# PMATH 365 Course Notes

Differential Geometry

Ruxandra Moraru • Winter 2023 • University of Waterloo

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# 1 Submanifolds of $\mathbb{R}^n$

## 1.1 Preliminaries

To begin, we'll recall some facts about the topology of $\mathbb{R}^n$ and vector-valued functions.

In this course, we'll be working with the metric topology with respect to the Euclidean norm (or metric). Let $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. The **Euclidean norm** is defined to be

$$\|x\| = \sqrt{x_1^2 + \cdots + x_n^2},$$

and **Euclidean distance** is given by

$$\operatorname{dist}(x, y) = \|y - x\| = \sqrt{(y_1 - x_1)^2 + \cdots + (y_n - x_n)^2}.$$

We define the **open ball** of radius $r > 0$ centered at $x \in \mathbb{R}^n$ by

$$B_r(x) := \{y \in \mathbb{R}^n : \operatorname{dist}(x, y) < r\} \subset \mathbb{R}^n.$$

A **topology** on $\mathbb{R}^n$ is a collection $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ of subsets $U_\alpha \subset \mathbb{R}^n$ that satisfy the following properties.

(i) $\varnothing$ and $\mathbb{R}^n$ are in $\mathcal{U}$.

(ii) For any subcollection $\mathcal{V} = \{U_\beta\}_{\beta \in B}$ with $U_\beta \in \mathcal{U}$ for all $\beta \in B$, we have $\bigcup_{\beta \in B} U_\beta \in \mathcal{U}$.

(iii) For any *finite* subcollection $\{U_{\alpha_1}, \ldots, U_{\alpha_m}\} \subset \mathcal{U}$, we have $\bigcap_{i=1}^m U_{\alpha_i} \in \mathcal{U}$.

The sets $U_\alpha \in \mathcal{U}$ are called the **open sets** of the topology; their complements $F_\alpha = \mathbb{R}^n \setminus U_\alpha$ are called the **closed sets**.

Note that the sets $\varnothing$ and $\mathbb{R}^n$ are both open and closed. Moreover, the notion of a topology can be extended to more general sets $X$, not just $\mathbb{R}^n$. A topology can also be defined starting with closed sets, but we prefer to work with open sets because many nice properties, such as differentiability, are better described with them.

Under the metric topology, we say that a set $A \subset \mathbb{R}^n$ is **open** if $A = \varnothing$ or if for all $p \in A$, there exists $r > 0$ such that $B_r(p) \subset A$. Moreover, $A$ is **closed** if its complement $A^c = \mathbb{R}^n \setminus A$ is open. (We leave it as an exercise to show that this is indeed a topology.)

For example, the open balls $B_r(x)$ are open sets for all $x \in \mathbb{R}^n$ and $r > 0$. Indeed, for any point $p \in B_r(x)$, one sees that by picking $r' = (r - \|p - x\|)/2$, we have $B_{r'}(p) \subset B_r(x)$.

In general, open sets are described with strict inequalities, while closed sets are described using equality or inclusive inequalities. However, note that most sets are neither open nor closed, such as the half-open interval $U = (-1, 1]$ over $\mathbb{R}$.

The metric topology is not the only topology on $\mathbb{R}^n$; one example is the one consisting of only the sets $\mathcal{U} = \{\varnothing, \mathbb{R}^n\}$. However, we generally want more open sets to work with since we might want to know the behaviour of functions around a point $p \in \mathbb{R}^n$. If the only non-empty open set we had was $\mathbb{R}^n$, then this would apply to all points in $\mathbb{R}^n$, which does not yield a lot of information.

Let $p \in \mathbb{R}^n$. The previous paragraph leads us to the definition of an **open neighbourhood** of $p$, which is just an open set $U \subset \mathbb{R}^n$ such that $p \in U$.

We now turn our discussion to vector-valued functions. Let $U \subset \mathbb{R}^n$ and consider the vector-valued function

$$\begin{aligned} F : U \subset \mathbb{R}^n &\to B \subset \mathbb{R}^m \\ x = (x_1, \ldots, x_n) &\mapsto (F_1(x), \ldots, F_m(x)). \end{aligned}$$

Then $F$ is continuous if and only if the component functions $F_i : U \to \mathbb{R}$ are continuous for all $i = 1, \ldots, m$.

We say that $F$ is a **homeomorphism** if it is a continuous bijection whose inverse

$$F^{-1} : B \subset \mathbb{R}^m \to U \subset \mathbb{R}^n$$

is also continuous. For example, the identity map $\mathrm{Id}_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n$ and the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$ are both homeomorphisms.

It is a known fact that homeomorphisms map open sets to open sets and closed sets to closed sets. This follows from the topological characterization of continuity, which states that $F$ is continuous if and only if for every open (respectively closed) set $V \subset \mathbb{R}^m$, we have that $F^{-1}(V)$ is open (respectively closed). In fact, homeomorphisms preserve much more structure than this, as we'll see later.

## 1.2 Topological submanifolds of $\mathbb{R}^n$

We now define the main object we'll be working with in this course.

**Definition 1.1**

A **$k$-dimensional topological submanifold** (or **topological $k$-submanifold**) of $\mathbb{R}^n$ is a subset $M \subset \mathbb{R}^n$ such that for every $p \in M$, there exists an open neighbourhood $V$ of $p$ in $\mathbb{R}^n$, an open set $U \subset \mathbb{R}^k$, and a homeomorphism

$$\alpha : U \subset \mathbb{R}^k \to V \cap M \subset \mathbb{R}^n.$$

The homeomorphism $\alpha$ is called a **coordinate chart** (or **patch**) on $M$.

Note that the open neighbourhood $V \subset \mathbb{R}^n$ of $p$, the open set $U \subset \mathbb{R}^k$, and the map $\alpha$ do not need to be unique. But we'll see later that the dimension $k$ must be unique and is completely determined by $M$.

For example, $\mathbb{R}^n$ is a topological $n$-submanifold of $\mathbb{R}^n$ by taking $U = V = \mathbb{R}^n$ and $\alpha = \mathrm{Id}_{\mathbb{R}^n}$. Any open set $W \subset \mathbb{R}^n$ is a topological $n$-submanifold of $\mathbb{R}^n$ by taking $U = V = W$ and $\alpha = \mathrm{Id}_W$.

Let's now consider some non-trivial examples. Consider

$$M = \{(x, y) \in \mathbb{R}^2 : y = x^2\} \subset \mathbb{R}^2,$$

which is the graph of the parabola $f(x) = x^2$. Then $M$ is a topological 1-submanifold of $\mathbb{R}^2$ by considering the map $\alpha : \mathbb{R}^1 \to M \subset \mathbb{R}^2$, $t \mapsto (t, t^2)$. The inverse $\alpha^{-1} : M \to \mathbb{R}^1$ is just the projection of the first coordinate, which is continuous.

More generally, let $U \subset \mathbb{R}^k$ be an open set. Consider the graph of a continuous function

$$F : U \subset \mathbb{R}^k \to \mathbb{R}^{n-k},\ x \mapsto (F_1(x), \ldots, F_{n-k}(x)).$$

In other words, we are looking at the set

$$G = \{(x, y) \in \mathbb{R}^k \times \mathbb{R}^{n-k} : y = F(x),\ x \in U\} \subset \mathbb{R}^n.$$

We claim that $G$ is a $k$-dimensional topological submanifold of $\mathbb{R}^n$. To see this, define $\alpha : U \subset \mathbb{R}^k \to G \subset \mathbb{R}^n$ by $x \mapsto (x, F(x))$. Then $\alpha$ is continuous since $F$ is continuous, and it is a bijection since we are restricted to $G$. Moreover, it has continuous inverse $\alpha^{-1} : G \subset \mathbb{R}^n \to U \subset \mathbb{R}^k$, $(x, y) \mapsto x$.

Here are two more examples of this in action.

(1) Let $M = \{(x, y, z) \in \mathbb{R}^3 : z = x^2 + y^2\} \subset \mathbb{R}^3$. Then $M$ is the graph of the continuous function $f(x, y) = x^2 + y^2$, so it is a 2-dimensional topological submanifold of $\mathbb{R}^3$.

(2) Observe that $M = \{(x, y, z) \in \mathbb{R}^3 : y = x^2,\ z = x^3\} \subset \mathbb{R}^3$ is the graph of the continuous function $F(t) = (t^2, t^3)$, so it is a 1-dimensional topological submanifold of $\mathbb{R}^3$.

In all the examples above, we only needed one coordinate chart which worked for all points. However, this is not always the case! Consider the unit circle

$$\mathbb{S}^1 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \subset \mathbb{R}^2.$$

Note that $\mathbb{S}^1$ is compact. Therefore, by Heine-Borel, it is closed and bounded. Recall that homeomorphisms preserve closed sets, so it is impossible to find a unique chart $\alpha$. Indeed, if we had such a homeomorphism $\alpha : U \subset \mathbb{R}^1 \to \mathbb{S}^1 \subset \mathbb{R}^2$ for some open set $U$, then $U = \alpha^{-1}(\mathbb{S}^1)$ would be a compact subset of $\mathbb{R}^1$. But the only open and compact subset of $\mathbb{R}^n$ is $\varnothing$, which is a contradiction!

Nonetheless, two coordinate charts are enough to cover all points on $\mathbb{S}^1$. Define

$$\begin{aligned} V_1 &= \mathbb{R}^2 \setminus \{(x, 0) \in \mathbb{R}^2 : x \leq 0\}, \\ V_2 &= \mathbb{R}^2 \setminus \{(x, 0) \in \mathbb{R}^2 : x \geq 0\}, \end{aligned}$$

which are both open sets. Then the homeomorphism

$$\alpha_1 : U_1 = (-\pi, \pi) \to \mathbb{S}^1 \cap V_1,\ t \mapsto (\cos t, \sin t)$$

covers all points on $\mathbb{S}^1$ except for $(-1, 0)$, while

$$\alpha_2 : U_2 = (0, 2\pi) \to \mathbb{S}^1 \cap V_2,\ t \mapsto (\cos t, \sin t)$$

covers all points on $\mathbb{S}^1$ except for $(1, 0)$.

## 1.3 More preliminaries

We now introduce another definition from topology.

**Definition 1.2**

Let $A \subset \mathbb{R}^n$. A subset $U \subset A$ is **relatively open** if it is of the form $U = A \cap U'$ for some open set $U' \subset \mathbb{R}^n$. Similarly, we say that $F \subset A$ is **relatively closed** if $F = A \cap F'$ for some closed set $F' \subset \mathbb{R}^n$.

For example, consider $\mathbb{R}$ equipped with the metric topology so that the open (respectively closed) sets are the unions of open intervals (respectively finite intersections of closed intervals). Let $A = [-1, 2) \subset \mathbb{R}$, which is neither open nor closed in $\mathbb{R}$. Take $U = [-1, 1) \subset \mathbb{R}$, which is again neither open nor closed in $\mathbb{R}$. But $U$ is relatively open in $A$ since $U = A \cap (-3, 1)$. Similarly, $F = [-1, 1] = A \cap [-1, 1]$ is relatively closed in $A$.

Using the language of relatively open and closed sets, a lot of statements can be made simpler.

(1) We define an **open neighbourhood of $p$ in $A$** to be a relatively open set $U$ containing $p$.

(2) The relatively open sets form a topology on $A$, called the **relative topology** (verify this as an exercise).

(3) We now have a more concise definition of a topological submanifold. Let $M \subseteq \mathbb{R}^n$. Then $M$ is a **$k$-dimensional topological submanifold** of $\mathbb{R}^n$ if for every $p \in M$, there exists an open neighbourhood $V$ of $p$ in $M$, an open set $U \subset \mathbb{R}^k$, and a homeomorphism $\alpha : U \subset \mathbb{R}^k \to V \subset \mathbb{R}^n$.

**Definition 1.3**

Let $A \subset \mathbb{R}^n$. Then $A$ is **connected** if it cannot be written in the form $A = U \cup V$ where $U, V \neq \varnothing$ are relatively open in $A$ and $U \cap V = \varnothing$. Otherwise, we say that $A$ is **disconnected**; we call $U$ and $V$ **disconnecting sets** for $A$.

Let's go over a few example of connected sets. It can be shown that an open set in $\mathbb{R}^n$ is connected if and only if it is path connected; that is, there is a path between any two points in the set. This result can help us build some intuition for what a connected set should look like.

(1) $\mathbb{R}^n$ is connected.

(2) Let $\alpha < \beta \in \mathbb{R}$. Then $(\alpha, \beta)$, $(\alpha, \beta]$, $[\alpha, \beta)$, and $[\alpha, \beta]$ are all connected.

(3) Observe that $A = (-1, 0] \cup [1, 2]$ is a disconnected set because $(-1, 0] = A \cap (-1.3, 0.3)$ and $[1, 2] = A \cap (0.9, 2.1)$ are both relatively open in $A$ and disjoint.

(4) The open ball $B_r(p)$ is connected for all $p \in \mathbb{R}^n$ and $r > 0$.

An important property of connected sets is that the continuous image of a connected set is connected! This can be used to prove that a subset $M \subset \mathbb{R}^n$ is not a submanifold.

(1) Consider the **$\alpha$-curve** $C := \{(x, y) \in \mathbb{R}^2 : y^2 = x^2(x + 1)\} \subset \mathbb{R}^2$. This can be parametrized by the map

$$\begin{aligned} \alpha : \mathbb{R} &\to C \subset \mathbb{R}^2 \\ t &\mapsto (t^2 - 1, t(t^2 - 1)). \end{aligned}$$

Note that $\alpha$ is not injective since $\alpha(-1) = \alpha(1) = (0, 0)$. That is, $\alpha$ is not a homeomorphism on $\mathbb{R}$, but it becomes one if we remove the points $t = \pm 1$, whose inverse is

$$\begin{aligned} \alpha^{-1} : C \setminus \{(0, 0)\} &\to \mathbb{R} \setminus \{\pm 1\} \subset \mathbb{R} \\ (x, y) &\mapsto 1/x. \end{aligned}$$

Thus, $C$ is a 1-dimensional submanifold away from the point $(0, 0)$.

Our goal now is to show that the whole of $C$ is not a topological submanifold of $\mathbb{R}^2$. By contradiction, suppose that it were. By our above discussion, it must have dimension 1 because it has dimension 1 away from $(0, 0)$. Since $(0, 0) \in C$, it follows from the definition that there exists an open neighbourhood $V$ of $(0, 0)$ in $C$, an open set $U \subset \mathbb{R}^1$, and a homeomorphism

$$\alpha : U \subset \mathbb{R}^2 \to V \subset C.$$

There must be a unique point $t_0 \in U$ such that $\alpha(t_0) = (0, 0)$ since $\alpha$ is a bijection. Since $U$ is open and $t_0 \in U$, there exists $\varepsilon > 0$ such that $B_\varepsilon(t_0) \subset U$. But $U \subset \mathbb{R}^1$, so $B_\varepsilon(t_0) = (t_0 - \varepsilon, t_0 + \varepsilon) =: U'$. Then $\alpha|_{U'}$ is also a homeomorphism. Let $V' = \alpha(U')$. Observe that $V' \setminus \{(0, 0)\}$ has three or four pieces depending on how large the open set $U'$ is: one on the top right quadrant, one on the bottom right quadrant, and one or two on the left of the $y$-axis. On the other hand, $U' \setminus \{t_0\}$ has only two components, contradicting the fact that homeomorphisms preserve the number of connected components.

(2) Consider the **double cone** $M = \{(x, y, z) \in \mathbb{R}^3 : z^2 = x^2 + y^2\} \subset \mathbb{R}^3$. Away from $(0, 0, 0)$, every point in $M$ lies on the graph of one of the continuous functions $f_1(x, y) = \sqrt{x^2 + y^2}$ or $f_2(x, y) = -\sqrt{x^2 + y^2}$. Therefore, $M \setminus \{(0, 0, 0)\}$ is a 2-dimensional topological submanifold of $\mathbb{R}^3$ since $f_1$ and $f_2$ are both functions of two variables.

However, there is a problem at the point $(0, 0, 0)$ since it lies on the graph of both $f_1$ and $f_2$. Suppose that $M$ is a topological submanifold of $\mathbb{R}^3$. Then $M$ must necessarily be of dimension 2 because $M \setminus \{(0, 0, 0)\}$ is of dimension 2. Then by definition, there exists an open neighbourhood $V$ of $(0, 0, 0)$, an open set $U \subset \mathbb{R}^2$, and a homeomorphism

$$\alpha : U \subset \mathbb{R}^2 \to V \subset M.$$

Since $\alpha$ is a bijection, there exists a unique point $(x_0, y_0) \in U$ such that $\alpha(x_0, y_0) = (0, 0, 0)$. After shrinking $U$ (by the same argument as above), we may take $U' = B_\varepsilon((x_0, y_0))$ to ensure that we have a connected set. Consider now the restriction $\alpha|_{U'} : U' \to V = \alpha(U')$. Then $U' \setminus \{(x_0, y_0)\}$ has one component, whereas $V' \setminus \{(0, 0, 0)\}$ has two components (that is, it is disconnected), which is a contradiction.

Now, we want to prove the invariance of dimension.

**THEOREM 1.4: INVARIANCE OF DIMENSION**

$\mathbb{R}^m$ is homeomorphic to $\mathbb{R}^n$ if and only if $m = n$.

If $m = n$, then $\mathbb{R}^m = \mathbb{R}^n$, so there is nothing to prove. The other implication is much harder, and we'll need the following result.

**THEOREM 1.5: BROUWER INVARIANCE OF DOMAIN**

Let $U \subset \mathbb{R}^n$ be open, and let $f : U \subset \mathbb{R}^n \to \mathbb{R}^n$ be an injective continuous map. Then $f(U) \subset \mathbb{R}^n$ is open. In particular, $f$ is a homeomorphism onto its image.

An elementary proof can be found on Terry Tao's blog where he uses the Brouwer Fixed Point Theorem to prove it. Nowadays, the standard proof uses algebraic topology.

It is important that both the domain and codomain involve the same dimension $n$. For example, consider the injective continuous map $f : \mathbb{R} \to \mathbb{R}^2$ defined by $x \mapsto (x, 0)$. Observe that $f(U)$ is the $x$-axis, which is not open in $\mathbb{R}^2$.

**Proof of Theorem 1.4.**

We proceed by contradiction. Suppose that there is a homeomorphism $f : \mathbb{R}^m \to \mathbb{R}^n$ and that $m > n$. Consider the inclusion

$$\begin{aligned} \iota : \mathbb{R}^n &\to \mathbb{R}^m \\ (x_1, \dots, x_n) &\mapsto (x_1, \dots, x_m, 0, \dots 0), \end{aligned}$$

which is an injective continuous map. Then $\iota \circ f : \mathbb{R}^m \to \mathbb{R}^m$ is also an injective continuous map since it is the composition of two injective continuous maps. By Theorem 1.5, we have that $\iota \circ f(\mathbb{R}^m)$ is an open set in $\mathbb{R}^m$. But this is impossible because if $(x_1, \dots, x_n, 0, \dots, 0) \in \iota \circ f(\mathbb{R}^m)$, then

$$(x_1, \dots, x_n, \varepsilon/2, 0, \dots, 0) \notin \iota \circ f(\mathbb{R}^m)$$

for all $\varepsilon > 0$. Then $B_\varepsilon((x_1, \dots, x_n, 0, \dots, 0)) \not\subset \iota \circ f(\mathbb{R}^m)$ for all $\varepsilon > 0$, implying that $\iota \circ f(\mathbb{R}^m)$ is not open in $\mathbb{R}^m$. Therefore, we must have $m \leq n$. If $n < m$, then we can repeat the same argument with $f^{-1} : \mathbb{R}^n \to \mathbb{R}^m$, which again leads to a contradiction. We conclude that $n = m$. ☐

Note that we actually proved something stronger: if $m > n$ and $U$ is a nonempty open subset of $\mathbb{R}^m$, then there is no continuous mapping from $U$ to $\mathbb{R}^n$. As a consequence, we get the following.

**PROPOSITION 1.6**

If $M \subset \mathbb{R}^n$ is a $k$-dimensional topological submanifold of $\mathbb{R}^n$, then $k \leq n$.

**Proof of Proposition 1.6.**

If $M \subset \mathbb{R}^n$ is a $k$-dimensional topological submanifold of $\mathbb{R}^n$, then for all $p \in M$, there exists an open set $U \subset \mathbb{R}^k$, an open neighbourhood $V \subset M$ of $p$, and a homeomorphism

$$\alpha : U \subset \mathbb{R}^k \to V \subset M \subset \mathbb{R}^n.$$

Since $\alpha$ is an injective continuous map, this forces $k \leq n$ by the above discussion. ☐

Finally, we must have the same $k$ for any chart $\alpha : U \subset \mathbb{R}^k \to V \subset M$. Indeed, let $p \in M$, and suppose that we have two different charts, say $\alpha : U \subset \mathbb{R}^k \to V \subset M$ and $\beta : U' \subset \mathbb{R}^{k'} \to V' \subset M$ where $p \in V \cap V'$. Then $V \cap V' \neq \varnothing$, so we can consider the restrictions

$$\begin{aligned} \alpha|_{\alpha^{-1}(V\cap V')} &: \alpha^{-1}(V \cap V') \to V \cap V', \\ \beta|_{\beta^{-1}(V\cap V')} &: \beta^{-1}(V \cap V') \to V \cap V'. \end{aligned}$$

Then $\beta^{-1} \circ \alpha : \alpha^{-1}(V \cap V') \subset \mathbb{R}^k \to \beta^{-1}(V \cap V') \subset \mathbb{R}^{k'}$ is a homeomorphism. Hence, $\alpha^{-1}(V \cap V')$ is a $k$-dimensional topological submanifold of $\mathbb{R}^{k'}$. By Proposition 1.6, we have $k \leq k'$. Similarly, $\alpha^{-1} \circ \beta : \beta^{-1}(V \cap V') \subset \mathbb{R}^{k'} \to \alpha^{-1}(V \cap V') \subset \mathbb{R}^k$ is a homeomorphism, so $\beta^{-1}(V \cap V')$ is a $k'$-dimensional submanifold of $\mathbb{R}^k$. It follows that $k' \leq k$ and so $k' = k$.

## 1.4 Submanifolds of $\mathbb{R}^n$ of class $C^r$

Let $U \subset \mathbb{R}^n$ be an open set and consider the vector-valued function

$$\begin{aligned} F : U \subset \mathbb{R}^n &\to \mathbb{R}^m \\ x = (x_1, \ldots, x_n) &\mapsto (F_1(x), \ldots, F_m(x)). \end{aligned}$$

Recall that $F$ is of **class** $C^r$ for $r \geq 1$ if each component function $F_i : U \subset \mathbb{R}^n \to \mathbb{R}$ is of class $C^r$. That is, the partial derivatives of $F_i$ exist and are continuous up to order $r$. Also, we say that $F$ is of class $C^\infty$ or **smooth** if each $F_i$ is smooth (the partial derivatives exist up to any order).

(1) All polynomials are smooth.

(2) The function $f(x) = x^{4/3}$ is of class $C^1$. Its derivative $f'(x) = \frac{4}{3}x^{1/3}$ is continuous, but the second derivative $f''(x) = \frac{4}{9}x^{-2/3}$ is not defined at $x = 0$.

(3) The vector-valued function $F(x, y) = (2\cos x, xy - 1, e^{2\sin y + x})$ is smooth on $\mathbb{R}^2$ because each component function is smooth.

The **partial derivative** of $F$ with respect to the variable $x_j$ is

$$\frac{\partial F}{\partial x_j} := \left(\frac{\partial F_1}{\partial x_j}, \ldots, \frac{\partial F_m}{\partial x_j}\right).$$

If we fix a component function $F_i$, its **gradient** is

$$\nabla F_i := \left(\frac{\partial F_i}{\partial x_1}, \ldots, \frac{\partial F_i}{\partial x_n}\right).$$

The **derivative matrix** or **Jacobian matrix** of $F$ is the $m \times n$ matrix

$$DF := \begin{bmatrix} \partial F_1/\partial x_1 & \cdots & \partial F_1/\partial x_n \\ \vdots & \ddots & \vdots \\ \partial F_m/\partial x_1 & \cdots & \partial F_m/\partial x_n \end{bmatrix}.$$

That is, the rows correspond to the component functions $F_i$, and the columns correspond to the variables $x_j$. We can also think of the rows as the gradients and the columns as the partial derivatives; that is, we have

$$DF = \left[\begin{array}{c|c|c} \dfrac{\partial F}{\partial x_1} & \cdots & \dfrac{\partial F}{\partial x_n} \end{array}\right] = \left[\begin{array}{c} \nabla F_1 \\ \hline \vdots \\ \hline \nabla F_m \end{array}\right]$$

In general, we want to work with some differentiability. This leads to the following definition.

**Definition 1.7**

Let $M \subset \mathbb{R}^n$. Suppose that for every $p \in M$, there exists an open neighbourhood $V$ of $p$ in $M$, an open subset $U \subset \mathbb{R}^k$, and a homeomorphism $\alpha : U \subset \mathbb{R}^k \to V \subset M$ such that

(1) $\alpha$ is of class $C^r$ for some $r \geq 1$;

(2) $D\alpha(x)$ has rank $k$ for all $x \in U$.

Then $M$ is called a **$k$-dimensional submanifold of $\mathbb{R}^n$ of class $C^r$**. We call $\alpha$ a **coordinate chart** (or **coordinate patch**) about $p$.

Note that every submanifold of class $C^r$ is a topological submanifold. We are only imposing the extra conditions (1) and (2) on the coordinate charts. We will see that condition (2) will allow us to define tangent spaces to the submanifolds at every point. A submanifold of class $C^\infty$ is called a **smooth submanifold**.

As usual, let's go over some examples.

(1) Let $U \subset \mathbb{R}^n$ be open. Then $\alpha : U \subset \mathbb{R}^n \to V = U \subset \mathbb{R}^n$ sending $x$ to itself is smooth. Since the component functions are $F_i(x) = x_i$ for all $i = 1, \ldots, n$, we have

$$D\alpha(x) = \left[\frac{\partial F_i}{\partial x_j}\right] = \left[\frac{\partial x_i}{\partial x_j}\right] = [\delta_{ij}],$$

where $\delta_{ij}$ is the Kronecker delta. In other words, $D\alpha(x)$ is the $n \times n$ identity matrix and has rank $n$ for all $x \in U$, so $U \subset \mathbb{R}^n$ is a smooth $n$-dimensional submanifold of $\mathbb{R}^n$.

(2) **Graphs of functions of class $C^r$.** Let $U \subset \mathbb{R}^k$ be an open set and consider a function

$$\begin{aligned} F : U \subset \mathbb{R}^k &\to \mathbb{R}^{n-k} \\ (x_1, \ldots, x_k) &\mapsto (F_1(x), \ldots, F_{n-k}(x)) \end{aligned}$$

of class $C^r$ (so each $F_i$ is of class $C^r$). Let

$$M = \{(x, F(x)) \in \mathbb{R}^k \times \mathbb{R}^{n-k} : x \in U\} \subset \mathbb{R}^n$$

be the graph of $F$. We have already seen that $M$ is a $k$-dimensional submanifold of $\mathbb{R}^n$ by taking $V = M$ and the homeomorphism $\alpha : U \subset \mathbb{R}^k \to V \subset \mathbb{R}^n$ defined by

$$\alpha(x) = (x, F(x)) = (x_1, \ldots, x_k, F_1(x), \ldots, F_{n-k}(x)).$$

In particular, $\alpha$ is of class $C^r$ since the identity components are smooth and the $F_i$ are of class $C^r$. Let's look at the derivative matrix in terms of the columns of partial derivatives. We have

$$\frac{\partial \alpha}{\partial x_j} = \left(0, \ldots, 0, 1, 0, \ldots, 0, \frac{\partial F_1}{\partial x_j}, \ldots, \frac{\partial F_{n-k}}{\partial x_j}\right)$$

where the 1 corresponds to the $j$-th component, and hence

$$D\alpha(x) = \left[\begin{array}{cccc} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ \hline \partial F_1/\partial x_1 & \partial F_1/\partial x_2 & \cdots & \partial F_1/\partial x_k \\ \vdots & \vdots & \ddots & \vdots \\ \partial F_{n-k}/\partial x_1 & \partial F_{n-k}/\partial x_2 & \cdots & \partial F_{n-k}/\partial x_k \end{array}\right] = \left[\begin{array}{c} I_{k\times k} \\ \hline DF(x) \end{array}\right].$$

This matrix has rank $k$ for all $x \in U$, so $M$ is a $k$-dimensional submanifold of class $C^r$.

(3) We saw that the circle $\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \subset \mathbb{R}^2$ was a 1-dimensional topological submanifold using the charts $\alpha_1 : U_1 = (-\pi, \pi) \subset \mathbb{R}^1 \to \mathbb{S}^1 \setminus \{(-1, 0)\} \subset \mathbb{R}^2$ and $\alpha_2 : U_2 = (0, 2\pi) \subset \mathbb{R}^1 \to \mathbb{S}^1 \setminus \{(1, 0)\} \subset \mathbb{R}^2$, both defined by $t \mapsto (\cos t, \sin t)$. Note that both $\alpha_i$ are smooth functions with derivative matrix

$$D\alpha_i = \begin{bmatrix} \frac{\mathrm{d}\alpha_i}{\mathrm{d}t} \end{bmatrix} = \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix},$$

which has rank 1 because $\sin t$ and $\cos t$ don't have the same zeroes, and hence $D\alpha_i$ is never the zero vector. Thus, $\mathbb{S}^1$ is a smooth 1-dimensional submanifold of $\mathbb{R}^2$.

Not every topological submanifold of $\mathbb{R}^n$ is of class $C^r$ for some $r \geq 1$. For example, consider the graph

$$M = \{(x, |x|) : x \in \mathbb{R}\}$$

of the function $f(x) = |x|$ on $\mathbb{R}$. Since $f$ is continuous, we know that $M$ is a 1-dimensional topological submanifold of $\mathbb{R}^2$. Note that $f$ is smooth away from $x = 0$, so $M \setminus \{(0, 0)\}$ is a smooth 1-dimensional submanifold of $\mathbb{R}^2$. However, we claim that it cannot be a submanifold of class $C^r$ on any neighbourhood of the point $(0, 0)$.

# 2 Curves in $\mathbb{R}^n$

Tangent spaces: Let $M \subset \mathbb{R}^n$ with class $C^r$ of dimension $k$. For a chart $\alpha : U \subset \mathbb{R}^k \to V \subset M \subset \mathbb{R}^n$, we have for all $p \in V$ that

$$T_p(M) := \alpha_*(T_{x_0}(\mathbb{R}^k)),$$

where $p = \alpha(x_0)$. We showed that

$$T_p(M) = \operatorname{span}_{\mathbb{R}} \left\{ \frac{\partial \alpha}{\partial x_1}(x_0), \dots, \frac{\partial \alpha}{\partial x_k}(x_0) \right\} \subset T_p(\mathbb{R}^n),$$

which is a $k$-dimensional subspace of $T_p(\mathbb{R}^n)$.

Examples:

(1) Let $U \subset \mathbb{R}^n$ be an open set. Then $\alpha : U \subset \mathbb{R}^n \to V = U \subset \mathbb{R}^n$ given by $x \mapsto x$ yields $D\alpha(x) = I_{n\times n}$ for all $x \in U$, so

$$T_x(U) = \alpha_*(T_x(\mathbb{R}^n))$$

by definition. Then for all $(x; v) \in T_x(\mathbb{R}^n)$, we get

$$\alpha_*(x; v) = (\alpha(x); D\alpha(x)v) = (x; v).$$

That is, we have $T_x(U) = T_x(\mathbb{R}^n)$.

(2) We saw two different ways of seeing if something is a submanifold. If $f : U \subset \mathbb{R}^k \to \mathbb{R}^{n-k}$ is a function of a class $C^r$ and

$$M = \{(x, f(x)) \in \mathbb{R}^n : x \in U\} \subset \mathbb{R}^n$$

its graph, then $M$ is a $k$-dimensional submanifold of $\mathbb{R}^n$ of class $C^r$. We can parameterize all points in $M$ with the map $\alpha : U \subset \mathbb{R}^k \to M \subset \mathbb{R}^n$ defined by $\alpha(x) = (x, f(x))$. The derivative matrix is

$$D\alpha(x) = \left[\begin{array}{c} I_{k\times k} \\ \hline Df(x) \end{array}\right]$$

for all $x \in U$. Let $p \in M$ so that $p = (x_0, f(x_0))$ for some $x_0 \in U$. Then

$$\begin{aligned} T_p(M) &= \alpha_*(T_{x_0}(\mathbb{R}^k)) \\ &= \{\alpha_*(x_0; v) : v \in T_{x_0}(\mathbb{R}^k)\} \\ &= \{(\alpha(x_0); D\alpha(x_0)v) : v \in \mathbb{R}^k\} \\ &= \{(p; w) : w = (v, Df(x_0)v),\ v \in \mathbb{R}^k\}, \end{aligned}$$

since $\alpha(x_0) = p$ and $D\alpha(x)$ is the block matrix with $I_{k\times k}$ upstairs and $Df(x)$ downstairs. Also,

$$T_p(M) = \operatorname{span}_{\mathbb{R}} \left\{ \frac{\partial \alpha}{\partial x_1}(x_0), \dots, \frac{\partial \alpha}{\partial x_k}(x_0) \right\} = \operatorname{span}_{\mathbb{R}} \left\{ \left[\begin{array}{c} e_i \\ \hline \partial f(x_0)/\partial x_i \end{array}\right] : i = 1, \dots, k \right\}.$$

(3) Let $U \subset \mathbb{R}^n$ be open. If $F : U \subset \mathbb{R}^n \to \mathbb{R}^{n-k}$ is a function of class $C^r$ with $DF(p)$ having rank $n - k$ for all $p \in U$, then

$$M = \{x \in U : F(x) = 0\}$$

is a $k$-dimensional submanifold of $\mathbb{R}^n$ of class $C^r$. In this case, we leave it as an exercise to show that

$$T_p(M) = \ker(DF(p)).$$

In particular, if $k = n - 1$, then $F : U \subset \mathbb{R}^n \to \mathbb{R}$ is a scalar function and

$$T_p(M) = \ker(\nabla F(p)).$$

Then $\nabla F(p)$ is the normal vector of $T_p(M)$.

For example, take the $n$-sphere

$$S^n = \{x \in \mathbb{R}^{n+1} : \|x\|^2 = 1\},$$

which is the zero set of the function $F : \mathbb{R}^{n+1} \to \mathbb{R}$ defined by $F(x) = \|x\|^2 - 1 = x_1^2 + \cdots + x_{n+1}^2 - 1$. The derivative matrix is just the gradient; that is,

$$DF(x) = \nabla F(x) = \begin{bmatrix} 2x_1 & \cdots & 2x_{n+1} \end{bmatrix} = 2x.$$

## 2.1

What is a curve? Intuitively, it is a 1-dimensional subset of $\mathbb{R}^n$.

The level sets $f(x, y) = k$ of a two variable function in $\mathbb{R}^2$ are curves. For example, for $f(x, y) = x^2 + y^2$, we see that $C\colon x^2 + y^2 = k$ for $k > 0$ is a circle centered at $(0, 0)$ of radius $k^{1/2}$.

The intersection of two surfaces in $\mathbb{R}^3$ is also a curve. For example, take $z = x^2 + y^2$ and $z = 2$. Their intersection is the circle $C\colon x^2 + y^2 = 2$ in the plane $z = 2$.

For our purposes, we'll work with parametrized curves $\gamma : I = (\alpha, \beta) \subset \mathbb{R} \to \mathbb{R}^n$ of class $C^r$. (In practice, we need $r \geq n$ when working over $\mathbb{R}^n$.)

Example: **Circular helix.** Let $\gamma(t) = (a \cos t, a \sin t, bt)$ for $t \in \mathbb{R}$ and $a, b > 0$. Note that $x^2 + y^2 = a^2$, so $\gamma(t)$ lies above the circle $x^2 + y^2 = a^2$ in the $xy$-plane. We have $\gamma(0) = (a, 0, 0)$ and $\gamma(\pi/2) = (0, a, b\pi/2)$. (The circular helix looks like a spiral along the cylinder.)

DEFINITION 2.1

A parameterized curve $\gamma : (\alpha, \beta) \to \mathbb{R}^n$ of class $C^r$ is called **regular** if for all $t \in (\alpha, \beta)$, we have

$$\gamma'(t) = \frac{\mathrm{d}\gamma}{\mathrm{d}t}(t) \neq 0.$$

We call $\|\gamma'(t)\|$ the **speed of** $\gamma$ **at** $\gamma(t)$ and we say that $\gamma$ is **unit speed** if $\|\gamma'(t)\| = 1$ for all $t \in (\alpha, \beta)$.

Note that unit speed implies regular because $\|x\| = 0$ if and only if $x = 0$.

Example: Let $a > 0$ and take $\gamma(t) = (a \cos t, a \sin t)$ for $t \in \mathbb{R}$, which is a parametrization of the circle of radius $a$. Then $\gamma'(t) = (-a \sin t, a \cos t) \neq (0, 0)$ for all $t \in \mathbb{R}$ and $\|\gamma'(t)\| = a$, so $\gamma$ is unit speed if and only if $a = 1$.