PMATH 365 Course Notes

Differential Geometry

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1 Submanifolds of $\mathbb{R}^n$

1.1 Preliminaries

To begin, we'll recall some facts about the topology of $\mathbb{R}^n$ and vector-valued functions.

In this course, we'll be working with the metric topology with respect to the Euclidean norm (or metric). Let $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$. The **Euclidean norm** is defined to be

$$\|\mathbf{x}\| = \sqrt{x_1^2 + \cdots + x_n^2},$$

and **Euclidean distance** is given by

$$\operatorname{dist}(\mathbf{x}, \mathbf{y}) = \|\mathbf{y} - \mathbf{x}\| = \sqrt{(y_1 - x_1)^2 + \cdots + (y_n - x_n)^2}.$$

We define the **open ball** of radius $r > 0$ centered at $\mathbf{x} \in \mathbb{R}^n$ by

$$B_r(\mathbf{x}) := \{\mathbf{y} \in \mathbb{R}^n : \operatorname{dist}(\mathbf{x}, \mathbf{y}) < r\} \subset \mathbb{R}^n.$$

A **topology** on $\mathbb{R}^n$ is a collection $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ of subsets $U_\alpha \subset \mathbb{R}^n$ that satisfy the following properties.

(i) $\varnothing$ and $\mathbb{R}^n$ are in $\mathcal{U}$.

(ii) For any subcollection $\mathcal{V} = \{U_\beta\}_{\beta \in B}$ with $U_\beta \in \mathcal{U}$ for all $\beta \in B$, we have $\bigcup_{\beta \in B} U_\beta \in \mathcal{U}$.

(iii) For any *finite* subcollection $\{U_{\alpha_1}, \ldots, U_{\alpha_m}\} \subset \mathcal{U}$, we have $\bigcap_{i=1}^m U_{\alpha_i} \in \mathcal{U}$.

The sets $U_\alpha \in \mathcal{U}$ are called the **open sets** of the topology; their complements $F_\alpha = \mathbb{R}^n \setminus U_\alpha$ are called the **closed sets**. Note that by properties (i) and (iii), the sets $\varnothing$ and $\mathbb{R}^n$ are both open and closed.

Under the metric topology, we say that a set $A \subset \mathbb{R}^n$ is **open** if $A = \varnothing$ or if for all $p \in A$, there exists $r > 0$ such that $B_r(p) \subset A$. Moreover, A is **closed** if its complement $A^c = \mathbb{R}^n \setminus A$ is open. (We leave it as an exercise to show that this is indeed a topology.)

For example, the open balls $B_r(\mathbf{x})$ are open sets for all $\mathbf{x} \in \mathbb{R}^n$ and $r > 0$. Indeed, for any point $\mathbf{p} \in B_r(\mathbf{x})$, one sees that by picking $r' = (r - \|\mathbf{p} - \mathbf{x}\|)/2$, we have $B_{r'}(\mathbf{p}) \subset B_r(\mathbf{x})$.