Appendix A. Linearized Alternating Direction Method of Multipliers (LADMM)

A.1. Proof of Lemma 1

It follows from the update of x^{k+1} that

$$\nabla l(x^k)-((B^\top B)^{-1}(B^\top A))^\top\lambda^k+\delta\left(x^{k+1}-x^k\right)+\beta((B^\top B)^{-1}(B^\top A))^\top\left(((B^\top B)^{-1}(B^\top A))x^{k+1}-z^k\right)=0. \tag{1}$$

Combining

$$\lambda^{k+1}=\lambda^k-\beta\left(((B^\top B)^{-1}(B^\top A))x^{k+1}-z^k\right)$$

yields that

$$\nabla l(x^k)+\delta\left(x^{k+1}-x^k\right)=((B^\top B)^{-1}(B^\top A))^\top\lambda^{k+1}.$$

Therefore, we conclude that

$$\begin{aligned}
\left\|\lambda^{k+1}\right\|^2 &\leq \frac{1}{\lambda_{\min}(((B^\top B)^{-1}(B^\top A))((B^\top B)^{-1}(B^\top A))^\top)}\left\|((B^\top B)^{-1}(B^\top A))^\top\lambda^{k+1}\right\|^2\\
&= \frac{1}{\lambda_{\min}(((B^\top B)^{-1}(B^\top A))((B^\top B)^{-1}(B^\top A))^\top)}\left\|\left(\nabla l(x^k)-\nabla l(x^{k+1})\right)+\delta\left(x^{k+1}-x^k\right)+\nabla l(x^{k+1})\right\|^2\\
&\leq \frac{3}{\lambda_{\min}(((B^\top B)^{-1}(B^\top A))((B^\top B)^{-1}(B^\top A))^\top)}\left\|\nabla l(x^{k+1})\right\|^2\\
&\quad+\frac{3L^2+3\delta^2}{\lambda_{\min}(((B^\top B)^{-1}(B^\top A))((B^\top B)^{-1}(B^\top A))^\top)}\left\|x^{k+1}-x^k\right\|^2,
\end{aligned} \tag{2}$$

and

$$\begin{aligned}
\left\|\lambda^{k+1}-\lambda^k\right\|^2 &\leq \frac{1}{\lambda_{\min}(((B^\top B)^{-1}(B^\top A))((B^\top B)^{-1}(B^\top A))^\top)}\left\|((B^\top B)^{-1}(B^\top A))^\top\lambda^{k+1}-((B^\top B)^{-1}(B^\top A))^\top\lambda^k\right\|^2\\
&= \frac{1}{\lambda_{\min}(((B^\top B)^{-1}(B^\top A))((B^\top B)^{-1}(B^\top A))^\top)}\left\|\left(\nabla l(x^k)-\nabla l(x^{k-1})\right)+\delta\left(x^{k+1}-x^k\right)-\delta\left(x^k-x^{k-1}\right)\right\|^2\\
&\leq \frac{3L^2+3\delta^2}{\lambda_{\min}(((B^\top B)^{-1}(B^\top A))((B^\top B)^{-1}(B^\top A))^\top)}\left\|x^k-x^{k-1}\right\|^2\\
&\quad+\frac{3\delta^2}{\lambda_{\min}(((B^\top B)^{-1}(B^\top A))((B^\top B)^{-1}(B^\top A))^\top)}\left\|x^{k+1}-x^k\right\|^2.
\end{aligned}$$

A.2. Proof of Lemma 2

Combining Eq. (1) and the following inequality,

$$\left(x^k-x^{k+1}\right)^\top\nabla l(x^k)-l(x^k)+l(x^{k+1})\leq\frac{L}{2}\left\|x^k-x^{k+1}\right\|^2,$$

we have

$$
\begin{aligned}
0 &= \left(x^k - x^{k+1}\right)^\top (\nabla l(x^k) - ((B^\top B)^{-1}(B^\top A))^\top \lambda^k + \delta\left(x^{k+1} - x^k\right) \\
&\quad +\beta((B^\top B)^{-1}(B^\top A))^\top \left(((B^\top B)^{-1}(B^\top A))x^{k+1} - z^k\right)) \\
&\leq l(x^k) - l(x^{k+1}) + \left(\frac{L}{2} - \delta\right)\left\|x^k - x^{k+1}\right\|^2 - \left\langle \lambda^k, ((B^\top B)^{-1}(B^\top A))x^k - z^k \right\rangle \\
&\quad + \left\langle \lambda^k, ((B^\top B)^{-1}(B^\top A))x^{k+1} - z^k \right\rangle + \frac{\beta}{2}\left\|((B^\top B)^{-1}(B^\top A))x^k - z^k\right\|^2 \\
&\quad -\frac{\beta}{2}\left\|((B^\top B)^{-1}(B^\top A))x^{k+1} - z^k\right\|^2 - \frac{\beta}{2}\left\|((B^\top B)^{-1}(B^\top A))x^{k+1} - ((B^\top B)^{-1}(B^\top A))x^k\right\|^2.
\end{aligned} \tag{3}
$$

Then it follows from the update of z^{k+1} that,

$$
\begin{aligned}
& r(z^{k+1}) - \left\langle \lambda^{k+1}, ((B^\top B)^{-1}(B^\top A))x^{k+1} - z^{k+1} \right\rangle + \frac{\beta}{2}\left\|((B^\top B)^{-1}(B^\top A))x^{k+1} - z^{k+1}\right\|^2 \\
\leq\ & r(z^k) - \left\langle \lambda^{k+1}, ((B^\top B)^{-1}(B^\top A))x^{k+1} - z^k \right\rangle + \frac{\beta}{2}\left\|((B^\top B)^{-1}(B^\top A))x^{k+1} - z^k\right\|^2.
\end{aligned} \tag{4}
$$

Combining Eq. (3), Eq. (4) and Lemma 1 yields that,

$$
\begin{aligned}
& r(z^{k+1}) + l(x^{k+1}) - \left\langle \lambda^{k+1}, ((B^\top B)^{-1}(B^\top A))x^{k+1} - z^{k+1} \right\rangle \\
& +\frac{\beta}{2}\left\|((B^\top B)^{-1}(B^\top A))x^{k+1} - z^{k+1}\right\|^2 + \left(\delta - \frac{L}{2}\right)\left\|x^k - x^{k+1}\right\|^2 \\
\leq\ & r(z^k) + l(x^k) - \left\langle \lambda^k, ((B^\top B)^{-1}(B^\top A))x^k - z^k \right\rangle + \frac{\beta}{2}\left\|((B^\top B)^{-1}(B^\top A))x^k - z^k\right\|^2 + \frac{1}{\beta}\left\|\lambda^{k+1} - \lambda^k\right\|^2 \\
\leq\ & r(z^k) + l(x^k) - \left\langle \lambda^k, ((B^\top B)^{-1}(B^\top A))x^k - z^k \right\rangle + \frac{\beta}{2}\left\|((B^\top B)^{-1}(B^\top A))x^k - z^k\right\|^2 \\
& +\frac{3L^2 + 3\delta^2}{\beta\lambda_{\min}(((B^\top B)^{-1}(B^\top A))((B^\top B)^{-1}(B^\top A))^\top)}\left\|x^k - x^{k-1}\right\|^2 \\
& +\frac{3\delta^2}{\beta\lambda_{\min}(((B^\top B)^{-1}(B^\top A))((B^\top B)^{-1}(B^\top A))^\top)}\left\|x^{k+1} - x^k\right\|^2
\end{aligned}
$$

which implies that

$$
\begin{aligned}
& \left(\delta - \frac{L}{2} - \frac{3L^2 + 6\delta^2}{\beta\lambda_{\min}(((B^\top B)^{-1}(B^\top A))((B^\top B)^{-1}(B^\top A))^\top)}\right)\left\|x^k - x^{k+1}\right\|^2 \\
\leq\ & \Phi(x^k, x^{k-1}, z^k, \lambda^k) - \Phi(x^{k+1}, x^k, z^{k+1}, \lambda^{k+1})
\end{aligned} \tag{5}
$$

where

$$
\begin{aligned}
\Phi(x, \hat{x}, z, \lambda) &= l(x) + r(z) - \left\langle \lambda, ((B^\top B)^{-1}(B^\top A))x - z \right\rangle + \frac{\beta}{2}\left\|((B^\top B)^{-1}(B^\top A))x - z\right\|^2 \\
&\quad +\frac{3L^2 + 3\delta^2}{\beta\lambda_{\min}(((B^\top B)^{-1}(B^\top A))((B^\top B)^{-1}(B^\top A))^\top)}\left\|x - \hat{x}\right\|^2.
\end{aligned} \tag{6}
$$

Since $\delta > \frac{L}{2}$ and $\beta > 0$ satisfies that

$$\beta > \left(3L^2 + 6\delta^2\right) / \lambda_{\min}(((B^\top B)^{-1}(B^\top A))((B^\top B)^{-1}(B^\top A))^\top) \left(\delta - \frac{L}{2}\right),$$

we conclude that $\Phi(x^{k+1}, x^k, z^{k+1}, \lambda^{k+1})$ is monotonically decreasing as k increases.

On the other hand, we have

$$\begin{aligned}
& \Phi(x^{k+1}, x^k, z^{k+1}, \lambda^{k+1}) \\
= \;& l(x^{k+1}) + r(z^{k+1}) - \left\langle \lambda^{k+1}, ((B^\top B)^{-1}(B^\top A))x^{k+1} - z^{k+1} \right\rangle + \frac{\beta}{2} \left\| ((B^\top B)^{-1}(B^\top A))x^{k+1} - z^{k+1} \right\|^2 \\
& + \frac{3L^2 + 3\delta^2}{\beta \lambda_{\min}(((B^\top B)^{-1}(B^\top A))((B^\top B)^{-1}(B^\top A))^\top)} \left\| x^{k+1} - x^k \right\|^2 \\
\geq \;& l(x^{k+1}) + r(z^{k+1}) - \frac{1}{2\beta} \left\| \lambda^{k+1} \right\|^2 - \frac{\beta}{2} \left\| ((B^\top B)^{-1}(B^\top A))x^{k+1} - z^{k+1} \right\|^2 + \frac{\beta}{2} \left\| ((B^\top B)^{-1}(B^\top A))x^{k+1} - z^{k+1} \right\|^2 \\
& + \frac{3L^2 + 3\delta^2}{\beta \lambda_{\min}(((B^\top B)^{-1}(B^\top A))((B^\top B)^{-1}(B^\top A))^\top)} \left\| x^{k+1} - x^k \right\|^2 \\
\geq \;& l(x^{k+1}) + r(z^{k+1}) - \frac{3}{2\beta \lambda_{\min}(((B^\top B)^{-1}(B^\top A))((B^\top B)^{-1}(B^\top A))^\top)} \left\| \nabla l(x^{k+1}) \right\|^2 \\
& - \frac{3L^2 + 3\delta^2}{2\beta \lambda_{\min}(((B^\top B)^{-1}(B^\top A))((B^\top B)^{-1}(B^\top A))^\top)} \left\| x^{k+1} - x^k \right\|^2 \\
& + \frac{3L^2 + 3\delta^2}{\beta \lambda_{\min}(((B^\top B)^{-1}(B^\top A))((B^\top B)^{-1}(B^\top A))^\top)} \left\| x^{k+1} - x^k \right\|^2 \\
\geq \;& l(x^{k+1}) + r(z^{k+1}) - \beta_0 \left\| \nabla l(x^{k+1}) \right\|^2 \\
= \;& \bar{l}(x^{k+1}) + r(z^{k+1}) \\
\geq \;& \bar{l}^* + r^* = \Phi^*,
\end{aligned} \tag{7}$$

where the second inequality holds due to Eq. (2) and the third inequality holds since

$$\beta \geq \frac{3}{2\beta_0 \lambda_{\min}(((B^\top B)^{-1}(B^\top A))((B^\top B)^{-1}(B^\top A))^\top)}.$$

Therefore, we conclude that $\Phi(x^{k+1}, x^k, z^{k+1}, \lambda^{k+1})$ is uniformly lower bounded.

A.3. Proof of Theorem 3

Combining Eq. (7) and the fact that $\bar{l}(x)$ is coercive, we conclude that $\{x^{k+1}\}$ is bounded. Then it directly follows from Eq. (2) that $\{\lambda^{k+1}\}$ is bounded. Furthermore, we obtain from Eq. (5) and Eq. (7) that

$$\left(\delta - \frac{L}{2} - \frac{3L^2 + 6\delta^2}{\beta\lambda_{\min}(((B^\top B)^{-1}(B^\top A))((B^\top B)^{-1}(B^\top A))^\top)}\right)\sum_{k=1}^{\infty}\left\|x^k - x^{k+1}\right\|^2 \leq \Phi(x^1, x^0, z^1, \lambda^1) - \Phi^* < +\infty, \tag{8}$$

which implies that $\left\|x^k - x^{k+1}\right\| \to 0$ and hence $\left\|\lambda^k - \lambda^{k+1}\right\| \to 0$ as $k \to +\infty$. Since

$$((B^\top B)^{-1}(B^\top A))x^{k+1} - z^{k+1} = \frac{1}{\beta}\left(\lambda^k - \lambda^{k+1}\right),$$

we have $\left\|((B^\top B)^{-1}(B^\top A))x^{k+1} - z^{k+1}\right\| \to 0$, which implies that $\{z^{k+1}\}$ is bounded and $\left\|z^k - z^{k+1}\right\| \to 0$ as $k \to +\infty$. In summary, we obtain that $\{x^{k+1}, z^{k+1}, \lambda^{k+1}\}$ is a bounded sequence, and

$$\left\|x^k - x^{k+1}\right\| \to 0, \quad \left\|z^k - z^{k+1}\right\| \to 0, \quad \left\|((B^\top B)^{-1}(B^\top A))x^{k+1} - z^{k+1}\right\| \to 0.$$

Since $\{x^{k+1}, z^{k+1}, \lambda^{k+1}\}$ is bounded, this sequence must have at least one limit point. Let $\{x^*, z^*, \lambda^*\}$ be a limit point, that is, there exits a subsequence $\{k_q\}_{q=1}^{\infty}$ such that

$$\lim_{q\to+\infty}\left(x^{k_q}, z^{k_q}, \lambda^{k_q}\right) = (x^*, z^*, \lambda^*).$$

and it holds true that

$$\left\|x^{k_q} - x^{k_q+1}\right\| \to 0, \quad \left\|z^{k_q} - z^{k_q+1}\right\| \to 0, \quad \left\|((B^\top B)^{-1}(B^\top A))x^{k_q+1} - z^{k_q+1}\right\| \to 0.$$

We consider the first-order optimality condition of updating x^{k_q+1} and z^{k_q+1} and $r(z) = r_1(z) - r_2(z)$, i.e.,

$$\begin{aligned} 0 &= \nabla l(x^{k_q}) - ((B^\top B)^{-1}(B^\top A))^\top\lambda^{k_q} + \delta\left(x^{k_q+1} - x^{k_q}\right) + \beta((B^\top B)^{-1}(B^\top A))^\top\left(((B^\top B)^{-1}(B^\top A))x^{k_q+1} - z^{k_q}\right), \\ 0 &\in \partial r_1(z^{k_q+1}) - \partial r_2(z^{k_q+1}) + \lambda^{k_q} - \beta\left(((B^\top B)^{-1}(B^\top A))x^{k_q+1} - z^{k_q+1}\right). \end{aligned}$$

Letting $q \to +\infty$, by considering the semi-continuity of $\partial r_1(\cdot)$ and $\partial r_2(\cdot)$, we obtain that

$$\begin{aligned} 0 &= \nabla l(x^*) - ((B^\top B)^{-1}(B^\top A))^\top\lambda^*, \\ 0 &\in \partial r_1(z^*) - \partial r_2(z^*) + \lambda^*. \end{aligned}$$

Therefore, (x^*, z^*, λ^*) is a critical point.

Moreover, it follows from Eq. (8) that

$$\min_{0\leq k\leq n}\left\|x^k - x^{k+1}\right\|^2 \leq \frac{\Phi(x^1, x^0, z^1, \lambda^1) - \Phi^*}{n\delta_{\min}},$$

where $\delta_{\min}$ is defined as

$$\delta_{\min} = \delta - \frac{L}{2} - \frac{3L^2 + 6\delta^2}{\beta\lambda_{\min}(((B^\top B)^{-1}(B^\top A))((B^\top B)^{-1}(B^\top A))^\top)} > 0.$$

This completes the proof of Theorem 3.