CS 754 Assignment 2 : Report

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Question 1

(1)

By the defination of the isometry constant δ_{2s} of the matrix Φ, δ_{2s} is the smallest number such that the below inequality holds for all 2s sparse h vectors

$$(1-\delta_{2s})||h||_{l_2}^2 \leq ||\Phi h||_{l_2}^2 \leq (1+\delta_{2s})||h||_{l_2}^2$$

If $\delta_{2s} = 1$ then $0 \leq ||\Phi h||_{l_2}^2 \leq 2||h||_{l_2}^2$. This implies there may exist a 2s sparse vector h such that $||\Phi h||_{l_2}^2 = 0 \Rightarrow \Phi h = 0$. Let Φ_i be the ith column of Φ and h_i be the ith element of h.

$$\Phi h = 0$$
$$\Rightarrow \Phi_1 h_1 + \Phi_2 h_2 + \Phi_2 h_2 + ... + \Phi_n h_n = 0$$

As h is a 2s-sparse vector (at most 2s elements of h are non-zero), this implies there may exist some set of 2s columns of Φ such that they are linearly dependent.

(2)

By triangle inequality theorem, vectors a and b obey below inequality

$$||a+b|| \leq ||a|| + ||b||$$

Now consider,

$$||\Phi(x^* - x)||_{l_2}^2 = ||(\Phi x^* - y) + (y - \Phi x^*)||_{l_2}^2$$

By triangle inequality theorem,

$$||(\Phi x^* - y) + (y - \Phi x^*)||_{l_2}^2 \leq ||\Phi x^* - y||_{l_2} + ||y - \Phi x^*||_{l_2}$$
$$\Rightarrow ||\Phi(x^* - x)||_{l_2}^2 \leq ||\Phi x^* - y||_{l_2} + ||\Phi x^* - y||_{l_2}$$

As, x^* is the feasible solution of min $||x||_{l_1}$ subject to $||y - \Phi x^*||_{l_2} \leq \varepsilon$, therefore $||y - \Phi x^*||_{l_2} \leq \varepsilon$. As, x is the true signal of $y = \Phi x + z$ where z is noise and ε is the upper bound of noise, $||z||_{l_2} \leq \varepsilon \Rightarrow ||y - \Phi x||_{l_2} \leq \varepsilon$.

$$||y - \Phi x^*||_{l_2} \leq \varepsilon \text{ and } ||y - \Phi x||_{l_2} \leq \varepsilon$$
$$\Rightarrow ||y - \Phi x^*||_{l_2} + ||y - \Phi x||_{l_2} \leq \varepsilon + \varepsilon$$
$$\Rightarrow ||y - \Phi x^*||_{l_2} + ||y - \Phi x||_{l_2} \leq 2\varepsilon$$

Therefore, $||\Phi(x^* - x)||_{l_2}^2 \leq ||\Phi x^* - y||_{l_2} + ||\Phi x^* - y||_{l_2} \leq 2\varepsilon$.

(3)

To justify, $\left\|h_{T_j}\right\|_2 \leq s^{1/2}\ \left\|h_{T_j}\right\|_\infty \leq s^{-1/2}\ \left\|h_{T_{j-1}}\right\|_1$
Each h_{T_j} is s-sparse vector(at most s elements of $\left\|h_{T_j}\right\|_2$ are non-zero), therefore

$$
\begin{gathered}
||h_{T_j}||_2^2 = \textstyle\sum_{i=1}^{s} h_{T_j,i}^2 \text{ and } h_{T_j,i}^2 \leq ||h_{T_j}||_\infty^2 \\
\Rightarrow ||h_{T_j}||_2^2 \leq \textstyle\sum_{i=1}^{s} ||h_{T_j}||_\infty^2 \\
\Rightarrow ||h_{T_j}||_2^2 \leq s||h_{T_j}||_\infty^2 \\
\Rightarrow ||h_{T_j}||_2 \leq s^{1/2}||h_{T_j}||_\infty
\end{gathered}
$$

As, T_j is the set of indices corresponding to the s largest absolute value elements of $h_{(T_0 \cup T_1 .... \cup T_{j-1})^c}$. Therefore, all elements of h_{T_j} are less or equal to all elements of $h_{T_{j-1}}$. This implies largest element of h_{T_j} is less than or equal to smallest element of $h_{T_{j-1}}$.

$$
\begin{gathered}
\max \{\ |h_{T_j,i}| : \forall 1 \leq i \leq s\ \} \leq \min \{|h_{T_{j-1},i}| : \forall\ 1 \leq i \leq s\ \} \\
\Rightarrow ||h_{T_j}||_\infty \leq \min \{\ |h_{T_{j-1},i}| : \forall\ 1 \leq i \leq s\ \}
\end{gathered}
$$

$$
\begin{gathered}
||h_{T_j}||_1 = \textstyle\sum_{i=1}^{s} |h_{T_j,i}| \\
\Rightarrow ||h_{T_j}||_1 \geq \text{s*}(\min \{\ |h_{T_{j-1},i}| : \forall\ 1 \leq i \leq s\ \}\ ) \\
\Rightarrow ||h_{T_j}||_1 \geq \text{s}||h_{T_j}||_\infty \\
\Rightarrow s^{1/2}||h_{T_j}||_\infty \leq s^{-1/2}||h_{T_j}||_1
\end{gathered}
$$

Therefore, $\left\|h_{T_j}\right\|_2 \leq s^{1/2}\ \left\|h_{T_j}\right\|_\infty \leq s^{-1/2}\ \left\|h_{T_{j-1}}\right\|_1$.

(4)

To justify, $\sum\limits_{j\geq 2} \left\|h_{T_j}\right\|_2 \leq s^{-1/2}(\|h_{T_1}\|_1 + \|h_{T_2}\|_1 + \ldots) \leq s^{-1/2}\ \left\|h_{T_0^c}\right\|_1$

$$
\begin{gathered}
\text{As, } h = h_{T_0} + h_{T_1} + h_{T_2} + \ldots \\
\Rightarrow h - h_{T_0} = h_{T_1} + h_{T_2} + \ldots \\
\Rightarrow h_{T_0^c} = h_{T_1} + h_{T_2} + \ldots \\
\Rightarrow \left\|h_{T_0^c}\right\|_1 = \|h_{T_1} + h_{T_2} + \ldots\|_1 \geq (\|h_{T_1}\|_1 + \|h_{T_2}\|_1 + \ldots) \\
\Rightarrow \left\|h_{T_0^c}\right\|_1 \geq (\|h_{T_1}\|_1 + \|h_{T_2}\|_1 + \ldots) \\
\Rightarrow s^{-1/2}(\|h_{T_1}\|_1 + \|h_{T_2}\|_1 + \ldots) \leq s^{-1/2}\ \left\|h_{T_0^c}\right\|_1
\end{gathered}
$$

Using the inequalities $\left\|h_{T_j}\right\|_2 \leq s^{1/2}\ \left\|h_{T_j}\right\|_\infty \leq s^{-1/2}\ \left\|h_{T_{j-1}}\right\|_1$ from part **(3)**

$$
\begin{gathered}
\left\|h_{T_j}\right\|_2 \leq s^{-1/2}\ \left\|h_{T_{j-1}}\right\|_1\ \forall\ 1 \leq j \\
\Rightarrow (\|h_{T_2}\|_2 + \|h_{T_3}\|_2 + \ldots) \leq s^{-1/2}(\|h_{T_1}\|_1 + \|h_{T_2}\|_1 + \ldots) \\
\Rightarrow \sum_{j\geq 2} \left\|h_{T_j}\right\|_2 \leq s^{-1/2}(\|h_{T_1}\|_1 + \|h_{T_2}\|_1 + \ldots)
\end{gathered}
$$

Therefore, $\sum\limits_{j\geq 2} \left\|h_{T_j}\right\|_2 \leq s^{-1/2}(\|h_{T_1}\|_1 + \|h_{T_2}\|_1 + \ldots) \leq s^{-1/2}\ \left\|h_{T_0^c}\right\|_1$

(5)

To justify, $\left\|h_{(T_0 \cup T_1)^c}\right\|_2 = \left\|\sum_{j\geq 2} h_{T_j}\right\|_2 \leq \sum_{j\geq 2} \left\|h_{T_j}\right\|_2 \leq s^{-1/2}\ \left\|h_{T_0^c}\right\|_1$

$$
\begin{gathered}
\text{As, } h = h_{T_0} + h_{T_1} + h_{T_2} + \ldots \\
\Rightarrow h - h_{T_0} - h_{T_1} = h_{T_2} + h_{T_3} + \ldots \\
\Rightarrow h_{(T_0 \cup T_1)^c} = h_{T_2} + h_{T_3} + \ldots \\
\Rightarrow \left\|h_{(T_0 \cup T_1)^c}\right\|_2 = \left\|\textstyle\sum_{j\geq 2} h_{T_j}\right\|_2
\end{gathered}
$$

By triangle inequality,

$$\|h_{T_2} + h_{T_3} + \ldots\|_2 \le (\|h_{T_2}\|_2 + \|h_{T_3}\|_2 + \ldots)$$
$$\Rightarrow \left\|\sum_{j\ge 2} h_{T_j}\right\|_2 \le \sum_{j\ge 2} \left\|h_{T_j}\right\|_2$$

Using the inequalities $\sum\limits_{j\ge 2} \left\|h_{T_j}\right\|_2 \le s^{-1/2}(\|h_{T_1}\|_1 + \|h_{T_2}\|_1 + \ldots) \le s^{-1/2}\ \left\|h_{T_0^c}\right\|_1$ from part **(4)**, we get $\sum\limits_{j\ge 2} \left\|h_{T_j}\right\|_2 \le s^{-1/2}\ \left\|h_{T_0^c}\right\|_1$.

Therefore, $\left\|h_{(T_0\cup T_1)^c}\right\|_2 = \left\|\sum_{j\ge 2} h_{T_j}\right\|_2 \le \sum_{j\ge 2} \left\|h_{T_j}\right\|_2 \le s^{-1/2}\ \left\|h_{T_0^c}\right\|_1$

(6)

To jusify, $\sum\limits_{i\in T_0} |x_i + h_i| + \sum\limits_{i\in T_0^c} |x_i + h_i| \ge \|x_{T_0}\|_{l_1} - \|h_{T_0}\|_{l_1} + \left\|h_{T_0^c}\right\|_{l_1} - \left\|x_{T_0^c}\right\|_{l_1}$

By triangle inequality theorem, $||a| - |b|| \le |a - b|$

Applying triangle inequality theorem,

$$|x_i - (-h_i)| \ge ||x_i| - |h_i|| \text{ and } ||x_i| - |h_i|| \ge |x_i| - |h_i|\ \forall\ i\epsilon T_0$$
$$\Rightarrow |x_i + h_i| \ge |x_i| - |h_i|\ \forall\ i\epsilon T_0$$
$$\Rightarrow \sum_{i\in T_0} |x_i + h_i| \ge \sum_{i\in T_0} (|x_i| - |h_i|)$$
$$\Rightarrow \sum_{i\in T_0} |x_i + h_i| \ge \|x_{T_0}\|_{l_1} - \|h_{T_0}\|_{l_1} \Rightarrow \textbf{equation (1)}$$

Applying triangle inequality theorem,

$$|h_i - (-x_i)| \ge ||h_i| - |x_i|| \text{ and } ||h_i| - |x_i|| \ge |h_i| - |x_i|\ \forall\ i\epsilon T_0^c$$
$$\Rightarrow |x_i + h_i| \ge |h_i| - |x_i|\ \forall\ i\epsilon T_0^c$$
$$\Rightarrow \sum_{i\in T_0^c} |x_i + h_i| \ge \sum_{i\in T_0^c} (|h_i| - |x_i|)$$
$$\Rightarrow \sum_{i\in T_0^c} |x_i + h_i| \ge \left\|h_{T_0^c}\right\|_{l_1} - \left\|x_{T_0^c}\right\|_{l_1} \Rightarrow \textbf{equation (2)}$$

Adding equations (1) and (2),

$$\sum_{i\in T_0} |x_i + h_i| + \sum_{i\in T_0^c} |x_i + h_i| \ge \|x_{T_0}\|_{l_1} - \|h_{T_0}\|_{l_1} + \left\|h_{T_0^c}\right\|_{l_1} - \left\|x_{T_0^c}\right\|_{l_1}$$

(7)

To justify, $\left\|h_{T_0^c}\right\|_{l_1} \le \|h_{T_0}\|_{l_1} + 2\ \left\|x_{T_0^c}\right\|_{l_1}$

By triangle inequality theorem, $||a + b|| \le ||a|| + ||b||$

Using **equation (2)** from part **(7)**,

$$\sum_{i\in T_0^c} |x_i + h_i| \ge \left\|h_{T_0^c}\right\|_{l_1} - \left\|x_{T_0^c}\right\|_{l_1}$$
$$\Rightarrow \sum_{i\in T_0^c} |x_i + h_i| + \left\|x_{T_0^c}\right\|_{l_1} \ge \left\|h_{T_0^c}\right\|_{l_1}$$
$$\Rightarrow \left\|x_{T_0^c} + h_{T_0^c}\right\|_{l_1} + \left\|x_{T_0^c}\right\|_{l_1} \ge \left\|h_{T_0^c}\right\|_{l_1} \Rightarrow \textbf{equation (3)}$$

Applying triangle inequality theorem,

$$\left\|x_{T_0^c}\right\|_{l_1} + \left\|h_{T_0^c}\right\|_{l_1} \ge \left\|x_{T_0^c} + h_{T_0^c}\right\|_{l_1}$$
$$\Rightarrow \left\|x_{T_0^c}\right\|_{l_1} + \left\|x_{T_0^c}\right\|_{l_1} + \left\|h_{T_0^c}\right\|_{l_1} \ge \left\|x_{T_0^c}\right\|_{l_1} + \left\|x_{T_0^c} + +h_{T_0^c}\right\|_{l_1}$$
$$\Rightarrow 2\left\|x_{T_0^c}\right\|_{l_1} + \left\|h_{T_0^c}\right\|_{l_1} \ge \left\|x_{T_0^c} + +h_{T_0^c}\right\|_{l_1} + \left\|x_{T_0^c}\right\|_{l_1} \Rightarrow \textbf{equation (4)}$$

Combining equations (3) and (4),

$$\left\|h_{T_0^c}\right\|_{l_1} \le \|h_{T_0}\|_{l_1} + 2\ \left\|x_{T_0^c}\right\|_{l_1}$$

(8)

To justify, $\left\|h_{(T_0\cup T_1)^c}\right\|_{l_2} \leq \|h_{T_0}\|_{l_2} + 2e_0, \quad e_0 \equiv s^{-1/2}\ \|x - x_s\|_{l_1}$
As, h_{T_0} is a s-sparse vector

$$||h_{T_0}||_2^2 = \sum_{i=1}^{s} h_{T_0,i} \text{ and } ||h_{T_0}||_1 = \sum_{i=1}^{s} |h_{T_0,i}|$$

By Cauchy–Schwarz inequality theorem,

$$(\sum_{i=1}^{s}(|h_{T_0,i}| * 1))^2 \leq (\sum_{i=1}^{s} 1)(\sum_{i=1}^{s} h_{T_0,i}^2)$$
$$\Rightarrow (\sum_{i=1}^{s} |h_{T_0,i}|)^2 \leq s * (\sum_{i=1}^{s} h_{T_0,i}^2)$$
$$\Rightarrow ||h_{T_0}||_1^2 \leq s||h_{T_0}||_2^2$$
$$\Rightarrow ||h_{T_0}||_1 \leq s^{1/2}||h_{T_0}||_2 \Rightarrow \textbf{equation (5)}$$

Combing inequalities inequalities, $\left\|h_{(T_0\cup T_1)^c}\right\|_2 = \left\|\sum_{j\geq 2} h_{T_j}\right\|_2 \leq \sum_{j\geq 2}\left\|h_{T_j}\right\|_2 \leq s^{-1/2}\ \left\|h_{T_0^c}\right\|_1$ from **part (5)** and $\left\|h_{T_0^c}\right\|_{l_1} \leq \|h_{T_0}\|_{l_1} + 2\ \left\|x_{T_0^c}\right\|_{l_1}$ from **part (7)**. We get,

$$\left\|h_{(T_0\cup T_1)^c}\right\|_{l_2} \leq s^{-1/2}(\|h_{T_0}\|_{l_1} + 2\ \left\|x_{T_0^c}\right\|_{l_1}) \Rightarrow \textbf{equation (6)}$$

Combining equations (5) and (6), we get

$$\left\|h_{(T_0\cup T_1)^c}\right\|_{l_2} \leq s^{-1/2}(s^{1/2}||h_{T_0}||_2 + 2\ \left\|x_{T_0^c}\right\|_{l_1})$$
$$\Rightarrow \left\|h_{(T_0\cup T_1)^c}\right\|_{l_2} \leq ||h_{T_0}||_2 + 2s^{-1/2}\ \left\|x_{T_0^c}\right\|_{l_1}$$
$$\Rightarrow \left\|h_{(T_0\cup T_1)^c}\right\|_{l_2} \leq \|h_{T_0}\|_{l_2} + 2e_0, \quad e_0 \equiv s^{-1/2}\ \|x - x_s\|_{l_1}$$

(9)

To justify, $|\langle \Phi h_{T_0\cup T_1}, \Phi h\rangle| \leq \|\Phi h_{T_0\cup T_1}\|_2\ \|\Phi h\|_2 \leq 2\epsilon\sqrt{1+\delta_{2s}}\ \|h_{T_0\cup T_1}\|_2$

By Cauchy–Schwarz inequality theorem, $|\langle \Phi h_{T_0\cup T_1}, \Phi h\rangle| \leq \|\Phi h_{T_0\cup T_1}\|_2\ \|\Phi h\|_2$
Using inequality $||\Phi(x^* - x)||_{l_2}^2 \leq ||\Phi x^* - y||_{l_2} + ||\Phi x^* - y||_{l_2} \leq 2\varepsilon$ from part (2)

$$|\langle \Phi h_{T_0\cup T_1}, \Phi h\rangle| \leq \|\Phi h_{T_0\cup T_1}\|_2\ \|\Phi h\|_2 \text{ and } h = x - x^*$$
$$\Rightarrow |\langle \Phi h_{T_0\cup T_1}, \Phi h\rangle| \leq \|\Phi h_{T_0\cup T_1}\|_2\ \|\Phi h\|_2 = \|\Phi h_{T_0\cup T_1}\|_2\ \|\Phi(x - x^*)\|_2$$
$$\Rightarrow |\langle \Phi h_{T_0\cup T_1}, \Phi h\rangle| \leq \|\Phi h_{T_0\cup T_1}\|_2\ \|\Phi h\|_2 \leq 2\epsilon\,\|\Phi h_{T_0\cup T_1}\|_2 \Rightarrow \textbf{equation (7)}$$

$h_{T_0\cup T_1}$ is a 2s-sparse vector because h_{T_0} and h_{T_1} are s-sparse vectors and $h_{T_0\cup T_1} = h_{T_0} + h_{T_1}$(As T_0 and T_1 are disjoint sets).
As Φ obeys RIP property of order 2s,

$$\sqrt{1-\delta_{2s}}\ \|h_{T_0\cup T_1}\|_2 \leq \|\Phi h_{T_0\cup T_1}\|_2 \leq \sqrt{1+\delta_{2s}}\ \|h_{T_0\cup T_1}\|_2$$
$$\Rightarrow 2\epsilon\,\|\Phi h_{T_0\cup T_1}\|_2\ \leq 2\epsilon\sqrt{1+\delta_{2s}}\ \|h_{T_0\cup T_1}\|_2 \Rightarrow \textbf{equation (8)}$$

Combining equations (7) and (8),

$$|\langle \Phi h_{T_0\cup T_1}, \Phi h\rangle| \leq \|\Phi h_{T_0\cup T_1}\|_2\ \|\Phi h\|_2 \leq 2\epsilon\sqrt{1+\delta_{2s}}\ \|h_{T_0\cup T_1}\|_2$$

(10)

To justify, $|\langle \Phi h_{T_0}, \Phi h_{T_j}\rangle| \leq \delta_{2s}\ \|h_{T_0}\|_2 \left\|h_{T_j}\right\|_2$
From Lemma 2.1 in the paper,

$$|\langle \Phi x, \Phi x'\rangle| \leq \delta_{s+s'}\ \|x\|_2 \|x'\|_2$$

for all x, x' supported on disjoint subsets $T, T' \subseteq \{1, \ldots, n\}$ with $|T| \leq s, |T'| \leq s'$.

Substituting h_{T_0} in place of x and h_{T_j} in place of x' in Lemma 2.1. As h_{T_0} and h_{T_j} are s-sparse vectors, therefore $\delta_{s+s'}$ in lemma 2.1 becomes δ_{2s}.
Therefore, we get

$$|\langle \Phi h_{T_0}, \Phi h_{T_j}\rangle| \leq \delta_{2s}\ \|h_{T_0}\|_2 \left\|h_{T_j}\right\|_2$$

(11)

To justify, $\|h_{T_0}\|_2 + \|h_{T_1}\|_2 \leq \sqrt{2}\ \|h_{T_0 \cup T_1}\|_2$
As, T_0 and T_1 are two disjoint sets, therefore

$$\begin{gathered} h_{T_0 \cup T_1} = h_{T_0} + h_{T_1} \\ \Rightarrow \|h_{T_0 \cup T_1}\|_2 = \|h_{T_0}\|_2 + \|h_{T_1}\|_2 \end{gathered}$$

By AM-GM inequality theorem, $\frac{a+b}{2} \geq \sqrt{ab}$
Putting $a = \|h_{T_0}\|_2^2$ and $b = \|h_{T_1}\|_2^2$ in AM-GM theorem, we get

$$\begin{gathered} \|h_{T_0}\|_2^2 + \|h_{T_0}\|_2^2 \geq 2\sqrt{\|h_{T_0}\|_2^2\ \|h_{T_1}\|_2^2} \\ \Rightarrow 2\|h_{T_0}\|_2\ \|h_{T_1}\|_2 \leq \|h_{T_0}\|_2^2 + \|h_{T_0}\|_2^2 \\ \Rightarrow 2\|h_{T_0}\|_2\ \|h_{T_1}\|_2 + \|h_{T_0}\|_2^2 + \|h_{T_0}\|_2^2 \leq 2(\|h_{T_0}\|_2^2 + \|h_{T_0}\|_2^2) \\ \Rightarrow (\|h_{T_0}\|_2 + \|h_{T_1}\|_2)^2 \leq 2\ \|h_{T_0 \cup T_1}\|_2^2 \\ \Rightarrow \|h_{T_0}\|_2 + \|h_{T_1}\|_2 \leq \sqrt{2}\ \|h_{T_0 \cup T_1}\|_2 \end{gathered}$$

Therefore, $\|h_{T_0}\|_2 + \|h_{T_1}\|_2 \leq \sqrt{2}\ \|h_{T_0 \cup T_1}\|_2$.

(12)

To justify, $(1 - \delta_{2s})\ \|h_{T_0 \cup T_1}\|_2^2 \leq \|\Phi h_{T_0 \cup T_1}\|_2^2 \leq \|h_{T_0 \cup T_1}\|_2\ (2\epsilon\sqrt{1+\delta_{2s}} + \sqrt{2}\,\delta_{2s} \sum_{j \geq 2} \|h_{T_j}\|_2)$
As, Φ obeys RIP of order 2s,

$$\begin{gathered} \sqrt{1-\delta_{2s}}\ \|h_{T_0 \cup T_1}\|_2 \leq \|\Phi h_{T_0 \cup T_1}\|_2 \leq \sqrt{1+\delta_{2s}}\ \|h_{T_0 \cup T_1}\|_2 \\ \Rightarrow (1 - \delta_{2s})\ \|h_{T_0 \cup T_1}\|_2^2 \leq \|\Phi h_{T_0 \cup T_1}\|_2^2 \Rightarrow \textbf{equation 9} \end{gathered}$$

As all T_j's are disjoint sets,

$$\begin{gathered} h = h_{T_0} + h_{T_1} + h_{T_2} + h_{T_3} \dots \\ \Rightarrow h = h_{T_0 \cup T_1} + \sum_{j \geq 2} h_{T_j} \\ \Rightarrow h_{T_0 \cup T_1} = h - \sum_{j \geq 2} h_{T_j} \\ \Rightarrow \Phi h_{T_0 \cup T_1} = \Phi h - \sum_{j \geq 2} \Phi h_{T_j} \\ \Rightarrow \|\Phi h_{T_0 \cup T_1}\|_2^2 = \langle \Phi h, \Phi h_{T_0 \cup T_1} \rangle - \langle \sum_{j \geq 2} \Phi h_{T_j}, \Phi h_{T_0 \cup T_1} \rangle \end{gathered}$$

By triangle inequality theorem,

$$\Rightarrow \|\Phi h_{T_0 \cup T_1}\|_2^2 \leq |\langle \Phi h, \Phi h_{T_0 \cup T_1} \rangle| + |\langle \sum_{j \geq 2} \Phi h_{T_j}, \Phi h_{T_0 \cup T_1} \rangle|$$

By using inequality $|\langle \Phi h_{T_0 \cup T_1}, \Phi h \rangle| \leq \|\Phi h_{T_0 \cup T_1}\|_2\ \|\Phi h\|_2 \leq 2\epsilon\sqrt{1+\delta_{2s}}\ \|h_{T_0 \cup T_1}\|_2$ from part (9),

$$\Rightarrow \|\Phi h_{T_0 \cup T_1}\|_2^2 \leq 2\epsilon\sqrt{1+\delta_{2s}}\ \|h_{T_0 \cup T_1}\|_2 + |\langle \sum_{j \geq 2} \Phi h_{T_j}, h_{T_0 \cup T_1} \rangle| \Rightarrow \textbf{equation 10}$$

As, T_0 and T_1 are disjoint sets,

$$\begin{gathered} h_{T_0 \cup T_1} = h_{T_0} + h_{T_1} \\ \Rightarrow \Phi h_{T_0 \cup T_1} = \Phi h_{T_0} + \Phi h_{T_1} \\ \Rightarrow \langle \Phi h_{T_0 \cup T_1} \sum_{j \geq 2} \Phi h_{T_j} \rangle = \langle \Phi h_{T_0}, \sum_{j \geq 2} \Phi h_{T_j} \rangle + \langle \Phi h_{T_1}, \sum_{j \geq 2} \Phi h_{T_j} \rangle \\ \Rightarrow \langle \Phi h_{T_0 \cup T_1} \sum_{j \geq 2} \Phi h_{T_j} \rangle = \sum_{j \geq 2} \langle \Phi h_{T_0}, \Phi h_{T_j} \rangle + \sum_{j \geq 2} \langle \Phi h_{T_1}, \Phi h_{T_j} \rangle \end{gathered}$$

By using triangle inequality,

$$\Rightarrow \langle \Phi h_{T_0 \cup T_1} \sum_{j \geq 2} \Phi h_{T_j} \rangle \leq \sum_{j \geq 2} |\langle \Phi h_{T_0}, \Phi h_{T_j} \rangle| + \sum_{j \geq 2} |\langle \Phi h_{T_1}, \Phi h_{T_j} \rangle| \Rightarrow \textbf{equation 11}$$

As all h_{T_j}'s are s-sparse and by using inequality $|\langle \Phi x, \Phi x' \rangle| \leq \delta_{s+s'} \ \|x\|_2 \, \|x'\|_2$ Lemma 2.1 from paper,

$$|\langle \Phi h_{T_0}, \Phi h_{T_j} \rangle| \leq \delta_{2s} \ \|h_{T_0}\|_2 \left\|h_{T_j}\right\|_2 \text{ and } |\langle \Phi h_{T_1}, \Phi h_{T_j} \rangle| \leq \delta_{2s} \ \|h_{T_1}\|_2 \left\|h_{T_j}\right\|_2 \ \forall \ j \geq 2$$
$$\Rightarrow \sum_{j \geq 2} |\langle \Phi h_{T_0}, \Phi h_{T_j} \rangle| \leq \sum_{j \geq 2} \delta_{2s} \ \|h_{T_0}\|_2 \left\|h_{T_j}\right\|_2 \text{ and } \sum_{j \geq 2} |\langle \Phi h_{T_1}, \Phi h_{T_j} \rangle| \leq \sum_{j \geq 2} \delta_{2s} \ \|h_{T_1}\|_2 \left\|h_{T_j}\right\|_2$$
$$\Rightarrow \sum_{j \geq 2} |\langle \Phi h_{T_0}, \Phi h_{T_j} \rangle| + \sum_{j \geq 2} |\langle \Phi h_{T_1}, \Phi h_{T_j} \rangle| \leq \sum_{j \geq 2} \delta_{2s} \ \|h_{T_0}\|_2 \left\|h_{T_j}\right\|_2 + \sum_{j \geq 2} \delta_{2s} \ \|h_{T_1}\|_2 \left\|h_{T_j}\right\|_2$$
$$\Rightarrow \sum_{j \geq 2} |\langle \Phi h_{T_0}, \Phi h_{T_j} \rangle| + \sum_{j \geq 2} |\langle \Phi h_{T_1}, \Phi h_{T_j} \rangle| \leq \sum_{j \geq 2} \delta_{2s} \left(\|h_{T_0}\|_2 + \|h_{T_1}\|_2\right) \left\|h_{T_j}\right\|_2$$

By using inequality $\|h_{T_0}\|_2 + \|h_{T_1}\|_2 \leq \sqrt{2} \ \|h_{T_0 \cup T_1}\|_2$ from part (11),

$$\Rightarrow \sum_{j \geq 2} |\langle \Phi h_{T_0}, \Phi h_{T_j} \rangle| + \sum_{j \geq 2} |\langle \Phi h_{T_1}, \Phi h_{T_j} \rangle| \leq \sum_{j \geq 2} \delta_{2s} \sqrt{2} \ \|h_{T_0 \cup T_1}\|_2 \left\|h_{T_j}\right\|_2 \Rightarrow \textbf{equation 12}$$

By combining equations (11) and (12),

$$\langle \Phi h_{T_0 \cup T_1} \sum_{j \geq 2} \Phi h_{T_j} \rangle \leq \sum_{j \geq 2} \delta_{2s} \sqrt{2} \ \|h_{T_0 \cup T_1}\|_2 \left\|h_{T_j}\right\|_2 \Rightarrow \textbf{equation 13}$$

By combining equations (10) and (13),

$$\|\Phi h_{T_0 \cup T_1}\|_2^2 \leq 2\epsilon\sqrt{1+\delta_{2s}} \ \|h_{T_0 \cup T_1}\|_2 + \sum_{j \geq 2} \delta_{2s} \sqrt{2} \ \|h_{T_0 \cup T_1}\|_2 \left\|h_{T_j}\right\|_2$$
$$\Rightarrow \|\Phi h_{T_0 \cup T_1}\|_2^2 \leq \|h_{T_0 \cup T_1}\|_2 \ (2\epsilon\sqrt{1+\delta_{2s}} + \sqrt{2}\,\delta_{2s} \sum_{j \geq 2} \left\|h_{T_j}\right\|_2) \Rightarrow \textbf{equation 14}$$

By combining equations (9) and (14),

$$(1-\delta_{2s}) \ \|h_{T_0 \cup T_1}\|_2^2 \leq \|\Phi h_{T_0 \cup T_1}\|_2^2 \leq \|h_{T_0 \cup T_1}\|_2 \ (2\epsilon\sqrt{1+\delta_{2s}} + \sqrt{2}\,\delta_{2s} \sum_{j \geq 2} \left\|h_{T_j}\right\|_2)$$

(13)

To justify, $\|h_{T_0 \cup T_1}\|_2 \leq \alpha\epsilon + \rho s^{-1/2} \left\|h_{T_0^c}\right\|_1 \quad \alpha \equiv \frac{2\sqrt{1+\delta_{2s}}}{1-\delta_{2s}}, \rho \equiv \frac{\sqrt{2}\delta_{2s}}{1-\delta_{2s}}$

Combining inequalities,

$$\sum_{j \geq 2} \left\|h_{T_j}\right\|_2 \leq s^{-1/2}(\|h_{T_1}\|_1 + \|h_{T_2}\|_1 + \ldots) \leq s^{-1/2} \ \left\|h_{T_0^c}\right\|_1 \text{ from part (3) and}$$
$$(1-\delta_{2s}) \ \|h_{T_0 \cup T_1}\|_2^2 \leq \|\Phi h_{T_0 \cup T_1}\|_2^2 \leq \|h_{T_0 \cup T_1}\|_2 \ (2\epsilon\sqrt{1+\delta_{2s}} + \sqrt{2}\,\delta_{2s} \sum_{j \geq 2} \left\|h_{T_j}\right\|_2) \text{ from part (12)}$$

We get,

$$\sum_{j \geq 2} \left\|h_{T_j}\right\|_2 \leq s^{-1/2} \ \left\|h_{T_0^c}\right\|_1 \text{ and } (1-\delta_{2s}) \ \|h_{T_0 \cup T_1}\|_2^2 \leq \|h_{T_0 \cup T_1}\|_2 \ (2\epsilon\sqrt{1+\delta_{2s}} + \sqrt{2}\,\delta_{2s} \sum_{j \geq 2} \left\|h_{T_j}\right\|_2)$$
$$\Rightarrow (1-\delta_{2s}) \ \|h_{T_0 \cup T_1}\|_2^2 \leq \|h_{T_0 \cup T_1}\|_2 \ (2\epsilon\sqrt{1+\delta_{2s}} + \sqrt{2}\,\delta_{2s}\, s^{-1/2} \ \left\|h_{T_0^c}\right\|_1)$$
$$\Rightarrow (1-\delta_{2s}) \ \|h_{T_0 \cup T_1}\|_2 \leq (2\epsilon\sqrt{1+\delta_{2s}} + \sqrt{2}\,\delta_{2s}\, s^{-1/2} \ \left\|h_{T_0^c}\right\|_1)$$
$$\Rightarrow \|h_{T_0 \cup T_1}\|_2 \leq (\tfrac{2\sqrt{1+\delta_{2s}}}{1-\delta_{2s}}\, \varepsilon + \tfrac{\sqrt{2}\delta_{2s}}{1-\delta_{2s}}\, s^{-1/2} \ \left\|h_{T_0^c}\right\|_1)$$

Therefore, $\|h_{T_0 \cup T_1}\|_2 \leq \alpha\epsilon + \rho s^{-1/2} \left\|h_{T_0^c}\right\|_1 \quad \alpha \equiv \frac{2\sqrt{1+\delta_{2s}}}{1-\delta_{2s}}, \rho \equiv \frac{\sqrt{2}\delta_{2s}}{1-\delta_{2s}}$

(14)

To justify, $\|h_{T_0 \cup T_1}\|_2 \leq (1-\rho)^{-1}(\alpha\epsilon + 2\rho e_0) \quad \alpha \equiv \frac{2\sqrt{1+\delta_{2s}}}{1-\delta_{2s}}, \rho \equiv \frac{\sqrt{2}\delta_{2s}}{1-\delta_{2s}}, e_0 \equiv s^{-1/2} \ \|x - x_s\|_{l_1}$

Combining inequalities,

$$\|h_{T_0 \cup T_1}\|_2 \leq \alpha\epsilon + \rho s^{-1/2} \left\|h_{T_0^c}\right\|_1 \quad \alpha \equiv \frac{2\sqrt{1+\delta_{2s}}}{1-\delta_{2s}}, \rho \equiv \frac{\sqrt{2}\delta_{2s}}{1-\delta_{2s}} \text{ from part (13) and}$$
$$\left\|h_{T_0^c}\right\|_{l_1} \leq \|h_{T_0}\|_{l_1} + 2 \ \left\|x_{T_0^c}\right\|_{l_1} \text{ from part (7)}$$

We get,

$$\|h_{T_0 \cup T_1}\|_2 \leq \alpha\epsilon + \rho s^{-1/2} \|h_{T_0}\|_{l_1} + 2\rho s^{-1/2} \ \left\|x_{T_0^c}\right\|_{l_1}$$
$$\Rightarrow \|h_{T_0 \cup T_1}\|_2 \leq \alpha\epsilon + \rho s^{-1/2} \|h_{T_0}\|_{l_1} + 2\rho e_0$$

By using equation (5) from part (8),

$$||h_{T_0}||_1 \leq s^{1/2}||h_{T_0}||_2 \text{ and } \|h_{T_0 \cup T_1}\|_2 \leq \alpha\epsilon + \rho s^{-1/2} \|h_{T_0}\|_{l_1} + 2\rho e_0$$
$$\Rightarrow \|h_{T_0 \cup T_1}\|_2 \leq \alpha\epsilon + \rho \|h_{T_0}\|_{l_2} + 2\rho e_0$$
$$\Rightarrow (1-\rho) \|h_{T_0 \cup T_1}\|_2 \leq \alpha\epsilon + 2\rho e_0$$
$$\Rightarrow \|h_{T_0 \cup T_1}\|_2 \leq (1-\rho)^{-1}(\alpha\epsilon + 2\rho e_0)$$

Therefore, $\|h_{T_0 \cup T_1}\|_2 \leq (1-\rho)^{-1}(\alpha\epsilon + 2\rho e_0)$.

(15)

To justify, $\|h\|_2 \leq \|h_{T_0 \cup T_1}\|_2 + \left\|h_{(T_0 \cup T_1)^c}\right\|_2 \leq 2\|h_{T_0 \cup T_1}\|_2 + 2e_0 \leq 2\,(1-\rho)^{-1}\,(\alpha\epsilon + (1+\rho)e_0)$

As all T_j's are disjoint sets,

$$h = h_{T_0} + h_{T_1} + h_{T_2} + h_{T_3} + \dots$$
$$\Rightarrow h = h_{T_0 \cup T_1} + h_{(T_0 \cup T_1)^c}$$
$$\Rightarrow \|h\|_2 = \left\|h_{T_0 \cup T_1} + h_{(T_0 \cup T_1)^c}\right\|_2$$

By triangle inequality theorem,

$$\Rightarrow \|h\|_2 \leq \|h_{T_0 \cup T_1}\|_2 + \left\|h_{(T_0 \cup T_1)^c}\right\|_2 \Rightarrow \textbf{equation 15}$$

By using inequality $\left\|h_{(T_0 \cup T_1)^c}\right\|_{l_2} \leq \|h_{T_0}\|_{l_2} + 2e_0, \quad e_0 \equiv s^{-1/2} \ \|x - x_s\|_{l_1}$ from part (8).
We get,

$$\|h_{T_0 \cup T_1}\|_2 + \left\|h_{(T_0 \cup T_1)^c}\right\|_2 \leq \|h_{T_0 \cup T_1}\|_2 + \|h_{T_0}\|_{l_2} + 2e_0$$

As, $\|h_{T_0 \cup T_1}\|_2^2 = \|h_{T_0}\|_2^2 + \|h_{T_1}\|_2^2 \Rightarrow \|h_{T_0}\|_2 \leq \|h_{T_0 \cup T_1}\|_2$

$$\Rightarrow \|h_{T_0 \cup T_1}\|_2 + \left\|h_{(T_0 \cup T_1)^c}\right\|_2 \leq \|h_{T_0 \cup T_1}\|_2 + \|h_{T_0 \cup T_1}\|_2 + 2e_0$$
$$\Rightarrow \|h_{T_0 \cup T_1}\|_2 + \left\|h_{(T_0 \cup T_1)^c}\right\|_2 \leq 2\|h_{T_0 \cup T_1}\|_2 + 2e_0 \Rightarrow \textbf{equation 16}$$

By using inequality $\|h_{T_0 \cup T_1}\|_2 \leq (1-\rho)^{-1}(\alpha\epsilon + 2\rho e_0)$ from part (14),

$$2\|h_{T_0 \cup T_1}\|_2 + 2e_0 \leq 2(1-\rho)^{-1}(\alpha\epsilon + 2\rho e_0) + 2e_0$$
$$\Rightarrow 2\|h_{T_0 \cup T_1}\|_2 + 2e_0 \leq 2\|h_{T_0 \cup T_1}\|_2 + 2e_0 \leq 2\,(1-\rho)^{-1}\,(\alpha\epsilon + (1+\rho)e_0) \Rightarrow \textbf{equation 17}$$

Combining equations 15, 16 and 17,

$$\|h\|_2 \leq \|h_{T_0 \cup T_1}\|_2 + \left\|h_{(T_0 \cup T_1)^c}\right\|_2 \leq 2\|h_{T_0 \cup T_1}\|_2 + 2e_0 \leq 2\,(1-\rho)^{-1}\,(\alpha\epsilon + (1+\rho)e_0)$$

(16)

To justify, $\|h\|_1 = \|h_{T_0}\|_1 + \|h_{T_0{}^c}\|_1 \leq 2(1+\rho)(1-\rho)^{-1} \left\|x_{T_0^c}\right\|_1$
As all T_j's are disjoint sets,

$$h = h_{T_0} + h_{T_1} + \dots$$
$$\Rightarrow h = h_{T_0} + h_{T_0{}^c}$$
$$\Rightarrow \|h\|_1 = \|h_{T_0}\|_1 + \|h_{T_0{}^c}\|_1 \Rightarrow \textbf{equation 18}$$

As, $\|h_{T_0 \cup T_1}\|_2^2 = \|h_{T_0}\|_2^2 + \|h_{T_1}\|_2^2 \Rightarrow \|h_{T_0}\|_2 \leq \|h_{T_0 \cup T_1}\|_2$ and by using equation (5) from part (8),

$$||h_{T_0}||_1 \leq s^{1/2} ||h_{T_0}||_2 \text{ and } \|h_{T_0}\|_2 \leq \|h_{T_0 \cup T_1}\|_2$$
$$\Rightarrow ||h_{T_0}||_1 \leq s^{1/2} \|h_{T_0 \cup T_1}\|_2 \Rightarrow \textbf{equation 19}$$

By using inequality $\|h_{T_0 \cup T_1}\|_2 \leq \alpha\epsilon + \rho s^{-1/2} \left\|h_{T_0^c}\right\|_1 \quad \alpha \equiv \frac{2\sqrt{1+\delta_{2s}}}{1-\delta_{2s}}, \rho \equiv \frac{\sqrt{2}\delta_{2s}}{1-\delta_{2s}}$ from part (13) with $\varepsilon = 0$

$$\|h_{T_0 \cup T_1}\|_2 \leq \alpha\epsilon + \rho s^{-1/2} \left\|h_{T_0^c}\right\|_1 \text{ and } \varepsilon = 0$$
$$\Rightarrow \|h_{T_0 \cup T_1}\|_2 \leq \rho s^{-1/2} \left\|h_{T_0^c}\right\|_1$$
$$\Rightarrow s^{1/2} \|h_{T_0 \cup T_1}\|_2 \leq \rho \left\|h_{T_0^c}\right\|_1 \Rightarrow \textbf{equation 20}$$

Combining equations 19 and 20,

$$||h_{T_0}||_1 \leq \rho \left\|h_{T_0^c}\right\|_1 \Rightarrow \textbf{equation 21}$$

By using inequality $\left\|h_{T_0^c}\right\|_{l_1} \leq \|h_{T_0}\|_{l_1} + 2 \left\|x_{T_0^c}\right\|_{l_1}$ from part (7) and equation 21,

$$||h_{T_0}||_1 \leq \rho \left\|h_{T_0^c}\right\|_1 \text{ and } \left\|h_{T_0^c}\right\|_{l_1} \leq \|h_{T_0}\|_{l_1} + 2 \left\|x_{T_0^c}\right\|_{l_1}$$
$$\Rightarrow \left\|h_{T_0^c}\right\|_{l_1} \leq \rho \left\|h_{T_0^c}\right\|_1 + 2 \left\|x_{T_0^c}\right\|_{l_1}$$
$$\Rightarrow (1-\rho) \left\|h_{T_0^c}\right\|_{l_1} \leq 2 \left\|x_{T_0^c}\right\|_{l_1}$$
$$\Rightarrow \left\|h_{T_0^c}\right\|_{l_1} \leq (1-\rho)^{-1} 2 \left\|x_{T_0^c}\right\|_{l_1}$$
$$\Rightarrow (1+\rho) \left\|h_{T_0^c}\right\|_{l_1} \leq 2(1+\rho)(1-\rho)^{-1} \left\|x_{T_0^c}\right\|_{l_1} \Rightarrow \textbf{equation 22}$$

Adding $\left\|h_{T_0^c}\right\|_1$ on both sides of equation 21,

$$||h_{T_0}||_1 + \left\|h_{T_0^c}\right\|_1 \leq (1+\rho) \left\|h_{T_0^c}\right\|_1 \Rightarrow \textbf{equation 23}$$

Combining equations 18,21 and 22,

$$\|h\|_1 = \|h_{T_0}\|_1 + \|h_{T_0{}^c}\|_1 \leq 2(1+\rho)(1-\rho)^{-1} \left\|x_{T_0^c}\right\|_1$$

Question 2

a)

Oracular solution $\tilde{x}$ using measurement submatrix Φ_S,

$$\tilde{x} = min_x \|y - \Phi_S x\|_2$$

Taking derivative to find the minimum w.r.t x,

$$\begin{aligned} \frac{\partial}{\partial x} \|y - \Phi_S x\|_2^2 = 0 \\ \Rightarrow 2(\frac{\partial}{\partial x}(y - \Phi_S x))(y - \Phi_S x) = 0 \\ \Rightarrow 2\Phi_S^T(y - \Phi_S x) = 0 \\ \Rightarrow \Phi_S^T \Phi_S x = \Phi_S^T y \end{aligned}$$

As, $\Phi_S^T \Phi_S$ is a invertible matrix,

$$\Rightarrow \tilde{x} = \left(\Phi_S^T \Phi_S\right)^{-1} \Phi_S^T y$$

b)

As, $\tilde{x} = \left(\Phi_S^T \Phi_S\right)^{-1} \Phi_S^T y$ and $y = \Phi x + \eta$,

$$\begin{aligned} \|x - \tilde{x}\|_2 &= \left\| x - \left(\Phi_S^T \Phi_S\right)^{-1} \Phi_S^T y \right\|_2 \\ &= \left\| x - \left(\Phi_S^T \Phi_S\right)^{-1} \Phi_S^T (\Phi x + \eta) \right\|_2 \\ &= \left\| x - x - \left(\Phi_S^T \Phi_S\right)^{-1} \Phi_S^T \eta \right\|_2 \\ &= \left\| \Phi_S^\dagger \eta \right\|_2 \end{aligned}$$

By Cauchy–Schwartz inequality theorem, $\left\| \Phi_S^\dagger \eta \right\|_2 \leq \left\| \Phi_S^\dagger \right\|_2 \|\eta\|_2$

$$\|x - \tilde{x}\|_2 = \left\| \Phi_S^\dagger \eta \right\|_2 \leq \left\| \Phi_S^\dagger \right\|_2 \|\eta\|_2$$

c)

Given, $k = |S|$ and δ_{2k} is the RIC of Φ of order 2k
For a 2k sparse vector θ, δ_{2k} is the minimum value such that,

$$\begin{aligned} (1 - \delta_{2k}) \|\theta\|_2^2 \leq \|\Phi\theta\|_2^2 \leq (1 + \delta_{2k}) \|\theta\|_2^2 \\ \Rightarrow (1 - \delta_{2k}) \leq \frac{\|\Phi\theta\|_2^2}{\|\theta\|_2^2} \leq (1 + \delta_{2k}) \end{aligned}$$

As, $\lambda_{max} = max_{\theta \epsilon R^s} \frac{\|A\theta\|^2}{\|\theta\|^2}$ and $\lambda_{min} = min_{\theta \epsilon R^s} \frac{\|A\theta\|^2}{\|\theta\|^2}$ (λ_{max} and λ_{min} are maximum and minimum eigenvalues of $A^T A$). Let $\lambda_{\Phi,max}$ and $\lambda_{\Phi,min}$ be the max and min eigenvalues of Φ.

$$\Rightarrow \sqrt{(1 - \delta_{2k})} \leq \lambda_{\Phi,min} \leq \lambda_{\Phi,max} \leq \sqrt{(1 + \delta_{2k})}$$

As, $\Phi_S^\dagger = (\Phi_S^T \Phi_S)^{-1}\Phi_S^T$
By Singular value decomposition, $\Phi = USV^T$

$$\begin{aligned}\Phi_S^\dagger &= (\Phi_S^T \Phi_S)^{-1}\Phi_S^T \\ &= (VSU^TUSV^T)^{-1}VSU^T \\ &= (VS^2V^T)-1VSU^T \\ &= VS^{-2}V^TVSU^T \\ &= VS^{-1}U^T\end{aligned}$$

Therefore, the singular values of $\Phi_S^\dagger$ are inverse of singular values of Φ. This implies the largest singular value of $\Phi_S^\dagger$ is between $\frac{1}{\sqrt{1+\delta_{2k}}}$ and $\frac{1}{\sqrt{1-\delta_{2k}}}$.

d)

As,

$$\begin{aligned}\|x-\tilde{x}\|_2 = \left\|\Phi_S^\dagger \eta\right\|_2 \leq \left\|\Phi_S^\dagger\right\|_2 \|\eta\|_2 \text{ and } \|\eta\|_2 \leq \epsilon \\ \Rightarrow \|x-\tilde{x}\|_2 = \left\|\Phi_S^\dagger \eta\right\|_2 \leq \left\|\Phi_S^\dagger\right\|_2 \epsilon \\ \Rightarrow \tfrac{\epsilon}{\sqrt{1+\delta_{2k}}} \leq \|x-\tilde{x}\|_2 \leq \tfrac{\epsilon}{\sqrt{1-\delta_{2k}}}\end{aligned}$$

By theorem 3 ($h = x^* - x_0$),

$$\begin{aligned}\|h\|_2 \leq \tfrac{4\epsilon\sqrt{1+\delta_{2k}}}{1-\delta_{2k}(\sqrt{2}+1)} \\ \Rightarrow \|x-x\|_2 \leq \tfrac{4\epsilon\sqrt{1+\delta_{2k}}}{1-\delta_{2k}(\sqrt{2}+1)} \\ \Rightarrow \|x-x\|_2 \leq (\tfrac{\epsilon}{\sqrt{1-\delta_{2k}}})(\tfrac{4\sqrt{1+\delta_{2k}}}{\sqrt{1-\delta_{2k}}(1+\sqrt{2})}) \\ \Rightarrow \|x-x\|_2 \leq (\tfrac{\epsilon}{\sqrt{1-\delta_{2k}}})C \text{ and } C = \tfrac{4\sqrt{1+\delta_{2k}}}{\sqrt{1-\delta_{2k}}(1+\sqrt{2})}\end{aligned}$$

As, $\delta_{2k} \geq 0 \Rightarrow \frac{\sqrt{1+\delta_{2k}}}{\sqrt{1-\delta_{2k}}} \geq 1$

$$\begin{aligned}C = \tfrac{4\sqrt{1+\delta_{2k}}}{\sqrt{1-\delta_{2k}}(1+\sqrt{2})} \\ \Rightarrow C \geq 1\end{aligned}$$

Therefore, the upper bound of theorem 3 is greater than the upper bound with oracle solutions.

Question 3

If sensing matrix Φ satisfies restricted isometry property of order k then δ_k is the minimum value such that,

$$\begin{aligned}(1-\delta_k)\|\theta\|^2 \leq \|\Phi\theta\|^2 \leq (1+\delta_k)\ \|\theta\|^2 \\ \Rightarrow (1-\delta_k) \leq \tfrac{\|\Phi\theta\|^2}{\|\theta\|^2} \leq (1+\delta_k) \\ \Rightarrow 1 - \tfrac{\|\Phi\theta\|^2}{\|\theta\|^2} \leq \delta_k \text{ and } \tfrac{\|\Phi\theta\|^2}{\|\theta\|^2} - 1 \leq \delta_k\end{aligned}$$

As δ_{2k} is the minimum value satisfying above inequalities, (Γ is the subset of columns of Φ whose maximum size is less than or equal to k)

$$\Rightarrow \delta_k = max\{1 - min_{\theta_\Gamma,|\Gamma|\leq K}\tfrac{\|\Phi_\Gamma\theta_\Gamma\|^2}{\|\theta_\Gamma\|^2}\ ,\ max_{\theta_\Gamma,|\Gamma|\leq K}\tfrac{\|\Phi_\Gamma\theta_\Gamma\|^2}{\|\theta_\Gamma\|^2} - 1\}$$

As, $\lambda_{max} = max_{\theta_\Gamma,|\Gamma|\leq K}\frac{\|\Phi_\Gamma\theta_\Gamma\|^2}{\|\theta_\Gamma\|^2}$ and $\lambda_{min} = min_{\theta_\Gamma,|\Gamma|\leq K}\frac{\|\Phi_\Gamma\theta_\Gamma\|^2}{\|\theta_\Gamma\|^2}$ where λ_{max} and λ_{min} are the maximum and minimum eigenvalues of matrix $(\Phi_\Gamma)^T(\Phi_\Gamma)$ over all sets Γ,

$$\Rightarrow \delta_k = max\{1-\lambda_{min}\ ,\ \lambda_{max}-1\}$$

Let, Γ_s and Γ_s be the set of columns of Φ whose maximum size is less than or equal to s and t respectively.
If $s \leq t$ then Γ_t covers all the sets covered by the Γ_s and some extra sets as Γ_k is the subset of columns of Φ whose maximum is less than or equal to k. Therefore,

$$\lambda_{max,t} = max_{\theta_\Gamma,|\Gamma|\leq t}\frac{\|\Phi_\Gamma\theta_\Gamma\|^2}{\|\theta_\Gamma\|^2} \text{ and } \lambda_{min,t} = min_{\theta_\Gamma,|\Gamma|\leq t}\frac{\|\Phi_\Gamma\theta_\Gamma\|^2}{\|\theta_\Gamma\|^2}$$
$$\Rightarrow \lambda_{max,t} = max\{max_{\theta_\Gamma,|\Gamma|\leq s}\frac{\|\Phi_\Gamma\theta_\Gamma\|^2}{\|\theta_\Gamma\|^2}, max_{\theta_\Gamma,s\leq|\Gamma|\leq t}\frac{\|\Phi_\Gamma\theta_\Gamma\|^2}{\|\theta_\Gamma\|^2}\} \text{ and}$$
$$\lambda_{min,t} = min\{min_{\theta_\Gamma,|\Gamma|\leq s}\frac{\|\Phi_\Gamma\theta_\Gamma\|^2}{\|\theta_\Gamma\|^2}, min_{\theta_\Gamma,s\leq|\Gamma|\leq t}\frac{\|\Phi_\Gamma\theta_\Gamma\|^2}{\|\theta_\Gamma\|^2}\}$$
$$\Rightarrow \lambda_{max,t} = max\{\lambda_{max,s}, max_{\theta_\Gamma,s\leq|\Gamma|\leq t}\frac{\|\Phi_\Gamma\theta_\Gamma\|^2}{\|\theta_\Gamma\|^2}\} \text{ and } \lambda_{min,t} = min\{\lambda_{min,s}, min_{\theta_\Gamma,s\leq|\Gamma|\leq t}\frac{\|\Phi_\Gamma\theta_\Gamma\|^2}{\|\theta_\Gamma\|^2}\}$$
$$\Rightarrow \lambda_{max,t} \geq \lambda_{max,s} \text{ and } \lambda_{min,t} \leq \lambda_{min,s}$$
$$\Rightarrow \lambda_{max,t} - 1 \geq \lambda_{max,s} - 1 \text{ and 1- } \lambda_{min,t} \geq 1 - \lambda_{min,s}$$
$$\Rightarrow \delta_t \geq \delta_s$$

Hence proved.

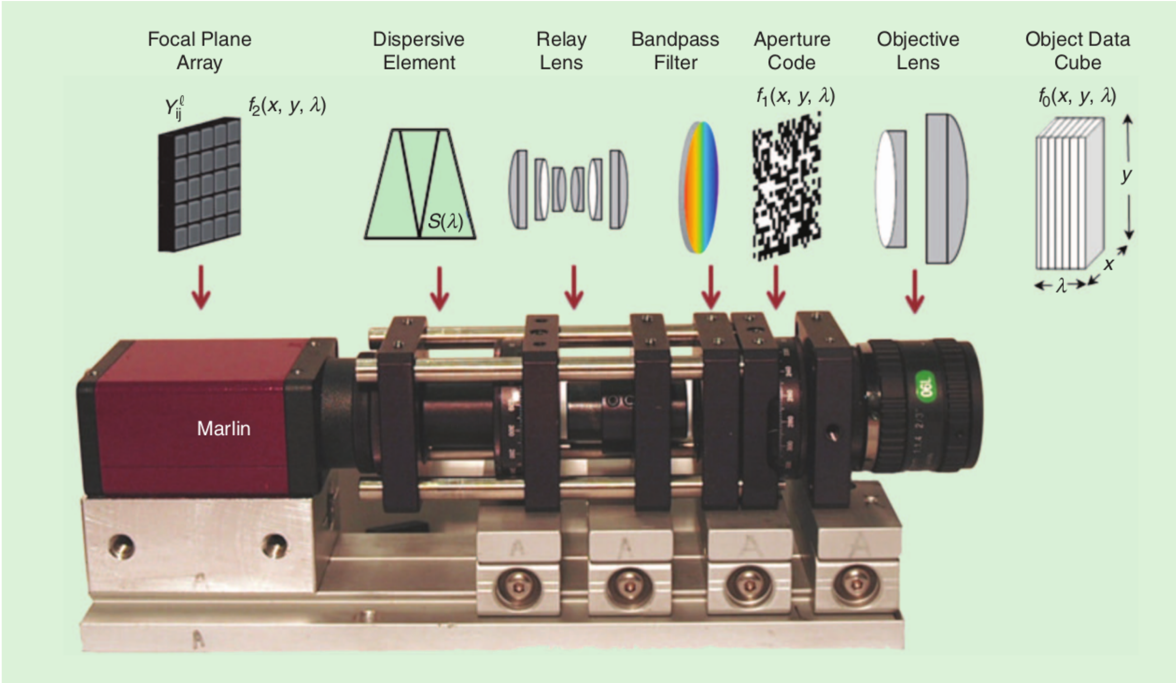
Question 4

a)

- **Title of the Paper :** Compressive Coded Aperture Spectral Imaging
- **Authors :** Gonzalo R Arce, David J Brady, Lawrence Carin, Henry Arguello, David S Kittle
- **Published on :** 03 December 2013
- **Venue :** Office of Naval Research
- **Link :** `https://ieeexplore.ieee.org/stamp/stamp.jsp?arnumber=6678264`

b)

The imaging system in this paper is coded aperture snapshot spectral imagers (CASSI). It has a coded aperture, a dispersive element such as a prism and an one focal plane array(FPA) detector. By means of a coded aperture as shown in the below image, the coding is applied to the image source density and this resulting coded field is subsequently modified by a dispersive element before it strikes the FPA detector. Finally, by the integration of the dispersed field, compressive measurements are realized across the FPA detector.



[FIG1] Compressive CASSI sensor components. (Image courtesy of David J. Brady and David S. Kittle.)

c)

Sensing matrix is a block diagonal matrix whose ith diagonal matrix for l-th spectral band is the vectorized form of the coded aperture for the ith snapshot at the shift for the l-th spectral band. If data at each wavelength is a 2D-image of size N_x x N_y where the number of wavelengths is N_λ then the sensing matrix size is N_xN_y x $N_xN_yN_\lambda$. RIC minimization is done for sensing matrix design.
RIP inequality is written as,

$$\|A\theta\|_2^2 - \|\theta\|_2^2 \leq \lambda_s \|\theta\|_2^2$$

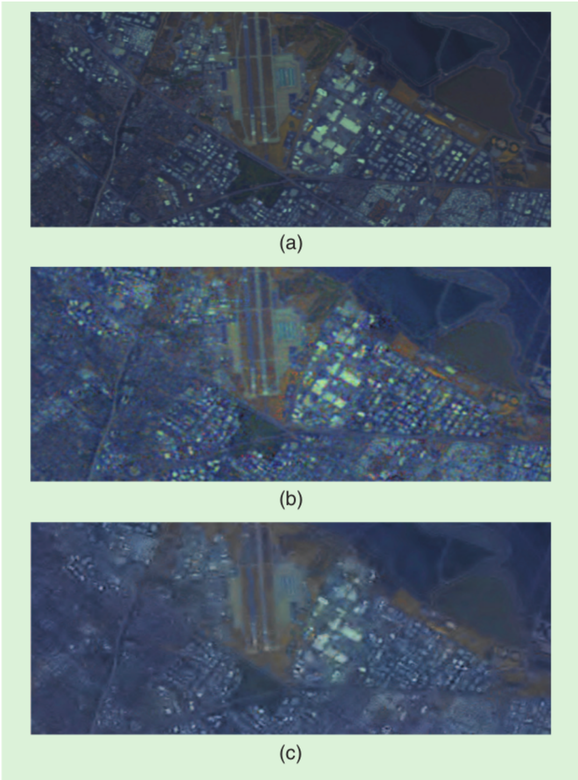
Constraining the vector θ to $\|\theta\|_2^2 = 1$, taking the supremum over all the vectors θ with $supp(\theta) \subset \Gamma$, $|\gamma| \leq S$, and taking the maximum with respect to all subsets γ leads to $\lambda_s = max_{\gamma\subset[n]|\gamma|\leq s}\lambda_{max}(A^T_{|\lambda|}A_{|\lambda|} - I_{|\lambda|})$ where $n = N^2L$.
The design strategy is then formulated as seeking the set of coded apertures $\{T^0_{j\,l}\}\dots\{T^{k-1}_{j\,l}\}$, such that

$$\{T^0_{j\,l}\}\dots\{T^{k-1}_{j\,l}\} = argmin_{\{T^0_{j\,l}\}\dots\{T^{k-1}_{j\,l}\}} max_{\gamma\subset[n]|\gamma|\leq s}\lambda_{max}(A^T_{|\lambda|}A_{|\lambda|} - I_{|\lambda|})$$

where $\{T^i_{j\,l_!}\}\{T^i_{j\,l_2}\}$ is the product of two elements of the ith coded aperture, both at jth row, but at the jth row, but at the column positions l_1 and l_2.

e)

Spectrometers based on optical bandpass filters sequentially scan the secene by tunning the bandpass filters in steps. In this method, spectral data cube is obtained by merging the underlying spectral scenes. The problem with spectrometers is that they require scanning a number of zones linearly proportional to the desired spatial and spectral resolution. Whereas CASSI gets the entrie data cube with a few FPA(focal plane Array) measurements by using the compressive sensing principles. Using CASSI model, reconstruction can be done faster compared to the block model. CASSI snapshot spectral images are simple and efficient compared to spectrometers based on optical bandpass filters. The below image illustrates the reconstruction quality attained by CASSI model.



[FIG8] (a) The original RGB and zoomed-in version of the $512 \times 512 \times 32$ data cube. Reconstructions for 10 FPA measurement shots using (b) the block approach with block size $B = 64$, 31.84 dB and (c) the traditional reconstruction approach, 30.99 dB. Block overlapped was used in this example [13].

Question 6

Given, n subjects are being tested by Dorfman pooling and only $k << n$ out of these are infected. These n subjects are divided into groups of size g each.

Average number of tests required

There are n subjects divided into groups of size g. Therefore there are n/g pools.
Let the p be the probablility of being atleast one positive case in each pool.
Where

$$p = 1 - (1 - \frac{R}{n})^g$$

The average no.of tests to be taken is sum of no.of pools to be tested in first round and expected no.of positives to tested in second round $Avg = \frac{n}{g} + pn/g * g$

$$\begin{aligned} Avg &= \frac{n}{g} + g * (p * n/g) \\ &= \frac{n}{g} + n(1 - (1 - \frac{R}{n})^g) \end{aligned}$$