PMATH 365 Course Notes

Differential Geometry

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1.1 Preliminaries

To begin, we'll recall some facts about the topology of $\mathbb{R}^n$ and vector-valued functions.

In this course, we'll be working with the metric topology with respect to the Euclidean norm (or metric). Let $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in \mathbb{R}^n$. The **Euclidean norm** is defined to be

$$\|x\| = \sqrt{x_1^2 + \cdots + x_n^2},$$

and **Euclidean distance** is given by

$$\operatorname{dist}(x, y) = \|y - x\| = \sqrt{(y_1 - x_1)^2 + \cdots + (y_n - x_n)^2}.$$

We define the **open ball** of radius $r > 0$ centered at $x \in \mathbb{R}^n$ by

$$B_r(x) := \{y \in \mathbb{R}^n : \operatorname{dist}(x, y) < r\} \subset \mathbb{R}^n.$$

A **topology** on $\mathbb{R}^n$ is a collection $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ of subsets $U_\alpha \subset \mathbb{R}^n$ that satisfy the following properties.

(i) $\varnothing$ and $\mathbb{R}^n$ are in $\mathcal{U}$.

(ii) For any subcollection $\mathcal{V} = \{U_\beta\}_{\beta \in B}$ with $U_\beta \in \mathcal{U}$ for all $\beta \in B$, we have $\bigcup_{\beta \in B} U_\beta \in \mathcal{U}$.

(iii) For any *finite* subcollection $\{U_{\alpha_1}, \dots, U_{\alpha_m}\} \subset \mathcal{U}$, we have $\bigcap_{i=1}^m U_{\alpha_i} \in \mathcal{U}$.

The sets $U_\alpha \in \mathcal{U}$ are called the **open sets** of the topology; their complements $F_\alpha = \mathbb{R}^n \setminus U_\alpha$ are called the **closed sets**.

Note that the sets $\varnothing$ and $\mathbb{R}^n$ are both open and closed. Moreover, the notion of a topology can be extended to more general sets X, not just $\mathbb{R}^n$. A topology can also be defined starting with closed sets, but we prefer to work with open sets because many nice properties, such as differentiability, are better described with them.

Under the metric topology, we say that a set $A \subset \mathbb{R}^n$ is **open** if $A = \varnothing$ or if for all $p \in A$, there exists $r > 0$ such that $B_r(p) \subset A$. Moreover, A is **closed** if its complement $A^c = \mathbb{R}^n \setminus A$ is open. (We leave it as an exercise to show that this is indeed a topology.)

For example, the open balls $B_r(x)$ are open sets for all $x \in \mathbb{R}^n$ and $r > 0$. Indeed, for any point $p \in B_r(x)$, one sees that by picking $r' = (r - \|p - x\|)/2$, we have $B_{r'}(p) \subset B_r(x)$.

In general, open sets are described with strict inequalities, while closed sets are described using equality or inclusive inequalities. However, note that most sets are neither open nor closed, such as the half-open interval $U = (-1, 1]$ over $\mathbb{R}$.

The metric topology is not the only topology on $\mathbb{R}^n$; one example is the one consisting of only the sets $\mathcal{U} = \{\varnothing, \mathbb{R}^n\}$. However, we generally want more open sets to work with since we might want to know the behaviour of functions around a point $p \in \mathbb{R}^n$. If the only non-empty open set we had was $\mathbb{R}^n$, then this would apply to all points in $\mathbb{R}^n$, which does not yield a lot of information.

Let $p \in \mathbb{R}^n$. The previous paragraph leads us to the definition of an **open neighbourhood** of p, which is just an open set $U \subset \mathbb{R}^n$ such that $p \in U$.

We now turn our discussion to vector-valued functions. Let $U \subset \mathbb{R}^n$ and consider the vector-valued function

$$\begin{aligned} F : U \subset \mathbb{R}^n &\to B \subset \mathbb{R}^m \\ x = (x_1, \dots, x_n) &\mapsto (F_1(x), \dots, F_m(x)). \end{aligned}$$

Then F is continuous if and only if the component functions $F_i : U \to \mathbb{R}$ are continuous for all $i = 1, \dots, m$.

We say that F is a **homeomorphism** if it is a continuous bijection whose inverse

$$F^{-1} : B \subset \mathbb{R}^m \to U \subset \mathbb{R}^n$$

is also continuous. For example, the identity map $\mathrm{Id}_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n$ and the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$ are both homeomorphisms.

It is a known fact that homeomorphisms map open sets to open sets and closed sets to closed sets. This follows from the topological characterization of continuity, which states that F is continuous if and only if for every open (respectively closed) set $V \subset \mathbb{R}^m$, we have that $F^{-1}(V)$ is open (respectively closed). In fact, homeomorphisms preserve much more structure than this, as we'll see later.

1.2 Topological submanifolds of $\mathbb{R}^n$

We now define the main object we'll be working with in this course.

Definition 1.1

A k**-dimensional topological submanifold** (or **topological** k**-submanifold**) of $\mathbb{R}^n$ is a subset $M \subset \mathbb{R}^n$ such that for every $p \in M$, there exists an open neighbourhood V of p in $\mathbb{R}^n$, an open set $U \subset \mathbb{R}^k$, and a homeomorphism

$$\alpha : U \subset \mathbb{R}^k \to V \cap M \subset \mathbb{R}^n.$$

The homeomorphism α is called a **coordinate chart** (or **patch**) on M.

Note that the open neighbourhood $V \subset \mathbb{R}^n$ of p, the open set $U \subset \mathbb{R}^k$, and the map α do not need to be unique. But we'll see later that the dimension k must be unique and is completely determined by M.

For example, $\mathbb{R}^n$ is a topological n-submanifold of $\mathbb{R}^n$ by taking $U = V = \mathbb{R}^n$ and $\alpha = \mathrm{Id}_{\mathbb{R}^n}$. Any open set $W \subset \mathbb{R}^n$ is a topological n-submanifold of $\mathbb{R}^n$ by taking $U = V = W$ and $\alpha = \mathrm{Id}_W$.

Let's now consider some non-trivial examples. Consider

$$M = \{(x, y) \in \mathbb{R}^2 : y = x^2\} \subset \mathbb{R}^2,$$

which is the graph of the parabola $f(x) = x^2$. Then M is a topological 1-submanifold of $\mathbb{R}^2$ by considering the map $\alpha : \mathbb{R}^1 \to M \subset \mathbb{R}^2$, $t \mapsto (t, t^2)$. The inverse $\alpha^{-1} : M \to \mathbb{R}^1$ is just the projection of the first coordinate, which is continuous.

More generally, let $U \subset \mathbb{R}^k$ be an open set. Consider the graph of a continuous function

$$F : U \subset \mathbb{R}^k \to \mathbb{R}^{n-k},\ x \mapsto (F_1(x), \ldots, F_{n-k}(x)).$$

In other words, we are looking at the set

$$G = \{(x, y) \in \mathbb{R}^k \times \mathbb{R}^{n-k} : y = F(x),\ x \in U\} \subset \mathbb{R}^n.$$

We claim that G is a k-dimensional topological submanifold of $\mathbb{R}^n$. To see this, define $\alpha : U \subset \mathbb{R}^k \to G \subset \mathbb{R}^n$ by $x \mapsto (x, F(x))$. Then α is continuous since F is continuous, and it is a bijection since we are restricted to G. Moreover, it has continuous inverse $\alpha^{-1} : G \subset \mathbb{R}^n \to U \subset \mathbb{R}^k$, $(x, y) \mapsto x$.

Here are two more examples of this in action.

(1) Let $M = \{(x, y, z) \in \mathbb{R}^3 : z = x^2 + y^2\} \subset \mathbb{R}^3$. Then M is the graph of the continuous function $f(x, y) = x^2 + y^2$, so it is a 2-dimensional topological submanifold of $\mathbb{R}^3$.

(2) Observe that $M = \{(x, y, z) \in \mathbb{R}^3 : y = x^2,\ z = x^3\} \subset \mathbb{R}^3$ is the graph of the continuous function $F(t) = (t^2, t^3)$, so it is a 1-dimensional topological submanifold of $\mathbb{R}^3$.

In all the examples above, we only needed one coordinate chart which worked for all points. However, this is not always the case! Consider the unit circle

$$\mathbb{S}^1 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \subset \mathbb{R}^2.$$

Note that $\mathbb{S}^1$ is compact. Therefore, by Heine-Borel, it is closed and bounded. Recall that homeomorphisms preserve closed sets, so it is impossible to find a unique chart α. Indeed, if we had such a homeomorphism $\alpha : U \subset \mathbb{R}^1 \to \mathbb{S}^1 \subset \mathbb{R}^2$ for some open set U, then $U = \alpha^{-1}(\mathbb{S}^1)$ would be a compact subset of $\mathbb{R}^1$. But the only open and compact subset of $\mathbb{R}^n$ is $\varnothing$, which is a contradiction!

Nonetheless, two coordinate charts are enough to cover all points on $\mathbb{S}^1$. Define

$$\begin{aligned} V_1 &= \mathbb{R}^2 \setminus \{(x, 0) \in \mathbb{R}^2 : x \leq 0\}, \\ V_2 &= \mathbb{R}^2 \setminus \{(x, 0) \in \mathbb{R}^2 : x \geq 0\}, \end{aligned}$$

which are both open sets. Then the homeomorphism

$$\alpha_1 : U_1 = (-\pi, \pi) \to \mathbb{S}^1 \cap V_1,\ t \mapsto (\cos t, \sin t)$$

covers all points on $\mathbb{S}^1$ except for $(-1, 0)$, while

$$\alpha_2 : U_2 = (0, 2\pi) \to \mathbb{S}^1 \cap V_2,\ t \mapsto (\cos t, \sin t)$$

covers all points on $\mathbb{S}^1$ except for $(1, 0)$.

1.3 More preliminaries

We now introduce another definition from topology.

Definition 1.2

Let $A \subset \mathbb{R}^n$. A subset $U \subset A$ is **relatively open** if it is of the form $U = A \cap U'$ for some open set $U' \subset \mathbb{R}^n$. Similarly, we say that $F \subset A$ is **relatively closed** if $F = A \cap F'$ for some closed set $F' \subset \mathbb{R}^n$.

For example, consider $\mathbb{R}$ equipped with the metric topology so that the open (respectively closed) sets are the unions of open intervals (respectively finite intersections of closed intervals). Let $A = [-1, 2) \subset \mathbb{R}$, which is neither open nor closed in $\mathbb{R}$. Take $U = [-1, 1) \subset \mathbb{R}$, which is again neither open nor closed in $\mathbb{R}$. But U is relatively open in A since $U = A \cap (-3, 1)$. Similarly, $F = [-1, 1] = A \cap [-1, 1]$ is relatively closed in A.

Using the language of relatively open and closed sets, a lot of statements can be made simpler.

(1) We define an **open neighbourhood of p in A** to be a relatively open set U containing p.

(2) The relatively open sets form a topology on A, called the **relative topology** (verify this as an exercise).

(3) We now have a more concise definition of a topological submanifold. Let $M \subseteq \mathbb{R}^n$. Then M is a **k-dimensional topological submanifold** of $\mathbb{R}^n$ if for every $p \in M$, there exists an open neighbourhood V of p in M, an open set $U \subset \mathbb{R}^k$, and a homeomorphism $\alpha : U \subset \mathbb{R}^k \to V \subset \mathbb{R}^n$.

Definition 1.3

Let $A \subset \mathbb{R}^n$. Then A is **connected** if it cannot be written in the form $A = U \cup V$ where $U, V \neq \varnothing$ are relatively open in A and $U \cap V = \varnothing$. Otherwise, we say that A is **disconnected**; we call U and V **disconnecting sets** for A.

Let's go over a few example of connected sets. It can be shown that an open set in $\mathbb{R}^n$ is connected if and only if it is path connected; that is, there is a path between any two points in the set. This result can help us build some intuition for what a connected set should look like.

(1) $\mathbb{R}^n$ is connected.

(2) Let $\alpha < \beta \in \mathbb{R}$. Then (α,β), $(\alpha,\beta]$, $[\alpha,\beta)$, and $[\alpha,\beta]$ are all connected.

(3) Observe that $A = (-1,0] \cup [1,2]$ is a disconnected set because $(-1,0] = A \cap (-1.3, 0.3)$ and $[1,2] = A \cap (0.9, 2.1)$ are both relatively open in A and disjoint.

(4) The open ball $B_r(p)$ is connected for all $p \in \mathbb{R}^n$ and $r > 0$.

An important property of connected sets is that the continuous image of a connected set is connected! This can be used to prove that a subset $M \subset \mathbb{R}^n$ is not a submanifold.

(1) Consider the **α-curve** $C := \{(x,y) \in \mathbb{R}^2 : y^2 = x^2(x+1)\} \subset \mathbb{R}^2$. This can be parametrized by the map

$$\begin{aligned} \alpha : \mathbb{R} &\to C \subset \mathbb{R}^2 \\ t &\mapsto (t^2-1, t(t^2-1)). \end{aligned}$$

Note that α is not injective since $\alpha(-1) = \alpha(1) = (0,0)$. That is, α is not a homeomorphism on $\mathbb{R}$, but it becomes one if we remove the points $t = \pm 1$, whose inverse is

$$\begin{aligned} \alpha^{-1} : C \setminus \{(0,0)\} &\to \mathbb{R} \setminus \{\pm 1\} \subset \mathbb{R} \\ (x,y) &\mapsto 1/x. \end{aligned}$$

Thus, C is a 1-dimensional submanifold away from the point $(0,0)$.

Our goal now is to show that the whole of C is not a topological submanifold of $\mathbb{R}^2$. By contradiction, suppose that it were. By our above discussion, it must have dimension 1 because it has dimension 1 away from $(0,0)$. Since $(0,0) \in C$, it follows from the definition that there exists an open neighbourhood V of $(0,0)$ in C, an open set $U \subset \mathbb{R}^1$, and a homeomorphism

$$\alpha : U \subset \mathbb{R}^2 \to V \subset C.$$

There must be a unique point $t_0 \in U$ such that $\alpha(t_0) = (0,0)$ since α is a bijection. Since U is open and $t_0 \in U$, there exists $\varepsilon > 0$ such that $B_\varepsilon(t_0) \subset U$. But $U \subset \mathbb{R}^1$, so $B_\varepsilon(t_0) = (t_0 - \varepsilon, t_0 + \varepsilon) =: U'$. Then $\alpha|_{U'}$ is also a homeomorphism. Let $V' = \alpha(U')$. Observe that $V' \setminus \{(0,0)\}$ has three or four pieces depending on how large the open set U' is: one on the top right quadrant, one on the bottom right quadrant, and one or two on the left of the y-axis. On the other hand, $U' \setminus \{t_0\}$ has only two components, contradicting the fact that homeomorphisms preserve the number of connected components.

(2) Consider the **double cone** $M = \{(x,y,z) \in \mathbb{R}^3 : z^2 = x^2 + y^2\} \subset \mathbb{R}^3$. Away from $(0,0,0)$, every point in M lies on the graph of one of the continuous functions $f_1(x,y) = \sqrt{x^2+y^2}$ or $f_2(x,y) = -\sqrt{x^2+y^2}$. Therefore, $M \setminus \{(0,0,0)\}$ is a 2-dimensional topological submanifold of $\mathbb{R}^3$ since f_1 and f_2 are both functions of two variables.

However, there is a problem at the point $(0,0,0)$ since it lies on the graph of both f_1 and f_2. Suppose that M is a topological submanifold of $\mathbb{R}^3$. Then M must necessarily be of dimension 2 because $M \setminus \{(0,0,0)\}$ is of dimension 2. Then by definition, there exists an open neighbourhood V of $(0,0,0)$, an open set $U \subset \mathbb{R}^2$, and a homeomorphism

$$\alpha : U \subset \mathbb{R}^2 \to V \subset M.$$

Since α is a bijection, there exists a unique point $(x_0,y_0) \in U$ such that $\alpha(x_0,y_0) = (0,0,0)$. After shrinking U (by the same argument as above), we may take $U' = B_\varepsilon((x_0,y_0))$ to ensure that we have a connected set. Consider now the restriction $\alpha|_{U'} : U' \to V = \alpha(U')$. Then $U' \setminus \{(x_0,y_0)\}$ has one component, whereas $V' \setminus \{(0,0,0)\}$ has two components (that is, it is disconnected), which is a contradiction.

Now, we want to prove the invariance of dimension.

Theorem 1.4: Invariance of Dimension

$\mathbb{R}^m$ is homeomorphic to $\mathbb{R}^n$ if and only if $m = n$.

If $m = n$, then $\mathbb{R}^m = \mathbb{R}^n$, so there is nothing to prove. The other implication is much harder, and we'll need the following result.

Theorem 1.5: Brouwer Invariance of Domain

Let $U \subset \mathbb{R}^n$ be open, and let $f : U \subset \mathbb{R}^n \to \mathbb{R}^n$ be an injective continuous map. Then $f(U) \subset \mathbb{R}^n$ is open. In particular, f is a homeomorphism onto its image.

An elementary proof can be found on Terry Tao's blog where he uses the Brouwer Fixed Point Theorem to prove it. Nowadays, the standard proof uses algebraic topology.

It is important that both the domain and codomain involve the same dimension n. For example, consider the injective continuous map $f : \mathbb{R} \to \mathbb{R}^2$ defined by $x \mapsto (x, 0)$. Observe that $f(U)$ is the x-axis, which is not open in $\mathbb{R}^2$.

Proof of Theorem 1.4.

We proceed by contradiction. Suppose that there is a homeomorphism $f : \mathbb{R}^m \to \mathbb{R}^n$ and that $m > n$. Consider the inclusion

$$\begin{aligned} \iota : \mathbb{R}^n &\to \mathbb{R}^m \\ (x_1, \ldots, x_n) &\mapsto (x_1, \ldots, x_m, 0, \ldots 0), \end{aligned}$$

which is an injective continuous map. Then $\iota \circ f : \mathbb{R}^m \to \mathbb{R}^m$ is also an injective continuous map since it is the composition of two injective continuous maps. By Theorem 1.5, we have that $\iota \circ f(\mathbb{R}^m)$ is an open set in $\mathbb{R}^m$. But this is impossible because if $(x_1, \ldots, x_n, 0, \ldots, 0) \in \iota \circ f(\mathbb{R}^m)$, then

$$(x_1, \ldots, x_n, \varepsilon/2, 0, \ldots, 0) \notin \iota \circ f(\mathbb{R}^m)$$

for all $\varepsilon > 0$. Then $B_\varepsilon((x_1, \ldots, x_n, 0, \ldots, 0)) \not\subset \iota \circ f(\mathbb{R}^m)$ for all $\varepsilon > 0$, implying that $\iota \circ f(\mathbb{R}^m)$ is not open in $\mathbb{R}^m$. Therefore, we must have $m \leq n$. If $n < m$, then we can repeat the same argument with $f^{-1} : \mathbb{R}^n \to \mathbb{R}^m$, which again leads to a contradiction. We conclude that $n = m$. $\square$

Note that we actually proved something stronger: if $m > n$ and U is a nonempty open subset of $\mathbb{R}^m$, then there is no continuous mapping from U to $\mathbb{R}^n$. As a consequence, we get the following.

Proposition 1.6

If $M \subset \mathbb{R}^n$ is a k-dimensional topological submanifold of $\mathbb{R}^n$, then $k \leq n$.

Proof of Proposition 1.6.

If $M \subset \mathbb{R}^n$ is a k-dimensional topological submanifold of $\mathbb{R}^n$, then for all $p \in M$, there exists an open set $U \subset \mathbb{R}^k$, an open neighbourhood $V \subset M$ of p, and a homeomorphism

$$\alpha : U \subset \mathbb{R}^k \to V \subset M \subset \mathbb{R}^n.$$

Since α is an injective continuous map, this forces $k \leq n$ by the above discussion. $\square$

Finally, we must have the same k for any chart $\alpha : U \subset \mathbb{R}^k \to V \subset M$. Indeed, let $p \in M$, and suppose that we have two different charts, say $\alpha : U \subset \mathbb{R}^k \to V \subset M$ and $\beta : U' \subset \mathbb{R}^{k'} \to V' \subset M$ where $p \in V \cap V'$. Then $V \cap V' \neq \varnothing$, so we can consider the restrictions

$$\alpha|_{\alpha^{-1}(V\cap V')} : \alpha^{-1}(V \cap V') \to V \cap V',$$
$$\beta|_{\beta^{-1}(V\cap V')} : \beta^{-1}(V \cap V') \to V \cap V'.$$

Then $\beta^{-1} \circ \alpha : \alpha^{-1}(V \cap V') \subset \mathbb{R}^k \to \beta^{-1}(V \cap V') \subset \mathbb{R}^{k'}$ is a homeomorphism. Hence, $\alpha^{-1}(V \cap V')$ is a k-dimensional topological submanifold of $\mathbb{R}^{k'}$. By Proposition 1.6, we have $k \leq k'$. Similarly, $\alpha^{-1} \circ \beta : \beta^{-1}(V \cap V') \subset \mathbb{R}^{k'} \to \alpha^{-1}(V \cap V') \subset \mathbb{R}^k$ is a homeomorphism, so $\beta^{-1}(V \cap V')$ is a k'-dimensional submanifold of $\mathbb{R}^k$. It follows that $k' \leq k$ and so $k' = k$.

1.4 Submanifolds of $\mathbb{R}^n$ of class C^r

Let $U \subset \mathbb{R}^n$ be an open set and consider the vector-valued function

$$F : U \subset \mathbb{R}^n \to \mathbb{R}^m$$
$$x = (x_1, \ldots, x_n) \mapsto (F_1(x), \ldots, F_m(x)).$$

Recall that F is of **class** C^r for $r \geq 1$ if each component function $F_i : U \subset \mathbb{R}^n \to \mathbb{R}$ is of class C^r. That is, the partial derivatives of F_i exist and are continuous up to order r. Also, we say that F is of class C^∞ or **smooth** if each F_i is smooth (the partial derivatives exist up to any order).

(1) All polynomials are smooth.

(2) The function $f(x) = x^{4/3}$ is of class C^1. Its derivative $f'(x) = \frac{4}{3}x^{1/3}$ is continuous, but the second derivative $f''(x) = \frac{4}{9}x^{-2/3}$ is not defined at $x = 0$.

(3) The vector-valued function $F(x, y) = (2\cos x, xy - 1, e^{2\sin y + x})$ is smooth on $\mathbb{R}^2$ because each component function is smooth.

The **partial derivative** of F with respect to the variable x_j is

$$\frac{\partial F}{\partial x_j} := \left(\frac{\partial F_1}{\partial x_j}, \ldots, \frac{\partial F_m}{\partial x_j}\right).$$

If we fix a component function F_i, its **gradient** is

$$\nabla F_i := \left(\frac{\partial F_i}{\partial x_1}, \ldots, \frac{\partial F_i}{\partial x_n}\right).$$

The **derivative matrix** or **Jacobian matrix** of F is the $m \times n$ matrix

$$DF := \begin{bmatrix} \partial F_1/\partial x_1 & \cdots & \partial F_1/\partial x_n \\ \vdots & \ddots & \vdots \\ \partial F_m/\partial x_1 & \cdots & \partial F_m/\partial x_n \end{bmatrix}.$$

That is, the rows correspond to the component functions F_i, and the columns correspond to the variables x_j. We can also think of the rows as the gradients and the columns as the partial derivatives; that is, we have

$$DF = \left[\begin{array}{c|c|c} \dfrac{\partial F}{\partial x_1} & \cdots & \dfrac{\partial F}{\partial x_n} \end{array}\right] = \left[\begin{array}{c} \nabla F_1 \\ \hline \vdots \\ \hline \nabla F_m \end{array}\right]$$

In general, we want to work with some differentiability. This leads to the following definition.

Definition 1.7

Let $M \subset \mathbb{R}^n$. Suppose that for every $p \in M$, there exists an open neighbourhood V of p in M, an open subset $U \subset \mathbb{R}^k$, and a homeomorphism $\alpha : U \subset \mathbb{R}^k \to V \subset M$ such that

(1) α is of class C^r for some $r \geq 1$;

(2) $D\alpha(x)$ has rank k for all $x \in U$.

Then M is called a **k-dimensional submanifold of $\mathbb{R}^n$ of class C^r**. We call α a **coordinate chart** (or **coordinate patch**) about p.

Note that every submanifold of class C^r is a topological submanifold. We are only imposing the extra conditions (1) and (2) on the coordinate charts. We will see that condition (2) will allow us to define tangent spaces to the submanifolds at every point. A submanifold of class C^∞ is called a **smooth submanifold**.

As usual, let's go over some examples.

(1) Let $U \subset \mathbb{R}^n$ be open. Then $\alpha : U \subset \mathbb{R}^n \to V = U \subset \mathbb{R}^n$ sending x to itself is smooth. Since the component functions are $F_i(x) = x_i$ for all $i = 1, \ldots, n$, we have

$$D\alpha(x) = \left[\frac{\partial F_i}{\partial x_j}\right] = \left[\frac{\partial x_i}{\partial x_j}\right] = [\delta_{ij}],$$

where δ_{ij} is the Kronecker delta. In other words, $D\alpha(x)$ is the $n \times n$ identity matrix and has rank n for all $x \in U$, so $U \subset \mathbb{R}^n$ is a smooth n-dimensional submanifold of $\mathbb{R}^n$.

(2) **Graphs of functions of class C^r.** Let $U \subset \mathbb{R}^k$ be an open set and consider a function

$$\begin{aligned} F : U \subset \mathbb{R}^k &\to \mathbb{R}^{n-k} \\ (x_1, \ldots, x_k) &\mapsto (F_1(x), \ldots, F_{n-k}(x)) \end{aligned}$$

of class C^r (so each F_i is of class C^r). Let

$$M = \{(x, F(x)) \in \mathbb{R}^k \times \mathbb{R}^{n-k} : x \in U\} \subset \mathbb{R}^n$$

be the graph of F. We have already seen that M is a k-dimensional submanifold of $\mathbb{R}^n$ by taking $V = M$ and the homeomorphism $\alpha : U \subset \mathbb{R}^k \to V \subset \mathbb{R}^n$ defined by

$$\alpha(x) = (x, F(x)) = (x_1, \ldots, x_k, F_1(x), \ldots, F_{n-k}(x)).$$

In particular, α is of class C^r since the identity components are smooth and the F_i are of class C^r. Let's look at the derivative matrix in terms of the columns of partial derivatives. We have

$$\frac{\partial \alpha}{\partial x_j} = \left(0, \ldots, 0, 1, 0, \ldots, 0, \frac{\partial F_1}{\partial x_j}, \ldots, \frac{\partial F_{n-k}}{\partial x_j}\right)$$

where the 1 corresponds to the j-th component, and hence

$$D\alpha(x) = \left[\begin{array}{cccc} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ \hline \partial F_1/\partial x_1 & \partial F_1/\partial x_2 & \cdots & \partial F_1/\partial x_k \\ \vdots & \vdots & \ddots & \vdots \\ \partial F_{n-k}/\partial x_1 & \partial F_{n-k}/\partial x_2 & \cdots & \partial F_{n-k}/\partial x_k \end{array}\right] = \left[\begin{array}{c} I_{k\times k} \\ \hline DF(x) \end{array}\right].$$

This matrix has rank k for all $x \in U$, so M is a k-dimensional submanifold of class C^r.

(3) We saw that the circle $\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \subset \mathbb{R}^2$ was a 1-dimensional topological submanifold using the charts $\alpha_1 : U_1 = (-\pi, \pi) \subset \mathbb{R}^1 \to \mathbb{S}^1 \setminus \{(-1, 0)\} \subset \mathbb{R}^2$ and $\alpha_2 : U_2 = (0, 2\pi) \subset \mathbb{R}^1 \to \mathbb{S}^1 \setminus \{(1, 0)\} \subset \mathbb{R}^2$, both defined by $t \mapsto (\cos t, \sin t)$. Note that both α_i are smooth functions with derivative matrix

$$D\alpha_i = \begin{bmatrix} \frac{\mathrm{d}\alpha_i}{\mathrm{d}t} \end{bmatrix} = \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix},$$

which has rank 1 because $\sin t$ and $\cos t$ don't have the same zeroes, and hence $D\alpha_i$ is never the zero vector. Thus, $\mathbb{S}^1$ is a smooth 1-dimensional submanifold of $\mathbb{R}^2$.

Not every topological submanifold of $\mathbb{R}^n$ is of class C^r for some $r \geq 1$. For example, consider the graph

$$M = \{(x, |x|) : x \in \mathbb{R}\}$$

of the function $f(x) = |x|$ on $\mathbb{R}$, which fails to be differentiable at $x = 0$. Since f is continuous, we know that M is a 1-dimensional topological submanifold of $\mathbb{R}^2$. Note that f is smooth away from $x = 0$, so $M \setminus \{(0, 0)\}$ is a smooth 1-dimensional submanifold of $\mathbb{R}^2$.

However, we claim that it cannot be a submanifold of class C^r on any neighbourhood of the point $(0, 0)$. Suppose otherwise, so there exists an open set $U \subset \mathbb{R}^1$, an open neighbourhood $V \subset M$ of $(0, 0)$, and a homeomorphism $\alpha : U \subset \mathbb{R}^1 \to V \subset M \subset \mathbb{R}^2$ of class C^r for some $r \geq 1$. Moreover, assume that $D\alpha(t)$ is of rank 1 for all $t \in U$. Since $k = 1$, we have

$$D\alpha(t) = \frac{\mathrm{d}\alpha}{\mathrm{d}t} = \alpha'(t) \neq 0$$

for all $t \in U$ using the rank 1 assumption. But $\alpha'(t)$ is tangent to M at $\alpha(t)$. There are two possibilities:

- If $\alpha(t)$ is on the line $y = x$, then $\alpha'(t)$ is a direction vector of $y = x$, so for some $c : I \subset \mathbb{R} \to \mathbb{R}$, we have

$$\alpha'(t) = c(t) \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{1.4.1}$$

  Since $\alpha(t)$ is of class C^r for some $r \geq 1$, we know that $\alpha'(t)$ is continuous, so $c : I \to \mathbb{R}$ is also continuous.

- If $\alpha(t)$ is on $y = -x$, then $\alpha'(t)$ is a direction vector of $y = -x$, so for some $d : I' \subset \mathbb{R} \to \mathbb{R}$, we have

$$\alpha'(t) = d(t) \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{1.4.2}$$

  The same argument as above shows that $d : I' \to \mathbb{R}$ is continuous.

However, since α is a bijection, we have $(0, 0) = \alpha(t_0)$ for some $t_0 \in U$. By the continuity of $\alpha'(t)$, we obtain

$$\lim_{t \to t_0^-} \alpha'(t) = \lim_{t \to t_0^+} \alpha'(t).$$

Without loss of generality, assume that $\alpha(t)$ is moving along M from left to right. (Otherwise, we can simply parametrize in the other direction.) Then if $t < t_0$, equation (1.4.2) holds, whereas if $t > t_0$, equation (1.4.1) holds. This means that

$$\lim_{t \to t_0^-} \alpha'(t) = \lim_{t \to t_0^-} d(t) \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \lim_{t \to t_0^+} c(t) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \lim_{t \to t_0^+} \alpha'(t).$$

But the above vectors are not parallel to each other, so the only way that these limits are equal is if $\lim_{t \to t_0^-} d(t) = \lim_{t \to t_0^+} c(t) = 0$. This implies that $\alpha'(t_0)$ is the zero vector, which is a contradiction to the fact that $D\alpha(t) = \alpha'(t)$ has rank 1 for all $t \in U$! This concludes the example that not every topological submanifold is a submanifold of class C^r for some $r \geq 1$.

In the definition of a submanifold of class C^r, it is important that α is a homeomorphism and not just a function of class C^r with $D\alpha(x)$ of rank k for all $x \in U$. These conditions alone don't even ensure that α is injective! For example, take the α-curve $C = \{(x,y) \in \mathbb{R}^2 : y^2 = x^2(x+1)\} \subset \mathbb{R}^2$ which we introduced back in Section 1.3, which could be parametrized using $\alpha(t) = (t^2-1, t(t^2-1))$ for $t \in \mathbb{R}$. We saw that this map was not injective, but it is smooth with derivative matrix

$$D\alpha(t) = \frac{\mathrm{d}\alpha}{\mathrm{d}t} = \begin{pmatrix} 2t \\ 3t^2-1 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

for all $t \in \mathbb{R}$. A map satisfying these two conditions is said to be an **immersion**, and a topological submanifold whose maps are immersions is called an **immersed manifold**. We record this in the definition below.

Definition 1.8

Let $U \subset \mathbb{R}^k$ be open with $k \leq n$. A map $\alpha : U \subset \mathbb{R}^k \to \mathbb{R}^n$ is an **immersion** (of class C^r) if

(1) α is of class C^r; and

(2) $D\alpha(x)$ has rank k for all $x \in U$.

We give some examples of immersions.

(1) **Canonical immersion.** The inclusion map $\iota : \mathbb{R}^k \to \mathbb{R}^n$ defined by $(x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_k, 0, \ldots, 0)$ is an immersion of class C^∞. Indeed, we see that ι is smooth and its derivative matrix is

$$D\iota = \left[\begin{array}{ccc} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \\ \hline 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{array}\right]$$

which has rank k because it contains the $k \times k$ identity matrix.

(2) The parametrization $\alpha(t) = (t^2-1, t(t^2-1))$ of the α-curve is an immersion of class C^∞.

(3) The charts $\alpha : U \to V$ of class C^r of a submanifold $M \subset \mathbb{R}^n$ of class C^r are immersions of class C^r.

Recall that a **diffeomorphism** is a differentiable bijection (of class C^r) whose inverse is also differentiable (of class C^r). The following proposition tells us that up to a diffeomorphism (by composition), every immersion is locally the canonical immersion.

Proposition 1.9

Let $U \subset \mathbb{R}^k$ be open and $\alpha : \mathbb{R}^k \to \mathbb{R}^n$ be an immersion of class C^r. Then up to a local diffeomorphism, α is the canonical immersion $\iota : \mathbb{R}^k \to \mathbb{R}^n$ defined by $\iota(x_1, \ldots, x_k) = (x_1, \ldots, x_k, 0, \ldots, 0)$.

In order to prove this, we require the Inverse Function Theorem.

Theorem 1.10: Inverse Function Theorem

Let $U \subset \mathbb{R}^\ell$ be open and let $F : U \subset \mathbb{R}^\ell \to \mathbb{R}^\ell$ be of class C^r. Suppose that for some $x_0 \in U$, the $\ell \times \ell$ derivative matrix $DF(x_0)$ is invertible (that is, $\det(DF(x_0)) \neq 0$). Then F is invertible in an open neighbourhood $U_0 \subset U$ of x_0 and $F^{-1} : V_0 = F(U_0) \subset \mathbb{R}^\ell \to U_0 \subset \mathbb{R}^\ell$ is also of class C^r.

Note that when we take $\ell = 1$ in the Inverse Function Theorem and we have a function $f : U \subset \mathbb{R}^1 \to \mathbb{R}^1$, then $Df(x) = f'(x)$. If $f'(x_0) \neq 0$, then either $f'(x) > 0$ around x_0 or $f'(x) < 0$. In particular, f is increasing or decreasing around x_0, which implies that it is strictly monotone and thus invertible around x_0.

Proof of Proposition 1.9.

Suppose that $\alpha : U \subset \mathbb{R}^k \to \mathbb{R}^n$ is an immersion given by

$$x = (x_1, \ldots, x_k) \mapsto (f_1(x), \ldots, f_n(x)).$$

The derivative matrix of α is the $n \times k$ matrix

$$D\alpha(x) = \begin{bmatrix} \nabla f_1(x) \\ \vdots \\ \nabla f_n(x) \end{bmatrix}$$

and $D\alpha(x)$ has rank k for all $x \in U$ by definition. Then k of the rows of $D\alpha(x)$ are linearly independent. Without loss of generality, we can assume that these are the first k rows after possibly permuting the variables in $\mathbb{R}^n$. We now divide α into the parts

$$\alpha(x) = (\alpha_1(x), \alpha_2(x)),$$

where $\alpha_1 : U \subset \mathbb{R}^k \to \mathbb{R}^k$ corresponds to the first k component functions of α, and $\alpha_2 : U \subset \mathbb{R}^k \to \mathbb{R}^{n-k}$ corresponds to the remaining $n-k$ component functions. Then we can write

$$D\alpha(x) = \left[\begin{array}{c} D\alpha_1(x) \\ \hline D\alpha_2(x) \end{array}\right]$$

where $D\alpha_1(x)$ is a $k \times k$ matrix and $D\alpha_2(x)$ is an $(n-k) \times k$ matrix. Notice that $D\alpha_1(x)$ has rank k, which implies that $D\alpha_1(x)$ is invertible. By the Inverse Function Theorem (Theorem 1.10), there exist open neighbourhoods $U_0 \subset U$ of x_0 and $V_0 \subset \mathbb{R}^k$ of $\alpha_1(x_0)$ such that

$$\alpha_1|_{U_0} : U_0 \subset U \subset \mathbb{R}^k \to V_0 \subset \mathbb{R}^k$$

is invertible with inverse $(\alpha_1|_{U_0})^{-1} : V_0 \subset \mathbb{R}^k \to U_0 \subset U \subset \mathbb{R}^k$ of class C^r. Hence, $\alpha_1|_{U_0}$ is a diffeomorphism of class C^r. Now, consider the composition $\alpha \circ \alpha_1^{-1} : V_0 \subset \mathbb{R}^k \to \mathbb{R}^n$ which yields

$$x \mapsto \alpha(\alpha_1^{-1}(x)) = (\alpha_1(\alpha_1^{-1}(x)), \alpha_2(\alpha_1^{-1}(x))) =: (x, f(x))$$

where $f : V_0 \subset \mathbb{R}^k \to \mathbb{R}^{n-k}$. So up to the diffeomorphism, α is the parametrization of the graph of a function of class C^r (because the composition is of class C^r).

If $f(x) = 0$ for all $x \in V_0$, then we are already done. Otherwise, compose the above function with $h : V_0 \times \mathbb{R}^{n-k} \subset \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n) \mapsto (x_1, \ldots, x_k, x_{k+1} - f_{k+1} \circ \alpha^{-1}(x), \ldots, x_n - f_n \circ \alpha^{-1}(x)),$$

which is a diffeomorphism of class C^r. For $x \in V_0$, we obtain $h \circ \alpha \circ \alpha^{-1}(x) = (x, 0) = \iota(x)$ as desired. $\square$

As a consequence, we have the following result.

Corollary 1.11

The image of an immersion of class C^r is locally the graph of a function of class C^r (up to diffeomorphism). In particular, any submanifold of $\mathbb{R}^n$ of class C^r is locally the graph of a function of class C^r.

Proof of Corollary 1.11.

From the proof of Proposition 1.9, we had a map $\alpha \circ \alpha_1^{-1} : V_0 \subset \mathbb{R}^k \to \mathbb{R}^n$ which sent $x \mapsto (x, f(x))$ by taking $f = \alpha_2 \circ \alpha_1^{-1}$ which was of class C^r. Then $\alpha(V_0)$ is the graph of f in $\mathbb{R}^n$.

If $M \subset \mathbb{R}^n$ is a k-dimensional submanifold of class C^r, then for all $p \in M$, there exists a coordinate chart

$$\alpha : U \subset \mathbb{R}^k \to V \subset M$$

with $p \in V$. Set $x_0 = \alpha^{-1}(p)$. Using the above notation, there exist open sets $U_0, V_0 \subset \mathbb{R}^k$ with $x_0 \in U_0$ and a diffeomorphism $\alpha_1 : U_0 \subset \mathbb{R}^k \to V_0 \subset \mathbb{R}^k$ such that

$$\begin{aligned} \tilde{\alpha} := \alpha \circ \alpha_1^{-1} : V_0 \subset \mathbb{R}^k &\to \alpha(U_0) \subset M \\ x &\mapsto (x, f(x)) \end{aligned}$$

is of class C^r since α and α_1^{-1} are of class C^r. (Note that $\alpha(U_0)$ is open in M since α is a homeomorphism.) This implies that f is of class C^r, so M is locally the graph of the C^r function $f : V_0 \subset \mathbb{R}^k \to \mathbb{R}^{n-k}$. ☐

We now introduce the notion of an embedding.

DEFINITION 1.12

Let $U \subset \mathbb{R}^k$ be open and $\alpha : U \subset \mathbb{R}^k \to \mathbb{R}^n$ be a map. Then α is an **embedding** of class C^r if

(1) α is a homeomorphism onto its image;

(2) α is of class C^r;

(3) $D\alpha(x)$ has rank k for all $x \in U$.

In other words, an embedding is just an immersion that is homeomorphic onto its image. In particular, submanifolds of $\mathbb{R}^n$ of class C^r are subsets of $\mathbb{R}^n$ that are locally the image of an embedding of class C^r.

The following proposition tells us that embeddings are in fact diffeomorphisms onto their images.

PROPOSITION 1.13

Let $U \subset \mathbb{R}^k$ be open, and let $\alpha : U \subset \mathbb{R}^k \to \mathbb{R}^n$ be an embedding of class C^r. Then

$$\alpha^{-1} : \alpha(U) \subset \mathbb{R}^n \to U \subset \mathbb{R}^k$$

is also of class C^r.

Proof of Proposition 1.13.

Note that α is an immersion of class C^r, so we can write it as $\alpha(x) = (\alpha_1(x), \alpha_2(x))$, where $\alpha_1 : U \subset \mathbb{R}^k \to \mathbb{R}^k$ is locally invertible. Letting $\pi : \mathbb{R}^n \to \mathbb{R}^k$ be the projection $(y_1, \ldots, y_n) \mapsto (y_1, \ldots, y_k)$, we have

$$\begin{aligned} \alpha^{-1} : \alpha(U) \subset \mathbb{R}^n &\to U \subset \mathbb{R}^k \\ (\alpha_1(x), \alpha_2(x)) &\mapsto x = \alpha_1^{-1} \circ \pi(x_1, \ldots, x_n). \end{aligned}$$

But α_1^{-1} and π are both of class C^r, so α^{-1} is also of class C^r. ☐

We now introduce atlases, which we could've done a while ago when we defined topological submanifolds at the beginning. However, we can now talk about atlases of class C^r.

Definition 1.14

Let $M \subset \mathbb{R}^n$ be a k-dimensional topological submanifold. An **atlas** of M is a collection of charts

$$\{\alpha_a : U_a \subset \mathbb{R}^k \to V_a \subset M\}_{a \in A}$$

such that $\bigcup_{a \in A} V_a = M$, where each $U_a \subset \mathbb{R}^k$ and $V_a \subset M$ is open. Moreover, if all the charts in the atlas are of class C^r, then it is called an **atlas of class** C^r, or a **smooth atlas** if it is of class C^∞.

We look at some examples of atlases.

(1) Let $U \subset \mathbb{R}^n$ be open. Then $\alpha = \mathrm{Id}_U$ is a smooth chart for U such that every point in U is contained in $\alpha(U) = U$. Therefore, $\{\alpha = \mathrm{Id}_U : U \to U\}$ is a smooth atlas for U.

(2) Consider the graph M of a function $F : U \subset \mathbb{R}^k \to \mathbb{R}^{n-k}$ of class C^r, where $U \subset \mathbb{R}^k$ is open. Then $M = \{(x, F(x)) : x \in U\}$ is a k-dimensional submanifold of class C^r. It admits the chart $\alpha : U \subset \mathbb{R}^k \to M \subset \mathbb{R}^n$ defined by $x \mapsto (x, F(x))$ of class C^r, so $\{\alpha : U \to M\}$ is an atlas of class C^r for M.

(3) Recall that the points on the circle $\mathbb{S}^1 = \{(x, y) : x^2 + y^2 = 1\} \subset \mathbb{R}^2$ can be described using the charts $\alpha_1 : (-\pi, \pi) \to \mathbb{S}^1 \setminus \{(-1, 0)\}$ and $\alpha_2 : (0, 2\pi) \to \mathbb{S}^1 \setminus \{(1, 0)\}$ via $t \mapsto (\cos t, \sin t)$. Then $\{\alpha_1, \alpha_2\}$ is a smooth atlas for $\mathbb{S}^1$.

Note that if M is compact (that is, closed and bounded), then any atlas of M must contain at least two charts. Moreover, atlases are not unique in general! For example, consider $\{\alpha : \mathbb{R} \to M,\ t \mapsto (t, 0)\}$ and $\{\beta : \mathbb{R} \to M,\ t \mapsto (-t, 0)\}$, which are both smooth atlases for the x-axis M in $\mathbb{R}^2$.

We now give an alternate definition of a k-dimensional submanifold of $\mathbb{R}^n$ of class C^r. We will soon prove that this is equivalent to the original definition.

Definition 1.15

Let $M \subset \mathbb{R}^n$ be such that M is locally given by the zero set $\{F \equiv 0\}$ of a C^r map $F : V \subset \mathbb{R}^n \to \mathbb{R}^{n-k}$ with maximal rank. That is, $DF(x)$ has rank $n - k$ for all $x \in V \cap M$, where $V \cap M = F^{-1}(0)$ holds for an appropriately chosen neighbourhood V of every point in M. Then M is called a **k-dimensional submanifold of $\mathbb{R}^n$ of class C^r**.

This alternate definition is useful, because it is generally easier to show that a given subset is locally the zero set of a function than to explicitly exhibit charts covering the space.

(1) Consider the circle $\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$, which is given by the equation $x^2 + y^2 = 1$, which we can rearrange as $x^2 + y^2 - 1 = 0$. Define $F : \mathbb{R}^2 \to \mathbb{R}$ via $F(x, y) = x^2 + y^2 - 1$. Then $\mathbb{S}^1 = F^{-1}(0)$, so we can take $V = \mathbb{R}^2$ in the definition. Also, we have

$$DF(x, y) = \begin{bmatrix} 2x & 2y \end{bmatrix},$$

which has rank 1 for every point since $(0, 0) \notin \mathbb{S}^1$. Thus, under the alternate definition, $\mathbb{S}^1$ is a smooth 1-dimensional submanifold of $\mathbb{R}^2$.

(2) More generally, the n-sphere $\mathbb{S}^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_{n+1}^2 = 1\} \subset \mathbb{R}^{n+1}$ can be viewed as the zero set of the smooth function $F : \mathbb{R}^{n+1} \to \mathbb{R}$ given by $F(x_1, \ldots, x_{n+1}) = x_1^2 + \cdots + x_{n+1}^2 - 1$. Here, we can take $V = \mathbb{R}^{n+1}$ in the definition. Since

$$DF(x) = \begin{bmatrix} 2x_1 & \cdots & 2x_{n+1} \end{bmatrix}$$

has rank 1 for all $x \in \mathbb{S}^n$, we see that $\mathbb{S}^n$ is a smooth n-dimensional submanifold of $\mathbb{R}^{n+1}$.

(3) Consider the twisted cubic $M = \{(x,y,z) : y = x^2, z = x^3\} \subset \mathbb{R}^3$. We have seen that this is a smooth 1-dimensional submanifold of $\mathbb{R}^3$. For all $(x,y,z) \in M$, we have $y = x^2$ and $z = x^3$. Rearranging gives $y - x^2 = 0$ and $z - x^3 = 0$, so we can view M has the zero set of the smooth function $F(x,y,z) = (y - x^2, z - x^3)$ defined for all $(x,y,z) \in \mathbb{R}^3$. This time, we have $V = \mathbb{R}^3$, and $F : \mathbb{R}^3 \to \mathbb{R}^2$ so that $n = 3$ and $k = 1$. The derivative matrix of F is

$$DF(x,y,z) = \begin{bmatrix} -2x & 1 & 0 \\ -3x^2 & 0 & 1 \end{bmatrix}$$

which is rank 2 for all $(x,y,z) \in M$ because the last two columns are the 2×2 identity matrix. Hence, M is a smooth 1-dimensional submanifold of $\mathbb{R}^3$.

Remark. Not all zero sets of functions of class C^r are submanifolds of class C^r. Take the α-curve $C = \{(x,y) \in \mathbb{R}^2 : y^2 = x^2(x+1)\}$, which is the zero set of the function $F : \mathbb{R}^2 \to \mathbb{R}$ defined by $(x,y) \mapsto y^2 - x^2(x+1)$. The derivative matrix of F is

$$DF(x,y) = \begin{bmatrix} -3x^2 - 2x & 2y \end{bmatrix},$$

which is equal to the zero vector if and only if $(x,y) \in \{(0,0), (-2/3, 0)\}$. We see that $(0,0) \in C$, and this is the problematic point.

(4) **(Graph of a function of class C^r.)** Let $f : U \subset \mathbb{R}^k \to \mathbb{R}^{n-k}$ be function of class C^r, where $U \subset \mathbb{R}^k$ is open. Let $M = \{(x, f(x)) \in \mathbb{R}^k \times \mathbb{R}^{n-k} : x \in U\} \subset \mathbb{R}^n$, which we have already seen is a k-dimensional submanifold of class C^r. We now view this using the zero set characterization.

For all $(x,y) \in M \subset \mathbb{R}^k \times \mathbb{R}^{n-k}$, we have $y = f(x)$ if and only if $F(x,y) := f(x) - y = 0 \in \mathbb{R}^{n-k}$. Note that $U \times \mathbb{R}^{n-k}$ is an open subset of $\mathbb{R}^k \times \mathbb{R}^{n-k}$. Then M is the zero set of the C^r function $F : U \times \mathbb{R}^{n-k} \subset \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$ given by $(x,y) \mapsto f(x) - y$.

It remains to check the rank condition. We have that

$$DF(x,y) = \left[\begin{array}{c|c} \dfrac{\partial F}{\partial x} & \dfrac{\partial F}{\partial y} \end{array} \right]$$

which has $n-k$ rows. Since $F(x,y) = f(x) - y$, we see that $\partial F/\partial x = Df(x)$ and $\partial F/\partial y = -I_{n-k}$, so $DF(x,y)$ has rank $n-k$ since I_{n-k} does.

Theorem 1.16

Let $M \subset \mathbb{R}^n$. The following are equivalent:

(i) M is a k-dimensional submanifold of class C^r (using Definition 1.7).

(ii) M is locally the graph of a function $f : U \subset \mathbb{R}^k \to \mathbb{R}^{n-k}$ of class C^r, where $U \subset \mathbb{R}^k$ is open.

(iii) M is locally the zero set of a C^r function $F : V \subset \mathbb{R}^n \to \mathbb{R}^{n-k}$ of maximal rank, where $V \subset \mathbb{R}^n$ is open (using Definition 1.15).

Due to Corollary 1.11, we already know that (i) and (ii) are equivalent. Example (4) above shows that (ii) implies (iii). Therefore, it suffices to prove that (iii) implies (ii), and we will see that this is a direct consequence of the Implicit Function Theorem. Let's recall what this says.

Suppose that $\mathbb{R}^n = \mathbb{R}^{k+m} = \mathbb{R}^k \times \mathbb{R}^m$ has coordinates $(x,y) = (x_1, \ldots, x_k, y_1, \ldots, y_m)$. Let $U \subset \mathbb{R}^n$ be an open subset and $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$ be a function of class C^r. Then the derivative matrix of F is

$$DF(x,y) = \left[\begin{array}{c|c} \dfrac{\partial F}{\partial x} & \dfrac{\partial F}{\partial y} \end{array} \right].$$

where if $F_1, \ldots, F_m$ are the component functions of F, then we have $\partial F/\partial x = (\partial F_i/\partial x_j)_{1 \le i \le m, 1 \le j \le n}$ and $\partial F/\partial y = (\partial F_i/\partial y_j)_{1 \le i,j \le m}$. In particular, $\partial F/\partial y$ is an $m \times m$ matrix.

Theorem 1.17: Implicit Function Theorem

Let $(x_0, y_0) \in U$ be such that $F(x_0, y_0) = 0$. Suppose that

$$\det\left(\frac{\partial F}{\partial y}(x_0, y_0)\right) \neq 0.$$

Then there exists an open neighbourhood $V_0 \subset \mathbb{R}^k$ of x_0 and a unique function $g : V_0 \to \mathbb{R}^m$ of class C^r such that $g(x_0) = y_0$ and $F(x, g(x)) = 0$ for all $x \in V_0$.

In other words, the Implicit Function Theorem tells us that if $\det(\partial F(x_0, y_0)/\partial y) \neq 0$, then for all points $(x, y) \in \{F \equiv 0\}$ in an open neighbourhood of (x_0, y_0), we have $y = g(x)$ for some function of class C^r. Thus, we can express the variables $(y_1, \ldots, y_m)$ as functions of $(x_1, \ldots, x_k)$ of class C^r near (x_0, y_0).

Before proving the theorem, we illustrate what the result tells us with a simple example. Let $F(x, y) = x^2 + y^2 - 1$ for all $(x, y) \in \mathbb{R}^2$. Then $F : \mathbb{R}^2 \to \mathbb{R}$ is smooth and has derivative matrix

$$DF(x, y) = \begin{bmatrix} 2x & 2y \end{bmatrix}.$$

In this case, we have $m = 1$ and $m + k = 2$ so that $k = 1$. We are writing $\mathbb{R}^2 = \mathbb{R}^1 \times \mathbb{R}^1$ where x corresponds to the first copy of $\mathbb{R}^1$ and y corresponds to the second copy. Then $\partial F/\partial y = 2y$, which is nonzero if and only if $y \neq 0$. By the Implicit Function Theorem, the points on

$$\{F \equiv 0\} = \{(x, y) : x^2 + y^2 - 1 = 0\}$$

have a y-coordinate that can be expressed locally as a function of x. This is indeed true since $x^2 + y^2 - 1 = 0$ if and only if $y = \pm\sqrt{1 - x^2}$, which is smooth for $x \notin \{\pm 1\}$ and hence for $y \neq 0$ on $\{F \equiv 0\}$. These parametrize $\mathbb{S}^1 \setminus \{(\pm 1, 0)\}$. Similarly, note that $\partial F/\partial x = 2x \neq 0$ if and only if $x \neq 0$, so this holds for the points $(x, y) \neq (0, \pm 1)$ on $\mathbb{S}^1$. We see that $\mathbb{S}^1$ can be expressed as $x = \pm\sqrt{1 - y^2}$ away from $(0, \pm 1)$.

Proof of Implicit Function Theorem (Theorem 1.17).

This follows from the Inverse Function Theorem (Theorem 1.10). Define $H : U \subset \mathbb{R}^n \to \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^m$ by $(x, y) \mapsto (x, F(x, y))$. The derivative matrix is

$$DH = \left[\begin{array}{c|c} I_{k\times k} & 0 \\ \hline \partial F/\partial x & \partial F/\partial y \end{array}\right],$$

which is an $n \times n$ matrix. Since $I_{k\times k}$ has rank k and $\partial F/\partial y$ has rank m at (x_0, y_0) using the fact that it is invertible there, it follows that $DH(x_0, y_0)$ has rank $k + m = n$. That is, $\det(DH(x_0, y_0)) \neq 0$ and H is locally invertible with some inverse G by the Inverse Function Theorem.

Write $G(u, v) = (G_1(u, v), G_2(u, v))$ where $u \in \mathbb{R}^k$ and $v \in \mathbb{R}^m$, and we separate G into the components $G_1(u, v) \in \mathbb{R}^k$ and $G_2(u, v) \in \mathbb{R}^m$. Then we have

$$\begin{aligned} (u, v) &= H \circ G(u, v) \\ &= H(G_1(u, v), G_2(u, v)) \\ &= (G_1(u, v), F(G_1(u, v), G_2(u, v))). \end{aligned}$$

This implies that $u = G_1(u, v)$, so $G(u, v) = (u, G_2(u, v))$ for some function $G : V_0 \subset \mathbb{R}^n \to \mathbb{R}^m$ of class C^r (where $V_0 \subset \mathbb{R}^n$ is open). Moreover, for all (x, y) with $F(x, y) = 0$, we have $H(x, y) = (x, F(x, y)) = (x, 0)$, and hence $(x, y) = G \circ H(x, y) = G(x, 0) = (x, G_2(x, 0))$. Then $y = G_2(x, 0)$ for all $(x, y) \in \{F \equiv 0\}$. By setting $g(x) := G_2(x, 0)$, we have $y = g(x)$ for all $(x, y) \in \{F \equiv 0\}$ near (x_0, y_0), and g is of class C^r. We see that $F(x, g(x)) = 0$. For the proof of uniqueness of g and more details, we refer to *Topology* by Munkres, Theorem 9.2 on page 74. $\square$

Finally, we prove that the definitions are equivalent. Recall from our earlier discussion that it suffices to show that (iii) implies (ii).

Proof of Theorem 1.16.

Suppose that M is locally the zero set of a C^r function $F : V \subset \mathbb{R}^n \to \mathbb{R}^{n-k}$ of maximal rank on $M \cap V$. Then for all $(x_0, y_0) \in M \cap V \subset \mathbb{R}^k \times \mathbb{R}^{n-k} = \mathbb{R}^n$, the derivative matrix

$$DF(x_0, y_0) = \left[\frac{\partial F}{\partial x}(x_0, y_0) \,\middle|\, \frac{\partial F}{\partial y}(x_0, y_0) \right]$$

has rank $n-k$, where $\frac{\partial F}{\partial y}(x_0, y_0)$ is an $(n-k) \times (n-k)$ matrix.

After possibly permuting the variables $(x_1, \dots, x_k, y_1, \dots, y_{n-k})$ and therefore the columns of $DF(x_0, y_0)$, we may assume that $\partial F(x_0, y_0)/\partial y$ has rank $n-k$. This implies that $\det(\partial F(x_0, y_0)/\partial y) \neq 0$. By the Implicit Function Theorem (Theorem 1.17), we see that y is a C^r function of x on $M \cap V$ near (x_0, y_0). Write $y = g(x)$ for some C^r function $g : V_0 \subset \mathbb{R}^k \to \mathbb{R}^m$, where V_0 is an open neighbourhood of x_0. Then M is locally of the form $\{(x, g(x)) : x \in V_0\}$, which implies that M is locally the graph of g. □

To end our discussion, we give one more example of how we can use the zero set characterization to find charts for a submanifold M using the Implicit Function Theorem. Consider the special linear group

$$\mathrm{SL}(2, \mathbb{R}) = \{A \in M_{2\times 2}(\mathbb{R}) : \det A = 1\}.$$

In a natural way, we can identify the matrix

$$A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \in M_{2\times 2}(\mathbb{R})$$

with the point $(a_1, a_2, a_3, a_4) \in \mathbb{R}^4$, and we have

$$\det A - 1 = a_1 a_4 - a_2 a_3 - 1 =: F(a_1, a_2, a_3, a_4)$$

so that $\mathrm{SL}(2, \mathbb{R}) = \{F \equiv 0\}$ where $F : \mathbb{R}^4 \to \mathbb{R}$ is smooth. The derivative matrix of F is

$$DF(a_1, a_2, a_3, a_4) = \begin{bmatrix} a_4 & -a_3 & -a_2 & a_1 \end{bmatrix},$$

which can have rank at most 1 since it is a 1×4 matrix. In particular, $DF(a_1, a_2, a_3, a_4)$ has rank 1 if and only if it is not the zero vector, which occurs if at least one of the a_i is nonzero. But for all $(a_1, a_2, a_3, a_4) \in \mathrm{SL}(2, \mathbb{R})$, we have $a_1 a_4 - a_2 a_3 = 1$ so that at least one of the a_i must be nonzero.

Suppose that $a_1 \neq 0$. Then we have $\partial F/\partial a_4 = a_1 \neq 0$, so the Implicit Function Theorem tells us that a_4 can be expressed as a smooth function of the remaining three variables on $\mathrm{SL}(2, \mathbb{R})$. Indeed, if $(a_1, a_2, a_3, a_4) \in \mathrm{SL}(2, \mathbb{R})$ with $a_1 \neq 0$, then $a_1 a_4 - a_2 a_3 = 1$ can be rearranged to obtain

$$a_4 = \frac{a_2 a_3 + 1}{a_1}.$$

A similar analysis can be done when the other variables are nonzero. Since DF has rank 1 everywhere, it follows that $\mathrm{SL}(2, \mathbb{R})$ is a smooth submanifold of $\mathbb{R}^4$ of dimension $4 - 1 = 3$.

1.5 Tangent vectors and tangent vector fields

The simplest example of a tangent vector is the velocity vector of a curve.

Definition 1.18

Let $\gamma : (a, b) \subset \mathbb{R} \to \mathbb{R}^n$ be a map of class C^r. We define the **velocity vector** of γ at $\gamma(t)$ to be

$$\gamma'(t) := D\gamma(t).$$

Note that if $\gamma(t) = (\gamma_1(t), \ldots, \gamma_n(t))$, then $\gamma'(t) = (\gamma_1'(t), \ldots, \gamma_n'(t))$. Moreover, we have

$$\gamma'(t_0) = \lim_{t \to t_0} \frac{\gamma(t) - \gamma(t_0)}{t - t_0}.$$

Observe that $(\gamma(t) - \gamma(t_0))/(t - t_0)$ is the velocity vector of the secant L passing through $\gamma(t)$ and $\gamma(t_0)$. Taking the limit as $t \to t_0$, it follows that $\gamma'(t_0)$ is the tangent vector to the curve in $\mathbb{R}^n$ given by $\gamma(t)$, under the assumption that $\gamma'(t_0) \neq 0$. Let's look at some examples.

(1) Let $x \in \mathbb{R}^n$ and $0 \neq v \in \mathbb{R}^n$. Set $\gamma(t) = x + tv$ for $t \in \mathbb{R}$, which parametrizes the line L in $\mathbb{R}^n$ passing through x with direction vector v. We have that $\gamma'(t) = v$ for all $t \in \mathbb{R}$, so at every point on the line, the velocity vector coincides with the direction vector.

(2) Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be defined by $t \mapsto (\cos t, \sin t)$, which parametrizes $\mathbb{S}^1$. Note that $\gamma'(t) = (-\sin t, \cos t) \neq (0,0)$ for all $t \in \mathbb{R}$.

(3) Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be given by $t \mapsto (t^2, t^3)$. This parametrizes the cusp curve $y^2 = x^3$. We have $\gamma'(t) = (2t, 3t^2)$. Observe that $\gamma'(0) = (0,0)$ so that the velocity vector is zero at $\gamma(0) = (0,0)$.

(4) Recall that the α-curve can be parametrized with $\gamma : \mathbb{R} \to \mathbb{R}^2$ defined by $t \mapsto (t^2 - 1, t(t^2 - 1))$. We have $\gamma'(t) = (2t, 3t^2 - 1) \neq (0,0)$ for all $t \in \mathbb{R}$.

Note that if γ is a homeomorphism onto its image

$$C = \{x \in \mathbb{R}^n : x = \gamma(t) \text{ for some } t \in (a,b)\}$$

and $D\gamma(t)$ has rank 1 for all $t \in (a,b)$, then C is a 1-dimensional submanifold of $\mathbb{R}^n$ of class C^r. In fact, we see that $D\gamma(t)$ has rank 1 if and only if $\gamma'(t) \neq 0$, so $\gamma'(t)$ is a direction vector for the tangent line to γ at $\gamma(t)$. In particular, the 1-dimensional submanifold of $\mathbb{R}^n$ determined by γ has a well-defined tangent line L at every point. We call $L := \{\gamma(t_0) + s\gamma'(t_0) : s \in \mathbb{R}\}$ the **tangent line to C at $\gamma(t_0)$**.

From example (1) above, we know that a line $\gamma(t) = x + tv$ coincides with its tangent line at every point.

Definition 1.19

Let $x \in \mathbb{R}^n$. A **tangent vector to $\mathbb{R}^n$ at x** is defined as a pair $(x; v)$ where $v \in \mathbb{R}^n$. We call

$$T_x(\mathbb{R}^n) := \{(x; v) : v \in \mathbb{R}^n\}$$

the **tangent space to $\mathbb{R}^n$ at x**. This is the set of all tangent vectors to $\mathbb{R}^n$ at x.

We can give $T_x(\mathbb{R}^n)$ a vector space structure with the operations $(x; v) + (x; w) = (x; v + w)$ and $c(x; v) = (x; cv)$ for all $c \in \mathbb{R}$ and $v, w \in \mathbb{R}^n$. Note that $T_x(\mathbb{R}^n) \simeq \mathbb{R}^n$ as vector spaces, where the isomorphism $T_x(\mathbb{R}^n) \to \mathbb{R}^n$ is given by $(x; v) \mapsto v$.

Lemma 1.20

For all $x \in \mathbb{R}^n$, we have

$$T_x(\mathbb{R}^n) = \{(x; v) : v \text{ is a velocity vector of some curve } \gamma(t) \text{ passing through } x\}.$$

Proof of Lemma 1.20.

Let $(x; v) \in T_x(\mathbb{R}^n)$. Set $\gamma(t) = x + tv$ for $t \in \mathbb{R}$, which satisfies $\gamma'(t) = v$ for all $t \in \mathbb{R}$. In particular, we have $\gamma(0) = x$ and $\gamma'(0) = v$. Note that if $v \neq 0$, then $\gamma(t)$ parametrizes the line through x with velocity vector v. $\square$

Can we get something similar for submanifolds $M \subset \mathbb{R}^n$? Towards this direction, we make a new definition.

Definition 1.21

Let $U \subset \mathbb{R}^k$ be open and let $\alpha : U \to \mathbb{R}^n$ be of class C^r. Also, let $x \in U$ and set $p = \alpha(x)$. The map

$$\begin{aligned} \alpha_* : T_x(\mathbb{R}^k) &\to T_p(\mathbb{R}^n) \\ (x; v) &\mapsto (p; D\alpha(x)v) \end{aligned}$$

is called the **pushforward of α at x**.

Note that $\alpha_* : T_x(\mathbb{R}^k) \to T_p(\mathbb{R}^n)$ is a vector space homomorphism since it is linear; in particular, it is given by multiplication by the $n \times k$ derivative matrix.

For all $(x; v) \in T_x(\mathbb{R}^k)$, consider the map $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ defined by $t \mapsto \alpha(x + tv)$ with $\varepsilon > 0$ such that $x + tv \in U$ for all $t \in (-\varepsilon, \varepsilon)$. That is, the line segment $L = \{x + tv : t \in (-\varepsilon, \varepsilon)\}$ is included in the open set U. Then for all $t_0 \in (-\varepsilon, \varepsilon)$, the chain rule gives

$$\gamma'(t_0) = D\alpha(x + t_0 v) D(x + tv)(t_0) = D\alpha(x + t_0 v)v$$

since $D(x + tv)(t_0) = v$ for all $t_0 \in \mathbb{R}$. In particular, we obtain

$$\begin{aligned} (\gamma(0); \gamma'(0)) &= (\alpha(x); D\alpha(x)v) \\ &= (p, D\alpha(x)v) \\ &= \alpha_*(x; v). \end{aligned}$$

This tells us that $\alpha_*(x; v)$ is the velocity vector of $\gamma(t) := \alpha(x + tv)$ at $p = \gamma(0)$.

Lemma 1.22

If $\alpha : U \subset \mathbb{R}^k \to \mathbb{R}^n$ and $\beta : V \subset \mathbb{R}^\ell \to \mathbb{R}^k$ are functions of class C^r where $U \subset \mathbb{R}^k$ and $V \subset \mathbb{R}^\ell$ are open with $\beta(V) \subset U$, then $(\alpha \circ \beta)_* = \alpha_* \circ \beta_*$ on V.

Proof of Lemma 1.22.

Note that $\alpha \circ \beta : V \subset \mathbb{R}^\ell \to \mathbb{R}^n$ is of class C^r. Let $x \in V$. Then for all $v \in \mathbb{R}^\ell$, we have

$$\begin{aligned} (\alpha \circ \beta)_*(x; v) &= (\alpha \circ \beta(x); D(\alpha \circ \beta)(x)v) \\ &= (\alpha(\beta(x)); D\alpha(\beta(x))D\beta(x)v) \\ &= \alpha_*(\beta(x); D\beta(x)v) \\ &= \alpha_*(\beta_*(x; v)), \end{aligned}$$

where the second equality follows from the chain rule. □

From Lemma 1.22, it now makes sense to define the following.

Definition 1.23

Let $M \subset \mathbb{R}^n$ be a k-dimensional submanifold of $\mathbb{R}^n$ of class C^r for some $r \geq 1$. Let $p \in M$ and let $\alpha : U \subset \mathbb{R}^k \to V \subset M$ be a coordinate chart of M about p (i.e. $p \in V$) of class C^r. We define the **tangent space of M at p** to be

$$T_p(M) := \alpha_*(T_{x_0}(\mathbb{R}^k)) \subset T_p(\mathbb{R}^n),$$

where $x_0 \in U$ is the unique point in U such that $\alpha(x_0) = p$.

2 Curves in $\mathbb{R}^n$

Tangent spaces: Let $M \subset \mathbb{R}^n$ with class C^r of dimension k. For a chart $\alpha : U \subset \mathbb{R}^k \to V \subset M \subset \mathbb{R}^n$, we have for all $p \in V$ that

$$T_p(M) := \alpha_*(T_{x_0}(\mathbb{R}^k)),$$

where $p = \alpha(x_0)$. We showed that

$$T_p(M) = \operatorname{span}_{\mathbb{R}} \left\{ \frac{\partial \alpha}{\partial x_1}(x_0), \ldots, \frac{\partial \alpha}{\partial x_k}(x_0) \right\} \subset T_p(\mathbb{R}^n),$$

which is a k-dimensional subspace of $T_p(\mathbb{R}^n)$.

Examples:

(1) Let $U \subset \mathbb{R}^n$ be an open set. Then $\alpha : U \subset \mathbb{R}^n \to V = U \subset \mathbb{R}^n$ given by $x \mapsto x$ yields $D\alpha(x) = I_{n\times n}$ for all $x \in U$, so

$$T_x(U) = \alpha_*(T_x(\mathbb{R}^n))$$

by definition. Then for all $(x; v) \in T_x(\mathbb{R}^n)$, we get

$$\alpha_*(x; v) = (\alpha(x); D\alpha(x)v) = (x; v).$$

That is, we have $T_x(U) = T_x(\mathbb{R}^n)$.

(2) We saw two different ways of seeing if something is a submanifold. If $f : U \subset \mathbb{R}^k \to \mathbb{R}^{n-k}$ is a function of a class C^r and

$$M = \{(x, f(x)) \in \mathbb{R}^n : x \in U\} \subset \mathbb{R}^n$$

its graph, then M is a k-dimensional submanifold of $\mathbb{R}^n$ of class C^r. We can parameterize all points in M with the map $\alpha : U \subset \mathbb{R}^k \to M \subset \mathbb{R}^n$ defined by $\alpha(x) = (x, f(x))$. The derivative matrix is

$$D\alpha(x) = \left[\begin{array}{c} I_{k\times k} \\ \hline Df(x) \end{array}\right]$$

for all $x \in U$. Let $p \in M$ so that $p = (x_0, f(x_0))$ for some $x_0 \in U$. Then

$$\begin{aligned} T_p(M) &= \alpha_*(T_{x_0}(\mathbb{R}^k)) \\ &= \{\alpha_*(x_0; v) : v \in T_{x_0}(\mathbb{R}^k)\} \\ &= \{(\alpha(x_0); D\alpha(x_0)v) : v \in \mathbb{R}^k\} \\ &= \{(p; w) : w = (v, Df(x_0)v),\ v \in \mathbb{R}^k\}, \end{aligned}$$

since $\alpha(x_0) = p$ and $D\alpha(x)$ is the block matrix with $I_{k\times k}$ upstairs and $Df(x)$ downstairs. Also,

$$T_p(M) = \operatorname{span}_{\mathbb{R}} \left\{ \frac{\partial \alpha}{\partial x_1}(x_0), \ldots, \frac{\partial \alpha}{\partial x_k}(x_0) \right\} = \operatorname{span}_{\mathbb{R}} \left\{ \left[\begin{array}{c} e_i \\ \hline \partial f(x_0)/\partial x_i \end{array}\right] : i = 1, \ldots, k \right\}.$$

(3) Let $U \subset \mathbb{R}^n$ be open. If $F : U \subset \mathbb{R}^n \to \mathbb{R}^{n-k}$ is a function of class C^r with $DF(p)$ having rank $n - k$ for all $p \in U$, then

$$M = \{x \in U : F(x) = 0\}$$

is a k-dimensional submanifold of $\mathbb{R}^n$ of class C^r. In this case, we leave it as an exercise to show that

$$T_p(M) = \ker(DF(p)).$$

In particular, if $k = n - 1$, then $F : U \subset \mathbb{R}^n \to \mathbb{R}$ is a scalar function and

$$T_p(M) = \ker(\nabla F(p)).$$

Then $\nabla F(p)$ is the normal vector of $T_p(M)$.

For example, take the n-sphere

$$S^n = \{x \in \mathbb{R}^{n+1} : \|x\|^2 = 1\},$$

which is the zero set of the function $F : \mathbb{R}^{n+1} \to \mathbb{R}$ defined by $F(x) = \|x\|^2 - 1 = x_1^2 + \cdots + x_{n+1}^2 - 1$. The derivative matrix is just the gradient; that is,

$$DF(x) = \nabla F(x) = \begin{bmatrix} 2x_1 & \cdots & 2x_{n+1} \end{bmatrix} = 2x.$$

2.1

What is a curve? Intuitively, it is a 1-dimensional subset of $\mathbb{R}^n$.

The level sets $f(x, y) = k$ of a two variable function in $\mathbb{R}^2$ are curves. For example, for $f(x, y) = x^2 + y^2$, we see that $C\colon x^2 + y^2 = k$ for $k > 0$ is a circle centered at $(0, 0)$ of radius $k^{1/2}$.

The intersection of two surfaces in $\mathbb{R}^3$ is also a curve. For example, take $z = x^2 + y^2$ and $z = 2$. Their intersection is the circle $C\colon x^2 + y^2 = 2$ in the plane $z = 2$.

For our purposes, we'll work with parametrized curves $\gamma : I = (\alpha, \beta) \subset \mathbb{R} \to \mathbb{R}^n$ of class C^r. (In practice, we need $r \geq n$ when working over $\mathbb{R}^n$.)

Example: **Circular helix.** Let $\gamma(t) = (a\cos t, a\sin t, bt)$ for $t \in \mathbb{R}$ and $a, b > 0$. Note that $x^2 + y^2 = a^2$, so $\gamma(t)$ lies above the circle $x^2 + y^2 = a^2$ in the xy-plane. We have $\gamma(0) = (a, 0, 0)$ and $\gamma(\pi/2) = (0, a, b\pi/2)$. (The circular helix looks like a spiral along the cylinder.)

Definition 2.1

A parameterized curve $\gamma : (\alpha, \beta) \to \mathbb{R}^n$ of class C^r is called **regular** if for all $t \in (\alpha, \beta)$, we have

$$\gamma'(t) = \frac{\mathrm{d}\gamma}{\mathrm{d}t}(t) \neq 0.$$

We call $\|\gamma'(t)\|$ the **speed of** γ **at** $\gamma(t)$ and we say that γ is **unit speed** if $\|\gamma'(t)\| = 1$ for all $t \in (\alpha, \beta)$.

Note that unit speed implies regular because $\|x\| = 0$ if and only if $x = 0$.

Example: Let $a > 0$ and take $\gamma(t) = (a\cos t, a\sin t)$ for $t \in \mathbb{R}$, which is a parametrization of the circle of radius a. Then $\gamma'(t) = (-a\sin t, a\cos t) \neq (0, 0)$ for all $t \in \mathbb{R}$ and $\|\gamma'(t)\| = a$, so γ is unit speed if and only if $a = 1$.