsilon'/2$ for $m > M_3$ (for some M_3) with the same m^{-2} failure probability. As such, similarly to before, $\sum_m \Pr(E_{m,\epsilon}) \leq \max\{M_1, M_2, M_3\} + \pi^2/6 < \infty$. \blacksquare