CS 602 Assignment 2

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Question 1

To prove that the tree T that is output by the algorithm given in question is a minimum weight directed tree rooted at r.

Proof:

This problem is a hitting set problem. A directed tree rooted at r is a minimum weight directed tree rooted at r if and only if $|T \cap \delta^{in}(S)| \geq 1$ for every subset S of vertices which do not contain the root vertex r. Here T is the set of edges in minimum weight rooted tree.

Complementary slackness conditions:
x and y are primal and dual optimal solutions if and only if they satisfy complementary slackness that is
1. for each edge $e \epsilon E$, $x_e > 0 \Rightarrow \sum_{S:e\epsilon\delta^{in}(S)} y_S = w_e$
2. for each subset $S \subseteq V - \{r\}$, $y_S \geq 0 \Rightarrow \sum_{e\epsilon\delta^{in}(S)} x_e = 1$

From the step 2 in algorithm, x_e is set to 1 only if the dual constraint for the edge e is tight i.e., if $\sum_{S:e\epsilon\delta^{in}(S)} y_S = w_e$. Hence, first condition of complementary slackness is satisfied by the algorithm.

The second complementary condition is also satisfied because in the step 2 of the algorithm we are choosing a subset S which is a inclusionwise minimal subset i.e., all its proper subsets of S the primal constraint is satisfied. This implies we are choosing the subset S which is a minimal strongly connected component to raise the dual variable y_S.

For the sake of contradiction, assume the second complementary slackness condition is not satisfied that is there is at least one subset S such that $y_S > 0 \Rightarrow \sum_{e\epsilon\delta^{in}(S)} x_e > 1$. This means there are at least two edges say e, e_p belonging to $\delta^{in}(S)$, such that $x_e = 1$ and $x_{e_p} = 1$. This implies the dual variable of a proper subset say S_p of S is tight for the edge belonging to $\delta^{in}(S)$. As, S is a strongly connected sub graph and there are two edges incoming to the set S, this means there is more than one path from every vertex in S to root r if we remove the edge e that is added in step 3 by the dual variable y_S becoming tight then there will be exactly one path. In step 5 of algorithm, that is in the pruning edges are removed in reverse order, if still has a path from the root to every other vertex. Hence, the edge e will be removed in the pruning step as S and S_p are strongly connected sub graphs and there is an edge e_p has already been added to T in step 3 of algorithm when y_{S_p} become tight. Hence, $y_S > 0 \Rightarrow \sum_{e\epsilon\delta^{in}(S)} x_e > 1$ and second complementary condition is also satisfied. Therefore, the tree T that is output by the above algorithm is a minimum weight directed tree rooted at r.

Question 2

Given, three 2-AND clauses, and each clause has an associated weight.

$$\begin{aligned} x_1 \wedge \overline{x}_2 &\rightarrow \textit{weight}\ 15 \\ x_2 \wedge x_3 &\rightarrow \textit{weight}\ 10 \\ \overline{x}_1 \wedge x_3 &\rightarrow \textit{weight}\ 10 \end{aligned}$$

Approximate linear program for this problem :

$$max\ 15z_1 + 10z_2 + 10z_3$$

$$0 \le z_i \le 1 \ \ \forall i \ \epsilon \ \{1,2,3\}$$

$$0 \le t_i, f_i \le 1 \ \ \forall i \ \epsilon \ \{1,2,3\}$$

$$z_1 \le t_1, \ \ z_1 \le f_2$$

$$z_2 \le t_2, \ \ z_2 \le t_3$$

$$z_3 \le f_1, \ \ z_3 \le t_2$$

$$t_i + f_i = 1 \ \ \forall i \ \epsilon \ \{1,2,3\}$$

Here, each z_i is for each clause given. As each each z_i is AND of the literals in the clause therefore it is minimum of values of the literals in the clause.

Randomized (1/2)-approximation algorithm based on LP rounding :

Let, (t^*, f^*, z^*) be the optimal solution of the above LP

Rounding Algorithm to get (1/2)-approximation,
set $x_1 = T$ with probability t_1' where $t_1' = \frac{1}{4}t_1^* + \frac{2}{3}$
set $x_2 = F$ with probability f_2' where $f_2' = \frac{1}{4}f_2^* + \frac{2}{3}$
set $x_3 = T$ with probability t_3' where $t_3' = \frac{1}{4}t_3^* + \frac{3}{4}$
The LP-optimal of approximate LP defined above,

$$LP\ opt = 15z_1^* + 10z_2^* + 10z_3^*$$

As, $z_1 \le t_1$ and $z_1 \le f_2 \ \Rightarrow \ 15z_1^* + 15z_1^* \le 15t_1^* + 15f_2^* \ \Rightarrow \ 15z_1^* \le \frac{15t_1^* + 15f_2^*}{2}$
As, $z_2 \le t_2$ and $z_1 \le t_3 \ \Rightarrow \ 10z_2^* + 10z_2^* \le 10t_2^* + 10t_3^* \ \Rightarrow \ 10z_2^* \le 5t_2^* + 5t_3^*$
As, $z_3 \le f_1$ and $z_1 \le t_3 \ \Rightarrow \ 10z_3^* + 10z_2^* \le 10f_1^* + 10t_3^* \ \Rightarrow \ 10z_3^* \le 5f_1^* + 5t_3^*$

Adding above three inequalities,

$$15z_1^* + 10z_2^* + 10z_3^* \le \frac{15t_1^* + 15f_2^*}{2} + 5t_2^* + 5t_3^* + 5f_1^* + 5t_3^*$$

$$\Rightarrow 15z_1^* + 10z_2^* + 10z_3^* \le \frac{15t_1^* + 15f_2^*}{2} + 5 - 5f_2^* + 5t_3^* + 5 - 5t_1^* + 5t_3^*$$

$$\Rightarrow 15z_1^* + 10z_2^* + 10z_3^* \le \frac{5t_1^* + 5f_2^*}{2} + 10t_3^* + 10$$

$$\Rightarrow LP\ opt \le 10t_3^* + 10 + \frac{5}{2}t_1^* + \frac{5}{2}f_2^* \qquad \rightarrow \textbf{equation (1)}$$

The expected weight of satisfied clauses based on the rounding,

$$
\begin{aligned}
E[w] &= \sum w_i * Probability(clause\ i\ is\ satisfied) \\
&= 15 * Pr(c_1 = T) + 10 * Pr(c_2 = T) + 10 * Pr(c_3 = T) \\
&= 15t'_1 f'_2 + 10t'_2 t'_3 + 10f'_1 t'_3 \\
&= 15t'_1 f'_2 + 10t'_2 t'_3 + 10(1 - t'_1)t'_3 \\
&= 15t'_1 f'_2 + 10(1 - f'_2)t'_3 + 10(1 - t'_1)t'_3 \\
&= 20t'_3 + 15t'_1 f'_2 - 10f'_2 t'_3 - 10t'_1 t'_3
\end{aligned}
$$

As, $t'_3 = \frac{1}{4}t^*_3 + \frac{3}{4}$ and $3/4 \leq t'_3 \leq 1$

$$10f'_2 t'_3 \leq 10f'_2 \ \ and \ \ 10t'_1 t'_3 \leq 10t'_1$$

$$\Rightarrow -10f'_2 t'_3 - 10t'_1 t'_3 \geq -10f'_2 - 10t'_1$$

$$\Rightarrow E[w] \geq 20t'_3 + 15t'_1 f'_2 - 10f'_2 - 10t'_1$$

$$\Rightarrow E[w] \geq 20(\frac{1}{4}t^*_3 + \frac{3}{4}) + 15t'_1 f'_2 - 10f'_2 - 10t'_1$$

$$\Rightarrow E[w] \geq 5t^*_3 + 15 + 15t'_1 f'_2 - 10f'_2 - 10t'_1$$

$$\Rightarrow E[w] \geq 5t^*_3 + 5 + 15t'_1 f'_2 - 5f'_2 - 5t'_1 + 5 - 5f'_2 + 5 - 5t'_1$$

As, $2/3 \leq f'_2 \leq 11/12$ and $2/3 \leq t'_1 \leq 11/12$

$$5 - 5f'_2 + 5 - 5t'_1 \geq 0$$

$$\Rightarrow E[w] \geq 5t^*_3 + 5 + 15t'_1 f'_2 - 5f'_2 - 5t'_1 \quad \rightarrow \textbf{equation (2)}$$

As, $t'_1 = \frac{1}{4}t^*_1 + \frac{2}{3}$ and $f'_2 = \frac{1}{4}f^*_2 + \frac{2}{3}$,

$$
\begin{aligned}
15t'_1 f'_2 - 5f'_2 - 5t'_1 &= 15(\frac{1}{4}t^*_1 + \frac{2}{3})(\frac{1}{4}f^*_2 + \frac{2}{3}) - 5(\frac{1}{4}f^*_2 + \frac{2}{3}) - 5(\frac{1}{4}t^*_1 + \frac{2}{3}) \\
&= \frac{15}{16}t^*_1 f^*_2 + (\frac{5}{2} - \frac{5}{4})t^*_1 + (\frac{5}{2} - \frac{5}{4})f^*_2 \\
&= \frac{15}{16}t^*_1 f^*_2 + \frac{5}{4}t^*_1 + \frac{5}{4}f^*_2 \qquad \rightarrow \textbf{equation (3)}
\end{aligned}
$$

Combining equations (2) and (3),

$$E[w] \geq 5t^*_3 + 5 + \frac{15}{16}t^*_1 f^*_2 + \frac{5}{4}t^*_1 + \frac{5}{4}f^*_2$$

$$\Rightarrow E[w] \geq 5t^*_3 + 5 + \frac{5}{4}t^*_1 + \frac{5}{4}f^*_2$$

$$\Rightarrow 2E[w] \geq 10t^*_3 + 10 + \frac{5}{2}t^*_1 + \frac{5}{2}f^*_2 \qquad \rightarrow \textbf{equation (4)}$$

Combining equations (1) and (4),

$$2E[w] \geq LPopt$$

$$E[w] \geq \frac{1}{2}LPopt \geq \frac{1}{2}Max\ 2\ AND\ optimum$$

Hence, this LP rounding gives (1/2)-approximation.

Question 3

Given, three 2-AND clauses, and each clause has an associated weight.

$$\begin{array}{lcccl} clause\ c_1 & \rightarrow & x_1 \wedge \overline{x}_2 & \rightarrow & weight\ 15 \\ clause\ c_2 & \rightarrow & x_2 \wedge x_3 & \rightarrow & weight\ 10 \\ clause\ c_3 & \rightarrow & \overline{x}_1 \wedge x_3 & \rightarrow & weight\ 10 \end{array}$$

Let, t_i be the value of the literal x_i for all $i \epsilon \{1,2,3\}$ that is if x_i is true then $t_i = 1$ else $t_i = 0$
Let, z_i be the value of the clause c_i for all $i \epsilon \{1,2,3\}$ that is if clause c_i is true then the z_i is 1 else z_i is 0.
Let $y_i = 1$ if $t_i = 1$ else $y_i = -1$ that is each $y_i = 2t_i - 1$ for all $i \epsilon \{1,2,3\}$

As each clause is AND of literals therefore z_i is product of the values of the literals in the clause and $t_i = \frac{1+y_i}{2}$ for all $i \epsilon \{1,2,3\}$.

$$\begin{aligned} z_1 &= t_1 * (1 - t_2) \\ &= (\frac{1+y_1}{2})(1 - \frac{1+y_2}{2}) \\ &= (\frac{1+y_1}{2})(\frac{1-y_2}{2}) \\ &= \frac{1 + y_1 - y_2 - y_1 y_2}{4} \end{aligned}$$

$$\begin{aligned} z_2 &= t_2 * t_3 \\ &= (\frac{1+y_2}{2})(\frac{1+y_3}{2}) \\ &= \frac{1 + y_2 + y_3 + y_2 y_3}{4} \end{aligned}$$

$$\begin{aligned} z_3 &= (1 - t_1) * t_3 \\ &= (1 - \frac{1+y_1}{2})(\frac{1+y_3}{2}) \\ &= (\frac{1-y_1}{2})(\frac{1+y_3}{2}) \\ &= \frac{1 - y_1 + y_3 - y_1 y_3}{4} \end{aligned}$$

Introducing an auxiliary variable y_0 where $y_0^2 = 1$ and replacing each y_i with $y_0 y_i$ to get only degree two terms in each z_i for all $i \epsilon \{1,2,3\}$

$$\begin{aligned} z_1 &= \frac{1 + y_0 y_1 - y_0 y_2 - y_1 y_2}{4} \\ z_2 &= \frac{1 + y_0 y_2 + y_0 y_3 + y_2 y_3}{4} \\ z_3 &= \frac{1 - y_0 y_1 + y_0 y_3 - y_1 y_3}{4} \end{aligned}$$

Let Y be a R^{4x4} matrix where $Y_{ij} = y_i y_j$ (by indexing the rows and columns from 0 to 4) and $Y_{ii} = 1$

Maximizing objective,

$$\begin{aligned}
\sum w_i z_i &= 15z_1 + 10z_2 + 10z_3 \\
&= 15\left(\frac{1+Y_{01}-Y_{02}-Y_{12}}{4}\right) + 10\left(\frac{1+Y_{02}+Y_{03}+Y_{23}}{4}\right) + 10\left(\frac{1-Y_{01}+Y_{03}-Y_{13}}{4}\right) \\
&= \frac{15+10+10}{4} + \frac{15-10}{4}Y_{01} + \frac{-15+10}{4}Y_{02} + \frac{10+10}{4}Y_{03} - \frac{15}{4}Y_{12} - \frac{10}{4}Y_{13} + \frac{10}{4}Y_{23} \\
&= \frac{35}{4} + \frac{5}{4}Y_{01} - \frac{5}{4}Y_{02} + 5Y_{03} - \frac{15}{4}Y_{12} - \frac{5}{2}Y_{13} + \frac{5}{2}Y_{23}
\end{aligned}$$

Now replace each y_i by $y'_i \epsilon R^k$ vector where $||y'_i|| = 1$ to get the approximate SDP for this problem

Approximate SDP for this problem

$$max \;\; \frac{35}{4} + \frac{5}{4}Y_{01} - \frac{5}{4}Y_{02} + 5Y_{03} - \frac{15}{4}Y_{12} - \frac{5}{2}Y_{13} + \frac{5}{2}Y_{23}$$
$$Y \succeq 0, \;\; Y \epsilon R^{4x4} \;\; where \;\; Y_{ij} = y_i'^T y_j' \;, \;\; y_i' \epsilon R^k$$

Randomized 0.79-approximation algorithm based on SDP rounding :

Let (z*,Y*) be the optimal solution of the above SDP

Rounding to get 0.79-approximation,

1. choose $h \epsilon R^k$ randomly uniformly
2. Hyperlane $h^T x = 0$
3. Set x_i to be True if and only if y'^*_i falls on the same side y'^*_0 that is if $(h^T y'^*_i)(h^T y'^*_0) \geq 0$. In particular we are setting y_0 to be 1.

The SDP-optimal of approximate SDP defined above,

Let θ_{ij} be the angle between the vectors y'^*_i and y'^*_j.

$$\begin{aligned}
SDP\ opt &= 15z_1^* + 10z_2^* + 10z_3^* \\
&= 15\left(\frac{1+Y_{01}-Y_{02}-Y_{12}}{4}\right) + 10\left(\frac{1+Y_{02}+Y_{03}+Y_{23}}{4}\right) + 10\left(\frac{1-Y_{01}+Y_{03}-Y_{13}}{4}\right) \\
&= 15\left(\frac{1+cos\theta_{01}-cos\theta_{02}-cos\theta_{12}}{4}\right) + 10\left(\frac{1+cos\theta_{02}+cos\theta_{03}+cos\theta_{23}}{4}\right) \\
&\quad + 10\left(\frac{1-cos\theta_{01}+cos\theta_{03}-cos\theta_{13}}{4}\right) \\
&= 15\left(\frac{1+cos\theta_{01}+cos(\pi-\theta_{02})+cos(\pi-\theta_{12})}{4}\right) + 10\left(\frac{1+cos\theta_{02}+cos\theta_{03}+cos\theta_{23}}{4}\right) \\
&\quad + 10\left(\frac{1+cos(\pi-\theta_{01})+cos\theta_{03}+cos(\pi-\theta_{13})}{4}\right)
\end{aligned}$$

The expected weight of satisfied clauses based on the rounding,

$$
\begin{aligned}
E[w] &= \sum w_i * Probability(clause\ i\ is\ satisfied) \\
&= 15Pr(z_1^* = 1) + 10Pr(z_2^* = T) + 10Pr(z_3^* = T) \\
&= 15Pr(x_1^* = T \wedge x_2^* = F) + 10Pr(x_2^* = T \wedge x_3^* = T) + 10Pr(x_1^* = F \wedge x_3^* = T) \\
&= 15Pr((h^T y'^*_1)(h^T y'^*_0) \geq 0 \wedge (h^T y'^*_2)(h^T y'^*_0) < 0) \\
&\quad + 10Pr((h^T y'^*_2)(h^T y'^*_0) \geq 0 \wedge (h^T y'^*_3)(h^T y'^*_0) \geq 0) \\
&\quad + 10Pr((h^T y'^*_1)(h^T y'^*_0) < 0 \wedge (h^T y'^*_3)(h^T y'^*_0) \geq 0) \\
&= \frac{15}{2}(Pr((h^T y'^*_1)(h^T y'^*_0) \geq 0) + Pr((h^T y'^*_2)(h^T y'^*_0) < 0) - Pr((h^T y'^*_1)(h^T y'^*_2) \geq 0) \\
&\quad + \frac{10}{2}(Pr((h^T y'^*_2)(h^T y'^*_0) \geq 0) + Pr(h^T y'^*_3)(h^T y'^*_0) \geq 0) - Pr((h^T y'^*_2)(h^T y'^*_3) < 0) \\
&\quad + \frac{10}{2}(Pr((h^T y'^*_1)(h^T y'^*_0) < 0) + Pr(h^T y'^*_3)(h^T y'^*_0) \geq 0) - Pr((h^T y'^*_1)(h^T y'^*_3) \geq 0) \\
&= \frac{15}{2}(1 - \frac{\theta_{01}}{\pi} + \frac{\theta_{02}}{\pi} - 1 + \frac{\theta_{12}}{\pi}) + \frac{10}{2}(1 - \frac{\theta_{02}}{\pi} + 1 - \frac{\theta_{03}}{\pi} - \frac{\theta_{23}}{\pi}) + \frac{10}{2}(\frac{\theta_{01}}{\pi} + 1 - \frac{\theta_{03}}{\pi} - 1 + \frac{\theta_{13}}{\pi}) \\
&= 15(\frac{-\theta_{01} + \theta_{02} + \theta_{12}}{2\pi}) + 10(1 - \frac{\theta_{01} + \theta_{02} + \theta_{12}}{2\pi}) + 10(\frac{\theta_{01} - \theta_{03} + \theta_{13}}{2\pi}) \\
&= 15(1 - \frac{\theta_{01} + \pi - \theta_{02} + \pi - \theta_{12}}{2\pi}) + 10(1 - \frac{\theta_{01} + \theta_{02} + \theta_{12}}{2\pi}) + 10(1 - \frac{\pi - \theta_{01} + \theta_{03} + \pi - \theta_{13}}{2\pi})
\end{aligned}
$$

By the theorem $1 - \frac{\theta_{01}+\theta_{02}+\theta_{12}}{2\pi} \geq 0.79(\frac{1+cos\theta_{02}+cos\theta_{03}+cos\theta_{23}}{4})$ given in question statement,

$$15(1 - \frac{\theta_{01} + \pi - \theta_{02} + \pi - \theta_{12}}{2\pi}) \geq 0.79 * 15(\frac{1 + cos\theta_{01} + cos(\pi - \theta_{02}) + cos(\pi - \theta_{12}}{4}))$$

$$10(1 - \frac{\theta_{01} + \theta_{02} + \theta_{12}}{2\pi}) \geq 0.79 * 10(\frac{1 + cos\theta_{02} + cos\theta_{03} + cos\theta_{23}}{4})$$

$$10(1 - \frac{\pi - \theta_{01} + \theta_{03} + \pi - \theta_{13}}{2\pi}) \geq 0.79 * 10(\frac{1 + cos(\pi - \theta_{01}) + cos\theta_{03} + cos(\pi - \theta_{13})}{4})$$

Adding above three inequalities we get,

$$15(1 - \frac{\theta_{01} + \pi - \theta_{02} + \pi - \theta_{12}}{2\pi}) + 10(1 - \frac{\theta_{01} + \theta_{02} + \theta_{12}}{2\pi}) + 10(1 - \frac{\pi - \theta_{01} + \theta_{03} + \pi - \theta_{13}}{2\pi})$$

$$\geq$$

$$0.79(15(\frac{1 + cos\theta_{01} + cos(\pi - \theta_{02}) + cos(\pi - \theta_{12}}{4})) + 10(\frac{1 + cos\theta_{02} + cos\theta_{03} + cos\theta_{23}}{4})$$

$$+10(\frac{1 + cos(\pi - \theta_{01}) + cos\theta_{03} + cos(\pi - \theta_{13})}{4}))$$

$$\Rightarrow E[w] \geq 0.79\ SDPopt$$

$$\Rightarrow E[w] \geq 0.79\ SDPopt \geq 0.79 Max\ 2\ AND\ optimum$$

Hence, this SDP rounding gives 0.79-approximation.