CS 602 Assignment 1

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Question 1

Given, q is not in the convex hull of $\{p_1, p_2, p_3, ..., p_k\}$.
By farkas lemma, $Ax = b$, $x \geq 0$ not feasible $\iff$ $y^T A \geq 0$, $y^T b < 0$ feasible

Proof:

To prove, $\exists a \epsilon R^n$, $b \epsilon R$ s.t $a^T q > b$ and $a^T p_i \leq b$ for each $1 \leq i \leq k$.

Converting above inequalities to the form of inequalities in Farkas lemma :

Let, $y \epsilon R^{(n+1)}$, y = $\begin{bmatrix} a \\ b \end{bmatrix}$ and $c \epsilon R^{(n+1)}$, c = $\begin{bmatrix} -q \\ 1 \end{bmatrix}$
$a^T q > b$
$\Rightarrow -a^T q < -b$
$\Rightarrow -a^T q + b < 0$
$\Rightarrow a^T * -q + b < 0$
$\Rightarrow \begin{bmatrix} a \\ b \end{bmatrix}^T \begin{bmatrix} -q \\ 1 \end{bmatrix} < 0$
$\Rightarrow y^T c < 0$

Let $A \epsilon R^{(n+1)xk}$ and A_i be ith column, $A_i = \begin{bmatrix} -p_i \\ 1 \end{bmatrix}$
$a^T p_i \leq b \ \forall i \epsilon [1, k]$
$\Rightarrow -a^T p_i + b \geq 0 \ \forall i \epsilon [1, k]$
$\Rightarrow \begin{bmatrix} a \\ b \end{bmatrix}^T A \geq 0$
$\Rightarrow y^T A \geq 0$

Hence, $\exists a \epsilon R^n$, $b \epsilon R$ s.t $a^T q > b$ and $a^T p_i \leq b$ for each $1 \leq i \leq k$ $\iff$ $y \epsilon R^{n+1}$, $y^T A \geq 0$, $y^T c < 0$ feasible

Applying farkas lemma :

By farkas lemma, $y \epsilon R^{n+1}$, $y^T A \geq 0$, $y^T c < 0$ feasible $\iff$ $Ax = c$, $x \geq 0$ not feasible. Therefore, showing there should not exist x, such that $x \epsilon R^{n+1}$, $Ax = c$, $x \geq 0$ is equivalent to proving there exists a separating hyperplane H such that q is on one side of H and the points p_1, p_2, p_3,...., p_k are on the other side of H.

Showing $Ax = c$, $x \geq 0$ not feasible :

$Ax = c$
$\Rightarrow \begin{bmatrix} -p_1 & -p_2 & . & . & . & -p_k \\ 1 & 1 & . & . & . & 1 \end{bmatrix} x = \begin{bmatrix} -q \\ 1 \end{bmatrix}$
$\Rightarrow \begin{bmatrix} -p_1 \\ 1 \end{bmatrix} x_1 + \begin{bmatrix} -p_2 \\ 1 \end{bmatrix} x_2 + ... + \begin{bmatrix} -p_k \\ 1 \end{bmatrix} x_k = \begin{bmatrix} -q \\ 1 \end{bmatrix}$
$\Rightarrow$ p_1x_1 + p_2x_2 +.....+p_kx_k = q and x_1 + x_2 + ...+ x_k = 1

Since q is not in the convex hull of $\{p_1, p_2, p_3,...., p_k\}$, there does not exist x, such that $x \epsilon R^k$, $x >= 0$, p_1x_1 + p_2x_2 +.....+p_kx_k = q and x_1 + x_2 + ...+ x_k = 1. Therefore, $Ax = c$, $x \geq 0$ is not feasible.

Conclusion :

By farkas lemma, $y \epsilon R^{n+1}$, $y^T A \geq 0$, $y^T c < 0$ feasible $\iff$ $Ax = c$, $x \geq 0$ not feasible. Therefore, there exists a separating hyperplane H such that q is on one side of H and the points p_1, p_2, p_3,...., p_k are on the other side of H.

Question 2

Let V1 and V2 be the set of vertices that are disjoint and independent sets and such that every edge (u,v) ϵE either $u \epsilon V1$ and $v \epsilon V2$ or $u \epsilon V2$ and $v \epsilon V1$.

Proof by contradiction:

To prove, the LP $0 \leq x_u \leq 1$ for each vertex u ϵ V and $x_u + x_v \geq 1$ for each edge (u,v) ϵ E describes the vertex cover polytope.

Let q be the feasible non-integral point that is at least one x_u in q is strictly between 0 and 1.
Let P be the set of vertices in V such that $0 < q(x_u) < 1$ for each u ϵ P .
Let P1 = P $\cap$ V1 and P2 = P $\cap$ V2.
Let ε = min{ q(x_u), 1-q(x_u) : u ϵ P}

Now, we define two points q1 and q2 such that

$$\text{q1} = \begin{cases} q(x_u) + \varepsilon & \text{if } u \epsilon P1 \\ q(x_u) - \varepsilon & \text{else if } u \epsilon P2 \\ q(x_u) & \text{else} \end{cases}$$
$$\text{q2} = \begin{cases} q(x_u) - \varepsilon & \text{if } u \epsilon P1 \\ q(x_u) + \varepsilon & \text{else if } u \epsilon P2 \\ q(x_u) & \text{else} \end{cases}$$

Both points q1 and q2 are the feasible points of the given LP because
As, $0 \leq$ q(x_u) , $\forall$ u ϵ V $\Rightarrow$ $0 \leq$ q(x_u) + ε, $\forall$ u ϵ V $\Rightarrow$ $0 \leq$ q1(x_u), $\forall$ u ϵ P1
As, $\varepsilon \leq$ 1-q(x_u), $\forall$ u ϵ P $\Rightarrow$ q(x_u) + $\varepsilon \leq 1$, $\forall$ u ϵ P $\Rightarrow$ $q1(x_u) \leq 1, \forall u \epsilon P1$
As, $\varepsilon \leq$ q(x_u), $\forall$ u ϵ P $\Rightarrow$ $0 \leq$ q(x_u) - ε, $\forall$ u ϵ P $\Rightarrow$ $0 \leq$ q1(x_u) $\forall$ u ϵ P2
As, q(x_u) ≤ 1, $\forall$ u ϵ V $\Rightarrow$ q(x_u) - $\varepsilon \leq 1$, $\forall$ u ϵ V $\Rightarrow$ q1(x_u) ≤ 1, $\forall$ u ϵ P2

As, $\varepsilon \leq$ q(x_u), $\forall$ u ϵ P $\Rightarrow$ $0 \leq$ q(x_u) - ε, $\forall$ u ϵ P $\Rightarrow$ $0 \leq$ q2(x_u) $\forall$ u ϵ P1
As, q(x_u) ≤ 1, $\forall$ u ϵ V $\Rightarrow$ q(x_u) - $\varepsilon \leq 1$, $\forall$ u ϵ V $\Rightarrow$ q2(x_u) ≤ 1, $\forall$ u ϵ P1
As, $0 \leq$ q(x_u) , $\forall$ u ϵ V $\Rightarrow$ $0 \leq$ q(x_u) + ε, $\forall$ u ϵ V $\Rightarrow$ $0 \leq$ q2(x_u), $\forall$ u ϵ P2
As, $\varepsilon \leq$ 1-q(x_u), $\forall$ u ϵ P $\Rightarrow$ q(x_u) + $\varepsilon \leq 1$, $\forall$ u ϵ P $\Rightarrow$ $q2(x_u) \leq 1, \forall u \epsilon P2$

For any edge (u,v) in E, either $u \epsilon V1$ and $v \epsilon V2$ or $u \epsilon V2$ and $v \epsilon V1$.
If $u \epsilon P1$ and $v \epsilon P2$ then
As, $q(x_u) + q(x_v) \geq 1 \quad \Rightarrow \quad q(x_u) + \varepsilon + q(x_v) - \varepsilon \geq 1 \quad \Rightarrow \quad q1(x_u) + q1(x_v) \geq 1$
As, $q(x_u) + q(x_v) \geq 1 \quad \Rightarrow \quad q(x_u) - \varepsilon + q(x_v) + \varepsilon \geq 1 \quad \Rightarrow \quad q2(x_u) + q2(x_v) \geq 1$
Else if $u \epsilon P2$ and $v \epsilon P1$ then
As, $q(x_u) + q(x_v) \geq 1 \quad \Rightarrow \quad q(x_u) - \varepsilon + q(x_v) + \varepsilon \geq 1 \quad \Rightarrow \quad q1(x_u) + q1(x_v) \geq 1$
As, $q(x_u) + q(x_v) \geq 1 \quad \Rightarrow \quad q(x_u) + \varepsilon + q(x_v) - \varepsilon \geq 1 \quad \Rightarrow \quad q2(x_u) + q2(x_v) \geq 1$

As we can see, q = (q1+q2)/2. By the defination of a corner(q is a corner in LP if it is not a midpoint of two points in LP), q cannot be a corner point as both q1 and q2 are feasible points of LP. Therefore, no non-integral feasible point can be a corner of this LP and every corner is integral.
Every integral feasible point is a vertex cover because for each edge (u,v) ϵ E at least one of it's endpoint is choosen as $x_u + x_v \geq 1$(that is $x_u = 1$ or/and x_v=1).
Therefore, the LP $0 \leq x_u \leq 1$ for each vertex u ϵ V and $x_u + x_v \geq 1$ for each edge (u,v) ϵ E describes the vertex cover polytope. Hence proved.

Question 3

Given, $A \epsilon R^{kxn}$, $b \epsilon R^k$, $C \epsilon R^{lxn}$, $d \epsilon R^l$
By strong duality theorem, if LP is max$\{\ w^T x : Ax \leq b\}$, LP^* is min$\{b^T y : y \geq 0,\ A^T y = w\}$ and LP^* is dual LP then opt(LP) = opt(LP^*)

<u>Proof:</u>

Let the first LP be LP_1 and the second LP be LP_2, that is

$$LP_1 = \min\{\ w^T x : Ax \leq b;\ x \geq 0;\ Cx = d\}$$
$$LP_2 = \max\{\ b^T y + d^T z : y \leq 0,\ z \epsilon R^l,\ A^T y + C^T z \leq w\}$$

The overall idea for the proof is to find the dual of LP_1 and then show LP2 is dual of LP_1.

Converting LP_1 into the form of LP in strong duality theorem :

$$LP_1 = \min\{\ w^T x : Ax \leq b;\ x \geq 0;\ Cx = d\}$$
$$\min\ w^T x \iff \max\ -w^T x$$
$$x \geq 0 \iff -x \leq 0$$
$$Cx = d \iff Cx \geq d \text{ and } Cx \leq d \iff -Cx \leq -d \text{ and } Cx \leq d$$

Hence, $LP_1 = \max\{\ -w^T x : Ax \leq b,\ -x \leq 0,\ Cx \leq d,\ -Cx \leq -d\ \}$

Let, $w_1 = -w$, $D \epsilon R^{nxn}$, $D = -I$, $A_1 = \begin{bmatrix} A \\ C \\ -C \\ D \end{bmatrix}$, $b_1 = \begin{bmatrix} b \\ d \\ -d \\ 0 \end{bmatrix}$.

(Here $0\ \epsilon\ R^n$)

Then we can say $LP_1 = \max\{\ w_1^T x : A_1 x \leq b_1\}$.

Applying Strong Duality theorem :

By strong duality theorem, LP_1 and dual LP_1 (LP_1^*) have same optimal value where $LP_1^* = \min\{b_1^T y_1 : y_1 \geq 0,\, A_1^T y_1 = w_1\}$.

$$LP_1^* = \min\{b_1^T y_1 : y_1 \geq 0,\, y_1 \epsilon R^{k+2l+n},\, A_1^T y_1 = w_1\}$$

Converting LP_1 into the form of LP^* in strong duality theorem :

Let, $z = q - p$ such that $p, q \epsilon R^l$, $p \geq 0$, $q \geq 0$

$$LP_2 = \max\{\ b^T y + d^T z : y \leq 0,\, z \epsilon R^l,\, A^T y + C^T z \leq w\}$$
$$y \leq 0 \iff -y \geq 0$$
$$A^T y + C^T z \leq w \iff -A^T * -y + C^T z + r = w,\, r \epsilon R^n,\, r \geq 0$$

$-A^T * -y + C^T z + r = w$
$\Rightarrow -1 * (-A^T * -y + C^T z + r) = -1 * w$
$\Rightarrow A^T * -y - C^T (q - p) - r = -w$
$\Rightarrow A^T * -y + C^T p - C^T q - r = -w$
$\Rightarrow \begin{bmatrix} A \\ C \\ -C \\ D \end{bmatrix}^T \begin{bmatrix} -y \\ p \\ q \\ r \end{bmatrix} = -w$
$\Rightarrow A_1^T \begin{bmatrix} -y \\ p \\ q \\ r \end{bmatrix} = w_1$

max $b^T y$ + $d^T z$
$\Rightarrow$ max $b^T y$ + $d^T q$ - $d^T p$
$\Rightarrow$ min $b^T * -y$ + $d^T p$ - $d^T q$ + $0 * r$
$\Rightarrow$ min $\begin{bmatrix} b \\ d \\ -d \\ 0 \end{bmatrix}^T \begin{bmatrix} -y \\ p \\ q \\ r \end{bmatrix}$

(Here $0 \ \epsilon\ R^n$)

$\Rightarrow$ min $b_1^T \begin{bmatrix} -y \\ p \\ q \\ r \end{bmatrix}$

Let, $y_2 \epsilon R^{k+2l+n}$, $y_2 = \begin{bmatrix} -y \\ p \\ q \\ r \end{bmatrix}$
As, $-y \geq 0$, $p \geq 0$, $q \geq 0$ and $r \geq 0$, this implies $y_2 \geq 0$

Therefore, $LP_2 = \min\{b_1^T y_2 : y_2 \geq 0,\, y_2 \epsilon R^{k+2l+n},\, A_1^T y_2 = w_1\}$

Conclusion :

As we can see, dual LP_1 (LP_1^*) and LP_2 are same. Therefore, by duality theorem optimal value of LP_1 and LP_2 are same.