

PMATH 365: Suggested problems for the final

To prepare for the final exam, I recommend reviewing Quizzes 1-10. I also recommend going through the following list of questions.

A. Differentiable maps, immersions, embeddings and submanifolds.

1. Define the following:

- (a) A k -dimensional topological submanifold of \mathbb{R}^n .

Solution. A set $M \subset \mathbb{R}^n$ such that for all $p \in M$, there exists an open neighbourhood V of p , an open set $U \subset \mathbb{R}^k$, and a homeomorphism $\alpha : U \subset \mathbb{R}^k \rightarrow V \subset M \subset \mathbb{R}^n$.

- (b) An immersion of class C^r , $r \geq 1$.

Solution. A map $\alpha : U \subset \mathbb{R}^k \rightarrow \mathbb{R}^n$ such that α is of class C^r and $D\alpha$ has maximal rank k everywhere on U .

- (c) An embedding of class C^r , $r \geq 1$.

Solution. An immersion that is homeomorphic onto its image.

- (d) A k -dimensional submanifold of \mathbb{R}^n of class C^r , $r \geq 1$.

Solution. A set $M \subset \mathbb{R}^n$ such that for all $p \in M$, there exists an open neighbourhood V of p , an open set $U \subset \mathbb{R}^k$, and a homeomorphism $\alpha : U \subset \mathbb{R}^k \rightarrow V \subset M \subset \mathbb{R}^n$ satisfying the following properties:

- (i) α is of class C^r ;
- (ii) $D\alpha(x)$ has rank k for all $x \in U$.

- (e) A coordinate chart (of class C^r) of a k -dimensional submanifold of \mathbb{R}^n (of class C^r , $r \geq 1$).

Solution. This is a homeomorphism as in (d).

- (f) An atlas (of class C^r) of a k -dimensional submanifold of \mathbb{R}^n (of class C^r , $r \geq 1$).

Solution. A set of coordinate charts $\{\alpha_i : U_i \subset \mathbb{R}^k \rightarrow V_i \subset M \subset \mathbb{R}^n\}_{i \in I}$ of class C^r for M (where I is some index set) such that $\bigcup_{i \in I} V_i = M$.

2. What is the difference between an immersion and an embedding? Give an example of both. In particular, give an example of an immersion that is *not* an embedding.

Solution. An embedding has the requirement that it is homeomorphic onto its image. The canonical immersion is the inclusion map $\iota : \mathbb{R}^k \rightarrow \mathbb{R}^n$ defined by $(x_1, \dots, x_k) \mapsto (x_1, \dots, x_k, 0, \dots, 0)$. This is an immersion of class C^∞ since ι is smooth with

$$D\iota = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$$

so that $D\iota$ has rank k everywhere. It is also an embedding as it is homeomorphic onto its image with inverse $\iota^{-1} : \iota(\mathbb{R}^k) \subset \mathbb{R}^n \rightarrow \mathbb{R}^k$ given by $(x_1, \dots, x_k, 0, \dots, 0) \mapsto (x_1, \dots, x_k)$.

For an example of an immersion that is not an embedding, consider the parametrization $\alpha(t) = (t^2 - 1, t(t^2 - 1))$ of the α -curve, where $t \in \mathbb{R}$. We see that α is smooth with derivative matrix $D\alpha(t) = (2t, 3t^2 - 1) \neq (0, 0)$ for all $t \in \mathbb{R}$, so α is an immersion of class C^∞ . However, it is not homeomorphic onto its image since it is not injective. Indeed, we have $\alpha(1) = \alpha(-1) = (0, 0)$.

3. Give an example of a subset of \mathbb{R}^n that is *not* a topological submanifold of \mathbb{R}^n . Explain!

Solution. We again consider the α -curve $C = \{(x, y) \in \mathbb{R}^2 : y^2 = x^2(x + 1)\} \subset \mathbb{R}^2$, which can be parametrized by the map $\alpha : \mathbb{R} \rightarrow C \subset \mathbb{R}^2$ given by $t \mapsto (t^2 - 1, t(t^2 - 1))$. Note that if we remove the points $t = \pm 1$, we see that α has inverse

$$\begin{aligned}\alpha^{-1} : C \setminus \{(0, 0)\} &\rightarrow \mathbb{R} \setminus \{\pm 1\} \\ (x, y) &\mapsto 1/x.\end{aligned}$$

That is, C is a 1-dimensional topological submanifold of \mathbb{R}^2 away from the point $(0, 0)$.

Now, suppose that C were a topological submanifold of \mathbb{R}^2 . Our discussion above means that C must be 1-dimensional. Since $(0, 0) \in C$, there exists an open neighbourhood V of $(0, 0)$, an open set $U \subset \mathbb{R}^1$, and a homeomorphism $\alpha : U \subset \mathbb{R}^1 \rightarrow V \subset C \subset \mathbb{R}^2$. Since α is a homeomorphism, there exists a unique point $t_0 \in U$ such that $\alpha(t_0) = (0, 0)$. Since U is open, we can pick $\varepsilon > 0$ such that $B_\varepsilon(t_0) = (t_0 - \varepsilon, t_0 + \varepsilon) \subset U$. Write $U' = B_\varepsilon(t_0)$ and let $V' = \alpha(U')$ so that $\alpha|_{U'} : U' \rightarrow V'$ is also a homeomorphism.

Observe that $V' \setminus \{(0, 0)\}$ consists of three or four connected components depending on how large V' is: one on the bottom right quadrant, one on the top right quadrant, and one or two on the left of the y -axis. On the other hand, $U' \setminus \{t_0\}$ has two components, which is a contradiction since homeomorphisms preserve the number of connected components. Therefore, C is not a topological submanifold of \mathbb{R}^2 .

4. Give an example of a subset of \mathbb{R}^n that is a topological submanifold but *not* a submanifold of class C^r for any $r \geq 1$. Explain!

Solution. Let $M = \{(x, |x|) : x \in \mathbb{R}\} \subset \mathbb{R}^2$, which is the graph of $f(x) = |x|$ over $x \in \mathbb{R}$. In particular, since $f(x) = |x|$ is continuous, it follows that M is a 1-dimensional topological submanifold of \mathbb{R}^2 . Note that f is smooth away from $x = 0$, so $M \setminus \{(0, 0)\}$ is a 1-dimensional submanifold of class C^∞ .

However, we claim that M is not a submanifold of class C^r for some $r \geq 1$. Suppose otherwise, so at the point $(0, 0) \in M$, there exists an open neighbourhood V of $(0, 0)$, an open set $U \subset \mathbb{R}$, and a homeomorphism $\alpha : U \subset \mathbb{R} \rightarrow V \subset M \subset \mathbb{R}^2$ such that α is of class C^r and $D\alpha$ has maximal rank 1 everywhere on U . In particular, this means that $D\alpha(t) = \alpha'(t) \neq (0, 0)$ for all $t \in U$. But $\alpha'(t)$ is tangent to M at $\alpha(t)$, which gives two possibilities:

- If $\alpha(t)$ is on the line $y = x$, then $\alpha'(t)$ is a direction vector of $y = x$. This means that for some $c : I \subset \mathbb{R} \rightarrow \mathbb{R}$, we have

$$\alpha'(t) = c(t) \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Since α is of class C^r where $r \geq 1$, this means $\alpha'(t)$ is continuous, so $c : I \rightarrow \mathbb{R}$ is continuous.

- If $\alpha(t)$ is on the line $y = -x$, then $\alpha'(t)$ is a direction vector of $y = -x$. Then for some $d : I' \subset \mathbb{R} \rightarrow \mathbb{R}$, we have

$$\alpha'(t) = d(t) \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The same argument shows that $d : I' \rightarrow \mathbb{R}$ is continuous.

But α is a bijection, so $(0, 0) = \alpha(t_0)$ for some $t_0 \in U$. By the continuity of $\alpha'(t)$, we obtain

$$\lim_{t \rightarrow t_0^-} \alpha'(t) = \lim_{t \rightarrow t_0^+} \alpha'(t).$$

Without loss of generality, suppose that $\alpha(t)$ is moving along M from left to right. (We can parametrize in the other direction otherwise.) Then if $t < t_0$, then we are in the second case above, whereas if $t > t_0$, then we are in the first case. This implies that

$$\lim_{t \rightarrow t_0^-} \alpha'(t) = \lim_{t \rightarrow t_0^-} d(t) \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \lim_{t \rightarrow t_0^+} c(t) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \lim_{t \rightarrow t_0^+} \alpha'(t).$$

But these vectors are not parallel to each other, so $\lim_{t \rightarrow t_0^-} d(t) = \lim_{t \rightarrow t_0^+} c(t) = 0$. This implies that $\alpha'(t_0) = (0, 0)$, which contradicts the fact that $D\alpha(t) = \alpha'(t)$ has rank 1 for all $t \in U$.

5. True or false?

- (a) Every topological submanifold of \mathbb{R}^n is also a submanifold of \mathbb{R}^n of class C^r for some $r \geq 1$.

Solution. False, take the graph of $f(x) = |x|$.

- (b) Every submanifold of \mathbb{R}^n of class C^r for some $r \geq 1$ is also a topological submanifold of \mathbb{R}^n .

Solution. True.

- (c) A subset of \mathbb{R}^n that is not a topological submanifold may be a submanifold of class C^r for some $r \geq 1$.

Solution. False; consider the contrapositive of (b).

- (d) Every topological submanifold of \mathbb{R}^n admits an atlas of class C^r for some $r \geq 1$.

Solution. False. If a topological submanifold of \mathbb{R}^n admits an atlas of class C^r for some $r \geq 1$, then it is a submanifold of \mathbb{R}^n of class C^r , but this is not true of all topological submanifolds by (a).

- (e) Every coordinate chart of class C^r , $r \geq 1$, of a submanifold is a homeomorphism.

Solution. True.

- (f) Every coordinate chart of class C^r , $r \geq 1$, of a submanifold is an embedding.

Solution. True.

- (g) Let $\alpha : U \subset \mathbb{R}^k \rightarrow V \subset M \subset \mathbb{R}^n$ be a coordinate chart of class C^r , $r \geq 1$, of a k -dimensional submanifold M of \mathbb{R}^n . Then, $\alpha^{-1} : V \subset \mathbb{R}^n \rightarrow U \subset \mathbb{R}^k$ is also of class C^r .

Solution. True.

- (h) Every immersion is a homeomorphism.

Solution. False; take the parametrization $\alpha(t) = (t^2 - 1, t(t^2 - 1))$ of the α -curve.

- (i) Every embedding is a homeomorphism.

Solution. True.

- (j) Every submanifold of \mathbb{R}^n is also an immersed submanifold of \mathbb{R}^n of class C^r .

Solution. True; the coordinate charts are embeddings and therefore immersions.

- (k) The n -sphere admits an atlas that consists of a single chart.

Solution. False. The n -sphere is compact, and hence closed and bounded by Heine-Borel. The domain of a coordinate chart is an open set. Since homeomorphisms send closed sets to closed sets and the only open and closed set in \mathbb{R}^{n+1} is \emptyset , it is impossible to use a single chart.

- (l) Every k -dimensional submanifold of \mathbb{R}^n of class C^r , $r \geq 1$, is locally the graph of a function $f : U \subset \mathbb{R}^r \rightarrow \mathbb{R}^{n-k}$ of class C^r .

Solution. True.

- (m) Every submanifold of \mathbb{R}^n of class C^r , $r \geq 1$, is locally the zero set of a function $F : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^{n-k}$ of class C^r of maximal rank (that is, DF has maximal rank $n - k$ on $F^{-1}(0)$).

Solution. True.

6. Determine whether the following sets are topological submanifolds of \mathbb{R}^n or submanifolds of \mathbb{R}^n of class C^r for some $r \geq 1$. Justify your answers!

- (a) \mathbb{R}^n .

Solution. \mathbb{R}^n is a submanifold of \mathbb{R}^n of class C^∞ since we can choose the identity map on \mathbb{R}^n as the coordinate chart.

- (b) An open subset U of \mathbb{R}^n .

Solution. This is a submanifold of \mathbb{R}^n of class C^∞ ; take the identity map on U as the coordinate chart.

- (c) The graph of a continuous function $f : U \subset \mathbb{R}^k \rightarrow \mathbb{R}^{n-k}$.

Solution. This is a k -dimensional topological submanifold of \mathbb{R}^n . If $M = \{(x, f(x)) \in \mathbb{R}^k \times \mathbb{R}^{n-k} : x \in U\}$ is the graph, then $\alpha : U \subset \mathbb{R}^k \rightarrow M \subset \mathbb{R}^n$ given by $x \mapsto (x, f(x))$ is a homeomorphism with inverse $(x_1, \dots, x_n) \mapsto (x_1, \dots, x_k)$.

- (d) The graph of a function $f : U \subset \mathbb{R}^k \rightarrow \mathbb{R}^{n-k}$ of class C^r for some $r \geq 1$.

Solution. This is a k -dimensional submanifold of class C^r . Take the same homeomorphism $\alpha : U \subset \mathbb{R}^k \rightarrow M \subset \mathbb{R}^n$ defined by $x \mapsto (x, f(x))$ as above. This is of class C^r since f is of class C^r and $D\alpha(x)$ has rank k for all $x \in U$ since the upper part of $D\alpha(x)$ is the $k \times k$ identity matrix.

- (e) The zero set of a continuous function $F : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^{n-k}$.

Solution. This is not a topological submanifold of \mathbb{R}^n in general. Consider the α -curve, which is the zero set of $f(x, y) = y^2 - x^2(x + 1)$ but is not a topological submanifold of \mathbb{R}^2 .

- (f) The zero set of a function $F : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^{n-k}$ of class C^r , $r \geq 1$, of maximal rank.

Solution. This is a k -dimensional submanifold of class C^r ; in fact, this is an equivalent characterization.

- (g) $\{(x, y) \in \mathbb{R}^2 : y = x^2, -1 \leq x \leq 1\} \cup \{(x, y) \in \mathbb{R}^2 : y = 1, -1 \leq x \leq 1\} \subset \mathbb{R}^2$.

Solution. This is a 1-dimensional topological submanifold of \mathbb{R}^2 , but not a submanifold of \mathbb{R}^2 of class C^r by considering the points $(\pm 1, 1)$, which lie on both curves.

- (h) The α -curve $\{(x, y) \in \mathbb{R}^2 : y^2 = x^2(x + 1)\} \subset \mathbb{R}^2$. For a picture, see (second picture).

Solution. Not a topological submanifold of \mathbb{R}^2 .

- (i) $\{(x, y) \in \mathbb{R}^2 : xy = 1\} \subset \mathbb{R}^2$.

Solution. This is a smooth submanifold of \mathbb{R}^2 since it is the graph of the smooth function $f(x) = 1/x$ on $x \in \mathbb{R} \setminus \{0\}$.

- (j) The cylinder $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\} \subset \mathbb{R}^3$.

Solution. Smooth submanifold of \mathbb{R}^3 since it is the graph of the smooth function $f(x) = x^2 + y^2 - 1$ on $x \in \mathbb{R}$.

- (k) The cone $\{(x, y, z) \in \mathbb{R}^3 : z = \sqrt{x^2 + y^2}\} \subset \mathbb{R}^3$.

Solution. Topological submanifold of \mathbb{R}^3 since it is the graph of the continuous function $f(x, y) = \sqrt{x^2 + y^2}$, but not a submanifold of \mathbb{R}^3 of class C^r for some $r \geq 1$ by considering the point $(0, 0, 0)$ in the cone.

- (l) The double cone $\{(x, y, z) \in \mathbb{R}^3 : z^2 = x^2 + y^2\} \subset \mathbb{R}^3$.

Solution. Not a topological submanifold of \mathbb{R}^3 by considering $(0, 0, 0)$.

- (m) The twisted cubic $\{(x, y, z) \in \mathbb{R}^3 : y = x^2, z = x^3\} \subset \mathbb{R}^3$.

Solution. Smooth submanifold of \mathbb{R}^3 , since it is the graph of $f(x) = (x^2, x^3)$.

- (n) $\{(x, y, z) \in \mathbb{R}^3 : y = -1\} \cup \{(x, y, z) \in \mathbb{R}^3 : x = y = 0\} \subset \mathbb{R}^3$.

Solution. Neither; one of the sets is 2-dimensional while the other is 1-dimensional.

- (o) The n -sphere $S^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \dots + x_{n+1}^2 = 1\} \subset \mathbb{R}^{n+1}$.

Solution. Smooth submanifold of \mathbb{R}^{n+1} since it is the zero set of the function $f(x_1, \dots, x_{n+1}) = x_1^2 + \dots + x_{n+1}^2 - 1$.

- (p) The general linear group $GL(n, \mathbb{R}) = \{A \in M_{n \times n}(\mathbb{R}) : \det A \neq 0\} \subset M_{n \times n} = \mathbb{R}^{n^2}$.

Solution. Note that \det is a smooth function as it is a polynomial of the matrix entries. Then $\mathbb{R} \setminus \{0\}$ is open, and therefore $GL(n, \mathbb{R})$ being the preimage of \det under $\mathbb{R} \setminus \{0\}$ is also open. Therefore, $GL(n, \mathbb{R})$ is a smooth submanifold of $M_{n \times n}(\mathbb{R})$.

- (q) The special linear group $SL(2, \mathbb{R}) = \{A \in M_{2 \times 2}(\mathbb{R}) : \det A = 1\} \subset M_{2 \times 2} = \mathbb{R}^4$.

Solution. Note that $\det A = a_1a_4 - a_2a_3$ and so $SL(2, \mathbb{R})$ is the zero set of the function $F(a_1, a_2, a_3, a_4) = a_1a_4 - a_2a_3 - 1$. Then $SL(2, \mathbb{R})$ is a smooth submanifold of \mathbb{R}^4 : we see that F is smooth and for any $A \in SL(2, \mathbb{R})$, we have

$$DF(A) = (a_4, -a_3, -a_2, a_1).$$

Note that A is not the zero matrix since $\det A = 1$, so $DF(A)$ has rank 1 for all $A \in SL(2, \mathbb{R})$.

7. Construct a smooth atlas for the following smooth submanifolds.

- (a) \mathbb{R}^n .

Solution. The identity map on \mathbb{R}^n .

- (b) An open subset U of \mathbb{R}^n .

Solution. The identity map on U .

- (c) The general linear group $GL(n, \mathbb{R}) = \{A \in M_{n \times n}(\mathbb{R}) : \det A \neq 0\} \subset M_{n \times n} = \mathbb{R}^{n^2}$.

Solution. The identity map on $GL(n, \mathbb{R})$ since it is open.

- (d) The graph of a function $F : U \subset \mathbb{R}^k \rightarrow \mathbb{R}^{n-k}$ of class C^r for some $r \geq 1$.

Solution. The map $U \subset \mathbb{R}^k \rightarrow \mathbb{R}^n$ defined by $x \mapsto (x, F(x))$.

- (e) The cubic $\{(x, y) \in \mathbb{R}^2 : x = y^3\} \subset \mathbb{R}^2$.

Solution. The map $\mathbb{R} \rightarrow \mathbb{R}^2$ defined by $x \mapsto (x, x^{1/3})$.

- (f) The hyperbola $\{(x, y) \in \mathbb{R}^2 : xy = 1\} \subset \mathbb{R}^2$.

Solution. The map $\mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}^2$ defined by $x \mapsto (x, 1/x)$.

- (g) The unit circle $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \subset \mathbb{R}^2$.

Solution. The maps $\alpha_1 : (0, 2\pi) \subset \mathbb{R} \rightarrow S^1 \setminus \{(1, 0)\}$ and $\alpha_2 : (-\pi, \pi) \subset \mathbb{R} \rightarrow S^1 \setminus \{(-1, 0)\}$, both defined by $t \mapsto (\cos t, \sin t)$.

- (h) The twisted cubic $\{(x, y, z) \in \mathbb{R}^3 : y = x^2, z = x^3\} \subset \mathbb{R}^3$.

Solution. The map $t \mapsto (t, t^2, t^3)$.

- (i) The plane $\{(x, y, z) \in \mathbb{R}^3 : x + y + z = 1\} \subset \mathbb{R}^3$.

Solution. This is the graph of the function $f(x, y) = 1 - x - y$, so we can take the chart $(x, y) \mapsto (x, y, 1 - x - y)$.

- (j) The paraboloid $\{(x, y, z) \in \mathbb{R}^3 : y = x^2 + z^2 + 1\} \subset \mathbb{R}^3$.

Solution. This is the graph of the function $f(x, z) = x^2 + z^2 + 1$, so we can take the chart $(x, z) \mapsto (x, x^2 + z^2 + 1, z)$.

- (k) The open unit ball $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 1\} \subset \mathbb{R}^3$.

Solution. Note that $B_1((0, 0, 0)) \subset \mathbb{R}^3$ is open, so we can take the identity map.

8. One version of the *Invariance of Dimension Theorem* states that if $m > n$ and U is an open subset of \mathbb{R}^m , then there exists no continuous injective mapping from U to \mathbb{R}^n . Use this to prove the following:

- (a) \mathbb{R}^m is homeomorphic to \mathbb{R}^n if and only if $m = n$.

Solution. We see that $m = n$ implies $\mathbb{R}^m = \mathbb{R}^n$, so there is nothing to prove. Conversely, suppose that \mathbb{R}^m is homeomorphic to \mathbb{R}^n . Then there is a continuous injective mapping $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$, so $m \leq n$ by the Invariance of Dimension Theorem. Similarly, we see that $f^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a continuous injective mapping, so $n \leq m$ and hence $m = n$.

- (b) If $M \subset \mathbb{R}^n$ is a k -dimensional submanifold of \mathbb{R}^n , then $k \leq n$.

Solution. Let $M \subset \mathbb{R}^n$ be a k -dimensional submanifold of \mathbb{R}^n . Then for all $p \in M$, there exists a neighbourhood V of p , an open set $U \subset \mathbb{R}^k$, and a homeomorphism $\alpha : U \subset \mathbb{R}^k \rightarrow V \subset M \subset \mathbb{R}^n$. In particular, we have that α is a continuous injective mapping from U to \mathbb{R}^n , so we must have $k \leq n$ by the Invariance of Dimension Theorem.

B. Velocity vectors, pushforwards, and tangents spaces.

1. Define the following:

- (a) The velocity vector of a parametrised curve $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^n$ of class C^r , $r \geq 1$.

Solution. The velocity vector is $\gamma'(t)$.

- (b) The tangent space $T_{\vec{x}}(\mathbb{R}^n)$ to \mathbb{R}^n at $\vec{x} \in \mathbb{R}^n$.

Solution. The tangent space $T_{\vec{x}}(\mathbb{R}^n)$ to \mathbb{R}^n at $\vec{x} \in \mathbb{R}^n$ is

$$T_{\vec{x}}(\mathbb{R}^n) = \{(\vec{x}; \vec{v}) : \vec{v} \in \mathbb{R}^n\}.$$

- (c) The pushforward α_* of a map $\alpha : U \rightarrow \mathbb{R}^n$ of class C^r , $r \geq 1$, where U is an open subset of \mathbb{R}^k .

Solution. The pushward α_* of the map $\alpha : U \rightarrow \mathbb{R}^n$ is defined by

$$\begin{aligned} \alpha_* : T_{\vec{x}} \mathbb{R}^k &\rightarrow T_p \mathbb{R}^n \\ (\vec{x}; \vec{v}) &\mapsto (p; D\alpha(\vec{x})\vec{v}), \end{aligned}$$

where $\vec{x} \in U$ and $p = \alpha(\vec{x})$.

- (d) The tangent space of $T_p M$ to a k -submanifold of \mathbb{R}^n of class C^r at $p \in M$.

Solution. Let $\alpha : U \subset \mathbb{R}^k \rightarrow V \subset M \subset \mathbb{R}^n$ be a coordinate chart about p . The tangent space of M at p is defined as

$$T_p M := \alpha_*(T_{x_0} \mathbb{R}^k) \subset T_p \mathbb{R}^n,$$

where $x_0 \in U$ is the unique point such that $p = \alpha(x_0)$.

2. Prove that, for all $\vec{x} \in \mathbb{R}^n$,

$$T_{\vec{x}} \mathbb{R}^n = \{(\vec{x}; \vec{v}) : \vec{v} \text{ is the velocity vector at } \vec{x} \text{ of a parametrised curve } \gamma \text{ passing through } \vec{x}\}.$$

Solution. Let $\vec{x} \in \mathbb{R}^n$. For all $\vec{v} \in \mathbb{R}^n$, consider the parametrized curve $\gamma(t) = \vec{x} + t\vec{v}$, where $t \in \mathbb{R}$. Observe that $\gamma'(t) = \vec{v}$. In particular, at $t = 0$, we have $\gamma(0) = \vec{x}$ and $\gamma'(0) = \vec{v}$, so \vec{v} is the velocity vector at \vec{x} .

3. Let M be a k -dimensional submanifold of \mathbb{R}^n of class C^r , $r \geq 1$. Show that $\dim_{\mathbb{R}} T_p M = k$ for all $p \in M$.

Solution. Let $p \in M$. Note that $T_p M = \alpha_*(T_{x_0} \mathbb{R}^k)$ where $x_0 \in U$ is the unique point such that $\alpha(x_0) = p$. We see that $D\alpha$ has rank k everywhere, so the image of the map

$$\alpha_*(x_0; \vec{v}) = (p; D\alpha(x_0)\vec{v}),$$

namely $T_p M$, must be of dimension k .

4. Compute the tangent space at the given point.

- (a) $T_p U$ for any point p in the open set $U \subset \mathbb{R}^n$.

Solution. The identity map $\alpha : U \rightarrow U$ has $D\alpha(x) = I_{n \times n}$ for all $x \in U$. By definition, we have $T_p U = \alpha_*(T_p \mathbb{R}^n)$ for all $p \in U$ since p is the unique point for which $\alpha(p) = p$. Then for all $(p; v) \in T_p \mathbb{R}^n$, we see that

$$\alpha_*(p; \vec{v}) = (p; D\alpha(p)\vec{v}) = (p; \vec{v}),$$

which implies that $T_p U = T_p \mathbb{R}^n$.

- (b) The graph of a function $f : U \subset \mathbb{R}^k \rightarrow \mathbb{R}^{n-k}$ of class C^r , where U is an open subset of \mathbb{R}^k . Take any $p \in M$.

Solution. Consider the C^r map $\alpha : U \subset \mathbb{R}^k \rightarrow \mathbb{R}^{n-k}$ given by $\vec{x} \mapsto (\vec{x}, f(\vec{x}))$. We have that

$$D\alpha(\vec{x}) = \begin{bmatrix} I_{k \times k} \\ Df(\vec{x}) \end{bmatrix},$$

where $Df(\vec{x})$ is an $(n-k) \times k$ matrix for all $\vec{x} \in U$. Now, let $p \in M$ so that $p = \alpha(\vec{x}_0) = (\vec{x}_0, f(\vec{x}_0))$ for some $\vec{x}_0 \in U$. Then we find that

$$T_p M = \text{span}_{\mathbb{R}} \left\{ \left[\frac{\vec{e}_i}{\partial f(\vec{x}_0)/\partial x_i} \right] : i = 1, \dots, k \right\},$$

where $\vec{e}_1, \dots, \vec{e}_k$ is the standard basis for \mathbb{R}^k . In particular, these are the columns of $D\alpha(\vec{x}_0)$.

- (c) The zero set M of a function $F : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^{n-k}$ of class C^r , where U is an open set in \mathbb{R}^n , such that $DF(p)$ has rank $n-k$ for all $p \in M$. Take any $p \in M$.

Solution. Here, we use Problem 1 of Assignment 3. Let M be a k -dimensional submanifold of \mathbb{R}^n of class C^r with $r \geq 1$ and let $p \in M$. Then $(p; \vec{v})$ is a tangent vector to M at p if and only if there exists a parametrized curve $\gamma : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}^n$ of class C^r whose image lies in M and is such that $(p; \vec{v}) = (\gamma(0); \gamma'(0))$.

We now show that $T_p M = \{(p; \vec{v}) \in M \times \mathbb{R}^n : DF(p)\vec{v} = \vec{0}\} = \ker(DF(p))$. Let $(p; \vec{v}) \in T_p M$. Then there exists a curve $\gamma : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}^n$ of class C^r on M such that $(p; \vec{v}) = (\gamma(0); \gamma'(0))$. We have $F(\gamma(t)) = \vec{0}$ for all $t \in (-\varepsilon, \varepsilon)$, so by the chain rule, we have

$$DF(p)\vec{v} = DF(\gamma(0))\gamma'(0) = \frac{d}{dt} F(\gamma(t)) \Big|_{t=0} = 0.$$

Then $T_p M \subset \ker(DF(p))$. But $T_p M$ has dimension k . Since $DF(p)$ has rank $n-k$ for all $p \in M$, we see that $\ker(DF(p))$ is a k -dimensional subspace of \mathbb{R}^n . Since $T_p M$ is a subspace of $\ker(DF(p))$ with both of dimension k , this implies that they are equal, as desired.

- (d) The sphere S^n in \mathbb{R}^{n+1} at any point in S^n .

Solution. Note that S^n is the zero set of the function $F(x_1, \dots, x_{n+1}) = x_1^2 + \dots + x_{n+1}^2 - 1$ with $DF(x_1, \dots, x_{n+1}) = (2x_1, \dots, 2x_{n+1})$. Let $\vec{x} = (x_1, \dots, x_{n+1}) \in S^1$. Then we have

$$\begin{aligned} T_{\vec{x}} S^n &= \ker(DF(\vec{x})) \\ &= \{\vec{v} \in \mathbb{R}^{n+1} : (2x_1, \dots, 2x_{n+1}) \cdot \vec{v} = 0\} \\ &= \{\vec{v} \in \mathbb{R}^{n+1} : x_1 v_1 + \dots + x_{n+1} v_{n+1} = 0\}. \end{aligned}$$

- (e) $T_I GL(n, \mathbb{R})$ where I is the identity $n \times n$ matrix.

Solution. We recall that $GL(n, \mathbb{R})$ is open in $M_{n \times n}(\mathbb{R})$, so using (a), we have that

$$T_I GL(n, \mathbb{R}) = T_I M_{n \times n}(\mathbb{R}) \simeq M_{n \times n}(\mathbb{R}).$$

- (f) $T_I SL(2, \mathbb{R})$ where I is the identity 2×2 matrix.

Solution. We see that $SL(2, \mathbb{R})$ is the zero set of the map $F(a_1, a_2, a_3, a_4) = a_1 a_4 - a_2 a_3 - 1$. Then $DF(a_1, a_2, a_3, a_4) = (a_4, -a_3, -a_2, a_1)$ with $DF(I) = (1, 0, 0, 1)$. Applying (c) yields

$$\begin{aligned} T_I SL(2, \mathbb{R}) &= \ker(DF(I)) \\ &= \{B \in M_{2 \times 2}(\mathbb{R}) : (1, 0, 0, 1) \cdot (b_1, b_2, b_3, b_4) = 0\} \\ &= \{B \in M_{2 \times 2}(\mathbb{R}) : b_1 + b_4 = 0\} \\ &= \{B \in M_{2 \times 2}(\mathbb{R}) : \text{tr } B = 0\}. \end{aligned}$$

C. Curves.

1. Let $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}$ be a parametrised curve of class C^r , $r \geq 3$. Define the following:

(a) The velocity, speed and acceleration of γ at $t \in (\alpha, \beta)$.

Solution. The velocity of γ at $t \in (\alpha, \beta)$ is $\gamma'(t)$. The speed is $\|\gamma'(t)\|$ and the acceleration is $\gamma''(t)$.

(b) The arclength of γ starting at the point $\gamma(t_0)$ for any $t_0 \in (\alpha, \beta)$.

Solution. The arclength of γ starting at the point $\gamma(t_0)$ is

$$s(t) := \int_{t_0}^t \|\gamma'(t)\| dt.$$

(c) The curve γ is regular.

Solution. A curve γ is regular if $\gamma'(t) \neq \vec{0}$ for all $t \in (\alpha, \beta)$.

(d) The curve γ is unit-speed.

Solution. A curve γ is unit-speed if $\|\gamma'(t)\| = 1$ for all $t \in (\alpha, \beta)$.

(e) The curvature $\kappa(t)$ of γ at $\gamma(t)$ for all $t \in (\alpha, \beta)$ (whether the parametrisation is unit-speed or just regular).

Solution. When γ is unit-speed, we have $\kappa(t) = \|\gamma''(t)\|$. If γ is regular, we have

$$\kappa(t) = \frac{\|\gamma'(t) \times \gamma''(t)\|}{\|\gamma'(t)\|^3}.$$

(f) The torsion $\tau(t)$ at points where $\kappa(t) > 0$ (whether the parametrisation is unit-speed or just regular).

Solution. When γ is unit-speed, we define $\tau : (\alpha, \beta) \rightarrow \mathbb{R}$ to be the scalar function such that $\frac{d}{dt}B(t) = -\tau(t)N(t)$. In the general case where γ is regular, we have

$$\tau(t) = \frac{(\gamma'(t) \times \gamma''(t)) \cdot \gamma'''(t)}{\|\gamma'(t) \times \gamma''(t)\|^2}.$$

(g) The Frenet frame of γ (whether the parametrisation is unit-speed or just regular).

Solution. We define the Frenet frame of γ to be $\{T, N, B\}$, where T is the unit tangent vector, N is the principal unit normal, and B is the unit binormal. When γ is unit-speed, we have $T(t) = \gamma'(t)$, $N(t) = \gamma''(t)/\kappa(t) = \gamma''(t)/\|\gamma''(t)\|$, and $B(t) = T(t) \times N(t)$. When γ is regular, we have

$$\begin{aligned} T(t) &= \frac{\gamma'(t)}{\|\gamma'(t)\|}, \\ B(t) &= \frac{\gamma'(t) \times \gamma''(t)}{\|\gamma'(t) \times \gamma''(t)\|}, \\ N(t) &= B(t) \times T(t). \end{aligned}$$

(h) Frenet-Serret equations for unit-speed curves.

Solution. The Frenet-Serret equations are $\frac{dT}{ds} = \kappa T$, $\frac{dN}{ds} = -\kappa T + \tau B$, and $\frac{dB}{ds} = -\tau N$. More compactly, they can be written as the matrix equation

$$\begin{bmatrix} \frac{dT}{ds} \\ \frac{dN}{ds} \\ \frac{dB}{ds} \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}.$$

2. Let $u : (\alpha, \beta) \rightarrow \mathbb{R}^n, t \mapsto u(t)$, be a vector-valued function of class $C^r, r \geq 1$, such that $\|u(t)\| = c$ for all $t \in (\alpha, \beta)$, for some fixed $c \in \mathbb{R}^{>0}$. Show that $u(t) \cdot u'(t) = 0$ for all $t \in (\alpha, \beta)$. In particular, if γ is a unit-speed curve, then $\gamma''(t)$ is either zero or perpendicular to $\gamma'(t)$ for all $t \in (\alpha, \beta)$.

Solution. Note that $u(t) \cdot u(t) = \|u(t)\|^2 = c^2$ for all $t \in (\alpha, \beta)$. Then

$$0 = \frac{d}{dt}(u(t) \cdot u(t)) = u'(t) \cdot u(t) + u(t) \cdot u'(t) = 2u(t) \cdot u'(t),$$

and therefore $u(t) \cdot u'(t) = 0$ for all $t \in (\alpha, \beta)$. For a unit-speed curve γ , we can set $u(t) = \gamma'(t)$ so that $\gamma'(t) \cdot \gamma''(t) = 0$ for all $t \in (\alpha, \beta)$. This implies that $\gamma''(t)$ is either zero or perpendicular to $\gamma'(t)$ for all $t \in (\alpha, \beta)$.

3. Prove that any regular curve $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^n$ of class $C^r, r \geq 1$, can be reparametrised in terms of arclength.

Solution. Note that the arclength function s of γ is smooth with $\frac{ds}{dt} = \|\gamma'(t)\| > 0$ for all $t \in (\alpha, \beta)$ since γ is regular. Then s is strictly increasing on (α, β) and its inverse is smooth by the Inverse Function Theorem. Then $s = s(t)$ and $t = t(s)$ (where $t = s^{-1}$) are both smooth on their domains, and one can parametrize s in terms of arclength via

$$\gamma(t) = \gamma(t(s)) = \tilde{\gamma}(s),$$

where we define $\tilde{\gamma} = \gamma \circ t$.

4. Compute the curvature and torsion of the circular helix $\gamma(t) = (4 \cos t, 4 \sin t, 3t), t \in \mathbb{R}$.

Solution. Note that $\gamma'(t) = (-4 \sin t, 4 \cos t, 3)$ with $\|\gamma'(t)\| = 5$, so γ is regular but not unit-speed. Next, we have $\gamma''(t) = (-4 \cos t, -4 \sin t, 0)$ so that

$$\gamma'(t) \times \gamma''(t) = (-4 \sin t, 4 \cos t, 3) \times (-4 \cos t, -4 \sin t, 0) = (12 \sin t, -12 \cos t, 16).$$

This gives us $\|\gamma'(t) \times \gamma''(t)\| = 20$, so the curvature of γ at $\gamma(t)$ is

$$\kappa(t) = \frac{\|\gamma'(t) \times \gamma''(t)\|}{\|\gamma'(t)\|^3} = \frac{20}{5^3} = \frac{4}{25}.$$

Next, we have $\gamma'''(t) = (4 \sin t, -4 \cos t, 0)$ so that

$$(\gamma'(t) \times \gamma''(t)) \cdot \gamma'''(t) = (12 \sin t, -12 \cos t, 16) \cdot (4 \sin t, -4 \cos t, 0) = 48.$$

It follows that the torsion of γ at $\gamma(t)$ is

$$\tau(t) = \frac{(\gamma'(t) \times \gamma''(t)) \cdot \gamma'''(t)}{\|\gamma'(t) \times \gamma''(t)\|^2} = \frac{48}{20^2} = \frac{3}{25}.$$

5. Consider the following unit-speed curve

$$\gamma(s) = \left(\frac{1}{3}(1+s)^{3/2}, \frac{1}{3}(1-s)^{3/2}, \frac{s}{\sqrt{2}} \right), s \in (-1, 1).$$

Find the Frenet frame $\{T, N, B\}$ and then compute the curvature and the torsion at a general point on the curve.

Solution. First, the unit tangent vector to γ at $\gamma(s)$ is

$$T(s) = \gamma'(s) = \left(\frac{1}{2}(1+s)^{1/2}, -\frac{1}{2}(1-s)^{1/2}, \frac{1}{\sqrt{2}} \right).$$

Next, we see that

$$\gamma''(s) = \left(\frac{1}{4}(1+s)^{-1/2}, \frac{1}{4}(1-s)^{1/2}, 0 \right),$$

so the curvature is

$$\kappa(s) = \|\gamma''(s)\| = \frac{1}{\sqrt{8(1-s^2)}} > 0.$$

Then the principal unit normal is

$$N(s) = \frac{\gamma''(s)}{\kappa(s)} = \frac{1}{\sqrt{2}} \left((1-s)^{1/2}, (1+s)^{1/2}, 0 \right).$$

The unit binormal is then

$$B(s) = T(s) \times N(s) = \left(-\frac{1}{2}(1+s)^{1/2}, \frac{1}{2}(1-s)^{1/2}, \frac{1}{\sqrt{2}} \right).$$

Note that

$$\frac{d}{ds}B(s) = \left(-\frac{1}{4}(1+s)^{-1/2}, -\frac{1}{4}(1-s)^{-1/2}, 0 \right) = -\gamma''(s) = -\kappa(s)N(s),$$

which implies that the torsion is

$$\tau(s) = \kappa(s) = \frac{1}{\sqrt{8(1-s^2)}}.$$

6. Let $\gamma(t)$, $t \in (\alpha, \beta)$, be a smooth regular curve in \mathbb{R}^3 . Prove that γ parametrises a straight line if and only if $\kappa(t) = 0$ for all $t \in (\alpha, \beta)$.

Solution. Without loss of generality, we assume that γ is parametrized using arclength so that it is unit-speed. Suppose that $\gamma(t) = \vec{x} + t\vec{v}$ is a straight line in \mathbb{R}^3 . Then we have $\gamma'(t) = \vec{v}$ and $\gamma''(t) = \vec{0}$ so that $\kappa(t) = \|\gamma''(t)\| = 0$ for all $t \in (\alpha, \beta)$. All of these steps are in fact reversible, so we are done.

7. Let $\gamma(t)$ be a smooth regular curve in \mathbb{R}^3 . Prove that $\gamma'(t)$ and $\gamma''(t)$ are linearly dependent (that is, $\gamma''(t) = c(t)\gamma'(t)$ for some smooth function c) if and only if $\gamma(t)$ parametrises a straight line.

Solution. Suppose that $\gamma''(t) = c(t)\gamma'(t)$ for some smooth scalar function c . Then $\gamma'(t) \times \gamma''(t) = \vec{0}$ for all $t \in (\alpha, \beta)$ (since they are parallel), which implies that

$$\kappa(t) = \frac{\|\gamma'(t) \times \gamma''(t)\|}{\|\gamma'(t)\|^3} = \frac{0}{\|\gamma'(t)\|^3} = 0$$

for all $t \in (\alpha, \beta)$. Applying the previous problem, this means that γ parametrizes a straight line. These steps are reversible and thus the converse holds.

8. Let $\gamma(t)$, $t \in (\alpha, \beta)$, be a smooth regular curve in \mathbb{R}^3 such that $\kappa(t) > 0$ for all $t \in (\alpha, \beta)$. Prove that $\gamma(t)$ is a plane curve if and only if its torsion $\tau(t)$ is equal to 0 at every $t \in (\alpha, \beta)$. Moreover, if it is a plane curve, then it is contained in the plane passing through $\gamma(t_0)$ with normal vector $B(t_0)$ for any $t_0 \in (\alpha, \beta)$.

Solution. (\Rightarrow) Suppose that $\gamma(t)$ is a plane curve. Let $t_0 \in (\alpha, \beta)$. Note that γ is a plane curve if and only if every point $\gamma(t)$ is contained in some fixed plane Π in \mathbb{R}^3 . Then $\gamma(t_0) \in \Pi$. If \mathbf{n}_0 is a unit normal to Π , then for any $t \in (\alpha, \beta)$, the vector going from $\gamma(t_0)$ to $\gamma(t)$ is parallel to Π and thus perpendicular to \mathbf{n}_0 . This implies that $\mathbf{n}_0 \cdot (\gamma(t) - \gamma(t_0)) = 0$, or equivalently $\mathbf{n}_0 \cdot \gamma(t) = \mathbf{n}_0 \cdot \gamma(t_0)$ for all $t \in (\alpha, \beta)$. The right-hand side is a constant, so we have

$$0 = \frac{d}{dt}(\mathbf{n}_0 \cdot \gamma(t)) = \frac{d}{dt}(\mathbf{n}_0) \cdot \gamma(t) + \mathbf{n}_0 \cdot \gamma'(t) = \mathbf{n}_0 \cdot T(t).$$

Similarly, we have $\mathbf{n}_0 \cdot \gamma''(t) = 0$ with $\gamma''(t) = \kappa(t)N(t)$ for all $t \in (\alpha, \beta)$, so

$$\mathbf{n}_0 \cdot N(t) = 0.$$

Then $\mathbf{n}_0 \perp T(t)$ and $\mathbf{n}_0 \perp N(t)$ implies that $\mathbf{n}_0 = \pm T(t) \times N(t) = \pm B(t)$. In particular, we have

$$\frac{dB}{dt}(t) = \vec{0}$$

for all $t \in (\alpha, \beta)$, which implies that $\tau(t) = 0$ for all $t \in (\alpha, \beta)$ since $\frac{dB}{dt}(t) = -\tau(t)N(t)$ by the Frenet-Serret equations with $N(t) \neq \vec{0}$ for all $t \in (\alpha, \beta)$.

(\Leftarrow) Suppose that $\tau(t) = 0$ for all $t \in (\alpha, \beta)$. Then $\frac{dB}{dt}(t) = \vec{0}$ and hence $B(t) = \mathbf{n}_0$ for all $t \in (\alpha, \beta)$ for some fixed $\mathbf{n}_0 \in \mathbb{R}^3$. We have $T(t) \perp \mathbf{n}_0$ for all $t \in (\alpha, \beta)$, which implies that $T(t) \cdot \mathbf{n}_0 = 0$. But $T(t) = \gamma'(t)$, and the product rule gives

$$\frac{d}{dt}(\gamma(t) \cdot \mathbf{n}_0) = \gamma'(t) \cdot \mathbf{n}_0 + \gamma(t) \cdot \frac{d}{dt}(\mathbf{n}_0) = T(t) \cdot \mathbf{n}_0 = 0$$

since $\frac{d}{dt}(\mathbf{n}_0) = \vec{0}$ and $T(t) \cdot \mathbf{n}_0 = 0$. It follows that $\gamma(t) \cdot \mathbf{n}_0 = c$ for some constant $c \in \mathbb{R}$. Fixing $t_0 \in (\alpha, \beta)$, we observe that

$$(\gamma(t) - \gamma(t_0)) \cdot \mathbf{n}_0 = \gamma(t) \cdot \mathbf{n}_0 - \gamma(t_0) \cdot \mathbf{n}_0 = c - c = 0$$

for all $t \in (\alpha, \beta)$. This means that γ is contained in the plane passing through $\gamma(t_0)$ normal to $B(t_0) = \mathbf{n}_0$, and γ is a plane curve.

9. Show that $\gamma(t) = (t, 1 + t^{-1}, t^{-1} - t)$, $t > 0$, is a plane curve and find the equation of the plane in which it lies.

Solution. By the previous problem, it is enough to show that $\tau(t) = 0$ for all $t > 0$. We have $\gamma'(t) = (1, -t^{-2}, -t^{-2} - 1)$, $\gamma''(t) = 2t^{-3}(0, 1, 1)$, and $\gamma'''(t) = -6t^{-4}(0, 1, 1)$. This gives us $\gamma'(t) \times \gamma''(t) = 2t^{-3}(1, -1, 1)$ and $(\gamma'(t) \times \gamma''(t)) \cdot \gamma'''(t) = 0$, which implies that $\tau(t) = 0$ for all $t > 0$. Therefore, γ is a plane curve.

To determine the equation of the plane, we solve for $a, b, c, d \in \mathbb{R}$ such that

$$at + b(1 + t^{-1}) + c(t^{-1} - t) = d.$$

This gives us $(b - d) + (a - c)t + (b + c)t^{-1} = 0$, so we have $b = d$, $a = c$, and $b = -c$. Setting $c = 1$, we have that $a = 1$ and $b = d = -1$. Then γ is contained in the plane $x - y + z = -1$.

Alternatively, the unit binormal vector is

$$B(t) = \frac{\gamma'(t) \times \gamma''(t)}{\|\gamma'(t) \times \gamma''(t)\|} = \frac{1}{\sqrt{3}}(1, -1, 1)$$

for all $t \in (\alpha, \beta)$, which is a normal vector to the plane that γ lies in. Since $\gamma(1) = (1, 2, 0)$ is a point on the curve and therefore the plane, we have that

$$(x - 1, y - 2, z) \cdot (1, -1, 1) = 0.$$

This gives the same plane $x - y + z = -1$.

10. State the Fundamental Theorem of Space Curves.

Solution. A rigid motion of \mathbb{R}^3 is a rotation followed by a translation. That is, a rigid motion of \mathbb{R}^3 is an affine map $M : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ of the form $M(v) = Av + b$ where A is an element of the special orthogonal group $\text{SO}(3) = \{C \in M_{3 \times 3}(\mathbb{R}) : C^T C = I_{3 \times 3}, \det C = 1\}$.

The Fundamental Theorem of Space Curves states that given two unit-speed curves γ_1 and γ_2 with the same curvature $\kappa(s)$ and the same torsion $\tau(s)$ for all $s \in (\alpha, \beta)$, there exists a rigid motion M of \mathbb{R}^3 such that $\gamma_2(s) = M(\gamma_1(s))$ for all $s \in (\alpha, \beta)$. Moreover, if k and t are functions of class C^3 with $k > 0$ everywhere, then there exists a unit-speed curve in \mathbb{R}^3 whose curvature is k and whose torsion is t .

11. Show that $\gamma(t) = (2 + \sqrt{2} \cos t, 1 - \sin t, 3 + \sin t)$, $t \in \mathbb{R}$, is a circle.

Solution. By the Fundamental Theorem of Space Curves, every circle has constant (positive) curvature and zero torsion. We verify that this is true for γ . Note that $\gamma'(t) = (-\sqrt{2} \sin t, -\cos t, \cos t)$, $\gamma''(t) = (-\sqrt{2} \cos t, \sin t, -\sin t)$, and $\gamma'''(t) = (\sqrt{2} \sin t, \cos t, -\cos t)$. We have that $\|\gamma'(t)\| = \sqrt{2}$ and $\gamma'(t) \times \gamma''(t) = (0, -\sqrt{2}, -\sqrt{2})$ so that $\|\gamma'(t) \times \gamma''(t)\| = 2$ and $(\gamma'(t) \times \gamma''(t)) \cdot \gamma'''(t) = 0$. This implies that the curvature is

$$\kappa(t) = \frac{\|\gamma'(t) \times \gamma''(t)\|}{\|\gamma'(t)\|^3} = \frac{2}{(\sqrt{2})^3} = \frac{1}{\sqrt{2}} > 0,$$

which is constant, and the torsion is

$$\tau(t) = \frac{(\gamma'(t) \times \gamma''(t)) \cdot \gamma'''(t)}{\|\gamma'(t) \times \gamma''(t)\|^2} = 0.$$

D. Surfaces.

1. Let S be a smooth surface in \mathbb{R}^3 and $\sigma : U \subset \mathbb{R}^2 \rightarrow V \subset S$ be a smooth coordinate chart. Define the following:

- (a) The standard unit normal, first and second fundamental forms of σ .

Solution. Let $p_0 = \sigma(u_0, v_0)$ for some $(u_0, v_0) \in U$. The standard unit normal is

$$N_\sigma(u_0, v_0) = \frac{\sigma_u(u_0, v_0) \times \sigma_v(u_0, v_0)}{\|\sigma_u(u_0, v_0) \times \sigma_v(u_0, v_0)\|}.$$

The first fundamental form of σ is

$$\mathcal{F}_I = \begin{bmatrix} \sigma_u \cdot \sigma_u & \sigma_u \cdot \sigma_v \\ \sigma_v \cdot \sigma_u & \sigma_v \cdot \sigma_v \end{bmatrix},$$

and the second fundamental form of σ is

$$\mathcal{F}_{II} = \begin{bmatrix} \sigma_{uu} \cdot N_\sigma & \sigma_{uv} \cdot N_\sigma \\ \sigma_{vu} \cdot N_\sigma & \sigma_{vv} \cdot N_\sigma \end{bmatrix}.$$

- (b) The principal, normal and geodesic curvatures of σ .

Solution. The principal curvatures of σ are the eigenvalues κ_1 and κ_2 of the shape operator $\mathcal{W} := \mathcal{F}_I^{-1} \mathcal{F}_{II}$. For a unit-speed curve $\gamma : (\alpha, \beta) \rightarrow V \subset S$ contained in the coordinate patch σ and a point $p_0 = \gamma(t_0)$, the normal curvature is

$$\kappa_n := \gamma''(t_0) \cdot N_\sigma,$$

whereas the geodesic curvature is

$$\kappa_g := \gamma''(t_0) \cdot (N_\sigma \times \gamma'(t_0)).$$

- (c) The Gaussian and mean curvatures of σ .

Solution. For the shape operator $\mathcal{W} = \mathcal{F}_I^{-1} \mathcal{F}_{II}$, the Gaussian curvature of σ is $K := \det \mathcal{W}$ and the mean curvature of σ is $H := \frac{1}{2} \operatorname{tr} \mathcal{W}$.

- (d) The geodesics of σ .

Solution. We call a unit-speed curve $\gamma : (\alpha, \beta) \rightarrow V \subset S$ that is contained in the coordinate patch σ a geodesic if $\kappa_g = 0$ everywhere. Equivalently, we have that $\gamma''(t) = \vec{0}$ or $\gamma''(t) \parallel N_\sigma$ for all $t \in (\alpha, \beta)$.

2. Let $\sigma : U \subset \mathbb{R}^2 \rightarrow V \subset S$ and $\tilde{\sigma} : \tilde{U} \subset \mathbb{R}^2 \rightarrow \tilde{V} \subset S$ be smooth coordinate charts of a surface S with $V \cap \tilde{V}$. Set $\Phi := \sigma^{-1} \circ \tilde{\sigma} : \tilde{\sigma}^{-1}(V \cap \tilde{V}) \rightarrow \sigma^{-1}(V \cap \tilde{V})$ so that $\tilde{\sigma} = \sigma \circ \Phi$.

- (a) Show that $N_{\tilde{\sigma}} = \pm N_\sigma$, where \pm is the sign of $\det(D\Phi)$.

Solution. Note that Φ is a smooth diffeomorphism with components

$$(\tilde{u}, \tilde{v}) \mapsto (u(\tilde{u}, \tilde{v}), v(\tilde{u}, \tilde{v})).$$

By the chain rule, we have $D\tilde{\sigma} = D\sigma D\Phi$, or more explicitly

$$[\tilde{\sigma}_{\tilde{u}} \mid \tilde{\sigma}_{\tilde{v}}] = [\sigma_u \mid \sigma_v] D\Phi.$$

Here, $D\Phi$ is the change of basis matrix of $T_p S$. Note that

$$D\Phi = \begin{bmatrix} \partial u / \partial \tilde{u} & \partial u / \partial \tilde{v} \\ \partial v / \partial \tilde{u} & \partial v / \partial \tilde{v} \end{bmatrix}.$$

Moreover, we have

$$\begin{aligned} \tilde{\sigma}_{\tilde{u}} &= \sigma_u \cdot \frac{\partial u}{\partial \tilde{u}} + \sigma_v \cdot \frac{\partial v}{\partial \tilde{u}}, \\ \tilde{\sigma}_{\tilde{v}} &= \sigma_u \cdot \frac{\partial u}{\partial \tilde{v}} + \sigma_v \cdot \frac{\partial v}{\partial \tilde{v}}, \end{aligned}$$

which implies that

$$\begin{aligned} \tilde{\sigma}_{\tilde{u}} \times \tilde{\sigma}_{\tilde{v}} &= \left(\sigma_u \cdot \frac{\partial u}{\partial \tilde{u}} + \sigma_v \cdot \frac{\partial v}{\partial \tilde{u}} \right) \times \left(\sigma_u \cdot \frac{\partial u}{\partial \tilde{v}} + \sigma_v \cdot \frac{\partial v}{\partial \tilde{v}} \right) \\ &= (\sigma_u \times \sigma_u) \left(\frac{\partial u}{\partial \tilde{u}} \frac{\partial v}{\partial \tilde{v}} \right) + (\sigma_u \times \sigma_v) \left(\frac{\partial u}{\partial \tilde{u}} \frac{\partial v}{\partial \tilde{u}} - \frac{\partial v}{\partial \tilde{u}} \frac{\partial u}{\partial \tilde{v}} \right) + (\sigma_v \times \sigma_v) \left(\frac{\partial v}{\partial \tilde{u}} \frac{\partial v}{\partial \tilde{v}} \right) \\ &= \left(\frac{\partial u}{\partial \tilde{u}} \frac{\partial v}{\partial \tilde{v}} - \frac{\partial v}{\partial \tilde{u}} \frac{\partial u}{\partial \tilde{v}} \right) (\sigma_u \times \sigma_v) = (\det D\Phi) (\sigma_u \times \sigma_v), \end{aligned}$$

where we used the fact that $\sigma_u \times \sigma_u = \sigma_v \times \sigma_v = \vec{0}$. It follows that

$$N_{\tilde{\sigma}} = \frac{\tilde{\sigma}_{\tilde{u}} \times \tilde{\sigma}_{\tilde{v}}}{\|\tilde{\sigma}_{\tilde{u}} \times \tilde{\sigma}_{\tilde{v}}\|} = \frac{\det D\Phi}{|\det D\Phi|} \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = \pm N_{\sigma},$$

where \pm is the sign of $\det D\Phi$.

- (b) Recall that if $\mathcal{F}_I, \mathcal{F}_{II}$ and $\tilde{\mathcal{F}}_I, \tilde{\mathcal{F}}_{II}$ are the first and second fundamental forms of σ and $\tilde{\sigma}$, respectively, then

$$\tilde{\mathcal{F}}_I = (D\Phi)^T \mathcal{F}_I (D\Phi)$$

and

$$\tilde{\mathcal{F}}_{II} = \pm (D\Phi)^T \mathcal{F}_{II} (D\Phi)$$

where \pm is the sign of $\det(D\Phi)$.

- i. Let \mathcal{W} and $\tilde{\mathcal{W}}$ be the shape operators associated to σ and $\tilde{\sigma}$ respectively. Show that $\tilde{\mathcal{W}} = \pm (D\Phi)^{-1} \mathcal{W} (D\Phi)$, where \pm is the sign of $\det(D\Phi)$.

Solution. Observe that

$$\begin{aligned} \tilde{\mathcal{W}} &= \tilde{\mathcal{F}}_I^{-1} \tilde{\mathcal{F}}_{II} = \pm [(D\Phi)^T \mathcal{F}_I (D\Phi)]^{-1} [(D\Phi)^T \mathcal{F}_{II} (D\Phi)] \\ &= \pm (D\Phi)^{-1} \mathcal{F}_I^{-1} ((D\Phi)^T)^{-1} (D\Phi)^T \mathcal{F}_{II} (D\Phi) \\ &= \pm (D\Phi)^{-1} \mathcal{F}_I^{-1} \mathcal{F}_{II} (D\Phi) \\ &= \pm (D\Phi)^{-1} \mathcal{W} (D\Phi), \end{aligned}$$

where \pm is the sign of $\det D\Phi$.

- ii. Show that $\tilde{K} = K$ and $\tilde{H} = \pm H$, i.e., Gaussian curvature is invariant under reparametrisations, whereas mean curvature may change sign under reparametrisations. Moreover, show that the principal curvatures change as follows under reparametrisations: $\tilde{\kappa}_1 = \pm \kappa_1$ and $\tilde{\kappa}_2 = \pm \kappa_2$.

Solution. Using the previous part and noting that we are working with 2×2 matrices, we have by the multiplicativity of determinant that

$$\tilde{K} = \det \tilde{\mathcal{W}} = (\pm 1)^2 \det((D\Phi)^{-1} \mathcal{W} (D\Phi)) = \det \mathcal{W} = K.$$

Similarly, we know that similar matrices have the same trace, so

$$\tilde{H} = \frac{1}{2} \operatorname{tr} \tilde{\mathcal{W}} = \frac{1}{2} \operatorname{tr}(\pm(D\Phi)^{-1}\mathcal{W}(D\Phi)) = \pm \frac{1}{2} \operatorname{tr} \mathcal{W} = H.$$

Next, we use the fact that similar matrices have the same eigenvalues. The principal curvatures are the eigenvalues of the shape operator, so κ_1 and κ_2 are the eigenvalues of \mathcal{W} whereas $\tilde{\kappa}_1$ and $\tilde{\kappa}_2$ are the eigenvalues of $\tilde{\mathcal{W}}$. If $\tilde{\mathcal{W}} = (D\Phi)^{-1}\mathcal{W}(D\Phi)$, then $\tilde{\mathcal{W}}$ has the same eigenvalues as \mathcal{W} so that (up to reordering) $\tilde{\kappa}_1 = \kappa_1$ and $\tilde{\kappa}_2 = \kappa_2$. If we had $\tilde{\mathcal{W}} = -(D\Phi)^{-1}\mathcal{W}(D\Phi)$, then this means that $\tilde{\mathcal{W}}$ and $-\mathcal{W}$ are similar and have the same eigenvalues. This implies that $\tilde{\kappa}_1 = -\kappa_1$ and $\tilde{\kappa}_2 = -\kappa_2$.

- (c) A smooth surface is called *flat* if its Gaussian curvature is everywhere zero. Use (b) to show that the notion of flat surface is independent of the atlas chosen.

Solution. Consider the two coordinate charts σ and $\tilde{\sigma}$ with $V \cap \tilde{V} \neq \emptyset$. By part (b), we know that $K = \tilde{K}$. Then the Gaussian curvature is independent of the choice of coordinate chart, and therefore the notion of flatness is independent of the choice of atlas.

- (d) A smooth surface is called *minimal* if its mean curvature is everywhere zero. Use (b) to show that the notion of minimal surface is independent of the atlas chosen.

Solution. By part (b), we have $H = \tilde{H}$. In particular, if $H = 0$ everywhere, then $\tilde{H} = 0$ everywhere as well. This means that the notion of minimal surface is independent of the choice of atlas.

- (e) Show that the Gaussian curvature of a minimal surface is ≤ 0 everywhere.

Solution. By part (d), the notion of minimal surface is independent of the choice of atlas. Therefore, take any atlas and let σ be a coordinate chart in the atlas. Let κ_1 and κ_2 be the principal curvatures of σ . For a minimal surface, we have that $H = \frac{1}{2}(\kappa_1 + \kappa_2) = 0$, which implies that $\kappa_1 = -\kappa_2$. This implies that $K = \kappa_1\kappa_2 = -\kappa_1^2 \leq 0$.

- (f) Show that if a unit-speed curve on a smooth surface has zero normal *and* geodesic curvatures everywhere, it is part of a straight line.

Solution. We know that $\kappa^2 = \kappa_n^2 + \kappa_g^2$ for a unit-speed curve, so if $\kappa_n = \kappa_g = 0$ everywhere, then $\kappa = 0$ everywhere. By Problem C.6, the curve parametrizes a straight line.

3. Let S be the graph of a smooth function $f : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ with U an open subset of \mathbb{R}^2 . Show that at a critical point $(x_0, y_0) \in U$ of f , the Gaussian curvature of S is equal to the determinant of the Hessian matrix $H(f)(x_0, y_0)$ of f at (x_0, y_0) .

Solution. Note that a coordinate chart for S is $\sigma : U \subset \mathbb{R}^2 \rightarrow S \subset \mathbb{R}^3$ defined by

$$(u, v) \mapsto (u, v, f(u, v)).$$

We have that $\sigma_u = (1, 0, f_u)$ and $\sigma_v = (0, 1, f_v)$, so the first fundamental form of σ is

$$\mathcal{F}_I = \begin{bmatrix} 1 + (f_u)^2 & f_u f_v \\ f_u f_v & 1 + (f_v)^2 \end{bmatrix}.$$

Next, we see that $\sigma_u \times \sigma_v = (-f_u, -f_v, 1)$ and $\|\sigma_u \times \sigma_v\| = \sqrt{1 + (f_u)^2 + (f_v)^2}$, so

$$N_\sigma = \frac{1}{\sqrt{1 + (f_u)^2 + (f_v)^2}}(-f_u, -f_v, 1).$$

Moreover, we have $\sigma_{uu} = (0, 0, f_{uu})$, $\sigma_{uv} = (0, 0, f_{uv})$ and $\sigma_{vv} = (0, 0, f_{vv})$ so that

$$\mathcal{F}_{II} = \frac{1}{\sqrt{1 + (f_u)^2 + (f_v)^2}} \begin{bmatrix} f_{uu} & f_{uv} \\ f_{vu} & f_{vv} \end{bmatrix} = \frac{1}{\sqrt{1 + (f_u)^2 + (f_v)^2}} H(f).$$

At a critical point $(u_0, v_0) \in U$ of f , we have that

$$f_u(u_0, v_0) = f_v(u_0, v_0) = 0.$$

At such a point, we have $\mathcal{F}_I = I_{2 \times 2}$ and $\mathcal{F}_{II} = H(f)$ so that the Gaussian curvature is

$$K = \det \mathcal{W} = \det H(f).$$

4. Let S be the smooth surface given by $z = ax^2 + by^2$, $a, b \in \mathbb{R}$. Show that:
 - (a) Every point on S is elliptic if and only if $ab > 0$, in which case S is an elliptic paraboloid.
 - (b) Every point on S is hyperbolic if and only if $ab < 0$, in which case S is a hyperbolic paraboloid (that is, a saddle surface).
 - (c) Every point on S is parabolic if and only if $ab = 0$ and $a \neq 0$ or $b \neq 0$, in which case S is a parabolic cylinder.
 - (d) Every point on S is planar if and only if $a = b = 0$, in which case S is the xy -plane ($z = 0$).
5. Classify all the points on the regular surface patch $\sigma(u, v) = (u, v, u^3 + v^2)$, $(u, v) \in \mathbb{R}^2$, as to being elliptic, hyperbolic, parabolic, or planar.
6. Let S be the circular cylinder $x^2 + y^2 = 1$. Consider the atlas $\{(S_i, \sigma_i)\}_{i=1}^2$ of S , where each S_i is the image in \mathbb{R}^3 of the smooth coordinate charts

$$\sigma_i(u, v) = (\cos u, \sin u, v),$$

with $(u, v) \in (0, 2\pi) \times \mathbb{R}$ when $i = 1$, and $(u, v) \in (-\pi, \pi) \times \mathbb{R}$ when $i = 2$.

- (a) Find the principal curvatures κ_1, κ_2 of S at every point.
 - (b) Show that the normal curvature κ_n of any unit-speed curve γ on S is such that

$$-1 \leq \kappa_n \leq 0.$$
 - (c) Is the cylinder S a flat surface? a minimal surface? Justify your answer. (See problem 2 for a definition of *flat* and *minimal*.)
7. Let $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^3$ be a smooth unit-speed curve whose curvature is everywhere non-zero, and is such that the map

$$\sigma(v, s) = \gamma(s) + v\gamma'(s), s \in (\alpha, \beta), v > 0,$$
 is a homeomorphism.
 - (a) Show that σ is a smooth embedding.
 - (b) Show that the smooth surface determined by σ is flat. (See problem 2 for a definition of *flat*.)
Hint: Start by showing that $L = M = 0$ everywhere. Conclude that $K = 0$ everywhere.
8. Let $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^3$ be a smooth whose curvature is everywhere non-zero, and let $\{T, N, B\}$ be its Frenet frame. Suppose that the map

$$\sigma(v, s) = \gamma(s) + vB(s),$$

$s \in (\alpha, \beta), v \in (-\epsilon, \epsilon)$, is a homeomorphism for small enough $\epsilon > 0$. Let S be image of σ in \mathbb{R}^3 .

- (a) Show that S is a smooth surface.
 - (b) Show that γ is a geodesic of σ and that every point on γ is either hyperbolic or parabolic.

9. (a) Let γ be a unit-speed curve on a smooth surface S . Show that γ is a geodesic if and only if $\kappa_g = 0$.
- (b) Prove or disprove (by giving a counterexample) the following statements:
 - i. If the unit-speed curve γ is a plane curve, then $\kappa_g = \pm\kappa$.
 - ii. If the unit-speed curve γ is a geodesic, then $\kappa_n = \pm\kappa$.
 - iii. Let p and q be two points on a smooth surface S , and let γ be a geodesic joining p and q . Then, γ is the path of shortest length from p to q .
- (c) Give an example of an isometry which is not a translation, rotation, or reflection.
- (d) Describe all geodesics on a plane, a sphere and a cylinder.

E. Integration.

1. Let M be a smooth k -dimensional submanifold of \mathbb{R}^n and $\alpha : U \subset \mathbb{R}^k \rightarrow V \subset M$ be a smooth coordinate chart of M . Assume that U is bounded. Define the following:
 - (a) The first fundamental form of α .
 - (b) The scalar integral of a smooth function f on U .
 - (c) The line integral of a smooth vector field on a smooth curve.
 - (d) A hypersurface in \mathbb{R}^n .
 - (e) An orientation of a hypersurface in \mathbb{R}^n .
 - (f) The integral of a smooth vector field on a hypersurface in \mathbb{R}^n .
2. Let $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^n$ be a smooth regular curve. Verify that the formula for the integral of a scalar function on γ is equivalent to

$$\int_{\gamma} f \, ds = \int_{\alpha}^{\beta} f(\gamma(t)) \|\gamma'(t)\| dt$$

for any smooth scalar function $f : (\alpha, \beta) \rightarrow \mathbb{R}$.

3. Let S be a surface in \mathbb{R}^3 and $\sigma : U \subset \mathbb{R}^2 \rightarrow V \subset S$ be a smooth coordinate chart with V bounded. Show that the formulas for the integrals of a scalar function or a vector field on V can be expressed in terms of σ as:
 - (a) $\int_V f \, dVol = \int_U f(\sigma(u, v)) \|\sigma_u \times \sigma_v\| du dv$ for any smooth scalar function $f : V \rightarrow \mathbb{R}$.
 - (b) $\int_V F \cdot dVol = \int_U F(\sigma(u, v)) \cdot (\sigma_u \times \sigma_v) du dv$ with respect to the orientation N_{σ} on V .
4. Prove that any smooth hypersurface M in \mathbb{R}^n is locally orientable. That is, for any point $p \in M$, there exists an open neighbourhood V of p in M such that V is an orientable hypersurface.
5. Prove that if M be a smooth bounded n -dimensional submanifold of \mathbb{R}^n , then $\int_M f dVol$ is just the standard multiple integral $\int_M f(x_1, \dots, x_n) dx_1 dx_2 \cdots dx_n$ for any smooth scalar function f on M .
6. (a) Evaluate the line integral $\int_{\gamma} x^3 y ds$ along part of the line $\gamma(t) = (t, -t)$, $t \in [-1, 2]$, in \mathbb{R}^2 .
- (b) Evaluate the line integral $\int_{\gamma} F \cdot dr$ of the vector field $F(x, y, z) = (yz, xz, xy)$ along part of the twisted cubic $\gamma(t) = (t, t^2, t^3)$, $t \in [0, 1]$, in \mathbb{R}^3 .
- (c) Evaluate the surface integral $\int_S 6xz dA$ where

$$S = \{(x, y, z) \in \mathbb{R}^3 : -1 \leq x \leq 3, 0 \leq y \leq 4, z = 1 - x - y\}$$

is the portion of the plane $x + y + z = 1$ lying above the rectangle $[-1, 3] \times [0, 4]$ in the xy -plane.

- (d) Evaluate the surface integral $\int_S y dA$ where S is the portion of the cylinder $x^2 + y^2 = 3$ that lies between $z = 0$ and $z = 6$.
- (e) Evaluate the integral $\int_R z dV$ where R is the rectangular box $[-1, 4] \times [0, 3] \times [-2, 5]$ in \mathbb{R}^3 .
- (f) Evaluate the integral of the vector field $F(x, y, z) = (-y, x, z)$ on the piece of paraboloid S in \mathbb{R}^3 parametrised $\sigma(u, v, u^2 + v^2)$, $(u, v) \in [0, 1] \times [0, 3]$, with respect to the orientation N_σ .
- (g) Evaluate the integral of the vector field $F(x, y, z) = (xy, 2, z - 1)$ on the half-cylinder C in \mathbb{R}^3 parametrised $\sigma(\cos u, \sin u, v)$, $(u, v) \in [0, \pi] \times [2, 4]$, with respect to the orientation N_σ .