5 Given :-

→ $J(x) = \|y - \phi x\|_2^2 + \lambda \|x\|_1$

And $\lambda > 0$, x is the minimizer of $J(.)$.

→ Problem is

$$\min_x \|x\|_1 \quad \text{s.t} \quad J(x) - \lambda \|x\|_1 \leq \epsilon \qquad \text{— } (P_1)$$

Say x_0 be minimizer of Lasso cost fn.

$$J'(x_0) = \underbrace{2\,\|y - \phi x\|_2 \Big|_{x=x_0}}_{= \epsilon' \text{ (say)}} \frac{d}{dx}\left(\|y - \phi x\|\right)\Big|_{x=x_0} + \lambda \frac{d}{dx}\|x\|_1 \Big|_{x=x_0} = 0$$

$$\Rightarrow \boxed{2\epsilon' \frac{d}{dx}\left(\|y - \phi x\|\right)\Big|_{x=x_0} = -\lambda \frac{d}{dx}\left(\|x\|_1\right)\Big|_{x=x_0}}$$

Let us look at the case split of the above equation

→ Case 1 :- both sides zero $\Rightarrow$ $\boxed{\|x_0\|_1 \text{ is minima at } x = x_0}$

→ Case a :-

$\epsilon' = 0$ $\Rightarrow$ this would solve our P_1

as well since $\epsilon' = 0 < \epsilon$

→ Case b :-

$\|y - \phi x\|$ has ~~global~~ min/max at $x = x_0$

In this case if we can set a value of ϵ greater than value of ϵ' → when finding solution for P_1 we can search and finally reach possible solution because we achieved best possible solution/reconstruction as $\frac{d}{dx}\|y - \phi x\|_2 = 0$ has global minima for $\|y - \phi x\|_2$ as a solution.

→ Case 2 :- non-zero sides.

i e $\epsilon' \neq 0$, both $\|x\|_1$ and $\|y - \phi x\|_2$ does not have extremas at $x = x_0$

$$\mathcal{L}(x, \lambda) = f(x) - \lambda g(x)$$

$$c \frac{d}{dx} \left(\|y - \phi x\|_2 - \epsilon \right) + \frac{d}{dx} \|x\|_1 = 0$$

→ here ϵ is some constant whereas c is the Lagraingian Multiplier Constant

Now if we put $\epsilon = \epsilon'$ and $c = \frac{2}{\lambda} \epsilon'$ then we are solving same eq^ns for x

⇒ Lasso sol^n will be sol^n for P_1.

Hence in Case 1, 2 ⇒ We proved solution of P_1 can be Lasso solution.