PMATH 352 Course Notes

Complex Analysis

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1 Introduction

Complex analysis is the study of complex functions

$$f : \mathbb{C} \to \mathbb{C}.$$

Recall that the complex plane $\mathbb{C}$ is defined as the set of all complex numbers; that is,

$$\mathbb{C} := \{z = x + yi : x, y \in \mathbb{R}\}.$$

Moreover, $\mathbb{C}$ can be identified with $\mathbb{R}^2$ via

$$\mathbb{C} \ni z = x + yi \leftrightarrow (x, y) \in \mathbb{R}^2.$$

To keep notation consistent, we denote any point in $\mathbb{R}^2$ by $z = (x, y)$, which corresponds to $z = x + yi$ in $\mathbb{C}$.

Also, recall that for any complex number $z = x + yi \in \mathbb{C}$, we have that $x = \Re(z)$ is the **real part** of z, while $y = \Im(z)$ is the **imaginary part** of z.

Given a complex function $f : \mathbb{C} \to \mathbb{C}$ and a complex number $z_0 = x_0 + y_0 i \in \mathbb{C}$, the value of f at z_0 is of the form

$$f(z_0) = w_0$$

for some $w_0 = u_0 + v_0 i \in \mathbb{C}$, where $u_0, v_0 \in \mathbb{R}$. In particular, we have

$$f(z) = u(z) + v(z)i$$

for some real-valued functions $u, v : \mathbb{C} \to \mathbb{R}$. Further identifying $\mathbb{C}$ with $\mathbb{R}^2$, we can then think of f as the mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x, y) = (u(x, y), v(x, y))$.

Hence, understanding complex functions means understanding mappings

$$f : \mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto (u(x, y), v(x, y)).$$

Indeed, many properties of complex functions $f : \mathbb{C} \to \mathbb{C}$ can be characterized in terms of the component functions u and v of their corresponding mapping $\mathbb{R}^2 \to \mathbb{R}^2$. For instance:

- f is defined to be **continuous** at the point $z_0 = x_0 + y_0 i$ if and only if $\lim_{z \to z_0} f(z) = f(z_0)$. We will see that this is equivalent to u and v being continuous at $z_0 = (x_0, y_0)$.
- f is defined to be **differentiable** at the point $z_0 = x_0 + y_0 i$ if and only if the limit

  $$f'(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

  exists. Moreover, f is said to be **analytic** at the point z_0 if it is differentiable in a neighbourhood Ω of z_0. We will see that this is equivalent to u and v being of class C^1 and satisfying the **Cauchy-Riemann equations**

  $$u_x = v_y, \quad u_y = -v_x$$

  everywhere in Ω, and that all higher order partials of u and v exist there.

Note that if f satisfies the Cauchy-Riemann equations, then $u_x = v_y$ and $u_y = -v_x$ so that

$$u_{xx} + u_{yy} = (u_x)_x + (u_y)_y = (v_y)_x + (-v_x)_y = v_{yx} - v_{xy} = 0,$$

and similarly, $v_{xx} + v_{yy} = 0$. Consequently, u and v also satisfy the **Laplace equation**

$$g_{xx} + g_{yy} = 0.$$

Solutions to the Laplace equation are said to be **harmonic**. In particular, the component functions u and v of an analytic function f are harmonic. We therefore begin the course by studying the properties of harmonic real two-variable functions before considering complex functions.

2 Topology, parametrized plane curves, and Jordan domains

We will be working in $\mathbb{R}^2$ and will denote any point in $\mathbb{R}^2$ by $z = (x, y)$ for some $x, y \in \mathbb{R}$, to reflect that we will also be thinking of $\mathbb{R}^2$ as the complex plane by identifying $z = (x, y) \in \mathbb{R}^2$ with $z = x + yi \in \mathbb{C}$.

We will also be working with the **Euclidean norm** given by

$$|z| = \sqrt{x^2 + y^2}$$

for all $z = (x, y) \in \mathbb{R}^2$. In particular, for any $z, z_0 \in \mathbb{R}^2$, the distance between z and z_0 is $|z - z_0|$.

Definition 2.1. Let $z_0 \in \mathbb{R}^2$ and $r > 0$. The **open disc of radius** r **centered at** z_0 is the set

$$D(z_0; r) := \{z \in \mathbb{R}^2 : |z - z_0| < r\}.$$

Definition 2.2. Let X be a set. Recall that a **topology** on X is a collection $\mathcal{C}$ of subsets of X which satisfy the following axioms:

(1) Both $\varnothing$ and X are in $\mathcal{C}$.

(2) Any union of elements in $\mathcal{C}$ is an element of $\mathcal{C}$.

(3) The intersection of *finitely many* elements in $\mathcal{C}$ is an element of $\mathcal{C}$.

The elements of $\mathcal{C}$ are said to be the **open** sets of the topology. In particular, property (1) says that $\varnothing$ and X are open, property (2) says that unions of open sets are open, and property (3) says that *finite* intersections of open sets are open. The **closed** sets of the topology are the complements of the open sets.

We will be working with the **standard topology** on $\mathbb{R}^2$, whose open sets correspond to unions of open balls. Let us review some of the concepts and results we will need.

Definition 2.3. Let Ω be a subset of $\mathbb{R}^2$ and $z \in \mathbb{R}^2$. Then z is said to be an **interior point** of Ω if there exists $r > 0$ with $D(z; r) \subseteq \Omega$. Since $z \in D(z; r)$, this means that the interior points of Ω are all contained in Ω. The **interior** of Ω is the set

$$\Omega^\circ := \{z \in \mathbb{R}^2 : z \text{ is an interior point of } \Omega\}.$$

Note that $\Omega^\circ \subseteq \Omega$ by our above observation.

Definition 2.4. Let $\Omega \subseteq \mathbb{R}^2$. We say that Ω is **open** if for every $z \in \Omega$, there exists $r > 0$ such that $D(z; r) \subseteq \Omega$; in other words, $\Omega \subseteq \Omega^\circ$.

Consequently, Ω is open if and only if $\Omega = \Omega^\circ$, which implies that Ω° is always open.

Exercise 2.5. Let $\Omega \subseteq \mathbb{R}^2$. One can show that Ω° is the largest open set contained in Ω. Moreover, Ω is open if and only if it is the union of open balls.

Example 2.6. The sets $\mathbb{R}^2$, the upper-half plane $\{z = (x, y) \in \mathbb{R}^2 : y > 0\}$, and the open disc $D(z_0; r)$ for all $z_0 \in \mathbb{R}^2$ and $r > 0$ are all open sets in $\mathbb{R}^2$.

Definition 2.7. Let $z_0 \in \mathbb{R}^2$ and $r > 0$. The **closed disc of radius** r **centered at** z_0 is the set

$$\overline{D}(z_0; r) := \{z \in \mathbb{R}^2 : |z - z_0| \leq r\}.$$

Example 2.8. Let $z_0 \in \mathbb{R}^2$ and $r > 0$. The interior of $\overline{D}(z_0; r)$ is the open disc $D(z_0; r)$.

Definition 2.9. A set $\Omega \subseteq \mathbb{R}^2$ is **closed** if its complement $\Omega^c := \mathbb{R}^2 \setminus \Omega$ is open. Moreover, a point $z \in \mathbb{R}^2$ is a **boundary point** of Ω if for all $r > 0$, $D(z; r)$ intersects both Ω and Ω^c. Note that unlike interior points, boundary points of Ω might not be included in Ω. The **boundary** of Ω is the set

$$\partial\Omega := \{z \in \mathbb{R}^2 : z \text{ is a boundary point of } \Omega\}.$$

We also define the **closure** of Ω to be the set

$$\overline{\Omega} := \Omega \cup \partial\Omega.$$

By definition, $\Omega \subseteq \overline{\Omega}$, and one can readily verify that $\overline{\Omega}^c$ is open. Thus, Ω is closed if and only if $\Omega = \overline{\Omega}$.

EXERCISE 2.10. Show that $\overline{\Omega}$ is the smallest closed set containing Ω.

EXAMPLE 2.11.

(1) Let $z_0 \in \mathbb{R}^2$ and $r > 0$. The closed disc $\overline{D}(z_0; r)$ is a closed set with interior the open disc $D(z_0; r)$ and boundary given by the circle $|z - z_0| = r$. The closure of $D(z_0; r)$ is $\overline{D}(z_0; r)$.

(2) The set $\Omega = \{z \in \mathbb{R}^2 : y \geq x\}$ is closed with interior given by $\Omega^\circ = \{z \in \mathbb{R}^2 : y > x\}$ and boundary $\partial\Omega = \{z \in \mathbb{R}^2 : y = x\}$.

DEFINITION 2.12. A set $\Omega \subseteq \mathbb{R}^2$ is **bounded** if there exists $r > 0$ such that $\Omega \subseteq D(0; r)$.

Recall that a set is **compact** if any open cover admits a finite subcover, and that by Heine-Borel, a set in $\mathbb{R}^2$ is compact if and only if it is closed and bounded.

EXAMPLE 2.13. The closed disc $\overline{D}(z_0; r)$ is bounded for any $z_0 \in \mathbb{R}^2$ and $r > 0$.

DEFINITION 2.14. An open subset $\Omega \subseteq \mathbb{R}^2$ is said to be **connected** if it cannot be written as the union of two disjoint non-empty sets, and **disconnected** if it is not connected. That is, Ω is disconnected if there exist two non-empty open sets U_1 and U_2 such that $U_1 \cup U_2 = \Omega$ and $U_1 \cap U_2 = \varnothing$.

DEFINITION 2.15. A connected open set is called a **domain**.

EXAMPLE 2.16. The entire plane $\mathbb{R}^2$, the right-half plane $\{z \in \mathbb{R}^2 : x > 0\}$, and open discs $D(z_0; r)$ are all domains.

It can be difficult to determine if a set is a domain straight from the definition. Nonetheless, domains can be characterized more geometrically as path-connected open sets, as will soon see.

DEFINITION 2.17. Let Ω be a non-empty subset of $\mathbb{R}^2$ and $z_1, z_2 \in \Omega$. A continuous function $\alpha : [a, b] \to \Omega$ (where $[a, b]$ is a closed interval in $\mathbb{R}$) with $\alpha(a) = z_1$ and $\alpha(b) = z_2$ is called a **path in Ω from z_1 to z_2**.

DEFINITION 2.18. Let $\Omega \subseteq \mathbb{R}^2$. Then Ω is said to be **path-connected** if for any two points $z_1, z_2 \in \Omega$, there exists a path from z_1 to z_2.

EXAMPLE 2.19. The following are examples of path-connected sets.

(1) **Convex sets.** Recall that a set $\Omega \subseteq \mathbb{R}^2$ is **convex** if for any two points $z_1, z_2 \in \Omega$, the line segment joining z_1 and z_2 is contained in Ω. Since we can continuously parametrize such a line segment with the map

$$\alpha : [0, 1] \to \Omega : t \mapsto tz_1 + (1 - t)z_2,$$

there exists a path from z_1 to z_2. Thus, convex sets are path-connected.

(2) **Annular regions.** Examples of non-convex path-connected sets are given by annular regions such as

$$\Omega = \{z \in \mathbb{R}^2 : c < |z| < d\}$$

for some constants $c, d \in \mathbb{R}^2$.

In the case where $\Omega \subseteq \mathbb{R}^2$ is open, it turns out that the notion of path-connectedness is equivalent to that of connectedness.

PROPOSITION 2.20. An open set $\Omega \subseteq \mathbb{R}^2$ is connected if and only if it is path-connected.

PROOF. Since $\varnothing$ is both connected and path-connected, we may assume that $\Omega \neq \varnothing$. We show that path-connectedness implies connectedness; the converse is left as an exercise. Suppose for a contradiction that Ω is path-connected but not connected. Since Ω is disconnected, there exists non-empty open sets U_1 and U_2 in $\mathbb{R}^2$ such that $U_1 \cup U_2 = \Omega$ and $U_1 \cap U_2 = \varnothing$. Then the function

$$f : \Omega \to \mathbb{R} : z \mapsto \begin{cases} 0 & z \in U_1 \\ 1 & z \in U_2 \end{cases}$$

is well-defined (since $U_1 \cap U_2 = \varnothing$) and continuous (since $f^{-1}(W)$ is open for any open set $W \subseteq \mathbb{R}^2$ as $f^{-1}(W)$ is either equal to $\varnothing$, U_1, U_2, or Ω, depending on whether $0 \in W$ or $1 \in W$). Let $z_1 \in U_1$ and $z_2 \in U_2$ so that $f(z_1) = 0$ and $f(z_2) = 1$. Since Ω is path-connected, there exists a continuous map $\alpha : [a, b] \to \Omega$ such that $\alpha(a) = z_1$ and $\alpha(b) = z_2$. Then $g := f \circ \alpha : [a, b] \to \mathbb{R}$ is a continuous function taking only the values 0 and 1. But this contradicts the Intermediate Value Theorem, since by continuity, g must take every value between $g(a) = 0$ and $g(b) = 1$. Hence, Ω is connected. $\square$

The above proposition then tells us that domains correspond to path-connected open sets.

Example 2.21. We give some more examples of domains.

(1) Open convex sets are domains since they are path-connected. For instance, $\mathbb{R}^2$, the upper-half plane $\{z \in \mathbb{R}^2 : y > 0\}$, and open discs are all domains.

(2) Open annular regions are also path-connected, and therefore domains. For example, $\Omega = \{z \in \mathbb{R}^2 : c < |z| < d\}$ is a domain for all constants $c, d \in \mathbb{R}^2$.

(3) The set $\Omega = D((-1, 0), 1) \cup D((1, 0), 1)$ is *not* a domain, since any path from $z = (-1/2, 0)$ to $z' = (1/2, 0)$ must pass through points that are not in Ω.

We now turn our attention to parametrized plane curves and Jordan domains. Consider the unit circle

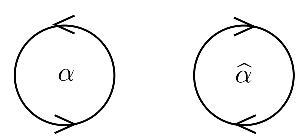
$$C = C(0; 1) : |z| = 1$$

in $\mathbb{R}^2$. Points on C can be parametrized by

$$\alpha : [0, 2\pi] \to \mathbb{R}^2 : t \mapsto (\cos t, \sin t).$$

Note that α traverses points on C counter-clockwise, whereas one can also parametrize C clockwise by

$$\widehat{\alpha} : [0, 2\pi] \to \mathbb{R}^2 : t \mapsto (\sin t, \cos t).$$



In general, we have the following definition.

Definition 2.22. Let $\Gamma \subseteq \mathbb{R}^2$ and $[a, b] \subseteq \mathbb{R}$ be a closed interval. A smooth vector-valued function

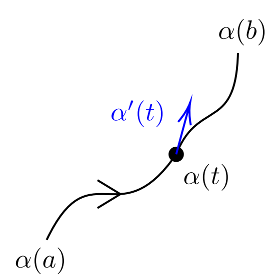
$$\alpha : [a, b] \to \Gamma : t \mapsto (\alpha_1(t), \alpha_2(t))$$

is called a **smooth parametrization of** Γ if the following conditions hold:

(1) The image of α is Γ; that is, $\alpha([a, b]) = \Gamma$.

(2) The **velocity vector** $\alpha'(t) = (\alpha_1'(t), \alpha_2'(t))$ is non-zero for all $t \in [a, b]$.

(3) If $\alpha(a) = \alpha(b)$, then $\alpha'(a) = \alpha'(b)$.

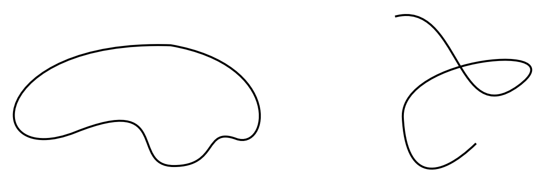


Recall that the velocity vector $\alpha'(t)$ is tangent to Γ at the point $\alpha(t)$ whenever it is non-zero. Condition (2) therefore ensures that one can specify a tangent direction to Γ at every point. Moreover, if $\alpha(a) = \alpha(b) = z_0$, then the curve Γ is **closed** (or a **loop**), and condition (3) ensures that there is a well-defined tangent direction to Γ at z_0.

Definition 2.23. A parametrization $\alpha : [a, b] \to \Gamma \subseteq \mathbb{R}^2$ is said to be **simple** if the restrictions of α to the intervals $[a, b)$ and $(a, b]$ are both one-to-one.

Note that if α is not closed, this is equivalent to α being injective on $[a, b]$.

If a parametrization α is simple, then the curve Γ does not self-intersect. This is a very important property of curves that admit simple parametrizations. The curve on the left is simple, whereas the curve on the right is not.

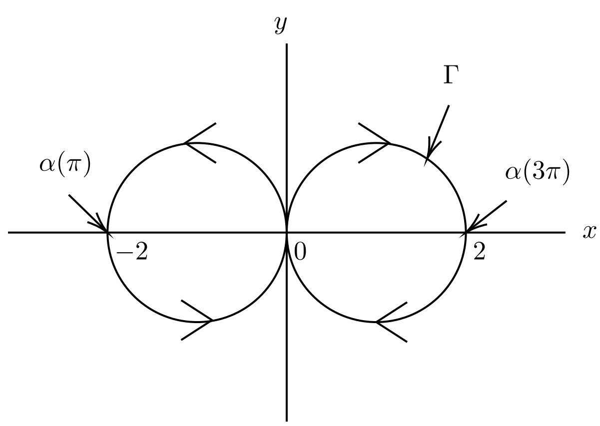


Note that if the parametrization α is only one-to-one on the open interval (a, b), then the curve Γ can still self-intersect; it is important that it is also one-to-one on a or b, as we will see in the following example.

Example 2.24. Consider the smooth parametrization of the curve α given by

$$\alpha : [0, 4\pi] \to \mathbb{R} : t \mapsto \begin{cases} (\cos t - 1, \sin t), & t \in [0, 2\pi] \\ (1 - \cos t, \sin t), & t \in (2\pi, 4\pi]. \end{cases}$$

It is a smooth parametrization of the figure-eight Γ, which self-intersects even though α is one-to-one on $(0, 4\pi)$ (check this).

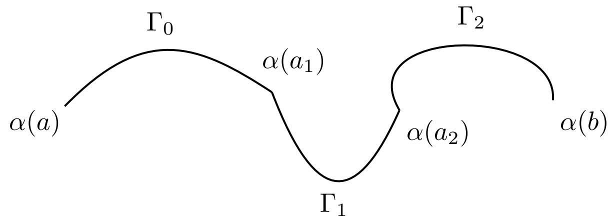


However, α fails to be one-to-one both on $[0, 4\pi)$ and $(0, 4\pi]$ since $\alpha(0) = \alpha(2\pi) = \alpha(4\pi) = (0, 0)$, which implies that α is not simple.

Definition 2.25. A **piecewise-smooth parametrization** of a subset Γ of $\mathbb{R}^2$ is a finite number of subsets $\Gamma_0, \ldots, \Gamma_{n-1}$ of Γ together with smooth parametrizations $\alpha_i : [a_i, a_{i+1}] \to \Gamma_i$ for each $0 \le i \le n - 1$ such that:

- $a = a_0 < a_1 < \cdots < a_n = b$,
- $\alpha_i(a_{i+1}) = \alpha_{i+1}(a_{i+1})$ for all $0 \le i \le n - 1$, and

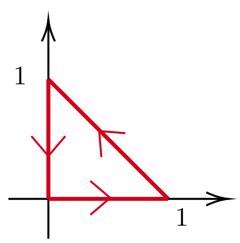
- $\Gamma = \bigcup_{i=0}^{n-1} \Gamma_i$.



Finally, a **(piecewise-)smooth parametrized curve** is a pair (Γ, α) consisting of a subset Γ of $\mathbb{R}^2$ together with a (piecewise-)smooth parametrization α of Γ.

EXAMPLE 2.26. For an example of a piecewise-smooth curve, consider the triangle Γ with vertices $(0,0)$, $(1,0)$ and $(0,1)$ together with the parametrization

$$\alpha : [0,3] \to \Gamma : t \mapsto \begin{cases} (t,0), & 0 \le t \le 1 \\ (2-t, t-1), & 1 \le t \le 2 \\ (0, 3-t), & 2 \le t \le 3. \end{cases}$$



DEFINITION 2.27. A **Jordan curve** is a simple, closed (piecewise-)smooth curve.

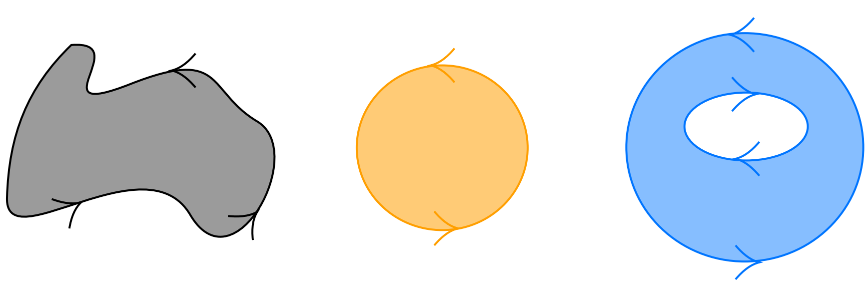
Jordan curves therefore look like loops that do not have self-intersections.

EXAMPLE 2.28. Circles, triangles, and rectangles are all Jordan curves.

THEOREM 2.29 (Jordan Curve Theorem). If Γ is a Jordan curve, its complement $\mathbb{R}^2 \setminus \Gamma$ consists of two disjoint domains: one bounded (the "inside"), and one unbounded (the "outside"), each domain having the curve Γ as its boundary. If a point inside Γ is joined by a path to a point outside Γ, then the path must cross Γ.

PROOF. Outside the scope of the course. $\square$

DEFINITION 2.30. A **Jordan domain** is a bounded domain $\Gamma \subseteq \mathbb{R}^2$ whose boundary is the union of a finite number of *positively oriented* Jordan curves, in the sense that when one walks along a given curve in the direction specified by its parametrization, then Γ always lies to the left.



EXAMPLE 2.31.

(1) Open discs with the boundary oriented counter-clockwise are Jordan domains.

(2) The annulus $\Gamma = \{z \in \mathbb{R}^2 : r_1 < |z| < r_2\}$ with the inner circle $|z| = r_1$ oriented clockwise and the outer circle $|z| = r_2$ oriented counter-clockwise is a Jordan domain.

3 Parametrized plane curves and integral calculus in the plane

Definition 3.1. Let $E : [c, d] \to [a, b]$ be a smooth bijection such that $E'(s) > 0$ for all $s \in [a, b]$. We say that E is an **equivalence**.

Let $E : [c, d] \to [a, b]$ be an equivalence. Note that since $E'(s) > 0$ for all $s \in [c, d]$, it follows that E is strictly increasing and has a smooth inverse by the Inverse Function Theorem.

Let $\alpha : [a, b] \to \Gamma \subseteq \mathbb{R}^2$ be a smooth parametrization of a plane curve Γ. Given an equivalence $E : [c, d] \to [a, b]$, we set $\beta = \alpha \circ E$. Then $\beta : [c, d] \to \Gamma$ is another smooth parametrization of Γ that traverses the curve in the same direction as α, since E is increasing.

Definition 3.2. We say that two smooth parametrizations $\alpha : [a, b] \to \Gamma$ and $\beta : [a, b] \to \Gamma$ of a curve $\Gamma \subseteq \mathbb{R}^2$ are **equivalent** if there exists an equivalence $E : [c, d] \to [a, b]$ such that $\beta = \alpha \circ E$.

Example 3.3. Let $\Gamma := \{(x, y) \in \mathbb{R}^2 : y = x^2, 1 \le x \le 2\}$. Then $\alpha : [1, 2] \to \Gamma : t \mapsto (t, t^2)$ and $\beta : [1, 4] \to \Gamma : s \mapsto (\sqrt{s}, s)$ are equivalent smooth parametrizations of Γ since we have $\beta = \alpha \circ E$, where $E : [1, 4] \to [1, 2] : s \mapsto \sqrt{s}$ is a smooth bijection with $E'(s) > 0$ for all $s \in [1, 4]$. Note that $\alpha'(t) = (1, 2t)$ and

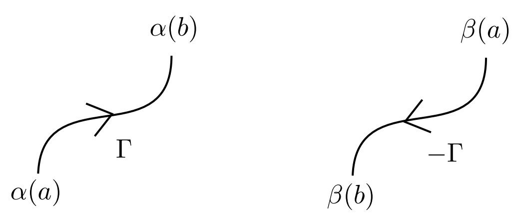
$$\beta'(s) = \left(\frac{1}{2\sqrt{s}}, 1\right) = \frac{1}{2\sqrt{s}}(1, 2\sqrt{s}) = E'(s)\alpha'(E(s))$$

point in the same direction at every point on Γ, as $E'(s) > 0$ everywhere.

As we see in the above example, the velocity vectors of equivalent parametrizations point in the same direction since $E'(s) > 0$ everywhere.

Definition 3.4. Let $\alpha : [a, b] \to \Gamma \subseteq \mathbb{R}^2$ be a smooth parametrized curve and set $\beta : [a, b] \to \mathbb{R}^2 : t \mapsto \alpha(a + b - t)$. We then denote by $-\Gamma$ the curve with **reverse parametrization** β.

Note that $\beta = \alpha \circ F$ where $F : [a, b] \to [a, b] : t \mapsto a + b - t$. Although F is a smooth bijection, we have $F'(t) = -1 < 0$ for all $t \in [a, b]$, and hence F is *not* an equivalence. Nonetheless, β is a smooth parametrization of Γ, but $\beta'(t) = \alpha'(a + b - t)$ for all $t \in [a, b]$ so that β traverses Γ in the opposite direction of α.



Example 3.5. Consider the unit circle $\Gamma = \{z \in \mathbb{R}^2 : |z| = 1\}$ in $\mathbb{R}^2$ with parametrization $\alpha : [0, 2\pi] \to \Gamma : t \mapsto (\cos t, \sin t)$, which traverses Γ counter-clockwise. Then $-\Gamma$ is the unit circle parametrized by $\beta : [0, 2\pi] \to \Gamma : t \mapsto (\cos(2\pi - t), \sin(2\pi - t)) = (\cos t, -\sin t)$, which traverses Γ clockwise.

Definition 3.6. Let $\alpha : [a, b] \to \Gamma \subseteq \mathbb{R}^2$ be a smooth parametrization of a plane curve. For any $s \in [a, b]$, we define the **length of the curve from** $\alpha(a)$ **to** $\alpha(s)$ to be

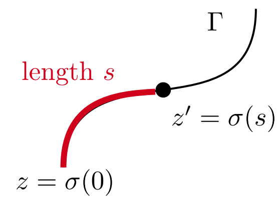
$$\int_a^s |\alpha'(t)| \,\mathrm{d}t.$$

Note that if $|\alpha'(t)| = 1$ for all $t \in [a, b]$, then the length of the curve from $\alpha(a)$ to $\alpha(s)$ is simply

$$\int_a^s |\alpha'(t)| \,\mathrm{d}t = \int_a^s 1 \,\mathrm{d}t = s - a.$$

In other words, $\alpha(s)$ is the point in Γ at a distance $s - a$ from the initial point $\alpha(a)$. In particular, if $a = 0$, then $\alpha(s)$ is the point on Γ at distance s from the initial point $\alpha(0)$. As it happens, such a parametrization always exists and is called the **arclength parametrization**.

PROPOSITION 3.7. Let $\alpha : [a, b] \to \Gamma \subseteq \mathbb{R}^2$ be a smooth parametrization of a plane curve of length L. Then there exists another parametrization $\sigma : [0, L] \to \Gamma$ that is equivalent to α and is such that $|\sigma'(s)| = 1$ for all $s \in [0, L]$.



PROOF. For all $\tau \in [a, b]$, set

$$F(\tau) = \int_a^\tau |\alpha'(t)| \, \mathrm{d}t.$$

Then $F(a) = 0$ and $F(b) = L$. Moreover, by the Fundamental Theorem of Calculus, we have $F'(\tau) = |\alpha'(\tau)|$ for all $\tau \in [a, b]$. In particular, this means that $F'(\tau) > 0$ for all τ since $\alpha'(\tau) \neq 0$ for all τ. Hence, by the Inverse Function Theorem, F is invertible with smooth inverse on $[a, b]$. Let $E : [0, L] \to [a, b]$ be its inverse, which is also smooth and is such that

$$E'(s) = \frac{1}{F'(E(s))} > 0$$

for all $s \in [0, L]$. That is, E is an equivalence. Now, define

$$\sigma(s) := \alpha(E(s)).$$

Then $\sigma'(s) = \alpha'(E(s))E'(s)$ by the Chain Rule, so that

$$|\sigma'(s)| = |\alpha'(E(s))| \cdot |E'(s)| = F'(E(s)) \cdot E'(s) = 1$$

for all $s \in [0, L]$, proving the result. $\square$

EXAMPLE 3.8. Let $r > 0$ and consider the circle $C = \{z \in \mathbb{R}^2 : |z| = r\}$, parametrized by $\alpha : [0, 2\pi] \to C : t \mapsto (r\cos t, r\sin t)$. Its arclength is $L = 2\pi r$ and the equivalent arclength parametrization is given by

$$\sigma : [0, 2\pi r] \to C : s \mapsto (r\cos(s/r), r\sin(s/r)).$$

Note that $\sigma'(s) = (-\sin(s/r), \cos(s/r))$ so that $|\sigma'(s)| = 1$ for all $s \in [0, 2\pi r]$.

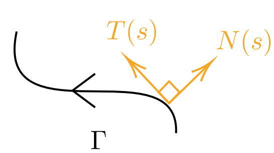
DEFINITION 3.9. Given an arclength parametrization $\sigma : [0, L] \to \Gamma \subseteq \mathbb{R}^2$ of a plane curve, we set

$$T(s) := \sigma'(s) = (\sigma_1'(s), \sigma_2'(s)),$$

which is the **unit tangent vector** to Γ at the point $\sigma(s)$, and

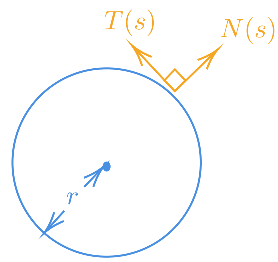
$$N(s) := (\sigma_2'(s), -\sigma_1'(s))$$

to be the **outward normal vector** to Γ at the point $\sigma(s)$.



Note that $N(s)$ and $T(s)$ are orthogonal with respect to the standard inner product $\mathbb{R}^2$, since $N(s)$ is obtained by rotating $T(s)$ clockwise by $90°$. Therefore, if Γ is the boundary of a Jordan domain, then $N(s)$ points outwards of the domain, which is why it is called the outward normal vector.

Example 3.10. Let $r > 0$ and consider the circle $C = \{z \in \mathbb{R}^2 : |z| = r\}$ with arclength parametrization given by $\sigma : [0, 2\pi r] \to C : s \mapsto (r\cos(s/r), r\sin(s/r))$. Then $T(s) = (-\sin(s/r), \cos(s/r))$ and $N(s) = (\cos(s/r), \sin(s/r))$.



Note that the unit normal vector will come up when we see Green's Theorem.

We now study integral calculus in the plane. Let $[a, b]$ be a closed interval in $\mathbb{R}$ and $f : [a, b] \to \mathbb{R}$ be a continuous real-valued function. Recall that f is then integral on $[a, b]$ so that

$$\int_a^b f(t)\,\mathrm{d}t$$

exists and is a real number called the **definite integral of f over** $[a, b]$. We list some properties of definite integrals. If $f, g \in C^0([a, b])$, then

(1) $\int_a^b f(t)\,\mathrm{d}t = \int_a^{t_0} f(t)\,\mathrm{d}t + \int_{t_0}^b f(t)\,\mathrm{d}t$ for all $t_0 \in [a, b]$;

(2) $\int_a^b (f + g)(t)\,\mathrm{d}t = \int_a^b f(t)\,\mathrm{d}t + \int_a^b g(t)\,\mathrm{d}t$;

(3) $\int_a^b Cf(t)\,\mathrm{d}t = C\int_a^b f(t)\,\mathrm{d}t$ for all $C \in \mathbb{R}$; and

(4) $\int_a^b f(t)\,\mathrm{d}t = -\int_b^a f(t)\,\mathrm{d}t$.

Properties (2) and (3) tell us that integrals are linear, whereas property (1) tells us that the integral over $[a, b]$ is equal to the sum of integrals over the subintervals $[a, t_0]$ and $[t_0, b]$.

Suppose now that we want to integrate a continuous vector-valued function $f : \mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto (p(x, y), q(x, y))$ along a curve. How can we set this up?

Definition 3.11. Let $\alpha : [a, b] \to \Gamma \subseteq \mathbb{R}^2$ be a smooth parametrization of a plane curve in a domain $\Omega \subseteq \mathbb{R}^2$, and suppose that $p, q : \Omega \to \mathbb{R}$ are continuous. We define

$$\int_\Gamma p\,\mathrm{d}x + q\,\mathrm{d}y := \int_a^b p(\alpha(t))\alpha_1'(t)\,\mathrm{d}t + \int_a^b q(\alpha(t))\alpha_2'(t)\,\mathrm{d}t.$$

This quantity is called the **line integral of the vector-valued function** $(p, q) : \Omega \to \mathbb{R}^2 : (x, y) \mapsto (p(x, y), q(x, y))$ **along** Γ.

Note that these integrals are well-defined since $p(\alpha(t))\alpha_1'(t)$ and $q(\alpha(t))\alpha_2'(t)$ are both smooth on $[a, b]$. Moreover, since $(p, q) : \Omega \to \mathbb{R}^2$ can be interpreted as a vector field in $\mathbb{R}^2$, the line integral $\int_\Gamma p\,\mathrm{d}x + q\,\mathrm{d}y$ is also called the **line integral of the vector field** $F = (p, q)$ **along** Γ.

Example 3.12. Let $(p, q) = (x^2, 1 - y)$ and $\Gamma = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ be parametrized by $\alpha : [0, 2\pi] \to \Gamma : t \mapsto (\cos t, \sin t)$. Then $\alpha'(t) = (-\sin t, \cos t)$, and

$$\int_\Gamma p\,\mathrm{d}x + q\,\mathrm{d}y = \int_0^{2\pi} (\cos t)^2(-\sin t)\,\mathrm{d}t + \int_0^{2\pi} (1 - \sin t)\cos t\,\mathrm{d}t = 0.$$

Note that one obtains the same value if one uses a parametrization $\beta : [c, d] \to \Gamma$ that is equivalent to α. Indeed, if β is such a parametrization, then $\beta = \alpha \circ E$ for some equivalence $E : [c, d] \to [a, b]$. We verify that

$$\int_a^b p(\alpha(t))\alpha_1'(t)\,\mathrm{d}t = \int_c^d p(\beta(s))\beta_1'(s)\,\mathrm{d}s. \tag{$\star$}$$

If $t = E(s)$, then $\mathrm{d}t = E'(s)\,\mathrm{d}s$ and $\beta'(s) = \alpha'(E(s))E'(s)$ so that

$$(\beta_1'(s), \beta_2'(s)) = (\alpha_1'(E(s))E'(s), \alpha_2'(E(s))E'(s)),$$

and moreover,

$$p(\alpha(t))\alpha_1'(t)\,\mathrm{d}t = p(\alpha(E(s)))\alpha_1'(E(s))E'(s)\,\mathrm{d}s = p(\beta(s))\beta_1'(s)\,\mathrm{d}s.$$

Since $a = E(c)$ and $b = E(d)$, the above change of variable formula gives ($\star$). A similar computation gives

$$\int_a^b q(\alpha(t))\alpha_2'(t)\,\mathrm{d}t = \int_c^d q(\beta(s))\beta_2'(s)\,\mathrm{d}s.$$

Thus, line integrals are independent of the parametrization up to equivalence.

Example 3.13. In Example 3.12, if Γ is parametrized instead by the equivalent parametrization $\beta : [0, \pi] \to \Gamma : t \mapsto (\cos 2t, \sin 2t)$, then $\beta'(t) = (-2\sin 2t, 2\cos 2t)$, so we obtain

$$\int_\Gamma p\,\mathrm{d}x + q\,\mathrm{d}y = \int_0^\pi (\cos 2t)^2(-2\sin 2t)\,\mathrm{d}t + \int_0^\pi (1 - \sin 2t)(2\cos 2t)\,\mathrm{d}t = 0.$$

Line integrals over piecewise-smooth curves are defined as the sum of the integrals over the smooth pieces. More precisely, we have the following definition.

Definition 3.14. Let (Γ, α) be a piecewise-smooth parametrized curve in a domain $\Omega \subseteq \mathbb{R}^2$ given by the smooth parametrizations $\alpha_i : [a_i, a_{i+1}] \to \Gamma_i$ for each $0 \le i \le n-1$. For any two continuous functions $p, q : \Omega \to \mathbb{R}$, the **line integral of** $(p, q) : \Omega \to \mathbb{R}^2$ **along** Γ is defined as

$$\int_\Gamma p\,\mathrm{d}x + q\,\mathrm{d}y := \int_{\Gamma_0} p\,\mathrm{d}x + q\,\mathrm{d}y + \int_{\Gamma_1} p\,\mathrm{d}x + q\,\mathrm{d}y + \cdots + \int_{\Gamma_{n-1}} p\,\mathrm{d}x + q\,\mathrm{d}y.$$

Example 3.15. Consider $(p, q) = (x^2y, x)$, and let Γ be the triangle with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$ with piecewise-smooth parametrization

$$\alpha : [0, 3] \to \Gamma : t \mapsto \begin{cases} (t, 0), & 0 \le t \le 1, \\ (2 - t, t - 1), & 1 \le t \le 2, \\ (0, 3 - t), & 2 \le t \le 3. \end{cases}$$

In this case, (Γ, α) is the union of the three line segments

$$\begin{aligned} &\alpha_0 : [0, 1] \to \Gamma_0 : t \mapsto (t, 0), \\ &\alpha_1 : [1, 2] \to \Gamma_1 : t \mapsto (2 - t, t - 1), \\ &\alpha_2 : [2, 3] \to \Gamma_2 : t \mapsto (0, 3 - t). \end{aligned}$$

We then have

$$\int_{\Gamma_0} p\,\mathrm{d}x + q\,\mathrm{d}y = \int_0^1 p(t, 0) \cdot 1\,\mathrm{d}t = 0$$

since $\alpha_0'(t) = (1, 0)$ and $p(t, 0) = 0$. Then, since $\alpha_1'(t) = (-1, 1)$, we have

$$\int_{\Gamma_1} p\,\mathrm{d}x + q\,\mathrm{d}y = \int_1^2 p(2 - t, t - 1) \cdot (-1)\,\mathrm{d}t + \int_1^2 q(2 - t, t - 1) \cdot 1\,\mathrm{d}t = 5/12.$$

Finally, we see that

$$\int_{\Gamma_2} p\,\mathrm{d}x + q\,\mathrm{d}y = \int_2^3 q(0,3-t)\cdot(-1)\,\mathrm{d}t = 0$$

since $\alpha_2'(t) = (0,-1)$ and $q(0,3-t) = 0$. Putting these together, we obtain

$$\int_{\Gamma} p\,\mathrm{d}x + q\,\mathrm{d}y = 0 + 5/12 + 0 = 5/12.$$

We list some properties of line integrals.

PROPOSITION 3.16. Let $\alpha : [a,b] \to \Gamma \subseteq \mathbb{R}^2$ be a piecewise-smooth parametrization of a plane curve in a domain $\Omega \subseteq \mathbb{R}^2$, and let $p, q, p_1, p_2, q_1, q_2 : \Omega \to \mathbb{R}$ be continuous. Moreover, let $C, C' \in \mathbb{R}$. Then

(1) $\int_\Gamma (p_1 + p_2)\,\mathrm{d}x + (q_1 + q_2)\,\mathrm{d}y = \int_\Gamma p_1\,\mathrm{d}x + q_1\,\mathrm{d}y + \int_\Gamma p_2\,\mathrm{d}x + q_2\,\mathrm{d}y$;

(2) $\int_\Gamma Cp\,\mathrm{d}x + C'q\,\mathrm{d}y = C\int_\Gamma p\,\mathrm{d}x + C'\int_\Gamma q\,\mathrm{d}y$; and

(3) $\int_{-\Gamma} p\,\mathrm{d}x + q\,\mathrm{d}y = -\int_\Gamma p\,\mathrm{d}x + q\,\mathrm{d}y$.

PROOF. This follows straight from the definition and the properties of definite integrals; we leave the proof as an exercise. $\square$

What happens if $p\,\mathrm{d}x + q\,\mathrm{d}y = \mathrm{d}u$ for some $u \in C^1(\Omega)$, in which case it is called **exact**? In other words,

$$p\,\mathrm{d}x + q\,\mathrm{d}y = u_x\,\mathrm{d}x = u_y\,\mathrm{d}y = \mathrm{d}u$$

so that $p = u_x$ and $q = u_y$. Then by the Chain Rule, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(u(\alpha(t))) &= \frac{\mathrm{d}}{\mathrm{d}t}(u(\alpha_1(t),\alpha_2(t)))\\ &= u_x(\alpha(t))\alpha_1'(t) + u_y(\alpha(t))\alpha_2'(t)\\ &= p(\alpha(t))\alpha_1'(t) + q(\alpha(t))\alpha_2'(t),\end{aligned}$$

which implies that

$$\int_\Gamma p\,\mathrm{d}x + q\,\mathrm{d}y = \int_a^b \frac{\mathrm{d}}{\mathrm{d}t}(u(\alpha(t)))\,\mathrm{d}t = u(\alpha(b)) - u(\alpha(a)) = u(z_2) - u(z_1).$$

Hence, $\int_\Gamma p\,\mathrm{d}x + q\,\mathrm{d}y$ depends only on the endpoints of Γ when $p\,\mathrm{d}x + q\,\mathrm{d}y$ is exact; that is, it is **independent of path**. We can summarize the above discussion with the following theorem.

THEOREM 3.17. Let $\Gamma \subseteq \mathbb{R}^2$ be a piecewise-smooth curve from z_1 to z_2 lying inside a domain $\Omega \subseteq \mathbb{R}^2$. Moreover, let $u \in C^1(\Omega)$. Then

$$\int_\Gamma \mathrm{d}u = u(z_2) - u(z_1).$$

In particular, if Γ is a loop so that $z_2 = z_1$, then $\int_\Gamma \mathrm{d}u = 0$.

EXAMPLE 3.18. Recall Example 3.12 where we had $(p,q) = (x^2, 1-y)$. Observe that $p\,\mathrm{d}x + q\,\mathrm{d}y$ is exact, since

$$p\,\mathrm{d}x + q\,\mathrm{d}y = x^2\,\mathrm{d}x = (1-y)\,\mathrm{d}y = \mathrm{d}\left(\frac{1}{3}x^3 - \frac{1}{2}(1-y)^2\right) = \mathrm{d}u.$$

Thus, it is not surprising that the integral we computed was 0 as we were integrating $\mathrm{d}u$ along a loop; namely, the circle $\Gamma = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.

As a consequence of Theorem 3.17, we see that if there exists a loop $\Gamma \subseteq \mathbb{R}^2$ along which

$$\int_\Gamma p\,\mathrm{d}x + q\,\mathrm{d}y \neq 0,$$

then $p\,\mathrm{d}x + q\,\mathrm{d}y$ cannot be exact. That is, there does not exist $u \in C^1(\Omega)$ such that $p\,\mathrm{d}x + q\,\mathrm{d}y = \mathrm{d}u$. For instance, we saw in Example 3.15 that with $(p,q) = (x^2y, x)$ and a loop Γ, we had $\int_\Gamma p\,\mathrm{d}x + q\,\mathrm{d}y = 5/12 \neq 0$. Hence, $p\,\mathrm{d}x + q\,\mathrm{d}y$ is not exact on $\mathbb{R}^2$.

4 More integral calculus on the plane

Let Ω be a bounded subset of $\mathbb{R}^2$ and let $f : \Omega \to \mathbb{R}$ be continuous. Then f is integrable on Ω so that

$$\iint_\Omega f(x, y)\,\mathrm{d}x\,\mathrm{d}y$$

exists and is a real number called the **double integral of f over Ω.**

We recall some properties of the double integral. Let $f, g \in C^0(\Omega)$ and $C \in \mathbb{R}$.

- We have $\iint_\Omega (Cf + g)(x, y)\,\mathrm{d}x\,\mathrm{d}y = C\iint_\Omega f(x, y)\,\mathrm{d}x\,\mathrm{d}y + \iint_\Omega g(x, y)\,\mathrm{d}x\,\mathrm{d}y$.
- If $f(z) > 0$ for all $z \in \Omega$, then $\iint_\Omega f(x, y)\,\mathrm{d}x\,\mathrm{d}y > 0$. This also holds by replacing $>$ with $\geq$.

Another useful property that will come up on more than one occasion when we study harmonic functions is the Bump Principle.

PROPOSITION 4.1 (Bump Principle). Let Ω be a bounded domain in $\mathbb{R}^2$ and $q : \Omega \to \mathbb{R}$ be a continuous function. Moreover, assume that $q(z) \geq 0$ for all $z \in \Omega$. Then q is identically zero on Ω if and only if

$$\iint_\Omega q(x, y)\,\mathrm{d}x\,\mathrm{d}y = 0.$$

PROOF. If q is identically zero on Ω, then clearly $\iint_\Omega q(x, y)\,\mathrm{d}x\,\mathrm{d}y = 0$. Conversely, suppose that we have $\iint_\Omega q(x, y)\,\mathrm{d}x\,\mathrm{d}y = 0$. Towards a contradiction, suppose that there exists $z_0 \in \Omega$ such that $q(z_0) > 0$. By the continuity of q, there exists an open neighbourhood W of z_0 contained in Ω such that $q(w) > 0$ for all $w \in W$. In particular, we have that $\iint_W q(x, y)\,\mathrm{d}x\,\mathrm{d}y > 0$. Moreover, $\iint_{\Omega\setminus W} q(x, y)\,\mathrm{d}x\,\mathrm{d}y \geq 0$ since $q(z) \geq 0$ for all $z \in \Omega \setminus W$. It follows that

$$0 = \iint_\Omega q(x, y)\,\mathrm{d}x\,\mathrm{d}y = \iint_W q(x, y)\,\mathrm{d}x\,\mathrm{d}y + \iint_{\Omega\setminus W} q(x, y)\,\mathrm{d}x\,\mathrm{d}y > 0,$$

which is a contradiction. Thus, q is identically zero on Ω. $\square$

DEFINITION 4.2. Let k be a positive integer. A Jordan domain $\Omega \subseteq \mathbb{R}^2$ is said to be ***k*-connected** if its boundary consists of k distinct Jordan curves. Note that this is equivalent to its complement $\mathbb{R}^2 \setminus \Omega$ consisting of k disjoint connected components. In particular, if Ω is 1-connected (so that it has "no holes" inside), then it is called **simply connected**.

EXAMPLE 4.3.

(1) Discs, triangles, and squares are simply connected.

(2) The annulus $\Omega = \{z \in \mathbb{R}^2 : r_1 < |z| < r_2\}$ is 2-connected.

(3) The domain $\Omega = D(0; 4) \setminus (\overline{D}(-2; 1) \cup \overline{D}(1; 2))$ is 3-connected.

THEOREM 4.4 (Green's Theorem). Let $\Omega \subseteq \mathbb{R}^2$ be a k-connected Jordan domain and suppose that $p, q \in C^1(\Omega^+)$, where Ω^+ is a domain containing Ω and $\partial\Omega$. Then

$$\int_{\partial\Omega} p\,\mathrm{d}x + q\,\mathrm{d}y = \iint_\Omega (q_x - p_y)\,\mathrm{d}x\,\mathrm{d}y.$$

PROOF. The proof reduces to proving the two equalities

$$\iint_{\partial\Omega} p\,\mathrm{d}x = -\iint_\Omega p_y\,\mathrm{d}x\,\mathrm{d}y, \tag{1}$$

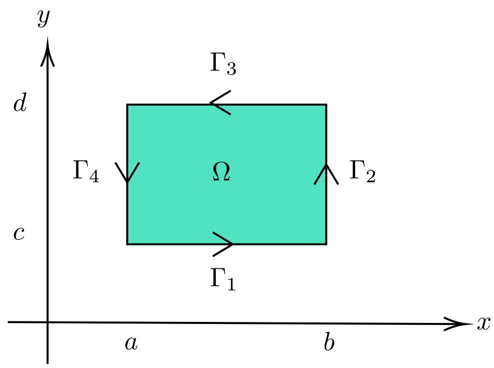
$$\iint_{\partial\Omega} q\,\mathrm{d}y = \iint_\Omega q_x\,\mathrm{d}x\,\mathrm{d}y, \tag{2}$$

since these must hold when $q = 0$ and $p = 0$, respectively.

We prove the theorem in the case where Ω is a rectangle; that is,

$$\Omega = \{z \in \mathbb{R}^2 : a < x < b,\, c < x < d\}.$$

The boundary $\partial\Omega$ of Ω consists of four line segments Γ_1, Γ_2, Γ_3, and Γ_4 oriented as in the following diagram so that $\partial\Omega$ is positively oriented.



Let us prove (1). First, note that

$$\iint_\Omega p_y \,\mathrm{d}x\,\mathrm{d}y = \int_a^b \left(\int_c^d p_y \,\mathrm{d}y\right) \mathrm{d}x = \int_a^b (p(x,d) - p(x,c))\,\mathrm{d}x.$$

Moreover, we have

$$\int_{\partial\Omega} p\,\mathrm{d}x = \int_{\Gamma_1} p\,\mathrm{d}x + \int_{\Gamma_2} p\,\mathrm{d}x + \int_{\Gamma_3} p\,\mathrm{d}x + \int_{\Gamma_4} p\,\mathrm{d}x.$$

To compute $\int_{\partial\Omega} p\,\mathrm{d}x$, we need to parametrize the line segments Γ_1, Γ_2, Γ_3, and Γ_4, and compute the corresponding line integrals. Indeed, consider the parametrizations

$$\begin{aligned}
\alpha &: [a,b] \to \Gamma_1 : x \mapsto (x,c),\\
\beta &: [c,d] \to \Gamma_2 : y \mapsto (b,y),\\
\gamma &: [a,b] \to \Gamma_3 : x \mapsto (a+b-x,d),\\
\delta &: [c,d] \to \Gamma_4 : y \mapsto (a,c+d-y).
\end{aligned}$$

Then $p(\alpha(x))\alpha_1'(x)\,\mathrm{d}x = p(x,c)\cdot 1\,\mathrm{d}x$ on Γ_1 so that

$$\int_{\Gamma_1} p\,\mathrm{d}x = \int_a^b p(x,c)\,\mathrm{d}x.$$

On the other hand, we have $p(\gamma(x))\gamma_1'(x)\,\mathrm{d}x = p(a+b-x,d)\cdot(-1)\,\mathrm{d}x$ on Γ_3, and hence

$$\int_{\Gamma_3} p\,\mathrm{d}x = -\int_a^b p(a+b-x,d)\,\mathrm{d}x = -\int_a^b p(x,d)\,\mathrm{d}x,$$

where the second equality follows from the substitution $u = a+b-x$. Finally, β_1 and δ_1 are constant on Γ_2 and Γ_4 respectively, so we have $\beta_1' = \delta_1' = 0$, which implies that $p(\beta(x))\beta_1'(x) = p(\delta(x))\delta_1'(x) = 0$ and $\int_{\Gamma_2} p\,\mathrm{d}x = \int_{\Gamma_4} p\,\mathrm{d}x = 0$. Putting it all together, we get

$$\int_{\partial\Omega} p\,\mathrm{d}x = \int_a^b p(x,c)\,\mathrm{d}x + 0 - \int_a^b p(x,d)\,\mathrm{d}x + 0 = \int_a^b (p(x,c) - p(x,d))\,\mathrm{d}x,$$

which yields (1). The equality (2) can be proved in a similar fashion, and is left as an exercise. $\square$

EXAMPLE 4.5. Let us verify Green's Theorem with an example. Let

$$\Omega = \{z \in \mathbb{R}^2 : x^2 \leq y \leq 1\}.$$

In particular, Ω is the domain bounded by $\Gamma_1 = \{z \in \mathbb{R}^2 : y = x^2,\ -1 \leq x \leq 1\}$ and $\Gamma_2 = \{z \in \mathbb{R}^2 : y = 1,\ -1 \leq x \leq 1\}$. Let us orient the boundary of Ω positively by parametrizing Γ_1 and Γ_2 with

$$\begin{aligned} \alpha &: [-1,1] \to \Gamma_1 : t \mapsto (t, t^2), \\ \beta &: \quad [1,3] \to \Gamma_2 : t \mapsto (2-t, 1). \end{aligned}$$

Consider the differential $p\,\mathrm{d}x + q\,\mathrm{d}y = x^2 y\,\mathrm{d}x - x\,\mathrm{d}y$, which satisfies the hypotheses of Green's Theorem since $p = x^2 y$ and $q = -x$ are both smooth in $\mathbb{R}^2$. We wish to verify that

$$\int_{\partial\Omega} p\,\mathrm{d}x + q\,\mathrm{d}y = \iint_{\Omega} (q_x - p_y)\,\mathrm{d}x\,\mathrm{d}y. \tag{$\star$}$$

First, we compute the right-hand side of ($\star$). Note that $q_x - p_y = -(1+x^2)$. It then follows that

$$\begin{aligned} \iint_{\Omega} (q_x - p_y)\,\mathrm{d}x\,\mathrm{d}y &= \int_{-1}^{1} \int_{x^2}^{1} -(1+x^2)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_{-1}^{1} -(1+x^2)y\,\Big|_{y=x^2}^{1}\,\mathrm{d}x \\ &= \int_{-1}^{1} x^4 - 1\,\mathrm{d}x \\ &= \frac{x^5}{5} - x\,\Big|_{x=-1}^{1} = -\frac{8}{5}. \end{aligned}$$

For the left-hand side of ($\star$), we have

$$\int_{\partial\Omega} p\,\mathrm{d}x + q\,\mathrm{d}y = \int_{\Gamma_1} p\,\mathrm{d}x + q\,\mathrm{d}y + \int_{\Gamma_2} p\,\mathrm{d}x + q\,\mathrm{d}y.$$

On Γ_1, we see that $x = t$ and $y = t^2$ so that $\mathrm{d}x = \mathrm{d}t$ and $\mathrm{d}y = 2t\,\mathrm{d}t$. Hence, we get

$$\begin{aligned} \int_{\Gamma_1} p\,\mathrm{d}x + q\,\mathrm{d}y &= \int_{-1}^{1} (t)^2(t^2)\,\mathrm{d}t + \int_{-1}^{1} (-t)\cdot 2t\,\mathrm{d}t \\ &= \int_{-1}^{1} t^4 - 2t^2\,\mathrm{d}t = -\frac{14}{15}. \end{aligned}$$

On the other hand, on Γ_2, we have $x = 2-t$ and $y = 1$ so that $\mathrm{d}x = (-1)\,\mathrm{d}t$ and $\mathrm{d}y = 0$. Thus, we obtain

$$\begin{aligned} \int_{\Gamma_2} p\,\mathrm{d}x + q\,\mathrm{d}y &= \int_{1}^{3} (2-t)(1)\cdot(-1)\,\mathrm{d}t + \int_{1}^{3} (t-2)\cdot 0\,\mathrm{d}t \\ &= -\int_{1}^{3} (2-t)^2\,\mathrm{d}t = -\frac{2}{3}. \end{aligned}$$

Combining these results, we therefore have

$$\int_{\partial\Omega} p\,\mathrm{d}x + q\,\mathrm{d}y = -\frac{14}{15} - \frac{2}{3} = -\frac{8}{5} = \iint_{\Omega} (q_x - p_y)\,\mathrm{d}x\,\mathrm{d}y,$$

which verifies Green's Theorem in this case.

We will see many consequences of Green's Theorem when we study harmonic functions. Nonetheless, we can already state two important consequences.

- **Path independence.** One can use Green's Theorem to obtain a shorter proof of the fact that the line integral of an exact form $\mathrm{d}u$ on a loop is always zero (as in Lecture 3), in the case where u is of class C^2.
- **Poincaré Lemma.** Another consequence is known as the Poincaré Lemma, which states that if $p, q \in C^1(\Omega)$ for a *simply connected* domain Ω, then $p\,\mathrm{d}x + q\,\mathrm{d}y$ is exact if and only if $p_y = q_x$.

We leave the proofs of both of these results to Assignment 2, but we note that the forward direction of the Poincaré Lemma was proved on Assignment 1.

5 Outward normal vector, harmonic functions in the plane

Given an arclength parametrization $\sigma : [0, L] \to \Gamma \subseteq \mathbb{R}^2$ of a plane curve, recall that the unit tangent and outward normal vectors to Γ at the point $\sigma(s)$ are defined, respectively, as $T(s) := \sigma'(s) = (\sigma_1'(s), \sigma_2'(s))$ and $N(s) := (\sigma_2'(s), -\sigma_1'(s))$.

If $z_0 = \sigma(s_0)$ for some $s_0 \in [0, L]$, we set $N(z_0) := N(s_0)$. Consider a function $u : \Omega \to \mathbb{R}$ on a domain $\Omega \subseteq \mathbb{R}^2$, and assume that its gradient

$$\nabla u = (u_x, u_y)$$

is defined on Ω. We set

$$\frac{\partial u}{\partial n}(z_0) := \nabla u(z_0) \cdot N(z_0).$$

REMARK 5.1. If u is differentiable at z_0, then $\frac{\partial u}{\partial n}(z_0)$ is the directional derivative of u in the direction $N(z_0)$ and is equal to the rate of change of u as one moves away from z_0 in the direction $N(z_0)$.

EXAMPLE 5.2. Consider the circle $C = \{z \in \mathbb{R}^2 : |z| = r\}$ of radius $r > 0$, with arclength parametrization $\sigma : [0, 2\pi r] \to C : s \mapsto (r\cos(s/r), r\sin(s/r))$. We saw in Example 3.10 that $N(s) = (\cos(s/r), \sin(s/r))$. Now, let $u \in C^1(\mathbb{R}^2)$. Then

$$\frac{\partial u}{\partial n} = u_x \cos(s/r) + u_y \sin(s/r).$$

In fact, we have $\frac{\partial u}{\partial n} = \frac{\partial u}{\partial r}$. Indeed, in terms of polar coordinates $z = (r\cos t, r\sin t)$, observe that

$$\frac{\partial u(r,t)}{\partial r} = \frac{\partial}{\partial r}(u(r\cos t, r\sin t)) = u_x \cos t + u_y \sin t.$$

It then follows that $\frac{\partial u}{\partial n} = u_x \cos(s/r) + u_y \sin(s/r) = \frac{\partial u}{\partial r}$ on C.

REMARK 5.3. In general, we have

$$\frac{\partial u}{\partial n}\,\mathrm{d}s = -u_y\,\mathrm{d}x + u_x\,\mathrm{d}y.$$

To see this, first note that

$$\begin{aligned}\frac{\partial u}{\partial n}(\sigma(s)) &= \nabla u(\sigma(s)) \cdot N(\sigma(s)) \\ &= (u_x(\sigma(s)), u_y(\sigma(s))) \cdot (\sigma_2'(s), -\sigma_1'(s)) \\ &= -u_y(\sigma(s))\sigma_1'(s) + u_x(\sigma(s))\sigma_2'(s).\end{aligned}$$

Therefore, we see that

$$\frac{\partial u}{\partial n}(\sigma(s))\,\mathrm{d}s = -u_y(\sigma(s))\sigma_1'(s)\,\mathrm{d}s + u_x(\sigma(s))\sigma_2'(s)\,\mathrm{d}s.$$

Now, notice that on Γ, we have $z = (\sigma_1(s), \sigma_2(s))$ so that $\mathrm{d}x = \sigma_1'(s)\,\mathrm{d}s$ and $\mathrm{d}y = \sigma_2'(s)\,\mathrm{d}s$. Thus, we obtain the desired formula

$$\frac{\partial u}{\partial n}\,\mathrm{d}s = -u_y\,\mathrm{d}x + u_x\,\mathrm{d}y.$$

As a direct corollary of Green's Theorem, we have the following result.

THEOREM 5.4 (Inside-Outside Theorem). Let $\Omega \subseteq \mathbb{R}^2$ be a k-connected Jordan domain, and let $u \in C^2(\Omega^+)$ for some domain Ω^+ containing Ω and $\partial\Omega$. Then

$$\int_{\partial\Omega} \frac{\partial u}{\partial n}\,\mathrm{d}s = \iint_{\Omega} \Delta u\,\mathrm{d}x\,\mathrm{d}y,$$

where $\Delta u = u_{xx} + u_{yy}$ is the Laplacian of u.

Proof. First, note that since $u \in C^2(\Omega^+)$, the partials u_x and u_y of u exist and are of class C^1 on Ω^+. Recall from Remark 5.3 that

$$\frac{\partial u}{\partial n}\,\mathrm{d}s = -u_y\,\mathrm{d}x + u_x\,\mathrm{d}y$$

on $\partial\Omega$. Moreover, by Green's Theorem with $p = -u_y$ and $q = u_x$, we obtain

$$\int_{\partial\Omega} -u_y\,\mathrm{d}x + u_x\,\mathrm{d}y = \iint_\Omega ((u_x)_x - (-u_y)_y)\,\mathrm{d}x\,\mathrm{d}y = \iint_\Omega \Delta x\,\mathrm{d}x\,\mathrm{d}y. \qquad \square$$

We will now focus on harmonic functions as we noted in Lecture 1. The Inside-Outside Theorem will be quite important in this regard.

Definition 5.5. Let $u \in C^2(\Omega)$ for some domain $\Omega \subseteq \mathbb{R}^2$. We say that u is **harmonic** if $\Delta u = 0$, where $\Delta u := u_{xx} + u_{yy}$ is the **Laplacian** of u.

In other words, harmonic functions are solutions to the **Laplace equation** $u_{xx} + u_{yy} = 0$.

Example 5.6.

(1) Constant functions are harmonic on $\mathbb{R}^2$.

(2) Linear polynomials are harmonic on $\mathbb{R}^2$. Indeed, suppose $u(x, y) = ax + by + c$ for some $a, b, c \in \mathbb{R}$. Then $u_{xx} = u_{yy} = 0$ so that $\Delta u = 0$.

(3) The function $u(x, y) = Ae^x \cos y + Be^x \sin y$ is harmonic on $\mathbb{R}^2$ for all $A, B \in \mathbb{R}$.

(4) Linear combinations of harmonic functions are harmonic. That is, if u and v are harmonic on a domain $\Omega \subseteq \mathbb{R}^2$, then so is $Au + v$ for all $A \in \mathbb{R}$.

(5) Products of harmonic functions might not be harmonic. For instance, it follows from (2) that $u(x, y) = x$ is harmonic, but $v(x, y) = (u(x, y))^2 = x^2$ is not.

We leave it as an exercise to check that (3), (4), and (5) hold.

The Laplace equation is a very important equation that has many applications in mathematics and physics. We will see later that the real and complex parts of differentiable complex functions are harmonic.

We now give a very useful characterization of harmonic functions in terms of line integrals.

Theorem 5.7. Let $\Omega \subseteq \mathbb{R}^2$ be a domain and let $u \in C^2(\Omega)$. Then u is harmonic in Ω if and only if for every Jordan curve $\Gamma \subseteq \mathbb{R}^2$ inside Ω whose interior lies in Ω, we have

$$\int_\Gamma \frac{\partial u}{\partial n}\,\mathrm{d}s = 0.$$

Proof. Suppose that u is harmonic. Then $\Delta u = 0$, and by the Inside-Outside Theorem, if Γ is any Jordan curve inside Ω with interior lying in Ω, then

$$\int_\Gamma \frac{\partial u}{\partial n}\,\mathrm{d}s = 0.$$

Conversely, suppose that $\int_\Gamma \frac{\partial u}{\partial n}\,\mathrm{d}s = 0$ for every Jordan curve inside Ω whose interior lies in Ω. Suppose to the contrary that there exists $z_0 \in \Omega$ such that $\Delta u(z_0) \neq 0$. Assume without loss of generality that $\Delta u(z_0) > 0$. Since $u \in C^2(\Omega)$, we see that Δu is continuous on Ω. By the continuity of Δu, there exists an open neighbourhood W of z_0 contained in Ω such that $\Delta u(w) > 0$ for all $w \in W$. In particular, since W is open, there exists $r > 0$ with $D(z_0; r) \subseteq W \subseteq \Omega$. Hence, $\Delta u(z) > 0$ for all $z \in D(z_0; r)$ so that

$$\iint_{D(z_0;r)} \Delta u\,\mathrm{d}x\,\mathrm{d}y > 0.$$

However, the boundary of $D(z_0; r)$ is the circle $C = \{z \in \mathbb{R}^2 : |z - z_0| = r\}$, which is a Jordan curve inside Ω whose interior $D(z_0; r)$ lies inside Ω. Moreover, orienting C counter-clockwise making $D(z_0; r)$ into a Jordan domain, it follows from the Inside-Outside Theorem that

$$\int_C \frac{\partial u}{\partial n}\, \mathrm{d}s = \iint_{D(z_0;r)} \Delta u \, \mathrm{d}x \, \mathrm{d}y > 0,$$

contradicting our hypothesis. Thus, $\Delta u = 0$ everywhere on Ω, and so u is harmonic. □

6 Mean Value Theorem

Recall that for a continuous real-valued function of one variable $f : [a, b] \to \mathbb{R}$, the Mean Value Theorem (MVT) states that if f is differentiable on (a, b), then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a},$$

and this value is called the **mean value of f on** $[a, b]$.

There is also an integral version of the MVT which states that if f is continuous on $[a, b]$, then there exists $c \in (a, b)$ such that

$$f(c) = \frac{1}{b - a} \int_a^b f(t)\, \mathrm{d}t.$$

We can see that this is a direct consequence of the MVT. Indeed, for $x \in [a, b]$, let

$$F(x) := \int_a^x f(t)\, \mathrm{d}t.$$

By the Fundamental Theorem of Calculus, we know that F is continuous on $[a, b]$ and differentiable on (a, b), with $F'(x) = f(x)$ for all $x \in (a, b)$. Thus, by the MVT, there exists $c \in (a, b)$ such that

$$F'(c) = \frac{F(b) - F(a)}{b - a}.$$

Since $F(a) = 0$, $F(b) = \int_a^b f(t)\, \mathrm{d}t$, and $F'(c) = f(c)$, this gives

$$f(c) = \frac{1}{b - a} \int_a^b f(t)\, \mathrm{d}t.$$

Note that $f(c)$ is again interpreted as the average or mean value of f on $[a, b]$. We will see that this theorem generalizes to harmonic functions as the Circumferential Mean Value Theorem.

The MVT and integral MVT can also be extended for real functions of two variables (and more generally, n variables). We only state the integral version of the MVT in two variables here. Let A be a connected, compact subset of $\mathbb{R}^2$, and let f be a continuous real-valued function on A. Then, there exists $z_0 \in A$ such that

$$f(z_0) = \frac{1}{\mathrm{area}(A)} \iint_A f(x, y)\, \mathrm{d}x\, \mathrm{d}y,$$

where area(A) is the area of A.

Example 6.1. If $A = \overline{D}(z_0; R)$, then area$(A) = \pi R^2$. Therefore, for any continuous real-valued function u on $\overline{D}(z_0; R)$, there exists $\zeta \in \overline{D}(z_0; R)$ such that

$$u(\zeta) = \frac{1}{\pi R^2} \iint_{D(z_0; r)} u\, \mathrm{d}x\, \mathrm{d}y. \tag{1}$$

We will see that for harmonic functions u, equation (1) holds with $\zeta = z_0$ for any z_0 in the domain Ω of u and any disc centered at z_0 whose domain is in Ω. This is known as the Solid Mean Value Theorem, which we will prove at the end of this lecture.

Before discussing the Circumferential Mean Value Theorem, we first need to define line integrals of continuous scalar functions.

Definition 6.2. Let $\alpha : [a, b] \to \Gamma$ be a smooth parametrization of a plane curve in a domain Ω, and let $f : \Omega \to \mathbb{R}$ be continuous. We define

$$\int_\Gamma f(x, y)\, \mathrm{d}s := \int_a^b f(\alpha(t)) |\alpha'(t)|\, \mathrm{d}t.$$

This quantity is called the **line integral of the scalar function** $f : \Omega \to \mathbb{R} : (x, y) \mapsto f(x, y)$ **along** Γ.

Note that these line integrals are well-defined since $f(\alpha(t))|\alpha'(t)|$ is continuous on $[a,b]$. Moreover, if f is the constant function 1, then $\int_\Gamma \mathrm{d}s = \int_a^b |\alpha'(t)|\,\mathrm{d}t$ is the arclength of Γ.

EXAMPLE 6.3. Consider the circle $C = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = R^2\}$ parametrized by $\alpha : [0, 2\pi] \to C : \theta \mapsto (R\cos\theta, R\sin\theta)$. Then $\alpha'(\theta) = (-R\sin\theta, R\cos\theta)$ and $|\alpha'(\theta)| = R$. Therefore, for any continuous function $f : \mathbb{R}^2 \to \mathbb{R}$, we have $f(\alpha(\theta)) = f(R,\theta)$, and

$$\int_C f(x,y)\,\mathrm{d}s = \int_0^{2\pi} f(R,\theta)R\,\mathrm{d}\theta = 0.$$

Note that unlike line integrals of vector-valued functions, one obtains the same value for $\int_\Gamma f(x,y)\,\mathrm{d}s$ for any parametrization $\alpha : [a,b] \to \Gamma$ of the plane curve (not even the orientation matters).

PROPOSITION 6.4. A line integral of a scalar function along a smooth curve is independent of the parametrization.

PROOF. Let $\alpha : [a,b] \to \Gamma$ be a smooth parametrization of the curve Γ inside the domain Ω, and let $f : \Omega \to \mathbb{R}$ be a continuous scalar function on Ω. We saw in Lecture 3 that α is equivalent to the arclength parametrization $\sigma : [0, L] \to \Gamma$, where L is the length of Γ and σ is such that $|\sigma'(s)| = 1$. Therefore, $\alpha = \sigma \circ E$ for some equivalence $E : [a,b] \to [0,L]$. Let us verify that

$$\int_a^b f(\alpha(t))|\alpha'(t)|\,\mathrm{d}t = \int_0^L f(\sigma(s))\,\mathrm{d}s. \tag{2}$$

Consider the change of variable $s = E(t)$. Then $\mathrm{d}s = E'(t)\,\mathrm{d}t$ and $\alpha'(t) = \sigma'(E(t))E'(t)$ so that

$$|\alpha'(t)| = |\sigma'(E(t))E'(t)| = |\sigma'(E(t))||E'(t)| = E'(t)$$

as $|\sigma'(E(t))| = 1$ and $E'(t) > 0$ on $[0, L]$. Therefore, we have

$$f(\alpha(t))|\alpha'(t)|\,\mathrm{d}t = f(\sigma(E(t)))E'(t)\,\mathrm{d}t = f(\sigma(s))\,\mathrm{d}s.$$

Since $0 = E(a)$ and $L = E(b)$, we obtain equation (2). Thus, every parametrization of Γ yields a line integral that is equal to $\int_0^L f(\sigma(s))\,\mathrm{d}s$. By construction, the arclength parametrization of α depends only on the orientation of α, since $\sigma(s)$ is the point on Γ at a distance s from $\sigma(0) = \alpha(a)$. Therefore, it only remains to check that if we change the orientation of σ, we still obtain the same value for the line integral.

Consider the curve $-\Gamma$ parametrized by $\tau(r) = \sigma(L - r)$, where $r \in [0, L]$. Then $\tau'(r) = -\sigma'(L - r)$ so that $|\tau'(r)| = |-\sigma'(L - r)|$ and

$$\int_{-\Gamma} f(x,y)\,\mathrm{d}s = \int_0^L f(\tau(r))|\tau'(r)|\,\mathrm{d}r = \int_0^L f(\sigma(L - r))\,\mathrm{d}r.$$

Performing the substitution $s = L - r$, we obtain

$$\int_{-\Gamma} f(x,y)\,\mathrm{d}s = \int_L^0 f(\sigma(s))(-1)\,\mathrm{d}s = \int_0^L f(\sigma(s))\,\mathrm{d}s.$$

Thus, the line integral of the scalar function $f(x,y)$ is independent of the parametrization α. □

We define line integrals of scalar functions over piecewise-smooth curves as the sum of the integrals over the smooth pieces. More precisely, if (Γ, α) is a piecewise-smooth parametrized curve over a domain Γ given by the smooth parametrizations $\alpha_i : [\alpha_i, \alpha_{i+1}] \to \Gamma_i$ for all $0 \le i \le n-1$, then for any continuous scalar function $f : \Omega \to \mathbb{R}$, the line integral of f along Γ is defined as

$$\int_\Gamma f(x,y)\,\mathrm{d}s := \int_{\Gamma_0} f(x,y)\,\mathrm{d}s + \int_{\Gamma_1} f(x,y)\,\mathrm{d}s + \cdots + \int_{\Gamma_{n-1}} f(x,y)\,\mathrm{d}s.$$

THEOREM 6.5 (Circumferential Mean Value Theorem). Let u be a harmonic function on a domain $\Omega \subseteq \mathbb{R}^2$, let z_0 be a point in Ω, and suppose that the closed disc $\overline{D} = \overline{D}(z_0; R)$ is contained in Ω. Set $C = \partial D = \{z \in \mathbb{R}^2 : |z - z_0| = R\}$. Then

$$u(z_0) = \frac{1}{2\pi R}\int_C u(z)\,\mathrm{d}s = \frac{1}{2\pi}\int_0^{2\pi} u(R,\theta)\,\mathrm{d}\theta,$$

where C is oriented counter-clockwise and (r, θ) denote polar coordinates centered at the point z_0. In other words, $u(z_0)$ is the mean value of u along the circle C.

PROOF. Before we proceed with the proof, we note that if $z_0 = (x_0, y_0)$, then the polar coordinates centered at z_0 are given by $x = x_0 + r\cos\theta$ and $y = y_0 + r\sin\theta$ with $r > 0$ and $0 \leq \theta \leq 2\pi$. Moreover, $2\pi R$ is the arclength of the circle C and plays the role of the length $b - a$ of $[a, b]$ in the integral MVT for real one variable functions.

First, observe that since u is harmonic on Ω, we have $\Delta u = 0$ on Ω. Let $0 < r \leq R$ and $C_r = \{z \in \mathbb{R}^2 : |z - z_0| = r\}$ be the circle centered at z_0 of radius r. Then C_r and its interior $D(z_0; r)$ are contained in Ω, so by the Inside-Outside Theorem, we have

$$\int_{C_r} \frac{\partial u}{\partial n}\,\mathrm{d}s = \iint_{D(z_0;r)} \Delta u\,\mathrm{d}x\,\mathrm{d}y = 0.$$

Note that the arclength parametrization of C_r is oriented counter-clockwise; that is, we have

$$\sigma : [0, 2\pi r] \to C_r : s \mapsto (x_0 + r\cos(s/r), y_0 + r\sin(s/r)),$$

where $z_0 = (x_0, y_0)$. If we use polar coordinates centered at z_0 so that $x = x_0 + r\cos\theta$ and $y = y_0 + r\sin\theta$, then

$$\frac{\partial u}{\partial n} = \frac{\partial u}{\partial r}(r,\theta)$$

on C_r (as we saw in Example 5.2). Moreover, since $\theta = s/r$ on C_r, we have that $0 \leq \theta \leq 2\pi$ and $\mathrm{d}s = r\,\mathrm{d}\theta$ on C_r. Therefore, we see that

$$0 = \int_{C_r} \frac{\partial u}{\partial n}\,\mathrm{d}s = \int_0^{2\pi} \frac{\partial u}{\partial r}(r,\theta) r\,\mathrm{d}\theta = r\frac{\mathrm{d}}{\mathrm{d}r}\int_0^{2\pi} u(r,\theta)\,\mathrm{d}\theta,$$

or equivalently

$$\frac{\mathrm{d}}{\mathrm{d}r}\int_0^{2\pi} u(r,\theta)\,\mathrm{d}\theta = 0,$$

which implies that $\int_0^{2\pi} u(r,\theta)\,\mathrm{d}\theta$ does not depend on r. In particular, this means that

$$\int_0^{2\pi} u(r,\theta)\,\mathrm{d}\theta = \int_0^{2\pi} u(R,\theta)\,\mathrm{d}\theta$$

since $0 < r \leq R$, and

$$\lim_{r\to 0}\int_0^{2\pi} u(r,\theta)\,\mathrm{d}\theta = \int_0^{2\pi} u(r,\theta)\,\mathrm{d}\theta.$$

On the other hand, by the integral version of the MVT, there exists $\theta_r \in [0, 2\pi]$ such that

$$u(r,\theta_r) = \frac{1}{2\pi}\int_0^{2\pi} u(r,\theta)\,\mathrm{d}\theta.$$

Taking the limit $r \to 0$ on both sides, we obtain $u(z_0)$ on the left, while the right side is unchanged. Putting it all together, we have

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(R,\theta)\,\mathrm{d}\theta. \qquad \square$$

We list some applications of the Circumferential Mean Value Theorem.

- We can compute line integrals of harmonic functions on circles. For example, suppose we want to evaluate $\int_C \log(x^2+y^2)\,\mathrm{d}s$ where C is the circle centered at $z_0=(1,5)$ with radius $R=3$, oriented counter-clockwise. Since $u(x,y)=\log(x^2+y^2)$ is harmonic on the domain $\mathbb{R}^2\setminus\{0\}$ (check this) and $\overline{D}((1,5);3)\subseteq\mathbb{R}^2\setminus\{0\}$, the Circumferential MVT tells us that

$$\int_C \log(x^2+y^2)\,\mathrm{d}s = 2\pi(3)u(1,5) = 6\pi\log(26).$$

- We can use the Circumferential MVT to give a new characterization of harmonic functions in terms of line integrals over circles, and to prove the Solid MVT.

Definition 6.6. Let $\Omega\subseteq\mathbb{R}^2$ be a domain and let $u\in C^2(\Omega)$. Then u is said to have the **circumferential mean value property** in Ω if for every $z_0\in\Omega$ and every disc $D=D(z_0;R)$ such that $\overline{D}\subseteq\Omega$, we have

$$u(z_0)=\frac{1}{2\pi R}\int_{\partial D} u(z)\,\mathrm{d}s.$$

By the Circumferential MVT, harmonic functions have the circumferential mean value property. In fact, this property is a sufficient condition for harmonicity.

Theorem 6.7. Let $\Omega\subseteq\mathbb{R}^2$ be a domain and let $u\in C^2(\Omega)$. Then u is harmonic if and only if it has the circumferential mean value property.

Proof. It suffices to prove the backward direction. Suppose that for every $z_0\in\Omega$ and every disc $D=D(z_0;R)$ such that $\overline{D}\subseteq\Omega$, we have

$$u(z_0)=\frac{1}{2\pi R}\int_{\partial D} u(z)\,\mathrm{d}s.$$

Let $z_0\in\Omega$ and suppose that $\Delta u(z_0)\neq 0$. Without loss of generality, we may assume that $\Delta u(z_0)>0$ (for otherwise we can work with $-\Delta u(z_0)$). By the continuity of u, there exists $r'>0$ such that $\Delta u>0$ on $D(z_0;r')$. In particular, since Ω is open, we can choose such an r' such that $D(z_0;r')\subseteq\Omega$. Choose $r<r'$ and set $D_r=D(z_0;r)$. Then $\overline{D}_r\subseteq\Omega$ and by assumption,

$$u(z_0)=\frac{1}{2\pi r}\int_{\partial D_r} u(z)\,\mathrm{d}s=\frac{1}{2\pi}\int_0^{2\pi} u(r,\theta)\,\mathrm{d}\theta,$$

where (r,θ) denote polar coordinates centered at the point z_0. Differentiating both sides with respect to r and multiplying by r, we obtain

$$0=r\frac{\mathrm{d}}{\mathrm{d}r}\int_0^{2\pi} u(r,\theta)\,\mathrm{d}\theta=\int_0^{2\pi}\frac{\partial u}{\partial r}(r,\theta)\,\mathrm{d}\theta=\int_{C_r}\frac{\partial u}{\partial n}\,\mathrm{d}s=\iint_{D_r}\Delta x\,\mathrm{d}x\,\mathrm{d}y,$$

where C_r is the circle of radius r and the last equality follows from the Inside-Outside Theorem. By the Bump Principle, we then have $\Delta u=0$ on $D_r=D(z_0;r)$, contradicting our assumption that $\Delta u(z_0)\neq 0$. Hence, u is harmonic on Ω. □

Remark 6.8. One can actually prove the stronger statement that if u satisfies the circumferential mean value property, then u is not only harmonic, but also smooth. Thus, all harmonic functions are smooth.

The proof requires some notions that we do not need in this course, so we will omit it. However, we will see a different proof of this fact when studying complex analytic functions.

Theorem 6.9 (Solid Mean Value Theorem). Let u be a harmonic function on a domain $\Omega\subseteq\mathbb{R}^2$, let z_0 be a point in Ω, and suppose the closed disc $\overline{D}=\overline{D}(z_0;R)$ is contained in Ω. Then

$$u(z_0)=\frac{1}{\pi R^2}\iint_D u\,\mathrm{d}x\,\mathrm{d}y.$$

In particular, $u(z_0)$ is the mean value of u over the disc D.

Proof. By the Circumferential MVT, we know that

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(r, \theta)\, \mathrm{d}\theta$$

for all $0 < r \leq R$. Multiplying both sides by $r\, \mathrm{d}r$ and integrating from $r = 0$ to $r = R$, we get

$$\int_0^R u(z_0) r\, \mathrm{d}r = \frac{1}{2\pi} \int_0^R \int_0^{2\pi} u(r, \theta) r\, \mathrm{d}\theta\, \mathrm{d}r.$$

The left side of the equation gives

$$\int_0^R u(z_0) r\, \mathrm{d}r = u(z_0) \int_0^R r\, \mathrm{d}r = u(z_0) \frac{R^2}{2}.$$

On the other hand, the right side of the equation is equal to

$$\frac{1}{2\pi} \int_0^R \int_0^{2\pi} u(r, \theta) r\, \mathrm{d}\theta\, \mathrm{d}r = \frac{1}{2\pi} \int_0^{2\pi} \int_0^R u(r, \theta) r\, \mathrm{d}r\, \mathrm{d}\theta$$

by Fubini's Theorem. Therefore, we get

$$u(z_0) = \frac{1}{\pi R^2} \int_0^{2\pi} \int_0^R u(r, \theta) r\, \mathrm{d}r\, \mathrm{d}\theta.$$

However, we have

$$\int_0^{2\pi} \int_0^R u(r, \theta) r\, \mathrm{d}r\, \mathrm{d}\theta = \iint_D u\, \mathrm{d}x\, \mathrm{d}y.$$

To see this, let us use a polar change of coordinates to express $\iint_D u\, \mathrm{d}x\, \mathrm{d}y$ as an integral in terms of r and θ. Suppose that $z_0 = (x_0, y_0)$. Then $D = D(z_0; R) = \{z \in \mathbb{R}^2 : |z - (x_0, y_0)| < R\}$, which can be described in polar coordinates by setting $x = x_0 + r\cos\theta$ and $y = y_0 + r\sin\theta$. Then $0 \leq \theta \leq 2\pi$, $0 \leq r \leq R$, and $\mathrm{d}x\, \mathrm{d}y = r\, \mathrm{d}r\, \mathrm{d}\theta$ by the change of variable formula. Hence, we obtain

$$u(z_0) = \frac{1}{\pi R^2} \iint_D u\, \mathrm{d}x\, \mathrm{d}y. \qquad \square$$

To finish this lecture, we give some applications of the Solid Mean Value Theorem.

- We can compute double integrals of harmonic functions over discs. For example, suppose we want to evaluate $\iint_D e^x \cos y\, \mathrm{d}x\, \mathrm{d}y$ where $D = D((-1, 3); 1/4)$. Since $u(x, y) = e^x \cos y$ is harmonic on $\mathbb{R}^2$ (as noted in Example 5.6), the Solid MVT implies that

$$\iint_D e^x \cos y\, \mathrm{d}x\, \mathrm{d}y = \pi(1/4)^2 u(-1, 3) = \frac{\pi\cos(3)}{16e}.$$

- The Solid MVT can be used to prove Harnack's Inequality, which is a key ingredient in the proof of Liouville's Theorem.

7 Maximum Principle

Let f be a continuous real-valued function on a compact subset E of $\mathbb{R}^2$. Recall that the Extreme Value Theorem states that f attains both its maximum and minimum values on E, and that they occur either on ∂E or at a critical point of f in E°.

For harmonic functions, which are always continuous, we can say much more. This is a result of the Strong Maximum Principle and its corollaries. To be precise, we will see the following results:

- If u is a non-constant harmonic function on a domain $\Omega \subseteq \mathbb{R}^2$, then u reaches its maximum and minimum on $\partial\Omega$ only.
- Let $\Omega \subseteq \mathbb{R}^2$ be a bounded domain. If u is continuous on $\overline{\Omega}$, harmonic on Ω, and vanishes on $\partial\Omega$, then u is identically zero on Ω.

The second point tells us that harmonic functions on bounded domains are completely determined by their behaviour on the boundary.

Theorem 7.1 (Strong Maximum Principle). Let u be a harmonic function on a domain $\Omega \subseteq \mathbb{R}^2$, and suppose that it has a maximum or minimum in Ω. Then u is constant.

Proof. Without loss of generality, suppose that u has a maximum at $z_0 \in \Omega$ with $u(z_0) = c$. Then $u(z) \leq c$ for all $z \in \Omega$. Note that by the continuity of u, the set $A := u^{-1}(c)$ of points in Ω where u is equal to c is closed. Moreover, we have $A \neq \varnothing$ since $z_0 \in A$. By connectedness of Ω, it suffices to show that A is also open, as the only non-empty subset of Ω which is both open and closed is Ω itself. Since the choice of the point $z_0 \in A$ was arbitrary, it is enough to show that there exists $r > 0$ such that $D(z_0; r) \subseteq A$.

Choose $r > 0$ such that $\overline{D}(z_0; r) \subseteq \Omega$ (which is always possible since Ω is open so that $D(z_0; r') \subseteq \Omega$ for some $r' > 0$, and we can take $r = r'/2$). Let us verify that $D(z_0; r) \subseteq A$. Suppose that there exists $z_1 \in D(z_0; r)$ such that $z_1 \notin A$. In particular, we have $u(z_1) < c$. By the continuity of u, we see that u is less than c in an open disc D' centered at z_1 and contained in $D(z_0; r)$. Therefore, $u(z) < c$ for all $z \in D'$, which implies that

$$\iint_{D'} u \,\mathrm{d}x\,\mathrm{d}y < \iint_{D'} c \,\mathrm{d}x\,\mathrm{d}y.$$

Moreover, since $u(z) \leq c$ for all $z \in D(z_0; r) \setminus D'$, we have

$$\iint_{D(z_0;r)\setminus D'} u \,\mathrm{d}x\,\mathrm{d}y \leq \iint_{D(z_0;r)\setminus D'} c \,\mathrm{d}x\,\mathrm{d}y.$$

Hence, we see that

$$\begin{aligned}\frac{1}{\pi r^2}\iint_{D(z_0;r)} u \,\mathrm{d}x\,\mathrm{d}y &= \frac{1}{\pi r^2}\left(\iint_{D'} u \,\mathrm{d}x\,\mathrm{d}y + \iint_{D(z_0;r)\setminus D'} u \,\mathrm{d}x\,\mathrm{d}y\right) \\ &< \frac{1}{\pi r^2}\iint_{D(z_0;r)} c \,\mathrm{d}x\,\mathrm{d}y = c = u(z_0),\end{aligned}$$

where the equality follows from the fact that

$$\iint_{D(z_0;r)} c \,\mathrm{d}x\,\mathrm{d}y = c\iint_{D(z_0;r)} 1 \,\mathrm{d}x\,\mathrm{d}y = c\dot{a}\mathrm{rea}(D(z_0; r)) = c\pi r^2.$$

But by the Solid MVT, we get

$$u(z_0) = \frac{1}{\pi r^2}\iint_{D(z_0;r)} u \,\mathrm{d}x\,\mathrm{d}y$$

since u is harmonic on Ω, leading to a contradiction. Hence, u is constantly equal to c on $D(z_0; r)$, which implies that $D(z_0; r) \subseteq A$. □

Remark 7.2. Equivalently, the Strong Maximum Principle states that a non-constant function in a domain $\Omega \subseteq \mathbb{R}^2$ does not assume its maximum or minimum values on Ω.

Remark 7.3. The fact that $\Omega \subseteq \mathbb{R}^2$ is connected (since it is a domain) is essential for the result to hold. Indeed, suppose that Ω is disconnected; that is, there exist disjoint open sets $\Omega_1, \Omega_2 \subseteq \mathbb{R}^2$ such that $\Omega = \Omega_1 \cup \Omega_2$. Consider the function

$$u : \Omega \to \mathbb{R} : z \mapsto \begin{cases} 0 & \text{if } z \in \Omega_1, \\ 1 & \text{if } z \in \Omega_2. \end{cases}$$

Then u is non-constant and satisfies the Laplace equation at every point in Ω. Moreover, u attains its maximum of 1 on Ω_2 and its minimum of 0 on Ω_1. In particular, u assumes both its maximum and minimum on Ω. Nonetheless, this does not contradict the Strong Maximum Principle as Ω is not connected.

Theorem 7.4 (Weak Maximum Principle). Let $\Omega \subseteq \mathbb{R}^2$ be a bounded domain. Let u be continuous on $\overline{\Omega}$ and harmonic on Ω. Then either u is constant, or u attains its maximum and minimum value over $\overline{\Omega}$ on $\partial\Omega$ only.

Proof. Note that since Ω is bounded, so is $\overline{\Omega}$, which implies that $\overline{\Omega}$ is compact. Since u is continuous on $\overline{\Omega}$, we know by the Extreme Value Theorem that u has a maximum and minimum on $\overline{\Omega}$, which occur either on $\partial\Omega$ or at a critical point of u in Ω. If u is non-constant, the Strong Maximum Principle implies that both the maximum and minimum must occur on $\partial\Omega$. □

Example 7.5. Consider $u(x, y) = xy$, which is harmonic on $\mathbb{R}^2$. Then u has a maximum of 1 and a minimum of 0 on the square $\Omega = \{z \in \mathbb{R}^2 : 0 < x, y < 1\}$, which both occur on the boundary; in particular, $u(1, 1) = 1$ and u is zero along the x-axis and y-axis.

Remark 7.6.

(1) The Weak Maximum Principle tells us that any non-constant harmonic function on the bounded domain $\Omega \subseteq \mathbb{R}^2$ that is continuous on $\overline{\Omega}$ does not have critical points in Ω.

(2) Equivalently, the Weak Maximum Principle states that if Ω is a bounded domain and u is a non-constant function which is continuous on $\overline{\Omega}$ and harmonic in Ω, then for every point $z_0 \in \Omega$, we have

$$\min_{z \in \partial\Omega} u(z) < u(z_0) < \max_{z \in \partial\Omega} u(z).$$

(3) The boundedness is crucial. For instance, suppose that $u(x, y) = 2x - y$, which is continuous and harmonic on $\mathbb{R}^2$. The maximum and minimum of u are not attained on the upper-half plane $H = \{z \in \mathbb{R}^2 : y > 0\}$, which is unbounded.

Theorem 7.7. Let $\Omega \subseteq \mathbb{R}^2$ be a bounded domain. Let u and v be functions which are continuous on $\overline{\Omega}$, harmonic on Ω, and equal on $\partial\Omega$ (that is, $u(z) = v(z)$ for all $z \in \partial\Omega$). Then $u = v$ on Ω. In particular, harmonic functions on a bounded domain are completely and uniquely determined by their behaviour on the boundary.

Proof. Since u and v are both continuous on $\overline{\Omega}$ and harmonic on Ω, so is $u - v$. Moreover, $u - v$ vanishes on $\partial\Omega$ since u and v are equal there. Suppose that $u - v$ does not vanish identically on Ω; that is, there exists $z_0 \in \Omega$ such that $(u - v)(z_0) \neq 0$. Without loss of generality, suppose that $(u - v)(z_0) > 0$. This implies that $u - v$ is not constant on $\overline{\Omega}$, so by the Weak Maximum Principle, $u - v$ attains its maximum and minimum values on $\partial\Omega$. However, $(u - v)(z) = 0$ for all $z \in \partial\Omega$ and $(u - v)(z_0) > 0$, so we see that the maximum value of $u - v$ must lie in Ω, a contradiction. □

Remark 7.8.

(1) Setting $v(x, y) = 0$, Theorem 7.7 implies that if u is harmonic on Ω, continuous on $\overline{\Omega}$, and zero on $\partial\Omega$, then u is identically zero on Ω.

(2) As with the Weak Maximum Principle, the boundedness of the domain is crucial. For example, take $u(x, y) = 0$ and $v(x, y) = x$ on the domain $\Omega = \{z \in \mathbb{R}^2 : x < 0\}$. Then u and v are both continuous on $\overline{\Omega}$, harmonic on Ω, and zero on $\partial\Omega$ (which is the y-axis). However, u and v are not equal on Ω.

(3) Theorem 7.7 is specific to harmonic functions (although we will see that complex analytic functions have similar properties). Indeed, continuous or smooth functions can agree on the boundary, but not be equal on the domain. For example, $u(x, y) = 1/(2 - x^2 - y^2)$ and $v(x, y) = x^2 + y^2$ are both smooth on the closed unit disc $\overline{D} = \overline{D}(0; 1)$, and agree on the boundary, which is the unit circle $|z| = 1$. However, they are not equal on the open disc $D = D(0; 1)$ since $u(0, 0) = 1/2$ but $v(0, 0) = 0$.

8 Liouville's Theorem

We consider one last property of harmonic functions before moving on to complex functions. This property deals with **entire** functions, which are functions that are harmonic on all of $\mathbb{R}^2$. Liouville's Theorem states that bounded entire functions are constant. In order to prove this result, we first require Harnack's Inequality.

THEOREM 8.1 (Harnack's Inequality). Let $D = D(z_0; R)$ be an open disc and let u be a harmonic function on D such that $u(z) \geq 0$ for all $z \in D$. Then for all $z \in D$, we have

$$0 \leq u(z) \leq \left(\frac{R}{R - |z - z_0|}\right)^2 u(z_0).$$

PROOF. Let $z \in D$ and set $D' = D(z; R - |z - z_0|)$. For all $z' \in D'$, we have $|z' - z| < R - |z - z_0|$ so that

$$|z' - z_0| \leq |z' - z| + |z - z_0| < (R - |z - z_0|) + |z - z_0| = R.$$

Hence, $D' \subseteq D$. Applying the Solid MVT to the disc D', we obtain

$$0 \leq u(z) = \frac{1}{\pi(R - |z - z_0|)^2} \iint_{D'} u \,\mathrm{d}x \,\mathrm{d}y.$$

Moreover, since $u \geq 0$ on D and $D' \subseteq D$, we see that

$$\iint_{D'} u \,\mathrm{d}x \,\mathrm{d}y \leq \iint_{D} u \,\mathrm{d}x \,\mathrm{d}y = \pi R^2 u(z_0),$$

where the last equality follows from the Solid MVT. Combining these inequalities gives the desired result. $\square$

THEOREM 8.2 (Liouville's Theorem). An entire function that is bounded above or below is constant.

PROOF. Suppose that u is an entire function. Suppose that u is bounded below, so there exists $M \in \mathbb{R}$ such that $u(z) \geq M$ for all $z \in \mathbb{R}^2$. Set $v = u - M$ so that $v \geq 0$ on $\mathbb{R}^2$. Moreover, v is harmonic on $\mathbb{R}^2$ and v is constant if and only if u is constant. Therefore, it suffices to prove that v is constant.

Indeed, since $v(z) \geq 0$ for all $z \in \mathbb{R}^2$, we can apply Harnack's Inequality. Fix $z_0, z_1 \in \mathbb{R}^2$. For any radius $R > |z_1 - z_0|$, we have that

$$0 \leq v(z_1) \leq \left(\frac{R}{R - |z_1 - z_0|}\right)^2 v(z_0).$$

Since v is harmonic on $\mathbb{R}^2$, Harnack's Inequality holds for any $R > 0$ as large as we want as long as $R > |z_1 - z_0|$. In particular, note that

$$\lim_{R \to \infty} \left(\frac{R}{R - |z_1 - z_0|}\right)^2 = 1.$$

Since $v(z_0)$ and $v(z_1)$ are fixed constants, we obtain $v(z_1) \leq v(z_0)$. Reversing the roles of z_0 and z_1, we also find that $v(z_0) \leq v(z_1)$, and hence $v(z_0) = v(z_1)$. Since z_0 and z_1 were arbitrary, it follows that v is constant, and hence so is u.

Finally, suppose u is bounded above. Then $-u$ is entire and bounded below. Hence, $-u$ is constant by what we just showed, which implies that u is also constant. $\square$

Liouville's Theorem is useful as it can help us determine whether or not a given non-constant function is harmonic without having to check if it satisfies the Laplace equation. In particular, if the non-constant function is defined on all of $\mathbb{R}^2$ and is bounded above or below, then it cannot be harmonic.

EXAMPLE 8.3. Consider $f(x, y) = 1/(1 + x^2 + y^2)$, which is defined for all $\mathbb{R}^2$, bounded below by 0, and bounded above by 1. By Liouville's Theorem, f is not harmonic. We can check this explicitly; we have

$$f_{xx} + f_{yy} = \frac{6x^2 - 2(1 + y^2)}{(1 + x^2 + y^2)^3} + \frac{6y^2 - 2(1 + x^2)}{(1 + x^2 + y^2)^3} = \frac{4(x^2 + y^2) - 4}{(1 + x^2 + y^2)^3} \neq 0.$$

9 Complex numbers

Why do we need complex numbers? One reason is to find solutions to certain polynomial equations, such as $x^2 + 1 = 0$. We can construct the complex numbers as a natural extension of the real numbers. Recall that $\mathbb{R}$ equipped with the usual operations of addition and multiplication is a field. In particular, for all $a, b, c \in \mathbb{R}$, we have

(1) $a + b = b + a$;

(2) $a + (b + c) = (a + b) + c$;

(3) $ab = ba$;

(4) $a(bc) = (ab)c$;

(5) $(a + b)c = ac + bc$.

Moreover, we know that the additive and multiplicative identities of $\mathbb{R}$ are 0 and 1 respectively.

As mentioned above, the polynomial $x^2 + 1 = 0$ has no real solution. Therefore, we need to extend our number system to allow such equations to have solutions. Indeed, let $i := \sqrt{-1}$ denote a solution to the equation $x^2 + 1 = 0$, and note that both i and $-i$ are solutions. Then, this gives rise to the definition of complex numbers.

Definition 9.1. A **complex number** is an expression of the form $z = a + bi$ where $a, b \in \mathbb{R}$. We say that a is the **real part** of z, written $a = \Re(z)$, and b is the **imaginary part** of z, written $b = \Im(z)$. The set of all complex numbers is denoted by $\mathbb{C}$.

Remark 9.2. Let $z = a + bi \in \mathbb{C}$.

(1) If $b = 0$, then $z = a$ is a real number. In particular, we see that $\mathbb{R} \subseteq \mathbb{C}$, since $a = a + 0i \in \mathbb{C}$ for all $a \in \mathbb{R}$. This is a strict subset as $i \notin \mathbb{R}$.

(2) If $a = 0$, then $z = bi$ is said to be **purely imaginary**.

We say that two complex numbers $z = a + bi$ and $w = c + di$ are equal if and only if their real and imaginary parts are equal; that is, $a = c$ and $b = d$.

Remark 9.3. The real numbers are ordered with respect to $>$ (and $\geq$), which gives us the notion of a real number being larger than another real number. On the other hand, there is no natural ordering on $\mathbb{C}$. Thus, if we write $a > b$, this will always imply that $a, b \in \mathbb{R}$.

Given $z = a + bi$, $w = c + di \in \mathbb{C}$, we define the following operations:

$$\begin{aligned}
\textbf{Addition: } & z + w = (a + bi) + (c + di) := (a + c) + (b + d)i; \\
\textbf{Subtraction: } & z - w = (a + bi) - (c + di) := (a - c) + (b - d)i; \\
\textbf{Multiplication: } & z \cdot w = (a + bi)(c + di) := (ac - bd) + (ad + bc)i; \\
\textbf{Division: } & \frac{z}{w} = \frac{a + bi}{c + di} := \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2} i, \text{ where } w \neq 0.
\end{aligned}$$

As expected, we have $i^2 = (0 + i)(0 + i) = -1$. Moreover, we have $(a + bi)(a - bi) = a^2 + b^2 = |a + bi|^2 \in \mathbb{R}$.

Remark 9.4.

(1) The complex numbers $\mathbb{C}$ are closed under the above operations.

(2) The additive and multiplicative identities of $\mathbb{C}$ are 0 and 1 respectively.

(3) Addition and multiplication of complex numbers is commutative, associative, and distributive.

Recall that the Fundamental Theorem of Algebra states that every polynomial with complex coefficients has at least one complex root. For example, the second order polynomial $az^2 + bz + c$ has real roots

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \in \mathbb{R}$$

if $b^2 - 4ac \geq 0$, and complex roots

$$z = \frac{-b \pm i\sqrt{4ac - b^2}}{2a} \in \mathbb{C}$$

if $b^2 - 4ac < 0$.

Note that every complex number $z = a + bi$ is uniquely determined by its real part a and imaginary part b, and hence can be uniquely identified with the ordered pair of real numbers (a, b). Thus, there is a one-to-one correspondence between complex numbers and points in the xy-plane. In this context, the xy-plane is called the **complex plane** or the **z-plane**.

Since real numbers $z = a \in \mathbb{R}$ correspond to points $(a, 0)$ on the x-axis, the x-axis is called the **real axis**. Similarly, the purely imaginary numbers $z = bi$ for $b \in \mathbb{R}$ correspond to points $(0, b)$ on the y-axis, so the y-axis is called the **imaginary axis**. From now on, we will refer to the point (a, b) representing the complex number $z = a + bi$ as simply the point z.

Definition 9.5. The **modulus** of a complex number $z = a + bi$ is denoted by $|z|$ and is given by

$$|z| := \sqrt{a^2 + b^2}.$$

Note that for all $z \in \mathbb{C}$, we have $|z| \geq 0$, and $|z| = 0$ if and only if $z = 0$.

Identifying $z = a + bi$ with the point (a, b) on the xy-plane, we see that

$$|z| = \operatorname{dist}(0, (a, b)) = \operatorname{dist}(0, z).$$

Moreover, if $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$, then

$$\begin{aligned} |z_1 - z_2| &= |(a_1 - a_2) + (b_1 - b_2)i| \\ &= \sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2} \\ &= \operatorname{dist}((a_1, b_1), (a_2, b_2)) \\ &= \operatorname{dist}(z_1, z_2). \end{aligned}$$

In particular, if $z_0 \in \mathbb{C}$ and R is a positive real number, then the set $\{z \in \mathbb{C} : |z - z_0| = R\}$ corresponds to the points in the z-plane at a distance R from z_0; that is, the circle of radius R centered at z_0.

Definition 9.6. Let $z = a + bi \in \mathbb{C}$. The **complex conjugate** (or simply the **conjugate**) of z is defined as the complex number

$$\bar{z} := a - bi.$$

Note that $\bar{z}$ corresponds to the point $(a, -b)$, which is the reflection of the point (a, b) corresponding to z over the real axis.

Proposition 9.7. For all $z \in \mathbb{C}$, we have

(1) $\bar{z} = z$ if $z \in \mathbb{R}$;

(2) $|\bar{z}| = |z|$;

(3) $z\bar{z} = |z|^2$;

(4) $z^{-1} = \bar{z}/|z|^2$ if $z \neq 0$;

(5) $\Re(z) = (z + \bar{z})/2$ and $\Im(z) = (z - \bar{z})/(2i)$.

Proposition 9.8. For all $z_1, z_2 \in \mathbb{C}$, we have

(1) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$;

(2) $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$;

(3) $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$;

(4) $\overline{z_1 / z_2} = \overline{z_1} / \overline{z_2}$.

The above properties suggest another way of computing the quotient of two complex numbers $z_1, z_2 \in \mathbb{C}$ where $z_2 \neq 0$; we have

$$\frac{z_1}{z_2} = \frac{z_1}{z_2} \cdot \frac{\overline{z_2}}{\overline{z_2}} = \frac{z_1 \overline{z_2}}{z_2 \overline{z_2}} = \frac{z_1 \overline{z_2}}{|z_2|^2}.$$

In particular, if $z \neq 0$, then

$$z^{-1} = \frac{1}{z} = \frac{\bar{z}}{|z|^2}.$$

Finally, we list two useful properties of the modulus.

Proposition 9.9.

(1) For all $z_1, z_2 \in \mathbb{C}$, we have

$$|z_1 z_2| = |z_1||z_2|.$$

In particular, for all $z \in \mathbb{C}$ and $n \in \mathbb{N}$, we see that $|z^n| = |z|^n$.

(2) For all $z_1, z_2 \in \mathbb{C}$ with $z_2 \neq 0$, we have

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}.$$

Consequently, we see that the modulus has similar properties to the absolute value over $\mathbb{R}$.

10 Complex functions

Definition 10.1. A **complex function** f of a complex variable z is a rule that assigns to each z in some set $S \subseteq \mathbb{C}$ a complex number. The set S is called the **domain** of f and is denoted $D(f)$; if it is not specified, then it is taken to be the largest possible set. The set of values w resulting from f is called the **range** of f and is denoted $R(f)$; that is, $R(f) := \{w \in \mathbb{C} : w = f(z) \text{ for some } z \in D(f)\}$.

By identifying $z = x + iy$ and $w = u + iv$ with $(x, y), (u, v) \in \mathbb{R}^2$, it is useful to represent f in the form

$$w = f(z) = f(x + iy) = u(x, y) + iv(x, y)$$

where $u, v : D(f) \subseteq C \to \mathbb{R}$ are real-valued functions.

Example 10.2.

(1) Let $z_0 = x_0 + iy_0 \in \mathbb{C}$ and set $f(z) = z + z_0$ for all $z \in \mathbb{C}$. Then $D(f) = R(f) = \mathbb{C}$ and

$$w = u_i v = (x + x_0) + i(y + y_0)$$

so that $u(x, y) = x + x_0$ and $v(x, y) = y + y_0$. Thus, f corresponds to the translation $\mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto (x + x_0, y + y_0)$.

(2) Let $g(z) = \bar{z}$ for all $z \in \mathbb{C}$. Then $D(g) = R(g) = \mathbb{C}$ and $w = u + iv = x - iy$. Hence, $u(x, y) = x$ and $v(x, y) = -y$, so we see that g corresponds to the mapping $\mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto (x, -y)$, which is the reflection with respect to the x-axis.

From these examples, we see that a complex function $w = f(z) = u(x, y) + iv(x, y)$ can be thought of as a mapping from the z-plane to the z-plane, or equivalently, as a mapping $\mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto (u(x, y), v(x, y))$. Consequently, $w = f(z)$ is also called the **image** of z.

Example 10.3.

(1) Let $f(z) = 1/z$. Then $D(f) = R(f) = \mathbb{C} \setminus \{0\}$, and f corresponds to the mapping

$$\mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\} : (x, y) \mapsto \left(\frac{x}{x^2 + y^2}, -\frac{y}{x^2 + y^2} \right).$$

Note that the image of the circle $|z| = R$ is the circle $|w| = 1/R$ for all $R > 0$.

(2) If $f(z) = z^2 = (x^2 - y^2) + i(2xy) = u(x, y) + iv(x, y)$, then $D(f) = R(f) = \mathbb{C}$, and f corresponds to the mapping $\mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto (x^2 - y^2, 2xy)$. Observe that the hyperbolas $x^2 - y^2 = k$ (respectively $xy = \ell$) get mapped to the lines $u = k$ (respectively $v = 2\ell$).

(3) The exponential function $f(z) = e^z := e^x \cos y + ie^x \sin y$ has $D(f) = \mathbb{C}$ and $R(f) = \mathbb{C} \setminus \{0\}$.

Remark 10.4. A complex function may be given in the form

$$f(x, y) = u(x, y) + iv(x, y)$$

for some real-valued functions u and v. To find the expression of f in terms of z, set $x = (z + \bar{z})/2$ and $y = (z - \bar{z})/(2i)$ in the expression of f.

For example, suppose we are given $f(x, y) = 2xy - iy$ for all $(x, y) \in \mathbb{R}^2$. Then

$$f(x, y) = 2xy - iy = 2\left(\frac{z + \bar{z}}{2}\right)\left(\frac{z - \bar{z}}{2i}\right) - i\left(\frac{z - \bar{z}}{2i}\right) = \frac{1}{2i}(z^2 - \bar{z}^2 - iz + i\bar{z}).$$

We now turn our attention to limits.

DEFINITION 10.5. Let $w_0 \in \mathbb{C}$ and let f be a complex function. If for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(z) - w_0| < \varepsilon$$

for all $z \in D(f)$ with $0 < |z - z_0| < \delta$, then we say that w_0 is the **limit of f at z_0** and write

$$\lim_{z \to z_0} f(z) = w_0.$$

In other words, we can make $f(z)$ as close as we like to w_0 by taking z sufficiently close to z_0.

Note that f does not necessarily need to be defined at z_0, but must be defined in the deleted neighbourhood $D(f) \cap \{z \in \mathbb{C} : 0 < |z - z_0| < \delta\}$.

Given the interpretation of complex functions as mappings, we have the following result.

THEOREM 10.6. Suppose that $f(x, y) = u(x, y) + iv(x, y)$. If $z_0 = x_0 + iy_0$ and $w_0 = u_0 + iv_0$, then $\lim_{z \to z_0} f(z) = w_0$ if and only if $\lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0$ and $\lim_{(x,y)\to(x_0,y_0)} v(x, y) = v_0$.

PROOF. Suppose that $\lim_{z \to z_0} f(z) = w_0$. Then for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $0 < |z - z_0| < \delta$, we have $|f(z) - w_0| < \varepsilon$. Hence, we see that

$$|u(x, y) - u_0| \leq \sqrt{(u(x, y) - u_0)^2 + (v(x, y) - v_0)^2} = |f(z) - w_0| < \varepsilon,$$

so we obtain $\lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0$. By the same argument, we have $\lim_{(x,y)\to(x_0,y_0)} v(x, y) = v_0$.

Conversely, suppose that $\lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0$ and $\lim_{(x,y)\to(x_0,y_0)} v(x, y) = v_0$. For all $\varepsilon > 0$, there exists $\delta > 0$ such that $|u(x, y) - u_0| < \varepsilon/2$ and $|v(x, y) - v_0| < \varepsilon/2$. Thus, we have

$$\begin{aligned} |f(z) - w_0| &= \sqrt{(u(x, y) - u_0)^2 + (v(x, y) - v_0)^2} \\ &\leq |u(x, y) - u_0| + |v(x, y) - v_0| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon, \end{aligned}$$

implying that $\lim_{z \to z_0} f(z) = w_0$. $\square$

In other words, if $\lim_{z \to z_0} f(z)$ exists, then

$$\lim_{z \to z_0} f(z) = \lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0 + i \lim_{(x,y)\to(x_0,y_0)} v(x, y) = v_0.$$

EXAMPLE 10.7.

(1) $\lim_{z \to z_0} c = c$ for all $c \in \mathbb{C}$.

(2) $\lim_{z \to z_0} z = \lim_{(x,y)\to(x_0,y_0)} x + iy = x_0 + iy_0 = z_0$.

(3) $\lim_{z \to z_0} z^2 = \lim_{(x,y)\to(x_0,y_0)} (x^2 - y^2) + i(2xy) = (x_0^2 - y_0^2) + i(2x_0y_0) = z_0^2$.

(4) $\lim_{z \to z_0} e^z = \lim_{(x,y)\to(x_0,y_0)} e^x \cos y + ie^x \sin y = e^{x_0} \cos y_0 + ie^{x_0} \sin y_0 = e^{z_0}$.

As a direct consequence of Theorem 10.6, we obtain the following.

PROPOSITION 10.8. If $\lim_{z \to z_0} f(z)$ exists, then it is unique.

PROOF. This follows from the uniqueness of $\lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0$ and $\lim_{(x,y)\to(x_0,y_0)} v(x, y) = v_0$. $\square$

As we would expect, we also have the following properties of limits.

THEOREM 10.9. Suppose that $\lim_{z \to z_0} f(z) = A$ and $\lim_{z \to z_0} g(z) = B$. Then

(i) $\lim_{z \to z_0} f(z) + g(z) = A + B$;

(ii) $\lim_{z \to z_0} f(z)g(z) = AB$; and

(iii) $\lim_{z\to z_0} f(z)/g(z) = A/B$, given that $B \neq 0$.

PROOF. This can be proved directly using the definition, or by using the fact that two-variable real-valued functions satisfy the same properties. Indeed, let us prove property (i). Suppose that $f = u_1 + iv_1$ and $g = u_2 + iv_2$. Then

$$\lim_{z\to z_0} f(z) + g(z) = \lim_{(x,y)\to(x_0,y_0)} (u_1 + u_2) + i(v_1 + v_2) = (\Re(A) + \Re(B)) + i(\Im(A) + \Im(B)) = A + B.$$

The proofs of properties (ii) and (iii) are left as an exercise. □

These properties can be used to compute some more complicated limits, as we will illustrate in the following example.

EXAMPLE 10.10.

(1) Since z^n is the product of z by itself n times, property (i) implies that $\lim_{z\to z_0} z^n = z_0^n$.

(2) By properties (i) and (ii), we have $\lim_{z\to z_0} p(z) = p(z_0)$ for any polynomial $p(z) = a_0 + a_1 z + \cdots + a_n z^n$.

(3) By property (iii), we obtain $\lim_{z\to z_0} p(z)/q(z) = p(z_0)/q(z_0)$ for any two polynomials p and q such that $q(z_0) \neq 0$.

DEFINITION 10.11. We say that a complex number z **approaches** ∞, denoted by $z \to \infty$, if its magnitude approaches ∞; that is, $z \to \infty$ if $|z| \to \infty$. Hence, we write $\lim_{z\to z_0} f(z) = \infty$ if $\lim_{z\to z_0} |f(z)| = \infty$.

More formally, we have $\lim_{z\to z_0} f(z) = \infty$ if for every positive real number $M > 0$, there exists $\delta > 0$ such that

$$|f(z)| > M$$

whenever $0 < |z - z_0| < \delta$, or equivalently, if $\lim_{z\to z_0} 1/f(z) = 0$.

Finally, by $\lim_{z\to\infty} f(z)$, we mean $\lim_{z\to 0} f(1/z)$. In other words, we consider the limit of f for complex numbers z such that $|z| \to \infty$.

EXAMPLE 10.12. We have $\lim_{z\to 1} \frac{z+2}{z^2+1} = \infty$ since $\lim_{z\to 1} \frac{z^2+1}{z+2} = \frac{0}{3} = 0$.

DEFINITION 10.13. Let f be a complex function and let $z_0 \in D(f)$. Then f is said to be **continuous** at z_0 if $\lim_{z\to z_0} f(z)$ exists and $\lim_{z\to z_0} f(z) = f(z_0)$.

In particular, by Theorem 10.6, we see that $f(z) = u(x, y) + iv(x, y)$ is continuous at $z_0 = x_0 + iy_0$ if and only if $u(x, y)$ and $v(x, y)$ are both continuous at (x_0, y_0).

EXAMPLE 10.14.

(1) Polynomials, quotients of polynomials, and e^z are all continuous on their domains.

(2) If $f(z) = \bar{z} = x - iy$, then $u(x, y) = x$ and $v(x, y) = -y$, which are both continuous on $\mathbb{R}^2$. Hence, f is continuous on $\mathbb{C}$.

(3) The function

$$f(z) = \begin{cases} z^2 + 1, & z \neq 0 \\ 2, & z = 0 \end{cases}$$

is not continuous at $z = 0$ since

$$\lim_{z\to 0} f(z) = \lim_{z\to 0} z^2 + 1 = 1 \neq 2 = f(0).$$

By Theorem 10.9, it is easy to see that the following properties hold.

THEOREM 10.15. Suppose that f and g are continuous at $z_0 \in \mathbb{C}$. Then

(i) $f + g$ is continuous at z_0;

(ii) fg is continuous at z_0; and

(iii) f/g is continuous at z_0 if $g(z_0) \neq 0$.

As a consequence, sums, products, and quotients of polynomials are continuous, as well as exponentials.

We now state another useful theorem.

Theorem 10.16. The composition of two continuous complex functions is continuous.

Proof. This is clear since the composition of two mappings whose components are continuous also has continuous components. □

Example 10.17. The functions $f(z) = e^{z^2+1}$ and $g(z) = (z^2 - 2z + 3)^3$ are both continuous on $\mathbb{C}$.

We end this lecture by stating two theorems that are similar to those we have seen in multivariable calculus. We leave the proofs as an exercise.

Theorem 10.18. Let f be a complex function which is continuous at $z_0 \in \mathbb{C}$, and suppose that $f(z_0) \neq 0$. Then there exists a neighbourhood D_{z_0} of z_0 such that $f(z) \neq 0$ for all $z \in D_{z_0}$.

Theorem 10.19 (Extreme Value Theorem). Suppose that f is a complex function which is continuous on the compact set $S \subseteq \mathbb{C}$. Then $|f(z)|$ attains a maximum and a minimum value on S.

11 Derivatives

DEFINITION 11.1. Suppose that f is a complex function defined in some open neighbourhood of $z_0 \in \mathbb{C}$. The **derivative of f at z_0** is given by

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

where $\Delta z = \Delta x + i\Delta y$, given that this limit exists. If this is the case, we say that f is **differentiable at z_0**.

Note that we can equivalently write the above limit as

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

EXAMPLE 11.2.

(1) $\dfrac{\mathrm{d}}{\mathrm{d}z}(c) = \lim_{\Delta z \to 0} \dfrac{c - c}{\Delta z} = 0$ for all $c \in \mathbb{C}$.

(2) $\dfrac{\mathrm{d}}{\mathrm{d}z}(z) = \lim_{\Delta z \to 0} \dfrac{(z + \Delta z) - z}{\Delta z} = \lim_{\Delta z \to 0} \dfrac{\Delta z}{\Delta z} = 1.$

(3)
$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}z}(z^n) &= \lim_{\Delta z \to 0} \frac{(z + \Delta z)^n - z^n}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{z^n + nz^{n-1} + \cdots + (\Delta z)^n - z^n}{\Delta z} \\ &= \lim_{\Delta z \to 0} nz^{n-1} + \frac{n(n-1)}{2} z^{n-2}\Delta z + \cdots + (\Delta z)^{n-1} = nz^{n-1}. \end{aligned}$$

EXAMPLE 11.3. The function $f(z) = \bar{z}$ is not differentiable at any $z_0 \in \mathbb{C}$. To prove this, we fix $z_0 \in \mathbb{C}$ and show that the limit

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{(\overline{z_0} + \overline{\Delta z}) - \overline{z_0}}{\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$$

does not exist. We compute this limit along two different paths leading to 0.

Along the line $y = 0$ (the x-axis), we have $z = \bar{z} = x$ and $\Delta z = \overline{\Delta z} = \Delta x$ so that

$$\lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1.$$

On the other hand, along the line $x = 0$ (the y-axis), we have $z = iy$ and $\bar{z} = -iy$, which gives $\Delta z = i\Delta y$ and $\overline{\Delta z} = -i\Delta y$. Hence, we see that

$$\lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = \lim_{\Delta y \to 0} \frac{-i\Delta y}{i\Delta y} = -1 \neq 1.$$

Since we obtain two different limits along the distinct paths $y = 0$ and $x = 0$, we see that the limit $\lim_{\Delta z \to 0}(f(z_0 + \Delta z) - f(z_0))/\Delta z$ does not exist. Since $z_0 \in \mathbb{C}$ was arbitrary, it follows that $f(z) = \bar{z}$ is not differentiable at any $z_0 \in \mathbb{C}$.

EXAMPLE 11.4. The function $f(z) = \Im(z) = y$ is not differentiable at any $z_0 \in \mathbb{C}$. In this case, we have

$$\begin{aligned} \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} &= \lim_{(\Delta x, \Delta y) \to (0,0)} \frac{\Im((x_0 + iy_0) + (\Delta x + i\Delta y)) - y_0}{\Delta x + i\Delta y} \\ &= \lim_{(\Delta x, \Delta y) \to (0,0)} \frac{(y_0 + \Delta y) - y_0}{\Delta x + i\Delta y} \\ &= \lim_{\Delta z \to 0} \frac{\Delta y}{\Delta z}. \end{aligned}$$

As in Example 11.3, we now compute this limit along two different paths leading to 0. Along the line $y = 0$, we have $\Delta z = \Delta x$ since $\Delta y = 0$, which gives

$$\lim_{\Delta z \to 0} \frac{\Delta y}{\Delta z} = \lim_{\Delta x \to 0} \frac{0}{\Delta x} = 0.$$

Along the line $x = 0$, we have $\Delta z = i\Delta y$ since $\Delta x = 0$, so

$$\lim_{\Delta z \to 0} \frac{\Delta y}{\Delta z} = \lim_{\Delta y \to 0} \frac{\Delta y}{i\Delta y} = \frac{1}{i} = -i \neq 0.$$

Thus, $f(z) = y$ is not differentiable at any $z_0 \in \mathbb{C}$.

Example 11.5. The function $f(z) = z\bar{z}$ is differentiable at $z = 0$, but not differentiable at any other point $0 \neq z \in \mathbb{C}$. Indeed, we have

$$f'(0) = \lim_{\Delta z \to 0} \frac{(0 + \Delta z)(\overline{0 + \Delta z})}{\Delta z} = \lim_{\Delta z \to 0} \overline{\Delta z} = \lim_{(\Delta x, \Delta y) \to (0,0)} \Delta x - i\Delta y = 0.$$

However, if $0 \neq z \in \mathbb{C}$, then

$$\begin{aligned} \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} &= \lim_{\Delta z \to 0} \frac{(z_0 + \Delta z)(\overline{z_0 + \Delta z}) - z_0\overline{z_0}}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{(z_0 + \Delta z)(\overline{z_0} + \overline{\Delta z}) - z_0\overline{z_0}}{\Delta z} \\ &= \lim_{\Delta z \to 0} z_0 \frac{\overline{\Delta z}}{\Delta z} + \overline{z_0} + \overline{\Delta z}, \end{aligned}$$

and this limit does not exist due to the $\overline{\Delta z}/\Delta z$ term, as we saw in Example 11.3.

From Example 11.5, we see that one can have a function that is differentiable at a point but not differentiable anywhere near it. To avoid this type of situation, we would like to impose a stronger condition on complex functions.

Definition 11.6. A complex function f is **analytic at** $z_0 \in \mathbb{C}$ if it is differentiable at z_0 and a neighbourhood of z_0.

Example 11.7.

(1) Since $f(z) = z^n$ is differentiable at every point in $\mathbb{C}$, it is analytic at every point in $\mathbb{C}$.

(2) As we saw in Example 11.5, $f(z) = z\bar{z}$ is not analytic at $z = 0$.

Definition 11.8. A complex function f is said to be **entire** if it is analytic on all of $\mathbb{C}$.

Example 11.9. The function $f(z) = z^n$ is entire.

From the limit properties, we obtain the following differentiation rules.

Proposition 11.10. Suppose that f and g are differentiable at $z_0 \in \mathbb{C}$. Then

(1) $(f + g)'(z) = f'(z) + g'(z)$;

(2) $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$;

(3) $\left(\frac{f}{g}\right)'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{(g(z))^2}$ if $g(z) \neq 0$; and

(4) **(Chain Rule)** if g is differentiable at z and f is differentiable at $g(z)$, then

$$\frac{\mathrm{d}}{\mathrm{d}z}(f(g(z))) = f'(g(z)) \cdot g'(z).$$

PROOF. Exercise. □

These rules are useful for determining the differentiability of functions.

EXAMPLE 11.11.

(a) Polynomials $p(z) = a_0 + a_1 x + \cdots + a_n z^n$ are entire, since z^m is differentiable at every $z \in \mathbb{C}$ for all non-negative integers m, and we can apply rules (1) and (2).

(b) The function $f(z) = \Im(z) = (z - \bar{z})/(2i)$ is not differentiable at any $z \in \mathbb{C}$. Indeed, suppose to the contrary that $f(z) = \Im(z)$ is differentiable at $z \in \mathbb{C}$. Then, since $-z/(2i)$ is also differentiable at z, we see that

$$\bar{z} = -2i\left(\left(\frac{z - \bar{z}}{2i}\right) + \frac{-z}{2i}\right)$$

is differentiable at z as it is a linear combination of differentiable functions. But we saw in Example 11.3 that $\bar{z}$ is nowhere differentiable, so $f(z) = \Im(z)$ must be nowhere differentiable as well.

(c) The function $f(z) = \bar{z}^{1/2}$ is nowhere differentiable. Suppose to the contrary that $f(z) = \bar{z}^{1/2}$ were differentiable at z. Then

$$\bar{z} = (\bar{z}^{1/2})(\bar{z}^{1/2})$$

is also differentiable at z, which is a contradiction. Thus, $f(z) = \bar{z}^{1/2}$ is not differentiable at any $z \in \mathbb{C}$.

We finish this lecture with a familiar property of real differentiable functions.

THEOREM 11.12. If f is differentiable at $z_0 \in \mathbb{C}$, then f is continuous at z_0.

PROOF. If f is differentiable at z_0, then we know that the limit

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

exists. To prove that f is continuous at z_0, we need to show that $\lim_{z \to z_0} f(z) = f(z_0)$, or equivalently, that $\lim_{z \to z_0} f(z) - f(z_0) = 0$. Indeed, we have

$$\begin{aligned}
\lim_{z \to z_0} f(z) - f(z_0) &= \lim_{\Delta z \to 0} f(z_0 + \Delta z) - f(z_0) \\
&= \lim_{\Delta z \to 0} \left(\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}\right) \cdot \Delta z \\
&= f'(z_0) \cdot 0 = 0.
\end{aligned}$$

□

This theorem gives us another way of testing the differentiability of a function. In particular, if f is not continuous at $z_0 \in \mathbb{C}$, then f cannot be differentiable at z_0.

In the next lecture, we will see that differentiable functions satisfy the Cauchy-Riemann equations, which gives is another method of testing for differentiability.

12 The Cauchy-Riemann equations

In Lecture 10, we saw that for any complex function $f(z) = u(x, y) + iv(x, y)$, we have

$$\lim_{z\to z_0} f(z) = w_0 = u_0 + iv_0 \iff \lim_{z\to z_0} u(x, y) = u_0 \text{ and } \lim_{z\to z_0} v(x, y) = v_0,$$

and that f is continuous at z_0 if and only if u and v are both continuous at z_0.

In this lecture, we will explore necessary and sufficient conditions for the differentiability of f at a point $z_0 \in \mathbb{C}$ which can be completely expressed in terms of u and v.

Let $f(z) = u(x, y) + iv(x, y)$ and suppose that $f'(z_0)$ exists at the point $z_0 = x_0 + iy_0$. Then, we see that

$$\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

converges to $f'(z_0)$ along any path leading to 0.

Consider the path $y = y_0$. In this case, we have $\Delta y = 0$ so that $\Delta z = \Delta x + i\Delta y = \Delta x$. It follows that

$$\begin{aligned}\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} &= \frac{[u(z_0 + \Delta x) + iv(z_0 + \Delta x)] - [u(z_0) + iv(z_0)]}{\Delta x} \\ &= \left(\frac{u(z_0 + \Delta x) - u(z_0)}{\Delta x}\right) + i\left(\frac{v(z_0 + \Delta x) - v(z_0)}{\Delta x}\right),\end{aligned}$$

and hence

$$\lim_{\Delta x\to 0} \frac{f(z_0 + \Delta x) - f(z_0)}{\Delta x} = f'(z_0)$$

is equivalent to both the equations

$$\begin{aligned}\lim_{\Delta x\to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} &= \Re(f'(z_0)), \\ \lim_{\Delta x\to 0} \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x} &= \Im(f'(z_0))\end{aligned}$$

holding; in particular, $u_x(z_0)$ and $v_x(z_0)$ exist, and we have $f'(z_0) = u_x(z_0) + iv_x(z_0)$.

Next, consider the path $x = x_0$. Here, we have $\Delta x = 0$ so that $\Delta z = i\Delta y$, and hence

$$\begin{aligned}\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} &= \frac{[u(z_0 + i\Delta y) + iv(z_0 + i\Delta y)] - [u(z_0) + iv(z_0)]}{i\Delta y} \\ &= \left(\frac{v(z_0 + i\Delta y) - v(z_0)}{\Delta y}\right) - i\left(\frac{u(z_0 + i\Delta y) - u(z_0)}{\Delta y}\right),\end{aligned}$$

Therefore, we see that

$$\lim_{i\Delta y\to 0} \frac{f(z_0 + i\Delta y) - f(z_0)}{i\Delta y} = f'(z_0)$$

is equivalent to the equations

$$\begin{aligned}\lim_{\Delta y\to 0} \frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{\Delta y} &= \Re(f'(z_0)), \\ \lim_{\Delta y\to 0} \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{\Delta y} &= -\Im(f'(z_0))\end{aligned}$$

holding, so $u_y(z_0)$ and $v_y(z_0)$ both exist with $f'(z_0) = v_y(z_0) - iu_y(z_0)$.

Thus, if $f'(z_0)$ exists, then the partials of u and v exist at (x_0, y_0) and satisfy the **Cauchy-Riemann equations**

$$\begin{aligned} u_x(x,y) &= v_y(x,y), \\ u_y(x,y) &= -v_x(x,y) \end{aligned}$$

at (x_0, y_0). We summarize these results with the following theorem.

THEOREM 12.1 (Necessary conditions for differentiability). Suppose that $f(z) = u(x,y) + iv(x,y)$ is differentiable at $z_0 = x_0 + iy_0$. Then the partials u_x, u_y, v_x, v_y exist at (x_0, y_0), and we have

$$f'(z_0) = u_x(x_0,y_0) + iv_x(x_0,y_0) = v_y(x_0,y_0) - iu_y(x_0,y_0).$$

Moreover, if f is analytic on an open set U, then the Cauchy-Riemann equations must hold at every point in U.

EXAMPLE 12.2. The function $f(z) = z^2 = (x^2 - y^2) + i(2xy)$ is entire as it is a polynomial. By Theorem 12.1, the Cauchy-Riemann equations hold everywhere. Indeed, we have $u(x,y) = x^2 - y^2$ and $v(x,y) = 2xy$, so we see that $u_x = v_y = 2x$ and $u_y = -v_x = -2y$ for all $(x,y) \in \mathbb{R}^2$.

REMARK 12.3. As a consequence of Theorem 12.1, if f does not satisfy the Cauchy-Riemann equations at z_0, then f is not differentiable at z_0.

EXAMPLE 12.4. Let $f(z) = \bar{z} = x - iy$ so that $u(x,y) = x$ and $v(x,y) = -y$. Then $u_x = 1 \neq -1 = v_y$ everywhere, so the Cauchy-Riemann equations are not satisfied for any $z \in \mathbb{C}$. This gives us another way of showing that $f(z) = \bar{z}$ is nowhere differentiable.

It turns out that the converse of Theorem 12.1 does not hold. Consider the function

$$f(z) = \begin{cases} \bar{z}^2/z & z \neq 0, \\ 0 & z = 0. \end{cases}$$

We show that the real and imaginary parts u and v of f have well-defined partials at $(0,0)$ and satisfy the Cauchy-Riemann equations at $(0,0)$, but f is not differentiable at $(0,0)$.

First, we obtain expressions for u and v. If $z \neq 0$, then

$$f(z) = \frac{\bar{z}^2}{z} = \frac{\bar{z}^3}{z\bar{z}} = \frac{(x-iy)^3}{x^2+y^2} = \left(\frac{x^3 - 3xy^2}{x^2+y^2}\right) + i\left(\frac{y^3 - 3x^2y}{x^2+y^2}\right),$$

so $u(x,y) = (x^3 - 3xy^2)/(x^2+y^2)$ and $v(x,y) = (y^3 - 3x^2y)/(x^2+y^2)$. Moreover, since $f(0) = 0$, we have $u(0,0) = v(0,0) = 0$. Thus, we obtain

$$u(x,y) = \begin{cases} (x^3 - 3xy^2)/(x^2+y^2) & (x,y) \neq (0,0), \\ 0 & (x,y) = (0,0); \end{cases}$$

$$v(x,y) = \begin{cases} (y^3 - 3x^2y)/(x^2+y^2) & (x,y) \neq (0,0), \\ 0 & (x,y) = (0,0). \end{cases}$$

Their partials at $(0,0)$ are given by

$$\begin{aligned} u_x &= \lim_{h\to 0} \frac{u(0+h,0) - u(0,0)}{h} = \lim_{h\to 0} \frac{h^3/h^2 - 0}{h} = 1, \\ u_y &= \lim_{h\to 0} \frac{u(0,0+h) - u(0,0)}{h} = \lim_{h\to 0} \frac{0/h^2 - 0}{h} = 0, \\ v_x &= \lim_{h\to 0} \frac{v(0+h,0) - v(0,0)}{h} = \lim_{h\to 0} \frac{0/h^2 - 0}{h} = 0, \\ v_y &= \lim_{h\to 0} \frac{v(0,0+h) - v(0,0)}{h} = \lim_{h\to 0} \frac{h^3/h^2 - 0}{h} = 1. \end{aligned}$$

Thus, we obtain $u_x(0,0) = v_y(0,0) = 1$ and $u_y(0,0) = -v_x(0,0) = 0$, so u and v satisfy the Cauchy-Riemann equations at $(0,0)$.

We now show that f is not differentiable at $z = 0$. Consider the limit

$$\lim_{\Delta z \to 0} \frac{f(0 + \Delta z) - f(0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{f(\Delta z)}{\Delta z}.$$

Taking the path $y = 0$, we have $\Delta z = \Delta x$ so that

$$\lim_{\Delta z \to 0} \frac{f(\Delta z)}{\Delta z} = \lim_{\Delta x \to 0} \frac{f(\Delta x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(\overline{\Delta x})^2/\Delta x}{\Delta x} = \lim_{\Delta x \to 0} \frac{(\Delta x)^2}{(\Delta x)^2} = 1,$$

since $\overline{\Delta x} = \Delta x$. Along the path $y = x$, we have $\Delta x = \Delta y$ so that $\Delta z = \Delta x + i\Delta x = (1+i)\Delta x$, and

$$\begin{aligned}
\lim_{\Delta z \to 0} \frac{f(\Delta z)}{\Delta z} &= \lim_{\Delta x \to 0} \frac{f((1+i)\Delta x)}{(1+i)\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(\overline{(1+i)\Delta x})^2/((1+i)\Delta x)}{(1+i)\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{((1-i)\Delta x)^2}{((1+i)\Delta x)^2} = -1 \neq 1.
\end{aligned}$$

Therefore, the above limit does not exist, so f is not differentiable at $z = 0$ as claimed.

This example shows us that we can have a complex function $f(z) = u(x,y) + iv(x,y)$ such that u_x, u_y, v_x, v_y exist and satisfy the Cauchy-Riemann equations at a point z_0, but is not differentiable at z_0. Therefore, we need to impose stronger conditions on u and v to ensure differentiability.

Theorem 12.5 (Sufficient conditions for differentiability). Suppose that $f(z) = u(x,y) + iv(x,y)$ is defined throughout an open neighbourhood U of the point $z_0 = x_0 + iy_0$, and suppose that u_x, u_y, v_x, v_y exist everywhere in U. If u_x, u_y, v_x, v_y are continuous at (x_0, y_0) and satisfy the Cauchy-Riemann equations at (x_0, y_0), then f is differentiable at z_0 with

$$f'(z_0) = u_x(x_0,y_0) + iv_x(x_0,y_0) = v_y(x_0,y_0) - iu_y(x_0,y_0).$$

Proof. We need to prove that the limit

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

exists, where $\Delta z = \Delta x + i\Delta y$. First, note that

$$\begin{aligned}
\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} &= \lim_{\Delta z \to 0} \frac{[u(z_0 + \Delta z) + iv(z_0 + \Delta z)] - [u(z_0) + iv(z_0)]}{\Delta z} \\
&= \lim_{\Delta z \to 0} \frac{[u(z_0 + \Delta z) - u(z_0)] + i[v(z_0 + \Delta z) - v(z_0)]}{\Delta z}
\end{aligned}$$

Moreover, observe that u and v are differentiable at (x_0, y_0) since u_x, u_y, v_x, v_y are continuous at (x_0, y_0). Therefore, if

$$\begin{aligned}
L_{(x_0,y_0)}(x,y) &= u(x_0,y_0) + u_x(x_0,y_0)(x - x_0) + u_y(x_0,y_0)(y - y_0), \\
L^*_{(x_0,y_0)}(x,y) &= v(x_0,y_0) + v_x(x_0,y_0)(x - x_0) + v_y(x_0,y_0)(y - y_0),
\end{aligned}$$

are the linear approximations of u and v at (x_0, y_0) respectively on U, we have that

$$\begin{aligned}
\lim_{(\Delta x, \Delta y) \to (0,0)} \frac{u(x,y) - L_{(x_0,y_0)}(x,y)}{|(\Delta x, \Delta y)|} &= 0, \\
\lim_{(\Delta x, \Delta y) \to (0,0)} \frac{v(x,y) - L^*_{(x_0,y_0)}(x,y)}{|(\Delta x, \Delta y)|} &= 0
\end{aligned}$$

by the definition of differentiability. Then, setting

$$R_1(x,y) := u(x,y) - L_{(x_0,y_0)}(x,y),$$
$$R_1^*(x,y) := v(x,y) - L^*_{(x_0,y_0)}(x,y),$$

we obtain

$$\lim_{(\Delta x,\Delta y)\to(0,0)} \frac{R_1(x,y)}{|(\Delta x,\Delta y)|} = \lim_{(\Delta x,\Delta y)\to(0,0)} \frac{R_1^*(x,y)}{|(\Delta x,\Delta y)|} = 0.$$

In particular, it follows that

$$\lim_{\Delta z\to 0} \frac{R_1(z_0+\Delta z)}{\Delta z} = \lim_{\Delta z\to 0} \frac{R_1^*(z_0+\Delta z)}{\Delta z} = 0.$$

Now, observe that

$$u(z_0+\Delta z) - u(z_0) = u_x(x_0,y_0)\Delta x - v_x(x_0,y_0)\Delta y + R_1(x_0+\Delta x, y_0+\Delta y),$$
$$v(z_0+\Delta z) - v(z_0) = v_x(x_0,y_0)\Delta x + u_x(x_0,y_0)\Delta y + R_1^*(x_0+\Delta x, y_0+\Delta y)$$

so we get

$$[u(z_0+\Delta z) - u(z_0)] - [v(z_0+\Delta z) - v(z_0)] = [u_x(z_0) + iv_x(z_0)]\Delta z + R_1(z_0+\Delta z) + iR_1^*(z_0+\Delta z).$$

Finally, we see that

$$\begin{aligned}\lim_{\Delta z\to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} &= u_x(x_0,y_0) + iv_x(x_0,y_0) + \lim_{\Delta z\to 0} \frac{R_1(z_0+\Delta z)}{\Delta z} + i\lim_{\Delta z\to 0} \frac{R_1^*(z_0+\Delta z)}{\Delta z} \\ &= u_x(x_0,y_0) + iv_x(x_0,y_0) + 0 + 0 \\ &= u_x(x_0,y_0) + iv_x(x_0,y_0).\end{aligned}$$

Therefore, f is differentiable at z_0, with

$$f'(z_0) = u_x(x_0,y_0) + iv_x(x_0,y_0) = v_y(x_0,y_0) - iu_x(x_0,y_0). \qquad \square$$

We see that Theorem 12.5 is a useful tool to prove that a complex function is differentiable, and we can also compute its derivative.

EXAMPLE 12.6. Consider the function $f(z) = e^z = e^x\cos y + ie^x \sin y$ which is defined on all of $\mathbb{C}$. Letting $u(x,y) = e^x\cos y$ and $v(x,y) = e^x \sin y$, we have

$$u_x = e^x\cos y, \qquad v_x = e^x \sin y,$$
$$u_y = -e^x \sin y, \qquad v_y = e^x\cos y,$$

which are all defined and continuous on all of $\mathbb{R}^2$. Moreover, the Cauchy-Riemann equations hold on all of $\mathbb{R}^2$, so f is differentiable on all of $\mathbb{C}$. Then, we see that

$$f'(z) = u_x(x,y) + iv_x(x,y) = e^x\cos y + ie^x\sin y = e^z$$

for all $z \in \mathbb{C}$.

13 Analytic functions

We end our discussion of the differentiable properties of complex functions by stating a few properties of analytic functions.

Recall that a complex function $f : \Omega \to \mathbb{C}$ on a domain $\Omega \subseteq \mathbb{C}$ is said to be analytic at $z_0 \in \mathbb{C}$ if it is differentiable on a neighbourhood U_{z_0} of z_0 in Ω. In particular, this means that f is analytic at every point in U_{z_0}. Consequently, if U is any open subset of Ω, then f is analytic on U if and only if it is differentiable on U. Furthermore, we saw in Lecture 12 that if f is analytic on U, then its real and imaginary parts satisfy the Cauchy-Riemann equations at every point in U. This allows us to derive serval important properties of analytic functions.

Proposition 13.1. Let $\Omega \subseteq \mathbb{C}$ be a domain, and let $f, g : \Omega \to \mathbb{C}$ be analytic functions.

(i) If $f' = 0$ on Ω, then f is constant on Ω. In particular, if $f' = g'$ on Ω, then $f = g + C$ for some constant $C \in \mathbb{C}$ on Ω.

(ii) If $f(z) = u(x, y) + iv(x, y)$, then u is constant on Ω if and only if v is constant on Ω.

(iii) If $|f|$ is constant on Ω, then f is constant on Ω.

Proof.

(i) If $f = u + iv$ on Ω, then $f' = u_x + iv_x = v_y - iu_y$ on Ω by the analyticity of f. Therefore, $f' = 0$ on Ω if and only if $u_x = u_y = 0$ and $v_x = v_y = 0$, if and only if u and v are constant on Ω (since Ω is connected). Thus, f is constant on Ω. Now, note that since f and g are analytic on Ω, so is $f - g$. If we have $f' = g'$ on Ω, then $(f - g)' = f' - g' = 0$, so $f - g$ is constant on Ω. This implies that $f = g + C$ for some constant $C \in \mathbb{C}$.

(ii) Suppose that $u = C \in \mathbb{C}$ on Ω. Then $u_x = u_y = 0$ on Ω. Note that u and v satisfy the Cauchy-Riemann equations by the analyticity of f, so $v_x = -u_y = 0$ and $v_y = u_x = 0$. Hence, v is constant on Ω by the connectedness of Ω. The converse can be shown analogously.

(iii) We leave this as an exercise. □

Let $\Omega \subseteq \mathbb{C}$ be a domain and let $f : \Omega \to \mathbb{C}$ be analytic. Then the real and imaginary parts u and v satisfy the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$ on Ω. We will see later that u and v are both smooth on Ω so that $u_{xy} = u_{yx}$ and $v_{xy} = v_{yx}$. Consequently, we obtain

$$u_{xx} = (u_x)_x = (v_y)_x = v_{yx} = v_{xy} = (v_x)_y = (-u_y)_y = -u_{yy},$$

so u is harmonic on Ω. Similarly, it can be shown that v is harmonic on Ω. Thus, the real and imaginary parts of analytic functions provide us with many examples of harmonic functions. For instance, we have seen that $z^2 = (x^2 - y^2) + i(2xy)$ and $e^z = e^x \cos y + ie^x \sin y$ are entire, and their real and imaginary parts $x^2 - y^2$, $2xy$, $e^x \cos y$, and $e^x \sin y$ are all harmonic on $\mathbb{R}^2$.

We now turn to elementary complex functions other than polynomials and rational functions. We will cover the complex exponential, trigonometric, logarithmic, and hyperbolic functions, as well as complex powers; for the remainder of this lecture, we will focus on the first two. To motivate their expressions, we recall Euler's formula and how to write complex numbers in exponential form.

Consider the expression

$$z(\theta) := \cos\theta + i \sin\theta$$

where $\theta \in \mathbb{R}$. Then for all $\theta, \theta_1, \theta_2 \in \mathbb{R}$, we have

- $z(\theta_1) \cdot z(\theta_2) = \cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2) = z(\theta_1 + \theta_2)$ and
- $(z(\theta))^{-1} = \cos\theta - i \sin\theta = \cos(-\theta) + i \sin(-\theta) = z(-\theta)$,

suggesting that $z(\theta)$ behaves like an exponential. This motivates the following formula.

DEFINITION 13.2 (Euler's formula). For all $\theta \in \mathbb{R}$, we define

$$e^{i\theta} := \cos\theta + i\sin\theta.$$

For all $\theta, \theta_1, \theta_2 \in \mathbb{R}$ and $n \in \mathbb{R}$, we see that

- $e^{i0} = 1$;
- $e^{i\theta_1} \cdot e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$, and we set $e^{i\theta_1+i\theta_2} := e^{i(\theta_1+\theta_2)}$;
- $(e^{i\theta})^{-1} = e^{i(-\theta)}$ since $e^{i\theta} \cdot e^{-i\theta} = e^{i0} = 1$, and we set $e^{-i\theta} := e^{i(-\theta)}$;
- $e^{i\theta_1}/e^{i\theta_2} = e^{i(\theta_1-\theta_2)}$;
- $(e^{i\theta})^n = e^{in\theta}$, and we set $e^{n(i\theta)} := e^{in\theta}$;
- $\overline{e^{i\theta}} = e^{-i\theta}$.

Therefore, $e^{i\theta}$ indeed behaves like an exponential.

REMARK 13.3.

(1) For all $n \in \mathbb{Z}$, we have
- $e^{2n\pi i} = \cos(2n\pi) + i\sin(2n\pi) = 1$;
- $e^{(2n+1)\pi i} = \cos((2n+1)\pi) + i\sin((2n+1)\pi) = -1$;
- $e^{(4n\pm1)/2\cdot\pi i} = \cos\left(\dfrac{4n\pm1}{2}\pi\right) + i\sin\left(\dfrac{4n\pm1}{2}\pi\right) = \pm i$.

(2) All points on the unit circle $|z| = 1$ can be written as $z = e^{i\theta}$ for some $\theta \in \mathbb{R}$.

(3) Since $e^{i\theta} = \cos\theta + i\sin\theta$ and $e^{-i\theta} = \cos\theta - i\sin\theta$, we can write

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \quad \text{and} \quad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}.$$

From (2) of Remark 13.3, it follows that any non-zero complex number z can be written in the form

$$z = re^{i\theta} = |z|e^{i\arg z},$$

where $r := |z|$ and $\theta =: \arg z$, called the **exponential form** of z.

EXAMPLE 13.4.

(1) $i = e^{\frac{\pi}{2}i}$;

(2) $1 + i = \sqrt{2}e^{\frac{\pi}{4}i}$;

(3) $\sqrt{3} - i = 2e^{-\frac{\pi}{6}i}$;

(4) $-1 = e^{\pi i}$;

(5) $\frac{-1+\sqrt{3}i}{4} = \frac{1}{2}e^{\frac{2\pi}{3}i}$.

The multiplication, division, conjugation, and inverse operation of complex numbers can then be written in exponential form as

- $z_1z_2 = |z_1||z_2|e^{i(\arg z_1+\arg z_2)}$;
- $z^{-1} = \dfrac{1}{|z|}e^{-i\arg z}$;

- $\bar{z} = |z|e^{-i\arg z}$;
- $\dfrac{z_1}{z_2} = \dfrac{|z_1|}{|z_2|}e^{i(\arg z_1 - \arg z_2)}$;
- $z^n = |z|^n e^{i(n \arg z)}$

for all non-zero $z, z_1, z_2 \in \mathbb{C}$.

The exponential form is often useful for computations. For instance, we have

$$\frac{i}{1+i} = \frac{e^{\frac{\pi}{2}i}}{\sqrt{2}e^{\frac{\pi}{4}i}} = \frac{\sqrt{2}}{2}e^{(\frac{\pi}{2}-\frac{\pi}{4})i} = \frac{\sqrt{2}}{2}e^{\frac{\pi}{4}i} = \frac{1}{2} + \frac{i}{2}.$$

For points $z \in \mathbb{C}$ on the unit circle (so that $z = e^{i\theta}$ for some $\theta \in \mathbb{R}$), the identity $z^n = e^{in\theta}$ gives the following.

Proposition 13.5 (de Moivre's formula). For all $n \in \mathbb{Z}$, we have

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta).$$

Proof. We have $(\cos\theta + i\sin\theta)^n = (e^{i\theta})^n = e^{in\theta} = \cos(n\theta) + i\sin(n\theta)$. $\square$

This formula is useful for writing powers of $\sin\theta$ and $\cos\theta$ in terms of $\cos(m\theta)$ and $\sin(n\theta)$. For example, we have

$$\cos^3\theta = \left(\frac{e^{i\theta} - e^{-i\theta}}{2}\right)^3 = \frac{1}{8}(e^{3i\theta} + 3e^{i\theta} + 3e^{-i\theta} + e^{-3i\theta}) = \frac{1}{4}(\cos(3\theta) + 3\cos\theta).$$

Definition 13.6. For any $z = x + iy \in \mathbb{C}$, we define the **exponential function** by

$$e^z := e^x e^{iy} = e^x \cos y + ie^x \sin y.$$

We have seen that $f(z) = e^z$ is continuous on $\mathbb{C}$, and in fact, entire with derivative

$$\frac{\mathrm{d}}{\mathrm{d}z}(e^z) = e^z.$$

Moreover, for any $z = x + iy \in \mathbb{C}$, we see that

$$|e^z| = e^x > 0$$

so that $e^z \neq 0$ for all $z \in \mathbb{C}$. In addition, we have

$$\arg(e^z) = y + 2k\pi,\ k \in \mathbb{Z}.$$

Finally, the range of $f(z) = e^z$ is $\mathbb{C} \setminus \{0\}$. Indeed, let $w \in \mathbb{C} \setminus \{0\}$. Then, if $w = re^{i\theta}$, set $z = \log r + i\theta$. Note that $\log r$ is well-defined since $r = |w| > 0$. Therefore, we have

$$f(z) = e^{\log r + i\theta} = e^{\log r}e^{i\theta} = re^{i\theta} = w,$$

proving that the range of the complex exponential is $\mathbb{C} \setminus \{0\}$.

Remark 13.7. Unlike the real exponential, the complex exponential can take on negative real values. For instance, we have $e^{\pi i} = -1 < 0$.

Remark 13.8. The complex exponential is not injective on $\mathbb{C}$. Indeed, for all $z \in \mathbb{C}$ and $k \in \mathbb{Z}$, notice that

$$\begin{aligned} e^{z+2\pi ki} = e^{x+i(y+2\pi k)} &= e^x(\cos(y + 2\pi k) + i\sin(y + 2\pi k)) \\ &= e^x(\cos y + i\sin y) \\ &= e^x e^{iy} = e^z. \end{aligned}$$

More precisely, we have the following.

THEOREM 13.9. For any $z_1, z_2 \in \mathbb{C}$, we have $e^{z_1} = e^{z_2}$ if and only if $z_1 = z_2 + 2\pi k i$ for some $k \in \mathbb{Z}$. In particular, $f(z) = e^z$ is periodic on $\mathbb{C}$ with pure imaginary period $2\pi i$.

PROOF. Suppose that $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ with $e^{z_1} = e^{z_2}$. Then, notice that

$$e^{x_1} = |e^{z_1}| = |e^{z_2}| = e^{x_2}$$

and $\arg(z_1) = \arg(z_2)$. Since the exponential function is injective on $\mathbb{R}$ and $x_1, x_2 \in \mathbb{R}$, it must be that $x_1 = x_2$. Moreover, $y_1 = y_2 + 2\pi k$ for some $k \in \mathbb{Z}$ since

$$y_1 + 2\pi\ell = \arg(z_1) = \arg(z_2) = y_2 + 2\pi m$$

for some $\ell, m \in \mathbb{Z}$. Consequently, $z_1 = z_2 + 2\pi k$ for some $k \in \mathbb{Z}$. Note that we proved the converse in Remark 13.8. In conclusion, since

$$f(z + 2\pi i) = e^{z+2\pi i} = e^z = f(z)$$

for all $z \in \mathbb{C}$, it follows that $f(z) = e^z$ is $2\pi i$-periodic on $\mathbb{C}$. □

We list some properties of e^z, of which the proofs will be left as an exercise.

(i) If $y = 0$, then $e^z = e^x$ so that e^x coincides with the real exponential when z is real. In particular, $e^0 = 1$.

(ii) If $x = 0$, then $e^z = e^{iy} = \cos y + i \sin y$. In particular, we have $|e^{iy}| = 1$ for all $y \in \mathbb{R}$.

(iii) For all $z_1, z_2 \in \mathbb{C}$, we have $e^{z_1} \cdot e^{z_2} = e^{z_1+z_2}$.

(iv) For all $z \in \mathbb{C}$ and $n \in \mathbb{Z}$, we have $(e^z)^n = e^{nz}$.

Note that even though e has n distinct n-th roots in $\mathbb{C}$, we know by the definition of e^z that $e^{1/n}$ is a *real* n-th root of $\mathbb{C}$.

Recall that we can write $\cos\theta = (e^{i\theta} + e^{-i\theta})/2$ and $\sin\theta = (e^{i\theta} - e^{-i\theta})/(2i)$ for all $\theta \in \mathbb{R}$ as noted in Remark 13.3 part (3). This motivates the following extensions of the cosine and sine functions to $\mathbb{C}$.

DEFINITION 13.10. Given $z \in \mathbb{C}$, we define

$$\cos z := \frac{e^{iz} + e^{-iz}}{2} \quad \text{and} \quad \sin z := \frac{e^{iz} - e^{-iz}}{2i}.$$

Since e^{iz} and e^{-iz} are entire, we see that $\cos z$ and $\sin z$ are also entire. In particular, for all $z \in \mathbb{C}$, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}z}(\cos z) &= \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{e^{iz} + e^{-iz}}{2}\right) = \frac{i}{2}(e^{iz} - e^{-iz}) = -\frac{e^{iz} - e^{-iz}}{2i} = -\sin z, \\
\frac{\mathrm{d}}{\mathrm{d}z}(\sin z) &= \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{e^{iz} - e^{-iz}}{2i}\right) = \frac{i}{2i}(e^{iz} + e^{-iz}) = -\frac{e^{iz} + e^{-iz}}{2} = -\cos z.
\end{aligned}$$

Note that for any z on the real axis, $\cos z$ and $\sin z$ coincide with the real cosine and sine functions. Moreover, the derivatives of $\cos z$ and $\sin z$ agree with the complex extensions of the derivatives of $\cos x$ and $\sin x$, where $x \in \mathbb{R}$. In fact, many of the properties of $\cos x$ and $\sin x$ extend to $\cos z$ and $\sin z$.

PROPOSITION 13.11. For all $z \in \mathbb{C}$, we have

(1) $\cos(-z) = \cos z$ and $\sin(-z) = -\sin z$;

(2) $\cos(z + 2\pi) = \cos z$ and $\sin(z + 2\pi) = \sin z$;

(3) $\cos(z + \pi) = -\cos z$ and $\sin(z + \pi) = -\sin z$;

(4) $\cos(z + \pi/2) = -\sin z$ and $\sin(z + \pi/2) = \cos z$.

Proof. We leave this as an exercise; use the fact that $e^{2\pi i} = e^{-2\pi i} = 1$, $e^{\pi i} = e^{-\pi i} = -1$, and $e^{\frac{\pi}{2} i} = i$. $\square$

We also have the following familiar identities.

Proposition 13.12. For all $z, z_1, z_2 \in \mathbb{C}$, we have

(i) $\cos^2 z + \sin^2 z = 1$;

(ii) $\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$;

(iii) $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$.

Proof. We prove (i) and leave (ii) and (iii) as an exercise. We have

$$\cos^2 z + \sin^2 z = \left(\frac{e^{iz} + e^{-iz}}{2}\right)^2 + \left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2 = \frac{e^{2iz} + 2 + e^{-2iz}}{4} + \frac{e^{2iz} - 2 + e^{-2iz}}{-4} = \frac{4}{4} = 1. \qquad \square$$

We end with a few more properties of the complex cosine and sine functions.

Proposition 13.13.

(i) We have $\cos z = 0$ if and only if $z = \pi/2 + k\pi$ for some $k \in \mathbb{Z}$.

(ii) We have $\sin z = 0$ if and only if $z = k\pi$ for some $k \in \mathbb{Z}$.

(iii) The functions $|\cos z|$ and $|\sin z|$ are both unbounded on $\mathbb{C}$.

Proof.

(i) For every $z \in \mathbb{C}$, we have

$$\begin{aligned} \cos z = 0 &\iff e^{iz} = e^{-iz} = 0 \\ &\iff e^{iz} = e^{-iz} \\ &\iff e^{2iz} = -1 = e^{\pi i} \\ &\iff 2iz = \pi i + 2\pi k i \text{ for some } k \in \mathbb{Z} \\ &\iff z = \pi/2 + k\pi \text{ for some } k \in \mathbb{Z}. \end{aligned}$$

(ii) Apply (i) using the fact that $\sin(z) = -\cos(z + \pi/2)$.

(iii) For all $y \in \mathbb{R}$, we have

$$|\cos(iy)| = \left|\frac{e^{-y} + e^{y}}{2}\right| = \frac{e^{y} + e^{-y}}{2} = \cosh y,$$

so $|\cos z|$ is unbounded on $\mathbb{C}$. Similarly, $|\sin z|$ is unbounded. $\square$

Finally, the other trigonometric functions are defined as in the real case; we have

- $\tan z := \sin z / \cos z$ and $\sec z := 1/\cos z$ where $z \neq \pi/2 + \pi k$ for some $k \in \mathbb{Z}$;
- $\cot z := \cos z / \sin z$ and $\csc z := 1/\sin z$ where $z \neq \pi k$ for some $k \in \mathbb{Z}$.

Note that these functions are all entire on their domains. Moreover, their derivatives are given by

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}z}(\tan z) &= \sec^2 z, & \frac{\mathrm{d}}{\mathrm{d}z}(\sec z) &= \sec z \tan z, \\ \frac{\mathrm{d}}{\mathrm{d}z}(\cot z) &= -\csc^2 z, & \frac{\mathrm{d}}{\mathrm{d}z}(\csc z) &= -\csc z \cot z, \end{aligned}$$

for all z in their respective domains, just as in the real case.

14 Argument functions and complex logarithms

We now turn our attention to logarithmic functions. Before defining them, we need to recall the polar form of complex numbers as well as define the argument function, which associates to any complex number the angles appearing in its polar form.

Recall that any point $z = (x, y) = x + iy \neq 0$ on the plane can be expressed in terms of polar coordinates r and θ. In particular, we have

$$z = re^{i\theta} = r(\cos\theta + i\sin\theta)$$

where $r = |z| > 0$ and θ is uniquely determined up to $2\pi k$ where $k \in \mathbb{Z}$. In this case, the **argument of** z is the multi-valued function

$$\arg z := \theta + 2\pi k,\ k \in \mathbb{Z}.$$

Note that there is some $k_0 \in \mathbb{Z}$ such that

$$\theta_0 = \theta + 2\pi k_0 \in (-\pi, \pi],$$

and we define the **principal argument of** z by

$$\operatorname{Arg} z := \theta_0.$$

The polar form of z is then $z = |z|(\cos\theta_0 + i\sin\theta_0)$ where $\theta_0 = \operatorname{Arg} z$, with multi-valued function $\arg(z) = \operatorname{Arg}(z) + 2\pi k$, $k \in \mathbb{Z}$.

Example 14.1. Let $z = -1 + i$. Then $|z| = \sqrt{2}$ so that

$$z = \sqrt{2}\left(\frac{-1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = \sqrt{2}(\cos\theta + i\sin\theta).$$

Hence, $\cos\theta = -1/\sqrt{2}$ and $\sin\theta = 1/\sqrt{2}$, which implies that $\theta = 3\pi/4 + 2\pi k$, $k \in \mathbb{Z}$. Therefore, we have $\operatorname{Arg}(z) = 3\pi/4 \in (-\pi, \pi]$ and $\arg(z) = 3\pi/4 + 2\pi k$, $k \in \mathbb{Z}$.

Example 14.2 (Polar form of the Cauchy-Riemann equations). Suppose that we write $z = r(\cos\theta + i\sin\theta)$ where $\theta \in (-\pi, \pi]$. Then any complex function can be expressed in terms of r and θ. In particular, if u and v are the real and imaginary parts of f, then

$$f(z) = u(r, \theta) + iv(r, \theta).$$

A direct computation shows that if f is differentiable at a point $z_0 \in \mathbb{C}$, then u and v satisfy the polar form of the Cauchy-Riemann equations

$$u_r = \frac{1}{r}v_\theta, \quad u_\theta = -rv_r$$

and hence

$$f'(z_0) = e^{-i\theta}(u_r + iv_r) = \frac{1}{r}e^{-i\theta}(v_\theta - iu_\theta).$$

Conversely, it can be shown that if u and v satisfy these equations at z_0 and are of class C^1 in a neighbourhood of z_0, then f is differentiable at z_0.

Proposition 14.3 (Properties of arg). For all $z, z_1, z_2 \in \mathbb{C} \setminus \{0\}$ and $c \in \mathbb{R} \setminus \{0\}$, we have

(i) $\arg(\bar{z}) = -\arg z$;

(ii) $\arg(cz) = \arg z$ if $c > 0$, and $\arg(cz) = \arg z + \pi$ if $c < 0$;

(iii) $\arg(z_1 z_2) = \arg z_1 + \arg z_2$;

(iv) $\arg(z^{-1}) = -\arg z$;

(v) $\arg(z_1/z_2) = \arg(z_1) - \arg(z_2)$.

Proof. Exercise. □

Note that properties (iii) and (v) only hold for arg, and not Arg. For instance, we have

$$\operatorname{Arg}(-1) + \operatorname{Arg}(i) = \pi + \frac{\pi}{2} = \frac{3\pi}{2} \neq -\frac{\pi}{2} = \operatorname{Arg}(-i) = \operatorname{Arg}(-1 \cdot i).$$

Definition 14.4 (Principal argument function). For all $z \in \mathbb{C} \setminus \{0\}$, we define the **principal argument function** by

$$\operatorname{Arg} z := \theta \in \mathbb{R}$$

where $z = |z|e^{i\theta}$ and $-\pi < \theta \leq \pi$.

Note that $\operatorname{Arg} z$ is continuous everywhere on its domain except along the negative x-axis. Indeed, for any $z_0 = c$ where $c \in \mathbb{R}^{<0}$, by approaching z_0 along the circle arc $(c\cos t, c\sin t)$ with $t \to \pi^-$, we have

$$\lim_{z \to z_0} \operatorname{Arg}(z) = \lim_{t \to \pi^-} t = \pi.$$

However, approaching z_0 along the same circle arc with $t \to -\pi^+$, we get

$$\lim_{z \to z_0} \operatorname{Arg}(z) = \lim_{t \to -\pi^+} t = -\pi.$$

More generally, we can single out other branches of the argument function which are again single-valued.

Definition 14.5 (Argument functions). Fix $\tau \in \mathbb{R}$. For $z \in \mathbb{C} \setminus \{0\}$, we define

$$\arg_\tau(z) := \theta \in \mathbb{R}$$

where $z = |z|e^{i\theta}$ and $\tau < \theta \leq \tau + 2\pi$.

By a similar argument to above, $\arg_\tau(z)$ is continuous everywhere except along the ray $\theta = \tau$. Moreover, note that $\operatorname{Arg} z = \arg_{-\pi}(z)$.

Recall that the real exponential function e^x is injective with range $\mathbb{R} \setminus \{0\}$, and whose inverse is $\ln x$ for $x \in \mathbb{R} \setminus \{0\}$. Can we find an inverse for the complex exponential function e^z?

Previously, we saw that e^z is an entire function with range $\mathbb{C} \setminus \{0\}$. However, since e^z is $2\pi i$-periodic over $\mathbb{C}$, it is clearly not injective on $\mathbb{C}$. Nonetheless, it is injective on any strip of the form

$$D := \{z \in \mathbb{C} : \alpha < \Im(z) \leq \alpha + 2\pi i\}$$

for any fixed $\alpha \in \mathbb{R}$, in which case it takes all possible values of $\mathbb{C} \setminus \{0\}$ on D.

Moreover, we know that if $w_0 = re^{i\theta}$ is of the form $w_0 = e^z$, then $z = \ln r + i\theta$. However, it is clear that θ here is not unique by the $2\pi i$-periodicity of e^z. Therefore, the inverse of e^z is a multi-valued function which is defined as follows.

Definition 14.6 (Complex logarithm). For all $z \in \mathbb{C} \setminus \{0\}$, we define the **complex logarithm** as

$$\log z := \ln|z| + i\arg z = \ln|z| + i\operatorname{Arg} z + 2\pi i k,\ k \in \mathbb{Z},$$

and the **principal branch of the logarithm** as

$$\operatorname{Log} z := \ln|z| + i\operatorname{Arg} z,$$

where $\operatorname{Arg} z \in (-\pi, \pi]$ is the principal argument of z.

EXAMPLE 14.7. Let $z = 1 + i$. Then $|z| = \sqrt{2}$ and

$$z = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \sqrt{2}(\cos\theta + i\sin\theta),$$

which implies that $\operatorname{Arg}(1+i) = \pi/4$. Therefore, we have

$$\log(1+i) = \ln\sqrt{2} + i\pi/4 + 2\pi i k,\ k \in \mathbb{Z},$$

as well as

$$\operatorname{Log}(1+i) = \ln\sqrt{2} + i\pi/4.$$

PROPOSITION 14.8 (Properties of the complex logarithm). For all $z_1, z_2 \in \mathbb{C}$, we have

(i) $\log(z_1 z_2) = \log z_1 + \log z_2$, and

(ii) $\log(z_1/z_2) = \log z_1 - \log z_2$.

PROOF. Exercise. □

REMARK 14.9. These properties only hold for $\log z$, and not $\operatorname{Log} z$. This is because properties (iii) and (v) of Proposition 14.3 do not generally hold.

Let us now examine where the complex logarithm is continuous, differentiable, and analytic. We begin by noting that these notions only make sense for single-valued functions. We therefore consider the principal branch of the logarithm $\operatorname{Log} z$.

Since $\operatorname{Arg} z$ is continuous everywhere on $\mathbb{C} \setminus \{0\}$ except for the negative real axis, then so is $\operatorname{Log} z$. In particular, $\operatorname{Log} z$ is continuous on D^*, where

$$D^* := \mathbb{C} \setminus \{z \in \mathbb{C} : \Re(z) \leq 0 \text{ and } \Im(z) = 0\}.$$

Certainly, $\operatorname{Log} z$ cannot be differentiable on the negative real axis as it is not even continuous there. Hence, $\operatorname{Log} z$ is only potentially differentiable on D^*. The following proposition tells us that it is in fact analytic on all of D^*.

PROPOSITION 14.10. The function $\operatorname{Log} z$ is analytic on D^* with derivative

$$\frac{\mathrm{d}}{\mathrm{d}z}(\operatorname{Log} z) = \frac{1}{z}$$

for all $z \in D^*$.

PROOF. If $z \in D^*$, then $z = re^{i\theta}$ with $-\pi < \theta < \pi$, and

$$\operatorname{Log} z = \ln r + i\theta = u(r,\theta) + iv(r,\theta).$$

Taking partials of u and v with respect to r and θ, we obtain $u_r = 1/r$, $u_\theta = 0$, $v_r = 0$, and $v_\theta = 1$. In particular, we see that $ru_r = r(1/r) = 1 = v_\theta$ and $v_\theta = 0 = -rv_r$ for all $z \in D^*$, so u and v satisfy the polar Cauchy-Riemann equations (see Example 14.2) everywhere on D^*. Hence, $\operatorname{Log} z$ is differentiable at every point in D^*, and therefore in the neighbourhood of any point in D^*, so $\operatorname{Log} z$ is analytic on D^*. Finally, for all $z \in D^*$, we obtain

$$f'(z) = e^{-i\theta}(u_r + iv_r) = e^{-i\theta}\left(\frac{1}{r}\right) = \frac{1}{re^{i\theta}} = \frac{1}{z}.$$ □

One can similarly define other analytic branches of $\log z$.

DEFINITION 14.11. Fix $\tau \in \mathbb{R}$. For all $z \in \mathbb{C} \setminus \{0\}$, we define

$$L_\tau(z) := \ln|z| + i\arg_\tau(z).$$

Note that $\operatorname{Log} z = L_{-\pi}(z)$. If we define

$$D_\tau^* := \mathbb{C} \setminus \{z \in \mathbb{C} : \operatorname{Arg}(z) = \tau + 2\pi\},$$

which is $\mathbb{C}$ with the ray $\theta = \tau$ removed, then $L_\tau(z)$ is continuous and analytic on D_τ^* with derivative

$$\frac{\mathrm{d}}{\mathrm{d}z}(L_\tau(z)) = \frac{1}{z}$$

for all $z \in D_\tau^*$. We call the functions L_τ^* **analytic branches** of the logarithm.

Corollary 14.12. For all $\tau \in \mathbb{R}$, the argument function $\arg_\tau(z)$ is harmonic on D_τ^*. In particular, the principal argument function $\operatorname{Arg}(z)$ is harmonic on D^*.

Proof. Let $\tau \in \mathbb{R}$. Then this follows from the fact that $\arg_\tau(z)$ is the imaginary part of the function $L_\tau(z)$, which is analytic on D_τ^*. $\square$

Corollary 14.13. The real function $\ln|z|$ is harmonic on $\mathbb{C} \setminus \{0\}$.

Proof. For all $z \in \mathbb{C} \setminus \{0\}$, there exists $\tau \in \mathbb{R}$ such that $z \in D_\tau^*$. Since $\ln|z|$ is the real part of the function $L_\tau(z)$ which is analytic, it follows that $\ln|z|$ is harmonic on z. $\square$

Note that unlike $\ln|z|$, the argument functions $\arg_\tau(z)$ cannot be harmonic on $\mathbb{C} \setminus \{0\}$ as they fail to be continuous on a ray in $\mathbb{C} \setminus \{0\}$.

Finally, we can use logarithms to define exponentials with base other than e. Given $c \in \mathbb{C} \setminus \{0\}$, we define

$$c^z := e^{z \log c}$$

for $z \in \mathbb{C}$, which is a multi-valued function. However, we get a single-valued entire function for a fixed value of $\log c$. For instance, we see that

$$c^z := e^{z \operatorname{Log} c}$$

is entire, with derivative

$$\frac{\mathrm{d}}{\mathrm{d}z}(c^z) = \frac{\mathrm{d}}{\mathrm{d}z}(e^{z \operatorname{Log} c}) = \operatorname{Log} c \cdot e^{z \operatorname{Log} c} = \operatorname{Log} c \cdot c^z$$

for all $z \in \mathbb{C}$.

15 Complex powers and harmonic conjugates

Previously, we saw that one can use logarithms to define exponentials with base other than e. Given $c \in \mathbb{C}\setminus\{0\}$, we defined $c^z := e^{z \log c}$ for all $z \in \mathbb{C}$, which is a multi-valued function, but becomes a single-valued entire function for a fixed value of $\log c$. For instance, we could take $c^z := e^{z \operatorname{Log} c}$, which is entire with derivative $\frac{\mathrm{d}}{\mathrm{d}z}(c^z) = \operatorname{Log} c \cdot c^z$ for all $z \in \mathbb{C}$.

We will now see that complex powers can be defined similarly by setting

$$z^c := e^{c \log z}$$

for any fixed $c \in \mathbb{C}$ and any $z \in \mathbb{C} \setminus \{0\}$. First, we will review some facts about n-th powers and n-th roots.

Recall that any non-zero complex number z can be written as $z = re^{i\theta}$, where $r = |z|$ and $\arg z = \theta + 2\pi k$, $k \in \mathbb{Z}$. In fact, if $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ are two non-zero complex numbers, then $z_1 = z_2$ if and only if $|z_1| = |z_2|$ and $\arg z_1 = \arg z_2$; namely, $r_1 = r_2$ and $\theta_1 = \theta_2 + 2\pi k$ for some $k \in \mathbb{Z}$. Moreover, for any $n \in \mathbb{Z}$, the n-th power of z has exponential form $z^n = r^n e^{in\theta}$. In particular, this means that

$$z^n = \left(e^{\ln|z|}\right)^n e^{in \arg z} = e^{n \ln|z|} \cdot e^{in \arg z} = e^{n \log z}.$$

Hence, for all $z \in \mathbb{C} \setminus \{0\}$ and $n \in \mathbb{Z}$, we have $z^n = e^{n \log z}$, and this number is unique.

Let us now recall how n-th roots of complex numbers are defined. Fix a non-zero complex number $z_0 = r_0 e^{i\theta_0}$, where $\theta_0 \in (-\pi, \pi]$.

Definition 15.1. An **n-th root** of z_0 is a non-zero complex number such that $z^n = z_0$.

In particular, if $z = re^{i\theta}$ is an n-th root of z_0, then

$$\begin{aligned} z^n = z_0 &\iff r^n e^{in\theta} = r_0 e^{i\theta_0} \\ &\iff r^n = r_0 \text{ and } n\theta = \theta_0 + 2\pi k,\ k \in \mathbb{Z} \\ &\iff r = \sqrt[n]{r_0} \text{ and } \theta = \theta_0/n + 2\pi k/n,\ k \in \mathbb{Z}. \end{aligned}$$

This implies that $|z| = \sqrt[n]{r_0} = e^{\frac{1}{n}\ln|z_0|}$ and

$$\arg z = \left(\frac{\theta_0}{n} + \frac{2\pi k}{n}\right) + 2\pi\ell,\ \ell \in \mathbb{Z}$$

for some $k \in \mathbb{Z}$. Now, since the complex exponential is $2\pi i$-periodic, we obtain n distinct n-th roots of $z_0 = r_0 e^{i\theta_0}$, which are given by

$$z_0^{1/n} = \sqrt[n]{r_0} e^{i(\theta_0 + 2\pi k)/n}$$

for each $0 \le k \le n-1$. Note that we chose $\theta_0 = \operatorname{Arg} z_0$, and moreover, $|z| = e^{\frac{1}{n}\ln|z_0|}$. Hence, we can rewrite these n distinct n-th roots as

$$z_0^{1/n} = e^{\frac{1}{n}(\operatorname{Log} z_0 + 2\pi i k)}$$

where $0 \le k \le n-1$. When $k = 0$, the n-th root

$$z_0^{1/n} = \sqrt[n]{r_0} e^{i\theta_0/n} = e^{\frac{1}{n}\operatorname{Log} z_0}$$

is sometimes called the **principal n-th root** of z_0.

Remark 15.2.

(1) Recall that $\log z_0 = \operatorname{Log} z_0 + 2\pi i k$, $k \in \mathbb{Z}$. Therefore, we can write the n-th roots of z_0 compactly as

$$z_0^{1/n} = e^{\frac{1}{n}\log z_0},$$

noting that this expression has n distinct values, each corresponding to a distinct n-th root of z_0.

(2) If z is any n-th root of z_0, then $|z| = \sqrt[n]{r_0}$ so that the n-th roots of z_0 all lie on the circle $|z| = \sqrt[n]{r_0}$ and are equally spaced by $2\pi/n$ radians, starting with θ_0/n. In other words, the n-th roots of z_0 are the vertices of a regular n-gon on the circle $|z| = \sqrt[n]{r_0}$ which has θ_0/n as a vertex.

EXAMPLE 15.3 (Roots of unity). Since $e^{2\pi ik} = 1$ for all $k \in \mathbb{Z}$, we have

$$1^{1/n} = e^{2\pi ik/n}$$

for all $0 \leq k \leq n-1$. Let $w_n = e^{2\pi i/n}$. Then $w_n^k = e^{2\pi ik/n}$, so the n distinct n-th roots of unity are

$$1, w_n, w_n^2, \ldots, w_n^{n-1}.$$

Moreover, we have $w_n^n = 1$. For instance, in the case where $n = 3$, we have $w_3 = e^{2\pi i/3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ and $w_3^2 = e^{4\pi i/3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$, so the third roots of unity are given by 1, $-\frac{1}{2} + i\frac{\sqrt{3}}{2}$, and $-\frac{1}{2} - i\frac{\sqrt{3}}{2}$.

EXAMPLE 15.4. Let $z_0 = 1 - i$. Then, we have $|z_0| = \sqrt{2}$ and $z_0 = \sqrt{2}(1/\sqrt{2} - i/\sqrt{2}) = \sqrt{2}e^{i\pi/4}$, so the fourth roots of z_0 are

$$z_0^{1/4} = 2^{1/8} \cdot e^{i(\pi/16+2\pi k/4)}$$

where $0 \leq k \leq 3$.

We have now seen that for $n \in \mathbb{Z}$, the n-th power $z^n = e^{n \log z}$ is single-valued, whereas the n-th roots of z can be written as $z^{1/n} = e^{\frac{1}{n}\log z}$, which is a multi-valued expression.

We can imagine taking more complicated powers of z. For instance, if $p/q \in \mathbb{Q}$, we can set

$$z^{p/q} := (z^{1/q})^p = (e^{\frac{1}{q}\log z})^p = e^{\frac{p}{q}\log z},$$

where the last equality follows from the properties of the exponential function. Note that this expression is again multi-valued. This motivates the following definition.

DEFINITION 15.5 (Complex powers). Given a fixed complex number c and a non-zero complex number z, we define the function

$$z^c := e^{c \log z},$$

which is multi-valued on $\mathbb{C} \setminus \{0\}$. Moreover, the **principal value of** z^c is

$$\mathrm{PV}(z^c) := e^{c \operatorname{Log} z},$$

giving rise to the **principal branch of** z^c, which is a single-valued function on $\mathbb{C} \setminus \{0\}$.

REMARK 15.6. For all $z \in \mathbb{C} \setminus \{0\}$, we have the properties $z^0 = 1$ and

$$z^{-c} = e^{-c\log z} = \frac{1}{e^{c\log z}} = \frac{1}{z^c}$$

as with real powers. However, due to the properties of $\arg z$ and $\operatorname{Arg} z$, the usual power rules do not always hold. In particular, we have $z^\alpha \cdot z^\beta = z^{\alpha+\beta}$ only for principal values, and $(z_1z_2)^\alpha = z_1^\alpha z_2^\alpha$ only for values which are not principal.

EXAMPLE 15.7.

(1) We have $i^i = e^{i\log i} = e^{i(\ln|i|+i\arg i)} = e^{-\arg i} = e^{-\pi/2-2\pi k}$, $k \in \mathbb{Z}$, and the principal value of i^i is

$$\mathrm{PV}(i^i) = e^{i(\ln|i|+i\operatorname{Arg} i)} = e^{-\pi/2}.$$

(2) We have $2^{1/3} = e^{\frac{1}{3}\log 2} = e^{\frac{1}{3}(\ln|2|+2\pi ik)} = \sqrt[3]{2} \cdot e^{2\pi ik/3}$, $k \in \mathbb{Z}$, and the principal value of $2^{1/3}$ is

$$\mathrm{PV}(2^{1/3}) = e^{\frac{1}{3}\ln 2} = \sqrt[3]{2},$$

which is simply the real cubic root of 2.

If one chooses an analytic branch $L_\tau(z)$ for some $\tau \in \mathbb{R}$ of $\log z$, then

$$z^c = e^{cL_\tau(z)} = e^{c(\ln r + i\theta)}$$

where $\tau < \theta \leq \tau + 2\pi$. Moreover, z^c is analytic on D_τ^*, which is $\mathbb{C}$ with the ray $\theta = \tau$ removed. The derivative is then given by

$$\frac{\mathrm{d}}{\mathrm{d}z}(z^c) = \frac{\mathrm{d}}{\mathrm{d}z}(e^{cL_\tau(z)}) = c\frac{\mathrm{d}}{\mathrm{d}z}(L_\tau(z)) \cdot e^{cL_\tau(z)} = \frac{c}{z} \cdot z^c = cz^{c-1}$$

for all $z \in D_\tau^*$, which coincides with the derivative of real powers.

Example 15.8. Let us define a branch of $(1-z^2)^{1/2}$ that is analytic on the unit disc $|z| < 1$. In particular, we need to find a single-valued function $f(z)$ which is analytic on $\Omega = \{z \in \mathbb{C} : |z| < 1\}$ and satisfies

$$f(z)^2 = 1 - z^2.$$

Thus, we must pick a branch L_τ for some $\tau \in \mathbb{R}$ of $\log w$ such that $1 - z^2 \in D_\tau^*$ for all $z \in \Omega$. Consider the principal branch $\operatorname{Log} w$ so that $\tau = -\pi$ and the ray $\theta = -\pi$ corresponds to the non-positive real axis. Then, we have to verify that $1 - z^2$ is never a non-positive real when $z \in \Omega$. Indeed, if $z \in \Omega$, then $|z| < 1$ so that $|z^2| = |z|^2 < 1$ and

$$1 - z^2 \in \{w \in \mathbb{C} : |w-1| < 1\} \subseteq D^* = D_{-\pi}^*,$$

which implies that

$$f(z) = e^{\frac{1}{2}\operatorname{Log}(1-z^2)}$$

is a branch of $(1-z)^{1/2}$ that is analytic on the unit disc.

Example 15.9. Suppose we want to define a branch of $(z^2-4)^{1/2}$ that is analytic on the exterior of the circle $|z| = 2$; that is, on the set $\Omega = \{z \in \mathbb{C} : |z| > 2\}$. We need to find a single-valued function $f(z)$ that is analytic on Ω and satisfies

$$f(z)^2 = z^2 - 4.$$

Unfortunately, no branch L_τ for any $\tau \in \mathbb{R}$ of $\log w$ will work here. Indeed, if $z \in \Omega$, then $|z| > 2$ so that $|z^2| = |z|^2 > 4$ and $z^2 - 4 \in \{w \in \mathbb{C} : |w+4| > 4\} =: \Omega'$. In fact, every $w_0 \in \Omega'$ is of this form, as we have $w_0 = z_0^2 - 4$ for $z_0 = \sqrt{|w_0|}e^{i\operatorname{Arg}(w_0)/2} \in \Omega$. In other words, $z^2 - 4$ maps Ω onto Ω'. However, for every $\tau \in \mathbb{R}$, there is a point in Ω' which is on the ray $\theta = \tau$, which implies that there is no $\tau \in \mathbb{R}$ such that $z^2 - 4 \in D_\tau^*$ for all $z \in \Omega$. Therefore, we must consider another approach.

We can think of $(z^2-4)^{1/2}$ as a solution to

$$w^2 = z^2 - 4.$$

Continuing in this way, we can rewrite $(z^2-4)^{1/2}$ as

$$z\left(1 - \frac{4}{z^2}\right)^{1/2},$$

where we have $|4/z^2| < 1$ for all $z \in \Omega$. Then, from the previous example, we know that $e^{\frac{1}{2}\operatorname{Log}(1-z^2/4)}$ is a branch of $(1-4/z^2)^{1/2}$ that is analytic on Ω. Hence, it follows that

$$f(z) = ze^{\frac{1}{2}\operatorname{Log}(1-4/z^2)}$$

is a branch of $(z^2-4)^{1/2}$ that is analytic on Ω.

We end our discussion of elementary functions by noting that one can define, as in the real case, complex hyperbolic and inverse trigonometric functions.

Definition 15.10. For any $z \in \mathbb{C}$, we define $\cosh z := \frac{1}{2}(e^z + e^{-z})$ and $\sinh z := \frac{1}{2}(e^z - e^{-z})$.

These functions coincide with their real counterparts for $z = x \in \mathbb{R}$. Moreover, they are entire with derivatives $\frac{\mathrm{d}}{\mathrm{d}z}(\cosh z) = \sinh z$ and $\frac{\mathrm{d}}{\mathrm{d}z}(\sinh z) = \cosh z$.

The other hyperbolic functions are defined by

- $\tanh z := \sinh z / \cosh z$ and $\operatorname{sech} z := 1/\cosh z$ for all $z \neq \frac{\pi}{2}i + \pi i k$, $k \in \mathbb{Z}$; and
- $\coth z := \cosh z / \sinh z$ and $\operatorname{csch} z := 1/\sinh z$ for all $z \neq \pi i k$, $k \in \mathbb{Z}$.

Definition 15.11. The inverse trigonometric function are defined by

$$\begin{aligned}
\arcsin z &:= -i \log\left(iz + (1-z^2)^{1/2}\right), \\
\arccos z &:= -i \log\left(z + i(1-z^2)^{1/2}\right), \\
\arctan z &:= \frac{i}{2} \log\left(\frac{i+z}{i-z}\right).
\end{aligned}$$

Note that these functions are multi-valued. Nonetheless, when specific analytic branches of $\log(\cdot)$ and $(\cdot)^{1/2}$ are chosen, they are single-valued and analytic. In particular, their derivatives are given by

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}z}(\arcsin z) &= \frac{1}{(1-z^2)^{1/2}}, \\
\frac{\mathrm{d}}{\mathrm{d}z}(\arccos z) &= -\frac{1}{(1-z^2)^{1/2}}, \\
\frac{\mathrm{d}}{\mathrm{d}z}(\arctan z) &= \frac{1}{1+z^2}.
\end{aligned}$$

We now focus our attention on the relationship between harmonic functions on $\mathbb{R}^2$ and analytic functions on $\mathbb{C}$. We have seen that if $f(z) = u(x,y) + iv(x,y)$ is analytic on a domain Ω, then its real and imaginary parts $u(x,y)$ and $v(x,y)$ are both harmonic on Ω. Now that we have many examples of analytic functions, we also have many examples of harmonic functions.

Example 15.12.

(1) Since $f(z) = z^3$ is entire, it follows that $u(x,y) = x^3 - 3xy^2$ and $v(x,y) = 3x^2y - y^3$ are both harmonic on $\mathbb{R}^2$, as they are the real and imaginary parts of f.

(2) Since $\operatorname{Log} z$ is analytic on $D^* = \mathbb{C} \setminus \{z \in \mathbb{C} : x \leq 0,\, y = 0\}$, we know that $u(x,y) = \ln|z|$ and $v(x,y) = \operatorname{Arg} z$ are both harmonic on D^*.

We can also ask the converse; namely, given a harmonic function $u(x,y)$ on a domain Ω, is there a harmonic function $v(x,y)$ on Ω such that $f = u + iv$ is analytic on Ω? This motivates the following definition.

Definition 15.13 (Harmonic conjugate). Given an analytic function $f = u + iv$ on a domain Ω, the imaginary part v is said to be a **harmonic conjugate** of the real part u. Equivalently, given two harmonic functions u and v in Ω, we say that v is a harmonic conjugate of u if $u + iv$ is analytic on Ω.

Example 15.14.

(1) The function $v(x,y) = 3x^2y - y^3$ is a harmonic conjugate of $u(x,y) = x^3 - 3xy^2$ on $\mathbb{R}^2$ since $z^3 = u + iv$ is entire.

(2) The function $v(x,y) = e^x \sin y$ is a harmonic conjugate of $u(x,y) = e^x \cos y$ on $\mathbb{R}^2$ since $e^z = u + iv$ is entire.

Remark 15.15.

(i) It follows immediately from the definition that v is a harmonic conjugate of u on Ω if and only if u and v satisfy the Cauchy-Riemann equations on Ω. Thus, given a harmonic function u on Ω, finding a harmonic conjugate of u on Ω boils down to solving the Cauchy-Riemann equations on Ω.

(ii) The order matters; namely, if v is a harmonic conjugate of u, it is not always the case that u is a harmonic conjugate of v (since $u + iv$ being analytic does not guarantee that $v + iu$ is analytic). For example, $v(x, y) = 2xy$ is a harmonic conjugate of $u(x, y) = x^2 - y^2$ since $u + iv = z^2$ is entire. However, u is not a harmonic conjugate of v since $v + iu = 2xy + i(x^2 - y^2)$ is nowhere analytic; it is only differentiable at $z = 0$.

How many harmonic conjugates does a given harmonic function have? For instance, $v(x, y) = e^x \sin y + c$ is a harmonic conjugate of $u(x, y) = e^x \cos y$ for any $c \in \mathbb{R}$, which gives us an infinite family of harmonic conjugates of u. However, it turns out that these are the only possibilities.

PROPOSITION 15.16 (Uniqueness of harmonic conjugates). Let $u(x, y)$ be a harmonic function on a domain Ω. If $v(x, y)$ is a harmonic conjugate of $u(x, y)$ on Ω, then it is unique up to an additive constant.

PROOF. Suppose that v and v^* are harmonic conjugates of u on Ω. Then $f = u + iv$ and $g = u + iv^*$ are both analytic on Ω, which implies that $f - g = i(v - v^*)$ is also analytic on Ω. However, since the real part of $f - g$ is constant (as it is 0 everywhere), so is its imaginary part $v - v^*$ by Proposition 13.1. Thus, $v^* = v + c$ for some $c \in \mathbb{R}$. $\square$

Do harmonic conjugates always exist? We will see that on a simply connected domain Ω, any harmonic function u has a harmonic conjugate v, so any harmonic function on a simply connected domain induces an analytic function.

For example, take $u(x, y) = xy - x + y$, which is harmonic on $\mathbb{R}^2$ since $u_{xx} = u_{yy} = 0$. Then any harmonic conjugate v of u on $\mathbb{R}^2$ must be such that u and v satisfy the Cauchy-Riemann equations on $\mathbb{R}^2$. Hence, we see that

$$\begin{aligned} v_x &= -u_y = -x - 1, \\ v_y &= u_x = y - 1. \end{aligned}$$

It follows that

$$v(x, y) = \int v_x \,\mathrm{d}x = \int -x - 1 \,\mathrm{d}x = -\frac{x^2}{2} - x + C(y),$$

where $C(y)$ is a real scalar function depending only on y and is such that $C'(y) = v_y = y - 1$. In particular, we have $C(y) = y^2/2 - y + c$ for some constant $c \in \mathbb{C}$, and hence

$$v(x, y) = -\frac{1}{2}(x^2 - y^2) - (x + y) + c$$

is the family of harmonic conjugates of u on $\mathbb{R}^2$.

THEOREM 15.17 (Existence of harmonic conjugates). Given a real harmonic function $u(x, y)$ on a simply connected domain Ω, there exist infinitely many harmonic conjugates $v(x, y)$ of $u(x, y)$ on Ω, each differing by an additive constant. Hence, $u(x, y)$ is the real part of infinitely many analytic functions

$$f(z) = u(z) + iv(z)$$

on Ω, each differing by a purely imaginary additive constant.

PROOF. We only need to show that there exists a harmonic function v on Ω such that $v_x = -u_y$ and $v_y = u_x$, in which case we have

$$-u_y \,\mathrm{d}x + u_x \,\mathrm{d}y = v_x \,\mathrm{d}x + v_y \,\mathrm{d}y = \mathrm{d}v.$$

In other words, we need to show that $p \,\mathrm{d}x + q \,\mathrm{d}y = -u_y \,\mathrm{d}x + u_x \,\mathrm{d}y$ is exact on Ω. Since Ω is simply connected and $p, q \in C^1(\Omega)$, we know from the Poincaré Lemma (see the end of Lecture 4 for a refresher) that $p \,\mathrm{d}x + q \,\mathrm{d}y$ is exact if and only if $p_y = q_x$. Indeed, we have

$$p_y - q_x = (-u_y)_y - (u_x)_x = -(u_{xx} + u_{yy}) = 0$$

by the harmonicity of u. Hence, v exists, and by Proposition 15.16, it is unique up to an additive constant. $\square$

Remark 15.18. In the previous theorem, it is crucial that Ω is simply connected. For example, we have seen that $\ln|z|$ is harmonic on $\mathbb{C} \setminus \{0\}$, which is not a simply connected domain. However, it does not admit a harmonic conjugate on all of $\mathbb{C} \setminus \{0\}$, even though it does admit harmonic conjugates $\arg_\tau(z)$ for all $\tau \in \mathbb{R}$ on D^*_τ, the complex plane with the ray $\theta = \tau$ omitted. Indeed, since harmonic conjugates are unique up to an additive constant, any harmonic conjugate of $\ln|z|$ on $\mathbb{C} \setminus \{0\}$ must be a continuous extension of one of the functions $\arg_\tau(z)$, none of which are continuous on $\mathbb{C} \setminus \{0\}$.

16 Complex integration

We want to find a meaningful way of integrating complex functions. If f is a complex-valued function of one *real* variable so that

$$f : [a,b] \to \mathbb{C} : t \mapsto u(t) + iv(t)$$

with $u, v : [a,b] \to \mathbb{R}$, we can then define

$$\int_a^b f(t)\,\mathrm{d}t := \int_a^b u(t)\,\mathrm{d}t + i\int_a^b v(t)\,\mathrm{d}t.$$

Let us start by deriving some more properties of complex-valued functions of one real variable.

Let $f : [a,b] \to \mathbb{C}$ be a complex-valued function of one real variable. Write

$$f(t) = u(t) + iv(t)$$

with $u, v : [a,b] \to \mathbb{R}$ so that u and v are the real and imaginary parts of f, respectively. Since we can identify $\mathbb{R}$ with the real axis in $\mathbb{C}$, we can think of f has a complex-valued function defined on the set $\{z \in \mathbb{C} : z = x \text{ with } a \leq x \leq b\}$. Using familiar properties of complex functions, we have $\lim_{t\to t_0} f(t) = w_0 = u_0 + iv_0$ if and only if $\lim_{t\to t_0} u(t) = u_0$ and $\lim_{t\to t_0} v(t) = v_0$, and hence

$$\lim_{t\to t_0} f(t) = \lim_{t\to t_0} u(t) + i \lim_{t\to t_0} v(t)$$

if $\lim_{t\to t_0} u(t)$ and $\lim_{t\to t_0} u(t)$ both exist. We immediately see that f is continuous at $t = t_0$ if and only if u and v are both continuous at $t = t_0$. Moreover, we define the **derivative** of $f(t)$ at $t = t_0$ as

$$f'(t_0) = \lim_{t\to t_0} \frac{f(t) - f(t_0)}{t - t_0},$$

which exists if and only if the limits

$$u'(t_0) = \lim_{t\to t_0} \frac{u(t) - u(t_0)}{t - t_0},$$
$$v'(t_0) = \lim_{t\to t_0} \frac{v(t) - v(t_0)}{t - t_0}$$

both exist, in which case we have $f'(t_0) = u'(t_0) + iv'(t_0)$. As usual, we say that f is **differentiable** at $t = t_0$ if $f'(t_0)$ exists.

Similarly, an **antiderivative** of f is a differentiable function $F : [a,b] \to \mathbb{C}$ such that $F'(t) = f(t)$ for all $t \in [a,b]$. Note that if $U, V : [a,b] \to \mathbb{C}$ are the real and imaginary parts of F respectively, then

$$F'(t) = U'(t) + iV'(t)$$

so that $U'(t) = u(t)$ and $V'(t) = v(t)$ for all $t \in [a,b]$. In other words, U and V are antiderivatives of u and v, respectively.

As we saw in the beginning of the lecture, the **definite integral of f on** $[a,b]$ is defined as

$$\int_a^b f(t)\,\mathrm{d}t := \int_a^b u(t)\,\mathrm{d}t + i\int_a^b v(t)\,\mathrm{d}t.$$

If this integral exists, we say that f is **integrable** on $[a,b]$.

Complex definite integrals have properties similar to those of definite integrals of real one-variable functions.

- If $c \in \mathbb{C}$ and $f, g : [a,b] \to \mathbb{C}$ are integrable, then

$$\int_a^b cf(t) + g(t)\,\mathrm{d}t = c\int_a^b f(t)\,\mathrm{d}t + \int_a^b g(t)\,\mathrm{d}t.$$

- We have $\int_b^a f(t)\,\mathrm{d}t = -\int_a^b f(t)\,\mathrm{d}t$.
- If f is continuous on $[a,b]$, then $\int_a^b f(t)\,\mathrm{d}t$ exists.
- **(Fundamental Theorem of Calculus)** If F is an antiderivative of F on $[a,b]$, then

$$\int_a^b f(t)\,\mathrm{d}t = F(b) - F(a).$$

Remark 16.1. From a practical point of view, taking limits, differentiating, and integrating *complex-valued* functions of one real variable is the same as taking limits, differentiating, and integrating *real-valued* functions of one variable; we simply treat i appearing in the expressions as a constant.

Example 16.2.

(1) $\lim_{t\to -3} 2t^2 - it\ln(t^2+1) = 18 + 3i\ln 10$.

(2) $\dfrac{\mathrm{d}}{\mathrm{d}t}(2t^3e^{4it} - i\sin^2(5t^2-it)) = 6t^2e^{4it} + 8it^3e^{4it} - 2i\sin(5t^2-it)\cos(5t^2-it)(10t-i)$.

(3) $\displaystyle\int_{-\pi/2}^{\pi} 3\cos t + 2ie^{-it}\,\mathrm{d}t = \left[3\sin t - 2e^{-it}\right]_{t=-\pi/2}^{t=\pi} = 5+2i.$

Let $\Gamma : [a,b] \to \mathbb{R}^2 : t \mapsto (\alpha_1(t), \alpha_2(t))$ be a smooth parametrized curve in $\mathbb{R}^2$. By identifying $\mathbb{R}^2$ with $\mathbb{C}$ in the usual way, we can then think of Γ as a parametrized curve in $\mathbb{C}$. Indeed, setting

$$z : [a,b] \to \mathbb{C} : t \mapsto \alpha_1(t) + i\alpha_2(t),$$

then z is a one-variable complex function with real part α_1 and imaginary part α_2. In particular, we obtain

$$z'(t) = \alpha_1'(t) + i\alpha_2'(t)$$

for all $t \in [a,b]$, and since $\alpha'(t) \neq 0$ for all $t \in [a,b]$, we also have $z'(t) \neq 0$ for all $t \in [a,b]$. Moreover, observe that $|z'(t)| = |\alpha'(t)|$ for all $t \in [a,b]$, which implies that the arclength of Γ is given by

$$\int_a^b |z'(t)|\,\mathrm{d}t = \int_a^b |\alpha'(t)|\,\mathrm{d}t.$$

This motivates the following definition.

Definition 16.3. Let Γ be a subset of $\mathbb{C}$, and let $[a,b] \subseteq \mathbb{R}$ be a closed interval. A smooth complex function

$$z : [a,b] \to \Gamma : t \mapsto z(t) = x(t) + iy(t)$$

is called a **smooth parametrization of** Γ if

(1) the image of z is Γ;

(2) the derivative $z'(t) = x'(t) + iy'(t)$ is non-zero for all $t \in [a,b]$; and

(3) if $z(a) = z(b)$, then $z'(a) = z'(b)$.

Moreover, if $z(a) = z(b) = z_0$, then Γ is said to be **closed** (or a **loop**).

Remark 16.4. Let $z : [a,b] \to \Gamma : t \mapsto z(t) = x(t) + iy(t)$ be a smooth parametrization of a curve $\Gamma \subseteq \mathbb{C}$. Setting $\alpha(t) = (x(t), y(t))$ for all $t \in [a,b]$, we see that $\alpha : [a,b] \to \Gamma \subseteq \mathbb{R}^2$ is a smooth parametrized curve in $\mathbb{R}^2$. Conversely, if $\alpha : [a,b] \to \Gamma \subseteq \mathbb{R}^2$ is a smooth parametrized curve in $\mathbb{R}^2$, then by our previous discussion, we have a smooth parametrized curve in $\mathbb{C}$. Therefore, we can think of smooth parametrized curves in $\mathbb{R}^2$ and $\mathbb{C}$ interchangeably.

EXAMPLE 16.5.

(1) Consider the unit circle $C = \{z \in \mathbb{C} : |z| = 1\}$ parametrized counter-clockwise by $z(t) = \cos t + i \sin t = e^{it}$ where $t \in [0, 2\pi]$, in which case we have $z'(t) = ie^{it}$. It can also be parametrized clockwise by $z(t) = \sin t + i \cos t = ie^{-it}$ for all $t \in [0, 2\pi]$, and in this case, we have $z'(t) = e^{-it}$.

(2) Similarly, the circle $C = \{z \in \mathbb{C} : |z - z_0| = R\}$ can be parametrized counter-clockwise by $z(t) = z_0 + Re^{it}$ for all $t \in [0, 2\pi]$.

(3) The line segment joining $-1 - i$ and $2 + 2i$ can be parametrized by $z(t) = t + it$ when $t \in [-1, 2]$, and we have $z'(t) = 1 + i$.

REMARK 16.6. Piecewise-smooth and simple parametrized curves in $\mathbb{C}$ are defined as in $\mathbb{R}^2$. Moreover, given a piecewise-smooth parametrized curve $\Gamma \subseteq \mathbb{C}$ with parametrization $z : [a, b] \to \Gamma$, we denote by $-\Gamma$ the curve parametrized by $z(a + b - t)$ for all $t \in [a, b]$ as in the real case. In particular, this is just Γ with the opposite orientation.

DEFINITION 16.7 (Contour integrals). Let $f : \Omega \to \mathbb{C}$ be a continuous function on a domain $\Omega \subseteq \mathbb{C}$. Moreover, let $z : [a, b] \to \Gamma \subseteq \mathbb{C}$ be a smooth parametrized curve in Ω. We define the **contour integral** of f on the curve Γ to be

$$\int_\Gamma f(z)\,\mathrm{d}z := \int_a^b f(z(t))z'(t)\,\mathrm{d}t.$$

Moreover, if Γ is piecewise-smooth with smooth pieces $\Gamma_1, \ldots, \Gamma_m$, we define

$$\int_\Gamma f(z)\,\mathrm{d}z := \sum_{i=1}^m \int_{\Gamma_i} f(z)\,\mathrm{d}z.$$

Note that if f is continuous and z is smooth, then $f(z(t))z'(t)$ is a continuous one-variable complex function and hence is integrable on $[a, b]$. Thus, the contour integral $\int_\Gamma f(z)\,\mathrm{d}z$ is well-defined.

PROPOSITION 16.8. Let Γ be a piecewise-smooth parametrized curve on a domain $\Omega \subseteq \mathbb{C}$, and let $f, g : \Omega \to \mathbb{C}$ be continuous functions.

(i) For all $\alpha \in \mathbb{C}$, we have $\int_\Gamma \alpha f(z) + g(z)\,\mathrm{d}z = \alpha \int_\Gamma f(z)\,\mathrm{d}z + \int_\Gamma g(z)\,\mathrm{d}z$.

(ii) We have $\int_{-\Gamma} f(z)\,\mathrm{d}z = -\int_\Gamma f(z)\,\mathrm{d}z$.

PROOF. These properties follow immediately from the definition. □

EXAMPLE 16.9. Consider $\int_\Gamma z\,\mathrm{d}z$, where Γ is parametrized by $z : [0, 1] \to \Gamma : t \mapsto t + it^2$. Then $f(z) = z$ is continuous on $\mathbb{C}$ and $z'(t) = 1 + i(2t)$, so we obtain

$$\begin{aligned}
\int_\Gamma z\,\mathrm{d}z &= \int_0^1 (t + it^2)(1 + i(2t))\,\mathrm{d}t \\
&= \int_0^1 (t - 2t^3) + i(3t^2)\,\mathrm{d}t \\
&= \int_0^1 t - 2t^3\,\mathrm{d}t + i\int_0^1 3t^2\,\mathrm{d}t = i.
\end{aligned}$$

Let $f : [a, b] \to \mathbb{R}$ be a real one-variable function, and let F be an antiderivative of f on $[a, b]$. By the Fundamental Theorem of Calculus, we have

$$\int_a^b f(t)\,\mathrm{d}t = F(b) - F(a).$$

In other words, $\int_a^b f(t)\,\mathrm{d}t$ depends only on the endpoints of $[a, b]$.

Is there a similar result to the Fundamental Theorem of Calculus for contour integrals? First, we introduce the notion of an antiderivative of a complex function.

DEFINITION 16.10. Let f be a continuous complex function defined on a domain $\Omega \subseteq \mathbb{C}$. A function F defined on Ω is an **antiderivative** of f if $F'(z) = f(z)$ for all $z \in \Omega$.

Note that antiderivatives are analytic by definition, so they must differ by a constant as their derivatives are equal on the domain Ω.

EXAMPLE 16.11.

(1) For all $c \in \mathbb{C}$, the function $2z^3 + \cos z + c$ is an antiderivative of $6z^2 - \sin z$ on $\mathbb{C}$.

(2) For all $\tau \in \mathbb{R}$, the function $L_\tau(z)$ is an antiderivative of $1/z$ on D_τ^*, which is $\mathbb{C}$ without the ray $\theta = \tau$.

THEOREM 16.12 (Independence of path). Suppose that f is a continuous complex function on a domain $\Omega \subseteq \mathbb{C}$ which has an antiderivative F throughout Ω. Then for any piecewise-smooth parametrized curve Γ lying in Ω with initial point z_I and terminal point z_T, we have

$$\int_\Gamma f(z)\,\mathrm{d}z = F(z_T) - F(z_I).$$

In other words, the contour integral $\int_\Gamma f(z)\,\mathrm{d}z$ is independent of path. In particular, if Γ is a loop, then $\int_\Gamma f(z)\,\mathrm{d}z = 0$.

PROOF. This is a direct consequence of the Fundamental Theorem of Calculus. Indeed, if Γ is smooth and parametrized by $z : [a, b] \to \Gamma$, then

$$\begin{aligned}
\int_\Gamma f(z)\,\mathrm{d}z &= \int_a^b f(z(t))z'(t)\,\mathrm{d}t \\
&= \int_a^b F'(z(t))z'(t)\,\mathrm{d}t \\
&= \int_a^b \frac{\mathrm{d}}{\mathrm{d}t}(F(z(t)))\,\mathrm{d}t \\
&= F(z(b)) - F(z(a)) = F(z_T) - F(z_I).
\end{aligned}$$

More generally, suppose that Γ is piecewise-smooth with smooth pieces $\Gamma_1, \ldots, \Gamma_m$. Let the initial points be $z_{I_1}, \ldots, z_{I_m}$, and the terminal points be $z_{T_1}, \ldots, z_{T_m}$. Then $z_T = z_{T_m}$ and $z_I = z_{I_1}$, and we have

$$\int_\Gamma f(z)\,\mathrm{d}z = \sum_{i=1}^m \int_{\Gamma_i} f(z)\,\mathrm{d}z = \sum_{i=1}^m F(z_{T_i}) - F(z_{I_i}) = F(z_{T_m}) - F(z_{I_1}) = F(z_T) - F(z_I). \qquad \square$$

EXAMPLE 16.13.

(1) Consider $\int_\Gamma z^2\,\mathrm{d}z$, where Γ is any smooth curve starting at 0 and ending at $1 + i$. Note that z^2 is entire with antiderivative $z^3/3$, so we have

$$\int_\Gamma z^2\,\mathrm{d}z = (1+i)^3/3 - 0 = 2(-1+i)/3.$$

(2) More generally, for all $n \neq -1$, we have $\int_\Gamma (z - z_0)^n\,\mathrm{d}z = 0$ where Γ is any loop not passing through z_0, since $(z - z_0)^n$ is entire and has antiderivative $(z - z_0)^{n+1}/(n+1)$. In particular, if $n \neq -1$ and C is a circle centered at z_0, then

$$\int_C (z - z_0)^n\,\mathrm{d}z = 0.$$

(3) For any loop Γ in $\mathbb{C}$, we have

$$\int_\Gamma e^z\,\mathrm{d}z = \int_\Gamma \cos(3z)\,\mathrm{d}z = \int_\Gamma \sin z + z\,\mathrm{d}z = 0$$

since each of the integrands are entire and have antiderivatives e^z, $\sin(3z)/3$, and $-\cos z + z^2/z$ respectively on $\mathbb{C}$.

(4) Let us now evaluate $\int_\Gamma z^{1/2}\,\mathrm{d}z$, where the integrand denotes the analytic branch $e^{\frac{1}{2}L_{\pi/2}(z)}$ of $z^{1/2}$ and Γ is the lower-half circle $|z|=3$ parametrized by $z:[-\pi,0]\to\Gamma: t\mapsto 3e^{it}$.

First, note that Γ lies entirely on $\{z\in\mathbb{C}: y\le 0\}$ with endpoints $z=-3$ and $z=3$, which are both on the real axis. Also, recall that
$$L_{\pi/2}(z)=\ln|z|+i\arg_{\pi/2}(z)$$
for all $z\in D^*_{\pi/2}$, where $\pi/2<\arg_{\pi/2}(z)\le 5\pi/2$. Therefore, we have $\Gamma\subseteq D^*_{\pi/2}$, and $z^{1/2}$ has antiderivative
$$\frac{2}{3}z^{3/2}=\frac{2}{3}e^{\frac{3}{2}L_{\pi/2}(z)}$$
on $D^*_{\pi/2}$. Since the initial and terminal points of Γ are $z=-3$ and $z=3$ respectively, we obtain
$$\int_\Gamma z^{1/2}\,\mathrm{d}z=\frac{2}{3}e^{\frac{3}{2}L_{\pi/2}(3)}-\frac{2}{3}e^{\frac{3}{2}L_{\pi/2}(-3)}.$$
It remains to compute $L_{\pi/2}(3)$ and $L_{\pi/2}(-3)$. Since $|\pm 3|=3$, $\arg(3)=2\pi k$ for $k\in\mathbb{Z}$, and $\arg(-3)=(2\ell+1)\pi$ for $\ell\in\mathbb{Z}$, we see that $L_{\pi/2}(3)=\ln 3+2\pi i$ and $L_{\pi/2}(-3)=\ln 3+\pi i$. Thus, we conclude that
$$\int_\Gamma z^{1/2}\,\mathrm{d}z=\frac{2}{3}e^{\frac{3}{2}(\ln 3+2\pi i)}-\frac{2}{3}e^{\frac{3}{2}(\ln 3+\pi i)}=\frac{2}{3}(3\sqrt{3})(e^{3\pi i}-e^{3\pi i/2})=2\sqrt{3}(-1+i).$$

Remark 16.14.

(i) Theorem 16.12 tells us that $\int_\Gamma f(z)\,\mathrm{d}z$ depends only on the endpoints of Γ and not on Γ, as long as f is continuous on Ω and has an antiderivative there.

(ii) The contrapositive of Theorem 16.12 tells us that if $\int_\Gamma f(z)\,\mathrm{d}z$ depends on a path or is non-zero along a loop in Γ in Ω, then f does not have an antiderivative there.

For example, consider $f(z)=1/z$ on $\mathbb{C}\setminus\{0\}$. We have seen that f has antiderivatives on each D^*_τ, which are given by $L_\tau(z)$ up to a constant. However, it does not have an antiderivative on all of $\mathbb{C}\setminus\{0\}$. Indeed, consider $\int_C(1/z)\,\mathrm{d}z$ where C is the unit circle oriented counter-clockwise; in particular, it is parametrized by $z:[0,2\pi]\to\mathbb{C}: t\mapsto e^{it}$. Then
$$\int_C\frac{1}{z}\,\mathrm{d}z=2\pi i\ne 0,$$
which proves that $1/z$ does not have an antiderivative on all of $\mathbb{C}\setminus\{0\}$.

We have now seen that if f has an antiderivative in a domain Ω, then the contour integrals of f in Ω are independent of path. It turns out that the converse is also true.

Theorem 16.15 (Existence of antiderivatives). Let $\Omega\subseteq\mathbb{C}$ be a domain and let f be a continuous function on Ω. The following are equivalent.

(i) f has an antiderivative on Ω.

(ii) $\int_\Gamma f(z)\,\mathrm{d}z$ is independent of path for any piecewise-smooth curve Γ in Ω.

(iii) $\int_\Gamma f(z)\,\mathrm{d}z=0$ for any closed curve Γ in Ω.

Proof. We have already seen that (ii) and (iii) are equivalent, and Theorem 16.12 establishes that (i) implies (ii). Therefore, we only need to check that (ii) implies (i).

Suppose that $\int_\Gamma f(z)\,\mathrm{d}z$ is independent of path for any piecewise-smooth curve Γ in Ω. Fix $z_0\in\Omega$ and let $z\in\Omega$. We define
$$F(z):=\int_\Gamma f(w)\,\mathrm{d}w$$

for any piecewise-smooth curve Γ with initial point z_0 and terminal point z. Since $\int_\Gamma f(w)\,\mathrm{d}w$ is independent of path, we see that $F(z)$ is well-defined and gives a function $F:\Omega\to\mathbb{C}$. Let us check that F is an antiderivative of f on Ω. In particular, we need to show that $F'(z)=f(z)$ for all $z\in\Omega$.

By definition, we have

$$F'(z)=\lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z}.$$

Let Γ and Γ' be piecewise-smooth curves with initial point z_0 and terminal points z and $z+\Delta z$, respectively. We then obtain

$$\frac{f(z+\Delta z)-f(z)}{\Delta z}=\frac{1}{\Delta z}\left(\int_{\Gamma'}f(w)\,\mathrm{d}w-\int_\Gamma f(w)\,\mathrm{d}w\right).$$

Let Γ'' be the line segment joining z to $z+\Delta z$, oriented from z to $z+\Delta z$. Then $\Gamma+\Gamma''-\Gamma'$ is a loop in Ω, implying that

$$\int_\Gamma f(w)\,\mathrm{d}w-\int_{\Gamma'}f(w)\,\mathrm{d}w+\int_{\Gamma''}f(w)\,\mathrm{d}w=0.$$

In other words, we have

$$\lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z}=\lim_{\Delta z\to 0}\frac{1}{\Delta z}\int_{\Gamma''}f(w)\,\mathrm{d}w.$$

To compute $\int_{\Gamma''}f(w)\,\mathrm{d}w$, set $z(t)=z+t\Delta z$ for $t\in[0,1]$ so that $z'(t)=\Delta z$. We see that

$$\begin{aligned}\lim_{\Delta z\to 0}\frac{1}{\Delta z}\int_{\Gamma''}f(w)\,\mathrm{d}w&=\lim_{\Delta z\to 0}\frac{1}{\Delta z}\int_0^1 f(z+t\Delta z)\Delta z\,\mathrm{d}t\\&=\lim_{\Delta z\to 0}\int_0^1 f(z+t\Delta z)\,\mathrm{d}t=f(z),\end{aligned}$$

so $F'(z)=f(z)$ as desired. $\square$

17 The ML-inequality and the Cauchy-Goursat Theorem

In this lecture, we consider several theorems related to complex integration. In particular, we will prove the ML-inequality, Cauchy's Integral Theorem and the Cauchy-Goursat Theorem. Note that Cauchy's Integral Theorem is a direct consequence of Green's Theorem and was initially proven by Cauchy. The conditions appearing in the statement of the theorem were later relaxed by Goursat to give what is now known as the Cauchy-Goursat Theorem.

We begin with a very useful inequality.

THEOREM 17.1 (ML-inequality). Let f be a continuous function defined on a curve $\Gamma \subseteq \mathbb{C}$. Suppose that $|f(z)| \leq M$ for all $z \in \Gamma$, and let L be the length of Γ. Then

$$\left| \int_\Gamma f(z)\,\mathrm{d}z \right| \leq \int_\Gamma |f(z)||\mathrm{d}z| \leq ML,$$

where $|\mathrm{d}z|$ denotes $|z'(t)|\,\mathrm{d}t$ for any parametrization $z(t)$ of Γ.

PROOF. Let $I = \int_\Gamma f(z)\,\mathrm{d}z$. Since $I \in \mathbb{C}$, we have $I = |I|e^{i\theta_0}$ for some angle $\theta_0 \in \mathbb{R}$. Hence, it follows that

$$|I| = e^{-i\theta_0} I = \int_\Gamma e^{-i\theta_0} f(z)\,\mathrm{d}z = \int_a^b e^{-i\theta_0} f(z(t))z'(t)\,\mathrm{d}t,$$

where $z(t)$ is a parametrization of Γ with $a \leq t \leq b$. We set

$$e^{-i\theta_0} f(z(t))z'(t) = U(t) + iV(t),$$

where U and V are the real and imaginary parts of $e^{-i\theta_0} f(z(t))z'(t)$ respectively. We then have

$$|I| = \int_a^b U(t)\,\mathrm{d}t + i\int_a^b V(t)\,\mathrm{d}t.$$

Note that $|I|$, U, and V are all real, so we must have $\int_a^b V(t)\,\mathrm{d}t = 0$ and hence

$$|I| = \int_a^b U(t)\,\mathrm{d}t = \left| \int_a^b U(t)\,\mathrm{d}t \right| \leq \int_a^b |U(t)|\,\mathrm{d}t.$$

Moreover, observe that

$$|U(t)| \leq |U(t) + iV(t)| = |e^{-i\theta_0} f(z(t))z'(t)| \leq |f(z(t))||z'(t)|,$$

which implies that

$$\begin{aligned} \left| \int_\Gamma f(z)\,\mathrm{d}z \right| = |I| &\leq \int_a^b |f(z(t))||z'(t)|\,\mathrm{d}t \\ &= \int_\Gamma |f(z)||\mathrm{d}z| \\ &\leq \int_a^b M|z'(t)|\,\mathrm{d}t \\ &= M\int_a^b |z'(t)|\,\mathrm{d}t = ML \end{aligned}$$

since $|f(z(t))| \leq M$ for all $t \in [a,b]$ and $L = \int_a^b |z'(t)|\,\mathrm{d}t$. $\square$

EXAMPLE 17.2.

(1) Let Γ be the circle $|z| = 2$ traversed once in the counter-clockwise direction. We can use the ML-inequality to find an upper bound for $|\int_\Gamma e^z/(z^2+1)\,\mathrm{d}z|$. Indeed, note that Γ has arclength $L = 4\pi$. We now find an upper bound M for $|e^z/(z^2+1)|$. First, observe that

$$|e^z| = |e^{x+iy}| = |e^x \cdot e^{iy}| = e^x \cdot |e^{iy}| = e^x \leq e^2,$$

since $-2 \leq x \leq 2$ on the circle $|z| = 2$. Moreover, by the triangle inequality, we obtain

$$|z^2+1| \geq |z^2| - |1| = |z|^2 - 1 = 4 - 1 = 3$$

on $|z| = 2$. Therefore, we have the upper bound $M = e^2/3$ for $|e^z/(z^2+1)|$. The ML-inequality then implies that

$$\left| \int_\Gamma \frac{e^z}{z^2+1}\,\mathrm{d}z \right| \leq ML = \frac{e^2}{3} \cdot 4\pi = \frac{4\pi e^2}{3}.$$

(2) Let Γ be the line segment from $z = 0$ to $z = i$. We claim that

$$\left| \int_\Gamma e^{\sin z}\,\mathrm{d}z \right| \leq 1.$$

First, we see that Γ has arclength $L = 1$. Moreover, every point on Γ is of the form $z = iy$ where $0 \leq y \leq 1$. Therefore, we obtain

$$\sin z = \sin(iy) = \frac{e^{i^2 y} - e^{-i^2 y}}{2i} = \frac{i}{2}(e^y - e^{-y}),$$

and hence we see that $|e^{\sin z}| = |e^{(i/2)(e^y - e^{-y})}| = 1$ on Γ, giving us $M = 1$. The claim then follows from the ML-inequality.

There are several versions of Cauchy's Integral Theorem. We begin with a more restrictive version that assumes the functions we are considering are analytic in a domain Ω, and that the derivatives of its real and imaginary parts are continuous on Ω. Suppose that f is analytic on Ω and its real and imaginary parts are u and v respectively. Since f is analytic on Ω, its derivative $f'(z)$ exists at every point $z \in \Omega$, and we have

$$f'(z) = u_x(z) + iv_x(z) = v_y(z) - iu_y(z).$$

We see that $u, v \in C^1(\Omega)$ if and only if f' is continuous on Ω.

THEOREM 17.3 (Cauchy's Integral Theorem). Let f be analytic in a domain $\Omega \subseteq \mathbb{C}$ and suppose that its derivative f' is continuous on Ω. Then for any closed Jordan curve Γ in Ω whose interior is contained in Ω, we have

$$\int_\Gamma f(z)\,\mathrm{d}z = 0.$$

PROOF. This is a direct consequence of Green's Theorem. Let u and v the real and imaginary parts of f respectively. Then $u, v \in C^1(\Omega)$ since f' is continuous on Ω (by our above discussion). Moreover, we have

$$f(z)\,\mathrm{d}z = (u+iv)(\mathrm{d}x + i\,\mathrm{d}y) = (u\,\mathrm{d}x - v\,\mathrm{d}y) + i(v\,\mathrm{d}x + u\,\mathrm{d}y).$$

Therefore, we obtain

$$\int_\Gamma f(z)\,\mathrm{d}z = \int_\Gamma (u\,\mathrm{d}x - v\,\mathrm{d}y) + i\int_\Gamma (v\,\mathrm{d}x + u\,\mathrm{d}y).$$

Let D be the interior of Γ. Then by Green's Theorem, we have

$$\begin{aligned}\int_\Gamma u\,\mathrm{d}x - v\,\mathrm{d}y &= \iint_D (-v)_x - u_y\,\mathrm{d}x\,\mathrm{d}y,\\ \int_\Gamma v\,\mathrm{d}x + u\,\mathrm{d}y &= \iint_D u_x - v_y\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

Since f is analytic on Ω, we see that u and v satisfy the Cauchy-Riemann equations on Ω. In particular, we have $u_x = v_y$ and $u_y = -v_x$, so both the double integrals evaluate to 0, and we are done. $\square$

EXAMPLE 17.4. Let Γ be any closed Jordan curve in $\mathbb{C}$. Then

$$\int_\Gamma e^{z^3}\,\mathrm{d}z = 0,$$

since e^{z^3} is the composition of entire functions and its derivative $f'(z) = 3z^2e^{z^3}$ is continuous everywhere.

It turns out that we can relax the condition of the derivative f' being continuous on Ω in the statement of Cauchy's Integral Theorem. This was proven by Goursat and is known as the Cauchy-Goursat Theorem.

THEOREM 17.5 (Cauchy-Goursat Theorem). Let f be analytic on a domain $\Omega \subseteq \mathbb{C}$. Then for any closed Jordan curve Γ in Ω whose interior is contained in Ω, we have

$$\int_\Gamma f(z)\,\mathrm{d}z = 0.$$

PROOF. Let R be the subset of Ω consisting of Γ and its interior. We subdivide R by drawing a grid consisting of a finite number of equally spaced lines parallel to the x-axis and the y-axis. These lines form squares whose sides all have the same length. We keep the squares that are contained in R, remove the squares that have no points in R, and remove the points that are not in R from the squares that contain points from R but are not contained in R. We are then left with n squares and partial squares that cover R, which we will denote by $\sigma_1, \dots, \sigma_n$.

Suppose that the squares appearing in the grid have sides equal to s. Then for any $1 \le j \le n$ and $z, z' \in \sigma_j$, we have

$$|z - z'| \le \sqrt{2}s$$

since the diagonal of the square with side s has length $\sqrt{2}s$, and each σ_j is contained in such a square. Moreover, if the boundary C_j of σ_j has length L_j, we set

$$L = \max_{1\le j\le n} L_j.$$

CLAIM. Let $\varepsilon > 0$. For every $1 \le j \le n$, there exists $z_j \in \sigma_j^\circ$ such that

$$\left|\frac{f(z) - f(z_j)}{z - z_j} - f'(z_j)\right| < \frac{\varepsilon}{n\sqrt{2}sL} \tag{$\star$}$$

for all $z \in \sigma_j \setminus \{z_j\}$.

PROOF OF CLAIM. We proceed by contradiction. Suppose that for some $1 \le j_0 \le n$, there does not exist $z_{j_0} \in \sigma_{j_0}$ such that

$$\left|\frac{f(z) - f(z_{j_0})}{z - z_{j_0}} - f'(z_{j_0})\right| < \frac{\varepsilon}{n\sqrt{2}sL}$$

for all $z \in \sigma_{j_0} \setminus \{z_{j_0}\}$. Let us subdivide σ_{j_0} into 4 subregions in the same way we subdivided R.

If $(\star)$ holds for each of the 4 regions of σ_{j_0}, then we simply replace σ_{j_0} in the statement of the claim by the 4 subregions covering it. Otherwise, there must exist $1 \le j_1 \le n$ for each inequality $(\star)$ does not hold. Again, subdivide σ_{j_1} into 4 subregions as we subdivided R. If $(\star)$ holds for each of the 4 subregions of σ_{j_1}, we replace σ_{j_1} by the 4 subregions containing it. Otherwise, there exists $1 \le j_2 \le n$ for which $(\star)$ does not hold, and we can continue to repeat the above argument.

If this process does not end after a finite number of steps, we obtain a nested infinite sequence

$$\sigma_{j_0} \supseteq \sigma_{j_1} \supseteq \sigma_{j_2} \supseteq \cdots \supseteq \sigma_{j_k} \supseteq \cdots$$

and by an analogue of the Nested Interval Theorem for rectangles, there exists $z_0 \in \bigcap_k \sigma_{j_k} \neq \varnothing$. Since the arclength of the boundary of each σ_{j_k} decreases at every step, if we choose $z_{j_k} \in \sigma_{j_k}$ for each k, we obtain a sequence $\{z_{j_k}\}$ of points in R such that $z_{j_k} \to z_0$ as $k \to \infty$. In other words, z_0 is a limit point of R. But R is closed, which implies that $z_0 \in R$.

Recall that f is analytic on R, so the derivative of f exists at z_0. In particular, there exists $\delta > 0$ such that

$$\left|\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right| < \frac{\varepsilon}{n\sqrt{2}sL}$$

for all $z \in R$ such that $0 < |z - z_0| < \delta$, contradicting the choice of the σ_{j_k}'s. ∎

Having proven the claim, set

$$\delta_j(z) := \frac{f(z) - f(z_j)}{z - z_j} - f'(z_j)$$

for all $z \in \sigma_j \setminus \{z_j\}$ and $1 \leq j \leq n$. Note that δ_j is continuous on each $z \in \sigma_j \setminus \{z_j\}$ since f is analytic. Moreover, since $f'(z_j)$ exists, we have

$$\lim_{z \to z_j} \delta_j(z) = 0.$$

Consequently, we can extend δ_j to the continuous function on σ_j given by

$$\delta_j(z) = \begin{cases} \dfrac{f(z) - f(z_j)}{z - z_j} - f'(z_j), & z \neq z_j \\ 0, & z = z_j \end{cases}$$

which is such that

$$|\delta_j(z)| < \frac{\varepsilon}{n\sqrt{2}sL}$$

for all $z \in \sigma_j$. Now, consider the boundary C_j of σ_j for each $1 \leq j \leq n$. Note that $z_j \notin C_j$ for all $1 \leq j \leq n$ since $z_j \in \sigma_j^\circ$. Therefore, given the definition of δ_j away from z_j, we can write

$$f(z) = [f(z_j) - f'(z_j)z_j] + f'(z_j)z + \delta_j(z)(z - z_j)$$

for all $z \in C_j$. Let us orient each C_j positively. Then we have

$$\int_{C_j} f(z)\,\mathrm{d}z = [f(z_j) - f'(z_j)z_j] \int_{C_j} \mathrm{d}z + f'(z_j) \int_{C_j} z\,\mathrm{d}z + \int_{C_j} \delta_j(z)(z - z_j)\,\mathrm{d}z.$$

Since 1 and z possess antiderivatives in the whole complex plane and C_j is a loop, we have

$$\int_{C_j} \mathrm{d}z = \int_{C_j} z\,\mathrm{d}z = 0,$$

which implies that

$$\int_{C_j} f(z)\,\mathrm{d}z = \int_{C_j} \delta_j(z)(z - z_j)\,\mathrm{d}z.$$

Moreover, since each C_j is oriented positively and two adjacent C_j's have common pieces of boundary with opposite orientations, we get

$$\sum_{j=1}^{n} \int_{C_j} f(z)\,\mathrm{d}z = \int_{\Gamma} f(z)\,\mathrm{d}z$$

because the contributions of these common pieces have opposite signs and thus cancel themselves out.

Putting all this together, we obtain

$$\int_{\Gamma} f(z)\,\mathrm{d}z = \sum_{j=1}^{n} \int_{C_j} \delta_j(z)(z - z_j)\,\mathrm{d}z.$$

If we can show that

$$\left|\int_{C_j} \delta_j(z)(z-z_j)\,\mathrm{d}z\right| < \frac{\varepsilon}{n}$$

for all $1 \le j \le n$, this will imply that

$$\left|\int_{\Gamma} f(z)\,\mathrm{d}z\right| \le \sum_{j=1}^{n}\left|\int_{C_j} \delta_j(z)(z-z_j)\,\mathrm{d}z\right| < n\cdot\frac{\varepsilon}{n} = \varepsilon,$$

and thus $\int_\Gamma f(z)\,\mathrm{d}z = 0$. Indeed, recall that by construction, C_j has arclength at most L, $|z-z_j| \le \sqrt{2}s$ for all $z \in C_j$, and $|\delta_j(z)| < \varepsilon/(n\sqrt{2}sL)$ for all $1 \le j \le n$. Therefore, we have

$$|\delta_j(z)(z-z_j)| = |\delta_j(z)||z-z_j| < \frac{\varepsilon}{n\sqrt{2}sL}\cdot\sqrt{2}s = \frac{\varepsilon}{nL}.$$

Finally, by the ML-inequality, it follows that

$$\left|\int_{C_j} \delta_j(z)(z-z_j)\,\mathrm{d}z\right| < \frac{\varepsilon}{nL}\cdot L = \frac{\varepsilon}{n}$$

for all $1 \le j \le n$. This completes the proof. $\square$

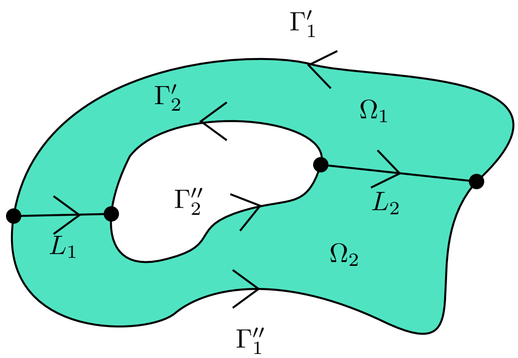
18 Deformation Principle, variations of Cauchy's Integral Theorem

We begin with a remarkable theorem which will be very useful in our study of complex analysis. Essentially, it states that the contour integral of an analytic function along a simple loop does not depend on the loop.

THEOREM 18.1 (Deformation Principle). Let $\Omega \subseteq \mathbb{C}$ be a domain, and let f be analytic on Ω. Moreover, let Γ_1 and Γ_2 be similarly oriented piecewise-smooth simple curves inside Ω such that Γ_2 lies in the interior of Γ_1, and f is analytic at all points between the curves. (Note that these assumptions ensure that Γ_1 and Γ_2 do not intersect.) Then we have

$$\int_{\Gamma_1} f(z)\,\mathrm{d}z = \int_{\Gamma_2} f(z)\,\mathrm{d}z.$$

PROOF. This is a consequence of the Cauchy-Goursat Theorem. Suppose without loss of generality that Γ_1 and Γ_2 are both positively oriented, and let Ω' be the region between the two curves. Then Ω' is a 2-connected Jordan domain with boundary $\partial\Omega = \Gamma_1 - \Gamma_2$, and can be broken into two simply connected domains Ω_1 and Ω_2.



We set $\Gamma_1 = \Gamma_1' + \Gamma_1''$ and $\Gamma_2 = \Gamma_2' + \Gamma_2''$. The boundaries of Ω_1 and Ω_2 are simple closed curves that are oriented positively by

$$\begin{aligned}\partial\Omega_1 &= \Gamma_1' + L_1 - \Gamma_2' + L_2,\\ \partial\Omega_2 &= \Gamma_1'' - L_2 - \Gamma_2'' - L_1.\end{aligned}$$

By the Cauchy-Goursat Theorem, we have

$$\int_{\partial\Omega_1} f(z)\,\mathrm{d}z = \int_{\partial\Omega_2} f(z)\,\mathrm{d}z = 0,$$

which implies that

$$\begin{aligned}\int_{\Gamma_1'} f(z)\,\mathrm{d}z &= \int_{\Gamma_2' - L_1 - L_2} f(z)\,\mathrm{d}z,\\ \int_{\Gamma_1''} f(z)\,\mathrm{d}z &= \int_{\Gamma_2'' + L_1 + L_2} f(z)\,\mathrm{d}z.\end{aligned}$$

Finally, it follows that

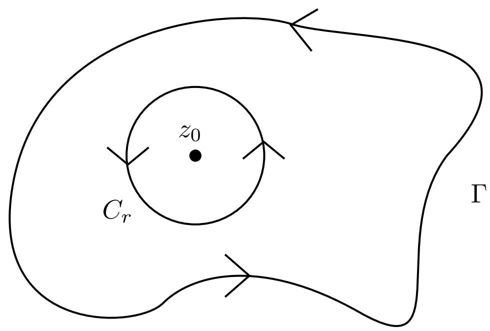
$$\int_{\Gamma_1} f(z)\,\mathrm{d}z = \int_{\Gamma_1' + \Gamma_1''} f(z)\,\mathrm{d}z = \int_{\Gamma_2' + \Gamma_2''} f(z)\,\mathrm{d}z = \int_{\Gamma_2} f(z)\,\mathrm{d}z. \qquad \square$$

REMARK 18.2. An important consequence of the Deformation Principle is that it allows us to compute contour integrals using simpler contours. For example, suppose that $\Omega \subseteq \mathbb{C}$ is a domain and $z_0 \in \Omega$. Consider a simple closed curve Γ in $\Omega \setminus \{z_0\}$ whose interior contains z_0 and an analytic function f on $\Omega \setminus \{z_0\}$. If f is analytic on all of Ω, then by the Cauchy-Goursat Theorem, we have

$$\int_\Gamma f(z)\,\mathrm{d}z = 0.$$

However, if f fails to be analytic at the point z_0, then one has to compute the integral directly, which can be quite complicated depending on the curve Γ. However, by the Deformation Principle, if $C_r = \{z \in \mathbb{C} : |z - z_0| = r\}$ is a circle centered at z_0 lying in the interior of Γ with the same orientation as Γ, then

$$\int_\Gamma f(z)\,\mathrm{d}z = \int_{C_r} f(z)\,\mathrm{d}z.$$



One can also use the Deformation Principle to prove a more general form of Cauchy's Integral Theorem. We leave this as an exercise.

Theorem 18.3 (Cauchy's Integral Theorem for Jordan domains). Let $\Omega \subseteq \mathbb{C}$ be a k-connected Jordan domain, and let f be analytic on some domain Ω^+ containing Ω and $\partial\Omega$. Then we have

$$\partial\Omega = \Gamma_1 + \cdots + \Gamma_k$$

for k positively oriented Jordan curves $\Gamma_1, \ldots, \Gamma_k \subseteq \Omega^+$, and

$$\int_{\partial\Omega} f(z)\,\mathrm{d}z := \sum_{i=1}^{k} \int_{\Gamma_i} f(z)\,\mathrm{d}z = 0.$$

Recall that a Jordan domain is simply connected if it is 1-connected. More generally, we have the following definition.

Definition 18.4 (Simply connected domain). A domain $\Omega \subseteq \mathbb{C}$ is **simply connected** if it has the property that if Γ is any simply closed curve in Ω, then the domain interior to Γ is included in Ω.

As usual, we can intuitively think of a simply connected domain as a domain which has "no holes".

On a simply connected domain, we have another version of Cauchy's Integral Theorem.

Theorem 18.5 (Cauchy's Integral Theorem for simply connected domains). Suppose that f is analytic on a simply connected domain $\Omega \subseteq \mathbb{C}$. Then for any closed curve Γ in Ω, we have

$$\int_\Gamma f(z)\,\mathrm{d}z = 0.$$

Note that in the above statement, we do not assume that the closed curve Γ is simple.

Remark 18.6. As a consequence of the previous theorem, we have that if f is analytic on a simply connected domain $\Omega \subseteq \mathbb{C}$, then

- f has an antiderivative throughout Ω;
- $\int_\Gamma f(z)\,\mathrm{d}z$ is independent of path for any contour Γ in Ω; and
- $\int_\Gamma f(z)\,\mathrm{d}z = 0$ for any loop Γ in Ω.

Remark 18.7. With the results we have seen so far, we give a general outline for computing integrals of complex-valued functions.

(i) For the definite integral of a complex-valued function of one real variable $f : [a, b] \to \mathbb{C}$, proceed as in the real case and treat i as a constant.

(ii) For the contour integral of an analytic function $f : \Omega \to \mathbb{C}$ over a curve Γ in a domain $\Omega \subseteq \mathbb{C}$ parametrized by $z : [a, b] \to \Gamma$:

- If Γ is a simple loop or a finite union of simple loops, then the contour integral is zero.
- If Γ is not closed and we can find an antiderivative F of f on Ω, then

$$\int_\Gamma f(z)\,\mathrm{d}z = F(z(b)) - F(z(a)).$$

- Otherwise, directly compute the indefinite integral

$$\int_\Gamma f(z)\,\mathrm{d}z = \int_a^b f(z(t))z'(t)\,\mathrm{d}t.$$

If necessary, use a simpler curve that can be deformed to Γ as in the Deformation Principle.

19 Cauchy's Integral Formula and its consequences

We now state an important theorem that will have many applications in this course.

Theorem 19.1 (Cauchy's Integral Formula). Let f be analytic on a domain $\Omega \subseteq \mathbb{C}$, and let Γ be a positively oriented simple closed curve inside Ω whose interior is also contained in Ω. For any z_0 in the interior of Γ, we have

$$f(z_0) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z - z_0} \, \mathrm{d}z.$$

Proof. Let U be the interior of Γ. First, we note that since $z_0 \in U$ and U is open, there exists $R > 0$ such that $D(z_0; R) \subseteq U \subseteq \Omega$. Choose $0 < r < R$ such that the circle $C_r = \{z \in \mathbb{C} : |z - z_0| = r\}$ is contained in $D(z_0; R)$, and hence also contained in U. Then, C_r is a simple closed curve in U. Moreover, since $f(z)/(z - z_0)$ is analytic on $\Omega \setminus \{z_0\}$ and z_0 lies in the interior of C_r (as it is the center of C_r), it follows from the Deformation Principle that

$$\int_\Gamma \frac{f(z)}{z - z_0} \, \mathrm{d}z = \int_{C_r} \frac{f(z)}{z - z_0} \, \mathrm{d}z,$$

where C_r is positively oriented. Thus, it is enough to show that

$$\frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{z - z_0} \, \mathrm{d}z = f(z_0).$$

Note that we can write $f(z) = (f(z) - f(z_0)) + f(z_0)$ so that

$$\int_{C_r} \frac{f(z)}{z - z_0} \, \mathrm{d}z = \int_{C_r} \frac{f(z) - f(z_0)}{z - z_0} \, \mathrm{d}z + \int_{C_r} \frac{f(z_0)}{z - z_0} \, \mathrm{d}z.$$

Moreover, we have

$$\int_{C_r} \frac{f(z_0)}{z - z_0} \, \mathrm{d}z = 2\pi i \cdot f(z_0).$$

Hence, it only remains to check that

$$I := \frac{1}{2\pi i} \int_{C_r} \frac{f(z) - f(z_0)}{z - z_0} \, \mathrm{d}z = 0.$$

Indeed, since f is analytic at z_0, it is continuous there. For any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(z) - f(z_0)| < \varepsilon$$

whenever $|z - z_0| < \delta$. Therefore, if $r < \delta$, then for any $z \in C_r$, we have

$$\left| \frac{f(z) - f(z_0)}{z - z_0} \right| = \frac{|f(z) - f(z_0)|}{|z - z_0|} = \frac{|f(z) - f(z_0)|}{r} < \frac{\varepsilon}{r}.$$

In addition, C_r has arclength $2\pi r$, so by the ML-inequality, we obtain

$$|I| < \frac{1}{2\pi} \cdot \frac{\varepsilon}{r} \cdot 2\pi r = \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, it follows that $I = 0$. □

Remark 19.2.

(i) A nice consequence of Cauchy's Integral Formula is that one can determine all the values of an analytic function in the interior of a simple loop Γ just by knowing its values on Γ. In particular, the behaviour of an analytic function on a region is completely determined by its behaviour on the boundary.

(ii) Cauchy's Integral Formula can be used to compute contour integrals whose integrands are of the form $f(z)/(z-z_0)$ where f is analytic, since we have

$$\int_\Gamma \frac{f(z)}{z-z_0}\,\mathrm{d}z = 2\pi i \cdot f(z_0).$$

Example 19.3.

(1) Consider the contour integral

$$\int_\Gamma \frac{e^z+\sin z}{z}\,\mathrm{d}z$$

where Γ is the circle $|z-2|=3$ traversed once in the counter-clockwise direction. Observe that the integrand is of the form $f(z)/(z-z_0)$ where $f(z)=e^z+\sin z$ and $z_0=0$. Moreover, f is entire and z_0 lies in Γ, so by Cauchy's Integral Formula, we have

$$\int_\Gamma \frac{e^z+\sin z}{z}\,\mathrm{d}z = 2\pi i\cdot f(0) = 2\pi i.$$

(2) Consider the contour integral

$$\int_\Gamma \frac{\cos z}{z^2-4}\,\mathrm{d}z$$

where Γ is a closed simple curve contained in $\Omega=\{z\in\mathbb{C} : x>-1\}$ such that $z=2$ lies in the interior of Γ. Note that the denominator of the integrand z^2-4 vanishes at $z=\pm 2$, but only $z=2$ lies inside Γ. Hence, we can write the integrand as $f(z)/(z-z_0)$ where $f(z)=\cos z/(z+2)$ and $z_0=2$. Then $f(z)$ is analytic on the domain Ω containing Γ and z_0 lies inside Γ. It follows from Cauchy's Integral Formula that

$$\int_\Gamma \frac{\cos z}{z^2-4}\,\mathrm{d}z = 2\pi i\cdot f(2) = 2\pi i\cdot\frac{\cos 2}{4} = \frac{\pi i\cos 2}{2}.$$

As a consequence of Cauchy's Integral Formula, we obtain the following important result.

Theorem 19.4. Let f be analytic on a domain $\Omega\subseteq\mathbb{C}$. Then the derivatives of f of every order exist and are analytic on Ω. Moreover, for all $z_0\in\Omega$, we have

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_\Gamma \frac{f(z)}{(z-z_0)^{n+1}}\,\mathrm{d}z,$$

where Γ is any positively oriented simple closed curve inside Ω such that the interior of Γ contains z_0 and is contained in Ω.

Proof. We prove the formula for $n=1$. As in the proof of Cauchy's Integral Formula, we have

$$\int_\Gamma \frac{f(z)}{(z-z_0)^2}\,\mathrm{d}z = \int_{C_r}\frac{f(z)}{(z-z_0)^2}\,\mathrm{d}z$$

for any circle $C_r=\{z\in\mathbb{C} : |z-z_0|=r\}$ centered at z_0 lying in the interior of Γ. We need to show that

$$\lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h} = \frac{1}{2\pi i}\int_{C_r}\frac{f(z)}{(z-z_0)^2}\,\mathrm{d}z.$$

By Cauchy's Integral Formula, we know that

$$f(z_0) = \frac{1}{2\pi i}\int_{C_r}\frac{f(z)}{z-z_0}\,\mathrm{d}z.$$

For h sufficiently close to 0, the point z_0+h also lies in C_r so that

$$f(z_0+h) = \frac{1}{2\pi i}\int_{C_r}\frac{f(z)}{z-(z_0+h)}\,\mathrm{d}z.$$

Moreover, we have

$$\begin{aligned}\frac{f(z_0+h)-f(z_0)}{h} &= \frac{1}{2\pi i}\int_{C_r}\frac{1}{h}\left(\frac{f(z)}{z-(z_0+h)}-\frac{f(z)}{z-z_0}\right)\mathrm{d}z\\ &= \frac{1}{2\pi i}\int_{C_r}\frac{f(z)}{(z-(z_0+h))(z-z_0)}\,\mathrm{d}z.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\frac{f(z_0+h)-f(z_0)}{h}-\frac{1}{2\pi i}\int_{C_r}\frac{f(z)}{(z-z_0)^2}\,\mathrm{d}z &= \frac{1}{2\pi i}\int_{C_r}\left[\frac{f(z)}{(z-(z_0+h))(z-z_0)}-\frac{f(z)}{(z-z_0)^2}\right]\mathrm{d}z\\ &= \frac{1}{2\pi i}\int_{C_r}\frac{f(z)h}{(z-(z_0+h))(z-z_0)^2}\,\mathrm{d}z.\end{aligned}$$

Note that C_r has arclength $2\pi r$ and $|z-z_0|=r$ since $z\in C_r$. It follows from the triangle inequality that

$$|z-(z_0+h)|\geq |z-z_0|-|h| = r-|h|>0.$$

whenever $|h|<r$. Thus, we have

$$\left|\frac{f(z)h}{(z-(z_0+h))(z-z_0)^2}\right|\leq \frac{|f(z)||h|}{(r-|h|)r^2}.$$

Finally, applying the ML-inequality gives

$$\left|\frac{f(z_0+h)-f(z_0)}{h}-\frac{1}{2\pi i}\int_{C_r}\frac{f(z)}{(z-z_0)^2}\,\mathrm{d}z\right|\leq \frac{1}{2\pi}\cdot\frac{|f(z)||h|}{(r-|h|)r^2}\cdot 2\pi r = \frac{|f(z)||h|}{(r-|h|)r},$$

which tends to 0 as h approaches 0. Thus, we conclude that

$$f'(z_0) = \frac{1}{2\pi i}\int_{C_r}\frac{f(z)}{(z-z_0)^2}\,\mathrm{d}z.$$

One can repeat this procedure for the general case; we leave the details as an exercise. $\square$

Remark 19.5. Due to Theorem 19.4, we see that for an analytic function, the existence of its derivative on an open set ensures that its derivatives exist up to any order. In particular, this implies that the real and imaginary parts of analytic functions are smooth.

20 Complex analogues of properties of harmonic functions

In the previous lecture, we finally proved that the real and imaginary parts of an analytic function are smooth, implying that they are harmonic since they satisfy the Cauchy-Riemann equations. Given this fact, several theorems we have seen for harmonic functions have counterparts for complex analytic functions.

First, we consider the complex analogues of the Circumferential and Solid MVTs.

THEOREM 20.1. Let f be analytic on a domain $\Omega \subseteq \mathbb{C}$ which contains the closed disc $\overline{D}(z_0; r)$.

(i) (Circumferential MVT) Then $f(z_0)$ is the arithmetic mean of the values that f assumes on the circumference of the disc; that is, we have

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\,\mathrm{d}\theta.$$

(ii) (Solid MVT) We have

$$f(z_0) = \frac{1}{\pi r^2}\iint_{D(z_0;r)} f(z)\,\mathrm{d}x\,\mathrm{d}y.$$

PROOF. Let u and v be the real and imaginary parts of f respectively so that $f = u + iv$. Since f is analytic on Ω, we know that u and v are harmonic on Ω.

(i) By the Circumferential MVT for harmonic functions (Theorem 6.5), we have

$$\begin{aligned} f(z_0) = u(z_0) + iv(z_0) &= \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})\,\mathrm{d}\theta + i\cdot\frac{1}{2\pi}\int_0^{2\pi} v(z_0 + re^{i\theta})\,\mathrm{d}\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\,\mathrm{d}\theta. \end{aligned}$$

(ii) By the Solid MVT for harmonic functions (Theorem 6.9), we similarly obtain

$$\begin{aligned} f(z_0) = u(z_0) + iv(z_0) &= \frac{1}{\pi r^2}\iint_{D(z_0;r)} u(z)\,\mathrm{d}x\,\mathrm{d}y + i\cdot\frac{1}{\pi r^2}\iint_{D(z_0;r)} v(z)\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{1}{\pi r^2}\iint_{D(z_0;r)} f(z)\,\mathrm{d}x\,\mathrm{d}y. \end{aligned}$$ □

We now turn to the complex analogue of the Maximum Principle for harmonic functions (which was the main topic of Lecture 7). Recall that the Maximum Principle tells us that a non-constant function harmonic function cannot assume a maximum on a domain. For a complex-valued function, it does not make sense to consider its maximum value since there is no ordering on the complex numbers. Nonetheless, we can consider its maximum modulus.

DEFINITION 20.2 (Maximum modulus). Let $\Omega \subseteq \mathbb{C}$ be a domain and let f be a complex-valued function on $\mathbb{C}$. We say that f **attains a maximum modulus at** z_0 **in** Ω if

$$|f(z)| \leq |f(z_0)|$$

for all $z \in \Omega$.

REMARK 20.3. Geometrically, if f attains a maximum modulus at z_0 in Ω, then $f(z_0)$ is the furthest point from the origin in $f(\Omega)$.

THEOREM 20.4 (Strong Maximum Modulus Principle). Let f be an analytic function on a domain $\Omega \subseteq \mathbb{C}$ and suppose that it attains a maximum modulus in Ω. Then f is constant.

PROOF. Suppose that f attains a maximum modulus at $z_0 \in \Omega$ and set $c_0 = |f(z_0)|$. Choose $c \in R^{>0}$ such that $c > c_0$ and $g(z) = f(z) + cf(z_0)$ is such that $g(\Omega) \subseteq D_\tau^* = \mathbb{C} \setminus \{re^{i\tau} : r \geq 0\}$ for some $\tau \in \mathbb{R}$. This is certainly possible since g is simply f translated by $cf(z_0)$.

The composition $L_\tau(g(z))$ is then defined and analytic on Ω, which implies that its real part $\ln|g(z)|$ is harmonic on Ω. Moreover, $\ln|g(z)|$ attains its maximum on z_0. Indeed, for all $z \in \Omega$, we have

$$|g(z)| \leq |f(z)| + c|f(z_0)| \leq |f(z_0)| + c|f(z_0)| = |g(z_0)|.$$

Since $\ln|\cdot|$ is monotone increasing, it follows that

$$\ln|g(z)| \leq \ln|g(z_0)|$$

for all $z \in \Omega$, so $\ln|g(z)|$ attains its maximum at z_0. This can only happen if $\ln|g(z)|$ is constant by the harmonicity of $\ln|g(z)|$ in Ω. By part (c) of Question 1 on Assignment 5, this in turn implies that $|g(z)|$ and $g(z)$ are constant in Ω, and hence so is f. □

REMARK 20.5. The Strong Maximum Modulus Principle equivalently states that a non-constant analytic function in a domain $\Omega \subseteq \mathbb{C}$ does not attain its maximum modulus at any point in Ω.

THEOREM 20.6 (Minimum Modulus Principle). Let f be a non-constant analytic function on the domain Ω with $f(z) \neq 0$ for all $z \in \mathbb{C}$. Then f does not attain its minimum modulus at any point in Ω.

PROOF. Apply the Strong Maximum Modulus Principle to $1/f(z)$. □

EXAMPLE 20.7. Consider the function $f(z) = (z+1)^2$, which is entire. Let Ω be the interior of the triangle with vertices $z = 0$, $z = 2$, and $z = i$. Note that

$$|f(z)| = |(x+1) + iy|^2 = (x+1)^2 + y^2,$$

which has maximum value 9 at $z = 2$ and minimum value 1 at $z = 0$ on $\overline{\Omega}$. Both the maximum and minimum occur on $\partial\Omega$, confirming the Maximum and Minimum Modulus Principles.

The fact that the maximum of $|f(z)|$ occurs on $\partial\Omega$ in Example 20.7 is due to the following theorem.

THEOREM 20.8 (Weak Maximum Modulus Principle). Let $\Omega \subseteq \mathbb{C}$ be a bounded domain such that f is continuous on $\overline{\Omega}$ and analytic on Ω. Then f is either constant or attains its maximum modulus over $\overline{\Omega}$ on $\partial\Omega$ only.

PROOF. Since $|f(z)|$ is continuous on the compact set $\overline{\Omega}$, it has a maximum on $\overline{\Omega}$ by the Extreme Value Theorem, which can only be attained on $\partial\Omega$ by the Strong Maximum Modulus Principle if f is a non-constant function. □

Recall that harmonic functions are completely and uniquely determined by their boundary behaviour (by Theorem 7.7). The same is also true of analytic functions.

THEOREM 20.9. Let f and g be analytic functions on a domain $\Omega \subseteq \mathbb{C}$, and let Γ be a simple closed curve whose interior lies inside Ω. If f and g are equal on Γ, then they are also equal on the interior of Γ.

PROOF. Let Ω' be the interior of Γ. Since Γ is simple and closed, we see that Ω' is a domain. Moreover, since $\overline{\Omega'} \subseteq \Omega$, we have that f and g are analytic on $\overline{\Omega'}$, so they are continuous on $\partial\Omega'$ and analytic on Ω'. Their real and imaginary parts are then continuous and equal on $\partial\Omega'$ and harmonic on Ω', implying that they are also equal on Ω'. □

REMARK 20.10. In fact, f and g are equal on Ω since they agree on an open set. This result is known as the Identity Theorem, which we will prove later.

We now consider the complex counterpart of Liouville's Theorem (Theorem 8.2), which states that bounded entire harmonic functions are constant.

THEOREM 20.11 (Liouville's Theorem). A bounded entire complex function is constant.

PROOF. If $f = u + iv$ is entire, then u and v are entire harmonic functions. Moreover, for all $z \in \mathbb{C}$, we have

$$|u(z)| \leq |u(z) + iv(z)| = |f(z)|,$$

and similarly $|v(z)| \leq |f(z)|$. Since f is bounded, so are u and v. By Liouville's Theorem for harmonic functions, u and v are constant. Consequently, f is constant. □

Finally, as an application of Liouville's Theorem, we prove the Fundamental Theorem of Algebra.

THEOREM 20.12 (Fundamental Theorem of Algebra). Let $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ be a polynomial of degree $n \geq 1$ with complex coefficients. Then $p(z)$ has a root in $\mathbb{C}$.

PROOF. Suppose to the contrary that $p(z)$ has no roots in $\mathbb{C}$ so that $p(z) \neq 0$ for all $z \in \mathbb{C}$. Then

$$f(z) = \frac{1}{p(z)}$$

is entire since it is a rational function whose denominator never vanishes. We show that f is bounded. First, note that $p(z) = (a_n + w)z^n$ where

$$w = \frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \cdots + \frac{a_0}{z^n}.$$

We claim that there exists $R > 1$ such that

$$\frac{|a_i|}{|z|^{n-i}} < \frac{|a_n|}{2n}$$

for all $0 \leq i \leq n-1$ whenever $|z| > R$. Indeed, let

$$s = \max_{0 \leq i \leq n-1} |a_i|$$

and choose $R > 1$ such that

$$\frac{s}{R} < \frac{|a_n|}{2n}.$$

Then $|z|^{n-i} > R^{n-i} \geq R$ for all $|z| > R$ since $R > 1$, which implies that

$$\frac{|a_i|}{|z|^{n-i}} < \frac{s}{R} < \frac{|a_n|}{2n}$$

for all $0 \leq i \leq n-1$. Moreover, by the triangle inequality, we have

$$|w| \leq \frac{|a_{n-1}|}{|z|} + \frac{|a_{n-2}|}{|z|^2} + \cdots + \frac{|a_0|}{|z|^n} < n \cdot \frac{|a_n|}{2n} = \frac{|a_n|}{2},$$

and so $|a_n - w| \geq ||a_n| - |w|| > |a_n|/2$. Hence, when $|z| > R$, we obtain

$$|p(z)| = |a_n + w||z|^n > \frac{|a_n|}{2} \cdot R^n = \frac{|a_n|R^n}{2},$$

so it follows that

$$|f(z)| = \frac{1}{|p(z)|} < \frac{2}{|a_n|R^n}.$$

Therefore, f is bounded on $|z| > R$. Moreover, since f is continuous, it is also bounded on the closed disc $|z| \leq R$ by the Extreme Value Theorem. Thus, f is bounded on $\mathbb{C}$.

Finally, we have by Liouville's Theorem that f is constant. But this implies that p is constant, which is a contradiction as p is a polynomial of degree $n \geq 1$. □

21 More consequences of Theorem 19.4

Recall that Theorem 19.4 states that if f is analytic on a domain $\Omega \subseteq \mathbb{C}$, then its derivatives of every order exist and are analytic on Ω. Moreover, for all $z_0 \in \Omega$ and every integer $n \geq 0$, we have

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_\Gamma \frac{f(z)}{(z - z_0)^{n+1}} \, \mathrm{d}z, \tag{$\star$}$$

where Γ is any positively oriented simple closed curve inside Ω whose interior contains z_0 and is contained in Ω. The formulae given by ($\star$) are known as the **Cauchy Integral Formulae**.

As a consequence of Theorem 19.4, we saw that the real and imaginary parts of an analytic function are smooth and hence harmonic. In Lecture 20, we used this fact to show that many of the properties of harmonic functions have analogues for complex analytic functions.

In this lecture, we will explore three more consequences of Theorem 19.4.

- (Morera's Theorem) This theorem can be viewed as the converse of the Cauchy-Goursat Theorem. It states that if a continuous function on a domain $\Omega \subseteq \mathbb{C}$ has the path independence property, then it is analytic on Ω.
- The Cauchy Integral Formulae in ($\star$) can be used to compute contour integrals whose integrands are of the form $f(z)/(z - z_0)^{n+1}$ since we have
$$\int_\Gamma \frac{f(z)}{(z - z_0)^{n+1}} \, \mathrm{d}z = \frac{2\pi i}{n!} f^{(n)}(z_0).$$
- (Cauchy's inequality) The Cauchy Integral Formulae can also be used to find local upper bounds for the derivatives of an analytic function up to any order.

Recall that the Cauchy-Goursat Theorem tells us that if a function is analytic on and inside a simply closed curve Γ, then

$$\int_\Gamma f(z) \, \mathrm{d}z = 0.$$

It turns out that the converse is true; this relies on the fact that the derivatives of an analytic function exist and are analytic up to any order.

THEOREM 21.1 (Morera's Theorem). Let f be a continuous complex-valued function on a domain $\Omega \subseteq \mathbb{C}$. If

$$\int_\Gamma f(z) \, \mathrm{d}z = 0$$

for every closed curve Γ in Ω, then f is analytic on Ω.

PROOF. First, note that if we have

$$\int_\Gamma f(z) \, \mathrm{d}z = 0$$

for every closed curve Γ in Ω, then f has an antiderivative F in Ω by independence of path. In particular, we have $F'(z) = f(z)$ for all $z \in \Omega$, so F is analytic on Ω. Since the derivatives of an analytic function exist and are analytic up to any order, we see that $f = F'$ is also analytic on Ω. $\square$

We mentioned that the Cauchy Integral Formulae can be used to compute contour integrals whose integrands are of the form $f(z)/(z - z_0)^{n+1}$, since we have

$$\int_\Gamma \frac{f(z)}{(z - z_0)^{n+1}} \, \mathrm{d}z = \frac{2\pi i}{n!} f^{(n)}(z_0)$$

as long as z_0 lies in the interior of Γ. We have already seen this in case that $n = 0$, so we will look at some examples where $n \geq 1$.

EXAMPLE 21.2. Consider the contour integral

$$\int_\Gamma \frac{5z^2+2z+1}{(z-i)^3}\,\mathrm{d}z$$

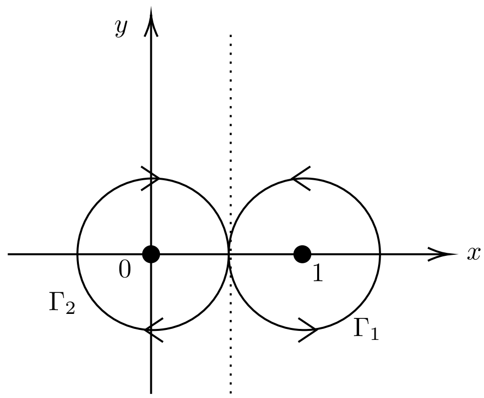
where Γ is the circle $|z|=2$ traversed once in the counter-clockwise direction. Then the integrand is of the form $f(z)/(z-z_0)^{n+1}$ where $f(z)=5z^2+2z+1$, $z_0=i$, and $n=2$. Moreover, f is entire and z_0 lies inside Γ. Therefore, by the Cauchy Integral Formulae, we have

$$\int_\Gamma \frac{5z^2+2z+1}{(z-i)^3}\,\mathrm{d}z = \frac{2\pi i}{2!}f''(i) = \frac{2\pi i}{2}(10) = 10\pi i.$$

EXAMPLE 21.3. Consider the contour integral

$$\int_\Gamma \frac{2z+1}{z(z-1)^2}\,\mathrm{d}z$$

where Γ is the sketched figure-eight curve below.



Note that integration along Γ is equivalent to integrating once around the positively oriented right loop Γ_1 and once around the negatively oriented left loop Γ_2. That is, we have

$$\int_\Gamma \frac{2z+1}{z(z-1)^2}\,\mathrm{d}z = \int_{\Gamma_1} \frac{(2z+1)/z}{(z-1)^2}\,\mathrm{d}z + \int_{\Gamma_2} \frac{(2z+1)/(z-1)^2}{z}\,\mathrm{d}z.$$

Moreover, the integrand $(2z+1)/[z(z-1)^2]$ is analytic everywhere except at $z=0$ and $z=1$, with $z=1$ in the interior of Γ_1 and $z=0$ in the interior of Γ_2.

We first focus on computing

$$\int_{\Gamma_1} \frac{(2z+1)/z}{(z-1)^2}\,\mathrm{d}z.$$

The integrand is of the form $f(z)/(z-z_0)^{n+1}$ where $f(z)=(2z+1)/z$, $z_0=1$, and $n=1$. Then f is analytic on a domain containing Γ_1 (such as $\Omega=\{z\in\mathbb{C}: x>0\}$) and z_0 lies inside Γ_1. Since Γ_1 is positively oriented, the Cauchy Integral Formulae implies that

$$\int_{\Gamma_1} \frac{(2z+1)/z}{(z-1)^2}\,\mathrm{d}z = \frac{2\pi i}{1!}f'(1) = 2\pi i\cdot(-1) = -2\pi i.$$

Now, we look at

$$\int_{\Gamma_2} \frac{(2z+1)/(z-1)^2}{z}\,\mathrm{d}z.$$

Observe that the integrand is of the form $f(z)/(z-z_0)^{n+1}$ where $f(z) = (2z+1)/(z-1)^2$, $z_0 = 0$, and $n = 0$. Note that f is analytic on a domain containing Γ_2 (such as $\Omega' = \{z \in \mathbb{C} : x < 1\}$ and z_0 lies in Γ_2. Since Γ_2 is negatively oriented, it follows from the Cauchy Integral Formulae that

$$\int_{\Gamma_2} \frac{(2z+1)/(z-1)^2}{z}\,\mathrm{d}z = -\frac{2\pi i}{0!} f(0) = -2\pi i \cdot 1 = -2\pi i.$$

Putting these together, we obtain

$$\int_{\Gamma} \frac{2z+1}{z(z-1)^2}\,\mathrm{d}z = -2\pi i - 2\pi i = -4\pi i.$$

We end our discussion on complex integration with a technical result that illustrates to what extent analyticity puts restrictions on the behaviour of a function.

Theorem 21.4 (Cauchy's inequality). Let f be analytic on the closed disc $\overline{D}(z_0;r)$, and let $M(z_0;r)$ be its maximum modulus on the boundary circle $C(z_0;r) = \{z \in \mathbb{C} : |z-z_0| = r\}$. (The maximum modulus $M(z_0;r)$ exists and is attained on $C(z_0;r)$ by the Maximum Modulus Principle.) Then we have

$$|f^{(n)}(z_0)| \leq \frac{n!M(z_0;r)}{r^n}.$$

Proof. By the Cauchy Integral Formulae, we have

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{C(z_0;r)} \frac{f(z)}{(z-z_0)^{n+1}}\,\mathrm{d}z.$$

Moreover, the circle $C(z_0;r)$ has arclength $2\pi r$ and

$$\left|\frac{f(z)}{(z-z_0)^{n+1}}\right| = \frac{|f(z)|}{|z-z_0|^{n+1}} = \frac{|f(z)|}{r^{n+1}} \leq \frac{M(z_0;r)}{r^{n+1}}$$

for all $z \in C(z_0;r)$. Hence, by the ML-inequality, we obtain

$$|f^{(n)}(z_0)| \leq \frac{n!}{2\pi} \cdot \frac{M(z_0;r)}{r^{n+1}} \cdot 2\pi r = \frac{n!M(z_0;r)}{r^n}. \qquad \square$$

Remark 21.5. One can use Cauchy's inequality to give an alternative proof of Liouville's Theorem. Indeed, let f be a bounded entire function. Then there exists $M > 0$ such that

$$|f(z)| \leq M$$

for all $z \in \mathbb{C}$. Then for any circle $C(z_0;r) = \{z \in \mathbb{C} : |z-z_0| = r\}$, we have

$$|f'(z_0)| \leq \frac{M(z_0;r)}{r} \leq \frac{M}{r}$$

by Cauchy's inequality with $n = 1$. Since $M/r \to 0$ as $r \to \infty$, it follows that $|f'(z_0)| = 0$, so $f'(z_0) = 0$ for all $z_0 \in \mathbb{C}$. Thus, f is constant.

22 Complex sequences and series

We now begin looking at sequences and series of complex numbers and functions. In this lecture, we will review the complex analogues of the main definitions and results of sequences and series of real numbers.

Definition 22.1. A **sequence** $\{z_n\}_{n=1}^{\infty}$ of complex numbers is a set of complex numbers indexed by the positive integers (or the natural numbers). A sequence $\{z_n\}_{n=1}^{\infty}$ is said to be **bounded** if there exists a real number $M > 0$ such that $|z_n| < M$ for all $n \in \mathbb{Z}^+$.

Example 22.2.

(1) The sequences $\{i^n\}_{n=0}^{\infty}$ and $\{i^n/n\}_{n=1}^{\infty}$ are both bounded.

(2) The sequence $\{1 + in\}_{n=0}^{\infty}$ is not bounded.

Definition 22.3. A sequence $\{z_n\}_{n=1}^{\infty}$ is said to **converge** to z if for every $\varepsilon > 0$, there exists an integer n_0 such for all $n > n_0$, we have $|z_n - z| < \varepsilon$, in which case we write $\lim_{n\to\infty} z_n = z$. If $\{z_n\}_{n=1}^{\infty}$ does not converge to any complex number, then the sequence is said to **diverge**.

Remark 22.4. It follows immediately from the definition that convergent sequences are bounded. Moreover, a sequence $\{z_n\}_{n=1}^{\infty}$ diverges if and only if for any complex number z, there exists $\varepsilon_0 > 0$ such that for all $N \in \mathbb{Z}^+$, we have $|z_n - z| \geq \varepsilon_0$ for some $n \geq N$.

Example 22.5.

(1) The sequence $\{i^n/n\}_{n=1}^{\infty}$ converges since for any $\varepsilon > 0$, we can choose $n_0 \in \mathbb{Z}^+$ such that $1/n_0 < \varepsilon$, in which case we have
$$|i^n/n| = 1/n < 1/n_0 < \varepsilon$$
for all $n > n_0$.

(2) The sequence $\{1 + in\}_{n=0}^{\infty}$ diverges since it is not bounded.

In the case where the sequence has a general term $z_n = x_n + iy_n$ where $x_n, y_n \in \mathbb{R}$ for all $n \in \mathbb{Z}^+$, one can determine whether it converges or diverges by looking at its real and imaginary parts.

Theorem 22.6. The sequence $\{z_n = x_n + iy_n\}_{n=1}^{\infty}$ converges to $z = x + iy$ if and only if $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$.

Proof. Exercise. □

Example 22.7.

(1) The sequence $\{1/n^2 + 2i/n^3\}_{n=1}^{\infty}$ converges to 0. Indeed, the general term of the sequence is $z_n = x_n + iy_n$ where $x_n = 1/n^2$ and $y_n = 2/n^3$, and we have $\lim_{n\to\infty} 1/n^2 = \lim_{n\to\infty} 2/n^3 = 0$.

(2) The sequence $\{(3 - n^2 i)/n\}_{n=1}^{\infty}$ diverges since the imaginary part of $(3 - n^2 i)/n$ is $-n$, and $\{-n\}_{n=1}^{\infty}$ diverges.

(3) The sequence $\{i^n\}_{n=0}^{\infty}$ diverges since $i^n = x_n + iy_n$ where we have $\{x_n\}_{n=0}^{\infty} = \{1, 0, -1, 0, 1, 0, \dots\}$ and $\{y_n\}_{n=0}^{\infty} = \{0, 1, 0, -1, 0, 1, \dots\}$, and both of these sequences are divergent.

Definition 22.8. An **infinite series** $\sum_{n=1}^{\infty} z_n$ of a complex sequence $\{z_n\}_{n=1}^{\infty}$ is said to **converge** to a sum S if and only if the **sequence of partial sums**
$$S_n := z_1 + \cdots + z_N$$
converges to S. Otherwise, we say that $\sum_{n=1}^{\infty} z_n$ is **divergent**.

Theorem 22.9. Let $\sum_{n=1}^{\infty} z_n$ be a complex series with $z_n = x_n + iy_n$.

(1) We have $\sum_{i=1}^{\infty} z_n = x + iy$ if and only if $\sum_{n=1}^{\infty} x_n = x$ and $\sum_{n=1}^{\infty} y_n = y$.

(2) (Divergence Test) If $\lim_{n\to\infty} z_n \neq 0$, then $\sum_{n=1}^{\infty} z_n$ diverges.

(3) (Absolute convergence) If $\sum_{n=1}^{\infty} |z_n|$ is convergent, then $\sum_{n=1}^{\infty} z_n$ is convergent.

PROOF. Exercise. □

EXAMPLE 22.10.

(1) Recall the p-series $\sum_{n=1}^{\infty} n^p$, which converges if and only if $p < -1$. Then by part (1) of Theorem 22.9, we see that
   - $\sum_{n=1}^{\infty}(1/n^2 - 2i/n^3)$ converges since $\sum_{n=1}^{\infty} 1/n^2$ and $\sum_{n=1}^{\infty} -2/n^3$ both converge, and
   - $\sum_{n=1}^{\infty}(1/n^2 - 2i/n)$ diverges since $\sum_{n=1}^{\infty} -2/n$ diverges.

(2) Observe that $\sum_{n=1}^{\infty}(n-i)/n$ diverges by the Divergence Test since

$$\frac{n-i}{n} = 1 - \frac{i}{n} \xrightarrow{n\to\infty} 1 \neq 0.$$

(3) (Complex geometric series) Fix $b \in \mathbb{C}$. We claim that

$$\sum_{n=0}^{\infty} b^n$$

converges if and only if $|b| < 1$. Indeed, recall the real geometric series $\sum_{n=0}^{\infty} a^n$ with $a \in \mathbb{R}$, which converges if and only if $|a| < 1$. Hence, $\sum_{n=0}^{\infty} |b|^n$ converges if and only if $|b| < 1$.
   - If $|b| < 1$, then $\sum_{n=0}^{\infty} |b^n| = \sum_{n=0}^{\infty} |b|^n$ converges by what we just noted, and hence $\sum_{n=0}^{\infty} b^n$ converges by absolute convergence.
   - If $|b| \geq 1$, then we see that $b^n \nrightarrow 0$ as $n \to \infty$ (since $|b|^n \nrightarrow 0$ as $n \to \infty$), so $\sum_{n=0}^{\infty} b^n$ diverges by the Divergence Test.

We give two more familiar tests for convergence.

THEOREM 22.11. Let $\sum_{n=1}^{\infty} z_n$ be a complex series.

(1) (Ratio Test) Suppose that $\lim_{n\to\infty} |z_{n+1}/z_n| = L$.
   - If $L < 1$, then $\sum_{n=1}^{\infty} z_n$ converges.
   - If $L > 1$, then $\sum_{n=1}^{\infty} z_n$ diverges.
   - If $L = 1$, then the test is inconclusive.

(2) (Comparison Test) Suppose there exists $n_0 \in \mathbb{Z}^+$ and a real sequence $\{r_n\}_{n=1}^{\infty}$ such that

$$|z_n| \leq r_n$$

for all $n \geq n_0$. If $\sum_{n=1}^{\infty} r_n$ converges, then $\sum_{n=1}^{\infty} z_n$ converges.

PROOF.

(1) First, suppose that $L < 1$. Then

$$\lim_{n\to\infty} \frac{|z_{n+1}|}{|z_n|} = L,$$

so by the real Ratio Test, we see that $\sum_{n=1}^{\infty} |z_n|$ is convergent. By absolute convergence, $\sum_{n=1}^{\infty} z_n$ is convergent.

Suppose now that $L > 1$. Then for sufficiently large $n_0 \in \mathbb{Z}^+$, we have $|z_{n+1}| > |z_n|$ for all $n \geq n_0$, which implies that $z_n \nrightarrow 0$ as $n \to \infty$. Hence, $\sum_{n=1}^{\infty} z_n$ diverges by the Divergence Test.

Finally, for $L = 1$, the test is inconclusive since the real Ratio Test is also inconclusive.

(2) By the real Comparison Test, since $|z_n| \leq r_n$ for all $n \geq n_0$ and $\sum_{n=1}^{\infty} r_n$ is convergent, it follows that $\sum_{n=1}^{\infty} |z_n|$ is convergent. The result follows from absolute convergence. $\square$

Remark 22.12. The complex Ratio Test is identical to the real version. However, the complex Comparison Test is more restrictive compared to its real counterpart. Indeed, recall that the real Comparison Test also states that given a real series $\sum_{n=1}^{\infty} a_n$, if there exists $n_0 \in \mathbb{Z}^+$ and a divergent real series $\sum_{n=1}^{\infty} r_n$ such that

$$a_n \geq r_n$$

for all $n \geq n_0$, then $\sum_{n=1}^{\infty} a_n$ also diverges. However, for a complex series $\sum_{n=1}^{\infty} z_n$ such that

$$|z_n| \geq r_n$$

for all $n \geq n_0$, we only know that $\sum_{n=1}^{\infty} |z_n|$ is divergent; it still may be the case that $\sum_{n=1}^{\infty} z_n$ converges. For instance, $\sum_{n=1}^{\infty} (-1)^n/n$ converges, whereas $\sum_{n=1}^{\infty} 1/n$ diverges.

We end our review of series by recalling the Cauchy Convergence Criterion, which states that a real series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if for every $\varepsilon > 0$, there is an index n_ε such that whenever $n > n_\varepsilon$ and p is any positive integer, then

$$|a_{n+1} + a_{n+2} + \cdots + a_{n+p}| < \varepsilon.$$

This has the following complex analogue.

Theorem 22.13 (Complex Cauchy Convergence Criterion). The complex series $\sum_{n=1}^{\infty} z_n$ is convergent if and only if given any $\varepsilon > 0$, there is an index n_ε such that whenever $n > n_\varepsilon$ and p is any positive integer, then

$$|z_{n+1} + z_{n+2} + \cdots + z_{n+p}| < \varepsilon.$$

Proof. Suppose that $z_n = x_n + iy_n$ for all $n \geq 1$. If $\sum_{n=1}^{\infty} z_n$ converges, then $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ also converge. By the real Cauchy Convergence Criterion, given any $\varepsilon > 0$, there is an index n_ε such that whenever $n > n_\varepsilon$ and p is any positive integer, then

$$\begin{aligned} |x_{n+1} + x_{n+2} + \cdots + x_{n+p}| &< \varepsilon/2, \\ |y_{n+1} + y_{n+2} + \cdots + y_{n+p}| &< \varepsilon/2. \end{aligned}$$

It follows that

$$|z_{n+1} + z_{n+2} + \cdots + z_{n+p}| \leq |x_{n+1} + x_{n+2} + \cdots + x_{n+p}| + |y_{n+1} + y_{n+2} + \cdots + y_{n+p}| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Conversely, suppose that given any $\varepsilon > 0$, we can find an index n_ε such that whenever $n > n_\varepsilon$ and p is any positive integer, then

$$|z_{n+1} + z_{n+2} + \cdots + z_{n+p}| < \varepsilon.$$

Observe that

$$\begin{aligned} |x_{n+1} + x_{n+2} + \cdots + x_{n+p}| &\leq |z_{n+1} + z_{n+2} + \cdots + z_{n+p}| < \varepsilon, \\ |y_{n+1} + y_{n+2} + \cdots + y_{n+p}| &\leq |z_{n+1} + z_{n+2} + \cdots + z_{n+p}| < \varepsilon, \end{aligned}$$

so $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ converge by the real Cauchy Convergence Criterion. Thus, $\sum_{n=1}^{\infty} z_n$ converges. $\square$

While one does not usually use the Cauchy Convergence Criterion to determine whether a given series converges, it is still a useful characterization of convergence. We will use it when we study the convergence properties of power series.

23 Power series

Definition 23.1. Fix $z_0 \in \mathbb{C}$. A **complex power series about** z_0 is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

where $z \in \mathbb{C}$ and $a_n \in \mathbb{C}$ for all $n \in \mathbb{N}$.

Example 23.2.

(1) The geometric series $\sum_{n=1}^{\infty}(z-z_0)^n$ is a power series.

(2) The infinite series $\sum_{n=1}^{\infty} 4^{-n}(z-z_0)^n$ is a power series.

Remark 23.3.

(1) We can think of power series as "infinite polynomials".

(2) At $z = z_0$, we have $\sum_{n=0}^{\infty} a_n(z-z_0)^n = a_n$. As a result, we call z_0 the **center** of the power series.

For fixed $z \in \mathbb{C}$, observe that $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is just an infinite series $\sum_{n=0}^{\infty} b_n$ where $b_n = a_n(z-z_0)^n$. Now, consider the limit

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|.$$

Let us first assume that this limit exists and is equal to L. Then

$$\left|\frac{b_{n+1}}{b_n}\right| = \left|\frac{a_{n+1}}{a_n}\right| \cdot |z-z_0| \xrightarrow{n\to\infty} L \cdot |z-z_0|.$$

By the Ratio Test, the series $\sum_{n=0}^{\infty} b_n$ converges if $L \cdot |z-z_0| < 1$, and diverges if $L \cdot |z-z_0| > 1$. We now consider two cases.

(i) If $L = 0$, then clearly $L \cdot |z-z_0| < 1$ for all $z \in \mathbb{C}$, so $\sum_{n=0}^{\infty} b_n$ converges for all $z \in \mathbb{C}$.

(ii) If $L > 0$, then $L \cdot |z-z_0| < 1$ if and only if $|z-z_0| < 1/L$, and $L \cdot |z-z_0| > 1$ if and only if $|z-z_0| > 1/L$. In other words, if we set $R := 1/L$, then $\sum_{n=0}^{\infty} b_n$ converges if $z \in D(z_0; R)$, and diverges if $z \in \mathbb{C} \setminus \overline{D}(z_0; R)$.

On the other hand, if $\lim_{n\to\infty} |a_{n+1}/a_n| = \infty$, then

$$\lim_{n\to\infty} \left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \cdot |z-z_0| = \infty$$

unless $z = z_0$. Therefore, $\sum_{n=0}^{\infty} b_n$ only converges when $z = z_0$ in this case. We summarize our results in the following proposition.

Proposition 23.4. Let $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a power series and consider the limit

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|.$$

(i) If the limit is ∞, then the power series only converges at $z = z_0$.

(ii) If the limit exists and is equal to 0, then the power series converges for all $z \in \mathbb{C}$.

(iii) If the limit exists and is equal to $L > 0$, then the power series converges for all $z \in D(z_0; R)$ and diverges for all $z \in \mathbb{C} \setminus \overline{D}(z_0; R)$, where $R = 1/L$.

In case (iii), we call R the **radius of convergence** of the power series, and the disc $D(z_0; R)$ is called its **disc of convergence**.

EXAMPLE 23.5.

(1) The radius of convergence of the geometric series $\sum_{n=0}^{\infty}(z-z_0)^n$ is $R=1$, and its disc of convergence is $D(z_0;1)$.

(2) Consider the power series $\sum_{n=0}^{\infty} 4^{-n}(z-z_0)^n$. In this case, we have $a_n = 4^{-n}$ and

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{4}.$$

The power series then has radius of convergence $R=4$ and disc of convergence $D(z_0;4)$. Moreover, the power series diverges for all $z \in \mathbb{C} \setminus \overline{D}(z_0;4)$. Finally, when $|z-z_0|=4$, we have

$$|4^{-n}(z-z_0)^n| = \left(\frac{|z-z_0|}{4}\right)^n = 1.$$

Therefore, $4^{-n}(z-z_0)^n \nrightarrow 0$ as $n\to\infty$, which implies that $\sum_{n=0}^{\infty} 4^{-n}(z-z_0)^n$ diverges by the Divergence Test. Thus, the power series only converges on $D(z_0;4)$.

(3) Consider the power series $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(z-z_0)^n$. Then $a_n = (-1)^n/((2n)!)$, so we obtain

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{1}{2n+1} = 0.$$

It follows that the power series converges for all $z\in\mathbb{C}$. In other words, we can say that it has radius of convergence $R=\infty$ and its disc of convergence is $\mathbb{C}$.

Observe that at points $z\in\mathbb{C}$ where a power series converges, it defines a function. In fact, it defines a continuous function on its disc of convergence. Before we prove this, we require the notion of uniform convergence.

DEFINITION 23.6. Let $\{f_n\}_{n=0}^{\infty}$ be a sequence of complex-valued functions, each defined on a domain $\Omega\subseteq\mathbb{C}$. We say that $\{f_n\}_{n=0}^{\infty}$ **converges to a function** f **on** Ω if $\lim_{n\to\infty} f_n(z)$ exists for all $z\in\Omega$, in which case we set $f(z) = \lim_{n\to\infty} f_n(z)$.

EXAMPLE 23.7. For any $n\geq 0$, let $f_n(z) = 1+z+\cdots+z^n$. Then we have

$$\lim_{n\to\infty} f_n(z) = \frac{1}{1-z}$$

for all $z\in D(0;1)$. If we set $\Omega = D(0;1)$, we see that $\{f_n\}_{n=0}^{\infty}$ is a sequence of complex-valued functions on Ω that converge to the function $f(z)=1/(1-z)$ on Ω.

Consider now a sequence of complex-valued functions $\{f_n\}_{n=0}^{\infty}$, each defined on a domain $\Omega\subseteq\mathbb{C}$, that converges to the function f on Ω. By definition, we have $f(z)=\lim_{n\to\infty} f_n(z)$ for all $z\in\Omega$. In particular, for all $\varepsilon>0$, there exists $N_{z,\varepsilon}\in\mathbb{N}$ such that $|f_n(z)-f(z)|<\varepsilon$ for all $n\geq N_{z,\varepsilon}$. Note that the choice of $N_{z,\varepsilon}$ may depend on z.

The previous example is one such case; we leave it as an exercise to check this. The reason that this happens is because the limit function $f(z)=1/(1-z)$ is undefined at $z=1$ with $\lim_{z\to1} 1/(1-z)=\infty$, whereas $f(z)$ is defined and continuous at every other point on the closed disc $\overline{D}(0;1)$.

We are interested in convergent sequences of functions for which one can choose $N_{z,\varepsilon}$ which depends only on ε. In this case, some of the properties of the functions in the sequence are preserved by the limit function.

DEFINITION 23.8. Let $\{f_n\}_{n=0}^{\infty}$ be a sequence of complex-valued functions, each defined on a domain $\Omega\subseteq\mathbb{C}$. Then $\{f_n\}_{n=0}^{\infty}$ is said to **converge uniformly** to the function f on a subset $S\subseteq\Omega$ if for every $\varepsilon>0$, there exists $N_\varepsilon\in\mathbb{N}$ such that $|f_n(z)-f(z)|<\varepsilon$ for all $n\geq N_\varepsilon$ and $z\in S$. In other words, N_ε depends only on ε. The function f is called the **uniform limit** of the sequence.

Theorem 23.9.

(1) The uniform limit of continuous functions is continuous.

(2) Let $\{f_n\}_{n=0}^{\infty}$ be a sequence of complex-valued functions that are continuous on a curve $\Gamma \subseteq \mathbb{C}$, and suppose that $\{f_n\}_{n=0}^{\infty}$ converges uniformly to f on Γ. Then

$$\lim_{n\to\infty} \int_\Gamma f_n(z)\,\mathrm{d}z = \int_\Gamma \lim_{n\to\infty} f_n(z)\,\mathrm{d}z = \int_\Gamma f(z)\,\mathrm{d}z.$$

Proof. The proof of (1) is identical to the real case and is left as an exercise. For (2), assume that $\{f_n\}_{n=0}^{\infty}$ converges uniformly to f on Γ. It suffices to show that

$$\lim_{n\to\infty} \left| \int_\Gamma f_n(z)\,\mathrm{d}z - \int_\Gamma f(z)\,\mathrm{d}z \right| = 0.. \qquad (\star)$$

Indeed, let L be the arclength of Γ. For every $\varepsilon > 0$, it follows from uniform convergence that there exists $N_\varepsilon \in \mathbb{N}$ such that

$$|f_n(z) - f(z)| < \varepsilon/L$$

for all $n \geq N_\varepsilon$ and $z \in \Gamma$. By the ML-inequality, we obtain

$$\left| \int_\Gamma f_n(z)\,\mathrm{d}z - \int_\Gamma f(z)\,\mathrm{d}z \right| = \left| \int_\Gamma (f_n(z) - f(z))\,\mathrm{d}z \right| < \frac{\varepsilon}{L} \cdot L = \varepsilon$$

for all $n \geq N_\varepsilon$. This implies that $(\star)$ holds, so we are done. $\square$

We are now ready to prove that every power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ defines a continuous function f on its disc of convergence $D(z_0;R)$. In fact, we will see that this function f is analytic on $D(z_0;R)$. For now, we will focus on continuity.

Theorem 23.10. Let $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a power series, which defines a function $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ on its disc of convergence $D(z_0;R)$. On each smaller closed disc $\overline{D}(z_0;r)$ with $r < R$, the sequence of partial sums $\{s_n\}$ converges uniformly to f. Consequently, f is continuous on $D(z_0;R)$.

Proof. Let $r < R$ and choose $z_1 \in \mathbb{C}$ such that $r < |z_1 - z_0| < R$. Since the power series converges at z_1, we have by the Divergence Test that

$$a_n(z_1-z_0)^n \xrightarrow{n\to\infty} 0,$$

which implies that the terms of the sequence $\{a_n(z_1-z_0)^n\}_{n=0}^{\infty}$ are bounded by some $M > 0$; that is,

$$|a_n(z_1-z_0)^n| < M$$

for all $n \geq 0$. Now, pick $z \in \overline{D}(z_0;r)$. We then have

$$|a_n(z-z_0)^n| = |a_n(z_1-z_0)^n| \cdot \left| \frac{z-z_0}{z_1-z_0} \right|^n < M \cdot \left(\frac{r}{|z_1-z_0|} \right)^n = M\rho^n,$$

where $\rho = r/|z_1-z_0|$. Note that $r < |z_1 - z_0|$, so $\rho < 1$. Therefore, for any $p \in \mathbb{Z}^+$, we have

$$\begin{aligned} |a_{n+1}(z-z_0)^{n+1} + \cdots + a_{n+p}(z-z_0)^{n+p}| &\leq |a_{n+1}(z-z_0)^{n+1}| + \cdots + |a_{n+p}(z-z_0)^{n+p}| \\ &< M\rho^{n+1} + \cdots + M\rho^{n+p} \\ &= M(\rho^{n+1} + \cdots + \rho^{n+p}). \end{aligned}$$

Since $\rho < 1$, the series $\sum_{n=0}^{\infty} \rho^n$ converges. By the Cauchy Convergence Criterion, for every $\varepsilon > 0$, there exists an index n_ε such that

$$\rho^{n+1} + \cdots + \rho^{n+p} < \varepsilon/M$$

for all $n > n_\varepsilon$ and $p \in \mathbb{Z}^+$. Hence, we see that

$$|a_{n+1}(z - z_0)^{n+1} + \cdots + a_{n+p}(z - z_0)^{n+p}| < M \cdot \frac{\varepsilon}{M} = \varepsilon$$

for all $n > n_\varepsilon$ and $p \in \mathbb{Z}^+$. By the Cauchy Convergence Criterion, the sequence of partial sums $\{s_n\}$ converges. Moreover, since the choice of n_ε depends on ρ, which is independent of z, this convergence is uniform. Each partial sum $s_n(z)$ is a complex polynomial in z and therefore continuous at every $z \in \mathbb{C}$. Since $\{s_n\}$ converges uniformly to f on $\overline{D}(z_0; r)$, it follows that f is continuous on $\overline{D}(z_0; r)$ for all $r < R$. Since every point $z \in D(z_0; R)$ is contained in a closed disc $\overline{D}(z_0; r)$ for some $r < R$, we have that f is continuous on all of $D(z_0; R)$. $\square$

24 More properties of power series

In this lecture, we will prove the following familiar facts about complex power series.

- Power series define analytic functions on their discs of convergence.
- One can differentiate and integrate power series term-by-term on their discs of convergence.
- Any analytic function admits a power series representation, which is called its **Taylor series**.

First, we require a few more facts about uniform limits of sequences of functions.

Previously, we saw that the uniform limit of a sequence of continuous complex-valued functions is continuous. A similar statement holds for sequences of analytic functions.

THEOREM 24.1. Let $\{f_n\}$ be a sequence of analytic functions on a domain $\Omega \subseteq \mathbb{C}$. Let $f(z) = \lim_{n\to\infty} f_n(z)$ uniformly on Ω. Then f is also analytic on Ω and $\{f_n'\}$ converges uniformly to f' on Ω.

PROOF. Since analytic functions are continuous, we know that f is the uniform limit of a sequence of continuous functions and hence is also continuous on Ω. We now show that f is analytic on Ω. Let $z_0 \in \Omega$. We need to show that f is differentiable in a neighbourhood of z_0. Since Ω is open, there exists $r > 0$ such that $D(z_0; r) \subseteq \Omega$. Note that $\{f_n\}$ converges uniformly to f to $\overline{D}(z_0; r)$. Let $C(r)$ be the boundary of $\overline{D}(z_0; r)$. Then for each $z \in D(z_0; r)$, we have

$$f_n(z) = \frac{1}{2\pi i}\int_{C(r)} \frac{f_n(w)}{w-z}\,\mathrm{d}w \tag{$\star$}$$

by the Cauchy Integral Formula. Then by uniform convergence, we obtain

$$\lim_{n\to\infty}\int_{C(r)} \frac{f_n(w)}{w-z}\,\mathrm{d}w = \int_{C(r)} \frac{f(w)}{w-z}\,\mathrm{d}w.$$

Since $\lim_{n\to\infty} f_n(z) = f(z)$, taking the limit as $n \to \infty$ on both sides of $(\star)$ gives

$$f(z) = \frac{1}{2\pi i}\int_{C(r)} \frac{f(w)}{w-z}\,\mathrm{d}w$$

for all $z \in D(z_0; r)$. By the Cauchy Integral Formulae for higher derivatives, it follows that $f'(z)$ exists and is equal to

$$f'(z) = \frac{1}{2\pi i}\int_{C(r)} \frac{f(w)}{(w-z)^2}\,\mathrm{d}w$$

for all $z \in D(z_0; r)$. Moreover, as

$$f_n'(z) = \frac{1}{2\pi i}\int_{C(r)} \frac{f_n(w)}{(w-z)^2}\,\mathrm{d}w$$

for all $z \in D(z_0; r)$ by the Cauchy Integral Formula, taking the limit as $n \to \infty$ on both sides yields $\lim_{n\to\infty} f_n'(z) = f'(z)$ for all $z \in D(z_0; r)$. We leave it as an exercise to show that this convergence is uniform. $\square$

As a consequence of this theorem, we obtain the important result that power series define analytic functions on their discs of convergence.

COROLLARY 24.2. A power series defines an analytic function on its disc of convergence.

PROOF. Apply Theorem 24.1 to the sequence of partial sums of the power series. $\square$

EXAMPLE 24.3.

(1) The geometric series $\sum_{n=0}^{\infty}(z-z_0)^n$ has disc of convergence $D(z_0;1)$. Therefore, it defines an analytic function on $D(z_0;1)$. In fact, we know that this series converges to $f(z)=1/(1-(z-z_0))$ on $D(z_0;1)$.

(2) We have seen that $\sum_{n=0}^{\infty}4^{-n}(z-z_0)^n$ converges only on $D(z_0;4)$, so it defines an analytic function on $D(z_0;4)$.

(3) We saw that $\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(z-z_0)^n$ converges for all $z\in\mathbb{C}$, so it defines an entire function.

PROPOSITION 24.4 (Term-by-term differentiation and integration). Let $\{f_n\}$ be a sequence of analytic functions on a domain $\Omega\subseteq\mathbb{C}$ and suppose that the sequence of partial sums $s_m(z):=\sum_{n=0}^{m}f_n(z)$ converges uniformly to $f(z)$ on the subset $S\subseteq\Omega$ so that

$$f(z)=\sum_{n=0}^{\infty}f_n(z)$$

for all $z\in S$. Then

(1) $f'(z)=\sum_{n=0}^{\infty}f_n'(z)$, and

(2) $\int_\Gamma f(z)\,\mathrm{d}z=\sum_{n=0}^{\infty}\int_\Gamma f_n(z)\,\mathrm{d}z$ for any curve Γ in S.

In other words, the infinite series $\sum_{n=0}^{\infty}f_n(z)$ can be differentiated and integrated term-by-term in S.

PROOF. We first note that $s_m'(z)=\sum_{n=0}^{m}f_n'(z)$ for all $z\in S$. Hence, by Theorem 24.1, since $s_m(z):=\sum_{n=0}^{m}f_n(z)$ converges uniformly to $f(z)$ on S, we have that $\{s_m'\}$ converges uniformly to f', proving (1).

Now, let Γ be a curve in S. Then we have

$$\int_\Gamma s_m(z)\,\mathrm{d}z=\sum_{n=0}^{m}\int_\Gamma f_n(z)\,\mathrm{d}z$$

for all $m\geq 0$. Moreover, since $s_m(z):=\sum_{n=0}^{m}f_n(z)$ converges uniformly to $f(z)$ on S, we have

$$\lim_{m\to\infty}\int_\Gamma s_m(z)\,\mathrm{d}z=\int_\Gamma f(z)\,\mathrm{d}z.$$

It follows that

$$\int_\Gamma f(z)\,\mathrm{d}z=\lim_{m\to\infty}\int_\Gamma s_m(z)\,\mathrm{d}z=\lim_{m\to\infty}\sum_{n=0}^{m}\int_\Gamma f_n(z)\,\mathrm{d}z=\sum_{n=0}^{\infty}\int_\Gamma f_n(z)\,\mathrm{d}z,$$

which proves (2). □

Due to this proposition, we see that power series can be differentiated and integrated term-by-term on their discs of convergence.

THEOREM 24.5. Suppose that $f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n$ has disc of convergence $D(z_0;r)$. Then

$$f^{(k)}(z)=\sum_{n=k}^{\infty}\frac{n!}{(n-k)!}a_n(z-z_0)^{n-k}$$

for all $z\in D(z_0;r)$ and $k\geq 0$. In particular, we have

$$a_n=\frac{f^{(n)}(z_0)}{n!}$$

for all $n\geq 0$. Moreover, for any curve Γ in $D(z_0;r)$, we have

$$\int_\Gamma f(z)\,\mathrm{d}z=\sum_{n=0}^{\infty}a_n\int_\Gamma(z-z_0)^n\,\mathrm{d}z.$$

PROOF. We leave the details as an exercise. □

EXAMPLE 24.6.

(1) The geometric series $\sum_{n=0}^{\infty}(z-z_0)^n$ converges to $f(z) = 1/(1-(z-z_0))$ on $D(z_0;1)$. Thus, we have

$$f'(z) = \frac{1}{(1-(z-z_0))^2} = \sum_{n=1}^{\infty} n(z-z_0)^{n-1}$$

on $D(z_0;1)$. Moreover, we have $f^{(n)}(z_0) = n!$ for all $n \geq 0$.

(2) Since $f(z) = \sum_{n=0}^{\infty} 4^{-n}(z-z_0)^n$ is analytic on $D(z_0;4)$, we have that

$$f'(z) = \sum_{n=1}^{\infty} \frac{n}{4^n}(z-z_0)^{n-1}$$

on $D(z_0;4)$, and $f^{(n)}(z_0) = n!/4^n$ for all $n \geq 0$.

(3) Since $f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(z-z_0)^n$ is entire, we have

$$f'(z) = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!}(z-z_0)^n$$

on $\mathbb{C}$, and $f^{(n)}(z_0) = (-1)^n n!/((2n)!)$ for all $n \geq 0$.

We have now seen that power series define analytic functions on their discs of convergence whose derivatives at the center are equal to the coefficients of the power series. The converse is also true, and is known as Taylor's Theorem.

THEOREM 24.7 (Taylor's Theorem). Suppose that f is analytic on the disc $D(z_0;r)$. Then f has a power series representation

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$$

where the coefficients are given by

$$a_n = \frac{f^{(n)}(z_0)}{n!}$$

for all $n \geq 0$. This is the expansion of f into a **Taylor series** about the point z_0. When $z_0 = 0$, we also call

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$$

the **Maclaurin series** expansion of f.

PROOF. First, assume that $z_0 = 0$. We will show that

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$$

for all $z \in D(0;r)$ by applying the Cauchy Integral Formulae. Recall that for any complex number $a \neq 1$, we have

$$\frac{1-a^{m+1}}{1-a} = 1 + a + a^2 + \cdots + a^m.$$

Hence, for any $z \in D(0; r)$, we see that

$$\begin{aligned}\frac{1}{s-z} = \frac{1}{s(1-z/s)} &= \frac{1-(z/s)^{m+1}}{s(1-z/s)} + \frac{(z/s)^{m+1}}{s(1-z/s)} \\ &= \frac{1}{s(1+z/s+(z/s)^2+\cdots+(z/s)^m)} + \frac{(z/s)^{m+1}}{s(1-z/s)}.\end{aligned}$$

Expanding and simplifying, we obtain

$$\frac{1}{s-z} = \frac{1}{s} + \frac{z}{s^2} + \frac{z^2}{s^3} + \cdots + \frac{z^m}{s^{m+1}} + \frac{z^{m+1}}{(s-z)s^{m+1}}.$$

In particular, multiplying by $f(s)$ gives

$$\frac{f(s)}{s-z} = \sum_{n=0}^{m} \frac{z^n f(s)}{s^{n-1}} + \frac{z^{m+1} f(s)}{(s-z)s^{m+1}}. \tag{1}$$

Choose $|z| < r_0 < r$ and let $C_0 = \{w \in \mathbb{C} : |w| = r_0\}$. Then z is in the interior of C_0 and

$$f(z) = \frac{1}{2\pi i} \int_{C_0} \frac{f(s)}{s-z} \, \mathrm{d}s$$

by the Cauchy Integral Formula. Replacing the integrand by the expression in (1), we have

$$\begin{aligned}f(z) &= \sum_{n=0}^{m} \frac{1}{2\pi i} \int_{C_0} \frac{z^n f(s)}{s^{n+1}} \, \mathrm{d}s + \frac{1}{2\pi i} \int_{C_0} \frac{z^{m+1} f(s)}{(s-z)s^{m+1}} \, \mathrm{d}s \\ &= \sum_{n=0}^{m} \frac{z^n}{n!} \cdot \frac{n!}{2\pi i} \int_{C_0} \frac{f(s)}{(s-0)^{n+1}} \, \mathrm{d}s + \frac{1}{2\pi i} \int_{C_0} \frac{f(s)}{s-z} \cdot \left(\frac{z}{s}\right)^{m+1} \mathrm{d}s.\end{aligned}$$

Since 0 is in the interior of C_0, it follows from the Cauchy Integral Formulae that

$$\frac{n!}{2\pi i} \int_{C_0} \frac{f(s)}{(s-0)^{n+1}} \, \mathrm{d}s = f^{(n)}(0).$$

Now, setting

$$R_m(z) := \frac{1}{2\pi i} \int_{C_0} \frac{f(s)}{s-z} \cdot \left(\frac{z}{s}\right)^{m+1} \mathrm{d}s,$$

we obtain

$$f(z) = \sum_{n=0}^{m} \frac{f^{(n)}(0)}{n!} z^n + R_m(z). \tag{2}$$

It only remains to check that $\lim_{m\to\infty} R_m(z) = 0$. Note that C_0 is compact and $f(s)/(s-z)$ is continuous on C_0, so there exists $M > 0$ such that

$$\left| \frac{f(s)}{s-z} \right| < M$$

for all $s \in C_0$ by the Extreme Value Theorem. Moreover, we have $|s| = r_0$ for all $s \in C_0$, which implies that $|z/s| < 1$ and $\lim_{m\to\infty} (z/s)^m = 0$. Hence, for any $\varepsilon > 0$, there exists m_ε such that

$$\left| \frac{z}{s} \right| < \frac{2\pi i \varepsilon}{ML}$$

for all $m \geq m_\varepsilon$, where L is the circumference of C_0. This implies that

$$\left| \frac{f(s)}{s-z} \cdot \left(\frac{z}{s}\right)^m \right| < \frac{2\pi i \varepsilon}{L},$$

so by the ML-inequality, we obtain

$$|R_m(z)| \leq \frac{1}{2\pi i} \cdot \frac{2\pi i\varepsilon}{L} \cdot L = \varepsilon.$$

Thus, $\lim_{m\to\infty} R_m(z) = 0$, and taking the limit as $m \to \infty$ on both sides of (2) gives

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n.$$

For the case where $z_0 \neq 0$, set $w = z - z_0$. Then $w = 0$ whenever $z = z_0$, and note that $g(w) := f(w + z_0)$ is analytic at $w = 0$ since f is analytic at z_0. Therefore, we obtain

$$f(z) = g(w) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} w^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

where we have $f^{(n)}(z_0) = g^{(n)}(0)$ due to the Chain Rule. □

Remark 24.8. In the proof of Taylor's Theorem, we saw that for any $m \geq 0$, we have $f(z) = P_m(z) + R_m(z)$ where

$$P_m(z) := \sum_{n=0}^{m} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

is the Taylor polynomial of degree m, and

$$R_m(z) := \frac{1}{2\pi i} \int_{C_0} \frac{f(s)}{s - (z - z_0)} \cdot \left(\frac{z}{s}\right)^{m+1} \mathrm{d}s$$

is the remainder of degree m for all $z \in D(z_0; r)$. As in the real case, the Taylor polynomial can be used to approximate the function f at points in $D(z_0; r)$.

We list some important Maclaurin series below. These can be obtained similarly to the real case by taking the derivatives of the given functions and using the formula of the Maclaurin series expansion.

- $\displaystyle \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ for all $|z| < 1$.
- $\displaystyle e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ for all $z \in \mathbb{C}$.
- $\displaystyle \cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$ for all $z \in \mathbb{C}$.
- $\displaystyle \sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$ for all $z \in \mathbb{C}$.
- $\displaystyle \mathrm{Log}(1-z) = \sum_{n=0}^{\infty} -\frac{z^{n+1}}{n+1}$ for all $|z| < 1$.

Remark 24.9. An important consequence of Taylor's Theorem is that the power series representation of an analytic function is unique. Indeed, we previously saw that a power series $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ is analytic on its disc of convergence with

$$a_n = \frac{f^{(n)}(z_0)}{n!}$$

for all $n \geq 0$. In other words, the power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ is actually the Taylor series of f on its disc of convergence $D(z_0; r)$.

Therefore, it does not matter how one manipulates an existing power series to obtain a power series of a given analytic function; the result will always correspond to the Taylor series.

25 Computing Taylor series and the Identity Theorem

In this lecture, we finish our discussion of Taylor series with the following topics.

- We will consider different ways of finding the Taylor series of a function and compute several examples.
- Recall that given a subset $S \subseteq \mathbb{C}$, a point $z_0 \in S$ is said to be an **isolated point** of S if there exists $r > 0$ such that $D(z_0; r) \cap S = \{z_0\}$. We will use Taylor's Theorem to prove that the zeroes of a non-constant analytic function are all isolated.
- Finally, we will prove the Identity Theorem, which states that if two analytic functions are equal on an open set, then they are equal everywhere. This is an extremely important result in complex analysis that is specific to analytic functions.

One can always use the definition to find the Taylor series of a given analytic function. However, it is often easier to algebraically manipulate a known Taylor series, or differentiate/integrate a known Taylor series term-by-term. We will work with the Maclaurin series given at the end of the previous lecture.

Example 25.1. We show that the Maclaurin series of $\operatorname{Log}(1-z)$ can be obtained by integrating the Maclaurin series of $1/(1-z)$ term-by-term. Indeed, note that

$$\frac{\mathrm{d}}{\mathrm{d}z}(\operatorname{Log}(1-z)) = -\frac{1}{1-z},$$

so if Γ is the line segment in $\mathbb{C} \setminus \{1\}$ joining 0 and z, we have that

$$\operatorname{Log}(1-z) = \operatorname{Log}(1-z) - \operatorname{Log}(1-0) = \int_\Gamma -\frac{1}{1-w}\,\mathrm{d}w.$$

Since the Maclaurin series of $1/(1-w)$ is $\sum_{n=0}^\infty w^n$ for $|w| < 1$, we obtain

$$\operatorname{Log}(1-z) = \int_\Gamma -\left(\sum_{n=0}^\infty w^n\right)\mathrm{d}w = -\sum_{n=0}^\infty \int_\Gamma w^n\,\mathrm{d}w = -\sum_{n=0}^\infty \left(\frac{z^{n+1}}{n+1} - 0\right) = \sum_{n=0}^\infty -\frac{z^{n+1}}{n+1},$$

and it converges on the same disc of convergence as the Maclaurin series of $1/(1-z)$, namely $|z| < 1$.

Example 25.2. We can find the Maclaurin series of $1/(1-z)^2$ by differentiating the Maclaurin series of $1/(1-z)$ term-by-term. Observe that

$$\frac{1}{(1-z)^2} = \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{1}{1-z}\right) = \frac{\mathrm{d}}{\mathrm{d}z}\left(\sum_{n=0}^\infty z^n\right) = \sum_{n=0}^\infty \frac{\mathrm{d}}{\mathrm{d}z}(z^n) = \sum_{n=1}^\infty nz^{n-1},$$

which converges for $|z| < 1$.

We now give some examples of finding Maclaurin and Taylor series of analytic functions by algebraically manipulating known power series.

Example 25.3. We find the Maclaurin series of $f(z) = e^{3z^2}$. Note that $f(z) = e^w$ where $w = 3z^2$, and whenever $z = 0$, we also have $w = 0$. Therefore, we can start by finding the Maclaurin series of e^w and substituting $w = 3z^2$. Recall that

$$e^w = \sum_{n=0}^\infty \frac{w^n}{n!}$$

for all $w \in \mathbb{C}$. Then, we obtain

$$e^{3z^2} = \sum_{n=0}^\infty \frac{(3z^2)^n}{n!} = \sum_{n=0}^\infty \frac{3^n}{n!} z^{2n}.$$

Moreover, since the Maclaurin series of e^w converges for all $w \in \mathbb{C}$, it follows from the uniqueness of Taylor series that this is the Maclaurin series of e^{3z^2} for all $z \in \mathbb{C}$.

EXAMPLE 25.4. We find the Taylor series of $\operatorname{Log} z$ about $z = 2$. Recall that the Maclaurin series of $\operatorname{Log}(1 - w)$ is given by

$$\operatorname{Log}(1-w) = \sum_{n=0}^{\infty} -\frac{w^{n+1}}{n+1}$$

where $|w| < 1$. We want a power series centered at $z = 2$, which means that we want powers of $z - 2$. If we can write $\operatorname{Log} z$ as $\operatorname{Log}(1 - w)$ with w being an expression in terms of $z - 2$, then we only have to substitute that expression in the Maclaurin series of $\operatorname{Log}(1 - w)$ to obtain the desired power series. Indeed, we have

$$\operatorname{Log} z = \operatorname{Log}(2 - (-(z-2))) = \operatorname{Log}\left(2\left[1 - \left(-\frac{z-2}{2}\right)\right]\right) = \operatorname{Log} 2 + \operatorname{Log}\left(1 - \left(-\frac{z-2}{2}\right)\right).$$

Setting $w = -(z - 2)/2$ in the Maclaurin series of $\operatorname{Log}(1 - w)$, we obtain

$$\begin{aligned}
\operatorname{Log} z &= \operatorname{Log} 2 + \sum_{n=0}^{\infty} -\frac{(-(z-2)/2)^{n+1}}{n+1} \\
&= \operatorname{Log} 2 + \sum_{n=0}^{\infty} \frac{(-1)^n (z-2)^{n+1}}{2^{n+1}(n+1)} \\
&= \operatorname{Log} 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(z-2)^n}{2^n n}.
\end{aligned}$$

Moreover, the Maclaurin series converges for $|w| < 1$, so the above power series for $\operatorname{Log} z$ converges if

$$\left| -\frac{z-2}{2} \right| < 1,$$

which occurs if and only if $|z - 2| < 2$. By the uniqueness of Taylor series, it follows that

$$\operatorname{Log} z = \operatorname{Log} 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(z-2)^n}{2^n n}$$

with $|z - 2| < 2$ is the Taylor series of $\operatorname{Log} z$ about $z = 2$.

EXAMPLE 25.5. We now find the Maclaurin series of $\sinh z$. Recall that $\sinh z = -i \sin(iz)$ for all $z \in \mathbb{C}$. Moreover, $\sin w$ has Maclaurin series expansion

$$\sin w = \sum_{n=0}^{\infty} \frac{(-1)^n w^{2n+1}}{(2n+1)!}$$

for all $w \in \mathbb{C}$. Setting $w = iz$, we obtain

$$\begin{aligned}
\sinh z &= -i \sum_{n=0}^{\infty} \frac{(-1)^n (iz)^{2n+1}}{(2n+1)!} \\
&= -i \sum_{n=0}^{\infty} \frac{(-1)^n i^{2n+1} z^{2n+1}}{(2n+1)!} \\
&= -i \sum_{n=0}^{\infty} \frac{(-1)^n (-1)^n i z^{2n+1}}{(2n+1)!} \\
&= \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}
\end{aligned}$$

for all $z \in \mathbb{C}$, which is the Maclaurin series of $\sinh z$ by the uniqueness of power series representations.

EXAMPLE 25.6. We find the Maclaurin series of $f(z) = z/(4+z^2)$. Since we are looking for the Maclaurin series of f, we want a series in terms of powers of z. Therefore, if we can find the power series of $1/(4+z^2)$, we only have to multiply it by z to get the Maclaurin series of f by the uniqueness of power series representations. The function $1/(1-w)$ is the most similar to $1/(4+z^2)$ of all the functions whose Maclaurin series we already know, with

$$\frac{1}{1-w} = \sum_{n=0}^{\infty} w^n$$

when $|w| < 1$. Then, we have

$$\frac{1}{4+z^2} = \frac{1}{4(1-(-z^2/4))} = \frac{1}{4}\sum_{n=0}^{\infty}\left(-\frac{z^2}{4}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n z^{2n}}{4^{n+1}},$$

where the second equality was obtained by setting $w = -z^2/4$ in the Maclaurin series of $1/(1-w)$. This power series converges for $|w| < 1$, which happens if and only if $|z| < 2$. Thus, the Maclaurin series of f is

$$\frac{z}{4+z^2} = \sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}}{4^{n+1}}$$

for all $|z| < 2$.

For our final example, we show that power series representations may be written with more than one sum.

EXAMPLE 25.7. To find the Maclaurin series of $(z+1)\cos(2z^3)$, we recall that

$$\cos w = \sum_{n=0}^{\infty}\frac{(-1)^n w^{2n}}{(2n)!}$$

for all $w \in \mathbb{C}$. Setting $w = 2z^3$, it follows that

$$\begin{aligned}(z+1)\cos(2z^3) &= (z+1)\sum_{n=0}^{\infty}\frac{(-1)^n(2z^3)^{2n}}{(2n)!}\\ &= (z+1)\sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n} z^{6n}}{(2n)!}\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n} z^{6n+1}}{(2n)!} + \sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n} z^{6n}}{(2n)!},\end{aligned}$$

which converges for all $z \in \mathbb{C}$.

When studying a function f, it is natural to consider its set of zeroes $\{z \in \mathbb{C} : f(z) = 0\}$. All the analytic functions we have seen so far, other than the constant zero function, have either been nowhere vanishing, such as e^z, or have isolated zeroes. For instance, we know that

- $\operatorname{Log} z$ has a single zero at $z = 1$;
- a complex polynomial of degree $n \geq 1$ has at least one zero and at least n distinct zeroes;
- the zeroes of $\sin z$ and $\cos z$ are isolated points on the x-axis; and
- the zeroes of $\sinh z$ and $\cosh z$ are isolated points on the y-axis.

It turns out that this is always the case. As a nice application of Taylor's Theorem, we will now show that if an analytic function is not identically zero on a domain, then it is either nowhere vanishing or its zeroes are isolated points.

THEOREM 25.8. Let f be analytic on a domain $\Omega \subseteq \mathbb{C}$ and suppose that $f(z_0) = 0$ for some $z_0 \in \Omega$. Then there exists a disc $D = D(z_0; r)$ centered at z_0 and contained in Ω such that either f is identically zero on D or

$$f(z) = (z - z_0)^{n_0} g(z)$$

on D, where n_0 is a positive integer and g is a nowhere vanishing analytic function on D. In particular, the zeroes of f either occur on solid discs or at isolated points.

PROOF. Since f is analytic at z_0, then by Taylor's Theorem, it has a Taylor series expansion about z_0 which is given by

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

with $a_n = f^{(n)}(z_0)/n!$ for all $n \geq 0$, which converges on some disc $D(z_0; R)$. Note that $a_0 = f(z_0) = 0$ so that

$$f(z) = a_1(z - z_0) + a_2(z - z_0)^2 + \cdots .$$

If f is non-zero for some $z \in D(z_0; R)$, then at least one of the a_n is non-zero; let n_0 be the smallest such index. Then we have

$$\begin{aligned} f(z) &= a_{n_0}(z - z_0)^{n_0} + a_{n_0+1}(z - z_0)^{n_0+1} + \cdots \\ &= (z - z_0)^{n_0}(a_{n_0} + a_{n_0+1}(z - z_0) + \cdots). \end{aligned}$$

If we set

$$g(z) = a_{n_0} + a_{n_0+1}(z - z_0) + \cdots = \sum_{k=0}^{\infty} a_{n_0+k}(z - z_0)^k,$$

then g is analytic on $D(z_0; R)$ since the power series defining it converges there. Indeed, it is equal to a_{n_0} at z_0, and it converges for $z \neq z_0$ as it is equal to $(z - z_0)^{-n_0} \sum_{n=0}^{\infty} a_n(z - z_0)^n$ with $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converging for all $z \in D(z_0; R)$. Finally, since $g(z_0) = a_{n_0} \neq 0$ and g is continuous on $D(z_0; R)$, we can choose $0 < r \leq R$ such that g is non-zero on $D(z_0; r)$. □

REMARK 25.9. This is very different from the real case, where one can construct smooth bump functions that are not constantly zero, but whose zeroes are not isolated. For instance, the function

$$f : \mathbb{R} \to \mathbb{R} : x \mapsto \begin{cases} e^{-1/x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

is smooth on $\mathbb{R}$ but vanishes on $(-\infty, 0]$.

The fact that the zeroes of a non-constant analytic function are isolated gives rise to one of the most important results in complex analysis, known as the Identity Theorem.

THEOREM 25.10 (Identity Theorem). Let f and g be analytic functions on a domain $\Omega \subseteq \mathbb{C}$. If f and g are equal on an open subset of Ω, then they are equal on all of Ω.

PROOF. Suppose that f and g are equal on the non-empty open subset $U \subseteq \Omega$. Set $h = f - g$, which is analytic on Ω and identically zero on U. If we show that h is identically zero on Ω, then f and g must be equal on Ω.

First, consider the set of non-isolated zeroes of h; namely, the set

$$A = \{z \in \Omega : h \text{ vanishes on } D(z; r) \text{ for some } r > 0\}.$$

By definition, A is open, and $A \neq \varnothing$ since $U \subseteq A$. Indeed, if $z \in U$, then $D(z; r) \subseteq U$ for some $r > 0$ as U is open, implying that h vanishes on $D(z; r)$. Hence, $z \in A$.

Note that $\Omega = A \cup (\Omega \setminus A)$, where $\Omega \setminus A$ consists of the isolated zeroes of h along with the non-zero points of h. We claim that $\Omega \setminus A$ is open. To see this, let $z' \in \Omega \setminus A$. If $h(z') \neq 0$, then z' is an element of $\{z \in \Omega : h(z) \neq 0\} \subseteq \Omega \setminus A$, which is an open subset of Ω by the continuity of h. Thus, we have

$$D(z'; r') \subseteq \{z \in \Omega : h(z) \neq 0\} \subseteq \Omega \setminus A$$

for some $r' > 0$. On the other hand, if $h(z') = 0$, then z' is an isolated zero of h', so there exists $r' > 0$ such that h is non-zero on $D(z'; r') \setminus \{z'\}$. Then we have $D(z'; r') \subseteq \Omega \setminus A$, so it follows that $\Omega \setminus A$ is open.

By the connectedness of Ω, this implies that $\Omega \setminus A = \varnothing$, so h is identically zero on Ω. ☐

Remark 25.11. This is a very striking result that certainly does not hold for real-valued smooth functions; for example, take the function f from Remark 25.9, which is certainly not identically zero despite its zeroes not being isolated points.

The Identity Theorem equivalently states that if f is an analytic function on a domain $\Omega \subseteq \mathbb{C}$, then f being zero on an open subset of Ω implies that f is zero on all of Ω.

One can actually prove a stronger version of the Identity Theorem.

Theorem 25.12 (Identity Theorem, stronger version). Let f be analytic on a domain $\Omega \subseteq \mathbb{C}$, and suppose that $\{z_n\}_{n=1}^{\infty}$ is an infinite sequence of distinct points of Ω such that

(i) $z^* = \lim_{n\to\infty} z_n$ exists and is a point in Ω (that is, z^* is a limit point of Ω); and

(ii) $f(z_n) = 0$ for all $n \in \mathbb{N}$.

Then f is identically zero on Ω.

Proof. Let A be the set of non-isolated zeroes of f; that is,

$$A = \{z \in \Omega : f \text{ vanishes on } D(z; r) \text{ for some } r > 0\}.$$

Note that A is open by definition. Moreover, we claim that $z^* \in A$. Indeed, z^* is a limit point of Ω and hence is not isolated because for any $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $z_n \in D(z^*; \varepsilon) \cap \Omega$ for all $n \geq n_\varepsilon$. This implies that $D(z^*; \varepsilon) \cap \Omega \neq \{z^*\}$. But $f(z^*) = \lim_{n\to\infty} f(z_n) = 0$ by the continuity of f, so z^* is a non-isolated zero of f.

We have $\Omega = A \cup (\Omega \setminus A)$ where $\Omega \setminus A$ consists of the isolated zeroes of f as well as the points where f is non-zero. Note that $\Omega \setminus A$ is also open. To see this, let $z' \in \Omega \setminus A$. If $f(z') \neq 0$, then z' is an element of $\{z \in \Omega : f(z) \neq 0\} \subseteq \Omega \setminus A$, which is an open subset of Ω by the continuity of f. On the other hand, if $f(z') = 0$, then f is non-zero in $D(z'; r) \setminus \{z'\}$ for some $r > 0$ as $z' \notin A$ so that it is an isolated zero of f. Thus, $D(z; r) \subseteq \Omega \setminus A$, which implies that $\Omega \setminus A$ is open. Finally, by the connectedness of Ω, it follows that $\Omega \setminus A = \varnothing$, so f is identically zero on Ω. ☐

Corollary 25.13. Let f be a non-constant analytic function on the domain $\Omega \subseteq \mathbb{C}$. Then the zeroes of f (if they exist) are isolated.

Proof. If f has no zeroes on Ω, then we are done. Suppose now that f vanishes at $z \in \Omega$. Then either f vanishes on an open disc $D(z; r)$ centered at z and contained in Ω, or z is an isolated zero. In the former case, the Identity Theorem implies that f is identically zero on Ω, which is impossible as we assumed that f is non-constant. Thus, z is an isolated zero. ☐

26 Isolated singularities

We now turn to studying isolated singularities of analytic functions on punctured domains.

DEFINITION 26.1 (Isolated singularity). Let $\Omega \subseteq \mathbb{C}$ be a domain, let $z_0 \in \Omega$, and let f be an analytic function on the punctured domain $\Omega \setminus \{z_0\}$. The point z_0 is said to be an **isolated singularity** of f.

There are three types of isolated singularities.

- **Removable singularities** are isolated singularities such that

$$\lim_{z \to z_0} f(z) = w_0$$

  exists for some $w_0 \in \mathbb{C}$. We will see that f can be extended to an analytic function on Ω by setting $f(z_0) = w_0$ in this case.

  For example, the function given by $f(z) = z$ for $z \in \mathbb{C} \setminus \{0\}$ has a removable singularity at $z = 0$ since $\lim_{z \to 0} z = 0$ exists. Moreover, we can extend f to an entire function by setting $f(0) = 0$.

- **Poles** are isolated singularities such that

$$\lim_{z \to z_0} |f(z)| = +\infty.$$

  We will see that z_0 is a pole if and only if f can be written in the form

$$f(z) = \frac{g(z)}{(z - z_0)^m}$$

  where g is an analytic function on Ω such that $g(z_0) \neq 0$ and m is a positive integer.

  For example, the function $f(z) = 1/z$ for $\mathbb{C} \setminus \{0\}$ has a pole at $z = 0$ since

$$\lim_{z \to 0} |f(z)| = \lim_{z \to 0} \left| \frac{1}{z} \right| = \lim_{z \to 0} \frac{1}{|z|} = +\infty.$$

  Note that $f(z) = g(z)/(z - z_0)^1$ where $g(z) = 1$ for all $z \in \mathbb{C}$ and $z_0 = 0$ so that $g(0) = 1 \neq 0$.

- **Essential singularities** are isolated singularities which are neither removable nor poles.

  For example, $e^{1/z}$ has an essential singularity at $z = 0$. Indeed, along the path $y = 0$, we have

$$e^{1/x} \xrightarrow{x \to 0} +\infty,$$

  which implies that $\lim_{z \to 0} e^{1/z}$ does not exist. Hence, $z = 0$ is not a removable singularity. Now, along the path $x = 0$, we have $z = iy$ so that

$$e^{1/(iy)} = e^{i(-1/y)} = 1,$$

  so $\lim_{z \to 0} |e^{1/z}| \neq +\infty$. In other words, $z = 0$ is not a pole.

In this lecture, we will begin by considering removable singularities. In particular, we will see that removable singularities can be characterized by the fact that $|f(z)|$ is always bounded in some punctured disc around the removable singularity. This is known as Riemann's Criterion for Removable Singularities.

For instance, the function $f(z) = 1/z$ where $z \in \mathbb{C} \setminus \{0\}$ does not have a removable singularity at $z = 0$ as $|1/z|$ is unbounded in any punctured disc about $z = 0$; that is, any set $D(0; r) \setminus \{0\}$ where $r > 0$.

To understand poles and essential singularities, we will require the notion of Laurent series, which we will introduce in the next lecture.

Removable singularities are often defined in terms of Laurent series. However, we will define them as follows.

Definition 26.2 (Removable singularity). Let $\Omega \subseteq \mathbb{C}$ be a domain and let $z_0 \in \mathbb{C}$. Moreover, let f be an analytic function on $\Omega \setminus \{z_0\}$ so that z_0 is an isolated singularity of f. Then z_0 is said to be a **removable singularity** of f if there exists $w_0 \in \mathbb{C}$ such that $\lim_{z\to z_0} f(z) = w_0$.

Remark 26.3. Suppose that f has a removable singularity at z_0 with $\lim_{z\to z_0} f(z) = w_0$. If we set

$$\hat{f}(z) = \begin{cases} f(z) & \text{if } z \in \Omega \setminus \{z_0\}, \\ w_0 & \text{if } z = z_0, \end{cases}$$

then $\hat{f}$ is continuous on Ω. In other words, f extends to a continuous function $\hat{f}$ on Ω. Indeed, recall that analytic functions are continuous, so f is continuous on $\Omega \setminus \{z_0\}$. Moreover, we have

$$\lim_{z\to z_0} \hat{f}(z) = \lim_{z\to z_0} f(z) = w_0 = \hat{f}(z_0).$$

Hence, a necessary condition for an isolated singularity to be removable is that f extends to a continuous function on Ω. We will see that this extension is also analytic on Ω in Riemann's Criterion for Removable Singularities. Before stating and proving the criterion, we give some more examples.

Example 26.4.

(1) If $\Omega \subseteq \mathbb{C}$ is a domain and f is analytic on Ω, then the restriction of f to $\Omega \setminus \{z_0\}$ has a removable singularity at z_0. However, this example is rather artificial as we already know that $f|_{\Omega\setminus\{z_0\}}$ extends to an analytic function on Ω.

(2) Consider the function $f(z) = (z^2-1)/(z-1)$ on $\mathbb{C} \setminus \{1\}$, which has an isolated singularity that $z = 1$. However, we have $\lim_{z\to 1} f(z) = 2$, so $z = 1$ is a removable singularity. In particular, f extends to the entire function $\hat{f}(z) = z + 1$ on $\mathbb{C}$.

(3) Consider the function $f(z) = \sin(z)/z$ on $\mathbb{C} \setminus \{0\}$. Since f is the quotient of two entire functions and the denominator vanishes only at $z = 0$, it is analytic on $\mathbb{C} \setminus \{0\}$. However, we know that $\lim_{z\to 0} \sin(z)/z = 1$ so that f extends to the continuous function

$$\hat{f}(z) = \begin{cases} \sin(z)/z & \text{if } z \neq 0, \\ 1 & \text{if } z = 0. \end{cases}$$

Hence, $z = 0$ is a removable singularity. Moreover, note that

$$\hat{f}'(0) = \lim_{z\to 0} \frac{\hat{f}(z) - \hat{f}(0)}{z - 0} = \lim_{z\to 0} \frac{\sin(z)/z - 1}{z} = \lim_{z\to 0} \frac{\sin z - z}{z^2} = \lim_{z\to 0} \frac{\cos z - 1}{2z} = \lim_{z\to 0} \frac{-\sin z}{2} = 0$$

from applying l'Hôpital's Rule twice. Therefore, $\hat{f}$ is also analytic at $z = 0$, so $\hat{f}$ is entire.

We now state and prove Riemann's Criterion for Removable Singularities.

Theorem 26.5 (Riemann's Criterion for Removable Singularities). Let f be analytic on the punctured domain $\Omega \setminus \{z_0\}$. The isolated singularity z_0 is removable if and only if $|f(z)|$ is bounded in some punctured disc about z_0. In this case, $\lim_{z\to z_0} f(z) = w_0$ exists, and defining $f(z_0) = w_0$ produces an analytic function on Ω.

Proof. We have already seen that setting $f(z_0) = w_0$ induces a continuous function Ω, but it is not yet clear that f is analytic at z_0. The bulk of the proof will consist of showing that f is also analytic at z_0. Moreover, note that analytic extensions are unique by the Identity Theorem since any two analytic extensions of f to z_0 must be equal on the open set $\Omega \setminus \{z_0\}$, which will force them to be equal on all of Ω.

Let $r > 0$ be such that $D(z_0; r) \subseteq \Omega$. Let $C(z_0; r) = \partial D(z_0; r)$. Then for all $z \in D(z_0; r)$, we define

$$F(z) = \frac{1}{2\pi i} \int_{C(z_0;r)} \frac{f(s)}{s - z}\, \mathrm{d}s.$$

As in the proof of the Cauchy Integral Formula, the function F is analytic on $D(z_0;r)$. If we can show that $F(z) = f(z)$ for all $z \in D(z_0;r) \setminus \{z_0\}$, this will imply that F is an analytic extension of f to $D(z_0;r)$ and thus to Ω by the uniqueness of analytic extensions.

Let $z_1 \in D(z_0;r) \setminus \{z_0\}$. In particular, we have $0 < |z_1 - z_0| < r$. We will show that $F(z_1) - f(z_1) = 0$, and hence $F(z_1) = f(z_1)$. Choose r_1 such that $r_1 < \min\{|z_1 - z_0|, r - |z_1 - z_0\}$ so that $\overline{D}(z_1;r_1) \subseteq D(z_0;r) \setminus \{z_0\}$. If $C(z_1;r_1) = \partial D(z_1;r_1)$, then f is analytic on and inside $C(z_1;r_1)$, so by the Cauchy Integral Formula, we have

$$f(z_1) = \frac{1}{2\pi i}\int_{C(z_1;r_1)} \frac{f(s)}{s - z_1}\,\mathrm{d}s.$$

This implies that

$$F(z_1) - f(z_1) = \frac{1}{2\pi i}\int_{C(z_0;r)} \frac{f(s)}{s - z_1}\,\mathrm{d}s - \frac{1}{2\pi i}\int_{C(z_1;r_1)} \frac{f(s)}{s - z_1}\,\mathrm{d}s.$$

The difference between the two integrals in the above expression is the circle on which they are defined. Choose $\rho > 0$ such that $D(z_0;\rho) \subseteq D(z_0;r) \setminus \overline{D}(z_1;r_1)$. Let $C(z_0;\rho) = \partial D(z_0;\rho)$ and orient the three circles $C(z_0;r)$, $C(z_1;r_1)$, and $C(z_0;\rho)$ positively. Then $\Omega' = D(z_0;r) \setminus (\overline{D}(z_1;r_1) \cup \overline{D}(z_0;\rho))$ is a 3-connected Jordan domain with boundary $\partial\Omega' = C(z_0;r) - C(z_1;r_1) - C(z_0;\rho)$. By the Cauchy Integral Theorem for k-connected domains, we have

$$\int_{\partial\Omega'} \frac{f(s)}{s - z_1}\,\mathrm{d}s = 0,$$

which is equivalent to

$$\int_{C(z_0;r)} \frac{f(s)}{s - z_1}\,\mathrm{d}s = \int_{C(z_1;r_1)} \frac{f(s)}{s - z_1}\,\mathrm{d}s + \int_{C(z_0;\rho)} \frac{f(s)}{s - z_1}\,\mathrm{d}s.$$

Therefore, we have

$$F(z_1) - f(z_1) = \frac{1}{2\pi i}\int_{C(z_0;\rho)} \frac{f(s)}{s - z_1}\,\mathrm{d}s.$$

It only remains to show that this quantity is equal to 0. By assumption, we have that $|f(z)|$ is bounded in a punctured disc about z_0. By potentially shrinking ρ, we may assume that it is bounded on $D(z_0;\rho) \setminus \{z_0\}$. Moreover, note that $|s - z_1| \geq |z_1 - z_0| - \rho$ for all $s \in C(z_0;\rho)$. Hence, there exists $M > 0$ such that

$$\left|\frac{f(s)}{s - z_1}\right| < M$$

for all $s \in C(z_0;\rho)$. Since $C(z_0;\rho)$ has arclength $2\pi\rho$, the ML-inequality gives

$$|F(z_1) - f(z_1)| = \frac{1}{2\pi}\left|\int_{C(z_0;\rho)} \frac{f(s)}{s - z_1}\,\mathrm{d}s\right| < \frac{1}{2\pi} \cdot M \cdot 2\pi\rho = M\rho.$$

We can take ρ as small as we like, so this forces $F(z_1) - f(z_1) = 0$. Thus, we have $F(z_1) = f(z_1)$ for all $z_1 \in D(z_0;r) \setminus \{z_0\}$. $\square$

27 Laurent series

In this lecture, we will look at Laurent series, which are power series expansions of analytic functions on punctured discs.

THEOREM 27.1 (Laurent's Theorem). Let f be analytic on $D = \{z \in \mathbb{C} : 0 < r < |z - z_0| < R\}$, and let Γ be any simple closed curve in D whose interior contains z_0, oriented positively. Then f can be expressed as the sum of two power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \tag{$\star$}$$

for all $z \in D$, where

$$\begin{aligned} a_n &= \frac{1}{2\pi i} \int_\Gamma \frac{f(s)}{(s - z_0)^{n+1}} \, \mathrm{d}s, \qquad n \geq 0, \\ b_n &= \frac{1}{2\pi i} \int_\Gamma \frac{f(s)}{(s - z_0)^{-n+1}} \, \mathrm{d}s, \quad\ n \geq 1. \end{aligned}$$

The series representation ($\star$) is called the **Laurent series** of f and converges on all of D.

PROOF. Suppose that $z_0 = 0$. Let $z \in D$ be any point. Let C and C_1 be circles in D centered at the origin of radii ρ and ρ_1 respectively, with $\rho_1 < \rho$. Moreover, let C_2 be a circle centered at z that is contained in the annular region bounded by C and C_1. Assume that the three circles are positively oriented. Note that C_1 and C_2 lie in the interior of C. Let Ω be the region bounded by C, C_1, and C_2. Since f is analytic on C, C_1, C_2, and inside Ω, and Ω is a 3-connected Jordan domain with boundary $C - C_1 - C_2$, the Cauchy Integral Theorem gives

$$\int_C \frac{f(s)}{s - z} \, \mathrm{d}s = \int_{C_1} \frac{f(s)}{s - z} \, \mathrm{d}s + \int_{C_2} \frac{f(s)}{s - z} \, \mathrm{d}s.$$

Now, the Cauchy Integral Formula implies that

$$\int_{C_2} \frac{f(s)}{s - z} \, \mathrm{d}s = 2\pi i \cdot f(z)$$

as z is inside C_2 and f is analytic on and inside C_2. Therefore, if we set

$$\begin{aligned} I &= \int_C \frac{f(s)}{s - z} \, \mathrm{d}s, \\ I_1 &= \int_{C_1} \frac{f(s)}{s - z} \, \mathrm{d}s, \end{aligned}$$

we have that

$$f(z) = \frac{1}{2\pi i} \int_{C_2} \frac{f(s)}{s - z} \, \mathrm{d}s = I - I_1.$$

We know that z is inside C and $|z/s| < 1$ for all $s \in C$, so as in the proof of Taylor's Theorem, we have

$$I = \sum_{n=0}^{\infty} a_n z^n$$

where the coefficients are given by

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(s)}{s^{n+1}} \, \mathrm{d}s.$$

We now turn to computing I_1. This time, z is outside C_1 and $|s/z| < 1$ for all $s \in C_1$, so as in the proof of Taylor's Theorem, we obtain

$$\begin{aligned}
\frac{1}{s-z} &= -\frac{1}{z(1-s/z)} \\
&= -\frac{1}{z}\left(1 + \frac{s}{z} + \left(\frac{s}{z}\right)^2 + \cdots + \left(\frac{s}{z}\right)^m + \frac{(s/z)^{m+1}}{1-s/z}\right) \\
&= -\left(\frac{1}{z} + \frac{s}{z^2} + \frac{s^2}{z^3} + \cdots + \frac{s^m}{z^{m+1}} + \frac{(s/z)^{m+1}}{z-s}\right) \\
&= -\left(\sum_{n=1}^{m+1} \frac{s^{n-1}}{z^n}\right) - \left(\frac{s}{z}\right)^{m+1} \cdot \frac{1}{z-s}.
\end{aligned}$$

It follows that

$$I_1 = -\frac{1}{2\pi i}\left(\sum_{n=1}^{m+1}\left(\int_{C_1} \frac{f(s)}{s^{-n+1}}\,\mathrm{d}s\right) \cdot \frac{1}{z^n}\right) - \frac{1}{2\pi i}\int_{C_1} \frac{f(s)}{z-s} \cdot \left(\frac{s}{z}\right)^{m+1} \mathrm{d}s.$$

For all $1 \leq n \leq m+1$, set

$$b_n = \frac{1}{2\pi i}\int_{C_1} \frac{f(s)}{s^{-n+1}}\,\mathrm{d}s.$$

Moreover, set

$$R_m = \frac{1}{2\pi i}\int_{C_1} \frac{f(s)}{z-s} \cdot \left(\frac{s}{z}\right)^{m+1} \mathrm{d}s.$$

Then we have

$$I_1 = -\sum_{n=1}^{m+1} b_n z^{-n} - R_m.$$

It can be shown that $\lim_{m\to\infty} R_m = 0$ as in the proof of Taylor's Theorem, so taking $m \to \infty$ in the expression of I_1, we get

$$I_1 = -\sum_{n=1}^{\infty} b_n z^{-n}.$$

Thus, we have

$$f(z) = I - I_1 = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^{-n}$$

where the coefficients are

$$\begin{aligned}
a_n &= \frac{1}{2\pi i}\int_C \frac{f(s)}{s^{n+1}}\,\mathrm{d}s, \qquad n \geq 0, \\
b_n &= \frac{1}{2\pi i}\int_{C_1} \frac{f(s)}{s^{-n+1}}\,\mathrm{d}s, \quad n \geq 1.
\end{aligned}$$

Finally, by the Deformation Principle, we have

$$\begin{aligned}
a_n &= \frac{1}{2\pi i}\int_C \frac{f(s)}{s^{n+1}}\,\mathrm{d}s = \frac{1}{2\pi i}\int_\Gamma \frac{f(s)}{s^{n+1}}\,\mathrm{d}s, \\
b_n &= \frac{1}{2\pi i}\int_{C_1} \frac{f(s)}{s^{-n+1}}\,\mathrm{d}s = \frac{1}{2\pi i}\int_\Gamma \frac{f(s)}{s^{-n+1}}\,\mathrm{d}s
\end{aligned}$$

for any positively oriented simple closed curve Γ whose interior contains the origin. The general case follows from replacing z with $z - z_0$. □

Remark 27.2. Note that the Laurent series representation of a function f can also be written as

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z-z_0)^n,$$

where for all $n \in \mathbb{Z}$, we have

$$c_n = \frac{1}{2\pi i}\int_\Gamma \frac{f(s)}{(s-z_0)^{n+1}}\,\mathrm{d}s.$$

Remark 27.3.

(1) Unlike Taylor's Theorem, we are only assuming that f is analytic on a deleted neighbourhood of z_0. Thus, it might be the case that f is not analytic at z_0. However, if f is in fact analytic at z_0 and in a disc $|z-z_0| < R$ centered at z_0, then for all $n \geq 1$, we have

$$b_n = \frac{1}{2\pi i}\int_\Gamma \frac{f(s)}{(s-z_0)^{-n+1}}\,\mathrm{d}s = \frac{1}{2\pi i}\int_\Gamma f(s)(s-z_0)^{n-1}\,\mathrm{d}s,$$

where $f(s)(s-z_0)^{n-1}$ is analytic on and inside Γ. By the Cauchy-Goursat Theorem, we obtain

$$\int_\Gamma f(s)(s-z_0)^{n-1}\,\mathrm{d}s = 0,$$

which implies that $b_n = 0$ for all $n \geq 1$. Therefore, we have

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n,$$

so the Laurent series of f and the Taylor series of f are the same in this case by the uniqueness of power series representations.

(2) As with Taylor series, one can show that the Laurent series is unique; that is, if

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z-z_0)^n$$

on D, then it must be the Laurent series. Therefore, we do not always need to compute the contour integrals

$$\int_\Gamma \frac{f(s)}{(s-z_0)^{n+1}}\,\mathrm{d}s$$

to find the coefficients c_n. For instance, one can find these coefficients by algebraically manipulating other known series.

(3) The coefficient

$$c_{-1} = \frac{1}{2\pi i}\int_\Gamma \frac{f(s)}{(s-z_0)^{-1+1}}\,\mathrm{d}s = \frac{1}{2\pi i}\int_\Gamma f(s)\,\mathrm{d}s$$

is called the **residue of f at z_0** and satisfies

$$\int_\Gamma f(s)\,\mathrm{d}s = 2\pi i \cdot c_{-1}.$$

The Residue Theorem is based on this fact. It is important to note that since z_0 might not be a removable singularity of f, then the function f might not be analytic at z_0. In particular, the Cauchy Integral Theorem might not hold in this case; indeed, the Cauchy Integral Theorem requires that f is analytic on and inside Γ. Therefore, it is possible that

$$\int_\Gamma f(s)\,\mathrm{d}s \neq 0,$$

in which case it follows that $c_{-1} \neq 0$.

EXAMPLE 27.4. Let us find the Laurent series of $f(z) = e^{1/z}$ about $z = 0$ which is valid for all $z \neq 0$.

Note that f is the composition of e^w with $w = 1/z$. Therefore, we can start by finding the Maclaurin series of e^w and compose the result with $w = 1/z$. Recall that

$$e^w = \sum_{n=0}^{\infty} \frac{w^n}{n!}$$

for all $w \in \mathbb{C}$. Then

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{(1/z)^n}{n!} = \sum_{n=0}^{\infty} \frac{z^{-n}}{n!}.$$

Moreover, since the Maclaurin series of e^w converges for all $w \in \mathbb{C}$, then by the uniqueness of Laurent series, this must be the Laurent series of $e^{1/z}$ for all $z \in \mathbb{C} \setminus \{0\}$.

EXAMPLE 27.5. We can evaluate

$$\int_\Gamma z^2 e^{1/z} \, \mathrm{d}z$$

where Γ is any positively oriented simple closed curve containing $z = 0$.

Indeed, since $e^{1/z}$ is analytic for all $z \neq 0$, then by Laurent's Theorem, we have

$$\int_\Gamma z^2 e^{1/z} \, \mathrm{d}z = 2\pi i \cdot c_{-1}.$$

From the previous example, we have

$$z^2 e^{1/z} = z^2 \sum_{n=0}^{\infty} \frac{z^{-n}}{n!} = \sum_{n=0}^{\infty} \frac{z^{-n+2}}{n!} = z^2 + z + \frac{1}{2} + \frac{1}{6z} + \frac{1}{24z^2} + \cdots$$

for all $z \neq 0$. In particular, we have $c_{-1} = 1/6$ so that

$$\int_\Gamma z^2 e^{1/z} \, \mathrm{d}z = \frac{2\pi i}{6} = \frac{\pi i}{3}.$$

EXAMPLE 27.6. We find the Laurent series of $e^{4z}/(z+2)^3$ about $z = -2$.

We want a power series involving powers of $z + 2$. Note that

$$e^{4z} = e^{4(z+2)-8} = e^{-8} e^{4(z+2)}.$$

Moreover, $e^{4(z+2)}$ is the composition of e^w with $w = 4(z+2)$. Since

$$e^w = \sum_{n=0}^{\infty} \frac{w^n}{n!}$$

for all $w \in \mathbb{C}$, it follows that

$$e^{4(z+2)} = \sum_{n=0}^{\infty} \frac{(4(z+2))^n}{n!} = \sum_{n=0}^{\infty} \frac{4^n}{n!} (z+2)^n$$

for all $z \in \mathbb{C}$. Thus, we have that

$$\frac{e^{4z}}{(z+2)^3} = \frac{1}{(z+2)^3} \cdot e^{-8} \cdot \sum_{n=0}^{\infty} \frac{4^n}{n!} (z+2)^n = \sum_{n=-3}^{\infty} \frac{4^{n+3}}{e^8 (n+3)!} (z+2)^n$$

is the Laurent series of $e^{4z}/(z+2)^3$ about $z = -2$, which converges for all $z \neq -2$.

EXAMPLE 27.7. We find the Laurent series of $\cos(3z)/z^2$ about $z = 0$.

Recall that the Maclaurin series expansion of $\cos w$ is

$$\cos w = \sum_{n=0}^{\infty} \frac{(-1)^n w^{2n}}{(2n)!}$$

for all $w \in \mathbb{C}$. Setting $w = 3z$, we obtain

$$\cos(3z) = \sum_{n=0}^{\infty} \frac{(-1)^n (3z)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 9^n}{(2n)!} z^{2n}$$

for all $z \in \mathbb{C}$. Hence, we have

$$\frac{\cos(3z)}{z^2} = \sum_{n=0}^{\infty} \frac{(-1)^n 9^n}{(2n)!} z^{2n-2} = \sum_{m=-1}^{\infty} \frac{(-1)^{m+1} 9^{m+1}}{(2m+2)!} z^{2m}$$

for all $z \in \mathbb{C} \setminus \{0\}$, which must be the Laurent series by the uniqueness of Laurent series representations.

EXAMPLE 27.8. Let us find a Laurent series expansion of $f(z) = (z-2)/[z(z+5)^2]$ that converges on $0 < |z+5| < 5$.

Since we want to find a power series that converges on a punctured disc centered at $z = -5$, we should start by considering a power series about $z = -5$, which means that we want powers of $z + 5$. Note that

$$\begin{aligned}
f(z) = \frac{z-2}{z(z+5)^2} &= \frac{1}{(z+5)^2} \cdot \left(1 - \frac{2}{z}\right) \\
&= \frac{1}{(z+5)^2} \cdot \left(1 + \frac{2}{5-(z-5)}\right) \\
&= \frac{1}{(z+5)^2} \cdot \left(1 + \frac{2}{5}\left(\frac{1}{1-(z+5)/5}\right)\right) \\
&= \frac{1}{(z+5)^2} \cdot \left(1 + \frac{2}{5}\sum_{n=0}^{\infty}\left(\frac{z+5}{5}\right)^n\right),
\end{aligned}$$

which converges whenever $0 < |(z+5)/5| < 1$ and hence on the punctured disc $0 < |z+5| < 5$. Simplifying the above expression, we obtain

$$\begin{aligned}
f(z) &= \frac{1}{(z+5)^2} \cdot \left(\frac{7}{5} + \sum_{n=1}^{\infty} \frac{2(z+5)^n}{5^{n+1}}\right) \\
&= \frac{7}{5(z+5)^2} + \sum_{n=1}^{\infty} \frac{2(z+5)^{n-2}}{5^{n+1}} \\
&= \frac{7}{5(z+5)^2} + \sum_{m=-1}^{\infty} \frac{2}{5^{m+3}}(z+5)^m,
\end{aligned}$$

which converges on $0 < |z+5| < 5$. By the uniqueness of Laurent series expansions, this must be the Laurent series of $f(z) = (z-2)/[z(z+5)^2]$ about $z = -5$.

28 Poles and essential singularities

In this lecture, we end our analysis of isolated singularities by studying poles and essential singularities.

Let $\Omega \subseteq \mathbb{C}$ be a domain. Recall that a point $z_0 \in \Omega$ is called a **pole** of f if f is analytic on $\Omega \setminus \{z_0\}$ and

$$\lim_{z \to z_0} |f(z)| = +\infty.$$

In fact, this is equivalent to $1/f$ having an isolated zero at z_0.

Proposition 28.1. If z_0 is a pole of f, then it is a removable singularity of $1/f$ and $1/f$ has an isolated zero there. Conversely, if g is analytic at z_0 and has an isolated zero there, then $1/g$ has a pole at z_0.

Proof. If z_0 is a pole of f, then $\lim_{z\to z_0} |f(z)| = +\infty$ so that $\lim_{z\to z_0} 1/|f(z)| = 0$, which happens if and only if $\lim_{z\to z_0} 1/f(z) = 0$. This means that z_0 is a removable singularity of $g := 1/f$ and g becomes analytic at z_0 by setting $g(z_0) = 0$. Moreover, this is an isolated zero because f is analytic and therefore defined at every $z \neq z_0$, and since $\lim_{z\to z_0} |f(z)| = +\infty$, there exists $r > 0$ such that $f(z) \neq 0$ for all $z \in D(z_0; r) \setminus \{z_0\}$.

Conversely, if g is analytic and has an isolated zero at z_0, then there exists $r > 0$ such that g is analytic on $D(z_0; r)$ and $g(z) \neq 0$ for all $z \in D(z_0; r) \setminus \{z_0\}$. Hence, $f := 1/g$ has an isolated singularity at z_0 and $\lim_{z\to z_0} |f(z)| = +\infty$ since $\lim_{z\to z_0} |g(z)| = 0$, implying that z_0 is a pole of $1/g$. □

This proposition is useful as it gives us a way of describing poles explicitly. Indeed, recall from Theorem 25.8 that if a function g is analytic at z_0 and has an isolated zero there, then it can be written in the form

$$g(z) = (z - z_0)^m h(z)$$

where h is analytic on an open disc $D(z_0; r)$ and non-zero at z_0, and m is a positive integer. Hence, if f has a pole at z_0 so that $1/f$ has an isolated zero at z_0 by the previous proposition, then there exists a function h which is analytic on $D(z_0; r)$ and non-zero at z_0 such that

$$\frac{1}{f(z)} = (z - z_0)^m h(z).$$

Rearranging yields

$$f(z) = \frac{1}{(z - z_0)^m h(z)},$$

and setting $\hat{h} = 1/h$, we can write

$$f(z) = \frac{\hat{h}(z)}{(z - z_0)^m}$$

for all $z \in D(z_0; r)$, where $\hat{h}$ is analytic and nowhere vanishing on $D(z_0; r)$. Summarizing the above discussion, we have the following characterization of poles.

Corollary 28.2. The function f has a pole at z_0 if and only if it can be written in the form

$$f(z) = \frac{g(z)}{(z - z_0)^m} \qquad (\star)$$

for some positive integer m and an analytic function g on $D(z_0; r)$ such that $g(z_0) \neq 0$.

Definition 28.3. Let z_0 be a pole of f. The positive integer m appearing in $(\star)$ is called the **order** of the pole z_0. A pole is said to be **simple** in the case that $m = 1$.

Example 28.4.

(1) The function $f(z) = 1/(z - z_0)^m$ has a pole of order m at z_0 for all $m \geq 1$.

(2) The function $f(z) = e^z/(z - 2)$ has a simple pole at $z = 2$.

(3) The function $f(z) = 1/(\sin z)$ has simple poles at $z = k\pi$ where $k \in \mathbb{Z}$ since $1/f(z) = \sin z$ has zeroes of order 1 there.

(4) The function $f(z) = 1/(\sin(1/z))$ has simple poles at $z = 1/(k\pi)$ where $k \in \mathbb{Z} \setminus \{0\}$ since $1/f(z) = \sin(1/z)$ has zeroes of order 1 there. However, note that although f is not defined at $z = 0$, it is not an isolated singularity since any neighbourhood of $z = 0$ contains poles of f.

One can also characterize poles in terms of Laurent series. Suppose that f has a pole of order m so that

$$f(z) = \frac{g(z)}{(z - z_0)^m}$$

where m is a positive integer and g is an analytic function on $D(z_0; r)$ such that $g(z_0) \neq 0$. Since g is analytic at z_0, it has a Taylor series expansion $g(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ about $z = z_0$ with $a_0 = g(z_0)$. Since $g(z_0) \neq 0$, we have $a_0 \neq 0$. Moreover, observe that

$$f(z) = \frac{1}{(z - z_0)^m} \cdot \sum_{n=0}^{\infty} a_n(z - z_0)^n = \sum_{n=0}^{\infty} a_n(z - z_0)^{n-m} = \sum_{n=-m}^{\infty} a_{n+m}(z - z_0)^n$$

with $a_0 \neq 0$. By the uniqueness of Laurent series expansions, the above expression must be the Laurent series of f about $z = z_0$. In particular, we see that it only has a finite number of negative powers of $z - z_0$.

Using the notation from the previous lecture, we have now shown that f has a pole of order m at z_0 if and only if its Laurent series expansion about $z = z_0$ is of the form

$$f(z) = \sum_{n=-m}^{\infty} c_n(z - z_0)^n = \frac{c_{-m}}{(z - z_0)^m} + \cdots + \frac{c_{-1}}{z - z_0} + c_0 + c_1(z - z_0) + \cdots .$$

Note that f has a removable singularity at z_0 if and only if

$$c_{-m} = \cdots = c_{-1} = 0.$$

Putting everything together, we have the following characterization of isolated singularities in terms of Laurent series.

Theorem 28.5. Let z_0 be an isolated singularity of f, and suppose that its Laurent series expansion at z_0 is

$$\sum_{n=-\infty}^{\infty} c_n(z - z_0)^n$$

on the punctured disc $0 < |z - z_0| < r$. Then

(i) z_0 is a removable singularity of f if and only if $c_n = 0$ for all $n \leq -1$;

(ii) z_0 is a pole of f of order m if and only if $c_{-m} \neq 0$ and $c_n = 0$ for all $n \leq -m - 1$;

(iii) z_0 is an essential singularity of f if and only if there are infinitely many non-zero coefficients c_n where $n \leq -1$.

Example 28.6. Recall that the Laurent series of $e^{1/z}$ about $z = 0$ is

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n} z^{-n}.$$

It has infinitely many negative powers of z, which reaffirms that $z = 0$ is an essential singularity of $e^{1/z}$.

EXAMPLE 28.7. Recall that $\sin(z)/z$ has a removable singularity at $z = 0$. Its Laurent series about $z = 0$ is obtained by dividing the Maclaurin series of $\sin z$ by z; that is, we have

$$\frac{\sin z}{z} = \frac{1}{z}\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{(-1)^n z^{2n}}{(2n+1)!},$$

which converges for all $z \neq 0$. As expected, it only consists of non-negative powers of z.

EXAMPLE 28.8. We saw in the previous lecture that the Laurent series of $f(z) = (z-2)/[z(z+5)^2]$ about $z = -5$ is given by

$$f(z) = \frac{7}{5(z+5)^2} + \sum_{m=-1}^{\infty}\frac{2}{5^{m+3}}(z+5)^m$$

on $0 < |z+5| < 5$. Note that $z = -5$ is a pole of order 2 as $1/f$ has a zero of order 2 there. Therefore, it is not surprising that the coefficients c_n of the Laurent series of f about $z = -5$ satisfy $c_2 = 7/5 \neq 0$ and $c_n = 0$ for all $n \leq -3$.

REMARK 28.9. In Theorem 28.5, it is important to note that the relationship between negative powers appearing in the Laurent series expansion of f and the type of isolated singularity of z_0 only holds if one is working with a punctured disc $0 < |z - z_0| < r$. A power series expansion on a different type of domain, such as an annular region centered at z_0, may yield a power series expansion with different types of powers. We illustrate this with the following example.

EXAMPLE 28.10. We will look at different power series expansions of the function

$$f(z) = \frac{1}{z^2-2z-3} = \frac{1}{(z-3)(z+1)} = \frac{1}{4-(z-1)^2}$$

which converge on different domains.

- We find a power series that converges on $0 < |z+1| < 4$.

  Since $0 < |z+1| < 4$ is a punctured disc centered at $z = -1$, we only need to find the Laurent series of f about $z = -1$ as it converges on a punctured disc centered at $z = -1$, which means that we require powers of $z+1$. We have

$$\begin{aligned} f(z) = \frac{1}{(z-3)(z+1)} &= \frac{1}{z+1}\cdot\frac{-1}{4-(z+1)} \\ &= -\frac{1}{z+1}\cdot\frac{1}{4}\cdot\frac{1}{1-(z+1)/4} \\ &= -\frac{1}{z+1}\cdot\frac{1}{4}\cdot\sum_{n=0}^{\infty}\left(\frac{z+1}{4}\right)^n \\ &= \sum_{n=0}^{\infty}\frac{-1}{4^{n+1}}(z+1)^{n-1} \\ &= \sum_{m=-1}^{\infty}\frac{-1}{4^{m+2}}(z+1)^m, \end{aligned}$$

  which converges when $0 < |(z+1)/4| < 1$ and hence on $0 < |z+1| < 4$. Here, we used the geometric series $1/(1-w) = \sum_{n=0}^{\infty} w^n$ with $w = (z+1)/4$, which converges for all $|w| < 1$.

  It is not surprising that the Laurent series of f about $z = -1$ on the punctured disc $0 < |z+1| < 4$ only has $(z+1)^{-1}$ as a negative power as $z = -1$ is a simple pole of f.

- We find a power series that converges on $|z+1| > 4$.

  Since this is the outside of the disc centered at $z = -1$, we again consider a power series centered at $z = -1$. Our goal now is to use the geometric series $\sum_{n=0}^{\infty} w^n$ with $w = 4/(z+1)$ as the resulting power series will converge on $|z+1| > 4$. Indeed, we have

$$\begin{aligned} f(z) &= \frac{1}{(z-3)(z+1)} = \frac{1}{z+1} \cdot \frac{1}{(z+1)-4} \\ &= \frac{1}{(z+1)^2} \cdot \frac{1}{1-4/(z+1)} \\ &= \frac{1}{(z+1)^2} \cdot \sum_{n=0}^{\infty} \left(\frac{4}{z+1}\right)^n \\ &= \sum_{n=0}^{\infty} 4^n (z+1)^{-n-2} \\ &= \sum_{m=-\infty}^{-2} \frac{(z+1)^m}{4^{m+2}}. \end{aligned}$$

  We see that although $z = -1$ is a simple pole of f, this power series has an infinite number of negative powers of $z+1$.

- We find a power series that converges on $|z-1| < 2$.

  First, note that f is analytic on the disc $|z-1| < 2$ as its singularities are $z = -1$ and $z = 3$. Moreover, since the disc is centered at $z = 1$, we want a power series involving powers of $z-1$, which will correspond to the Taylor series about $z = 1$. We write

$$f(z) = \frac{1}{4-(z-1)^2} = \frac{1/4}{1-[(z-1)/2]^2} = \frac{1}{4}\sum_{n=0}^{\infty} \left[\left(\frac{z-1}{2}\right)^2\right]^n = \sum_{n=0}^{\infty} \frac{(z-1)^{2n}}{4^{n+1}},$$

  which converges on $|z-1| < 2$.

  This power series only has non-negative powers of $z-1$, which is to be expected as f is analytic on the disc $|z-1| < 2$ whose center is $z = 1$.

We end this lecture with the Casorati-Weierstrass Theorem, which is another useful characterization of essential singularities.

THEOREM 28.11 (Casorati-Weierstrass). Let z_0 be an essential singularity of f. Then for every $\varepsilon > 0$, $r > 0$, and $w \in \mathbb{C}$, there exists $z \in D(z_0; r) \setminus \{z_0\}$ such that $f(z) \in D(w; \varepsilon)$. In other words, the function f assumes values arbitrarily close to any complex number $w \in \mathbb{C}$.

PROOF. Assume towards a contradiction that we have

$$|f(z) - w| \geq \varepsilon > 0$$

for all $z \in D(z_0; r) \setminus \{z_0\}$. Set $g(z) = 1/(f(z) - w)$. Since z_0 is an isolated singularity of f, it is also an isolated singularity of $f(z) - w$. But

$$|g(z)| = \frac{1}{|f(z) - w|} \leq \frac{1}{\varepsilon}$$

for all $z \in D(z_0; r) \setminus \{z_0\}$, implying that z_0 is a removable singularity of g by Riemann's Criterion for Removable Singularities (Theorem 26.5). Hence, the limit

$$\lim_{z \to z_0} g(z) = w_0$$

exists. Setting $g(z_0) = w_0$, we have that g is also analytic at z_0. Now, if $w_0 = 0$, then z_0 is an isolated zero of g, implying that z_0 is a pole of $1/g(z) = f(z) - w$ and hence also of f, a contradiction. If $w_0 \neq 0$, then $1/g$ has a removable singularity at z_0, so f also has a removable singularity at z_0, another contradiction. $\square$

29 Boundary behaviour of power series and analytic continuation

In this lecture, we consider the behaviour of a function on the boundary of its disc of convergence.

Recall Proposition 23.4, which gives us a method of determining the radius convergence R and the disc of convergence $D(z_0; R)$ of a given power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ by computing the limit $\lim_{n\to\infty} |a_{n+1}/a_n|$. In particular, we have a description of where the power series converges in $\mathbb{C}$ except on the boundary $|z-z_0| = R$ of the disc of convergence. What happens at the boundary? The answer is that anything can happen; however, some things can be said in certain cases. We begin by looking at some examples.

Example 29.1. Consider the geometric series $\sum_{n=0}^{\infty} z^n$, which converges to $1/(1-z)$ on $|z| < 1$, but diverges for all $|z| \geq 1$ as $|z^n| \nrightarrow 0$ as $n \to \infty$. Nonetheless, we have

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

on $|z| < 1$. However, note that $1/(1-z)$ is analytic everywhere except at $z = 1$, so we can think of $1/(1-z)$ as the "analytic continuation" of the geometric series to $\mathbb{C} \setminus \{1\}$.

Example 29.2. Consider the power series $\sum_{n=1}^{\infty} z^n/n$. One readily checks that the radius of convergence of the power series is $R = 1$. Moreover, when $z = 1$, we obtain the harmonic series $\sum_{n=1}^{\infty} 1/n$, which diverges. However, it converges for every point on the boundary $|z| = 1$ which is not equal to 1. This is a consequence of the following test, which is due to Abel.

Proposition 29.3 (Abel's Test). Consider the power series

$$\sum_{n=1}^{\infty} a_n z^n,$$

and suppose that $\{a_n\}$ is a decreasing sequence of positive real numbers such that $\lim_{n\to\infty} a_n = 0$. Then the power series converges on $|z| = 1$ except possibly at $z = 1$.

Proof. Exercise. $\square$

Example 29.4. Consider the power series $\sum_{n=1}^{\infty} z^n/n^2$. The radius of convergence is again $R = 1$. When $z = 1$, we obtain the p-series $\sum_{n=1}^{\infty} 1/n^2$ with $p = 2$, which converges. Finally, the power series converges at every point on $|z| = 1$ by Abel's Test.

In fact, we have an even stronger result than Abel's Test.

Theorem 29.5 (Abel's Theorem). Suppose that the power series

$$f(z) = \sum_{n=1}^{\infty} a_n(z-z_0)^n$$

has radius of convergence $0 < R < \infty$, and that $\sum_{n=1}^{\infty} a_n$ converges. Then for any z_1 on the boundary $|z-z_0| = R$, we have $f(z) \to f(z_1)$ as $z \to z_1$ along a circle arc of $|z-z_0| = R$.

Consider the power series $f(z) = \sum_{n=1}^{\infty} a_n(z-z_0)^n$ and suppose that it has radius of convergence $0 < R < \infty$ so that it converges on the open disc $D(z_0; R)$. Then for any $z_0 \neq z_1 \in D(z_0; R)$, we have $0 < r := R - |z_1 - z_0| < R$, so the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_1)}{n!}(z-z_1)^n$$

converges to f on $D(z_1; r) \subseteq D(z_0; R)$ by Taylor's Theorem. However, in some cases, the above power series may have a larger radius of convergence than $r = R - |z_1 - z_0|$ so that it defines an **analytic continuation of** f to points outside of $|z-z_0| < R$. Note that analytic continuations, if they exist, are unique by the Identity Theorem.

Example 29.6. Consider the geometric series

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

on $|z| < 1$ centered at $z_0 = 0$. Then $z_1 = -1/2$ lies in $D(z_0; 1)$ with $r = 1 - |z_1 - z_0| = 1/2$. Moreover, we have

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} \frac{2^{n+1}}{3^{n+1}} (z + 1/2)^n$$

on $D(z_1; r) = D(-1/2; 1/2)$. However, it is easily checked that this power series has a larger radius of convergence than $1/2$; namely, it converges on $D(-1/2, 3/2)$, which contains points outside of $D(z_0; 1)$. Therefore, it is an analytic continuation of the geometric series outside $|z| < 1$.

Weierstrass' original method of continuation of an analytic function defined by a power series was initially to pick points inside the disc of convergence and take the Taylor series expansion about those points to obtain an analytic function on a bigger domain. However, note that one cannot use analytic continuation to increase the radius of convergence of the initial power series.

Proposition 29.7. Let R be the radius of convergence of the power series $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ and suppose that $0 < R < \infty$. Then there must exist at least one point z_1 on the boundary $|z - z_0| = R$ such that f cannot be analytically continued on any disc containing z_1.

Proof. Let $D = D(z_0; R)$ be the disc of convergence of the power series, and let $B = \{z \in \mathbb{C} : |z - z_0| = R\}$ be its boundary circle. Suppose to the contrary that for every $b \in B$, there exists an analytic continuation g_b of f to a domain D_b containing b. Let $\hat{D} = \bigcup_{b \in B} D_b$.

For any $b_1, b_2 \in B$, since g_{b_1} and g_{b_2} must agree on $D_{b_1} \cap D_{b_2} \cap D$, they must also agree on $D_{b_1} \cap D_{b_2}$ by the Identity Theorem. Thus, the functions g_b glue together to form a well-defined function g on $\hat{D}$. Since $\hat{D}$ is open and B is closed, the distance from B to the boundary $\partial\hat{D}$ of $\hat{D}$ is positive, which implies that the distance from z_0 to $\partial\hat{D}$ is greater than R. In particular, this means that there are points on $|z - z_0| > R$ where the power series convergences, contradicting the fact that R is the radius of convergence of the power series. Thus, there must exist at least one point z_1 on the boundary B such that f cannot be analytically continued on any disc containing z_1. $\square$

At the end of the course, we will revisit analytic continuation and discuss it in more detail.

30 Residue Theorem

In this lecture, we will see how residues can be used to compute certain contour integrals.

Recall that if z_0 is an isolated singularity of f, then f admits a Laurent series $\sum_{n=-\infty}^{\infty} c_n(z-z_0)^n$ on a punctured disc $D = \{z \in \mathbb{C} : 0 < |z-z_0| < r\}$ for some $r > 0$, where the coefficients are given by

$$c_n = \frac{1}{2\pi i}\int_\Gamma \frac{f(s)}{(s-z_0)^{n+1}}\,\mathrm{d}s$$

for all $n \in \mathbb{Z}$ and Γ is any positively oriented simple closed curve in D whose interior contains z_0. The coefficient c_{-1} was called the residue of f at z_0. More generally, we have the following definition.

Definition 30.1 (Residue). Let f be analytic on the punctured domain $\Omega \setminus \{z_0\}$. The **residue of** f **at** z_0 is defined to be

$$\mathrm{Res}(f; z_0) := \frac{1}{2\pi i}\int_\Gamma f(s)\,\mathrm{d}s$$

for any positively oriented simple closed curve Γ in $\Omega \setminus \{z_0\}$ whose interior contains z_0. We also denote it by $\mathrm{Res}_{z=z_0}(f(z))$ or $\mathrm{Res}(z_0)$.

Note that by the Deformation Principle, the residue of f at z_0 is equal to the coefficient c_{-1} of the Laurent series of f about z_0.

Example 30.2.

(1) If $f(z) = 1/(z-z_0)^m$, then $\mathrm{Res}(z_0) = 1$ if $m = 1$ and $\mathrm{Res}(z_0) = 0$ if $m \neq 1$.

(2) The function $f(z) = z^2e^{1/z}$ has an isolated singularity at $z = 0$ with Laurent series expansion

$$z^2e^{1/z} = z^2\sum_{n=0}^{\infty}\frac{(1/z)^n}{n!} = z^2 + z + \frac{1}{2!} + \frac{1}{3!z} + \frac{1}{4!z^2} + \cdots.$$

In particular, we have

$$\mathrm{Res}_{z=0}(z^2e^{1/z}) = \frac{1}{3!} = \frac{1}{6}.$$

Note that $z = 0$ is an essential singularity of f since the Laurent series of $z^2e^{1/2}$ about $z = 0$ has infinitely many negative powers of z.

(3) The function $f(z) = e^{2z}/(z-1)$ has an isolated singularity at $z = 1$ with Laurent series expansion

$$\begin{aligned} f(z) = \frac{e^{2(z-1)+2}}{z-1} &= \frac{e^2}{z-1}\sum_{n=0}^{\infty}\frac{(2(n-1))^n}{n!} \\ &= \sum_{n=0}^{\infty}\frac{e^2 2^n(z-1)^{n-1}}{n!} \\ &= \frac{e^2}{z-1} + \frac{e^2}{2!} + \frac{e^2}{3!}(z-1) + \frac{e^2}{4!}(z-1)^2 + \cdots. \end{aligned}$$

Hence, we have

$$\mathrm{Res}_{z=1}(f(z)) = e^2.$$

Observe that $z = 1$ is a pole of f of order 1 since the Laurent series of f about $z = 1$ has a single negative power of $z-1$, namely $(z-1)^{-1}$.

By definition, we have that

$$\int_\Gamma f(s)\,\mathrm{d}s = 2\pi i\,\mathrm{Res}(f; z_0)$$

for any positively oriented simple closed curve Γ in $\Omega \setminus \{z_0\}$ whose interior contains z_0. Therefore, residues give us a way of computing integrals around isolated singularities.

Example 30.3.

(1) If Γ is the unit circle traversed once counter-clockwise, then

$$\int_\Gamma z^2 e^{1/z}\,\mathrm{d}z = 2\pi i\,\mathrm{Res}_{z=0}(z^2 e^{1/2}) = 2\pi i \cdot \frac{1}{6} = \frac{\pi i}{3}.$$

(2) If Γ is the circle $|z-1| = 3$ traversed once counter-clockwise, then

$$\int_\Gamma \frac{e^{2z}}{z-1}\,\mathrm{d}z = 2\pi i\,\mathrm{Res}_{z=1}\left(\frac{e^{2z}}{z-1}\right) = 2\pi i \cdot e^2 = 2\pi e^2 i.$$

This can be extended to the case where there are finitely many isolated singularities, which gives rise to the Residue Theorem.

Theorem 30.4 (Residue Theorem). If f has finitely many isolated singularities $z_1, \ldots, z_n$ which are interior to a positively oriented simple closed curve Γ, then

$$\int_\Gamma f(z)\,\mathrm{d}z = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f; z_k).$$

Proof. For all $1 \le k \le n$, let C_k be a circle centered at z_k traversed once counter-clockwise, whose radius is chosen small enough to ensure that it is contained in Γ and does not intersect any of the other circles. First, note that

$$\int_{C_k} f(z)\,\mathrm{d}z = 2\pi i\,\mathrm{Res}(f; z_k)$$

for all $1 \le k \le n$. Moreover, $\Gamma - C_1 - \cdots - C_n$ is the boundary of an $(n+1)$-connected Jordan curve on which f is analytic. Hence, by the Cauchy Integral Theorem, we have

$$\int_{\Gamma - C_1 - \cdots - C_n} f(z)\,\mathrm{d}z = 0,$$

so it follows that

$$\int_\Gamma f(z)\,\mathrm{d}z = \sum_{k=1}^{n} \int_{C_k} f(z)\,\mathrm{d}z = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f; z_k). \qquad \square$$

Example 30.5. Let $f(z) = 1/[(3-z)(1+z)]$, and let Γ be the circle $|z| = 4$ traversed once counter-clockwise. Note that f has two isolated singularities at $z = -1$ and $z = 2$. Moreover, its Laurent series representation about these points are

$$f(z) = \sum_{n=-1}^{\infty} \frac{(z+1)^n}{4^{n+2}},$$

$$f(z) = \sum_{n=-1}^{\infty} \frac{(-1)^n (z-3)^n}{4^{n+2}}$$

respectively. Therefore, we see that $\mathrm{Res}_{z=-1}(f(z)) = 1/4$ and $\mathrm{Res}_{z=3}(f(z)) = -1/4$. Moreover, since both of these singularities lie inside Γ, the Residue Theorem gives

$$\int_\Gamma f(z)\,\mathrm{d}z = 2\pi i(\mathrm{Res}_{z=-1}(f(z)) + \mathrm{Res}_{z=3}(f(z))) = 2\pi i\left(\frac{1}{4} - \frac{1}{4}\right) = 0.$$

In some cases, it is not necessary to find the Laurent series of a function to find the residue. We will look at two specific cases.

PROPOSITION 30.6. Suppose that f has a pole at z_0 of order m so that

$$f(z) = \frac{g(z)}{(z-z_0)^m}$$

where g is analytic and non-zero at z_0. Then we have

$$\mathrm{Res}_{z=z_0}(f(z)) = \frac{g^{(m-1)}(z_0)}{(m-1)!} = \frac{1}{(m-1)!}\lim_{z\to z_0}\frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}}((z-z_0)^m f(z)).$$

PROOF. Note that $g(z) = (z-z_0)^m f(z)$. Moreover, since f has a pole of order m at z_0, its Laurent series about z_0 is of the form

$$f(z) = \frac{c_{-m}}{(z-z_0)^m} + \cdots + \frac{c_{-1}}{z-z_0} + \sum_{n=0}^{\infty} c_n(z-z_0)^n$$

with $c_{-m} \neq 0$. Hence, we have

$$g(z) = c_{-m} + \cdots + c_{-1}(z-z_0)^{m-1} + \sum_{n=0}^{\infty} c_n(z-z_0)^{n+m}.$$

Then, we see that

$$g^{(m-1)}(z) = c_{-1}(m-1)! + \sum_{n=0}^{\infty} c_n(n+m)\cdots(n+2)(z-z_0)^{n+1}$$

so that $g^{(m-1)}(z_0) = c_{-1}(m-1)!$. We conclude that

$$\mathrm{Res}_{z=z_0}(f(z)) = c_{-1} = \frac{g^{(m-1)}(z_0)}{(m-1)!}.$$

□

EXAMPLE 30.7.

(1) Consider the function $f(z) = \cos(z)/z^3$. We have $f(z) = g(z)/z^3$ where $g(z) = \cos z$ is analytic and non-zero at $z = 0$, so f has a pole of order 3 at $z = 0$ with

$$\mathrm{Res}_{z=0}(f(z)) = \frac{g''(0)}{2!} = \frac{-\cos(0)}{2!} = -\frac{1}{2}.$$

(2) The function $f(z) = e^z/(z^2-1)^2$ has poles of order 2 at $z = \pm 1$. At $z = -1$, we have $f(z) = g(z)/(z+1)^2$ where $g(z) = e^2/(z-1)^2$ so that $g'(z) = e^z(z+1)/(z-1)^3$. Hence, we obtain

$$\mathrm{Res}_{z=-1}(f(z)) = \frac{g'(-1)}{1!} = 0.$$

Similarly, we have $\mathrm{Res}_{z=1}(f(z)) = 0$.

(3) **Simple poles.** If z_0 is a simple pole of f, then we have

$$\mathrm{Res}_{z=z_0}(f(z)) = \lim_{z\to z_0}(z-z_0)f(z).$$

In this case, residues are easy to compute. For instance, the function $f(z) = 1/[(z-3)(z+1)]$ has simple poles at $z = -1$ and $z = 3$ with residues

$$\mathrm{Res}_{z=-1}(f(z)) = \lim_{z\to -1}(z+1)f(z) = \lim_{z\to -1}\frac{1}{z-3} = -\frac{1}{4},$$
$$\mathrm{Res}_{z=3}(f(z)) = \lim_{z\to 3}(z-3)f(z) = \lim_{z\to 3}\frac{1}{z+1} = \frac{1}{4}.$$

Another case where the residue is straightforward to compute is given by the following proposition.

PROPOSITION 30.8. Suppose that $f(z) = p(z)/q(z)$ is quotient of functions that are analytic at z_0. Moreover, suppose that $p(z_0)$ and $q'(z_0)$ are non-zero, whereas $q(z_0) = 0$ so that f has a simple pole at z_0. Then we have

$$\mathrm{Res}_{z=z_0}(f(z)) = \frac{p(z_0)}{q'(z_0)}.$$

PROOF. Exercise. $\square$

EXAMPLE 30.9. Let $f(z) = e^{2z}/(z-1)$. We have $f(z) = p(z)/q(z)$ where $p(z) = e^{2z}$ and $q(z) = z-1$. Since $p(1) = e^2$ and $q'(1) = 1$ are both non-zero and $q(1) = 1$, we obtain

$$\mathrm{Res}_{z=1}(f(z)) = \frac{p(1)}{q'(1)} = e^2.$$

31 Computing improper integrals with the Residue Theorem

In this lecture, we will see how the Residue Theorem can be used to compute certain real improper integrals. We begin by recalling some facts about improper integrals.

Let $f : (a, \infty) \to \mathbb{R}$ be an integrable function. The **improper integral** of f on (a, ∞) is defined to be

$$\int_a^\infty f(x)\,\mathrm{d}x := \lim_{R_1\to\infty} \int_a^{R_1} f(x)\,\mathrm{d}x,$$

if it exists. Similarly, if $f : (-\infty, a)$ is integrable, then

$$\int_{-\infty}^a f(x)\,\mathrm{d}x := \lim_{R_2\to\infty} \int_{-R_2}^{a} f(x)\,\mathrm{d}x,$$

provided it exists. Finally, if f is integrable on $(-\infty, \infty)$ and both limits exist, then

$$\int_{-\infty}^\infty f(x)\,\mathrm{d}x := \int_{-\infty}^a f(x)\,\mathrm{d}x + \int_a^\infty f(x)\,\mathrm{d}x = \lim_{R_2\to\infty} \int_{-R_2}^{a} f(x)\,\mathrm{d}x + \lim_{R_1\to\infty} \int_a^{R_1} f(x)\,\mathrm{d}x.$$

Example 31.1. Consider the improper integral

$$\int_{-\infty}^\infty \frac{1}{1+x^2}\,\mathrm{d}x.$$

In this case, the integrand $f(x) = 1/(1+x^2)$ is integrable on $(-\infty, \infty)$ with antiderivative $\arctan x$. Moreover, since $f(-x) = f(x)$ for all $x \in \mathbb{R}$, we have

$$\int_{-R}^0 f(x)\,\mathrm{d}x = \int_0^R f(x)\,\mathrm{d}x = [\arctan x]_0^R = \arctan R$$

for all $R \geq 0$. Therefore, we obtain

$$\lim_{R\to\infty} \int_{-R}^0 f(x)\,\mathrm{d}x = \lim_{R\to\infty} \int_0^R f(x)\,\mathrm{d}x = \lim_{R\to\infty} \arctan R = \frac{\pi}{2}.$$

Finally, since both limits exist, we have

$$\int_{-\infty}^\infty \frac{1}{1+x^2}\,\mathrm{d}x = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

Note that in this example, we were able to easily compute the integral because we had an explicit expression for the antiderivative of the integrand.

However, there are many important examples where we do not have explicit antiderivatives. In such cases, one typically has to be more creative.

Example 31.2. Consider the improper integral

$$I = \int_{-\infty}^\infty e^{-x^2}\,\mathrm{d}x.$$

Observe that $f(x) = e^{-x^2}$ is integrable on $(-\infty, \infty)$ as it is continuous there, but it does not have an explicit antiderivative. However, note that

$$\begin{aligned} I^2 &= \left(\int_{-\infty}^\infty e^{-x^2}\,\mathrm{d}x\right)\left(\int_{-\infty}^\infty e^{-y^2}\,\mathrm{d}y\right) = \iint_{\mathbb{R}^2} e^{-x^2-y^2}\,\mathrm{d}y \\ &= \lim_{R\to\infty} \int_0^{2\pi}\int_0^R e^{-r^2} r\,\mathrm{d}r\,\mathrm{d}\theta = \lim_{R\to\infty} 2\pi\left(1 - e^{-R^2}\right) = 2\pi. \end{aligned}$$

Consequently, we have $I = \sqrt{2\pi}$.

For the previous example, we were able to use polar coordinates to compute the double integral because the integrand was the composition of an elementary function with $x^2 + y^2 = r^2$.

In the examples we have discussed so far, the functions we considered were both even. Notice that if f is an even function, then

$$\int_{-R}^{0} f(x)\,\mathrm{d}x = \int_{0}^{R} f(x)\,\mathrm{d}x$$

for all $R \geq 0$, implying that $\int_{-\infty}^{0} f(x)\,\mathrm{d}x = \int_{0}^{\infty} f(x)\,\mathrm{d}x$ and

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 2\int_{0}^{\infty} f(x)\,\mathrm{d}x = \lim_{R\to\infty} 2\int_{0}^{R} f(x)\,\mathrm{d}x = \lim_{R\to\infty}\int_{-R}^{R} f(x)\,\mathrm{d}x.$$

Thus, another way to tackle the problem is to view the interval $[-R, R]$ as a line segment on the real axis. Consider the simple closed curve

$$\Gamma = [-R, R] + C_R^+$$

in $\mathbb{C}$ which is positively oriented, with

$$\begin{aligned} [-R, R] &= \{z \in \mathbb{C} : -R \leq x \leq R,\ y = 0\}, \\ C_R^+ &= \{z \in \mathbb{C} : |z| = R,\ y \geq 0\}. \end{aligned}$$

Then, we have

$$\int_{-R}^{R} f(x)\,\mathrm{d}x = \int_{-R}^{R} f(z)\,\mathrm{d}z = \int_{\Gamma} f(z)\,\mathrm{d}z - \int_{C_R^+} f(z)\,\mathrm{d}z,$$

where we can think of f as a complex function by replacing the real variable x with the complex variable z in its expression. We now wish to use the Residue Theorem to compute $\int_\Gamma f(z)\,\mathrm{d}z$, and apply the ML-inequality to obtain an upper bound for $\int_{C_R^+} f(z)\,\mathrm{d}z$. This will allow us to compute the limit

$$\lim_{R\to\infty}\int_{-R}^{R} f(x)\,\mathrm{d}x = \lim_{R\to\infty}\int_{-R}^{R} f(z)\,\mathrm{d}z.$$

This method can be used to compute improper integrals of the form

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\,\mathrm{d}x,\ \int_{-\infty}^{\infty} \sin(ax)\frac{p(x)}{q(x)}\,\mathrm{d}x,\ \int_{-\infty}^{\infty} \cos(ax)\frac{p(x)}{q(x)}\,\mathrm{d}x,$$

where $p(x)/q(x)$ is a rational function. First, we make the following definition.

DEFINITION 31.3 (Cauchy principal value). The **Cauchy principal value** of an improper integral of a function f on $(-\infty, \infty)$ is defined as

$$\text{p.v.}\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x := \lim_{R\to\infty}\int_{-R}^{R} f(x)\,\mathrm{d}x.$$

It is important to note that the Cauchy principal value might not be equal to the actual improper integral. For instance, consider

$$\int_{-\infty}^{\infty} x\,\mathrm{d}x,$$

which does not exist since

$$\int_{0}^{\infty} x\,\mathrm{d}x = \lim_{R\to\infty}\int_{0}^{R} x\,\mathrm{d}x = \lim_{R\to\infty}\frac{R^2}{2} = \infty,$$

and similarly $\int_{-\infty}^{0} x\,\mathrm{d}x = -\infty$, which both do not exist. However, we have

$$\int_{-R}^{R} x\,\mathrm{d}x = 0$$

for all $R \geq 0$ so that

$$\text{p.v.} \int_{-\infty}^{\infty} x \, \mathrm{d}x = \lim_{R \to \infty} \int_{-R}^{R} x \, \mathrm{d}x = 0.$$

However, we mentioned previously that if f is even, then

$$\int_{-\infty}^{\infty} f(x) \, \mathrm{d}x = \lim_{R \to \infty} \int_{-R}^{R} f(x) \, \mathrm{d}x = \text{p.v.} \int_{-\infty}^{\infty} f(x) \, \mathrm{d}x.$$

In particular, the Cauchy principal value coincides with the actual improper integral in the case that f is an even function. Since many important examples involve even functions, this gives us an effective method for computing their improper integrals.

For the remainder of this lecture, we will set $\Gamma = [-R, R] + C_R^+$, where

$$\begin{aligned} [-R, R] &= \{z \in \mathbb{C} : -R \leq x \leq R, \, y = 0\}, \\ C_R^+ &= \{z \in \mathbb{C} : |z| = R, \, y \geq 0\}, \end{aligned}$$

which we will orient positively.

We start by considering integrals of the form

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \, \mathrm{d}x,$$

where $p(x)/q(x)$ is a rational function.

Example 31.4. Consider the integral

$$\int_{-\infty}^{\infty} \frac{1}{x^4 + 4} \, \mathrm{d}x = \lim_{R \to \infty} \int_{-R}^{R} \frac{1}{x^4 + 4} \, \mathrm{d}x,$$

where the equality holds since $f(x) = 1/(x^4 + 4)$ is an even function. Note that f is defined and continuous everywhere, but this is an improper integral as we are integrating over $(-\infty, \infty)$. Moreover, we do not have an explicit expression for the antiderivative of f. However, we know that

$$\int_{-R}^{R} f(x) \, \mathrm{d}x = \int_{\Gamma} f(z) \, \mathrm{d}z - \int_{C_R^+} f(z) \, \mathrm{d}z.$$

We now focus on computing $\int_\Gamma f(z) \, \mathrm{d}z$. The integrand

$$f(z) = \frac{1}{z^4 + 4}$$

is a rational function with simple poles at the four zeroes of the denominator, given by

$$z_0 = \sqrt{2} e^{k\pi i/4}, \; k = 1, 3, 5, 7$$

and each with multiplicity one. More explicitly, we have

$$z_0 = 1 + i, \, -1 + i, \, -1 - i, \, 1 - i.$$

Note that for sufficiently large R, only two of these poles lie inside Γ, namely $z_1 = 1 + i = \sqrt{2} e^{\pi i/4}$ and $z_2 = -1 + i = \sqrt{2} e^{3\pi i/4}$. Moreover, since $f(z) = p(z)/q(z)$ with $p(z) = 1$ and $q(z) = z^4 + 4$ both entire, and $q'(z_0) = 4z_0^3 \neq 0$, we have that

$$\operatorname{Res}(f; z_0) = \frac{p(z_0)}{q'(z_0)} = \frac{1}{4z_0^3} = \frac{1}{8\sqrt{2} e^{3k\pi i/4}} = \frac{\sqrt{2} e^{-3k\pi i/4}}{16}$$

if $z_0 = \sqrt{2}e^{k\pi i/4}$. Hence, by the Residue Theorem, we obtain

$$\begin{aligned}\int_\Gamma \frac{1}{z^4+4}\,\mathrm{d}z &= 2\pi(\mathrm{Res}(f;z_1)+\mathrm{Res}(f;z_2)) \\ &= 2\pi i\left(\frac{1}{4z_1^3}+\frac{1}{4z_2^3}\right) \\ &= 2\pi i\left(\frac{\sqrt{2}e^{-3\pi i/4}}{16}+\frac{\sqrt{2}e^{-9\pi i/4}}{16}\right) \\ &= \frac{2\pi i}{16}(-1+i+(1+i)) = \frac{\pi}{4}.\end{aligned}$$

We now consider $\int_{C_R^+} f(z)\,\mathrm{d}z$. Note that C_R^+ is not a closed curve, so we cannot apply the Cauchy Integral Theorem, the Cauchy Integral Formula, or the Residue Theorem here. However, we can use the ML-inequality to attain an upper bound on its modulus. Indeed, note that $|z^4+4| \geq R^4-4$ on C_R^+ so that

$$\left|\frac{1}{z^4+4}\right| \leq \frac{1}{R^4-4} =: M.$$

Moreover, C_R^+ has arclength $L = \pi R$. Hence, we see that

$$\left|\int_{C_R^+}\frac{1}{z^4+4}\,\mathrm{d}z\right| \leq ML = \frac{\pi R}{R^4-4} \xrightarrow{R\to\infty} 0,$$

which implies that

$$\lim_{R\to\infty}\int_{C_R^+}\frac{1}{z^4+4}\,\mathrm{d}z = 0.$$

Finally, we obtain

$$\int_{-\infty}^{\infty}\frac{1}{x^4+4}\,\mathrm{d}x = \lim_{R\to\infty}\int_{-R}^{R}\frac{1}{x^4+4}\,\mathrm{d}x = \lim_{R\to\infty}\left(\int_\Gamma\frac{1}{z^4+4}\,\mathrm{d}z - \int_{C_R^+}\frac{1}{z^4+4}\,\mathrm{d}z\right) = \frac{\pi}{4}-0 = \frac{\pi}{4}.$$

Example 31.5. Consider the improper integral

$$I = \int_0^\infty \frac{x^2}{(x^2+1)(x^2+4)}\,\mathrm{d}x.$$

The function

$$f(x) = \frac{x^2}{(x^2+1)(x^2+4)}$$

is even and defined on all of $(-\infty,\infty)$, so we find that

$$I = \lim_{R\to\infty}\int_0^R f(x)\,\mathrm{d}x = \frac{1}{2}\lim_{R\to\infty}\int_{-R}^R f(x)\,\mathrm{d}x$$

where as usual, we have

$$\int_{-R}^R f(x)\,\mathrm{d}x = \int_\Gamma f(z)\,\mathrm{d}z - \int_{C_R^+} f(z)\,\mathrm{d}z.$$

To compute $\int_\Gamma f(z)\,\mathrm{d}z$, observe that

$$f(z) = \frac{z^2}{(z^2+1)(z^2+4)}$$

has simple poles at $z_0 = \pm i$, $\pm 2i$. Moreover, we have $f(z) = p(z)/q(z)$ with $p(z) = z^2$ and $q(z) = (z^2+1)(z^2+4)$ so that $p(z_0) \neq 0$ and $q'(z_0) \neq 0$. Hence, we obtain

$$\operatorname{Res}(f; z_0) = \frac{p(z_0)}{q'(z_0)} = \frac{z_0^2}{2z_0(2z_0^2+5)} = \frac{z_0}{2(2z_0^2+5)}.$$

For R sufficiently large, f is analytic on Γ and the only singularities inside Γ are i and $2i$. By the Residue Theorem, we get

$$\int_\Gamma \frac{z^2}{(z^2+1)(z^2+4)}\,\mathrm{d}z = 2\pi(\operatorname{Res}(f;i) + \operatorname{Res}(f;2i)) = 2\pi i\left(\frac{i}{2(2i^2+5)} + \frac{2i}{2(2(2i)^2+5)}\right) = \frac{\pi}{3}.$$

We now consider $\int_{C_R^+} f(z)\,\mathrm{d}z$. Again, we use the ML-inequality to find an upper bound on its modulus. Note that $|z|^2 = R^2$, $|z^2+1| \geq R^2 - 1$, and $|z^2+4| \geq R^2 - 4$ on C_R^+ so that

$$|f(z)| \leq \frac{R^2}{(R^2-1)(R^4-4)} =: M.$$

We also know that C_R^+ has arclength $L = \pi R$, so by the ML-inequality, we obtain

$$\left|\int_{C_R^+} \frac{z^2}{(z^2+1)(z^2+4)}\,\mathrm{d}z\right| \leq ML = \frac{\pi R^3}{(R^2-1)(R^2-4)} \xrightarrow{R\to\infty} 0.$$

It follows that

$$\begin{aligned}
\int_0^\infty \frac{x^2}{(x^2+1)(x^2+4)}\,\mathrm{d}x &= \frac{1}{2}\lim_{R\to\infty}\int_{-R}^{R} \frac{x^2}{(x^2+1)(x^2+4)}\,\mathrm{d}x \\
&= \frac{1}{2}\lim_{R\to\infty}\left(\int_\Gamma \frac{z^2}{(z^2+1)(z^2+4)}\,\mathrm{d}z - \int_{C_R^+} \frac{z^2}{(z^2+1)(z^2+4)}\,\mathrm{d}z\right) \\
&= \frac{1}{2}\left(\frac{\pi}{3} - 0\right) = \frac{\pi}{6}.
\end{aligned}$$

In general, we have the following result.

Proposition 31.6. If $f(z) = p(z)/q(z)$ is a rational function with $\deg(p) \leq \deg(q) - 2$, then

$$\lim_{R\to\infty}\int_{C_R^+} f(z)\,\mathrm{d}z = 0.$$

Proof. Suppose that $p(z) = \sum_{i=0}^m a_i z^i$ and $q(z) = \sum_{j=0}^n b_j z^j$ with $m \leq n-2$ so that $m+1-n < 0$. Then on C_R^+, we have

$$|f(z)| = \frac{|p(z)|}{|q(z)|} \leq \frac{|a_m|R^m + \cdots + |a_1|R + |a_0|}{|b_n|R^n - \cdots - |b_1|R - |b_0|} =: M.$$

Moreover, C_R^+ has arclength $L = \pi R$, so the ML-inequality gives

$$\left|\int_{C_R^+} f(z)\,\mathrm{d}z\right| \leq ML = \frac{\pi(|a_m|R^{m+1} + \cdots + |a_0|R)}{|b_n|R^n - \cdots - |b_0|} \xrightarrow{R\to\infty} 0$$

since $m+1-n < 0$, and the result follows. □

Due to this proposition, we immediately obtain the following corollary.

Corollary 31.7. Suppose that $f(z) = p(z)/q(z)$ is a rational function that has no poles on the real axis and is such that $\deg(p) \le \deg(q) - 2$. Then

$$\text{p.v.} \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f; z_k),$$

where $z_1, \ldots, z_n$ are the poles of f in the upper-half plane.

Example 31.8. Consider the improper integral

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2}\,\mathrm{d}x.$$

Then the integrand $f(x) = x^2/(x^2+1)^2$ is an even function so that

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \text{p.v.} \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x.$$

Moreover, $f(z) = p(z)/q(z)$ is a rational function with $p(z) = z^2$ and $q(z) = (z^2+1)^2$. Since $z_1 = i$ is its only pole inside Γ and $\deg(p) \le \deg(q) - 2$, we have

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2}\,\mathrm{d}x = 2\pi i \operatorname{Res}(f; i) = \frac{\pi}{2}.$$

32 Improper and definite integrals involving sine and cosine

We now turn to improper integrals of the form

$$\int_{-\infty}^{\infty} \sin(ax)\frac{p(x)}{q(x)}\,\mathrm{d}x,\ \int_{-\infty}^{\infty} \cos(ax)\frac{p(x)}{q(x)}\,\mathrm{d}x,$$

where $p(x)/q(x)$ is a rational function. These improper integrals often appear in Fourier analysis. Before giving a general method of computing such integrals, we begin with an example.

Example 32.1. Consider the improper integral

$$I = \int_{-\infty}^{\infty} \frac{x\sin(2x)}{(x^2+4)^2}\,\mathrm{d}x.$$

Notice that

$$\frac{x\sin(2x)}{(x^2+4)^2} = \Im\left(\frac{xe^{i2x}}{(x^2+4)^2}\right)$$

is an even function which is defined on $(-\infty,\infty)$, so we have

$$I = \Im\left(\lim_{R\to\infty}\int_{-R}^{R} \frac{xe^{i2x}}{(x^2+4)^2}\,\mathrm{d}x\right).$$

Now, set

$$f(z) = \frac{ze^{i2z}}{(z^2+4)^2}.$$

Then f has a pole of order 2 at $z_0 = \pm 2i$. Note that $z_0 = 2i$ is the only pole inside Γ for sufficiently large R. Moreover, we have

$$\mathrm{Res}(f;2i) = \lim_{z\to 2i}\frac{\mathrm{d}}{\mathrm{d}z}((z-2i)^2 f(z)) = \frac{1}{4e^4}.$$

As usual, we have

$$\int_{-R}^{R} f(x)\,\mathrm{d}x = \int_{\Gamma} f(z)\,\mathrm{d}z - \int_{C_R^+} f(z)\,\mathrm{d}z.$$

By the Residue Theorem, we obtain

$$\int_{\Gamma} f(z)\,\mathrm{d}z = 2\pi i\,\mathrm{Res}(f;2i) = \frac{\pi i}{2e^4}.$$

Moreover, C_R^+ has arclength $L = \pi R$ and

$$|f(z)| = \frac{|z||e^{i2z}|}{|z^2+4|^2} = \frac{Re^{-2y}}{|z^2+4|^2} \le \frac{R}{(R^2-4)^2} =: M$$

on C_R^+ since $|z| = R$ and $y \ge 0$ there. It follows from the ML-inequality that

$$\left|\int_{C_R^+} f(z)\,\mathrm{d}z\right| \le ML = \frac{\pi R^2}{(R^2-4)^2} \xrightarrow{R\to\infty} 0.$$

Finally, we get

$$I = \Im\left(\int_{\Gamma} f(z)\,\mathrm{d}z\right) = \Im\left(\frac{\pi i}{2e^4}\right) = \frac{\pi}{2e^4}.$$

In general, we have the following result.

PROPOSITION 32.2 (Jordan's Lemma). Let g be an analytic function on a domain containing the closed upper-half plane $\{z \in \mathbb{C} : y \geq 0\}$, except possibly at finitely many poles in the upper-half plane $\{z \in \mathbb{C} : y > 0\}$. Moreover, suppose that $g(z) \to 0$ uniformly as $|z| \to \infty$ in the closed upper-half plane, and let $a \in \mathbb{R}^{>0}$. Then we have

$$\lim_{R\to\infty} \int_{C_R^+} g(z)e^{iaz}\,\mathrm{d}z = 0.$$

In particular, if $g(z) = p(z)/q(z)$ is a rational function such that $\deg(q) \geq 1 + \deg(p)$, then

$$\lim_{R\to\infty} \int_{C_R^+} e^{iaz}\frac{p(z)}{q(z)}\,\mathrm{d}z = 0.$$

PROOF. Let $\varepsilon > 0$. Since $g(z) \to 0$ uniformly as $|z| \to \infty$, we have $|g(z)| < \varepsilon$ on $|z| = R$ for sufficiently large R. Moreover, if we parametrize C_R^+ by

$$z : [0, \pi] \to C_R^+ : t \mapsto Re^{it} = R\cos t + iR\sin t,$$

we have $|\mathrm{d}z| = |z'(t)|\,\mathrm{d}t = R\,\mathrm{d}t$ and

$$|e^{iaz}g(z)| = |e^{-aR\sin t + iaR\cos t}| \cdot |g(z)| = e^{-aR\sin t}|g(z)| < \varepsilon e^{-aR\sin t}.$$

By the ML-inequality, we obtain

$$\left|\int_{C_R^+} e^{iaz}g(z)\,\mathrm{d}z\right| < \int_0^\pi \varepsilon e^{-aR\sin t}R\,\mathrm{d}t = \varepsilon R\int_0^\pi e^{-aR\sin t}\,\mathrm{d}t.$$

Since $\sin(\pi - t) = \sin t$ for all $t \in [0, \pi]$, we have

$$\int_0^\pi e^{-aR\sin t}\,\mathrm{d}t = 2\int_0^{\pi/2} e^{-aR\sin t}\,\mathrm{d}t.$$

Moreover, the graph of $\sin t$ is concave down on $[0, \pi/2]$. Thus, it lies above the line segent joining $(0, \sin 0)$ and $(\pi/2, \sin(\pi/2))$. In other words, we have $\sin t \geq 2t/\pi$ for all $t \in [0, \pi/2]$. Furthermore, since $a > 0$, we obtain $-aR\sin t \leq -2aRt/\pi$ for all $t \in [0, \pi/2]$. Therefore, we have

$$\int_0^{\pi/2} e^{-aR\sin t}\,\mathrm{d}t \leq \int_0^{\pi/2} e^{-2aRt/\pi}\,\mathrm{d}t = \left[-\frac{\pi}{2aR}e^{-2aRt/\pi}\right]_0^{\pi/2} = \frac{\pi}{2aR}(1 - e^{-aR}) \leq \frac{\pi}{2aR},$$

so it follows that

$$\left|\int_{C_R^+} e^{iaz}g(z)\,\mathrm{d}z\right| < \frac{\pi\varepsilon}{2a}.$$

Since $\varepsilon > 0$ was arbitrary, we have

$$\lim_{R\to\infty} \int_{C_R^+} g(z)e^{iaz}\,\mathrm{d}z = 0. \qquad \square$$

COROLLARY 32.3. Let g be a function satisfying the conditions of Jordan's Lemma, and let $a \in \mathbb{R}^{>0}$. Suppose that $z_1, \ldots, z_n$ are the poles of $g(z)e^{iaz}$ on the upper-half plane. Then we have

$$\text{p.v.}\int_{-\infty}^{\infty} g(x)e^{iax}\,\mathrm{d}x = 2\pi i\sum_{k=1}^{n} \operatorname{Res}(g(z)e^{iaz}; z_k).$$

EXAMPLE 32.4. Consider the improper integral

$$I = \text{p.v.}\int_{-\infty}^{\infty} \frac{e^{2ix}}{x^2 + 4}\,\mathrm{d}x.$$

In this case, we have $g(z) = 1/(z^2+4)$, which is analytic everywhere in the closed upper-half plane except at the simple pole $z_0 = 2i$, and we have $a = 2 > 0$ here. By Corollary 32.3, we obtain

$$I = 2\pi i \operatorname{Res}\left(\frac{e^{2iz}}{z^2+4}; 2i\right) = \frac{\pi e^4}{2}.$$

The Residue Theorem can also be used to compute certain definite integrals of the form

$$\int_0^{2\pi} F(\sin\theta, \cos\theta)\,\mathrm{d}\theta.$$

The idea is to think of this definite integral as the integral obtained from a contour integral on the unit circle $C = \{z \in \mathbb{C} : |z| = 1\}$ parametrized by $z : [0, 2\pi] \to C : t \mapsto e^{it}$. Then we have

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - z^{-1}}{2i},$$
$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + z^{-1}}{2},$$

as well as

$$\mathrm{d}z = ie^{i\theta}\,\mathrm{d}\theta = iz\,\mathrm{d}\theta \iff \mathrm{d}\theta = \frac{\mathrm{d}z}{iz}.$$

Substituting this into the above integral yields

$$I = \int_0^{2\pi} F(\sin\theta, \cos\theta)\,\mathrm{d}\theta = \int_C F\left(\frac{z - z^{-1}}{2i}, \frac{z + z^{-1}}{2}\right)\frac{\mathrm{d}z}{iz},$$

and the Residue Theorem can be used to compute the integral on the right-hand side. This allows us to compute I in the case where there is no explicit expression for the antiderivative of $F(\sin\theta, \cos\theta)$. We illustrate this with an example.

Example 32.5. We show that for all $-1 < a < 1$, we have

$$I = \int_0^{2\pi} \frac{\mathrm{d}\theta}{1 + a\sin\theta} = \frac{2\pi}{\sqrt{1-a^2}}.$$

Indeed, by making the above substitution, we have

$$I = \int_C \frac{2/a}{z^2 + (2i/a)z - 1}\,\mathrm{d}z$$

after simplification. Notice that the integrand

$$f(z) = \frac{2/a}{z^2 + (2i/a)z - 1}$$

has two simple poles at the zeroes of the denominator $z^2 + (2i/a)z - 1$. Namely, these are

$$z_1 = \left(\frac{-1 + \sqrt{1-a^2}}{a}\right)i, \qquad z_2 = \left(\frac{-1 - \sqrt{1-a^2}}{a}\right)i.$$

Therefore, we can write the integrand in the form

$$f(z) = \frac{2/a}{(z - z_1)(z - z_2)}.$$

Now, observe that

$$|z_2| = \frac{1+\sqrt{1-a^2}}{|a|} > 1,$$
$$|z_1| = \frac{1-\sqrt{1-a^2}}{|a|} < 1,$$

so z_1 is the only pole of f lying inside C. Since it is a simple pole, we have

$$\operatorname{Res}(f; z_1) = \lim_{z \to z_1} (z - z_1) f(z) = \frac{2/a}{z_1 - z_2} = \frac{1}{i\sqrt{1-a^2}}.$$

It follows from the Residue Theorem that

$$I = 2\pi i \operatorname{Res}(f; z_1) = \frac{2\pi}{\sqrt{1-a^2}}.$$

33 The Argument Principle and Rouché's Theorem

In this lecture, we will take a look at the Argument Principle and Rouché's Theorem, which are applications of the Residue Theorem.

THEOREM 33.1 (Argument Principle). If f is analytic and non-zero everywhere on a positively oriented simple closed curve Γ, and is either analytic or has poles inside Γ, then

$$\frac{1}{2\pi i}\int_\Gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z = N_0(f) - N_p(f),$$

where $N_0(f)$ and $N_p(f)$ are the number of zeroes and poles of f inside Γ (counting multiplicity), respectively.

PROOF. The idea of the proof is fairly simple. We only have to determine what the singularities of the integrand $G(z) = f'(z)/f(z)$ are, and then compute their residues. First, note that G is analytic at every point on Γ since f is analytic and non-zero there. In addition, the singularities of G inside Γ occur at the zeroes and poles of f. Now, suppose that z_0 is a zero of f inside Γ. Then there exists a function h that is analytic and non-zero at z_0 such that

$$f(z) = (z - z_0)^m h(z)$$

for some positive integer m on some disc $D(z_0; r)$. In particular, this gives

$$\begin{aligned} f'(z) &= m(z - z_0)^{m-1}h(z) + (z - z_0)^m h'(z) \\ &= (z - z_0)^{m-1}(mh(z) - (z - z_0)h'(z)), \end{aligned}$$

which implies that

$$G(z) = \frac{(z - z_0)^{m-1}(mh(z) - (z - z_0)h'(z))}{(z - z_0)^m h(z)} = \frac{m}{z - z_0} + \frac{h'(z)}{h(z)}.$$

Note that $h'(z)/h(z)$ is analytic at z_0 since h is analytic and non-zero there. Thus, G must have a simple pole with residue m at z_0. On the other hand, let z_p be a pole of order k of f. Then there exists a function H that is analytic and non-zero at z_0 such that

$$f(z) = \frac{H(z)}{(z - z_p)^k}$$

on some disc $D(z_p; r)$. Hence, we have

$$\begin{aligned} f'(z) &= \frac{H'(z)(z - z_p)^k - kH(z)(z - z_p)^{k-1}}{(z - z_p)^{2k}} \\ &= \frac{H'(z)(z - z_p) - kH(z)}{(z - z_p)^{k+1}}, \end{aligned}$$

which implies that

$$G(z) = \frac{[H'(z)(z - z_p) - kH(z)]/(z - z_p)^{k+1}}{H(z)/(z - z_p)^k} = -\frac{k}{z - z_p} + \frac{H'(z)}{H(z)}.$$

Now, $H'(z)/H(z)$ is analytic at z_0 since H is analytic and non-zero there. Hence, G has a simple pole with residue $-k$ at z_p. Therefore, the sum of all the residues at the poles of G inside Γ is equal to $N_0(f) - N_p(f)$. It follows from the Residue Theorem that

$$\int_\Gamma G(z)\,\mathrm{d}z = 2\pi i(N_0(f) - N_p(f)). \qquad \square$$

Example 33.2. Consider the integral
$$\int_C \frac{f'(z)}{f(z)}\,\mathrm{d}z$$
where C is the circle $|z| = 2$ traversed once clockwise, and
$$f(z) = \frac{(z+1)^3(z-i)}{(z+3i)^2(2z-1)^7\sin z}.$$
Note that f is the quotient of two entire functions that have distinct zeroes. The zeroes of the numerator are -1 (with multiplicity 3) and i (with multiplicity 1), which both lie inside C, so $N_0(f) = 4$. The zeroes of the denominator are $-3i$ (with multiplicity 2), $1/2$ (with multiplicity 7), and 0 (with multiplicity 1). Since only $1/2$ and 0 lie inside C, we have $N_p(f) = 8$. Finally, since C is negatively oriented, it follows from the Argument Principle that
$$\int_C \frac{f'(z)}{f(z)}\,\mathrm{d}z = -2\pi i(N_0(f) - N_p(f)) = -2\pi(4-8) = 8\pi i.$$

Notice that if f has no poles, then $N_p(f) = 0$. Thus, we immediately obtain the following corollary of the Argument Principle.

Corollary 33.3. If f is analytic and non-zero everywhere on a positively oriented simple closed curve Γ and is also analytic inside all of Γ, then
$$\frac{1}{2\pi i}\int_\Gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z = N_0(f).$$

Example 33.4. Suppose we want to compute
$$I = \int_\Gamma \frac{2z-3}{z^2-3z-4}\,\mathrm{d}z,$$
where Γ is the circle $|z| = 5$ traversed once counter-clockwise. Notice that the integrand is of the form $f'(z)/f(z)$ where
$$f(z) = z^2 - 3z - 4 = (z+1)(z-4),$$
and the zeroes -1 and 4 both lie inside Γ. Hence, we obtain $N_0(f) = 2$ and $I = 2\pi i \cdot 2 = 4\pi i$.

As a direct consequence of the Argument Principle, we obtain the following result.

Theorem 33.5 (Rouché's Theorem). If f and h are both functions that are analytic on and inside a simple closed curve C, and
$$|h(z)| < |f(z)| \tag{$\star$}$$
for all $z \in C$, then f and $f + h$ must have the same number of zeroes (counting multiplicity) inside C.

Proof. Suppose that C is positively oriented. Note that the inequality ($\star$) only needs to hold on C, and it forces both f and $f + h$ to be non-zero on C. Since f and $f + h$ are analytic inside C, we have
$$\int_C \frac{f'(z)}{f(z)}\,\mathrm{d}z = N_0(f), \qquad \int_C \frac{(f+h)'(z)}{(f+h)(z)}\,\mathrm{d}z = N_0(f+h)$$
by Corollary 33.3. Now, define
$$F(z) = 1 + \frac{h(z)}{f(z)}.$$
Then we have
$$F'(z) = \frac{h'(z)f(z) - h(z)f'(z)}{(f(z))^2},$$

which implies that
$$\frac{F'(z)}{F(z)} = \frac{h'(z)f(z) - h(z)f'(z)}{(f(z)+h(z))f(z)} = \frac{(f+h)'(z)}{(f+h)(z)} - \frac{f'(z)}{f(z)}.$$
Consequently, we have
$$\int_C \frac{F'(z)}{F(z)}\,\mathrm{d}z = N_0(f+h) - N_0(f).$$
However, note that $|F(z)-1| = |h(z)|/|f(z)| < 1$ on C, which means that F maps C onto the open disc $|w-1| < 1$, which does not contain the non-negative real axis. Therefore, $F'(z)/F(z)$ has antiderivative $\mathrm{Log}(F(z))$ on C, a closed curve, which implies that
$$\int_C \frac{F'(z)}{F(z)} = 0.$$
Thus, we have $N_0(f) = N_0(f+h)$, so f and $f+h$ have the same number of zeroes inside C. □

Rouché's Theorem is a very useful result for determining the number of zeroes of a function in a given region of the plane by comparing it to a known function. We illustrate this with some examples.

Example 33.6. We can show that all the zeroes of the polynomial
$$g(z) = z^5 + 3z + 1$$
lie inside the disc $|z| < 2$. Indeed, let C be the circle $|z| = 2$. Set $f(z) = z^5$ and $h(z) = 3z+1$. Note that the only zero of f is 0, which has multiplicity 5 and lies inside C. Moreover, we have
$$|h(z)| = |3z+1| \leq 3|z| + 1 = 3(2) + 1 = 7 < 32 = |z|^5 = |f(z)|$$
on C. Thus, f and $f+h = g$ have the same number of zeroes inside C by Rouché's Theorem. That is, g has 5 zeroes inside C, and since g has 5 roots (counting multiplicity) as it is a polynomial of degree 5, it follows that all of its roots lie inside C.

Example 33.7. We can determine the number of roots (counting multiplicity) of the equation
$$2z^5 - 6z^2 + z + 1 = 0$$
in the annulus $1 \leq |z| < 2$. We begin by setting $f(z) = 2z^5$ and $h(z) = -6z^2 + z + 1$. Take C to be the circle $|z| = 2$. The only zero of f is 0 which has multiplicity 5 and lies inside C, and
$$|h(z)| \leq 6|z|^2 + |z| + 1 = 27 < 64 = 2|z|^5 = |f(z)|$$
on C. By Rouché's Theorem, f and $f+h = g$ have the same number of zeroes inside C, so all 5 zeroes of g lie inside C. Next, take C' to be the circle $|z| = 1$. The roots of h are $-1/3$ and $1/2$, which both lie inside C'. Moreover, we have
$$|f(z)| = 2|z|^5 = 2 < 4 = 6|z|^2 - (|z|+1) \leq 6|z|^2 - |z+1| \leq |h(z)|$$
on C', so by Rouché's Theorem, h and $h+f = g$ have the same number of zeroes inside C'. In particular, g has 2 zeroes inside C', so it follows that g has 3 roots inside the annulus $1 \leq |z| < 2$.

Example 33.8. One can show that the equation
$$z + 3 + 2e^z = 0$$
has precisely one root in the left-half plane. Since Rouché's Theorem can only be directly applied to a region bounded by a simple closed curve, and the left-half plane is unbounded, we instead consider the half-disc bounded by C_R, where C_R is the union of the half-circle
$$\{z \in \mathbb{C} : |z| = R,\, x \leq 0\}$$

with the line segment joining $-iR$ and iR on the imaginary axis. As $R \to \infty$, the region bounded by C_R will cover the whole left-half plane. Moreover, note that $f(z) = z + 3$ has exactly one zero, namely $z = -3$, which lies in the left-half plane. Set $h(z) = 2e^z$. On C_R, we then have $\Re(z) = x \leq 0$ so that

$$|h(z)| = |2e^z| = 2e^{\Re(z)} = 2e^x \leq 2e^0 = 2.$$

Moreover, note that

$$|f(z)| = |z+3| \geq \begin{cases} 3 & \text{if } z = iy, \\ |z| - 3 = R - 3 & \text{if } |z| = R. \end{cases}$$

Thus, for all $R > 5$, we have $|h(z)| < |f(z)|$ on C_R, and f has precisely one zero inside C_R. By taking $R \to \infty$, we obtain the desired result.

34 The Open Mapping Theorem, Brouwer's Fixed Point Theorem

In this lecture, we will discuss two important applications of Rouché's Theorem.

Recall that a function is said to be **open** if it maps open sets to open sets. It turns out that non-constant complex analytic functions are always open, and this result is known as the Open Mapping Theorem.

THEOREM 34.1 (Open Mapping Theorem). A non-constant analytic function is an open mapping.

PROOF. Let f be non-constant and analytic on the domain $\Omega \subseteq \mathbb{C}$. Let U be an open subset of Ω and pick any $w_0 \in f(U)$. To prove that $f(U)$ is open, it suffices to show that there exists $r > 0$ such that $D(w_0; r) \subseteq f(U)$. Indeed, since $w_0 \in f(U)$, there exists $z_0 \in U$ such that $f(z_0) = w_0$. Moreover, since U is open and $z_0 \in U$, there exists $a > 0$ such that $D(z_0; a) \subseteq U$. Choose $\delta' = a/2$ so that

$$\overline{D}(z_0; \delta') \subseteq D(z_0; a) \subseteq U.$$

Since f is analytic, so is $g(z) = f(z) - w_0$, with z_0 being a zero of g. However, by the Identity Theorem, the zeroes of g are isolated, as g would be identically zero on Ω otherwise. Hence, we see that z_0 is an isolated zero of g. We can choose $\delta \leq \delta'$ small enough so that $\overline{D}(z_0; \delta)$ contains only one zero of g, namely z_0. Then g cannot be zero at any point on the boundary circle $C(z_0; \delta)$.

Note that $|g(z)|$ is continuous and $C(z_0; \delta)$ is a compact set, so by the Extreme Value Theorem, $|g(z)|$ attains a minimum value ε on $C(z_0; \delta)$. We must have $\varepsilon > 0$ since g is non-zero on $C(z_0; \delta)$, which gives

$$|g(z)| \geq \varepsilon > 0$$

for all $z \in C(z_0; \delta)$. Now, we claim that $D(w_0; \varepsilon) \subseteq f(U)$. Let $w_1 \in D(w_0; \varepsilon)$ so that $|w_1 - w_0| < \varepsilon$. Set $h(z) = w_0 - w_1$, which is entire as it is a constant function. Then we have

$$|h(z)| = |w_0 - w_1| < \varepsilon \leq |g(z)|$$

on $C(z_0; \delta)$. By Rouché's Theorem, $g(z)$ and $g(z) + h(z) = f(z) - w_1$ have the same number of zeroes inside $C(z_0; \delta)$. But g only has one zero inside $C(z_0; \delta)$ by construction, so there is a unique $z_1 \in D(z_0; \delta) \subseteq U$ such that $f(z_1) = w_1$. It follows that $w_1 \in f(U)$, so $D(w_0; \varepsilon) \subseteq f(U)$ as claimed. ☐

REMARK 34.2. The Open Mapping Theorem is very specific to complex analytic functions. For instance, consider the smooth function $f : \mathbb{R} \to \mathbb{R} : x \mapsto x^2$. Then the image of the open interval $(-1, 1)$ is the set $[0, 1)$, so f fails to be an open mapping.

We are now ready to prove the Open Mapping Theorem for harmonic functions. Recall that on Assignment 3, we showed that if u is a non-constant harmonic function on a domain $\Omega \subseteq \mathbb{R}^2$, then $u(\Omega)$ is an open subset of $\mathbb{R}$. More generally, we have the following result.

THEOREM 34.3 (Open Mapping Theorem for harmonic functions). A non-constant harmonic function is an open mapping.

PROOF. Let u be a non-constant harmonic function on a domain $\Omega \subseteq \mathbb{R}^2$. Let U be an open subset of Ω, which can be covered by open discs. Let D be such a disc, and note that D is simply connected. Then the restriction of u to D is harmonic, and since D is simply connected, it admits a harmonic conjugate v in D (by Theorem 15.17). It follows that $f = u + iv$ is analytic on D, and it is non-constant on D since u is non-constant. By the Open Mapping Theorem, $f(D)$ is open in $\mathbb{C}$. Finally, $u(D)$ is the projection of $f(D)$ onto the real axis. It is straightforward to show that projections are open mappings, so $u(D)$ is open in $\mathbb{R}$. This implies that $u(U)$ is the union of open sets, and hence open. ☐

Let D be an open disc. Brouwer's Fixed Point Theorem is a famous theorem of topology which states that any continuous function $g : \overline{D} \to \overline{D}$ has at least one fixed point; that is, $g(z_0) = z_0$ for some $z_0 \in \overline{D}$. For instance, rotations leave their centers z_0 fixed, and map closed discs centered at z_0 to themselves. This result has many applications in other areas of mathematics as well as various real world problems. For example, it can be used to prove the existence of solutions to certain systems of differential equations, or to calculate economic equilibria.

We now state and prove a weaker version of this theorem for complex analytic functions (instead of general continuous functions).

THEOREM 34.4 (Brouwer's Fixed Point Theorem). Let g be an analytic function which maps the closed disc $\overline{D}(0;r)$ to the open disc $D(0;r)$. Then g has exactly one fixed point in $D(0;r)$.

PROOF. Note that the fixed points of g correspond to the zeroes of $g(z)-z$. Letting $f(z)=-z$, we are now considering the zeroes of $f+g$ inside $D(0;r)$. Since $g(\overline{D}(0;r)) \subseteq D(0;r)$, we have

$$|g(z)| < r = |z| = |-z| = |f(z)|$$

for all $z \in C(0;r) = \{z \in \mathbb{C} : |z| = r\}$. Therefore, by Rouché's Theorem, f and $f+g$ have the same number of zeroes inside $C(0;r)$, namely on $D(0;r)$. But f only has one zero on $D(0;r)$ at $z=0$, so $g(z)-z$ has precisely one zero on $D(0;r)$. $\square$

35 Conformal mappings

In this lecture, we study conformal mappings, which are locally invertible analytic functions. An important class of conformal mappings is given by linear fractional transformations, which appear in applications such as solving Dirichlet problems. We present some properties of linear fractional transformations and show how they can be used to solve Dirichlet problems. Moreover, we explain how linear fractional transformations can be interpreted as mappings from the Riemann sphere $\Sigma := \mathbb{C} \cup \{\infty\}$ to itself. Finally, we will state without proof the Riemann Mapping Theorem, which is one of the most important theorems of complex analysis. It states that any simply connected domain that is not $\mathbb{C}$ itself can always be mapped to the open unit disc via an injective analytic function.

Definition 35.1 (Conformal mapping). A transformation $w = f(z)$ is said to be **conformal at** z_0 if it is analytic at z_0 and $f'(z_0) \neq 0$. For a domain $\Omega \subseteq \mathbb{C}$, we say that f is **conformal on** Ω if it is conformal at each point in Ω.

Remark 35.2. If f is conformal at z_0, then it is in fact conformal on an open neighbourhood of z_0. Indeed, since f is analytic at z_0 with $f'(z_0) \neq 0$, there exists an open neighbourhood of z_0 for which f is analytic with $f'(z) \neq 0$ there.

Example 35.3.

(1) The functions $f(z) = z + a$ where $a \in \mathbb{C}$ and $g(z) = e^z$ are conformal on $\mathbb{C}$.

(2) The function $f(z) = az$, where $a \in \mathbb{C} \setminus \{0\}$, is conformal on $\mathbb{C}$.

(3) Note that $f(z) = z^2$ is entire, but $f'(z) = 2z$ so that $f'(z) \neq 0$ if and only if $z \neq 0$. Hence, f is only conformal on $\mathbb{C} \setminus \{0\}$.

(4) The function $f(z) = 1/z$ is analytic on $\mathbb{C} \setminus \{0\}$ and $f'(z) = -1/z^2$ is non-zero there, so f is conformal on $\mathbb{C} \setminus \{0\}$.

(5) More generally, the function
$$f(z) = \frac{az+b}{cz+d}$$
for $a, b, c, d \in \mathbb{C}$ is conformal on $\mathbb{C} \setminus \{-d/c\}$ if and only if $ad - bc \neq 0$. These mappings are called **linear fractional transformations**, which we will return to at the end of the lecture.

To show that conformal mappings are locally invertible, we recall that any complex function $f : \mathbb{C} \to \mathbb{C} : z \mapsto u + iv = w$ induces a mapping (or transformation) given by $F : \mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto (u(x, y), v(x, y))$.

Proposition 35.4. Let $f = u + iv : \mathbb{C} \to \mathbb{C}$ be conformal at the point $z_0 = x_0 + iy_0$. Then f has an inverse in a neighbourhood of z_0.

Proof. Suppose that f is conformal at z_0 so that f is analytic at z_0 and $f'(z_0) \neq 0$. Then u and v are smooth and satisfy the Cauchy-Riemann equations at (x_0, y_0), with $f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0)$. Moreover, by setting $F : \mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto (u(x, y), v(x, y))$, we have

$$\begin{aligned} \mathrm{Jac}(F)(x_0, y_0) &= \begin{vmatrix} u_x(x_0, y_0) & u_y(x_0, y_0) \\ v_x(x_0, y_0) & v_y(x_0, y_0) \end{vmatrix} \\ &= u_x(x_0, y_0)v_y(x_0, y_0) - v_x(x_0, y_0)u_y(x_0, y_0) \\ &= [u_x(x_0, y_0)]^2 + [v_x(x_0, y_0)]^2 \\ &= |f'(z_0)|^2 \neq 0, \end{aligned}$$

where the third equality follows from the Cauchy-Riemann equations. By the Inverse Function Theorem, it follows that F has a smooth inverse in a neighbourhood of (x_0, y_0). This implies that f has an inverse in a neighbourhood of z_0. □

We now state the Riemann Mapping Theorem, one of the most important theorems in complex analysis.

THEOREM 35.5 (Riemann Mapping Theorem). Let $\Omega \subsetneq \mathbb{C}$ be a simply connected domain other than $\mathbb{C}$ itself. Then there is an injective analytic function that maps Ω onto the open unit disc.

REMARK 35.6. Injective analytic functions are conformal (although conformal maps are not necessarily injective on their whole domain, take for instance $f(z) = e^z$). In particular, the Riemann Mapping Theorem tells us that there exists an injective conformal map $\Omega \to D(0;1)$ for any simply connected domain $\Omega \subsetneq \mathbb{C}$.

The Riemann Mapping Theorem is an existence result whose proof is rather technical and is beyond the scope of the course. In practice, even though we are assured that such a map exists, we still need to explicitly construct them for specific applications. We now look at some examples.

- The easiest example of an injective analytic map is a **translation**

$$z \mapsto z + d$$

  for fixed $d \in \mathbb{C}$. Clearly, it is analytic with inverse $z \mapsto z - d$, so it is injective. Moreover, it maps $\mathbb{C}$ to itself since every $w \in \mathbb{C}$ can be written in the form $(w - d) + d$. In fact, it maps any geometric object in $\mathbb{C}$ to a congruent object. For instance, it maps circles to circles, lines to lines, and discs to discs. However, note that only the open disc $D(-d;1)$ gets mapped onto $D(0;1)$ under the translation $z \mapsto z + d$. Thus, we surely need to consider more complicated maps in order to find one that maps a general domain $\Omega \subsetneq \mathbb{C}$ onto $D(0;1)$.

- Another simple example of an injective analytic map of $\mathbb{C}$ onto itself is given by a **rotation**, which is a transformation of the form

$$z \mapsto e^{i\theta} z = |z| e^{i(\operatorname{Arg} z + \theta)}$$

  where $\theta \in \mathbb{R}$. Such a map sends 0 to itself, and rotates any $z \neq 0$ about the origin through the angle θ. Similarly, they map geometric objects in $\mathbb{C}$ onto congruent objects. However, only $D(0;1)$ gets mapped onto itself by rotations.

- A **magnification** is another example of an injective analytic function on $\mathbb{C}$. They are defined as

$$z \mapsto \alpha z = \alpha |z| e^{i \operatorname{Arg} z}$$

  for some fixed $\alpha \in \mathbb{R}^{>0}$. Note that if $\alpha \neq 1$, a magnification either enlarges (if $\alpha > 1$) or contracts (if $\alpha < 1$) the distance of points in $\mathbb{C}$ to the origin.

- An **affine transformation** is a map of the form

$$z \mapsto az + b$$

  where $a, b \in \mathbb{C}$ and $a \neq 0$. When $b = 0$, observe that this is simply a linear transformation $z \mapsto az$. Moreover, any affine transformation is the composition of a rotation $z \mapsto e^{i \operatorname{Arg} a}$ with the magnification $w \mapsto |a| w$ and the translation $\zeta \mapsto \zeta + b$. Thus, they are injective analytic functions of $\mathbb{C}$ onto itself that map geometric objects in $\mathbb{C}$ onto congruent objects.

- Consider the **inversion** map

$$z \mapsto \frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{x}{x^2 + y^2} - i \frac{y}{x^2 + y^2},$$

  which maps $\mathbb{C} \setminus \{0\}$ injectively onto itself and is analytic there. However, in this case, the image of a line may either be a line or a circle, and the same is true of circles. Indeed, any line or circle can be described with the equation

$$A(x^2 + y^2) + Bx + Cy + D = 0$$

for some $A, B, C, D \in \mathbb{R}$, with lines corresponding to the case where $A = 0$. By setting $u+iv = w = 1/z$, we have $x = u/(u^2+v^2)$ and $y = -v/(u^2+v^2)$ so that a line or circle gets mapped to

$$A + Bu - Cv + D(u^2 + v^2) = 0,$$

which is again a line or circle. In particular, a line of the form

$$Bx + Cy + D = 0$$

with $D \neq 0$ gets mapped to the circle

$$Bu - Cv + D(u^2 + v^2) = 0.$$

Nonetheless, the unit disc $|z| < 1$ gets mapped to the domain $|w| > 1$ under the inversion map $z \mapsto 1/z$, which is not simply connected. Thus, we will have to look further for an injective analytic function that maps general simply connected domains to the unit disc $|z| < 1$.

Remark 35.7. Notice that $f(z) = 1/z$ has a pole at 0. In fact, we have $\lim_{z\to 0} 1/z = \infty$ and $\lim_{z\to\infty} 1/z = 0$. Therefore, we can think of f as a map from the **extended complex plane** $\Sigma := \mathbb{C} \cup \{\infty\}$ to itself by setting $f(0) = \infty$ and $f(\infty) = 0$.

Definition 35.8 (Riemann sphere). The **Riemann sphere** or **extended complex plane** is defined to be

$$\Sigma := \mathbb{C} \cup \{\infty\}.$$

Moreover, we can extend the usual algebraic operations on $\mathbb{C}$ to Σ by defining

- $a + \infty = \infty$ for all $a \in \Sigma$;
- $a \cdot \infty = \infty$ and $a/\infty = 0$ for all $a \neq \infty$;
- $a/0 = \infty$ for all $a \neq 0$.

Remark 35.9. One can define a topology on Σ whose open sets consist of the open sets in $\mathbb{C}$ together with the subsets of Σ of the form $A \cup \{\infty\}$ where $\mathbb{C} \setminus A$ is compact. Moreover, it can be shown that the inversion map $z \mapsto 1/z$ is a homeomorphism from Σ to itself with respect to this topology.

For the remainder of the lecture, we will focus on linear fractional transformations.

Definition 35.10 (Linear fractional transformation). A **linear fractional transformation** (or **Möbius transformation**) is a function of the form

$$f(z) = \frac{az + b}{cz + d}$$

with $a, b, c, d \in \mathbb{C}$ such that $ad - bc \neq 0$.

Remark 35.11. If $c \neq 0$, then f has a pole at $-d/c$. Moreover, we have

$$f'(z) = \frac{ad - bc}{(cz + d)^2} \neq 0$$

for all z where it is defined, which implies that f is a conformal map. Finally, observe that

$$\lim_{z\to\infty} f(z) = \begin{cases} a/c & \text{if } c \neq 0, \\ \infty & \text{if } c = 0. \end{cases}$$

Thus, one can think of linear fractional transformations as maps from the Riemann sphere Σ to itself by setting $f(\infty) = \lim_{z\to\infty} f(z)$.

We now look more closely at what linear fractional transformations look like. Let

$$f(z) = \frac{az+b}{cz+d}$$

with $a, b, c, d \in \mathbb{C}$ such that $ad - bc \neq 0$.

- If $c = 0$, then f is simply an affine transformation

  $$z \mapsto a'z + b'$$

  where $a' = a/d$ and $b' = b/d$.

- If $c \neq 0$, then we can write

  $$\frac{az+b}{cz+d} = \frac{a}{c} + \frac{(bc-ad)/c}{cz+d}$$

  so that f is the composition of the affine transformation $z \mapsto cz + d$ with the inversion $w \mapsto 1/w$ and the affine transformation

  $$\zeta \mapsto \frac{bc-ad}{c}\zeta + \frac{a}{c}.$$

  It follows that f is injective with inverse

  $$z \mapsto \frac{1}{c}\left(\frac{1}{z - a/c} - d\right) = \frac{1}{cz-a} - \frac{d}{c},$$

  which is again a linear fractional transformation.

- Linear fractional transformations map lines and circles to either lines or circles.

Our goal now is to find a linear fractional transformation which maps the unit disc $|z| < 1$ to a simply connected domain other than itself.

Example 35.12. Consider the linear fractional transformation

$$f(z) = \frac{z+1}{-z+1}.$$

We claim that f maps the unit disc $|z| < 1$ onto the right-half plane $\{z \in \mathbb{C} : x > 0\}$. First, note that the points i, -1, and $-i$ all lie on $|z| = 1$. Moreover, f must map $|z| = 1$ to either a circle or a line. Since $f(i) = i$, $f(-1) = 0$, and $f(-i) = -i$ are colinear (they all lie on the same straight line), the image of $|z| = 1$ under f must be a line that contains i, 0, and $-i$. In particular, it must be the imaginary axis. Finally, since $f(0) = 1 > 0$, we see that f maps $|z| < 1$ to the right-half plane. Thus, the inverse of f, which is given by

$$z \mapsto -\frac{z+2}{z-1},$$

is an injective analytic function which maps the open right-half plane to the open unit disc.

An important application of conformal mappings is to solve Dirichlet problems, which involves finding solutions to the Laplace equation which certain boundary conditions. First, we prove the following proposition, which states that solutions to the Laplace equation are preserved by conformal transformations.

Proposition 35.13. Let $f = u + iv : \mathbb{C} \to \mathbb{C}$ be conformal at the point $z_0 = x_0 + iy_0$, and let $h : \mathbb{R}^2 \to \mathbb{R}$ be a scalar function of class C^2. Set $H(x, y) = h(u(x, y), v(x, y))$ and $f(z_0) = u_0 + iv_0$. Then $H(x, y)$ is harmonic at (x_0, y_0) if and only if $h(u, v)$ is harmonic at (u_0, v_0).

Proof. A direct computation gives

$$H_{xx} + H_{yy} = (h_{uu} + h_{vv}) \cdot |f'(z_0)|.$$

Since f is conformal at z_0, we have $f'(z_0) \neq 0$, which implies that $H_{xx} + H_{yy} = 0$ if and only if $h_{uu} + h_{vv} = 0$. In particular, $H(x, y)$ is harmonic at (x_0, y_0) if and only if $h(u, v)$ is harmonic at (u_0, v_0). $\square$

Although we have seen many examples of harmonic functions by considering the real and imaginary parts of analytic functions, the difficulty of solving Dirichlet problems lies in finding examples that satisfy initial conditions. With the previous proposition, we can try to find a map that transforms given boundary conditions into simpler ones. We now look at an example.

Example 35.14. Suppose we want to find the solutions to the Laplace equation

$$H_{xx} + H_{yy} = 0$$

with boundary conditions $H = 0$ on the circle

$$|z - 1/2| = 1/2 \iff (x^2 + y^2) - x = 0$$

and $H = 100$ on

$$|z - 1/4| = 1/4 \iff 2(x^2 + y^2) - x = 0$$

whenever $(x, y) \neq (0, 0)$. Recall that the inversion $z \mapsto 1/z$ maps the circles $A(x^2 + y^2) + Bx = 0$ to the lines $A + Bu = 0$. Thus, we are looking for a harmonic function $h(u, v)$ such that $h = 0$ on $u = 1$ and $h = 100$ on $u = 2$. In particular, this means that some of the level curves of $h(u, v)$ must be vertical lines. For $h(u, v) = av + b$ for $a, b \in \mathbb{R}$, we see that $h(1, v) = a + b = 0$ and $h(2, v) = 2a + b = 100$ by taking $a = 100$ and $b = -100$, so $h(u, v) = 100u - 100$ satisfies the desired properties. Finally, the function

$$H(x, y) = \frac{100x}{x^2 + y^2} - 100$$

is the solution to the initial boundary problem.

36 Analytic continuation revisited

For the final lecture, we will make a few more comments on analytic continuation. Given a function that is analytic on a domain, we would like to determine if it is possible to extend it to an analytic function on a bigger domain. This leads us to the notion of a Riemann surface, and we will finish the course by describing the Riemann surface of the complex logarithmic function.

Consider the power series $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ and suppose that it has radius of convergence $0 < R < \infty$ so that it converges on the open disc $D(z_0; R)$. Recall that for any $z_0 \neq z_1 \in D(z_0; R)$, we have $0 < r := R - |z_1 - z_0| < R$, so the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_1)}{n!}(z-z_1)^n$$

converges to f on $D(z_1; r) \subseteq D(z_0; R)$ by Taylor's Theorem. However, this power series may have a larger radius of convergence than $r = R - |z_1 - z_0|$, in which case it defines an analytic continuation of f to points outside of $D(z_0; R)$. Moreover, analytic continuations, if they exist, are unique by the Identity Theorem.

Let f be analytic throughout the domain $\Omega \subseteq \mathbb{C}$. At any point $z_0 \in \Omega$, we can expand f as a power series about z_0 using Taylor's Theorem. Suppose that $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ has radius of convergence R and that R is at least as large as the distance $d_{z_0} > 0$ from z_0 to the boundary of Ω. If there exists $z_0 \in \Omega$ such that $d_{z_0} < R$, thhen we can analytically continue f to $\Omega \cup D(z_0; R)$ by setting

$$g(z) = \begin{cases} f(z) & z \in \Omega, \\ \sum_{n=0}^{\infty} a_n(z-z_0)^n & z \in D(z_0; R). \end{cases}$$

By applying this procedure to more points in the extended domain $\Omega \cup D(z_0; R)$, one might be able to extend the domain further. One approach would be to consider points along curves. As long as the domain is simply connected, one can do this to obtain a uniquely defined analytic function. To be precise, we have the following theorem.

Theorem 36.1 (Monodromy Theorem). Let f be analytic on a domain Ω, and let X be a simply connected domain containing Ω. Suppose that for every curve $\Gamma \subseteq X$, there is an analytic continuation of f to some domain containing Γ. Then there exists a unique analytic continuation of f to X.

Proof. We only give a sketch of the proof. First, note that if the analytic continuation exists, then it must be unique by the Identity Theorem. This is because if g_1 and g_2 are both analytic continuations of f to X, then they must be equal on Ω, which is an open subset of X, implying that they are equal on all of X.

Pick $z_0 \in \Omega$. Given $z_1 \in X$, we pick a path Γ_1 from z_0 to z_1. By hypothesis, there exists a unique analytic continuation g_1 to a domain containing Γ_1. One can then show that the value $g_1(z_1)$ is independent of the curve Γ_1 chosen from z_0 to z_1, but we will omit the details here. Then, by using every point $x \in X$ and picking paths from z_0 to each x, we obtain analytic continuations of f to domains $\tilde{\Omega}_x$ in X that agree on intersections $\tilde{\Omega}_x \cap \tilde{\Omega}_{x'}$ for points $x, x' \in X$ by the Identity Theorem. This gives rise to an analytic function on X whose restriction to Ω is equal to f. □

Remark 36.2. If X is not simply connected, then analytic continuations along curves may lead to a multi-valued function. For instance, let $X = \mathbb{C} \setminus \{0\}$ and consider

$$f(z) = \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j}(z-1)^j,$$

which is the Taylor series of $\operatorname{Log} z$ about $z = 1$. Take the analytic continuations of f along the curves Γ_1 and Γ_2 from 1 and -1, with the first in the upper-half plane, and the second in the lower-half plane. It can be verified that we obtain two different values of f at $z = -1$, which implies that the analytic continuations correspond to different analytic branches of $\log z$.

This leads to the notion of a Riemann surface, which is a topological space that can be locally described using one complex coordinate. Such spaces were initially introduced to extend the domain of certain multi-valued complex functions in a way that makes them single-valued and analytic.

We now describe the Riemann surface of the logarithmic function $f(z) = \log z$ whose domain is $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Consider the disjoint union

$$R := \bigsqcup_{k \in \mathbb{Z}} (\mathbb{C}^* \times \{k\}).$$

One can then put a topology on R whose open sets are unions of sets of the form

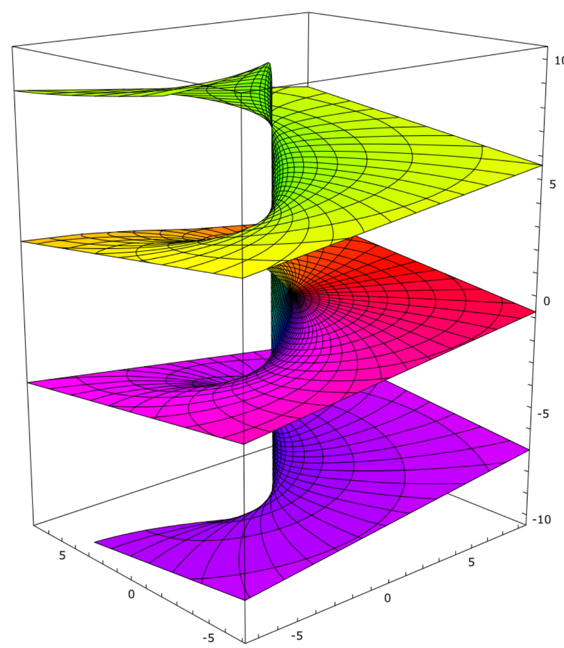
$$D((z_0, k); r) := \{(z, k) \in R : |z - z_0| < r\}$$

if $z_0 \in \mathbb{C}^*$ and the distance from z_0 to the positive real axis is greater than r, or

$$A((z_0, k); r) := \{(z, k) \in R : |z - z_0| < r,\, \Im(z) \geq 0\} \sqcup \{(z, k-1) \in R : |z - z_0| < r,\, \Im(z) < 0\}$$

if z_0 is a point on the positive real axis with $z_0 > r$, where $k \in \mathbb{Z}$ and $r > 0$.

The Riemann surface of $\log z$ is the topological space obtained by endowing R with this topology. A picture of it is given below.



If we set

$$\hat{f} : R \to \mathbb{C} : (z, k) \mapsto \ln|z| + i(\arg_0 z + 2\pi k),$$

we obtain a single-valued function that is analytic with respect to the "natural complex structure" on R that is equal to the analytic branch $L_{2\pi k}(z)$ of $\log z$ on the slice $\mathbb{C}^* \times \{k\}$. Of course, one would have the clearly describe what is meant by a "complex structure" on R, but this is beyond the scope of the course.

Finally, we note that more information on Riemann surfaces can be found in the following books.

1. O. Forster, *Lectures on Riemann Surfaces*, Springer-Verlag, 1981.

2. S. K. Donaldson, *Riemann Surfaces*, Oxford University Press, 2011.

3. R. Narasimhan, *Compact Riemann Surfaces*, Birhäuser, 1992.