PMATH 365 Course Notes

Differential Geometry

Ruxandra Moraru • Winter 2023 • University of Waterloo

Table of Contents

1 Submanifolds of $\mathbb{R}^n$

1.1 Preliminaries

To begin, we'll recall some facts about the topology of $\mathbb{R}^n$ and vector-valued functions.

In this course, we'll be working with the metric topology with respect to the Euclidean norm (or metric). Let $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in \mathbb{R}^n$. The **Euclidean norm** is defined to be

$$\|x\| = \sqrt{x_1^2 + \cdots + x_n^2},$$

and **Euclidean distance** is given by

$$\operatorname{dist}(x, y) = \|y - x\| = \sqrt{(y_1 - x_1)^2 + \cdots + (y_n - x_n)^2}.$$

We define the **open ball** of radius $r > 0$ centered at $x \in \mathbb{R}^n$ by

$$B_r(x) := \{y \in \mathbb{R}^n : \operatorname{dist}(x, y) < r\} \subset \mathbb{R}^n.$$

A **topology** on $\mathbb{R}^n$ is a collection $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ of subsets $U_\alpha \subset \mathbb{R}^n$ that satisfy the following properties.

(i) $\varnothing$ and $\mathbb{R}^n$ are in $\mathcal{U}$.

(ii) For any subcollection $\mathcal{V} = \{U_\beta\}_{\beta \in B}$ with $U_\beta \in \mathcal{U}$ for all $\beta \in B$, we have $\bigcup_{\beta \in B} U_\beta \in \mathcal{U}$.

(iii) For any *finite* subcollection $\{U_{\alpha_1}, \dots, U_{\alpha_m}\} \subset \mathcal{U}$, we have $\bigcap_{i=1}^m U_{\alpha_i} \in \mathcal{U}$.

The sets $U_\alpha \in \mathcal{U}$ are called the **open sets** of the topology; their complements $F_\alpha = \mathbb{R}^n \setminus U_\alpha$ are called the **closed sets**.

Note that the sets $\varnothing$ and $\mathbb{R}^n$ are both open and closed. Moreover, the notion of a topology can be extended to more general sets X, not just $\mathbb{R}^n$. A topology can also be defined starting with closed sets, but we prefer to work with open sets because many nice properties, such as differentiability, are better described with them.

Under the metric topology, we say that a set $A \subset \mathbb{R}^n$ is **open** if $A = \varnothing$ or if for all $p \in A$, there exists $r > 0$ such that $B_r(p) \subset A$. Moreover, A is **closed** if its complement $A^c = \mathbb{R}^n \setminus A$ is open. (We leave it as an exercise to show that this is indeed a topology.)

For example, the open balls $B_r(x)$ are open sets for all $x \in \mathbb{R}^n$ and $r > 0$. Indeed, for any point $p \in B_r(x)$, one sees that by picking $r' = (r - \|p - x\|)/2$, we have $B_{r'}(p) \subset B_r(x)$.

In general, open sets are described with strict inequalities, while closed sets are described using equality or inclusive inequalities. However, note that most sets are neither open nor closed, such as the half-open interval $U = (-1, 1]$ over $\mathbb{R}$.

The metric topology is not the only topology on $\mathbb{R}^n$; one example is the one consisting of only the sets $\mathcal{U} = \{\varnothing, \mathbb{R}^n\}$. However, we generally want more open sets to work with since we might want to know the behaviour of functions around a point $p \in \mathbb{R}^n$. If the only non-empty open set we had was $\mathbb{R}^n$, then this would apply to all points in $\mathbb{R}^n$, which does not yield a lot of information.

Let $p \in \mathbb{R}^n$. The previous paragraph leads us to the definition of an **open neighbourhood** of p, which is just an open set $U \subset \mathbb{R}^n$ such that $p \in U$.

We now turn our discussion to vector-valued functions. Let $U \subset \mathbb{R}^n$ and consider the vector-valued function

$$\begin{aligned} F : U \subset \mathbb{R}^n &\to B \subset \mathbb{R}^m \\ x = (x_1, \dots, x_n) &\mapsto (F_1(x), \dots, F_m(x)). \end{aligned}$$

Then F is continuous if and only if the component functions $F_i : U \to \mathbb{R}$ are continuous for all $i = 1, \dots, m$.

We say that F is a **homeomorphism** if it is a continuous bijection whose inverse

$$F^{-1} : B \subset \mathbb{R}^m \to U \subset \mathbb{R}^n$$

is also continuous. For example, the identity map $\mathrm{Id}_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n$ and the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$ are both homeomorphisms.

It is a known fact that homeomorphisms map open sets to open sets and closed sets to closed sets. This follows from the topological characterization of continuity, which states that F is continuous if and only if for every open (respectively closed) set $V \subset \mathbb{R}^m$, we have that $F^{-1}(V)$ is open (respectively closed). In fact, homeomorphisms preserve much more structure than this, as we'll see later.

1.2 Topological Submanifolds of $\mathbb{R}^n$

We now define the main object we'll be working with in this course.

Definition 1.1

A **k-dimensional topological submanifold** (or **topological k-submanifold**) of $\mathbb{R}^n$ is a subset $M \subset \mathbb{R}^n$ such that for every $p \in M$, there exists an open neighbourhood V of p in $\mathbb{R}^n$, an open set $U \subset \mathbb{R}^k$, and a homeomorphism

$$\alpha : U \subset \mathbb{R}^k \to V \cap M \subset \mathbb{R}^n.$$

The homeomorphism α is called a **coordinate chart** (or **patch**) on M.

Note that the open neighbourhood $V \subset \mathbb{R}^n$ of p, the open set $U \subset \mathbb{R}^k$, and the map α do not need to be unique. But we'll see later that the dimension k must be unique and is completely determined by M.

For example, $\mathbb{R}^n$ is a topological n-submanifold of $\mathbb{R}^n$ by taking $U = V = \mathbb{R}^n$ and $\alpha = \mathrm{Id}_{\mathbb{R}^n}$. Any open set $W \subset \mathbb{R}^n$ is a topological n-submanifold of $\mathbb{R}^n$ by taking $U = V = W$ and $\alpha = \mathrm{Id}_W$.

Let's now consider some non-trivial examples. Consider

$$M = \{(x, y) \in \mathbb{R}^2 : y = x^2\} \subset \mathbb{R}^2,$$

which is the graph of the parabola $f(x) = x^2$. Then M is a topological 1-submanifold of $\mathbb{R}^2$ by considering the map $\alpha : \mathbb{R}^1 \to M \subset \mathbb{R}^2$, $t \mapsto (t, t^2)$. The inverse $\alpha^{-1} : M \to \mathbb{R}^1$ is just the projection of the first coordinate, which is continuous.

More generally, let $U \subset \mathbb{R}^k$ be an open set. Consider the graph of a continuous function

$$F : U \subset \mathbb{R}^k \to \mathbb{R}^{n-k},\ x \mapsto (F_1(x), \ldots, F_{n-k}(x)).$$

In other words, we are looking at the set

$$G = \{(x, y) \in \mathbb{R}^k \times \mathbb{R}^{n-k} : y = F(x),\ x \in U\} \subset \mathbb{R}^n.$$

We claim that G is a k-dimensional topological submanifold of $\mathbb{R}^n$. To see this, define $\alpha : U \subset \mathbb{R}^k \to G \subset \mathbb{R}^n$ by $x \mapsto (x, F(x))$. Then α is continuous since F is continuous, and it is a bijection since we are restricted to G. Moreover, it has continuous inverse $\alpha^{-1} : G \subset \mathbb{R}^n \to U \subset \mathbb{R}^k$, $(x, y) \mapsto x$.

Here are two more examples of this in action.

(1) Let $M = \{(x, y, z) \in \mathbb{R}^3 : z = x^2 + y^2\} \subset \mathbb{R}^3$. Then M is the graph of the continuous function $f(x, y) = x^2 + y^2$, so it is a 2-dimensional topological submanifold of $\mathbb{R}^3$.

(2) Observe that $M = \{(x, y, z) \in \mathbb{R}^3 : y = x^2,\ z = x^3\} \subset \mathbb{R}^3$ is the graph of the continuous function $F(t) = (t^2, t^3)$, so it is a 1-dimensional topological submanifold of $\mathbb{R}^3$.

In all the examples above, we only needed one coordinate chart which worked for all points. However, this is not always the case! Consider the unit circle

$$\mathbb{S}^1 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \subset \mathbb{R}^2.$$

Note that $\mathbb{S}^1$ is compact. Therefore, by Heine-Borel, it is closed and bounded. Recall that homeomorphisms preserve closed sets, so it is impossible to find a unique chart α. Indeed, if we had such a homeomorphism $\alpha : U \subset \mathbb{R}^1 \to \mathbb{S}^1 \subset \mathbb{R}^2$ for some open set U, then $U = \alpha^{-1}(\mathbb{S}^1)$ would be a compact subset of $\mathbb{R}^1$. But the only open and compact subset of $\mathbb{R}^n$ is $\varnothing$, which is a contradiction!

Nonetheless, two coordinate charts are enough to cover all points on $\mathbb{S}^1$. Define

$$\begin{aligned} V_1 &= \mathbb{R}^2 \setminus \{(x, 0) \in \mathbb{R}^2 : x \leq 0\}, \\ V_2 &= \mathbb{R}^2 \setminus \{(x, 0) \in \mathbb{R}^2 : x \geq 0\}, \end{aligned}$$

which are both open sets. Then the homeomorphism

$$\alpha_1 : U_1 = (-\pi, \pi) \to \mathbb{S}^1 \cap V_1,\ t \mapsto (\cos t, \sin t)$$

covers all points on $\mathbb{S}^1$ except for $(-1, 0)$, while

$$\alpha_2 : U_2 = (0, 2\pi) \to \mathbb{S}^1 \cap V_2,\ t \mapsto (\cos t, \sin t)$$

covers all points on $\mathbb{S}^1$ except for $(1, 0)$.

2 Curves in $\mathbb{R}^n$

Tangent spaces: Let $M \subset \mathbb{R}^n$ with class C^r of dimension k. For a chart $\alpha : U \subset \mathbb{R}^k \to V \subset M \subset \mathbb{R}^n$, we have for all $p \in V$ that

$$T_p(M) := \alpha_*(T_{x_0}(\mathbb{R}^k)),$$

where $p = \alpha(x_0)$. We showed that

$$T_p(M) = \operatorname{span}_{\mathbb{R}} \left\{ \frac{\partial \alpha}{\partial x_1}(x_0), \ldots, \frac{\partial \alpha}{\partial x_k}(x_0) \right\} \subset T_p(\mathbb{R}^n),$$

which is a k-dimensional subspace of $T_p(\mathbb{R}^n)$.

Examples:

(1) Let $U \subset \mathbb{R}^n$ be an open set. Then $\alpha : U \subset \mathbb{R}^n \to V = U \subset \mathbb{R}^n$ given by $x \mapsto x$ yields $D\alpha(x) = I_{n\times n}$ for all $x \in U$, so

$$T_x(U) = \alpha_*(T_x(\mathbb{R}^n))$$

by definition. Then for all $(x; v) \in T_x(\mathbb{R}^n)$, we get

$$\alpha_*(x; v) = (\alpha(x); D\alpha(x)v) = (x; v).$$

That is, we have $T_x(U) = T_x(\mathbb{R}^n)$.

(2) We saw two different ways of seeing if something is a submanifold. If $f : U \subset \mathbb{R}^k \to \mathbb{R}^{n-k}$ is a function of a class C^r and

$$M = \{(x, f(x)) \in \mathbb{R}^n : x \in U\} \subset \mathbb{R}^n$$

its graph, then M is a k-dimensional submanifold of $\mathbb{R}^n$ of class C^r. We can parameterize all points in M with the map $\alpha : U \subset \mathbb{R}^k \to M \subset \mathbb{R}^n$ defined by $\alpha(x) = (x, f(x))$. The derivative matrix is

$$D\alpha(x) = \left[\begin{array}{c} I_{k\times k} \\ \hline Df(x) \end{array}\right]$$

for all $x \in U$. Let $p \in M$ so that $p = (x_0, f(x_0))$ for some $x_0 \in U$. Then

$$\begin{aligned} T_p(M) &= \alpha_*(T_{x_0}(\mathbb{R}^k)) \\ &= \{\alpha_*(x_0; v) : v \in T_{x_0}(\mathbb{R}^k)\} \\ &= \{(\alpha(x_0); D\alpha(x_0)v) : v \in \mathbb{R}^k\} \\ &= \{(p; w) : w = (v, Df(x_0)v),\ v \in \mathbb{R}^k\}, \end{aligned}$$

since $\alpha(x_0) = p$ and $D\alpha(x)$ is the block matrix with $I_{k\times k}$ upstairs and $Df(x)$ downstairs. Also,

$$T_p(M) = \operatorname{span}_{\mathbb{R}} \left\{ \frac{\partial \alpha}{\partial x_1}(x_0), \ldots, \frac{\partial \alpha}{\partial x_k}(x_0) \right\} = \operatorname{span}_{\mathbb{R}} \left\{ \left[\begin{array}{c} e_i \\ \hline \partial f(x_0)/\partial x_i \end{array}\right] : i = 1, \ldots, k \right\}.$$

(3) Let $U \subset \mathbb{R}^n$ be open. If $F : U \subset \mathbb{R}^n \to \mathbb{R}^{n-k}$ is a function of class C^r with $DF(p)$ having rank $n-k$ for all $p \in U$, then

$$M = \{x \in U : F(x) = 0\}$$

is a k-dimensional submanifold of $\mathbb{R}^n$ of class C^r. In this case, we leave it as an exercise to show that

$$T_p(M) = \ker(DF(p)).$$

In particular, if $k = n-1$, then $F : U \subset \mathbb{R}^n \to \mathbb{R}$ is a scalar function and

$$T_p(M) = \ker(\nabla F(p)).$$

Then $\nabla F(p)$ is the normal vector of $T_p(M)$.

For example, take the n-sphere

$$S^n = \{x \in \mathbb{R}^{n+1} : \|x\|^2 = 1\},$$

which is the zero set of the function $F : \mathbb{R}^{n+1} \to \mathbb{R}$ defined by $F(x) = \|x\|^2 - 1 = x_1^2 + \cdots + x_{n+1}^2 - 1$. The derivative matrix is just the gradient; that is,

$$DF(x) = \nabla F(x) = \begin{bmatrix} 2x_1 & \cdots & 2x_{n+1} \end{bmatrix} = 2x.$$

2.1

What is a curve? Intuitively, it is a 1-dimensional subset of $\mathbb{R}^n$.

The level sets $f(x, y) = k$ of a two variable function in $\mathbb{R}^2$ are curves. For example, for $f(x, y) = x^2 + y^2$, we see that $C\colon x^2 + y^2 = k$ for $k > 0$ is a circle centered at $(0, 0)$ of radius $k^{1/2}$.

The intersection of two surfaces in $\mathbb{R}^3$ is also a curve. For example, take $z = x^2 + y^2$ and $z = 2$. Their intersection is the circle $C\colon x^2 + y^2 = 2$ in the plane $z = 2$.

For our purposes, we'll work with parametrized curves $\gamma : I = (\alpha, \beta) \subset \mathbb{R} \to \mathbb{R}^n$ of class C^r. (In practice, we need $r \geq n$ when working over $\mathbb{R}^n$.)

Example: **Circular helix.** Let $\gamma(t) = (a \cos t, a \sin t, bt)$ for $t \in \mathbb{R}$ and $a, b > 0$. Note that $x^2 + y^2 = a^2$, so $\gamma(t)$ lies above the circle $x^2 + y^2 = a^2$ in the xy-plane. We have $\gamma(0) = (a, 0, 0)$ and $\gamma(\pi/2) = (0, a, b\pi/2)$. (The circular helix looks like a spiral along the cylinder.)

Definition 2.1

A parameterized curve $\gamma : (\alpha, \beta) \to \mathbb{R}^n$ of class C^r is called **regular** if for all $t \in (\alpha, \beta)$, we have

$$\gamma'(t) = \frac{\mathrm{d}\gamma}{\mathrm{d}t}(t) \neq 0.$$

We call $\|\gamma'(t)\|$ the **speed of** γ **at** $\gamma(t)$ and we say that γ is **unit speed** if $\|\gamma'(t)\| = 1$ for all $t \in (\alpha, \beta)$.

Note that unit speed implies regular because $\|x\| = 0$ if and only if $x = 0$.

Example: Let $a > 0$ and take $\gamma(t) = (a \cos t, a \sin t)$ for $t \in \mathbb{R}$, which is a parametrization of the circle of radius a. Then $\gamma'(t) = (-a \sin t, a \cos t) \neq (0, 0)$ for all $t \in \mathbb{R}$ and $\|\gamma'(t)\| = a$, so γ is unit speed if and only if $a = 1$.