PMATH 365: Suggested problems for the final

To prepare for the final exam, I recommend reviewing Quizzes 1-10. I also recommend going through the following list of questions.

A. Differentiable maps, immersions, embeddings and submanifolds.

1. Define the following:

   (a) A k-dimensional topological submanifold of $\mathbb{R}^n$.

   Solution. A set $M \subset \mathbb{R}^n$ such that for all $p \in M$, there exists an open neighbourhood V of p, an open set $U \subset \mathbb{R}^k$, and a homeomorphism $\alpha : U \subset \mathbb{R}^k \to V \subset M \subset \mathbb{R}^n$.

   (b) An immersion of class C^r, $r \geq 1$.

   Solution. A map $\alpha : U \subset \mathbb{R}^k \to \mathbb{R}^n$ such that α is of class C^r and $D\alpha$ has maximal rank k everywhere on U.

   (c) An embedding of class C^r, $r \geq 1$.

   Solution. An immersion that is homeomorphic onto its image.

   (d) A k-dimensional submanifold of $\mathbb{R}^n$ of class C^r, $r \geq 1$.

   Solution. A set $M \subset \mathbb{R}^n$ such that for all $p \in M$, there exists an open neighbourhood V of p, an open set $U \subset \mathbb{R}^k$, and a homeomorphism $\alpha : U \subset \mathbb{R}^k \to V \subset M \subset \mathbb{R}^n$ satisfying the following properties:

   (i) α is of class C^r;

   (ii) $D\alpha(x)$ has rank k for all $x \in U$.

   (e) A coordinate chart (of class C^r) of a k-dimensional submanifold of $\mathbb{R}^n$ (of class C^r, $r \geq 1$).

   Solution. This is a homeomorphism as in (d).

   (f) An atlas (of class C^r) of a k-dimensional submanifold of $\mathbb{R}^n$ (of class C^r, $r \geq 1$).

   Solution. A set of coordinate charts $\{\alpha_i : U_i \subset \mathbb{R}^k \to V_i \subset M \subset \mathbb{R}^n\}_{i \in I}$ of class C^r for M (where I is some index set) such that $\bigcup_{i \in I} V_i = M$.

2. What is the difference between an immersion and an embedding? Give an example of both. In particular, give an example of an immersion that is *not* an embedding.

   Solution. An embedding has the requirement that it is homeomorphic onto its image. The canonical immersion is the inclusion map $\iota : \mathbb{R}^k \to \mathbb{R}^n$ defined by $(x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_k, 0, \ldots, 0)$. This is an immersion of class C^∞ since ι is smooth with

$$D\iota = \left[\begin{array}{ccc} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \\ \hline 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{array}\right]$$

   so that $D\iota$ has rank k everywhere. It is also an embedding as it is homeomorphic onto its image with inverse $\iota^{-1} : \iota(\mathbb{R}^k) \subset \mathbb{R}^n \to \mathbb{R}^k$ given by $(x_1, \ldots, x_k, 0, \ldots, 0) \mapsto (x_1, \ldots, x_k)$.

   For an example of an immersion that is not an embedding, consider the parametrization $\alpha(t) = (t^2 - 1, t(t^2 - 1))$ of the α-curve, where $t \in \mathbb{R}$. We see that α is smooth with derivative matrix $D\alpha(t) = (2t, 3t^2 - 1) \neq (0, 0)$ for all $t \in \mathbb{R}$, so α is an immersion of class C^∞. However, it is not homeomorphic onto its image since it is not injective. Indeed, we have $\alpha(1) = \alpha(-1) = (0, 0)$.

3. Give an example of a subset of $\mathbb{R}^n$ that is *not* a topological submanifold of $\mathbb{R}^n$. Explain!

Solution. We again consider the α-curve $C = \{(x, y) \in \mathbb{R}^2 : y^2 = x^2(x + 1)\} \subset \mathbb{R}^2$, which can be parametrized by the map $\alpha : \mathbb{R} \to C \subset \mathbb{R}^2$ given by $t \mapsto (t^2 - 1, t(t^2 - 1))$. Note that if we remove the points $t = \pm 1$, we see that α has inverse

$$\begin{aligned} \alpha^{-1} : C \setminus \{(0,0)\} &\to \mathbb{R} \setminus \{\pm 1\} \\ (x, y) &\mapsto 1/x. \end{aligned}$$

That is, C is a 1-dimensional topological submanifold of $\mathbb{R}^2$ away from the point $(0, 0)$.

Now, suppose that C were a topological submanifold of $\mathbb{R}^2$. Our discussion above means that C must be 1-dimensional. Since $(0, 0) \in C$, there exists an open neighbourhood V of $(0, 0)$, an open set $U \subset \mathbb{R}^1$, and a homeomorphism $\alpha : U \subset \mathbb{R}^1 \to V \subset C \subset \mathbb{R}^2$. Since α is a homeomorphism, there exists a unique point $t_0 \in U$ such that $\alpha(t_0) = (0, 0)$. Since U is open, we can pick $\varepsilon > 0$ such that $B_\varepsilon(t_0) = (t_0 - \varepsilon, t_0 + \varepsilon) \subset U$. Write $U' = B_\varepsilon(t_0)$ and let $V' = \alpha(U')$ so that $\alpha|_{U'} : U' \to V'$ is also a homeomorphism.

Observe that $V' \setminus \{(0, 0)\}$ consists of three or four connected components depending on how large V' is: one on the bottom right quadrant, one on the top right quadrant, and one or two on the left of the y-axis. On the other hand, $U' \setminus \{t_0\}$ has two components, which is a contradiction since homeomorphisms preserve the number of connected components. Therefore, C is not a topological submanifold of $\mathbb{R}^2$.

4. Give an example of a subset of $\mathbb{R}^n$ that is a topological submanifold but *not* a submanifold of class C^r for any $r \geq 1$. Explain!

Solution. Let $M = \{(x, |x|) : x \in \mathbb{R}\} \subset \mathbb{R}^2$, which is the graph of $f(x) = |x|$ over $x \in \mathbb{R}$. In particular, since $f(x) = |x|$ is continuous, it follows that M is a 1-dimensional topological submanifold of $\mathbb{R}^2$. Note that f is smooth away from $x = 0$, so $M \setminus \{(0, 0)\}$ is a 1-dimensional submanifold of class C^∞.

However, we claim that M is not a submanifold of class C^r for some $r \geq 1$. Suppose otherwise, so at the point $(0, 0) \in M$, there exists an open neighbourhood V of $(0, 0)$, an open set $U \subset \mathbb{R}$, and a homeomorphism $\alpha : U \subset \mathbb{R} \to V \subset M \subset \mathbb{R}^2$ such that α is of class C^r and $D\alpha$ has maximal rank 1 everywhere on U. In particular, this means that $D\alpha(t) = \alpha'(t) \neq (0, 0)$ for all $t \in U$. But $\alpha'(t)$ is tangent to M at $\alpha(t)$, which gives two possibilities:

- If $\alpha(t)$ is on the line $y = x$, then $\alpha'(t)$ is a direction vector of $y = x$. This means that for some $c : I \subset \mathbb{R} \to \mathbb{R}$, we have
$$\alpha'(t) = c(t) \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$
Since α is of class C^r where $r \geq 1$, this means $\alpha'(t)$ is continuous, so $c : I \to \mathbb{R}$ is continuous.
- If $\alpha(t)$ is on the line $y = -x$, then $\alpha'(t)$ is a direction vector of $y = -x$. Then for some $d : I' \subset \mathbb{R} \to \mathbb{R}$, we have
$$\alpha'(t) = d(t) \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$
The same argument shows that $d : I' \to \mathbb{R}$ is continuous.

But α is a bijection, so $(0, 0) = \alpha(t_0)$ for some $t_0 \in U$. By the continuity of $\alpha'(t)$, we obtain

$$\lim_{t \to t_0^-} \alpha'(t) = \lim_{t \to t_0^+} \alpha'(t).$$

Without loss of generality, suppose that $\alpha(t)$ is moving along M from left to right. (We can parametrize in the other direction otherwise.) Then if $t < t_0$, then we are in the second case above, whereas if $t > t_0$, then we are in the first case. This implies that

$$\lim_{t \to t_0^-} \alpha'(t) = \lim_{t \to t_0^-} d(t) \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \lim_{t \to t_0^+} c(t) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \lim_{t \to t_0^+} \alpha'(t).$$

But these vectors are not parallel to each other, so $\lim_{t \to t_0^-} d(t) = \lim_{t \to t_0^+} c(t) = 0$. This implies that $\alpha'(t_0) = (0, 0)$, which contradicts the fact that $D\alpha(t) = \alpha'(t)$ has rank 1 for all $t \in U$.

5. True of false?

   (a) Every topological submanifold of $\mathbb{R}^n$ is also a submanifold of $\mathbb{R}^n$ of class C^r for some $r \geq 1$.

   Solution. False, take the graph of $f(x) = |x|$.

   (b) Every submanifold of $\mathbb{R}^n$ of class C^r for some $r \geq 1$ is also a topological submanifold of $\mathbb{R}^n$.

   Solution. True.

   (c) A subset of $\mathbb{R}^n$ that is not a topological submanifold may be a submanifold of class C^r for some $r \geq 1$.

   Solution. False; consider the contrapositive of (b).

   (d) Every topological submanifold of $\mathbb{R}^n$ admits an atlas of class C^r for some $r \geq 1$.

   Solution. False. If a topological submanifold of $\mathbb{R}^n$ admits an atlas of class C^r for some $r \geq 1$, then it is a submanifold of $\mathbb{R}^n$ of class C^r, but this is not true of all topological submanifolds by (a).

   (e) Every coordinate chart of class C^r, $r \geq 1$, of a submanifold is a homeomorphism.

   Solution. True.

   (f) Every coordinate chart of class C^r, $r \geq 1$, of a submanifold is an embedding.

   Solution. True.

   (g) Let $\alpha : U \subset \mathbb{R}^k \to V \subset M \subset \mathbb{R}^n$ be a coordinate chart of class C^r, $r \geq 1$, of a k-dimensional submanifold M of $\mathbb{R}^n$. Then, $\alpha^{-1} : V \subset \mathbb{R}^n \to U \subset \mathbb{R}^k$ is also of class C^r.

   Solution. True.

   (h) Every immersion is a homeomorphism.

   Solution. False; take the parametrization $\alpha(t) = (t^2 - 1, t(t^2 - 1))$ of the α-curve.

   (i) Every embedding is a homeomorphism.

   Solution. True.

   (j) Every submanifold of $\mathbb{R}^n$ is also an immersed submanifold of $\mathbb{R}^n$ of class C^r.

   Solution. True; the coordinate charts are embeddings and therefore immersions.

   (k) The n-sphere admits an atlas that consists of a single chart.

   Solution. False. The n-sphere is compact, and hence closed and bounded by Heine-Borel. The domain of a coordinate chart is an open set. Since homeomorphisms send closed sets to closed sets and the only open and closed set in $\mathbb{R}^{n+1}$ is $\varnothing$, it is impossible to use a single chart.

   (l) Every k-dimensional submanifold of $\mathbb{R}^n$ of class C^r , $r \geq 1$, is locally the graph of a function $f : U \subset \mathbb{R}^r \to \mathbb{R}^{n-k}$ of class C^r.

   Solution. True.

(m) Every submanifold of $\mathbb{R}^n$ of class C^r , $r \geq 1$, is locally the zero set a function $F : U \subset \mathbb{R}^n \to \mathbb{R}^{n-k}$ of class C^r of maximal rank (that is, DF has maximal rank $n-k$ on $F^{-1}(0)$).

Solution. True.

6. Determine whether the following sets are topological submanifolds of $\mathbb{R}^n$ or submanifolds of $\mathbb{R}^n$ of class C^r for some $r \geq 1$. Justify your answers!

(a) $\mathbb{R}^n$.

Solution. $\mathbb{R}^n$ is a submanifold of $\mathbb{R}^n$ of class C^∞ since we can choose the identity map on $\mathbb{R}^n$ as the coordinate chart.

(b) An open subset U of $\mathbb{R}^n$.

Solution. This is a submanifold of $\mathbb{R}^n$ of class C^∞; take the identity map on U as the coordinate chart.

(c) The graph of a continuous function $f : U \subset \mathbb{R}^k \to \mathbb{R}^{n-k}$.

Solution. This is a k-dimensional topological submanifold of $\mathbb{R}^n$. If $M = \{(x, f(x)) \in \mathbb{R}^k \times \mathbb{R}^{n-k} : x \in U\}$ is the graph, then $\alpha : U \subset \mathbb{R}^k \to M \subset \mathbb{R}^n$ given by $x \mapsto (x, f(x))$ is a homeomorphism with inverse $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_k)$.

(d) The graph of a function $f : U \subset \mathbb{R}^k \to \mathbb{R}^{n-k}$ of class C^r for some $r \geq 1$.

Solution. This is a k-dimensional submanifold of class C^r. Take the same homeomorphism $\alpha : U \subset \mathbb{R}^k \to M \subset \mathbb{R}^n$ defined by $x \mapsto (x, f(x))$ as above. This is of class C^r since f is of class C^r and $D\alpha(x)$ has rank k for all $x \in U$ since the upper part of $D\alpha(x)$ is the $k \times k$ identity matrix.

(e) The zero set of a continuous function $F : U \subset \mathbb{R}^n \to \mathbb{R}^{n-k}$.

Solution. This is not a topological submanifold of $\mathbb{R}^n$ in general. Consider the α-curve, which is the zero set of $f(x, y) = y^2 - x^2(x + 1)$ but is not a topological submanifold of $\mathbb{R}^2$.

(f) The zero set of a function $F : U \subset \mathbb{R}^n \to \mathbb{R}^{n-k}$ of class C^r, $r \geq 1$, of maximal rank.

Solution. This is a k-dimensional submanifold of class C^r; in fact, this is an equivalent characterization.

(g) $\{(x, y) \in \mathbb{R}^2 : y = x^2, -1 \leq x \leq 1\} \cup \{(x, y) \in \mathbb{R}^2 : y = 1, -1 \leq x \leq 1\} \subset \mathbb{R}^2$.

Solution. This is a 1-dimensional topological submanifold of $\mathbb{R}^2$, but not a submanifold of $\mathbb{R}^2$ of class C^r by considering the points $(\pm 1, 1)$, which lie on both curves.

(h) The α-curve $\{(x, y) \in \mathbb{R}^2 : y^2 = x^2(x + 1)\} \subset \mathbb{R}^2$. For a picture, see (second picture).

Solution. Not a topological submanifold of $\mathbb{R}^2$.

(i) $\{(x, y) \in \mathbb{R}^2 : xy = 1\} \subset \mathbb{R}^2$.

Solution. This is a smooth submanifold of $\mathbb{R}^2$ since it is the graph of the smooth function $f(x) = 1/x$ on $x \in \mathbb{R} \setminus \{0\}$.

(j) The cylinder $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\} \subset \mathbb{R}^3$.

Solution. Smooth submanifold of $\mathbb{R}^3$ since it is the graph of the smooth function $f(x) = x^2 + y^2 - 1$ on $x \in \mathbb{R}$.

(k) The cone $\{(x, y, z) \in \mathbb{R}^3 : z = \sqrt{x^2 + y^2}\} \subset \mathbb{R}^3$.

Solution. Topological submanifold of $\mathbb{R}^3$ since it is the graph of the continuous function $f(x, y) = \sqrt{x^2 + y^2}$, but not a submanifold of $\mathbb{R}^3$ of class C^r for some $r \geq 1$ by considering the point $(0, 0, 0)$ in the cone.

(l) The double cone $\{(x,y,z) \in \mathbb{R}^3 : z^2 = x^2 + y^2\} \subset \mathbb{R}^3$.

Solution. Not a topological submanifold of $\mathbb{R}^3$ by considering $(0,0,0)$.

(m) The twisted cubic $\{(x,y,z) \in \mathbb{R}^3 : y = x^2, z = x^3\} \subset \mathbb{R}^3$.

Solution. Smooth submanifold of $\mathbb{R}^3$, since it is the graph of $f(x) = (x^2, x^3)$.

(n) $\{(x,y,z) \in \mathbb{R}^3 : y = -1\} \cup \{(x,y,z) \in \mathbb{R}^3 : x = y = 0\} \subset \mathbb{R}^3$.

Solution. Neither; one of the sets is 2-dimensional while the other is 1-dimensional.

(o) The n-sphere $S^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_{n+1}^2 = 1\} \subset \mathbb{R}^{n+1}$.

Solution. Smooth submanifold of $\mathbb{R}^{n+1}$ since it is the zero set of the function $f(x_1, \ldots, x_{n+1}) = x_1^2 + \cdots + x_{n+1}^2 - 1$.

(p) The general linear group $GL(n, \mathbb{R}) = \{A \in M_{n\times n}(\mathbb{R}) : \det A \neq 0\} \subset M_{n\times n} = \mathbb{R}^{n^2}$.

Solution. Note that det is a smooth function as it is a polynomial of the matrix entries. Then $\mathbb{R} \setminus \{0\}$ is open, and therefore $\mathrm{GL}(n, \mathbb{R})$ being the preimage of det under $\mathbb{R} \setminus \{0\}$ is also open. Therefore, $\mathrm{GL}(n, \mathbb{R})$ is a smooth submanifold of $M_{n\times n}(\mathbb{R})$.

(q) The special linear group $SL(2, \mathbb{R}) = \{A \in M_{2\times 2}(\mathbb{R}) : \det A = 1\} \subset M_{2\times 2} = \mathbb{R}^4$.

Solution. Note that $\det A = a_1 a_4 - a_2 a_3$ and so $\mathrm{SL}(2, \mathbb{R})$ is the zero set of the function $F(a_1, a_2, a_3, a_4) = a_1 a_4 - a_2 a_3 - 1$. Then $\mathrm{SL}(2, \mathbb{R})$ is a smooth submanifold of $\mathbb{R}^4$: we see that F is smooth and for any $A \in \mathrm{SL}(2, \mathbb{R})$, we have

$$DF(A) = (a_4, -a_3, -a_2, a_1).$$

Note that A is not the zero matrix since $\det A = 1$, so $DF(A)$ has rank 1 for all $A \in \mathrm{SL}(2, \mathbb{R})$.

7. Construct a smooth atlas for the following smooth submanifolds.

(a) $\mathbb{R}^n$.

Solution. The identity map on $\mathbb{R}^n$.

(b) An open subset U of $\mathbb{R}^n$.

Solution. The identity map on U.

(c) The general linear group $GL(n, \mathbb{R}) = \{A \in M_{n\times n}(\mathbb{R}) : \det A \neq 0\} \subset M_{n\times n} = \mathbb{R}^{n^2}$.

Solution. The identity map on $\mathrm{GL}(n, \mathbb{R})$ since it is open.

(d) The graph of a function $F : U \subset \mathbb{R}^k \to \mathbb{R}^{n-k}$ of class C^r for some $r \geq 1$.

Solution. The map $U \subset \mathbb{R}^k \to \mathbb{R}^n$ defined by $x \mapsto (x, F(x))$.

(e) The cubic $\{(x,y) \in \mathbb{R}^2 : x = y^3\} \subset \mathbb{R}^2$.

Solution. The map $\mathbb{R} \to \mathbb{R}^2$ defined by $x \mapsto (x, x^{1/3})$.

(f) The hyperbola $\{(x,y) \in \mathbb{R}^2 : xy = 1\} \subset \mathbb{R}^2$.

Solution. The map $\mathbb{R} \setminus \{0\} \to \mathbb{R}^2$ defined by $x \mapsto (x, 1/x)$.

(g) The unit circle $S^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \subset \mathbb{R}^2$.

Solution. The maps $\alpha_1 : (0, 2\pi) \subset \mathbb{R} \to S^1 \setminus \{(1,0)\}$ and $\alpha_2 : (-\pi, \pi) \subset \mathbb{R} \to S^1 \setminus \{(-1,0)\}$, both defined by $t \mapsto (\cos t, \sin t)$.

(h) The twisted cubic $\{(x,y,z) \in \mathbb{R}^3 : y = x^2, z = x^3\} \subset \mathbb{R}^3$.

Solution. The map $t \mapsto (t, t^2, t^3)$.

(i) The plane $\{(x, y, z) \in \mathbb{R}^3 : x + y + z = 1\} \subset \mathbb{R}^3$.

Solution. This is the graph of the function $f(x, y) = 1 - x - y$, so we can take the chart $(x, y) \mapsto (x, y, 1 - x - y)$.

(j) The paraboloid $\{(x, y, z) \in \mathbb{R}^3 : y = x^2 + z^2 + 1\} \subset \mathbb{R}^3$.

Solution. This is the graph of the function $f(x, z) = x^2 + z^2 + 1$, so we can take the chart $(x, z) \mapsto (x, x^2 + z^2 + 1, z)$.

(k) The open unit ball $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 1\} \subset \mathbb{R}^3$.

Solution. Note that $B_1((0, 0, 0)) \subset \mathbb{R}^3$ is open, so we can take the identity map.

8. One version of the *Invariance of Dimension Theorem* states that if $m > n$ and U is an open subset of $\mathbb{R}^m$, then there exists no continuous injective mapping from U to $\mathbb{R}^n$. Use this to prove the following:

(a) $\mathbb{R}^m$ is homeomorphic to $\mathbb{R}^n$ if and only if $m = n$.

Solution. We see that $m = n$ implies $\mathbb{R}^m = \mathbb{R}^n$, so there is nothing to prove. Conversely, suppose that $\mathbb{R}^m$ is homeomorphic to $\mathbb{R}^n$. Then there is a continuous injective mapping $f : \mathbb{R}^m \to \mathbb{R}^n$, so $m \leq n$ by the Invariance of Dimension Theorem. Similarly, we see that $f^{-1} : \mathbb{R}^n \to \mathbb{R}^m$ is a continuous injective mapping, so $n \leq m$ and hence $m = n$.

(b) If $M \subset \mathbb{R}^n$ is a k-dimensional submanifold of $\mathbb{R}^n$, then $k \leq n$.

Solution. Let $M \subset \mathbb{R}^n$ be a k-dimensional submanifold of $\mathbb{R}^n$. Then for all $p \in M$, there exists a neighbourhood V of p, an open set $U \subset \mathbb{R}^k$, and a homeomorphism $\alpha : U \subset \mathbb{R}^k \to V \subset M \subset \mathbb{R}^n$. In particular, we have that α is a continuous injective mapping from U to $\mathbb{R}^n$, so we must have $k \leq n$ by the Invariance of Dimension Theorem.

B. Velocity vectors, pushforwards, and tangents spaces.

1. Define the following:

   (a) The velocity vector of a parametrised curve $\gamma : (\alpha, \beta) \to \mathbb{R}^n$ of class C^r, $r \geq 1$.

   Solution. The velocity vector is $\gamma'(t)$.

   (b) The tangent space $T_{\vec{x}}(\mathbb{R}^n)$ to $\mathbb{R}^n$ at $\vec{x} \in \mathbb{R}^n$.

   Solution. The tangent space $T_{\vec{x}}(\mathbb{R}^n)$ to $\mathbb{R}^n$ at $\vec{x} \in \mathbb{R}^n$ is

   $$T_{\vec{x}}(\mathbb{R}^n) = \{(\vec{x}; \vec{v}) : \vec{v} \in \mathbb{R}^n\}.$$

   (c) The pushforward α_* of a map $\alpha : U \to \mathbb{R}^n$ of class C^r, $r \geq 1$, where U is an open subset of $\mathbb{R}^k$.

   Solution. The pushward α_* of the map $\alpha : U \to \mathbb{R}^n$ is defined by

   $$\begin{aligned} \alpha_* : T_{\vec{x}}\,\mathbb{R}^k &\to T_p\,\mathbb{R}^n \\ (\vec{x}; \vec{v}) &\mapsto (p; D\alpha(\vec{x})\vec{v}), \end{aligned}$$

   where $\vec{x} \in U$ and $p = \alpha(\vec{x})$.

   (d) The tangent space of T_pM to a k-submanifold of $\mathbb{R}^n$ of class C^r at $p \in M$.

   Solution. Let $\alpha : U \subset \mathbb{R}^k \to V \subset M \subset \mathbb{R}^n$ be a coordinate chart about p. The tangent space of M at p is defined as

   $$T_pM := \alpha_*(T_{x_0}\,\mathbb{R}^k) \subset T_p\,\mathbb{R}^n,$$

   where $x_0 \in U$ is the unique point such that $p = \alpha(x_0)$.

2. Prove that, for all $\vec{x} \in \mathbb{R}^n$,

   $$T_{\vec{x}}\,\mathbb{R}^n = \{(\vec{x}; \vec{v}) : \ \vec{v} \text{ is the velocity vector at } \vec{x} \text{ of a parametrised curve } \gamma \text{ passing through } \vec{x}\}.$$

   Solution. Let $\vec{x} \in \mathbb{R}^n$. For all $\vec{v} \in \mathbb{R}^n$, consider the parametrized curve $\gamma(t) = \vec{x} + t\vec{v}$, where $t \in \mathbb{R}$. Observe that $\gamma'(t) = \vec{v}$. In particular, at $t = 0$, we have $\gamma(0) = \vec{x}$ and $\gamma'(0) = \vec{v}$, so $\vec{v}$ is the velocity vector at $\vec{x}$.

3. Let M be a k-dimensional submanifold of $\mathbb{R}^n$ of class C^r, $r \geq 1$. Show that $\dim_{\mathbb{R}} T_pM = k$ for all $p \in M$.

   Solution. Let $p \in M$. Note that $T_pM = \alpha_*(T_{x_0}\,\mathbb{R}^k)$ where $x_0 \in U$ is the unique point such that $\alpha(x_0) = p$. We see that $D\alpha$ has rank k everywhere, so the image of the map

   $$\alpha_*(x_0; \vec{v}) = (p; D\alpha(x_0)\vec{v}),$$

   namely T_pM, must be of dimension k.

4. Compute the tangent space at the given point.

   (a) T_pU for any point p in the open set $U \subset \mathbb{R}^n$.

   Solution. The identity map $\alpha : U \to U$ has $D\alpha(x) = I_{n\times n}$ for all $x \in U$. By definition, we have $T_pU = \alpha_*(T_p\,\mathbb{R}^n)$ for all $p \in U$ since p is the unique point for which $\alpha(p) = p$. Then for all $(p; v) \in T_p\,\mathbb{R}^n$, we see that

   $$\alpha_*(p; \vec{v}) = (p; D\alpha(p)\vec{v}) = (p; \vec{v}),$$

   which implies that $T_pU = T_p\,\mathbb{R}^n$.

(b) The graph of a function $f : U \subset \mathbb{R}^k \to \mathbb{R}^{n-k}$ of class C^r, where U is an open subset of $\mathbb{R}^k$. Take any $p \in M$.

Solution. Consider the C^r map $\alpha : U \subset \mathbb{R}^k \to \mathbb{R}^{n-k}$ given by $\vec{x} \mapsto (\vec{x}, f(\vec{x}))$. We have that

$$D\alpha(\vec{x}) = \left[\begin{array}{c} I_{k\times k} \\ \hline Df(\vec{x}) \end{array}\right],$$

where $Df(\vec{x})$ is an $(n-k) \times k$ matrix for all $\vec{x} \in U$. Now, let $p \in M$ so that $p = \alpha(\vec{x}_0) = (\vec{x}_0, f(\vec{x}_0))$ for some $\vec{x}_0 \in U$. Then we find that

$$T_pM = \operatorname{span}_{\mathbb{R}} \left\{ \left[\begin{array}{c} \vec{e}_i \\ \hline \partial f(\vec{x}_0)/\partial x_i \end{array}\right] : i = 1, \ldots, k \right\},$$

where $\vec{e}_1, \ldots, \vec{e}_k$ is the standard basis for $\mathbb{R}^k$. In particular, these are the columns of $D\alpha(\vec{x}_0)$.

(c) The zero set M of a function $F : U \subset \mathbb{R}^n \to \mathbb{R}^{n-k}$ of class C^r, where U is an open set in $\mathbb{R}^n$, such that $DF(p)$ has rank $n-k$ for all $p \in M$. Take any $p \in M$.

Solution. Here, we use Problem 1 of Assignment 3. Let M be a k-dimensional submanifold of $\mathbb{R}^n$ of class C^r with $r \geq 1$ and let $p \in M$. Then $(p; \vec{v})$ is a tangent vector to M at p if and only if there exists a parametrized curve $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ of class C^r whose image lies in M and is such that $(p; \vec{v}) = (\gamma(0); \gamma'(0))$.
We now show that $T_pM = \{(p; \vec{v}) \in M \times \mathbb{R}^n : DF(p)\vec{v} = \vec{0}\} = \ker(DF(p))$. Let $(p; \vec{v}) \in T_pM$. Then there exists a curve $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ of class C^r on M such that $(p; \vec{v}) = (\gamma(0); \gamma'(0))$. We have $F(\gamma(t)) = \vec{0}$ for all $t \in (-\varepsilon, \varepsilon)$, so by the chain rule, we have

$$DF(p)\vec{v} = DF(\gamma(0))\gamma'(0) = \frac{\mathrm{d}}{\mathrm{d}t} F(\gamma(t))\Big|_{t=0} = 0.$$

Then $T_pM \subset \ker(DF(p))$. But T_pM has dimension k. Since $DF(p)$ has rank $n-k$ for all $p \in M$, we see that $\ker(DF(p))$ is a k-dimensional subspace of $\mathbb{R}^n$. Since T_pM is a subspace of $\ker(DF(p))$ with both of dimension k, this implies that they are equal, as desired.

(d) The sphere S^n in $\mathbb{R}^{n+1}$ at any point in S^n.

Solution. Note that S^n is the zero set of the function $F(x_1, \ldots, x_{n+1}) = x_1^2 + \cdots + x_{n+1}^2 - 1$ with $DF(x_1, \ldots, x_{n+1}) = (2x_1, \ldots, 2x_{n+1})$. Let $\vec{x} = (x_1, \ldots, x_{n+1}) \in S^1$. Then we have

$$\begin{aligned} T_{\vec{x}}S^n &= \ker(DF(\vec{x})) \\ &= \{\vec{v} \in \mathbb{R}^{n+1} : (2x_1, \ldots, 2x_{n+1}) \cdot \vec{v} = 0\} \\ &= \{\vec{v} \in \mathbb{R}^{n+1} : x_1v_1 + \cdots + x_{n+1}v_{n+1} = 0\}. \end{aligned}$$

(e) $T_I GL(n, \mathbb{R})$ where I is the identity $n \times n$ matrix.

Solution. We recall that $\mathrm{GL}(n, \mathbb{R})$ is open in $M_{n\times n}(\mathbb{R})$, so using (a), we have that

$$T_I\mathrm{GL}(n, \mathbb{R}) = T_I M_{n\times n}(\mathbb{R}) \simeq M_{n\times n}(\mathbb{R}).$$

(f) $T_I SL(2, \mathbb{R})$ where I is the identity 2×2 matrix.

Solution. We see that $\mathrm{SL}(2, \mathbb{R})$ is the zero set of the map $F(a_1, a_2, a_3, a_4) = a_1a_4 - a_2a_3 - 1$. Then $DF(a_1, a_2, a_3, a_4) = (a_4, -a_3, -a_2, a_1)$ with $DF(I) = (1, 0, 0, 1)$. Applying (c) yields

$$\begin{aligned} T_I\mathrm{SL}(2, \mathbb{R}) &= \ker(DF(I)) \\ &= \{B \in M_{2\times 2}(\mathbb{R}) : (1, 0, 0, 1) \cdot (b_1, b_2, b_3, b_4) = 0\} \\ &= \{B \in M_{2\times 2}(\mathbb{R}) : b_1 + b_4 = 0\} \\ &= \{B \in M_{2\times 2}(\mathbb{R}) : \operatorname{tr} B = 0\}. \end{aligned}$$

C. Curves.

1. Let $\gamma : (\alpha, \beta) \to \mathbb{R}$ be a parametrised curve of class C^r, $r \geq 3$. Define the following:

   (a) The velocity, speed and acceleration of γ at $t \in (\alpha, \beta)$.

   Solution. The velocity of γ at $t \in (\alpha, \beta)$ is $\gamma'(t)$. The speed is $\|\gamma'(t)\|$ and the acceleration is $\gamma''(t)$.

   (b) The arclength of γ starting that the point $\gamma(t_0)$ for any $t_0 \in (\alpha, \beta)$.

   Solution. The arclength of γ starting at the point $\gamma(t_0)$ is

$$s(t) := \int_{t_0}^{t} \|\gamma'(t)\| \, \mathrm{d}t.$$

   (c) The curve γ is regular.

   Solution. A curve γ is regular if $\gamma'(t) \neq \vec{0}$ for all $t \in (\alpha, \beta)$.

   (d) The curve γ is unit-speed.

   Solution. A curve γ is unit-speed if $\|\gamma'(t)\| = 1$ for all $t \in (\alpha, \beta)$.

   (e) The curvature $\kappa(t)$ of γ at $\gamma(t)$ for all $t \in (\alpha, \beta)$ (whether the parametrisation is unit-speed or just regular).

   Solution. When γ is unit-speed, we have $\kappa(t) = \|\gamma''(t)\|$. If γ is regular, we have

$$\kappa(t) = \frac{\|\gamma'(t) \times \gamma''(t)\|}{\|\gamma'(t)\|^3}.$$

   (f) The torsion $\tau(t)$ at points where $\kappa(t) > 0$ (whether the parametrisation is unit-speed or just regular).

   Solution. When γ is unit-speed, we define $\tau : (\alpha, \beta) \to \mathbb{R}$ to be the scalar function such that $\frac{\mathrm{d}}{\mathrm{d}t} B(t) = -\tau(t) N(t)$. In the general case where γ is regular, we have

$$\tau(t) = \frac{(\gamma'(t) \times \gamma''(t)) \cdot \gamma'''(t)}{\|\gamma'(t) \times \gamma''(t)\|^2}.$$

   (g) The Frenet frame of γ (whether the parametrisation is unit-speed or just regular).

   Solution. We define the Frenet frame of γ to be $\{T, N, B\}$, where T is the unit tangent vector, N is the principal unit normal, and B is the unit binormal. When γ is unit-speed, we have $T(t) = \gamma'(t)$, $N(t) = \gamma''(t)/\kappa(t) = \gamma''(t)/\|\gamma''(t)\|$, and $B(t) = T(t) \times N(t)$. When γ is regular, we have

$$T(t) = \frac{\gamma'(t)}{\|\gamma'(t)\|},$$
$$B(t) = \frac{\gamma'(t) \times \gamma''(t)}{\|\gamma'(t) \times \gamma''(t)\|},$$
$$N(t) = B(t) \times T(t).$$

(h) Frenet-Serret equations for unit-speed curves.

Solution. The Frenet-Serret equations are $\frac{\mathrm{d}T}{\mathrm{d}s} = \kappa T$, $\frac{\mathrm{d}N}{\mathrm{d}s} = -\kappa T + \tau B$, and $\frac{\mathrm{d}B}{\mathrm{d}s} = -\tau N$. More compactly, they can be written as the matrix equation

$$\begin{bmatrix} \frac{\mathrm{d}T}{\mathrm{d}s} \\ \frac{\mathrm{d}N}{\mathrm{d}s} \\ \frac{\mathrm{d}B}{\mathrm{d}s} \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}.$$

2. Let $u : (\alpha, \beta) \to \mathbb{R}^n, t \mapsto u(t)$, be a vector-valued function of class C^r, $r \geq 1$, such that $\|u(t)\| = c$ for all $t \in (\alpha, \beta)$, for some fixed $c \in \mathbb{R}^{>0}$. Show that $u(t) \cdot \dot{u}(t) = 0$ for all $t \in (\alpha, \beta)$. In particular, if γ is a unit-speed curve, then $\gamma''(t)$ is either zero or perpendicular to $\gamma'(t)$ for all $t \in (\alpha, \beta)$.

   Solution. Note that $u(t) \cdot u(t) = \|u(t)\|^2 = c^2$ for all $t \in (\alpha, \beta)$. Then

   $$0 = \frac{\mathrm{d}}{\mathrm{d}t}(u(t) \cdot u(t)) = u'(t) \cdot u(t) + u(t) \cdot u'(t) = 2u(t) \cdot u'(t),$$

   and therefore $u(t) \cdot u'(t) = 0$ for all $t \in (\alpha, \beta)$. For a unit-speed curve γ, we can set $u(t) = \gamma'(t)$ so that $\gamma'(t) \cdot \gamma''(t) = 0$ for all $t \in (\alpha, \beta)$. This implies that $\gamma''(t)$ is either zero or perpendicular to $\gamma'(t)$ for all $t \in (\alpha, \beta)$.

3. Prove that any regular curve $\gamma : (\alpha, \beta) \to \mathbb{R}$ of class C^r, $r \geq 1$, can be reparametrised in terms of arclength.

   Solution. Note that the arclength function s of γ is smooth with $\frac{\mathrm{d}s}{\mathrm{d}t} = \|\gamma'(t)\| > 0$ for all $t \in (\alpha, \beta)$ since γ is regular. Then s is strictly increasing on (α, β) and its inverse is smooth by the Inverse Function Theorem. Then $s = s(t)$ and $t = t(s)$ (where $t = s^{-1}$) are both smooth on their domains, and one can parametrize s in terms of arclength via

   $$\gamma(t) = \gamma(t(s)) = \tilde{\gamma}(s),$$

   where we define $\tilde{\gamma} = \gamma \circ t$.

4. Compute the curvature and torsion of the circular helix $\gamma(t) = (4\cos t, 4\sin t, 3t), t \in \mathbb{R}$.

   Solution. Note that $\gamma'(t) = (-4\sin t, 4\cos t, 3)$ with $\|\gamma'(t)\| = 5$, so γ is regular but not unit-speed. Next, we have $\gamma''(t) = (-4\cos t, -4\sin t, 0)$ so that

   $$\gamma'(t) \times \gamma''(t) = (-4\sin t, 4\cos t, 3) \times (-4\cos t, -4\sin t, 0) = (12\sin t, -12\cos t, 16).$$

   This gives us $\|\gamma'(t) \times \gamma''(t)\| = 20$, so the curvature of γ at $\gamma(t)$ is

   $$\kappa(t) = \frac{\|\gamma'(t) \times \gamma''(t)\|}{\|\gamma'(t)\|^3} = \frac{20}{5^3} = \frac{4}{25}.$$

   Next, we have $\gamma'''(t) = (4\sin t, -4\cos t, 0)$ so that

   $$(\gamma'(t) \times \gamma''(t)) \cdot \gamma'''(t) = (12\sin t, -12\cos t, 16) \cdot (4\sin t, -4\cos t, 0) = 48.$$

   It follows that the torsion of γ at $\gamma(t)$ is

   $$\tau(t) = \frac{(\gamma'(t) \times \gamma''(t)) \cdot \gamma'''(t)}{\|\gamma'(t) \times \gamma''(t)\|^2} = \frac{48}{20^2} = \frac{3}{25}.$$

5. Consider the following unit-speed curve

$$\gamma(s) = \left(\frac{1}{3}(1+s)^{3/2}, \frac{1}{3}(1-s)^{3/2}, \frac{s}{\sqrt{2}}\right), s \in (-1, 1).$$

Find the Frenet frame $\{T, N, B\}$ and the compute the curvature and the torsion at a general point on the curve.

Solution. First, the unit tangent vector to γ at $\gamma(s)$ is

$$T(s) = \gamma'(s) = \left(\frac{1}{2}(1+s)^{1/2}, -\frac{1}{2}(1-s)^{1/2}, \frac{1}{\sqrt{2}}\right).$$

Next, we see that

$$\gamma''(s) = \left(\frac{1}{4}(1+s)^{-1/2}, \frac{1}{4}(1-s)^{1/2}, 0\right),$$

so the curvature is

$$\kappa(s) = \|\gamma''(s)\| = \frac{1}{\sqrt{8(1-s^2)}} > 0.$$

Then the principal unit normal is

$$N(s) = \frac{\gamma''(s)}{\kappa(s)} = \frac{1}{\sqrt{2}}\left((1-s)^{1/2}, (1+s)^{1/2}, 0\right).$$

The unit binormal is then

$$B(s) = T(s) \times N(s) = \left(-\frac{1}{2}(1+s)^{1/2}, \frac{1}{2}(1-s)^{1/2}, \frac{1}{\sqrt{2}}\right).$$

Note that

$$\frac{\mathrm{d}}{\mathrm{d}s}B(s) = \left(-\frac{1}{4}(1+s)^{-1/2}, -\frac{1}{4}(1+s)^{-1/2}, 0\right) = -\gamma''(s) = -\kappa(s)N(s),$$

which implies that the torsion is

$$\tau(s) = \kappa(s) = \frac{1}{\sqrt{8(1-s^2)}}.$$

6. Let $\gamma(t)$, $t \in (\alpha, \beta)$, be a smooth regular curve in $\mathbb{R}^3$. Prove that γ parametrises a straight line if and only if $\kappa(t) = 0$ for all $t \in (\alpha, \beta)$.

Solution. Without loss of generality, we assume that γ is parametrized using arclength so that it is unit-speed. Suppose that $\gamma(t) = \vec{x} + t\vec{v}$ is a straight line in $\mathbb{R}^3$. Then we have $\gamma'(t) = \vec{v}$ and $\gamma''(t) = \vec{0}$ so that $\kappa(t) = \|\gamma''(t)\| = 0$ for all $t \in (\alpha, \beta)$. All of these steps are in fact reversible, so we are done.

7. Let $\gamma(t)$ be a smooth regular curve in $\mathbb{R}^3$. Prove that $\gamma'(t)$ and $\gamma''(t)$ are linearly dependent (that is, $\gamma''(t) = c(t)\gamma'(t)$ for some smooth function c) if and only if $\gamma(t)$ parametrises a straight line.

Solution. Suppose that $\gamma''(t) = c(t)\gamma'(t)$ for some smooth scalar function c. Then $\gamma'(t) \times \gamma''(t) = \vec{0}$ for all $t \in (\alpha, \beta)$ (since they are parallel), which implies that

$$\kappa(t) = \frac{\|\gamma'(t) \times \gamma''(t)\|}{\|\gamma'(t)\|^3} = \frac{0}{\|\gamma'(t)\|^3} = 0$$

for all $t \in (\alpha, \beta)$. Applying the previous problem, this means that γ parametrizes a straight line. These steps are reversible and thus the converse holds.

8. Let $\gamma(t)$, $t \in (\alpha, \beta)$, be a smooth regular curve in $\mathbb{R}^3$ such that $\kappa(t) > 0$ for all $t \in (\alpha, \beta)$. Prove that $\gamma(t)$ is a plane curve if and only if its torsion $\tau(t)$ is equal to 0 at every $t \in (\alpha, \beta)$. Moreover, if it is a plane curve, then it is contained in the plane passing through $\gamma(t_0)$ with normal vector $B(t_0)$ for any $t_0 \in (\alpha, \beta)$.

Solution. ($\Rightarrow$) Suppose that $\gamma(t)$ is a plane curve. Let $t_0 \in (\alpha, \beta)$. Note that γ is a plane curve if and only if every point $\gamma(t)$ is contained in some fixed plane Π in $\mathbb{R}^3$. Then $\gamma(t_0) \in \Pi$. If $\mathbf{n}_0$ is a unit normal to Π, then for any $t \in (\alpha, \beta)$, the vector going from $\gamma(t_0)$ to $\gamma(t)$ is parallel to Π and thus perpendicular to $\mathbf{n}_0$. This implies that $\mathbf{n}_0 \cdot (\gamma(t) - \gamma(t_0)) = 0$, or equivalently $\mathbf{n}_0 \cdot \gamma(t) = \mathbf{n}_0 \cdot \gamma(t_0)$ for all $t \in (\alpha, \beta)$. The right-hand side is a constant, so we have

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{n}_0 \cdot \gamma(t)) = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{n}_0) \cdot \gamma(t) + \mathbf{n}_0 \cdot \gamma'(t) = \mathbf{n}_0 \cdot T(t).$$

Similarly, we have $\mathbf{n}_0 \cdot \gamma''(t) = 0$ with $\gamma''(t) = \kappa(t) N(t)$ for all $t \in (\alpha, \beta)$, so

$$\mathbf{n}_0 \cdot N(t) = 0.$$

Then $\mathbf{n}_0 \perp T(t)$ and $\mathbf{n}_0 \perp N(t)$ implies that $\mathbf{n}_0 = \pm T(t) \times N(t) = \pm B(t)$. In particular, we have

$$\frac{\mathrm{d}B}{\mathrm{d}t}(t) = \vec{0}$$

for all $t \in (\alpha, \beta)$, which implies that $\tau(t) = 0$ for all $t \in (\alpha, \beta)$ since $\frac{\mathrm{d}B}{\mathrm{d}t}(t) = -\tau(t) N(t)$ by the Frenet-Serret equations with $N(t) \neq \vec{0}$ for all $t \in (\alpha, \beta)$.

($\Leftarrow$) Suppose that $\tau(t) = 0$ for all $t \in (\alpha, \beta)$. Then $\frac{\mathrm{d}B}{\mathrm{d}t}(t) = \vec{0}$ and hence $B(t) = \mathbf{n}_0$ for all $t \in (\alpha, \beta)$ for some fixed $\mathbf{n}_0 \in \mathbb{R}^3$. We have $T(t) \perp \mathbf{n}_0$ for all $t \in (\alpha, \beta)$, which implies that $T(t) \cdot \mathbf{n}_0 = 0$. But $T(t) = \gamma'(t)$, and the product rule gives

$$\frac{\mathrm{d}}{\mathrm{d}t}(\gamma(t) \cdot \mathbf{n}_0) = \gamma'(t) \cdot \mathbf{n}_0 + \gamma(t) \cdot \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{n}_0) = T(t) \cdot \mathbf{n}_0 = 0$$

since $\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{n}_0) = \vec{0}$ and $T(t) \cdot \mathbf{n}_0 = 0$. It follows that $\gamma(t) \cdot \mathbf{n}_0 = c$ for some constant $c \in \mathbb{R}$. Fixing $t_0 \in (\alpha, \beta)$, we observe that

$$(\gamma(t) - \gamma(t_0)) \cdot \mathbf{n}_0 = \gamma(t) \cdot \mathbf{n}_0 - \gamma(t_0) \cdot \mathbf{n}_0 = c - c = 0$$

for all $t \in (\alpha, \beta)$. This means that γ is contained in the plane passing through $\gamma(t_0)$ normal to $B(t_0) = \mathbf{n}_0$, and γ is a plane curve.

9. Show that $\gamma(t) = (t, 1 + t^{-1}, t^{-1} - t)$, $t > 0$, is a plane curve and find the equation of the plane in which it lies.

Solution. By the previous problem, it is enough to show that $\tau(t) = 0$ for all $t > 0$. We have $\gamma'(t) = (1, -t^{-2}, -t^{-2} - 1)$, $\gamma''(t) = 2t^{-3}(0, 1, 1)$, and $\gamma'''(t) = -6t^{-4}(0, 1, 1)$. This gives us $\gamma'(t) \times \gamma''(t) = 2t^{-3}(1, -1, 1)$ and $(\gamma'(t) \times \gamma''(t)) \cdot \gamma'''(t) = 0$, which implies that $\tau(t) = 0$ for all $t > 0$. Therefore, γ is a plane curve.

To determine the equation of the plane, we solve for $a, b, c, d \in \mathbb{R}$ such that

$$at + b(1 + t^{-1}) + c(t^{-1} - t) = d.$$

This gives us $(b - d) + (a - c)t + (b + c)t^{-1} = 0$, so we have $b = d$, $a = c$, and $b = -c$. Setting $c = 1$, we have that $a = 1$ and $b = d = -1$. Then γ is contained in the plane $x - y + z = -1$.

Alternatively, the unit binormal vector is

$$B(t) = \frac{\gamma'(t) \times \gamma''(t)}{\|\gamma'(t) \times \gamma''(t)\|} = \frac{1}{\sqrt{3}}(1, -1, 1)$$

for all $t \in (\alpha, \beta)$, which is a normal vector to the plane that γ lies in. Since $\gamma(1) = (1, 2, 0)$ is a point on the curve and therefore the plane, we have that

$$(x - 1, y - 2, z) \cdot (1, -1, 1) = 0.$$

This gives the same plane $x - y + z = -1$.

10. State the Fundamental Theorem of Space Curves.

Solution. A rigid motion of $\mathbb{R}^3$ is a rotation followed by a translation. That is, a rigid motion of $\mathbb{R}^3$ is an affine map $M : \mathbb{R}^3 \to \mathbb{R}^3$ of the form $M(v) = Av + b$ where A is an element of the special orthogonal group $\mathrm{SO}(3) = \{C \in M_{3\times3}(\mathbb{R}) : C^T C = I_{3\times3},\ \det C = 1\}$.

The Fundamental Theorem of Space Curves states that given two unit-speed curves γ_1 and γ_2 with the same curvature $\kappa(s)$ and the same torsion $\tau(s)$ for all $s \in (\alpha, \beta)$, there exists a rigid motion M of $\mathbb{R}^3$ such that $\gamma_2(s) = M(\gamma_1(s))$ for all $s \in (\alpha, \beta)$. Moreover, if k and t are functions of class C^3 with $k > 0$ everywhere, then there exists a unit-speed curve in $\mathbb{R}^3$ whose curvature is k and whose torsion is t.

11. Show that $\gamma(t) = (2 + \sqrt{2}\cos t, 1 - \sin t, 3 + \sin t)$, $t \in \mathbb{R}$, is a circle.

Solution. By the Fundamental Theorem of Space Curves, every circle has constant (positive) curvature and zero torsion. We verify that this is true for γ. Note that $\gamma'(t) = (-\sqrt{2}\sin t, -\cos t, \cos t)$, $\gamma''(t) = (-\sqrt{2}\cos t, \sin t, -\sin t)$, and $\gamma'''(t) = (\sqrt{2}\sin t, \cos t, -\cos t)$. We have that $\|\gamma'(t)\| = \sqrt{2}$ and $\gamma'(t) \times \gamma''(t) = (0, -\sqrt{2}, -\sqrt{2})$ so that $\|\gamma'(t) \times \gamma''(t)\| = 2$ and $(\gamma'(t) \times \gamma''(t)) \cdot \gamma'''(t) = 0$. This implies that the curvature is

$$\kappa(t) = \frac{\|\gamma'(t) \times \gamma''(t)\|}{\|\gamma'(t)\|^3} = \frac{2}{(\sqrt{2})^3} = \frac{1}{\sqrt{2}} > 0,$$

which is constant, and the torsion is

$$\tau(t) = \frac{(\gamma'(t) \times \gamma''(t)) \times \gamma'''(t)}{\|\gamma'(t) \times \gamma''(t)\|^2} = 0.$$

D. Surfaces.

1. Let S be a smooth surface in $\mathbb{R}^3$ and $\sigma : U \subset \mathbb{R}^2 \to V \subset S$ be a smooth coordinate chart. Define the following:

   (a) The standard unit normal, first and second fundamental forms of σ.

   Solution. Let $p_0 = \sigma(u_0, v_0)$ for some $(u_0, v_0) \in U$. The standard unit normal is

   $$N_\sigma(u_0, v_0) = \frac{\sigma_u(u_0, v_0) \times \sigma_v(u_0, v_0)}{\|\sigma_u(u_0, v_0) \times \sigma_v(u_0, v_0)\|}.$$

   The first fundamental form of σ is

   $$\mathcal{F}_{\mathrm{I}} = \begin{bmatrix} \sigma_u \cdot \sigma_u & \sigma_u \cdot \sigma_v \\ \sigma_v \cdot \sigma_u & \sigma_v \cdot \sigma_v \end{bmatrix},$$

   and the second fundamental form of σ is

   $$\mathcal{F}_{\mathrm{II}} = \begin{bmatrix} \sigma_{uu} \cdot N_\sigma & \sigma_{uv} \cdot N_\sigma \\ \sigma_{vu} \cdot N_\sigma & \sigma_{vv} \cdot N_\sigma \end{bmatrix}.$$

   (b) The principal, normal and geodesic curvatures of σ.

   Solution. The principal curvatures of σ are the eigenvalues κ_1 and κ_2 of the shape operator $\mathcal{W} := \mathcal{F}_{\mathrm{I}}^{-1}\mathcal{F}_{\mathrm{II}}$. For a unit-speed curve $\gamma : (\alpha, \beta) \to V \subset S$ contained in the coordinate patch σ and a point $p_0 = \gamma(t_0)$, the normal curvature is

   $$\kappa_n := \gamma''(t_0) \cdot N_\sigma,$$

   whereas the geodesic curvature is

   $$\kappa_g := \gamma''(t_0) \cdot (N_\sigma \times \gamma'(t_0)).$$

   (c) The Gaussian and mean curvatures of σ.

   Solution. For the shape operator $\mathcal{W} = \mathcal{F}_{\mathrm{I}}^{-1}\mathcal{F}_{\mathrm{II}}$, the Gaussian curvature of σ is $K := \det \mathcal{W}$ and the mean curvature of σ is $H := \frac{1}{2} \operatorname{tr} \mathcal{W}$.

   (d) The geodesics of σ.

   Solution. We call a unit-speed curve $\gamma : (\alpha, \beta) \to V \subset S$ that is contained in the coordinate patch σ a geodesic if $\kappa_g = 0$ everywhere. Equivalently, we have that $\gamma''(t) = \vec{0}$ or $\gamma''(t) \parallel N_\sigma$ for all $t \in (\alpha, \beta)$.

2. Let $\sigma : U \subset \mathbb{R}^2 \to V \subset S$ and $\tilde{\sigma} : \tilde{U} \subset \mathbb{R}^2 \to \tilde{V} \subset S$ be smooth coordinate charts of a surface S with $V \cap \tilde{V}$. Set $\Phi := \sigma^{-1} \circ \tilde{\sigma} : \tilde{\sigma}^{-1}(V \cap \tilde{V}) \to \sigma^{-1}(V \cap \tilde{V})$ so that $\tilde{\sigma} = \sigma \circ \Phi$.

   (a) Show that $N_{\tilde{\sigma}} = \pm N_\sigma$, where $\pm$ is the sign of $\det(D\Phi)$.

   Solution. Note that Φ is a smooth diffeomorphism with components

   $$(\tilde{u}, \tilde{v}) \mapsto (u(\tilde{u}, \tilde{v}), v(\tilde{u}, \tilde{v})).$$

   By the chain rule, we have $D\tilde{\sigma} = D\sigma D\Phi$, or more explicitly

   $$\begin{bmatrix} \tilde{\sigma}_{\tilde{u}} \mid \tilde{\sigma}_{\tilde{v}} \end{bmatrix} = \begin{bmatrix} \sigma_u \mid \sigma_v \end{bmatrix} D\Phi.$$

Here, $D\Phi$ is the change of basis matrix of T_pS. Note that

$$D\Phi = \begin{bmatrix} \partial u/\partial \tilde{u} & \partial u/\partial \tilde{v} \\ \partial v/\partial \tilde{u} & \partial v/\partial \tilde{v} \end{bmatrix}.$$

Moreover, we have

$$\begin{aligned} \tilde{\sigma}_{\tilde{u}} &= \sigma_u \cdot \frac{\partial u}{\partial \tilde{u}} + \sigma_v \cdot \frac{\partial v}{\partial \tilde{u}}, \\ \tilde{\sigma}_{\tilde{v}} &= \sigma_u \cdot \frac{\partial u}{\partial \tilde{v}} + \sigma_v \cdot \frac{\partial v}{\partial \tilde{v}}, \end{aligned}$$

which implies that

$$\begin{aligned} \tilde{\sigma}_{\tilde{u}} \times \tilde{\sigma}_{\tilde{v}} &= \left(\sigma_u \cdot \frac{\partial u}{\partial \tilde{u}} + \sigma_v \cdot \frac{\partial v}{\partial \tilde{u}} \right) \times \left(\sigma_u \cdot \frac{\partial u}{\partial \tilde{v}} + \sigma_v \cdot \frac{\partial v}{\partial \tilde{v}} \right) \\ &= (\sigma_u \times \sigma_u) \left(\frac{\partial u}{\partial \tilde{u}} \frac{\partial v}{\partial \tilde{v}} \right) + (\sigma_u \times \sigma_v) \left(\frac{\partial u}{\partial \tilde{u}} \frac{\partial v}{\partial \tilde{v}} - \frac{\partial v}{\partial \tilde{u}} \frac{\partial u}{\partial \tilde{v}} \right) + (\sigma_v \times \sigma_v) \left(\frac{\partial v}{\partial \tilde{u}} \frac{\partial v}{\partial \tilde{v}} \right) \\ &= \left(\frac{\partial u}{\partial \tilde{u}} \frac{\partial v}{\partial \tilde{v}} - \frac{\partial v}{\partial \tilde{u}} \frac{\partial u}{\partial \tilde{v}} \right) (\sigma_u \times \sigma_v) = (\det D\Phi)(\sigma_u \times \sigma_v), \end{aligned}$$

where we used the fact that $\sigma_u \times \sigma_u = \sigma_v \times \sigma_v = \vec{0}$. It follows that

$$N_{\tilde{\sigma}} = \frac{\tilde{\sigma}_{\tilde{u}} \times \tilde{\sigma}_{\tilde{v}}}{\|\tilde{\sigma}_{\tilde{u}} \times \tilde{\sigma}_{\tilde{v}}\|} = \frac{\det D\Phi}{|\det D\Phi|} \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = \pm N_\sigma,$$

where $\pm$ is the sign of $\det D\Phi$.

(b) Recall that if $\mathcal{F}_I, \mathcal{F}_{II}$ and $\widetilde{\mathcal{F}}_I, \widetilde{\mathcal{F}}_{II}$ are the first and second fundamental forms of σ and $\tilde{\sigma}$, respectively, then

$$\widetilde{\mathcal{F}}_I = (D\Phi)^T \mathcal{F}_I (D\Phi)$$

and

$$\widetilde{\mathcal{F}}_{II} = \pm (D\Phi)^T \mathcal{F}_{II} (D\Phi)$$

where $\pm$ is the sign of $\det(D\Phi)$.

i. Let $\mathcal{W}$ and $\tilde{\mathcal{W}}$ be the shape operators associated to σ and $\tilde{\sigma}$ respectively. Show that $\tilde{\mathcal{W}} = \pm (D\Phi)^{-1} \mathcal{W} (D\Phi)$, where $\pm$ is the sign of $\det(D\Phi)$.

Solution. Observe that

$$\begin{aligned} \tilde{\mathcal{W}} = \tilde{\mathcal{F}}_{\mathrm{I}}^{-1} \tilde{\mathcal{F}}_{\mathrm{II}} &= \pm [(D\Phi)^T \mathcal{F}_{\mathrm{I}} (D\Phi)]^{-1} [(D\Phi)^T \mathcal{F}_{\mathrm{II}} (D\Phi)] \\ &= \pm (D\Phi)^{-1} \mathcal{F}_{\mathrm{I}}^{-1} ((D\Phi)^T)^{-1} (D\Phi)^T \mathcal{F}_{\mathrm{II}} (D\Phi) \\ &= \pm (D\Phi)^{-1} \mathcal{F}_{\mathrm{I}}^{-1} \mathcal{F}_{\mathrm{II}} (D\Phi) \\ &= \pm (D\Phi)^{-1} \mathcal{W} (D\Phi), \end{aligned}$$

where $\pm$ is the sign of $\det D\Phi$.

ii. Show that $\tilde{K} = K$ and $\tilde{H} = \pm H$, i.e., Gaussian curvature is invariant under reparametrisations, whereas mean curvature may change sign under reparametrisations. Moreover, show that the principal curvatures change as follows under reparametrisations: $\widetilde{\kappa}_1 = \pm \kappa_1$ and $\widetilde{\kappa}_2 = \pm \kappa_2$.

Solution. Using the previous part and noting that we are working with 2×2 matrices, we have by the multiplicativity of determinant that

$$\tilde{K} = \det \tilde{\mathcal{W}} = (\pm 1)^2 \det((D\Phi)^{-1} \mathcal{W} (D\Phi)) = \det \mathcal{W} = K.$$

Similarly, we know that similar matrices have the same trace, so

$$\tilde{H} = \frac{1}{2}\operatorname{tr}\tilde{\mathcal{W}} = \frac{1}{2}\operatorname{tr}(\pm(D\Phi)^{-1}\mathcal{W}(D\Phi)) = \pm\frac{1}{2}\operatorname{tr}\mathcal{W} = H.$$

Next, we use the fact that similar matrices have the same eigenvalues. The principal curvatures are the eigenvalues of the shape operator, so κ_1 and κ_2 are the eigenvalues of $\mathcal{W}$ whereas $\tilde{\kappa}_1$ and $\tilde{\kappa}_2$ are the eigenvalues of $\tilde{\mathcal{W}}$. If $\tilde{\mathcal{W}} = (D\Phi)^{-1}\mathcal{W}(D\Phi)$, then $\tilde{\mathcal{W}}$ has the same eigenvalues as $\mathcal{W}$ so that (up to reordering) $\tilde{\kappa}_1 = \kappa_1$ and $\tilde{\kappa}_2 = \kappa_2$. If we had $\tilde{\mathcal{W}} = -(D\Phi)^{-1}\mathcal{W}(D\Phi)$, then this means that $\tilde{\mathcal{W}}$ and $-\mathcal{W}$ are similar and have the same eigenvalues. This implies that $\tilde{\kappa}_1 = -\kappa_1$ and $\tilde{\kappa}_2 = -\kappa_2$.

(c) A smooth surface is called *flat* if its Gaussian curvature is everywhere zero. Use (b) to show that the notion of flat surface is independent of the atlas chosen.

Solution. Consider the two coordinate charts σ and $\tilde{\sigma}$ with $V \cap \tilde{V} \neq \varnothing$. By part (b), we know that $K = \tilde{K}$. Then the Gaussian curvature is independent of the choice of coordinate chart, and therefore the notion of flatness is independent of the choice of atlas.

(d) A smooth surface is called *minimal* if its mean curvature is everywhere zero. Use (b) to show that the notion of minimal surface is independent of the atlas chosen.

Solution. By part (b), we have $H = \tilde{H}$. In particular, if $H = 0$ everywhere, then $\tilde{H} = 0$ everywhere as well. This means that the notion of minimal surface is independent of the choice of atlas.

(e) Show that the Gaussian curvature of a minimal surface is ≤ 0 everywhere.

Solution. By part (d), the notion of minimal surface is independent of the choice of atlas. Therefore, take any atlas and let σ be a coordinate chart in the atlas. Let κ_1 and κ_2 be the principal curvatures of σ. For a minimal surface, we have that $H = \frac{1}{2}(\kappa_1 + \kappa_2) = 0$, which implies that $\kappa_1 = -\kappa_2$. This implies that $K = \kappa_1\kappa_2 = -\kappa_1^2 \leq 0$.

(f) Show that if a unit-speed curve on a smooth surface has zero normal *and* geodesic curvatures everywhere, it is part of a straight line.

Solution. We know that $\kappa^2 = \kappa_n^2 + \kappa_g^2$ for a unit-speed curve, so if $\kappa_n = \kappa_g = 0$ everywhere, then $\kappa = 0$ everywhere. By Problem C.6, the curve parametrizes a straight line.

3. Let S be the graph of a smooth function $f : U \subset \mathbb{R}^2 \to \mathbb{R}$ with U an open subset of $\mathbb{R}^2$. Show that at a critical point $(x_0, y_0) \in U$ of f, the Gaussian curvature of S is equal to the determinant of the Hessian matrix $H(f)(x_0, y_0)$ of f at (x_0, y_0).

Solution. Note that a coordinate chart for S is $\sigma : U \subset \mathbb{R}^2 \to S \subset \mathbb{R}^3$ defined by

$$(u, v) \mapsto (u, v, f(u, v)).$$

We have that $\sigma_u = (1, 0, f_u)$ and $\sigma_v = (0, 1, f_v)$, so the first fundamental form of σ is

$$\mathcal{F}_{\mathrm{I}} = \begin{bmatrix} 1 + (f_u)^2 & f_u f_v \\ f_u f_v & 1 + (f_v)^2 \end{bmatrix}.$$

Next, we see that $\sigma_u \times \sigma_v = (-f_u, -f_v, 1)$ and $\|\sigma_u \times \sigma_v\| = \sqrt{1 + (f_u)^2 + (f_v)^2}$, so

$$N_\sigma = \frac{1}{\sqrt{1 + (f_u)^2 + (f_v)^2}}(-f_u, -f_v, 1).$$

Moreover, we have $\sigma_{uu} = (0, 0, f_{uu})$, $\sigma_{uv} = (0, 0, f_{uv})$ and $\sigma_{vv} = (0, 0, f_{vv})$ so that

$$\mathcal{F}_{\mathrm{II}} = \frac{1}{\sqrt{1 + (f_u)^2 + (f_v)^2}} \begin{bmatrix} f_{uu} & f_{uv} \\ f_{vu} & f_{vv} \end{bmatrix} = \frac{1}{\sqrt{1 + (f_u)^2 + (f_v)^2}} H(f).$$

At a critical point $(u_0, v_0) \in U$ of f, we have that

$$f_u(u_0, v_0) = f_v(u_0, v_0) = 0.$$

At such a point, we have $\mathcal{F}_{\mathrm{I}} = I_{2\times 2}$ and $\mathcal{F}_{\mathrm{II}} = H(f)$ so that the Gaussian curvature is

$$K = \det \mathcal{W} = \det H(f).$$

4. Let S be the smooth surface given by $z = ax^2 + by^2$, $a, b \in \mathbb{R}$. Show that:
   (a) Every point on S is elliptic if and only if $ab > 0$, in which case S is an elliptic paraboloid.
   (b) Every point on S is hyperbolic if and only if $ab < 0$, in which case S is a hyperbolic paraboloid (that is, a saddle surface).
   (c) Every point on S is parabolic if and only if $ab = 0$ and $a \neq 0$ or $b \neq 0$, in which case S is a parabolic cylinder.
   (d) Every point on S is planar if and only if $a = b = 0$, in which case S is the xy-plane ($z = 0$).

Solution. We first compute the shape operator. Note that $\sigma(u, v) = (u, v, au^2 + bv^2)$ is a smooth coordinate chart for S, so we have $\sigma_u = (1, 0, 2au)$ and $\sigma_v = (0, 1, 2bv)$. This implies that

$$\mathcal{F}_{\mathrm{I}} = \begin{bmatrix} 1 + 4a^2u^2 & 4abuv \\ 4abuv & 1 + 4b^2v^2 \end{bmatrix}.$$

Note that $\det \mathcal{F}_{\mathrm{I}} = (1 + 4a^2u^2)(1 + 4b^2v^2) - (4abuv)^2 = 1 + 4a^2u^2 + 4b^2v^2$, so we have

$$\mathcal{F}_{\mathrm{I}}^{-1} = \frac{1}{1 + 4a^2u^2 + 4b^2v^2} \begin{bmatrix} 1 + 4a^2u^2 & -4abuv \\ -4abuv & 1 + 4b^2v^2 \end{bmatrix}.$$

Next, we have $\sigma_u \times \sigma_v = (-2au, -2bv, 1)$, so $\|\sigma_u \times \sigma_v\| = \sqrt{1 + 4a^2u^2 + 4b^2v^2}$ and

$$N_\sigma = \frac{1}{\sqrt{1 + 4a^2u^2 + 4b^2v^2}}(-2au, -2bv, 1).$$

Then $\sigma_{uu} = (0, 0, 2a)$, $\sigma_{uv} = (0, 0, 0)$ and $\sigma_{vv} = (0, 0, 2b)$, giving us

$$\mathcal{F}_{\mathrm{II}} = \frac{2}{\sqrt{1 + 4a^2u^2 + 4b^2v^2}} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}.$$

Therefore, the shape operator is

$$\mathcal{W} = \mathcal{F}_{\mathrm{I}}^{-1}\mathcal{F}_{\mathrm{II}} = \frac{2}{(1 + 4a^2u^2 + 4b^2v^2)^{3/2}} \begin{bmatrix} a(1 + 4a^2u^2) & -4ab^2uv \\ -4a^2buv & b(1 + 4b^2v^2) \end{bmatrix}.$$

The Gaussian curvature is

$$\begin{aligned} K = \det \mathcal{W} &= \frac{4}{(1 + 4a^2u^2 + 4b^2v^2)^3}[ab(1 + 4a^2u^2)(1 + 4b^2v^2) - (-4ab^2uv)(-4a^2buv)] \\ &= \frac{4ab}{(1 + 4a^2u^2 + 4b^2v^2)^3}[(1 + 4a^2u^2)(1 + 4b^2v^2) - (4abuv)^2] \\ &= \frac{4ab}{(1 + 4a^2u^2 + 4b^2v^2)^2}, \end{aligned}$$

and the mean curvature is

$$H = \frac{1}{2}\operatorname{tr} \mathcal{W} = \frac{a(1 + 4a^2u^2) + b(1 + 4b^2v^2)}{(1 + 4a^2u^2 + 4b^2v^2)^{3/2}}.$$

(a) Observe that $K > 0$ if and only if $ab > 0$, with points on S being elliptic there.

(b) We have $K < 0$ if and only if $ab < 0$, in which we have hyperbolic points.

(c) Suppose every point on S is parabolic so that $K = 0$ but $H \neq 0$. Then $ab = 0$ since $K = 0$, and we must have $a \neq 0$ or $b \neq 0$ for otherwise $H = 0$. Conversely, if $ab = 0$ with $a \neq 0$ or $b \neq 0$, then $H \neq 0$ and $K = 0$.

(d) Note that $K = H = 0$ if and only if $a = b = 0$.

5. Classify all the points on the regular surface patch $\sigma(u, v) = (u, v, u^3 + v^2)$, $(u, v) \in \mathbb{R}^2$, as to being elliptic, hyperbolic, parabolic, or planar.

Solution. We will use Problem D.3 to simplify computations a little. Note that σ corresponds to the graph of the function $f(u, v) = u^3 + v^2$ with $f_u = 3u^2$ and $f_v = 2v$, as well as $f_{uu} = 6u$, $f_{uv} = 0$, and $f_{vv} = 2$. This tells us that

$$\mathcal{F}_{\mathrm{I}} = \begin{bmatrix} 1 + (f_u)^2 & f_u f_v \\ f_u f_v & 1 + (f_v)^2 \end{bmatrix} = \begin{bmatrix} 1 + 9u^4 & 6u^2 v \\ 6u^2 v & 1 + 4v^2 \end{bmatrix}$$

and we also saw in that problem that

$$\mathcal{F}_{\mathrm{II}} = \frac{1}{\sqrt{1 + (f_u)^2 + (f_v)^2}} \begin{bmatrix} f_{uu} & f_{uv} \\ f_{vu} & f_{vv} \end{bmatrix} = \frac{1}{\sqrt{1 + 9u^4 + 4v^2}} \begin{bmatrix} 6u & 0 \\ 0 & 2 \end{bmatrix}.$$

Now, observe that $\det \mathcal{F}_{\mathrm{I}} = (1 + 9u^4)(1 + 4v^2) - (6u^2 v)^2 = 1 + 9u^4 + 4v^2$, which implies that

$$\mathcal{F}_{\mathrm{I}}^{-1} = \frac{1}{1 + 9u^4 + 4v^2} \begin{bmatrix} 1 + 9u^4 & -6u^2 v \\ -6u^2 v & 1 + 4v^2 \end{bmatrix}.$$

Then the shape operator is

$$\mathcal{W} = \mathcal{F}_{\mathrm{I}}^{-1} \mathcal{F}_{\mathrm{II}} = \frac{1}{(1 + 9u^4 + 4v^2)^{3/2}} \begin{bmatrix} 6u(1 + 9u^4) & -12u^2 v \\ -36u^3 v & 2(1 + 4v^2) \end{bmatrix}.$$

The Gaussian curvature is

$$\begin{aligned} K = \det \mathcal{W} &= \frac{1}{(1 + 9u^4 + 4v^2)^3} [12u(1 + 9u^4)(1 + 4v^2) - (-12u^2 v)(-36u^3 v)] \\ &= \frac{12u}{(1 + 9u^4 + 4v^2)^3} [(1 + 9u^4)(1 + 4v^2) - 36u^4 v^2] \\ &= \frac{12u}{(1 + 9u^4 + 4v^2)^2}, \end{aligned}$$

and the mean curvature is

$$H = \frac{1}{2} \operatorname{tr} \mathcal{W} = \frac{3u(1 + 9u^4) + (1 + 4v^2)}{(1 + 9u^4 + 4v^2)^{3/2}}.$$

Let $(u, v) \in \mathbb{R}^2$. We see that $\sigma(u, v)$ is elliptic for points where $u > 0$ and hyperbolic for points where $u < 0$. When $u = 0$, note that $H > 0$ for all $v \in \mathbb{R}$, so $\sigma(u, v)$ is parabolic there.

6. Let S be the circular cylinder $x^2 + y^2 = 1$. Consider the atlas $\{(S_i, \sigma_i)\}_{i=1}^2$ of S, where each S_i is the image in $\mathbb{R}^3$ of the smooth coordinate charts

$$\sigma_i(u, v) = (\cos u, \sin u, v),$$

with $(u, v) \in (0, 2\pi) \times \mathbb{R}$ when $i = 1$, and $(u, v) \in (-\pi, \pi) \times \mathbb{R}$ when $i = 2$.

(a) Find the principal curvatures κ_1, κ_2 of S at every point.

Solution. We first compute the shape operator. Since σ_1 and σ_2 are described using the same function but over different domains, we will denote them simply by σ for now. We have $\sigma_u = (-\sin u, \cos u, 0)$ and $\sigma_v = (0, 0, 1)$ so that

$$\mathcal{F}_{\mathrm{I}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Next, we see that $N_\sigma = \sigma_u \times \sigma_v = (\cos u, \sin u, 0)$ with $\sigma_{uu} = (-\cos u, -\sin u, 0)$ and $\sigma_{uv} = \sigma_{vv} = (0, 0, 0)$, so

$$\mathcal{F}_{\mathrm{II}} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}.$$

It follows that the shape operator is

$$\mathcal{W} = \mathcal{F}_{\mathrm{I}}^{-1} \mathcal{F}_{\mathrm{II}} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}.$$

The principal curvatures are the eigenvalues of $\mathcal{W}$, namely $\kappa_1 = 0$ and $\kappa_2 = -1$.

(b) Show that the normal curvature κ_n of any unit-speed curve γ on S is such that

$$-1 \leq \kappa_n \leq 0.$$

Solution. We recall that the principal curvatures κ_1 and κ_2 are the maximum and minimum values of the normal curvature κ_n. This implies that $-1 \leq \kappa_n \leq 0$.

(c) Is the cylinder S a flat surface? a minimal surface? Justify your answer. (See problem 2 for a definition of *flat* and *minimal*.)

Solution. For convenience, we restate the definitions here. A smooth surface is flat if its Gaussian curvature is everywhere zero, and minimal if its mean curvature is everywhere zero. In this case, we have $K = \det \mathcal{W} = 0$ and $H = \frac{1}{2} \operatorname{tr} \mathcal{W} = -\frac{1}{2}$, so the cylinder S is a flat surface but not minimal.

7. Let $\gamma : (\alpha, \beta) \to \mathbb{R}^3$ be a smooth unit-speed curve whose curvature is everywhere non-zero, and is such that the map

$$\sigma(v, s) = \gamma(s) + v\gamma'(s), s \in (\alpha, \beta), v > 0,$$

is a homeomorphism.

(a) Show that σ is a smooth embedding.

Solution. Since γ is smooth, we know that σ is smooth. Note that

$$D\sigma(v, s) = (\gamma'(s), \gamma'(s) + v\gamma''(s)).$$

This is nonzero everywhere since γ is unit-speed. Moreover, we know that $\gamma'(s) \cdot \gamma''(s) = 0$. Since the curvature of γ is everywhere nonzero, this implies that $\gamma'(s) \neq \vec{0}$ for all $s \in (\alpha, \beta)$ and therefore $\gamma'(s) \perp \gamma''(s)$. In particular, we know that $\gamma'(s)$ and $\gamma'(s) + v\gamma''(s)$ are linearly independent since $v > 0$, so $D\sigma$ has maximal rank 2 everywhere. Finally, it is given that σ is a homeomorphism, so σ is an embedding.

(b) Show that the smooth surface determined by σ is flat. (See problem 2 for a definition of *flat*.)
Hint: Start by showing that $L = M = 0$ everywhere. Conclude that $K = 0$ everywhere.

Solution. Since $K = \det \mathcal{W} = (\det \mathcal{F}_{\mathrm{I}})^{-1} \det \mathcal{F}_{\mathrm{II}}$ by the multiplicativity of the determinant, it is enough to show that $\det \mathcal{F}_{\mathrm{II}} = 0$. We first compute

$$\mathcal{F}_{\mathrm{II}} = \begin{bmatrix} \sigma_{vv} \cdot N_\sigma & \sigma_{vs} \cdot N_\sigma \\ \sigma_{sv} \cdot N_\sigma & \sigma_{ss} \cdot N_\sigma \end{bmatrix} = \begin{bmatrix} L & M \\ M & N \end{bmatrix}.$$

We have $\sigma_v = \gamma'(s)$ and $\sigma_s = \gamma'(s) + v\gamma''(s)$. Note that $T(s) = \gamma'(s)$ and $N(s) = \gamma''(s)/\|\gamma''(s)\|$, and moreover, we know that

$$B(s) = T(s) \times N(s) = \frac{\gamma'(s) \times \gamma''(s)}{\|\gamma''(s)\|}$$

with $\|B(s)\| = 1/\|\gamma''(s)\|$ since $\|T(s)\| = \|N(s)\| = 1$. In particular, we have

$$\sigma_v \times \sigma_s = \gamma'(s) \times (\gamma'(s) + v\gamma''(s)) = v\gamma'(s) \times \gamma''(s) = v\frac{B(s)}{\|B(s)\|} = vB(s)$$

and therefore $N_\sigma = B(s)$ since $\|B(s)\| = 1$ and $\|\sigma_v \times \sigma_s\| = v$. Finally, we compute that $\sigma_{vv} = \vec{0}$ and $\sigma_{vs} = \gamma''(s) = \kappa(s)N(s)$. Observe that $L = \sigma_{vv} \cdot N_\sigma = 0$ and $M = \sigma_{vs} \cdot N_\sigma = \gamma''(s) \cdot B(s) = \kappa(s)N(s) \cdot B(s) = 0$, so it follows that $\det \mathcal{F}_{\mathrm{II}} = 0$.

8. Let $\gamma : (\alpha, \beta) \to \mathbb{R}^3$ be a smooth whose curvature is everywhere non-zero, and let $\{T, N, B\}$ be its Frenet frame. Suppose that the map

$$\sigma(v, s) = \gamma(s) + vB(s),$$

$s \in (\alpha, \beta), v \in (-\epsilon, \epsilon)$, is a homeomorphism for small enough $\epsilon > 0$. Let S be image of σ in $\mathbb{R}^3$.

(a) Show that S is a smooth surface.

Solution. Since σ is a smooth homeomorphism with S as its image, it is enough to show that $D\sigma$ has maximal rank everywhere. Note that

$$D\sigma(v, s) = \left(B(s), \gamma'(s) + v\frac{\mathrm{d}}{\mathrm{d}s}B(s) \right) = (B(s), T(s) - v\tau(s)N(s)),$$

where we used the Frenet-Serret equations. To show that $\sigma_v = B(s)$ and $\sigma_s = T(s) - v\tau(s)N(s)$ are linearly independent, we can show that the area of the parallelogram spanned by them is nonzero. Indeed, we have that

$$\begin{aligned} \sigma_v \times \sigma_s &= B(s) \times T(s) - v\tau(s)B(s) \times N(s) \\ &= N(s) + v\tau(s)N(s) \times B(s) \\ &= N(s) + v\tau(s)T(s). \end{aligned}$$

Since $N(s) \perp T(s)$, this implies that

$$\|\sigma_v \times \sigma_s\| = 1 + v^2\tau(s)^2 > 0.$$

Therefore, σ is a smooth embedding and S is a smooth surface.

(b) Show that γ is a geodesic of σ and that every point on γ is either hyperbolic or parabolic.

Solution. Note that $\gamma(s) = \sigma(0, s)$ with $\gamma''(s) = \kappa(s)N(s) = \kappa(s)N_\sigma(0, s)$, so γ is a geodesic of σ. Next, we have $\sigma_v = B(s)$ and $\sigma_s = T(s)$ on γ, which implies that $\sigma_v \cdot \sigma_v = \sigma_s \cdot \sigma_s = 1$ and $\sigma_v \cdot \sigma_s = 0$. Thus, the first fundamental form is

$$\mathcal{F}_{\mathrm{I}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Recall that $N_\sigma(0,s) = N(s)$. Moreover, we have $\sigma_{vv} = \vec{0}$. The Frenet-Serret equations imply that $\sigma_{vs} = \frac{\mathrm{d}}{\mathrm{d}s}B(s) = -\tau(s)N(s)$ and $\sigma_{ss} = \frac{\mathrm{d}}{\mathrm{d}s}T(s) = \kappa(s)N(s)$. Then

$$\mathcal{W} = \mathcal{F}_{\mathrm{II}} = \begin{bmatrix} 0 & -\tau(s) \\ -\tau(s) & \kappa(s) \end{bmatrix}$$

since $\mathcal{F}_{\mathrm{I}} = I_{2\times 2}$. Notice that $K = \det \mathcal{W} = -\tau(s)^2 \leq 0$ and $H = \frac{1}{2}\operatorname{tr}\mathcal{W} = \frac{1}{2}\kappa(s) > 0$ since the curvature is everywhere nonzero. In particular, we either have $H < 0$ so that the point on γ is hyperbolic, or $H = 0$ and $K \neq 0$ so that the point is parabolic.

9. (a) Let γ be a unit-speed curve on a smooth surface S. Show that γ is a geodesic if and only if $\kappa_g = 0$.

   Solution. By definition, γ is a geodesic if for all $s \in (\alpha, \beta)$, we have $\gamma''(s) = \vec{0}$ or $\gamma''(s) \parallel N_\sigma$. This implies that $\kappa_g = \gamma''(s) \cdot (N_\sigma \times \gamma'(s)) = 0$. In the case that $\gamma''(s) = \vec{0}$, this is clear. Otherwise, we have $\gamma''(s) \parallel N_\sigma$. Since $N_\sigma \perp N_\sigma \times \gamma'(s)$ by the definition of the cross product, we have $\gamma''(s) \perp (N_\sigma \times \gamma'(s))$ as well.

   (b) Prove or disprove (by giving a counterexample) the following statements:

   i. If the unit-speed curve γ is a plane curve, then $\kappa_g = \pm\kappa$.

   Solution. False. The great circles on a sphere are geodesics, but are also plane curves as they are obtained by intersecting the sphere with a plane passing through the origin. By (a), we have $\kappa_g = 0$ everywhere. However, we have seen that the curvature of a sphere of radius a is $\kappa = 1/a$, so $\kappa_g \neq \pm\kappa$.

   ii. If the unit-speed curve γ is a geodesic, then $\kappa_n = \pm\kappa$.

   Solution. True. We have that $\kappa^2 = \kappa_n^2 + \kappa_g^2$. Since γ is a geodesic, we have by (a) that $\kappa_g = 0$, so $\kappa^2 = \kappa_n^2$ and $\kappa_n = \pm\kappa$.

   iii. Let p and q be two points on a smooth surface S, and let γ be a geodesic joining p and q. Then, γ is the path of shortest length from p to q.

   Solution. False. Pick two points p and q on a great circle of a sphere which are not antipodal. Then there are two arcs from p to q which are both geodesics, but one has greater length than the other.

   (c) Give an example of an isometry which is not a translation, rotation, or reflection.

   Solution. Let $S_1 = \{(x,y,z) \in \mathbb{R}^3 : z = 0,\ 0 < x < 2\pi\}$ be a subset of the xy-plane and $S_2 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = 1,\ x \neq 1\}$ be the cylinder of radius 1 minus the line $x = 1$. Consider the diffeomorphism

$$\begin{aligned} \Phi : S_1 &\to S_2 \\ (x,y,0) &\mapsto (\cos x, \sin x, y). \end{aligned}$$

   Define $\sigma : \mathbb{R}^2 \to S_1$ by $(u,v) \mapsto (u,v,0)$ and let

$$\begin{aligned} \tilde{\sigma} = \Phi \circ \sigma : \mathbb{R}^2 &\to S_2 \\ (u,v) &\mapsto (\cos u, \sin u, v). \end{aligned}$$

   Then $\tilde{\sigma}$ is the usual parametrization of the cylinder. We have seen in class that $\mathcal{F}_{\mathrm{I}} = \tilde{\mathcal{F}}_{\mathrm{I}} = I_{2\times 2}$, so Φ is an isometry.

   (d) Describe all geodesics on a plane, a sphere and a cylinder.

   Solution. The geodesics on a plane with coordinate $\sigma(u,v) = p_0 + uw_1 + vw_2$ are the lines

$$\gamma(t) = \sigma(at + b, ct + d).$$

The geodesics on a sphere are the great circles. To find the geodesics of a cylinder, one can use the isometry we gave in part (c), which gives a one-to-one correspondence between geodesics on the plane and geodesics on the cylinder. Here, the geodesics on the cylinder are the images of the lines in the plane under the isometry Φ, so we have

$$\gamma(t) = \Phi(at + b, ct + d, 0) = (\cos(at + b), \sin(at + b), ct + d).$$

This is a line if $a = 0$, a circle of $c = 0$, and a helix if a and c are both nonzero.

E. Integration.

1. Let M be a smooth k-dimensional submanifold of $\mathbb{R}^n$ and $\alpha : U \subset \mathbb{R}^k \to V \subset M$ be a smooth coordinate chart of M. Assume that V is bounded. Define the following:

   (a) The first fundamental form of α.

   Solution. The first fundamental form of α is defined to be

   $$\mathcal{F}_{\mathrm{I}} := D\alpha^T D\alpha.$$

   (b) The scalar integral of a smooth function f on V.

   Solution. The scalar integral of a smooth function f on V is

   $$\int_V f \,\mathrm{dVol} := \int_U f(\alpha(u))\sqrt{\det \mathcal{F}_{\mathrm{I}}}\,\mathrm{d}u_1 \cdots \mathrm{d}u_k.$$

   (c) The line integral of a smooth vector field on a smooth curve.

   Solution. Let $\gamma : [a, b] \to \mathbb{R}^n$ be a smooth regular curve contained in $\mathbb{R}^n$, and $F : U \to \mathbb{R}^n$ be a smooth vector field defined on an open set $U \subset \mathbb{R}^n$ containing γ. The line integral of F on γ is defined to be

   $$\int_\gamma F \cdot \mathrm{d}r := \int_a^b F(\gamma(t)) \cdot \gamma'(t)\,\mathrm{d}t.$$

   (d) A hypersurface in $\mathbb{R}^n$.

   Solution. A hypersurface is a smooth $(n-1)$-dimensional submanifold of $\mathbb{R}^n$, where $n > 3$.

   (e) An orientation of a hypersurface in $\mathbb{R}^n$.

   Solution. A hypersurface M in $\mathbb{R}^n$ is orientable if it admits a smooth unit normal vector N. There are two possible choices for the smooth unit normal vector N; such a choice of N is called an orientation of M.

   (f) The integral of a smooth vector field on a hypersurface in $\mathbb{R}^n$.

   Solution. Let $F : V \to \mathbb{R}^n$ be a smooth vector field, and let B be a bounded subset of V. Let N be a unit normal vector to M (i.e. an orientation). Then we define

   $$\int_B F \cdot \mathrm{dVol} = \int_B (F \cdot N)\,\mathrm{dVol},$$

   where the latter integral is the scalar integral of the smooth function $F \cdot N$ as defined in (b).

2. Let $\gamma : (\alpha, \beta) \to \mathbb{R}^n$ be a smooth regular curve. Verify that the formula for the integral of a scalar function on γ is equivalent to

   $$\int_\gamma f \; ds = \int_\alpha^\beta f(\gamma(t))\|\gamma'(t)\|dt$$

   for any smooth scalar function $f : (\alpha, \beta) \to \mathbb{R}$.

   Solution. Note that $D\gamma(t) = \gamma'(t)$, so $\mathcal{F}_{\mathrm{I}}$ is a 1×1 matrix with

   $$\mathcal{F}_{\mathrm{I}} = D\gamma(t)^T D\gamma(t) = \gamma'(t)^T\gamma'(t) = \gamma'(t) \cdot \gamma'(t) = \|\gamma'(t)\|^2.$$

   It follows that $\det \mathcal{F}_{\mathrm{I}} = \mathcal{F}_{\mathrm{I}} = \|\gamma'(t)\|^2$, and therefore

   $$\int_\gamma f \,\mathrm{d}s = \int_\alpha^\beta f(\gamma(t))\sqrt{\det \mathcal{F}_{\mathrm{I}}}\,\mathrm{d}t = \int_\alpha^\beta f(\gamma(t))\|\gamma'(t)\|\,\mathrm{d}t.$$

3. Let S be a surface in $\mathbb{R}^3$ and $\sigma : U \subset \mathbb{R}^2 \to V \subset S$ be a smooth coordinate chart with V bounded. Show that the formulas for the integrals of a scalar function or a vector field on V can be expressed in terms of σ as:

   (a) $\int_V f \; dVol = \int_U f(\sigma(u,v))\|\sigma_u \times \sigma_v\| dudv$ for any smooth scalar function $f : V \to \mathbb{R}$.

   Solution. From our study of surfaces, we have $\det \mathcal{F}_{\mathrm{I}} = \|\sigma_u \times \sigma_v\|^2$, so $\sqrt{\det \mathcal{F}_{\mathrm{I}}} = \|\sigma_u \times \sigma_v\|$ and the formula follows by definition.

   (b) $\int_V F \cdot dVol = \int_U F(\sigma(u,v)) \cdot (\sigma_u \times \sigma_v) dudv$ with respect to the orientation N_σ on V.

   Solution. Using the definition and part (a), we have that

$$\begin{aligned}\int_V F \cdot \mathrm{dVol} &= \int_V (F \cdot N_\sigma)\, \mathrm{d}A \\ &= \int_U F(\sigma(u,v)) \cdot N_\sigma \|\sigma_u \times \sigma_v\|\, \mathrm{d}u\, \mathrm{d}v \\ &= \int_U F(\sigma(u,v)) \cdot \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} \|\sigma_u \times \sigma_v\|\, \mathrm{d}u\, \mathrm{d}v \\ &= \int_U F(\sigma(u,v)) \cdot (\sigma_u \times \sigma_v)\, \mathrm{d}u\, \mathrm{d}v.\end{aligned}$$

4. Prove that any smooth hypersurface M in $\mathbb{R}^n$ is locally orientable. That is, for any point $p \in M$, there exists an open neighbourhood V of p in M such that V is an orientable hypersurface.

   Solution. The zero set of a smooth scalar function $G : V \subset \mathbb{R}^n \to \mathbb{R}$ with ∇G of maximal rank everywhere on an open set $V \subset \mathbb{R}^n$ is orientable with possible orientations $\pm \nabla G / \|\nabla G\|$. Since any smooth hypersurface M in $\mathbb{R}^n$ is locally the zero set of such a function, it follows that M is locally orientable.

5. Prove that if M be a smooth bounded n-dimensional submanifold of $\mathbb{R}^n$, then $\int_M f dVol$ is just the standard multiple integral $\int_M f(x_1, \ldots, x_n) dx_1 dx_2 \cdots dx_n$ for any smooth scalar function f on M.

   Solution. We know that the definition of the scalar integral is independent of the choice of coordinate chart. Let M be a smooth bounded n-dimensional submanifold of $\mathbb{R}^n$. Then M is an open set in $\mathbb{R}^n$, and we can simply take $\alpha = \mathrm{Id}_M$ as the coordinate chart which has $D\alpha = I_{n \times n}$ and therefore $\mathcal{F}_{\mathrm{I}} = D\alpha^T D\alpha = I_{n\times n}$. Then $\sqrt{\det \mathcal{F}_{\mathrm{I}}} = 1$, so

$$\int_M f\, \mathrm{dVol} = \int_M f(\alpha(x_1, \ldots, x_n)) \sqrt{\det \mathcal{F}_{\mathrm{I}}}\, \mathrm{d}x_1 \cdots \mathrm{d}x_n = \int_M f(x_1, \ldots, x_n)\, \mathrm{d}x_1 \cdots \mathrm{d}x_n.$$

6. (a) Evaluate the line integral $\int_\gamma x^3 y ds$ along part of the line $\gamma(t) = (t, -t)$, $t \in [-1, 2]$, in $\mathbb{R}^2$.

   Solution. Let $f(x,y) = x^3 y$. We have $f(\gamma(t)) = -t^4$ and $\gamma'(t) = (1, -1)$ so that $\|\gamma'(t)\| = \sqrt{2}$. Therefore, using the result from Problem E.2, we obtain

$$\begin{aligned}\int_\gamma x^3 y\, \mathrm{d}s &= \int_{-1}^{2} f(\gamma(t))\|\gamma'(t)\|\, \mathrm{d}t \\ &= -\sqrt{2} \int_{-1}^{2} t^4\, \mathrm{d}t \\ &= -\sqrt{2} \left[\frac{t^5}{5}\right]_{-1}^{2} = -\sqrt{2}\left(\frac{32}{5} - \frac{1}{5}\right) = -\frac{31\sqrt{2}}{5}.\end{aligned}$$

(b) Evaluate the line integral $\int_\gamma F \cdot dr$ of the vector field $F(x,y,z) = (yz, xz, xy)$ along part of the twisted cubic $\gamma(t) = (t, t^2, t^3)$, $t \in [0,1]$, in $\mathbb{R}^3$.

Solution. We have $F(\gamma(t)) = (t^5, t^4, t^3)$ and $\gamma'(t) = (1, 2t, 3t^2)$ so that

$$F(\gamma(t)) \cdot \gamma'(t) = (t^5, t^4, t^3) \cdot (1, 2t, 3t^2) = 6t^5.$$

From this, we obtain

$$\int_\gamma F \cdot \mathrm{d}r = \int_0^1 F(\gamma(t)) \cdot \gamma'(t)\,\mathrm{d}t = \int_0^1 6t^5\,\mathrm{d}t = 1^6 - 0^6 = 1.$$

(c) Evaluate the surface integral $\int_S 6xz dA$ where

$$S = \{(x,y,z) \in \mathbb{R}^3 : -1 \le x \le 3, 0 \le y \le 4, z = 1 - x - y\}$$

is the portion of the plane $x+y+z = 1$ lying above the rectangle $[-1,3] \times [0,4]$ in the xy-plane.

Solution. We can parametrize the surface S with the coordinate chart

$$\sigma(u,v) = (u, v, 1 - u - v),$$

where $(u,v) \in U = [-1,3] \times [0,4]$. Note that $\sigma_u = (1,0,-1)$ and $\sigma_v = (0,1,-1)$, so $\sigma_u \times \sigma_v = (1,1,1)$ and $\|\sigma_u \times \sigma_v\| = \sqrt{3}$. Next, for the smooth scalar function $f(x,y,z) = 6xz$, we have

$$f(\sigma(u,v)) = 6u(1 - u - v).$$

Applying part (a) of Problem E.3 gives

$$\begin{aligned}
\int_S 6xz\,\mathrm{d}A &= \int_0^4 \int_{-1}^3 f(\sigma(u,v))\|\sigma_u \times \sigma_v\|\,\mathrm{d}u\,\mathrm{d}v \\
&= 6\sqrt{3} \int_0^4 \int_{-1}^3 (u - u^2 - uv)\,\mathrm{d}u\,\mathrm{d}v \\
&= 6\sqrt{3} \int_0^4 \left[\frac{u^2}{2} - \frac{u^3}{3} - \frac{u^2 v}{2}\right]_{-1}^3 \mathrm{d}v \\
&= 6\sqrt{3} \int_0^4 \left[\left(\frac{9}{2} - 9 - \frac{9v}{2}\right) - \left(\frac{1}{2} + \frac{1}{3} - \frac{v}{2}\right)\right] \mathrm{d}v \\
&= 6\sqrt{3} \int_0^4 \left(-\frac{16}{3} - 4v\right) \mathrm{d}v \\
&= 6\sqrt{3} \left[-\frac{16}{3}v - 2v^2\right]_0^4 = -320\sqrt{3}.
\end{aligned}$$

(d) Evaluate the surface integral $\int_S y dA$ where S is the portion of the cylinder $x^2 + y^2 = 3$ that lies between $z = 0$ and $z = 6$.

Solution. Here, we can parametrize S using the chart

$$\sigma(u,v) = (\sqrt{3}\cos u, \sqrt{3}\sin u, v)$$

where $(u,v) \in U = [0, 2\pi] \times [0,6]$. We have $\sigma_u = (-\sqrt{3}\sin u, \sqrt{3}\cos u, 0)$ and $\sigma_v = (0,0,1)$ so that $\sigma_u \times \sigma_v = (\sqrt{3}\cos u, \sqrt{3}\sin u, 0)$ with $\|\sigma_u \times \sigma_v\| = \sqrt{3}$. Moreover, we have the scalar function $f(x,y,z) = y$ and $f(\sigma(u,v)) = \sqrt{3}\sin u$. Putting everything together,we have

$$\begin{aligned}
\int_S y\,\mathrm{d}A &= \int_0^6 \int_0^{2\pi} f(\sigma(u,v))\|\sigma_u \times \sigma_v\|\,\mathrm{d}u\,\mathrm{d}v = 3\int_0^6 \int_0^{2\pi} \sin u\,\mathrm{d}u\,\mathrm{d}v \\
&= 3\int_0^6 (-\cos 2\pi + \cos 0)\,\mathrm{d}v = 3\int_0^6 0\,\mathrm{d}v = 0.
\end{aligned}$$

(e) Evaluate the integral $\int_R z dV$ where R is the rectangular box $[-1,4]\times[0,3]\times[-2,5]$ in $\mathbb{R}^3$.

Solution. Since R is a 3-dimensional smooth submanifold of $\mathbb{R}^3$, we can perform a standard triple integral by Problem E.5. Then we have

$$\begin{aligned}\int_R z\,\mathrm{d}V &= \int_{-2}^{5}\int_0^3\int_{-1}^{4} z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = 5\int_{-2}^{5}\int_0^3 z\,\mathrm{d}y\,\mathrm{d}z\\ &= 15\int_{-2}^{5} z\,\mathrm{d}z = 15\left[\frac{z^2}{2}\right]_{-2}^{5} = 15\left(\frac{25}{2}-2\right) = \frac{315}{2}.\end{aligned}$$

(f) Evaluate the integral of the vector field $F(x,y,z) = (-y,x,z)$ on the piece of paraboloid S in $\mathbb{R}^3$ parametrised by $\sigma(u,v) = (u,v,u^2+v^2)$, $(u,v)\in[0,1]\times[0,3]$, with respect to the orientation N_σ.

Solution. First, we have that $F(\sigma(u,v)) = (-v,u,u^2+v^2)$. Next, we have $\sigma_u = (1,0,2u)$ and $\sigma_v = (0,1,2v)$, so $\sigma_u\times\sigma_v = (-2u,-2v,1)$. Then

$$F(\sigma(u,v))\cdot(\sigma_u\times\sigma_v) = (-v,u,u^2+v^2)\cdot(-2u,-2v,1) = u^2+v^2.$$

Note that S is bounded, and applying part (b) of Problem E.3 gives

$$\begin{aligned}\int_S F\cdot\mathrm{dVol} &= \int_0^3\int_0^1 F(\sigma(u,v))\cdot(\sigma_u\times\sigma_v)\,\mathrm{d}u\,\mathrm{d}v\\ &= \int_0^3\int_0^1 (u^2+v^2)\,\mathrm{d}u\,\mathrm{d}v\\ &= \int_0^3\left[\frac{u^3}{3}+uv^2\right]_0^1\mathrm{d}v\\ &= \int_0^3\left(\frac{1}{3}+v^2\right)\mathrm{d}v\\ &= \left[\frac{v}{3}+\frac{v^3}{3}\right]_0^3 = 1+9 = 10.\end{aligned}$$

(g) Evaluate the integral of the vector field $F(x,y,z) = (xy,2,z-1)$ on the half-cylinder C in $\mathbb{R}^3$ parametrised by $\sigma(u,v) = (\cos u,\sin u,v)$, $(u,v)\in[0,\pi]\times[2,4]$, with respect to the orientation N_σ.

Solution. We have $F(x,y,z) = (\cos u\sin u, 2, v-1)$. Moreover, we see that $\sigma_u = (-\sin u,\cos u,0)$ and $\sigma_v = (0,0,1)$, so $\sigma_u\times\sigma_v = (\cos u,\sin u,0)$ and

$$F(\sigma(u,v))\cdot(\sigma_u\times\sigma_v) = (\cos u\sin u,2,v-1)\cdot(\cos u,\sin u,0) = \cos^2 u\sin u + 2\sin u.$$

Moreover, C is bounded. Therefore, we can apply part (b) of Problem E.3 to obtain

$$\begin{aligned}\int_C F\cdot\mathrm{dVol} &= \int_2^4\int_0^\pi(\cos^2 u\sin u+2\sin u)\,\mathrm{d}u\,\mathrm{d}v\\ &= \int_2^4\left[-\frac{1}{3}\cos^3 u - 2\cos u\right]_0^\pi\mathrm{d}v\\ &= \int_2^4\left[\left(\frac{1}{3}+2\right)-\left(-\frac{1}{3}-2\right)\right]\mathrm{d}v\\ &= \int_2^4\frac{14}{3}\,\mathrm{d}v = \frac{28}{3}.\end{aligned}$$