MATH 148 Course Notes

Calculus 2 (Advanced Level)

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Preface

These notes were created as a resource for personal use, and for current, past, and future students of the course. They may contain errors and inconsistencies. If you notice any errors or have other inquiries regarding these notes, feel free to e-mail me at `mltlee@uwaterloo.ca`.

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0 Motivating the Riemann Integration

0.1. The concept of Riemann integration has many applications. We study two such applications to motivate the definition.

Example 0.2. Consider a person who is walking for 2 hours, equipped with a watch and a speedometer (which measures speed but not distance). The problem is to determine the total distance walked over the 2 hour period.

We attack this problem by partitioning the time interval from $t = 0$ to $t = 2$ into smaller intervals.

$$0 = t_0 < t_1 < \cdots < t_N = 2$$

Let V_i be the maximum speed over the time interval $[t_{i-1}, t_i]$, and v_i be the minimum speed over $[t_{i-1}, t_i]$, where $1 \leq i \leq N$.

If d_i is the distance covered in the time interval $[t_{i-1}, t_i]$, $1 \leq i \leq N$, then

$$v_i(t_i - t_{i-1}) \leq d_i \leq V_i(t_i - t_{i-1})$$

Thus the total distance covered satisfies

$$\sum_{i=1}^{N} v_i(t_i - t_{i-1}) \leq d = \sum_{i=1}^{N} d_i \leq \sum_{i=1}^{N} V_i(t_i - t_{i-1})$$

given that the speed of a person walking is a continuous function of time (by hypothesis), and we know that the speed function is uniformly continuous on $[0, 2]$.

Thus, given $\varepsilon > 0$, we can find $\delta > 0$ such that if $\max_{1\leq i\leq N} t_i - t_{i-1} < \delta$, then $\max_{1\leq i\leq N} V_i - v_i < \varepsilon$, in which case

$$\begin{aligned}
\left(\sum_{i=1}^{N} V_i(t_i - t_{i-1})\right) - \left(\sum_{i=1}^{N} v_i(t_i - t_{i-1})\right) &= \sum_{i=1}^{N} (V_i - v_i)(t_i - t_{i-1}) \\
&\leq \sum_{i=1}^{N} \varepsilon(t_i - t_{i-1}) \\
&= \varepsilon \sum_{i=1}^{N} (t_i - t_{i-1}) \\
&= \varepsilon\Big((t_N - \cancel{t_{N-1}}) + (\cancel{t_{N-1}} - \cancel{t_{N-2}}) + \cdots + (\cancel{t_1} - t_0)\Big) \\
&= \varepsilon(t_N - t_0) = \varepsilon(2 - 0) = 2\varepsilon
\end{aligned}$$

Thus we can estimate d to within 2ε using this argument.

Example 0.3. Suppose $f : [0, 2] \to [0, \infty)$ is continuous. We wish to calculate the area A under the curve $y = f(x)$ from $x = 0$ to $x = 2$.

As before, we partition the interval $[0, 2]$ as follows:

We set

$$0 = t_0 < t_1 < \cdots < t_N = 2$$

and let $M_i = \max_{x\in[t_{i-1},t_i]} f(x)$ and $m_i = \min_{x\in[t_{i-1},t_i]} f(x)$, where $1 \le i \le N$, which exist because f is continuous on $[t_{i-1}, t_i]$, $1 \le i \le N$.

Then we see that

$$\sum_{i=1}^{N} m_i(t_i - t_{i-1}) \le A \le \sum_{i=1}^{N} M_i(t_i - t_{i-1})$$

Arguing as in Example 0.2, since f is continuous on $[0, 2]$, it is uniformly continuous on $[0, 2]$, so given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\max_{1\le i\le N} t_i - t_{i-1} < \delta \implies \max_{1\le i\le N} M_i - m_i < \varepsilon$$

hence A can be approximated to within 2ε by either $\sum_{i=1}^{N} m_i(t_i - t_{i-1})$ or $\sum_{i=1}^{N} M_i(t_i - t_{i-1})$.

1 An Introduction to Riemann Integration

Notation 1.1. Throughout this course, we shall use the following notation:

- $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$; it is assumed that $a \leq b \in \mathbb{R}$.
- We denote an arbitrary (not necessarily bounded or closed) interval by $\mathbb{I}$.

Definition 1.2. Let $[a, b] \subseteq \mathbb{R}$ be an interval.

A **partition** of $[a, b]$ is a finite set of the form

$$P = \{a = p_0 < p_1 < \cdots < p_N = b\}$$

The **norm** of a partition P is

$$\|x\| = \max_{1 \leq i \leq N} p_i - p_{i-1}$$

We say that a partition P is **uniform** if $p_i - p_{i-1} = \frac{b-a}{N}$ with $1 \leq i \leq N$.

Equivalently, P is uniform if $\|P\| = \frac{b-a}{N}$.

We set $\mathcal{P}[a, b] = \{P : P \text{ is a partition of } [a, b]\}$.

Notation 1.3. Let $f : [a, b] \to \mathbb{R}$ be a **bounded** function.

Let $P \in \mathcal{P}[a, b]$, say $P = \{a = p_0 < p_1 < \cdots < p_N = b\}$.

We define

$$M_i (= M_i(f, P)) = \sup_{x \in [p_{i-1}, p_i]} f(x)$$

$$m_i (= m_i(f, P)) = \inf_{x \in [p_{i-1}, p_i]} f(x)$$

where $1 \leq i \leq N$.

Note: M_i and m_i depend on the choice of f and P.

Example 1.4. Some applications:

(a) Let $f : [a, b] \to \mathbb{R}$.

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \cap [a, b] \\ 0 & x \in [a, b] \setminus \mathbb{Q} \end{cases}$$

If $P \in \mathcal{P}[a, b]$, say $P = \{a = p_0 < p_1 < \ldots < p_N = b\}$, then $M_i = 1$ and $m_i = 0$, with $1 \leq i \leq N$.

(b) Suppose that $f : [a, b] \to \mathbb{R}$ is <u>decreasing</u> (i.e. $a \leq x \leq y \leq b \implies f(x) \geq f(y)$).

If $P = \{a = p_0 < p_1 < \ldots < p_N = b\} \in \mathcal{P}[a, b]$, then $M_i = f(p_{i-1})$ and $m_i = f(p_i)$, $1 \leq i \leq N$.

Definition 1.5. We define the **lower Riemann sum** to be

$$L(f, P) = \sum_{k=1}^{N} m_k (p_k - p_{k-1})$$

and the **upper Riemann sum** to be

$$U(f,P)=\sum_{k=1}^{N} M_k(p_k-p_{k-1}).$$

Note that $m_k \leq M_k$, and $p_k-p_{k-1}>0$, $1\leq k\leq N$, which implies that

$$L(f,P)\leq U(f,P).$$

Example 1.6.

(a) Consider the function $f:[0,1]\to\mathbb{R}$.

$$f(x)=\begin{cases}\frac{1}{2}x & x\in[0,\frac{1}{2}) \\ 0 & x=\frac{1}{2} \\ \frac{1}{2}x-\frac{1}{2} & x\in(\frac{1}{2},1]\end{cases}$$

Consider the partition

$$P=\{0,1/3,2/3,1\}.$$

Then,

$$\begin{aligned} m_1&=0 & m_2&=-1/4 & m_3&=-1/6 \\ M_1&=1/6 & M_2&=1/4 & M_3&=0 \end{aligned}$$

so that

$$\begin{aligned} L(f,P)&=m_1\big(1/3-0\big)+m_2\big(2/3-1/3\big)+m_3\big(1-2/3\big) \\ &=0\big(1/3\big)+\big(-1/4\big)\big(1/3\big)+\big(-1/6\big)\big(1/3\big) \\ &=-1/12-1/18=-5/36 \\ U(f,P)&=M_1\big(1/3-0\big)+M_2\big(2/3-1/3\big)+M_3\big(1-2/3\big) \\ &=(1/6)\big(1/3\big)+(1/4)\big(1/3\big)+0\big(1/3\big) \\ &=5/36 \end{aligned}$$

(b) Let $f:[a,b]\to\mathbb{R}$.

$$x\mapsto\begin{cases}1 & x\in\mathbb{Q}\cap[a,b] \\ 0 & x\in[a,b]\setminus\mathbb{Q}\end{cases}$$

Let $P=\{a=p_0<p_1<\ldots<p_N=b\}\in\mathcal{P}[a,b]$. Then $m_k=0$, $M_k=1$, $1\leq k\leq N$. Thus,

$$\begin{aligned} L(f,P)&=\sum_{k=1}^{N} m_k(p_k-p_{k-1})=\sum_{k=1}^{N} 0(p_k-p_{k-1})=0 \\ U(f,P)&=\sum_{k=1}^{N} M_k(p_k-p_{k-1})=\sum_{k=1}^{N} 1(p_k-p_{k-1}) \\ &=(p_1-p_0)+(p_2-p_1)+\ldots+(p_N-p_{N-1}) \\ &=p_N-p_0 \\ &=b-a \end{aligned}$$

Note: These are independent of the choice of P.

(c) Consider

$$\begin{aligned} f : [0,1] &\to \mathbb{R} \\ x &\mapsto x^2 \end{aligned}$$

Let $P = \{0 < \frac{1}{N} < \frac{2}{N} < \ldots < \frac{N-1}{N} < 1 = p_N\}$ (i.e. $p_k = \frac{k}{N}, 0 \leq k \leq N$) be a uniform partition.

Since f is increasing on $[0,1]$,

$$\begin{aligned} m_k &= f(p_{k-1}) = f\left(\frac{k-1}{N}\right) = \left(\frac{k-1}{N}\right)^2 \\ M_k &= f(p_k) = f\left(\frac{k}{N}\right) = \left(\frac{k}{N}\right)^2 \end{aligned}$$

Hence, we have

$$\begin{aligned} U(f,P) &= \sum_{k=1}^{N} M_k(p_k - p_{k-1}) \\ &= \sum_{k=1}^{N} \left(\frac{k}{N}\right)^2 \left(\frac{1}{N}\right) \\ &= \frac{1}{N^3} \sum_{k=1}^{N} k^2 \\ &= \frac{1}{N^3} \left(\frac{N(N+1)(2N+1)}{6}\right) \\ &= \frac{1(1+\frac{1}{N})(2+\frac{1}{N})}{6} \end{aligned}$$

$$\begin{aligned} L(f,P) &= \sum_{k=1}^{N} m_k(p_k - p_{k-1}) \\ &= \sum_{k=1}^{N} \left(\frac{k-1}{N}\right)^2 \left(\frac{1}{N}\right) \\ &= \frac{1}{N^3} \sum_{k=1}^{N} (k-1)^2 \\ &= \frac{1}{N^3} \sum_{k=0}^{N-1} k^2 \\ &= \frac{1}{N^3} \sum_{k=1}^{N-1} k^2 \\ &= \frac{1}{N^3} \left(\frac{(N-1)(N)(2N-1)}{6}\right) \\ &= \frac{(1-\frac{1}{N})(1)(2-\frac{1}{N})}{6} \end{aligned}$$

It follows that

$$\lim_{N\to\infty} L(f, P = P_N) = \frac{1}{3} = \lim_{N\to\infty} U(f, P = P_N).$$

Definition 1.7. Let $[a,b] \in \mathbb{R}$ and $P \in \mathcal{P}[a,b]$.

We say that $Q \in \mathcal{P}[a,b]$ is a **refinement** of P, or that Q **refines** P, if $P \subseteq Q$.

Example 1.8. Consider $P = \{0, \frac{1}{4}, \frac{1}{2}, \frac{9}{10}, 1\} \in \mathcal{P}[0,1]$.

Let

$$\begin{aligned} Q_1 &= \left\{0, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{3}{4}, \frac{9}{10}, 1 - \frac{\pi}{10^{23}}, 1\right\} \\ Q_2 &= \left\{0, \frac{1}{9}, \frac{2}{9}, \ldots, \frac{8}{9}, 1\right\} \\ Q_3 &= \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{9}{10}, 1, \frac{3}{2}\right\} \end{aligned}$$

Q_1 is a refinement of P.

Q_2 is **not** a refinement of P since $\frac{1}{4} \in P$ but $\frac{1}{4} \notin Q_2$.

Q_3 is **not** a refinement of P since $Q_3 \notin \mathcal{P}[0,1]$.

Proposition 1.9. Let $f : [a,b] \to \mathbb{R}$ be a bounded function, and let $P \in \mathcal{P}[a,b]$. If Q is a refinement of P, then

$$L(f,P) \leq L(f,Q) \leq U(f,Q) \leq U(f,P).$$

Proof.

Case 1. Q has exactly 1 more point than P.

Consider $P = \{a = p_0 < p_1 < ... < p_N = b\}$ and $Q = P \cup \{r\}$, where $p_{k-1} < r < p_k$ for some fixed $1 \leq k \leq N$.

Let $m_j = \inf\{f(x) : x \in [p_{j-1}, p_j]\}, 1 \leq j \leq N$.

We define the following:

$$\begin{aligned} {m_k}' &:= \inf\{f(x) : x \in [p_{k-1}, r]\} \\ {m_k}'' &:= \inf\{f(x) : x \in [r, p_k]\} \end{aligned}$$

Then

$$L(f,P) = \sum_{j=1}^{N} m_j(p_j - p_{j-1})$$

while

$$L(f,Q) = \sum_{j=1}^{k-1} m_j(p_j - p_{j-1}) + {m_k}'(r - p_{k-1}) + {m_k}''(p_k - r) + \sum_{j=k+1}^{N} m_j(p_j - p_{j-1})$$

and so

$$\begin{aligned} L(f,Q) - L(f,P) &= {m_k}'(r - p_{k-1}) + {m_k}''(p_k - r) - m_k\Big((p_k - r) + (r - p_{k-1})\Big) \\ &= ({m_k}' - m_k)(r - p_{k-1}) + ({m_k}'' - m_k)(p_k - r) \end{aligned}$$

Since ${m_k}' \geq m_k$ and ${m_k}'' \geq m_k$, all of the above terms are greater than or equal to 0, which implies that

$$L(f,Q) - L(f,P) \geq 0 \implies L(f,P) \leq L(f,Q).$$

Exercise: Prove that $U(f,Q) \leq U(f,P)$.

Hence, $L(f,P) \leq L(f,Q) \leq U(f,Q) \leq U(f,P)$.

Case 2. The general case.

Suppose $Q = P \cup \{r_1, r_2, \ldots, r_m\}$.

Set $Q_0 = P$, and for $1 \leq j \leq m$, set $Q_j = Q_{j-1} \cup \{r_j\}$.

Then, by Case 1, noting that each Q_j is a refinement of Q_{j-1} with exactly 1 more point, we find that

$$\begin{aligned} L(f,P) = L(f,Q_0) &\leq L(f,Q_1) \\ &\leq L(f,Q_2) \\ &\leq \dots \\ &\leq L(f,Q_m) \\ &= L(f,Q) \\ &\leq U(f,Q) \\ &= U(f,Q_m) \\ &\leq U(f,Q_{m-1}) \\ &\leq \dots \\ &\leq U(f,Q_0) = U(f,P) \ \square \end{aligned}$$

Corollary 1.10. Let $f : [a,b] \to \mathbb{R}$ be a bounded function.

If $P, Q \in \mathcal{P}[a,b]$, then

$$L(f,P) \leq U(f,Q).$$

Proof. Let $R = P \cup Q \in \mathcal{P}[a,b]$, so R is a <u>common</u> refinement of P and Q.

By Proposition 1.9, we have that

$$L(f,P) \leq L(f,R) \leq U(f,R) \leq U(f,Q). \ \square$$

Definition 1.11. Let $f : [a,b] \to \mathbb{R}$ be a bounded function.

We define the **lower Riemann integral** of f over $[a,b]$ to

$$L\int_a^b f := \sup\{L(f,P) : P \in \mathcal{P}[a,b]\} < \infty.$$

Similarly, we define the **upper Riemann integral** of f over $[a,b]$ to be

$$U\int_a^b f := \inf\{U(f,Q) : Q \in \mathcal{P}[a,b]\} \in \mathbb{R}.$$

We observe that

$$L\int_a^b f \leq U\int_a^b f.$$

To see this, note that if we fix $Q \in \mathcal{P}[a,b]$, then

$$L(f,P) \leq U(f,Q) \quad \forall P \in \mathcal{P}[a,b]$$

implies that

$$L\int_a^b f = \sup\{L(f,P) : P \in \mathcal{P}[a,b]\} \leq U(f,Q).$$

That is,

$$L\int_a^b f \leq U(f,Q).$$

But this holds for all $Q \in \mathcal{P}[a,b]$, whence

$$L\int_a^b f \leq \inf\{U(f,Q) : Q \in \mathcal{P}[a,b]\} = U\int_a^b f.$$

Example 1.12.

(a) Consider

$$\begin{aligned} f : [a,b] &\to \mathbb{R} \\ x &\mapsto \begin{cases} 1 & x \in \mathbb{Q} \cap [a,b] \\ 0 & x \in [a,b] \setminus \mathbb{Q} \end{cases} \end{aligned}$$

We saw in Example 1.6 that

$$\begin{aligned} L(f,P) &= 0 \\ U(f,P) &= b-a \end{aligned}$$

for all $P \in \mathcal{P}[a,b]$. Hence,

$$\begin{aligned} L\int_a^b f &= \sup\{0 : P \in \mathcal{P}[a,b]\} = 0 \\ U\int_a^b f &= \inf\{b-a : P \in \mathcal{P}[a,b]\} = b-a \end{aligned}$$

(b) Consider

$$\begin{aligned} f : [0,1] &\to \mathbb{R} \\ x &\mapsto x^2 \end{aligned}$$

Recall from Example 1.6 that if for each $N \geq 1$, we set

$$P_N = \left\{0 < \frac{1}{N} < \frac{2}{N} < \ldots < \frac{N-1}{N} < 1\right\} \in \mathcal{P}[0,1]$$

then

$$\begin{aligned} L(f,P_N) &= \frac{1}{6}\left(1-\frac{1}{N}\right)\left(2-\frac{1}{N}\right) \\ U(f,P_N) &= \frac{1}{6}\left(1+\frac{1}{N}\right)\left(2+\frac{1}{N}\right) \end{aligned}$$

Thus,

$$\begin{aligned} L\int_0^1 f &= \sup\{L(f,P) : P \in \mathcal{P}[0,1]\} \\ &\geq \sup\{L(f,P_N) : N \geq 1\} \\ &= \sup\left\{\frac{1}{6}\left(1-\frac{1}{N}\right)\left(2-\frac{1}{N}\right) : N \geq 1\right\} = \frac{1}{3} \end{aligned}$$

while

$$\begin{aligned} U\int_0^1 f &= \inf\{U(f,P) : P \in \mathcal{P}[0,1]\} \\ &\le \inf\{U(f,P_N) : N \ge 1\} \\ &= \inf\left\{\frac{1}{6}\left(1+\frac{1}{N}\right)\left(2+\frac{1}{N}\right) : N \ge 1\right\} = \frac{1}{3} \end{aligned}$$

It follows that

$$\frac{1}{3} \le L\int_0^1 f \le U\int_0^1 f \le \frac{1}{3}$$

so that

$$L\int_0^1 f = U\int_0^1 f = \frac{1}{3}.$$

Definition 1.13. Let $f : [a,b] \to \mathbb{R}$ be a bounded function.

We say f is **Riemann integrable** over $[a,b]$ if

$$L\int_a^b f = U\int_a^b f$$

in which case we denote the common value by

$$\int_a^b f.$$

We also write

$$\mathcal{R}[a,b] = \{f : [a,b] \to \mathbb{R} \mid f \text{ is Riemann integrable}\}.$$

Example 1.14.

(a) Consider

$$\begin{aligned} f : [a,b] &\to \mathbb{R} \\ x &\mapsto \begin{cases} 1 & x \in \mathbb{Q} \cap [a,b] \\ 0 & x \in [a,b] \setminus \mathbb{Q} \end{cases} \end{aligned}$$

with $a < b$ (because this would otherwise be uninteresting).

From Example 1.12,

$$L\int_a^b f = 0 \ne b - a = U\int_a^b f.$$

Thus f is **not** Riemann integrable over $[a,b]$.

(b) Let $f(x) = x^2$, $x \in [0,1]$. Then

$$L\int_0^1 f = U\int_0^1 f = \frac{1}{3}$$

by Example 1.12, so $f \in \mathcal{R}[0,1]$, and $\int_0^1 f = \frac{1}{3}$.

1.15. Classifying all Riemann integrable functions is beyond the scope of the course. Instead, we show that $\mathcal{R}[a,b]$ contains a large class of functions, and we develop techniques to evaluate $\int_a^b f$ when possible.

Theorem 1.16 (The Cauchy Criterion)**.** Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Then the following are equivalent:

(a) f is Riemann integrable over $[a,b]$.

(b) For every $\varepsilon > 0$, there exists a partition $P \in \mathcal{P}[a,b]$ such that

$$U(f,P) - L(f,P) < \varepsilon.$$

Proof. (a) $\implies$ (b). Suppose that $f \in \mathcal{R}[a,b]$, that is, $L\int_a^b f = U\int_a^b f$.

Let $\varepsilon > 0$.

Since $L\int_a^b f = \sup\{L(f,Q) : Q \in \mathcal{P}[a,b]\}$, we can find $Q \in \mathcal{P}[a,b]$ such that

$$L\int_a^b f - L(f,Q) < \frac{\varepsilon}{2}.$$

Similarly, since $U\int_a^b f = \inf\{U(f,R) : R \in \mathcal{P}[a,b]\}$, we can find $R \in \mathcal{P}[a,b]$ such that

$$U(f,R) - U\int_a^b f < \frac{\varepsilon}{2}.$$

Thus

$$\begin{aligned} U(f,R) - L(f,Q) &\leq U(f,R) - U\int_a^b f + L\int_a^b f - L(f,Q) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Let $P = Q \cup R \in \mathcal{P}[a,b]$ so that P is a common refinement of Q and R. Thus,

$$\begin{aligned} L(f,Q) \leq L(f,P) \leq L\int_a^b f \\ = U\int_a^b f \leq U(f,P) \leq U(f,R) \end{aligned}$$

and

$$\begin{aligned} U(f,P) - L(f,P) &\leq U(f,R) - L(f,Q) \\ &< \varepsilon \end{aligned}$$

which is what we wanted to show.

(b) $\implies$ (a). Suppose for every $\varepsilon > 0$, there exists a partition $P \in \mathcal{P}[a,b]$ such that $U(f,P) - L(f,P) < \varepsilon$.

For each $N \geq 1$, we can choose $P_N \in \mathcal{P}[a,b]$ such that

$$U(f,P_N) - L(f,P_N) < \frac{1}{N}.$$

Recall that

$$L(f, P_N) \leq L\int_a^b f - U\int_a^b f \leq U(f, P_N) \quad \forall N \geq 1$$

and so

$$\begin{aligned} 0 \leq U\int_a^b f - L\int_a^b f &\leq U(f, P_N) - L(f, P_N) \\ &< \frac{1}{N}, N \geq 1 \end{aligned}$$

Hence

$$U\int_a^b f - L\int_a^b f = 0,$$

that is,

$$L\int_a^b f = U\int_a^b f,$$

so $f \in \mathcal{R}[a, b]$. $\square$

Example 1.17. Suppose $a < c < b$ in $\mathbb{R}$ and that $0 < r \in \mathbb{R}$.

Consider the function

$$\begin{aligned} f : [a, b] &\to \mathbb{R} \\ x &\mapsto \begin{cases} 0 & x \in [a, c) \\ 2r & x = c \\ r & x \in (a, b] \end{cases} \end{aligned}$$

We shall use Cauchy's Criterion to prove that $f \in \mathcal{R}[a, b]$.

Proof. Let $\varepsilon > 0$.

Let $\delta = \min\left\{\frac{c-a}{2}, \frac{b-c}{2}, \frac{\varepsilon}{4r}\right\}$.

Consider the partition $P_\delta = \{a, c - \delta, c + \delta, b\}$.

Then we have

$$\begin{aligned} m_1 &= 0 & M_1 &= 0 \\ m_2 &= 0 & M_2 &= 2r \\ m_3 &= r & M_3 &= r \end{aligned}$$

Thus, we obtain

$$\begin{aligned} L(f, P_\delta) &= m_1\Big((c - \delta) - a\Big) + m_2\Big((c + \delta) - (c - \delta)\Big) + m_3\Big(b - (c + \delta)\Big) \\ &= 0 + 0 + r\Big(b - (c + \delta)\Big) \\ U(f, P_\delta) &= M_1\Big((c - \delta) - a\Big) + M_2\Big((c + \delta) - (c - \delta)\Big) + M_3\Big(b - (c + \delta)\Big) \\ &= 2r(2\delta) + r\Big(b - (c + \delta)\Big) \end{aligned}$$

Therefore,

$$U(f, P_\delta) - L(f, P_\delta) = 2r(2\delta) = \delta(4r) < \varepsilon.$$

Then by the Cauchy Criterion, $f \in \mathcal{R}[a, b]$. $\square$

Theorem 1.18. Let $a < b \in \mathbb{R}$ and suppose that $f : [a, b] \to \mathbb{R}$ is an increasing function. Then $f \in \mathcal{R}[a, b]$.

Proof. We will use the Cauchy Criterion to prove this.

Let $\varepsilon > 0$. Let $P \in \mathcal{P}[a, b]$, say

$$P = \{a = p_0 < p_1 < \cdots < p_N\}$$

with $\|P\| := \max_{1 \le k \le N} p_k - p_{k-1} < \frac{\varepsilon}{\left(f(b)-f(a)\right)+1}$.

Let

$$\begin{aligned} m_k &= \inf\{f(x) : x \in [p_{k-1}, p_k]\} = f(p_{k-1}) \\ M_k &= \sup\{f(x) : x \in [p_{k-1}, p_k\} = f(p_k) \end{aligned}$$

$1 \le k \le N$.

Thus,

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{k=1}^{N} M_k(p_k - p_{k-1}) - \sum_{k-1}^{N} m_k(p_k - p_{k-1}) \\ &= \sum_{k=1}^{N} (M_k - m_k)(p_k - p_{k-1}) \\ &\le \sum_{k=1}^{N} (M_k - m_k)\|P\| \\ &= \|P\| \sum_{k=1}^{N} \left(f(p_k) - f(p_{k-1})\right) \\ &= \|P\| \left(f(p_N) - f(p_0)\right) \\ &= \|P\| \left(f(b) - f(a)\right) \\ &< \varepsilon \end{aligned}$$

By the Cauchy Criterion, $f \in \mathcal{R}[a, b]$. $\square$

Example 1.19. Consider the function

$$f(x) = \begin{cases} 1 - \frac{1}{N} & 1 - \frac{1}{N} \le x \le 1 - \frac{1}{N+1}, N \ge 1 \\ 1 & x = 1 \end{cases}$$

Note that f has infinitely many jump discontinuities.

Nevertheless, it is increasing on $[0, 1]$, so by Theorem 1.18, $f \in \mathcal{R}[0, 1]$.

Proposition 1.20. Let $f, g : [a, b] \to \mathbb{R}$ be Riemann integrable and $\kappa \in \mathbb{R}$.

Then $f + g$ and κf are Riemann integrable and

$$\int_a^b \kappa f + g = \kappa \int_a^b f + \int_a^b g.$$

Proof. On the assignment. $\square$

Remark 1.21. The above result is showing that the main parts required to prove that $\mathcal{R}[a, b]$ is a vector space over $\mathbb{R}$.

The last statement asserts that the map

$$\begin{aligned} J : \mathcal{R}[a, b] &\to \mathbb{R} \\ f &\mapsto \int_a^b f \end{aligned}$$

is <u>linear</u>.

Proposition 1.22. Suppose that $f : [a, b] \to \mathbb{R}$ is Riemann integrable and that $c \in (a, b)$. Then

$$\begin{aligned} f|_{[a,c]} &\in \mathcal{R}[a, c] \\ f|_{[c,b]} &\in \mathcal{R}[c, b] \end{aligned}$$

and

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Proof. We shall use the Cauchy Criterion to prove this.

Let $\varepsilon > 0$. Since $f \in \mathcal{R}[a, b]$, by Cauchy's Criterion, there exists a partition $P \in \mathcal{P}[a, b]$ such that

$$U(f, P) - L(f, P) < \varepsilon.$$

Let $Q = P \cup \{c\}$ so that Q is a refinement of P. Thus

$$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P)$$

and so

$$U(f, Q) - L(f, Q) \leq U(f, P) - L(f, P) < \varepsilon.$$

Let $Q = \{a = q_0 < q_1 < \cdots < q_m = c < q_{m+1} < \cdots < q_N = b\}$.

Set

$$\begin{aligned} Q_\ell &:= \{a = q_0 < q_1 < \cdots < q_m = c\} \\ Q_r &:= \{c = q_m < q_{m+1} < \cdots < q_N = b\} \end{aligned}$$

Then

$$\begin{aligned} U(f|_{[a,c]}, Q_\ell) - L(f|_{[a,c]}, Q_\ell) &\leq \Big(U(f|_{[a,c]}, Q_\ell) - L(f|_{[a,c]}, Q_\ell)\Big) + \Big(U(f|_{[c,b]}, Q_r) - L(f|_{[c,b]}, Q_r)\Big) \\ &= U(f, Q) - L(f, Q) < \varepsilon \end{aligned}$$

By the Cauchy Criterion, $f|_{[a,c]} \in \mathcal{R}[a,c]$.

Exercise: Prove that $f|_{[c,b]} \in \mathcal{R}[c,b]$.

Then, since $f \in \mathcal{R}[a,b]$,

$$\begin{aligned}
\int_a^b f = U\int_a^b f &= \inf\{U(f,P) : P \in \mathcal{P}[a,b]\} \\
&= \inf\{U(f,P\cup\{c\}) : P \in \mathcal{P}[a,b]\} \\
&= \inf\{U(f|_{[a,c]},Q_\ell) + U(f|_{[c,b]},Q_r) : Q_\ell \in \mathcal{P}[a,c], Q_r \in \mathcal{P}[c,b]\} \\
&\geq \inf\{U(f|_{[a,c]},Q_\ell) : Q_\ell \in \mathcal{P}[a,c]\} + \inf\{U(f|_{[c,b]},Q_r) : Q_r \in \mathcal{P}[c,b]\} \\
&= U\int_a^c f + U\int_c^b f \\
&= \int_a^c f + \int_c^b f
\end{aligned}$$

Thus

$$\int_a^b f \geq \int_a^c f + \int_c^b f$$

while

$$\begin{aligned}
\int_a^b f = L\int_a^b f &= \sup\{L(f,P) : P \in \mathcal{P}[a,b]\} \\
&= \sup\{L(f,P\cup\{c\}) : P \in \mathcal{P}[a,b]\} \\
&= \sup\{L(f|_{[a,c]},Q_\ell) + L(f|_{[c,b]},Q_r) : Q_\ell \in \mathcal{P}[a,c], Q_r \in \mathcal{P}[c,b]\} \\
&\leq \sup\{L(f|_{[a,c]},Q_\ell) : Q_\ell \in \mathcal{P}[a,c]\} + \sup\{L(f|_{[c,b]},Q_r) : Q_r \in \mathcal{P}[c,b]\} \\
&= L\int_a^c f + L\int_c^b f \\
&= \int_a^c f + \int_c^b f
\end{aligned}$$

which implies

$$\int_a^b f \leq \int_a^c f + \int_c^b f.$$

Therefore, combining the inequalities yields

$$\int_a^b f = \int_a^c f + \int_c^b f. \ \square$$

Notation 1.23. Suppose that $f \in \mathcal{R}[a,b]$ and that $d \in [a,b]$, where $a < b \in \mathbb{R}$. We define

$$\int_d^d f = 0.$$

Furthermore, we define (once again for $a < b \in \mathbb{R}$)

$$\int_b^a f := -\int_a^b f.$$

Exercise: Let $a, b, c \in \mathbb{R}$, and let

$$\begin{aligned} \alpha &:= \min\{a, b, c\} \\ \beta &:= \max\{a, b, c\} \end{aligned}$$

and suppose that $g : [\alpha, \beta] \to \mathbb{R}$ is Riemann integrable.

Prove that

$$\int_a^b g = \int_a^c g + \int_c^b g.$$

2 Integration of Continuous Functions

Notation 2.1. Let $a < b \in \mathbb{R}$. We write

$$\zeta([a,b],\mathbb{R}) = \{f : [a,b] \to \mathbb{R} \mid f \text{ is continuous on } [a,b]\}$$

Note that this is a vector space over $\mathbb{R}$.

Definition 2.2. Let $\emptyset \neq E \subseteq \mathbb{R}$ be a non-empty set and let $f : E \to \mathbb{R}$ be a function. We say that f is **uniformly continuous** on E if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that $x, y \in E$ and $|x - y| < \delta$ implies that $|f(x) - f(y)| < \varepsilon$.

That is, $\delta > 0$ is chosen independently of $x, y \in E$ (but it still depends on $\varepsilon > 0$).

Theorem 2.3. Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Then f is uniformly continuous on $[a,b]$.

Proof. On assignment 2. $\square$

Remark 2.4. The fact that $[a,b]$ is closed and bounded is crucial to the above theorem. For example, if

$$\begin{aligned} f : [0,\infty) &\to \mathbb{R} \\ x &\mapsto x^2 \end{aligned}$$

then f is continuous on $[0,\infty)$, but it is **not** uniformly continuous.

Proof. Suppose $\varepsilon = 1$. Let $\delta > 0$ and consider $0 < x$, and set $y = x + \delta/2$.

Then

$$|x - y| < \delta/2$$

while

$$\begin{aligned} |f(y) - f(x)| &= |(x + \delta/2)^2 - x^2| \\ &= |x^2 + \delta x + \delta^2/4 - x^2| \\ &= \delta x + \delta^2/4 \\ &> \delta x \end{aligned}$$

But if $x > 1/\delta$, then $|x - (x + \delta/2)| = \delta/2 < \delta$, but

$$|f(x + \delta/2) - f(x)| > \delta x > 1 = \varepsilon$$

Thus f is not uniformly continuous on $[0,\infty)$. $\square$

Theorem 2.5. Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Then f is Riemann integrable. That is,

$$\zeta([a,b],\mathbb{R}) \subseteq \mathcal{R}[a,b].$$

Proof. Let $f \in \zeta([a,b],\mathbb{R})$. Let $\varepsilon > 0$.

By Theorem 2.3, since $f : [a,b] \to \mathbb{R}$ is continuous, it is uniformly continuous on $[a,b]$.

Thus, there exists a $\delta > 0$ such that $x, y \in [a,b]$ and $|x-y| < \delta$ implies that $|f(x) - f(y)| < \varepsilon$.

Let $P \in \mathcal{P}[a,b]$, say

$$P = \{a = p_0 < p_1 < \cdots < p_N = b\}$$

such that $\|P\| < \delta$.

(For example, consider the uniform partition with $\delta > \frac{b-a}{N}$, so that $\|P\| = \frac{b-a}{N} < \delta$.)

Now let

$$\begin{aligned} m_k &= \inf\{f(x) : x \in [p_{k-1}, p_k]\} \\ &= \min\{f(x) : x \in [p_{k-1}, p_k]\} && \text{(by continuity)} \\ &= f(x_k^*), \ \ x_k^* \in [p_{k-1}, p_k] \\ M_k &= \sup\{f(x) : x \in [p_{k-1}, p_k]\} \\ &= \max\{f(x) : x \in [p_{k-1}, p_k]\} && \text{(by continuity)} \\ &= f(y_k^*), \ \ y_k^* \in [p_{k-1}, p_k] \end{aligned}$$

Thus

$$\begin{aligned} U(f,P) - L(f,P) &= \sum_{k=1}^{N} (M_k - m_k)(p_k - p_{k-1}) \\ &= \sum_{k=1}^{N} \left(f(y_k^*) - f(x_k^*)\right)(p_k - p_{k-1}) \end{aligned}$$

But $f(x_k^*), f(y_k^*) \in [p_{k-1}, p_k]$ implies that

$$|x_k^* - y_k^*| \leq p_k - p_{k-1} \leq \|P\| < \delta$$

Thus $|f(x_k^*) - f(y_k^*)| < \delta$, $1 \leq k \leq N$.

Hence

$$\begin{aligned} U(f,P) - L(f,P) &< \sum_{k=1}^{N} \varepsilon(p_k - p_{k-1}) \\ &= \varepsilon(p_N - p_0) \\ &= \varepsilon(b-a) \end{aligned}$$

Since $b - a$ is constant, this suffices to show that $f \in \mathcal{R}[a,b]$. $\square$

2.6 (Estimating Integrals). Let $f : [a,b] \to \mathbb{R}$ be an integrable function. Let

$$P = \{a = p_0 < p_1 < \cdots < p_N = b\} \in \mathcal{P}[a,b]$$

and define m_k, M_k as usual.

Suppose that $p_k^* \in [p_{k-1}, p_k]$, $1 \leq k \leq N$.

Then $m_k \leq f(p_k^*) \leq M_k$ for all $1 \leq k \leq N$, and so

$$\begin{aligned} L(f,P) &= \sum_{k=1}^{N} m_k(p_k - p_{k-1}) \\ &\leq \sum_{k=1}^{N} f(p_k^*)(p_k - p_{k-1}) \\ &\leq \sum_{k=1}^{N} M_k(p_k - p_{k-1}) \\ &= U(f,P) \end{aligned}$$

Note that if $\varepsilon > 0$, then by the Cauchy Criterion, there exists a partition $P \in \mathcal{P}[a,b]$ such that

$$U(f,P) - L(f,P) < \varepsilon.$$

Observe that

$$\int_a^b f, \sum_{k=1}^{N} f(p_k^*)(p_k - p_{k-1}) \in [L(f,P), U(f,P)]$$

and so

$$\left| \int_a^b f - \sum_{k=1}^{N} f(p_k^*)(p_k - p_{k-1}) \right| \leq U(f,P) - L(f,P) < \varepsilon$$

In the case where f is continuous on $[a,b]$, then we saw in Theorem 2.5 that by choosing $\delta > 0$ such that $|x-y| < \delta$ implies $|f(x) - f(y)| < \frac{\varepsilon}{(b-a)+1}$, we have that $\|P\| < \delta$ implies that

$$U(f,P) - L(f,P) < \varepsilon$$

and so for such a partition, choosing **any** $p_k^* \in [p_{k-1}, p_k]$ yields

$$\left| \int_a^b f - \sum_{k=1}^{N} f(p_k^*)(p_k - p_{k-1}) \right| < \varepsilon.$$

Example 2.7. Let $f(x) = 1/x$ on $[2,3]$.
Estimate $\int_2^3 \frac{1}{x}\,dx$ to within an error of $\frac{1}{20}$.

Note that f is continuous on $[2,3]$, so we can use the "special case" of Remark 2.6 to estimate.

Here, we want $\varepsilon = \frac{1}{20}$, and $[a,b] = [2,3]$, so that $b - a = 1$.

We need to find $\delta > 0$ such that if $|x-y| < \delta$, then $|f(x) - f(y)| < \frac{\varepsilon}{(b-a)+1} = \frac{1}{20}$.

$$\begin{aligned} |f(x) - f(y)| = \left| \frac{1}{x} - \frac{1}{y} \right| = \left| \frac{x-y}{xy} \right| &= \frac{|x-y|}{xy} \\ &\leq \frac{|x-y|}{2 \cdot 2} \\ &= \frac{|x-y|}{4} < \frac{1}{20} \end{aligned}$$

Then $|x-y| < \frac{4}{20} = \frac{1}{5}$, so we can take $\delta = \frac{1}{6}$.

We need a partition $P \in \mathcal{P}[a,b]$ with $\|P\| \leq \delta = \frac{1}{6}$.

Consider the uniform partition $P = \left\{2, 2\frac{1}{6}, 2\frac{1}{3}, 2\frac{1}{2}, 2\frac{2}{3}, 2\frac{5}{6}, 3\right\}$.

We now pick points $p_k^* \in [p_{k-1}, p_k]$, $1 \leq k \leq 6$.

For example, let us take $p_k^* = p_k$, the right endpoints. Then our estimate is

$$\begin{aligned}\sum_{k=1}^{6} f(p_k^*)(p_k - p_{k-1}) &= \left(\frac{1}{2\frac{1}{6}} + \frac{1}{2\frac{1}{3}} + \cdots + \frac{1}{3}\right)\left(\frac{1}{6}\right) \\ &\approx 0.3918974\end{aligned}$$

Note that $f(x) = 1/x$ is decreasing, so this estimate is actually $L(f,P)$.

Exercise: Try $p_k^* = p_{k-1}$ and $p_k^* = \frac{p_k + p_{k-1}}{2}$.

Theorem 2.8. If $f : [a,b] \to \mathbb{R}$ is Riemann integrable, so is $|f|$. Moreover,

$$\left|\int_a^b f\right| \leq \int_a^b |f|$$

Proof. We will use the Cauchy Criterion.

Given $\varepsilon > 0$, choose a partition $P = \{a = p_0 < p_1 < \cdots < p_N = b\}$ so that

$$U(f,P) - L(f,P) = \sum_{k=1}^{N} \big(M_k(f,P) - m_k(f,P)\big)(p_k - p_{k-1}) < \varepsilon$$

Then

$$U(|f|,P) - L(|f|,P) = \sum_{k=1}^{N} \big(M_k(|f|,P) - m_k(|f|,P)\big)(p_k - p_{k-1})$$

We have

$$M_k(|f|,P) - m_k(|f|,P) = \sup_{x,y \in [p_{k-1},p_k]} \big||f(x)| - |f(y)|\big|$$

Recall: $\big||w| - |z|\big| \leq |w - z|$ (the reverse triangle inequality)

$$\begin{aligned} &\leq \sup_{x,y \in [p_{k-1},p_k]} |f(x) - f(y)| \\ &= M_k(f,P) - m_k(f,P)\end{aligned}$$

Therefore

$$U(|f|,P) - L(|f|,P) \leq U(f,P) - L(f,P) < \varepsilon$$

Hence $|f| \in \mathcal{R}[a,b]$ by the Cauchy Criterion.

Now, we have

$$\int_a^b f = \inf\{U(f,P) : P \in \mathcal{P}[a,b]\}$$
$$\int_a^b |f| = \inf\{U(|f|,P) : P \in \mathcal{P}[a,b]\}$$

We will show that $U(|f|,P) \geq U(f,P)$ for all $P \in \mathcal{P}[a,b]$.

$$U(|f|,P) = \sum_{k=1}^{N} M_k(|f|,P)(p_k - p_{k-1})$$

and

$$\begin{aligned} M_k(|f|,P) &= \sup_{x\in[p_{k-1},p_k]} |f(x)| \\ &\geq \sup_{x\in[p_{k-1},p_k]} f(x) \\ &= M_k(f,P) \end{aligned}$$

Hence

$$U(|f|,P) \geq U(f,P)$$

which implies

$$\int_a^b f \leq \int_a^b |f|$$

Also,

$$-\int_a^b f = \int_a^b (-f) \leq \int_a^b |-f| = \int_a^b |f|$$

where the inequality is the result of what was just shown above.

Therefore,

$$\int_a^b |f| \geq \int_a^b f \text{ and } \int_a^b |f| \geq -\int_a^b f$$

which implies that

$$\left|\int_a^b f\right| \leq \int_a^b |f|. \ \square$$

Corollary 2.9. If $f, g \in \mathcal{R}[a,b]$, then $fg \in \mathcal{R}[a,b]$.

Proof.

Step 1. Prove that if $h \in \mathcal{R}[a,b]$, then $h^2 \in \mathcal{R}[a,b]$.

Suppose $h \in \mathcal{R}[a,b]$. Note that h^2 is bounded because h is also bounded. Furthermore, $|h| \in \mathcal{R}[a,b]$ by Theorem 2.8.

Let $\varepsilon > 0$. By the Cauchy Criterion, there exists a partition $P \in \mathcal{P}[a,b]$ such that

$$U(|h|,P) - L(|h|,P) < \frac{\varepsilon}{2\|h\|_\infty + 1}.$$

Consider

$$\begin{aligned} M_k(h^2, P) &= \sup\left\{\big(h(x)\big)^2 : x \in [p_{k-1}, p_k]\right\} \\ &= \sup\left\{|h(x)|^2 : x \in [p_{k-1}, p_k]\right\} \\ &= \Big(\sup\left\{|h(x)| : x \in [p_{k-1}, p_k]\right\}\Big)^2 \\ &= \big(M_k(h, P)\big)^2, \ 1 \le k \le N \end{aligned}$$

and similarly

$$m_k(h^2, P) = \big(m_k(h, P)\big)^2, \ 1 \le k \le N$$

Observe that

$$\begin{aligned} m_k(|h|, P) &\le M_k(|h|, P) \\ &\le \sup\{|h(x)| : x \in [a,b]\} =: \|h\|_\infty \end{aligned}$$

Thus

$$\begin{aligned} U(h^2, P) - L(h^2, P) &= \sum_{k=1}^{N} \Big(M_k(h^2, P) - m_k(h^2, P)\Big)(p_k - p_{k-1}) \\ &= \sum_{k=1}^{N} \Big[\big(M_k(|h|, P)\big)^2 - \big(m_k(|h|, P)\big)^2\Big](p_k - p_{k-1}) \\ &= \sum_{k=1}^{N} \big[M_k(|h|, P) + m_k(|h|, P)\big]\big[M_k(|h|, P) - m_k(|h|, P)\big](p_k - p_{k-1}) \\ &\le \sum_{k=1}^{N} 2\|h\|_\infty \big[M_k(|h|, P) - m_k(|h|, P)\big](p_k - p_{k-1}) \\ &= 2\|h\|_\infty \big(U(|h|, P) - L(|h|, P)\big) \\ &< 2\|h\|_\infty \left(\frac{\varepsilon}{2\|h\|_\infty + 1}\right) \\ &< \varepsilon \end{aligned}$$

By the Cauchy Criterion, $h^2 \in \mathcal{R}[a,b]$.

Step 2. Show that $fg \in \mathcal{R}[a,b]$.

Note that

$$fg = \underbrace{\frac{1}{2}(f+g)^2}_{\in\mathcal{R}[a,b]} - \underbrace{\frac{1}{2}f^2}_{\in\mathcal{R}[a,b]} - \underbrace{\frac{1}{2}g^2}_{\in\mathcal{R}[a,b]}$$

Hence $fg \in \mathcal{R}[a,b]$. ■

Theorem 2.10 (First Mean Value Theorem For Integrals)**.** Suppose that $f, g \in \mathcal{R}[a,b]$ and that $0 \le g$. Let

$$\begin{aligned} m &= \inf\{f(x) : x \in [a,b]\} \\ M &= \sup\{f(x) : x \in [a,b]\} \end{aligned}$$

Then

(a) there exists $\beta \in [m, M]$ such that
$$\int_a^b fg = \beta \int_a^b g$$

(b) if f is continuous on $[a, b]$, then there exists $c \in [a, b]$ such that
$$\int_a^b fg = f(c) \int_a^b g$$

Proof. Recall that if $f, g \in \mathcal{R}[a, b]$ and $f \leq g$, then
$$\int_a^b f \leq \int_a^b g.$$

(a) Since $m \leq f(x) \leq M$ for all $x \in [a, b]$, and since $0 \leq g(x)$ for all $x \in [a, b]$, we see that
$$mg \leq fg \leq Mg$$
on $[a, b]$ and so
$$m \int_a^b g = \int_a^b mg \leq \int_a^b fg \leq \int_a^b Mg = M \int_a^b g$$
Hence there exists $m \leq \beta \leq M$ such that
$$\int_a^b fg = \beta \int_a^b g$$

(b) Suppose also that f is continuous on $[a, b]$.

We find that there must exists $d, e \in [a, b]$ such that $m = f(d)$ and $M = f(e)$.

By the Intermediate Value Theorem, there exists $c \in [\min(d, e), \max(d, e)] \subseteq [a, b]$ such that with β as in part (a), $\beta = f(c)$.

Hence by part (a),
$$\int_a^b fg = \beta \int_a^b g = f(c) \int_a^b g. \square$$

Notation 2.11. Recall that if $0 \leq f \in \mathcal{R}[a, b]$, then we defined the "area under the curve $y = f(x)$ from $x = a$ to $x = b$" to be
$$\int_a^b f.$$

It can be shown that $\int_a^b f$ is a "limit" of $L(f, P)$ where $\|P\| \to 0$.

Note: uniform partitions do not always suffice.

Leibniz thought of this limiting process as taking a generalized sum of "areas of rectangles" whose base is the point "x" (with infinitesimally small width dx) and height $f(x)$. The sum he expressed was
$$\int_a^b f(x)\, dx.$$

We define

$$\int_a^b f(x)\,dx := \int_a^b f$$

This "x" is a dummy variable – it can be replaced by anything (other than a, b, and f, which are already defined). Hence

$$\int_a^b f(x)\,dx = \int_a^b f(y)\,dy.$$

We also introduce the following notation:

Suppose $\emptyset \neq E \subseteq \mathbb{R}$. $F : E \to \mathbb{R}$ is a function with $a, b \in E$. We write

$$F|_{x=a}^{x=b} = F(x)|_{x=a}^{x=b} = F(x)\big]_{x=a}^{x=b} = \big[F(x)\big]_{x=a}^{x=b}$$

to mean $F(b) - F(a)$.

Definition 2.12. Recall that if $a < c < b \in \mathbb{R}$ and $f \in \mathcal{R}[a, b]$, then $f \in \mathcal{R}[a, c]$.

Thinking of c as a variable, and writing "x" in the place of "c", we can define the **definite integral** of f over $[a, b]$ as

$$F(x) = \int_a^x f.$$

Example 2.13. Let

$$f(x) = \begin{cases} 2 & x \in [0, 1] \\ -1 & x \in [-1, 0) \end{cases}$$

and set

$$F(x) = \int_0^x f.$$

For $0 \leq x \leq 1$,

$$F(x) = \int_0^x f = \int_0^x 2 = 2x.$$

For $-1 \leq x < 0$,

$$F(x) = \int_0^x (-1) = -\int_0^x 1 = \int_x^0 1 = -x$$

Hence

$$F(x) = \begin{cases} 2x & x \in [0, 1] \\ -x & x \in [-1, 0) \end{cases}$$

Note that F is continuous at 0, even though f is not.

Theorem 2.14. Let $a < b \in \mathbb{R}$, $f \in \mathcal{R}[a, b]$ and $F(x) = \int_a^x f$, $x \in [a, b]$. Then F is continuous on $[a, b]$.

Proof. Since $f \in \mathcal{R}[a, b]$, f is bounded on $[a, b]$.

Choose $M > 0$ such that $|f(x)| \leq M$ for all $x \in [a, b]$.

Let $x_0 \in [a, b]$ and $\varepsilon > 0$. Set $\delta = \varepsilon / M$.

If $x \in [a, b]$ and $|x - x_0| < \delta$, then

$$\begin{aligned}
|F(x) - F(x_0)| &= \left| \int_a^x f - \int_a^{x_0} f \right| \\
&= \left| \int_{x_0}^x f \right| \\
&\leq \int_{\min(x_0,x)}^{\max(x_0,x)} |f| \\
&\leq \int_{\min(x_0,x)}^{\max(x_0,x)} M \\
&= M|x_0 - x| \\
&< M\delta = \varepsilon
\end{aligned}$$

By definition, F is continuous at x_0, but $x_0 \in [a, b]$ was arbitrary, so F is continuous on $[a, b]$. □

Theorem 2.15 (Second Mean Value Theorem for Integrals)**.** Suppose $f, g \in \mathcal{R}[a, b]$, $0 \leq g$, and $m \leq f(x) \leq M$ for all $x \in [a, b]$. Then

(a) there exists $c \in [a, b]$ such that

$$\int_a^b fg = m \int_a^c g + M \int_c^b g$$

(b) if $m = 0$, then there exists $c \in [a, b]$ such that

$$\int_a^b fg = M \int_b^c g.$$

Proof.

(a) Note that $0 \leq g$ and $m \leq f \leq M$ on $[a, b]$ implies that $mg \leq fg \leq Mg$ on $[a, b]$, and since $f, g \in \mathcal{R}[a, b]$ implies $fg \in \mathcal{R}[a, b]$ by Corollary 2.9, we obtain

$$m \int_a^b g = \int_a^b mg \leq \int_a^b fg \leq \int_a^b Mg = M \int_a^b g.$$

Let

$$H(x) = m \underbrace{\int_a^x g}_{\text{cont. on } x} + M \underbrace{\int_x^b g}_{\text{cont. on } x}$$

for all $x \in [a, b]$. By Theorem 2.14, it follows that H is continuous on $[a, b]$. Now,

$$\begin{aligned}
H(a) &= m \int_a^a g + M \int_a^b g = M \int_a^b g \geq \int_a^b fg \\
H(b) &= m \int_a^b g + M \int_b^b g = m \int_a^b g \leq \int_a^b fg
\end{aligned}$$

By the Intermediate Value Theorem, there exists $c \in [a, b]$ such that

$$H(c) = m \int_a^c g + M \int_c^b g = \int_a^b fg.$$

(b) This follows immediately from part (a). $\square$

Remark 2.16. In the special case where $0 \leq f$ and $g \equiv 1$ on $[a, b]$, Theorem 2.15 compares the area under the curve $y = f(x)$, $x \in [a, b]$ with the area of a rectangle with height M and base $[c, b]$ for some $c \in [a, b]$.

3 The Fundamental Theorem of Calculus

Remark 3.1. The following observation, while simple, is very useful.

Suppose $a < b \in \mathbb{R}$, $f \in \mathcal{R}[a,b]$, and $\beta \in \mathbb{R}$. If

$$L(f,P) \leq \beta \leq U(f,P) \ \forall P \in \mathcal{P}[a,b],$$

then $\beta = \int_a^b f$.

Indeed,

$$\begin{aligned}\int_a^b f &= L\int_a^b f \\ &= \sup\{L(f,P) : P \in \mathcal{P}[a,b]\} \\ &\leq \beta \\ &\leq \inf\{U(f,P) : P \in \mathcal{P}[a,b]\} \\ &= U\int_a^b f \\ &= \int_a^b f\end{aligned}$$

whence $\beta = \int_a^b f$.

Theorem 3.2 (The Fundamental Theorem of Calculus, Part I)**.** Let $a < b \in \mathbb{R}$, $f, F : [a,b] \to \mathbb{R}$ be functions satisfying

(a) $f \in \mathcal{R}[a,b]$

(b) F is continuous on $[a,b]$

(c) F is differentiable on (a,b) with $F'(x) = f(x)$, $x \in (a,b)$

Then $\int_a^b f = F(b) - F(a)$.

Proof. We shall apply Remark 3.1 with $\beta = F(b) - F(a)$; that is, we shall show that

$$L(f,P) \leq F(b) - F(a) \leq U(f,P) \ \forall P \in \mathcal{P}[a,b].$$

Let $P \in \mathcal{P}[a,b]$, say $P = \{a = p_0 < p_1 < \cdots < p_N = b\}$. Then

$$\begin{aligned}F(b) - F(a) &= F(p_N) - F(0) \\ &= \sum_{k=1}^{N} F(p_k) - F(p_{k-1})\end{aligned}$$

Now, for each $1 \leq k \leq N$, F is continuous on $[p_{k-1}, p_k]$ and differentiable on (p_{k-1}, p_k), so by the Mean Value Theorem, for some $p_k^* \in (p_{k-1}, p_k)$,

$$\begin{aligned}F(p_k) - F(p_{k-1}) &= F'(p_k^*)(p_k - p_{k-1}) \\ &= f(p_k^*)(p_k - p_{k-1})\end{aligned}$$

Now $m_k \leq f(p_k^*) \leq M_k$ for all $1 \leq k \leq N$ and so

$$\begin{aligned} L(f,P) &= \sum_{k=1}^{N} m_k(p_k - p_{k-1}) \\ &\leq \sum_{k=1}^{N} f(p_k^*)(p_k - p_{k-1}) \\ &= \sum_{k=1}^{N} F(p_k) - F(p_{k-1}) \\ &= F(b) - F(a) \\ &= \sum_{k=1}^{N} f(p_k^*)(p_k - p_{k-1}) \\ &\leq \sum_{k=1}^{N} M_k(p_k - p_{k-1}) \\ &= U(f,P) \end{aligned}$$

By Remark 3.1, since $P \in \mathcal{P}[a,b]$ was arbitrary,

$$\int_a^b f = F(b) - F(a). \ \square$$

Example 3.3. Let $f(x) = \sin x$, $x \in [0,\pi]$. Let $F(x) = -\cos x + \sqrt[17]{e^{\pi}\left(\frac{1}{\sqrt{2}}\right)^{63}}$.

Then f and F satisfy the conditions of FTC-1 for $[0,\pi]$, so by that theorem,

$$\begin{aligned} \int_0^{\pi} \sin x\, dx &= F(\pi) - F(0) \\ &= \left(-\cos\pi + \sqrt[17]{e^{\pi}\left(\frac{1}{\sqrt{2}}\right)^{63}}\right) - \left(-\cos 0 + \sqrt[17]{e^{\pi}\left(\frac{1}{\sqrt{2}}\right)^{63}}\right) \\ &= -(-1) - (-1) \\ &= 2 \end{aligned}$$

Remark 3.4. The choice of F in Example 3.3 is **not** unique, as the choice of F in FTC-1 is **not** unique.

Suppose f, F satisfy the conditions of FTC-1. If $\kappa \in \mathbb{R}$ is a constant, and

$$G(x) := F(x) + \kappa,\ x \in [a,b]$$

then f, G also satisfy FTC-1, so

$$\begin{aligned} \int_a^b f &= G(b) - G(a) \\ &= \big(F(b) + \kappa\big) - \big(F(a) + \kappa\big) \\ &= F(b) - F(a) \end{aligned}$$

Conversely, if (f, F) and (f, G) both satisfy the conditions of FTC-1, then by (c),

$$F'(x) = f(x) = G'(x)$$

so there exists $\kappa \in \mathbb{R}$ such that

$$G(x) = F(x) + \kappa,\ x \in (a, b)$$

But F, G are continuous, so

$$G(x) = F(x) + \kappa,\ x \in [a, b].$$

Example 3.5. Evaluate the following:

(a) $\displaystyle\int_0^1 \frac{1}{1+x^2}\,dx$

Let $f(x) = \frac{1}{1+x^2}$, $x \in [0, 1]$.

Note that $f \in \zeta([0, 1], \mathbb{R}) \subseteq \mathcal{R}[0, 1]$.

Let $F(x) = \arctan x$, $x \in [0, 1]$, with F continuous on $[0, 1]$ and $F'(x) = \frac{1}{1+x^2}$, $x \in (0, 1)$.

Then by FTC-1,

$$\begin{aligned}\int_0^1 \frac{1}{1+x^2} dx &= F(1) - F(0) \\ &= \arctan(1) - \arctan(0) \\ &= \pi/4 - 0 \\ &= \pi/4\end{aligned}$$

(b) $\displaystyle\int_2^3 \frac{1}{x} dx \overset{\text{FTC-1}}{=} \ln x\Big|_{x=2}^{x=3} = \ln 3 - \ln 2 = \ln 3/2$

(c) $\displaystyle\int_{-1}^0 (x^3 + 2x^2) dx = \frac{x^4}{4} + \frac{2}{3}x^3\Big|_{x=-1}^{x=0} = (0 + 0) - \left(\frac{1}{4} - \frac{2}{3}\right) = \frac{5}{12}$

(d) $\displaystyle\int_0^{1/2} \frac{1}{\sqrt{1-x^2}} dx = \arcsin x\Big|_{x=0}^{x=1/2} = \arcsin(1/2) - \arcsin(0) = \frac{\pi}{6} - 0 = \frac{\pi}{6}$

(e) $\displaystyle\int_{-1}^1 e^{-x} dx = -e^{-x}\Big|_{x=-1}^{x=1} = -e^{-1} - (-e^1) = e - \frac{1}{e}$

Remark 3.6. Although FTC-1 is a great theorem, it is not perfect.

Given $f \in \mathcal{R}[a, b]$, why should f admit an antiderivative F? (In fact, it need not.) Even if it does, how do we find F? Also, can we always express F in a "nice" (computationally speaking) way?

For example, what should F be if f is one of:

$$f_1(x) = \frac{\sin x}{x} \qquad f_2(x) = \sin(x^2)$$
$$f_3(x) = e^{-x^2} \qquad f_4(x) = \sqrt{1 + x^3}$$

Theorem 3.7 (Darboux's Theorem)**.** Suppose that $F : [a, b] \to \mathbb{R}$ is differentiable on $[a, b]$ (i.e. right-differentiable at a, left-differentiable at b, and differentiable on (a, b)).

Suppose that $y_0 \in \mathbb{R}$ satisfies

$$F'(a) < y_0 < F'(b).$$

Then there exists $c \in (a, b)$ such that $F'(c) = y_0$.

Note: If F' is continuous on $[a, b]$, then this is a trivial application of the Intermediate Value Theorem. But we are **not** assuming that F' is continuous.

Proof. Since F' exists for all $x \in [a, b]$, F is continuous on $[a, b]$.

Let $G(x) = F(x) - y_0 x$, $x \in [a, b]$. Then G is continuous on $[a, b]$. By the Extreme Value Theorem, G attains its minimum at some point $c \in [a, b]$.

Note that

$$G'(a) = F'(a) - y_0 < 0$$

hence the minimum does not occur at a, and

$$G'(b) = F'(b) - y_0 > 0$$

the minimum does not occur at b.

Hence $a < c < b$, but $G'(c)$ exists, so it must be the case that $G'(c) = 0$. That is,

$$0 = G'(c) = F'(c) - y_0 \implies F'(c) = y_0. \ \square$$

Example 3.8. Consider the function

$$F(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

For $x \neq 0$,

$$\begin{aligned} F'(x) &= 2x \sin(1/x) + x^2(\cos 1/x)(-1/x^2) \\ &= 2x \sin(1/x) - \cos(1/x) \end{aligned}$$

Observe that $\lim_{x\to 0} 2x \sin(1/x) = 0$, while $\lim_{x\to 0} \cos(1/x)$ does not exist. Hence, it follows that $\lim_{x\to 0} F'(x)$ does not exist.

In particular, F' is not continuous at 0. Moreover,

$$\begin{aligned} F'(0) &= \lim_{h\to 0} \frac{F(0+h) - F(0)}{h} \\ &= \lim_{h\to 0} \frac{h^2 \sin(1/h) - 0}{h} \\ &= \lim_{h\to 0} h \sin(1/h) = 0 \end{aligned}$$

Hence F' exists for all $x \in \mathbb{R}$, but F' is not continuous at 0.

Example 3.9. Consider the function

$$f(x) = \begin{cases} 2 & x \in [0,1] \\ -1 & x \in [-1,0) \end{cases}$$

Recall that $f \in \mathcal{R}[-1,1]$.

Suppose $F : [-1,1] \to \mathbb{R}$ is continuous and $F'(x) = f(x)$, $x \in [-1,1]$. Then

$$F'(-1) = f(-1) = -1 < 0$$
$$F'(1) = f(1) = 2 > 0$$

so by Darboux's Theorem, there must exist $c \in (-1,1)$ such that

$$f(c) = F'(c) = 0.$$

But $f(x) \neq 0$ for all $x \in [-1,1]$, a contradiction.

Hence, such an F does not exist.

Theorem 3.10 (Fundamental Theorem of Calculus, Part II)**.** Let $f \in \zeta([a,b],\mathbb{R})$ and set

$$F(x) = \int_a^x f(t)dt,\ x \in [a,b].$$

Then F is

(a) continuous on $[a,b]$

(b) differentiable on $[a,b]$, and

(c) $F'(x) = f(x)$, $x \in [a,b]$.

Proof. We leave the right-differentiability of F at a and the left-differentiability of F at b as exercises.

Suppose $x_0 \in (a,b)$. We prove that

$$F'(x_0) = f(x_0).$$

Let $\varepsilon > 0$. Since f is continuous at x_0, there exists $0 < \delta < \min(x_0 - a, b - x_0)$ such that

$$|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon.$$

Consider, for $0 < |h| < \delta$,

$$\begin{aligned}
\left|\frac{F(x_0+h)-F(x_0)}{h} - f(x_0)\right| &= \left|\frac{\int_a^{x_0+h} f(t)dt - \int_a^{x_0} f(t)dt}{h} - f(x_0)\right| \\
&= \left|\frac{\int_{x_0}^{x_0+h} f(t)dt}{h} - \frac{\int_{x_0}^{x_0+h} f(x_0)dt}{h}\right| \\
&= \frac{1}{|h|}\left|\int_{x_0}^{x_0+h} \Big(f(t)-f(x_0)\Big)dt\right| \\
&\le \frac{1}{|h|}\int_{\min(x_0,x_0+h)}^{\max(x_0,x_0+h)} |f(t)-f(x_0)|dt \\
&\le \frac{1}{|h|}\int_{\min(x_0,x_0+h)}^{\max(x_0,x_0+h)} \varepsilon dt \\
&= \frac{|h|}{|h|}\varepsilon \\
&= \varepsilon
\end{aligned}$$

By the definition of $F'(x_0)$,

$$F'(x_0) = f(x_0). \ \square$$

Remark 3.11. This result tells us that the antiderivative of a continuous function f on $[a, b]$ always exists; namely

$$F(x) = \int_a^x f(t)dt.$$

3.12. The following is a short list of antiderivatives you should know. C is a constant.

(a) $\displaystyle\int x^r dx = \frac{x^{r+1}}{r+1} + C$, $r \neq -1$

(b) $\displaystyle\int x^{-1}dx = \ln|x| + C$

(c) $\displaystyle\int \sin x dx = -\cos x + C$

(d) $\displaystyle\int \cos x dx = \sin x + C$

(e) $\displaystyle\int \sec^2 x dx = \tan x + C$

(f) $\displaystyle\int \frac{1}{1+x^2}dx = \arctan x + C$

(g) $\displaystyle\int \frac{1}{\sqrt{1-x^2}}dx = \arcsin x + C$

(h) $\displaystyle\int e^x dx = e^x + C$

3.13. Integration is, in essence, "reverse engineering" differentiation. Here, we are required to recreate the steps we used in differentiation, but in reverse order. For example, consider $f(x) = \tan x$. Then

$$\begin{aligned} f'(x) &= \left(\frac{\sin x}{\cos x}\right)' \\ &= \frac{(\sin x)' \cos x - \sin x (\cos x)'}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ &= \frac{1}{\cos^2 x} \\ &= \sec^2 x \end{aligned}$$

Proposition 3.14 (The Substitution Rule)**.** Let $g : [a, b] \to \mathbb{R}$ be a continuously differentiable function. That is, $g'(x)$ exists for all $x \in [a, b]$ and $g' : [a, b] \to \mathbb{R}$ is continuous. Then there exists $c \leq d \in \mathbb{R}$ such that

$$g([a, b]) = \{g(x) : x \in [a, b]\} = [c, d].$$

If $f : [c, d] \to \mathbb{R}$ is continuous, then

$$\int_{g(a)}^{g(b)} f(s)ds = \int_a^b f\left(g(t)\right) g'(t)dt.$$

Proof. Note that g is differentiable on $[a, b]$, so g is continuous on $[a, b]$.

As such, g achieves its minimum, say c, and its maximum, say d, at the points $x_0, x_1 \in [a, b]$ respectively. That is,

$$\begin{aligned} c = g(x_0) &= \min\{g(x) : x \in [a, b]\} \\ d = g(x_1) &= \max\{g(x) : x \in [a, b]\} \end{aligned}$$

so that

$$g([a, b]) = \{g(x) : x \in [a, b]\} \subseteq [c, d].$$

But g is continuous on $[a, b]$, hence g is continuous on $[\min(x_0, x_1), \max(x_0, x_1)]$.

By the Intermediate Value Theorem, for each $\beta \in [c, d]$, there exists $x_\beta \in [\min(x_0, x_1), \max(x_0, x_1)]$ such that

$$g(x_\beta) = \beta.$$

Hence $g([a, b]) \subseteq [c, d]$, whence

$$g([a, b]) = [c, d].$$

Now define $G(x) = \displaystyle\int_a^x f\left(g(t)\right) g'(t)\, dt$ for all $x \in [a, b]$. Noting that $f\left(g(t)\right) g'(t)$ is continuous, then

$$\begin{aligned} F(y) &= \int_{g(a)}^{y} f(s)\, ds,\ y \in [c, d] \\ &= \int_c^y f(s)\, ds - \underbrace{\int_c^{g(a)} f(s)ds}_{\text{constant}} \end{aligned}$$

Then by FTC-II,

$$G'(x) = f\left(g(x)\right) g'(x),\ x \in [a, b]$$

while

$$\frac{dF}{dy} = f(y),\ y \in [c, d].$$

Let $y = g(x)$, $x \in [a, b]$. Then

$$\begin{aligned}(F \circ g)'(x) &= \frac{d}{dx} F\left(g(x)\right) \\ &= \frac{d}{dx} F(y) \\ &= \frac{dF}{dy}\frac{dy}{dx} \\ &= f(y)g'(x) \\ &= f\left(g(x)\right) g'(x) \\ &= G'(x)\end{aligned}$$

and thus there exists $\kappa \in \mathbb{R}$ such that

$$F \circ g(x) = G(x) + \kappa,\ x \in [a, b].$$

But

$$\begin{aligned}F \circ g(a) = F\left(g(a)\right) &= \int_{g(a)}^{g(a)} f(s)\, ds = 0 \\ G(a) &= \int_a^a f\left(g(t)\right) g'(t) dt = 0\end{aligned}$$

so $\kappa = F \circ g(a) - G(a) = 0 - 0 = 0$. That is, $F \circ g = G$. From this, we obtain

$$F \circ g(b) = G(b),$$

that is,

$$\begin{aligned}F \circ g(b) = F\left(g(b)\right) &= \int_{g(a)}^{g(b)} f(s)\, ds \\ &= G(b) = \int_a^b f\left(g(t)\right) g'(t) dt. \ \square\end{aligned}$$

3.15. Our strategy for computing integrals via the Substitution Rule is as follows.

To find $J = \displaystyle\int f\left(g(x)\right) g'(x\, dx$,

(I) We set $u = g(x)$ so that $\dfrac{du}{dx} = g'(x)$, and we write $du = g'(x)\, dx$.

(II) Find $F(u) = \displaystyle\int f(u)\, du$.

(III) We conclude that $J = F(u) + C = F\left(g(x)\right) + C$, where C is a constant.

3.16 (Integration by Parts)**.** Recall that if $f, g : [a, b] \to \mathbb{R}$ are differentiable, then

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x),$$

called the product rule. The inverse process for integration is called **integration by parts**.

$$\begin{aligned} fg(x) &= \int (fg)'(x)\,dx \\ &= \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx \end{aligned}$$

and so

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx.$$

In applications, we write

$$\begin{aligned} u &= f(x) & dv &= g'(x)\,dx \\ du &= f'(x)\,dx & v &= g(x) \end{aligned}$$

which yields

$$\int u\,dv = uv - \int v\,du.$$

3.17 (Integrating Rational Functions)**.** A function r is said to be a **rational function** if $r(x) = p(x)/q(x)$, where p, q are polynomials.

Theorem. The set Rat of rational functions forms a vector space over $\mathbb{R}$.

Moreover, the set of functions

(i) $1, x, x^2, \ldots$

(ii) $\dfrac{1}{x-a}, \dfrac{1}{(x-a)^2}, \dfrac{1}{(x-a)^3}, \ldots,\ a \in \mathbb{R}$

(iii) $\dfrac{1}{x^2+bx+c}, \dfrac{1}{(x^2+bx+c)^2}, \dfrac{1}{(x^2+bx+c)^3}, \ldots,\ b, c \in \mathbb{R},\ b^2 - 4c < 0$

forms a basis for this vector space.

Proof. In linear algebra. □

As such, <u>any</u> rational function can be expressed as a <u>finite</u> linear combination of functions of types (i), (ii), and (iii).

3.18 (Improper Riemann Integration)**.** So far, we have defined Riemann integrals for bounded functions on closed, bounded intervals satisfying certain properties (i.e. the Cauchy Criterion). Now we extend the notion of integrability to not necessarily bounded functions on open, but not necessarily bounded intervals.

The key is the following observation from the Fundamental Theorem of Calculus: If $a < b \in \mathbb{R}$, $f \in \mathcal{R}[a,b]$ and if $F(x) = \int_a^x f(t)\,dt$, then F is continuous on $[a,b)$. Hence

$$\begin{aligned}
\int_a^b f(t)dt &= F(b) - F(a) \\
&= \lim_{d\to b^-} F(d) - \lim_{c\to a^+} F(c) \\
&= \lim_{d\to b^-} \lim_{c\to a^+} F(d) - F(c) \\
&= \lim_{d\to b^-} \lim_{c\to a^+} \int_c^d f(t)dt \\
&= \lim_{c\to a^+} \lim_{d\to b^-} F(d) - F(c).
\end{aligned}$$

This motivates the following definition.

Definition 3.19. Let $\mathbb{I} = (a,b)$ be an open (not necessarily bounded) interval, and let $f : \mathbb{I} \to \mathbb{R}$ be a function.

(a) We say that f is **locally integrable** on $\mathbb{I}$ if for all $[c,d] \subseteq \mathbb{I}$, $f \in \mathcal{R}[c,d]$.

(b) We say that f is **improperly integrable** on $\mathbb{I} = (a,b)$ if

  (i) f is locally integrable on $\mathbb{I}$.

  (ii) $\displaystyle\int_a^b f := \lim_{d\to b^-} \lim_{c\to a^+} \int_c^d f$ exists.

When this is the case, we refer to

$$\int_a^b f$$

as the **improper** (Riemann) integral of f over $\mathbb{I}$.

Example 3.20. If f is improperly integrable on $\mathbb{I} = (a,b)$, then

$$\begin{aligned}
\int_a^b f &= \lim_{d\to b^-} \lim_{c\to a^+} \int_c^d f \\
&= \lim_{c\to a^+} \lim_{d\to b^-} \int_c^d f.
\end{aligned}$$

Remark 3.21. We may also consider the case of half-open intervals $(a,b]$ or $[a,b)$. For example, we say that $f : (a,b] \to \mathbb{R}$ is **locally integrable** on $(a,b]$ if $f \in \mathcal{R}[c,d]$ for all $[c,d] \subseteq (a,b]$ and that f is **improperly integrable** on $(a,b]$ if f is locally integrable on $(a,b]$ and

$$\int_a^b f := \lim_{c\to a^+} \int_c^b f \text{ exists.}$$

Example 3.22.

(a) Consider $f(x) = x^{-1/3}$, $x \in (0,1]$.

Observe that f is **not** bounded on $(0,1]$. But for any $[c,d] \subseteq (0,1]$, then $f|_{[c,d]}$ is continuous, so $f \in \mathcal{R}[c,d]$. Hence f is locally integrable on $(0,1]$. Moreover,

$$\begin{aligned}
\int_0^1 x^{-1/3}\,dx &= \lim_{c\to 0^+}\int_c^1 x^{-1/3}\,dx \\
&= \lim_{c\to 0^+} \frac{3}{2}x^{2/3}\Big|_c^1 \\
&= \frac{3}{2}\lim_{c\to 0^+} 1 - c^{2/3} \\
&= \frac{3}{2}(1-0) = \frac{3}{2}
\end{aligned}$$

(b) Let $f(x) = 1/x^p$, $x \in [1,\infty)$, where $p > 1$ is a fixed real number.

We claim that f is improperly integrable over $[1,\infty)$.

Indeed, if $[c,d] \subseteq [1,\infty)$, then $f|_{[c,d]}$ is continuous, hence $f \in \mathcal{R}[c,d]$. That is, f is locally integrable on $[1,\infty)$. Moreover,

$$\begin{aligned}
\int_1^\infty \frac{1}{x^p}\,dx &= \lim_{d\to\infty}\int_1^d \frac{1}{x^p}\,dx \\
&= \lim_{d\to\infty}\int_1^d x^{-p}\,dx \\
&= \lim_{d\to\infty} \frac{x^{1-p}}{1-p}\Big|_1^d \\
&= \frac{1}{1-p}\lim_{d\to\infty}\frac{1}{d^{p-1}} - \frac{1}{1} \\
&= \frac{1}{1-p}(0-1) \\
&= \frac{1}{p-1}
\end{aligned}$$

Theorem 3.23 (The Comparison Theorem for Improper Integrals)**.** Suppose that f, g are locally integrable on an interval $\mathbb{I} = (a,b)$ and that $0 \le f \le g$ on $\mathbb{I}$.

If g is improperly integrable on $\mathbb{I}$, then so is f, and

$$0 \le \int_a^b f \le \int_a^b g.$$

Proof. Note that f is locally integrable by hypothesis.

Now, for any $[c,d] \subseteq (a,b) = \mathbb{I}$, $f, g \in \mathcal{R}[c,d]$ and $0 \le f \le g$, so from our previous work,

$$0 \le \int_c^d f \le \int_c^d g.$$

Consider

$$F_1(x) = \int_c^x f(t)dt \qquad\qquad G_1(x) = \int_c^x g(t)dt$$

then from above, $0 \leq F_1(d) \leq G_1(d)$ for all $d \in (c, b)$.

But F_1 and G_1 are increasing on (c, b), and $\lim_{d \to b^-} G_1(d) = \int_c^b g$ exists, since g is improperly integrable on (a, b).

Since F_1 is increasing and bounded (by $\lim_{d \to b^-} G_1(d)$), $\lim_{d \to b^-} F_1(d)$ exists.

Exercise: Similarly,

$$0 \leq \lim_{c \to a^+} \int_c^b f \leq \lim_{c \to a^+} \int_c^b g = \int_a^b g \in \mathbb{R}.$$

Hence $\int_a^b f$ exists and

$$0 \leq \int_a^b f \leq \int_a^b g. \ \square$$

Remark 3.24. Two common applications of this result arise from the following estimates:

(a) $|\sin x| \leq |x|$ for all $x \in \mathbb{R}$

(b) For any $\alpha > 0$, there exists $\beta_\alpha > 1$ such that $|\ln x| \leq x^\alpha$ for all $x \in [\beta_\alpha, \infty)$

Example 3.25. Consider $f(x) = \frac{\sin x}{x^{4/3}}$, $x \in (0, 1]$.

Again, for any $[c, d] \subseteq (0, 1]$, f is continuous on $[c, d]$, so $f \in \mathcal{R}[c, d]$, and thus f is locally integrable on $(0, 1]$. Moreover,

$$0 \leq f(x) = \frac{\sin x}{x^{4/3}} \leq \frac{|x|}{x^{4/3}} = \frac{x}{x^{4/3}} = x^{-1/3}.$$

But as we saw in Example 3.22(a), $g(x) = x^{-1/3}$ is improperly integrable on $(0, 1]$. By the Comparison Theorem, f is improperly integrable on $(0, 1]$, and

$$0 \leq \int_a^b f \leq \frac{3}{2}.$$

Remark 3.26. Recall that if $f \in \mathcal{R}[a, b]$, then $|f| \in \mathcal{R}[a, b]$, and

$$\left| \int_a^b f \right| \leq \int_a^b |f|.$$

This condition is not so nice for improper integrals.

Definition 3.27. Let $\mathbb{I} = (a, b)$ be a non-empty, open interval in $\mathbb{R}$ and let $f : \mathbb{I} \to \mathbb{R}$ be a function.

(a) We say that f is **absolutely integrable** over (a, b) if

  (i) f is locally integrable over (a, b)

  (ii) $|f|$ is improperly integrable over (a, b).

(b) We say that f is **conditionally integrable** over (a, b) if

  (i) f is improperly integrable over (a, b)

  (ii) f is **not** absolutely integrable over (a, b).

Theorem 3.28. If f is absolutely integrable over (a,b), then f is improperly integrable over (a,b), and

$$\left|\int_a^b f\right| \leq \int_a^b |f|.$$

Proof. Observe that $|f|+f$ and $|f|-f$ are locally integrable over (a,b) and

$$0 \leq |f|+f \leq 2|f|$$
$$0 \leq |f|-f \leq 2|f|$$

so by the Comparison Theorem, $|f|+f$ and $|f|-f$ are improperly integrable over (a,b).

Exercise: Thus $f = \frac{1}{2}\big((|f|+f)-(|f|-f)\big)$ is improperly integrable.

The next result follows immediately. $\square$

4 An Application of Integration Theory

4.1. We now turn our attention to the problem of finding volumes of "solids of revolution".

4.2 (The Disk Method)**.** Let $0 \leq f : [a,b] \to \mathbb{R}$ be a function.

Consider the solid V of revolution obtained by revolving the region R bounded by

$$\begin{array}{ll} y = f(x) & x = a \\ y = 0 & x = b \end{array}$$

about the x-axis.

Let $P = \{a = p_0 < p_1 < \cdots < p_N = b\} \in \mathcal{P}[a,b]$. Also let

$$\begin{aligned} m_k &= \inf\{f(x) : x \in [p_{k-1}, p_k]\} \\ M_k &= \sup\{f(x) : x \in [p_{k-1}, p_k]\} \end{aligned}$$

for $1 \leq k \leq N$.

Then $\pi m_k^2(p_k - p_{k-1})$ is the volume of a disk contained in the solid, while $\pi M_k^2(p_k - p_{k-1})$ is the volume of a disk which contains that section of the solid from $x = p_{k-1}$ to $x = p_k$. Thus,

$$\sum_{k=1}^{N} \pi m_k^2(p_k - p_{k-1}) \leq \text{Volume}(V) \leq \sum_{k=1}^{N} \pi M_k^2(p_k - p_{k-1}).$$

That is,

$$\pi L(f^2, P) \leq \text{Volume}(V) \leq \pi U(f^2, P) \ \ \forall P \in \mathcal{P}[a,b]$$

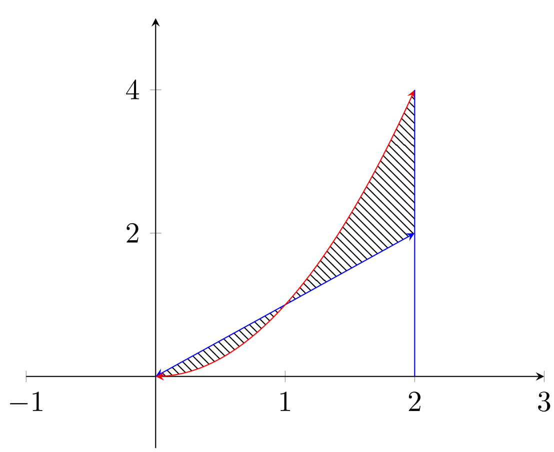
Note that if $f \in \mathcal{R}[a,b]$, then $f^2 \in \mathcal{R}[a,b]$, and then

$$\text{Volume}(V) = \pi \int_a^b f^2.$$

In particular, this works if $0 \leq f \in \zeta\left([a,b], \mathbb{R}\right)$.

Example 4.3. Let R be the region bounded by the curves

$$\begin{array}{ll} y = f_1(x) = x & x = 0 \\ y = f_2(x) = x^2 & x = 2 \end{array}$$



Let V be the solid of revolution obtained by rotating R about the x-axis.

Note that f_1, f_2 are continuous on $[0, 2]$, hence $f_1, f_2 \in \mathcal{R}[0, 2]$.

The volume of the part of V between $x = 0$ and $x = 1$ is the difference of the volumes of the solids of revolution determined by $y = f_1$ and $y = f_2$.

More precisely, since $f_1 \geq f_2$ on $[0, 1]$,

$$\begin{aligned}
\text{Volume}(V \text{ restricted to } 0 \leq x \leq 1) &= \pi \int_0^1 f_1^2 - \pi \int_0^1 f_2^2 \\
&= \pi \int_0^1 x^2 dx - \pi \int_0^1 (x^2)^2\, dx \\
&= \pi \int_0^1 (x^2 - x^4)\, dx \\
&= \pi \left[\frac{x^3}{3} - \frac{x^5}{5} \right]_{x=0}^{x=1} \\
&= \frac{2}{15}\pi
\end{aligned}$$

Since $f_2 \geq f_1$ on $[1, 2]$,

$$\begin{aligned}
\text{Volume}(V \text{ restricted to } 1 \leq x \leq 2) &= \pi \int_1^2 f_2^2 - \pi \int_1^2 f_1^2 \\
&= \pi \int_1^2 (x^4 - x^2)\, dx \\
&= \pi \left[\frac{x^5}{5} - \frac{x^3}{3} \right]_{x=1}^{x=3} \\
&= \pi \left[\left(\frac{32}{5} - \frac{8}{3} \right) - \left(\frac{1}{5} - \frac{1}{3} \right) \right] \\
&= \pi \left(\frac{31}{5} - \frac{7}{3} \right) = \frac{58}{15}\pi
\end{aligned}$$

and so

$$\text{Volume}(V) = \frac{2}{15}\pi + \frac{58}{15}\pi = 4\pi.$$

Note that in general, we use

$$\begin{aligned}
&\pi \int_a^b f_1^2 - f_2^2 \text{ where } f_1 \geq f_2 \geq 0 \\
&\pi \int_a^b f_2^2 - f_1^2 \text{ where } f_2 \geq f_1 \geq 0
\end{aligned}$$

so in general,

$$\text{Volume}(V) = \pi \int_a^b \left| f_1^2 - f_2^2 \right|.$$

Example 4.4. Find the volume of a ball of radius $r > 0$ in $\mathbb{R}^3$.

Since the volume of the ball B remains unchanged if we translate it, without loss of generality, we may assume that it is centred at the origin. In fact, if $\mathbb{R}$ is the region bounded by the curves

$$\begin{aligned} y &= \sqrt{r^2 - x^2} \\ y &= 0 \end{aligned}$$

then the ball B is the solid of revolution obtained by revolving R about the x-axis.

Hence

$$\begin{aligned} \text{Volume}(B) &= \pi \int_{-r}^{r} f^2 \\ &= \pi \int_{-r}^{r} (r^2 - x^2)\, dx \\ &= \pi \left[r^2 x - \frac{x^3}{3} \right]_{x=-r}^{x=r} \\ &= \pi \left[\left(r^3 - \frac{r^3}{3} \right) - \left(-r + \frac{r^3}{3} \right) \right] \\ &= \frac{4}{3}\pi r^3 \end{aligned}$$

Note that we can also consider the ball B as the solid of revolution obtained by revolving the region R' about the y-axis, where R' is the region bounded by the curves

$$\begin{aligned} x &= \sqrt{r^2 - y^2}, \;\; y \in [-r, r] \\ x &= 0 \end{aligned}$$

(in the x-y plane) in which case we obtain

$$\begin{aligned} \text{Volume}(B) &= \pi \int_{-r}^{r} (\sqrt{r^2 - y^2})^2\, dy \\ &= \frac{4}{3}\pi r^3. \end{aligned}$$

Observe that in both cases, the variable with respect to which we are integrating coincides with the axis about which we revolve around the region.

4.5 (The Shell Method)**.** In this method, we begin with a <u>continuous</u> function $0 \leq f \in \zeta([a, b], \mathbb{R})$, where $0 \leq a < b$.

Let R be the region bounded by the curves

$$\begin{aligned} y &= f(x) \qquad\qquad & x &= a \\ y &= 0 & x &= b \end{aligned}$$

Let S be the solid of revolution obtained by revolving the region R about the <u>y-axis</u>.

We approximate the volume of S by the volume of a union of cylindrical shells obtained as follows:

Let $P = \{a = p_0 < p_1 < \cdots < p_N = b\} \in \mathcal{P}[a, b]$ and set

$$\begin{aligned} m_k &= \inf\{f(x) : x \in [p_{k-1}, p_k]\} \\ &= \min\{f(x) : x \in [p_{k-1}, p_k]\} = f(q_k^*) \text{ for some } q_k^* \in [p_{k-1}, p_k]. \\ M_k &= \sup\{f(x) : x \in [p_{k-1}, p_k]\} \\ &= \max\{f(x) : x \in [p_{k-1}, p_k]\} = f(q_k^{**}) \text{ for some } q_k^{**} \in [p_{k-1}, p_k]. \end{aligned}$$

for $1 \le k \le N$.

Note that the union of the cylindical shells with base $[p_{k-1}, p_k]$ and height m_k (respectively M_k) is contained in S (respectively containing S). Thus,

$$\sum_{k=1}^{N} \pi m_k (p_k^2 - p_{k-1}^2) \le \text{Volume}(S) \le \sum_{k=1}^{N} \pi M_k (p_k^2 - p_{k-1}^2) \tag{*}$$

Neither the left hand side nor the right hand side look like a Riemann sum.

Recall that if $g \in \mathcal{R}[a, b]$ and $P \in \mathcal{P}[a, b]$, then

$$\sum_{k=1}^{N} m_k(g, P)(p_k - p_{k-1}) \le \int_a^b g \le \sum_{k=1}^{N} M_k(g, P)(p_k - p_{k-1}).$$

If $q_k \in [p_{k-1}, p_k]$ then $m_k \le g(q_k) \le M_k$, $1 \le k \le N$. Thus

$$L(g, P) \le \sum_{k=1}^{N} g(q_k)(p_k - p_{k-1}) \le U(g, P).$$

Now consider the equation (*). We can rewrite it as

$$\pi \sum_{k=1}^{N} m_k (p_k + p_{k-1})(p_k - p_{k-1}) \le \text{Volume}(S) \le \pi \sum_{k=1}^{N} M_k (p_k + p_{k-1})(p_k - p_{k-1}).$$

Consider

$$\pi \sum_{k=1}^{N} m_k (p_k + p_{k-1})(p_k - p_{k-1}) = \pi \left[\sum_{k=1}^{N} m_k p_k (p_k - p_{k-1}) + \sum_{k=1}^{N} m_k p_{k-1} (p_k - p_{k-1}) \right].$$

Let $g(x) = x$, $x \in [a, b]$. Then $p_k = M_k(g, P) = g(p_k)$, $p_{k-1} = g(p_{k-1})$.

Set $t_k^* = p_k$, $t_k^{**} = p_{k-1}$, so that $t_k^*, t_k^{**} \in [p_{k-1}, p_k]$. Then

$$\begin{aligned} \sum_{k=1}^{N} m_k p_k (p_k - p_{k-1}) &= \sum_{k=1}^{N} f(q_k^*) g(t_k^*)(p_k - p_{k-1}) \\ \sum_{k=1}^{N} m_k p_{k-1} (p_k - p_{k-1}) &= \sum_{k=1}^{N} f(q_k^*) g(t_k^{**})(p_k - p_{k-1}) \end{aligned}$$

Now consider the following proposition:

Proposition. Let $f, g \in \zeta([a,b],\mathbb{R})$ and $\varepsilon > 0$. Then there exists $\delta > 0$ such that if $P \in \mathcal{P}[a,b]$ with $\|P\| < \delta$, and if $y_k^*, z_k^* \in [p_{k-1}, p_k]$, $1 \leq k \leq N$, then

$$\left| \int_a^b fg - \sum_{k=1}^{N} f(y_k^*)g(z_k^*)(p_k - p_{k-1}) \right| < \varepsilon.$$

Proof. On the assignment. □

From this, it follows that

$$\text{Volume}(S) = 2\pi \int_a^b fg = 2\pi \int_a^b xf(x)\,dx.$$

Example 4.6. Find the volume of S, where S is the solid of revolution obtained by revolving the region R about the y-axis, where R is the region bounded by

$$\begin{aligned} &y = f(x) = \sin x, \;\; x \in [0,\pi] \\ &y = 0. \end{aligned}$$

From our work in 4.5,

$$\text{Volume}(S) = 2\pi \int_0^\pi x \sin x\,dx.$$

We solve this using Integration by Parts.

$$\begin{aligned} \text{Let } u &= x \qquad dv = \sin x\,dx \\ du &= dx \qquad v = -\cos x \end{aligned}$$

Then

$$\begin{aligned} \int x\sin x\,dx &= x(-\cos x) - \int -\cos x\,dx \\ &= -x\cos x + \int \cos x\,dx \\ &= -x\cos x + \sin x + \kappa, \;\; \kappa \in \mathbb{R} \end{aligned}$$

and thus

$$\begin{aligned} \text{Volume}(S) &= 2\pi \int_0^\pi x\sin x\,dx \\ &= 2\pi \Big[-x\cos x + \sin x \Big]_{x=0}^{x=\pi} \\ &= 2\pi^2. \end{aligned}$$

Remark 4.7. Integration can also be used to calculate volumes of other objects, not necessarily obtained by revolving a region about an axis.

For example, consider a pyramid with a square base with sides of length b and height h.

Again, the idea is to approximate the volume using "slices"; in this case, we might want to use horizontal slices, since these are easiest to calculate. Using Leibniz's notation, and thinking of a slice as having thickness dx, we need to figure out the volume of the slice at height $0 \leq x \leq h$.

If A_x is the area of the slice at height h, then by comparing triangles,

$$\frac{h-x}{b_x} = \frac{h}{b}$$

so $b_x = \dfrac{b}{h}(h-x)$ and $A_x = b_x^2 = \dfrac{b^2}{h^2}(h-x)^2$.

The volume of each slice is $A_x \mathrm{d}x$, and so the volume of the pyramid is

$$\begin{aligned}\int_0^h A_x \mathrm{dx} &= \int_0^h \frac{b^2}{h^2}(h-x)^2 \, \mathrm{d}x \\ &= \frac{b^2}{h^2}\int_{t=h}^{t=0} t^2(-\mathrm{d}t) \\ &= \frac{b^2}{h^2}\int_0^h t^2 \, \mathrm{d}t \\ &= \frac{b^2}{h^2}\left(\frac{h^3}{3}\right) \\ &= \frac{1}{3}b^2 h.\end{aligned}$$

5 Series

5.1. In this section, we are given an infinite sequence $(x_k)_{k=1}^{\infty}$ of real numbers which we would like to "add". Unfortunately, there is no way to do this. However, we can add finitely many at a time and take limits. Unlike finite sums, however, limits don't always behave nicely; first, the limit might not even exist, and secondly, even if it does, it might depend upon the order of the terms in the sequence.

Definition 5.2. Let $(x_k)_{k=1}^{\infty}$ be a sequence of real numbers. The symbol

$$\sum_{k=1}^{\infty} x_k$$

is **notation only**. It designates the sequence

$$\sum_{k=1}^{\infty} x_k \equiv (s_N)_{N=1}^{\infty}$$

of **partial sums** $s_N = \sum_{k=1}^{N} x_k$ of the x_k's, in that specific order. We refer to $\sum_{k=1}^{N} x_k$ as a series in $(x_k)_{k=1}^{\infty}$.

We say that the series $\sum_{k=1}^{\infty} x_k$ **converges** if $\lim_{N\to\infty} s_N$ exists. That is, if $(s_N)_{N=1}^{\infty}$ converges. We write $\sum_{k=1}^{\infty} x_k = \lim_{N\to\infty} s_N$.

We also say that $\sum_{k=1}^{\infty} x_k$ **diverges** to $+\infty$ if $\lim_{N\to\infty} s_N = \infty$, and it diverges to $-\infty$ if $\lim_{N\to\infty} s_N = -\infty$.

If $\lim_{N\to\infty} s_N$ simply does not exist, we say that $\sum_{k=1}^{\infty} x_k$ diverges.

Example 5.3.

(a) Let $0 \le |p| \le 1$. Observe that for $N > 1$,

$$1 + p + p^2 + \cdots + p^N = \frac{1 - p^{N+1}}{1 - p}$$

for $p \neq 1$.

Thus

$$\sum_{k=0}^{N} p^k = 1 + \sum_{k=1}^{N} p_k = \frac{1 - p^{N+1}}{1 - p},$$

so

$$\lim_{N\to\infty} \sum_{k=0}^{N} p^k = \lim_{N\to\infty} \frac{1 - p^{N+1}}{1 - p} = \frac{1}{1 - p}.$$

We write $\sum_{k=0}^{\infty} p^k = \frac{1}{1-p}$.

In particular, we have

$$\sum_{k=0}^{\infty} \frac{1}{2^k} = \frac{1}{1/2} = 2.$$

That is,
$$1+\sum_{k=1}^{\infty}\frac{1}{2^k}=2,$$
or
$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots=1.$$
Similarly,
$$\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots=\frac{1}{2}.$$

(b) $\sum_{k=1}^{\infty}(-1)^k$ diverges.

Let $s_N=\sum_{k=1}^{N}(-1)^k=\begin{cases}-1 & \text{if } N \text{ is odd}\\ 0 & \text{if } N \text{ is even}\end{cases}$

Clearly $\lim_{N\to\infty} s_N$ does not exist.

(c) If $p\geq 1$, then $\sum_{k=1}^{\infty}p^k$ diverges to $+\infty$.

- For $p=1$, $s_N=N$, $N\geq 1$ so $\lim_{N\to\infty}s_N=\infty$.
- For $p>1$, $s_N=\frac{1-p^{N+1}}{1-p}$ and $\lim_{N\to\infty}s_N=\infty$.

Proposition 5.4 (The Divergence Test)**.** Let $(x_k)_{k=1}^{\infty}$ be a sequence in $\mathbb{R}$. If $\sum_{k=1}^{\infty}x_k$ converges, then $\lim_{k\to\infty}x_k=0$.

Proof. Let $s_N=\sum_{k=1}^{N}x_k$, $N\geq 1$. If $\sum_{k=1}^{\infty}x_k$ converges, then $\lim_{N\to\infty}s_N=S$ exists. Hence $(s_N)_{N=1}^{\infty}$ is a Cauchy sequence, so given $\varepsilon>0$ there exists $N_0>1$ such that $m,n\geq N_0$ implies $|s_m-s_n|<\varepsilon$.

In particular, for $n\geq N_0$,
$$|x_{n+1}|=|s_{n+1}-s_n|<\varepsilon$$
so that $\lim_{n\to\infty}x_n=0$. $\square$

Caveat 5.5. The converse of Proposition 5.4 is false. If $(x_k)_{k=1}^{\infty}$ is a sequence in $\mathbb{R}$ and $\lim_{k\to\infty}x_k=0$, it does not necessarily follow that $\sum_{k=1}^{\infty}x_k$ converges.

For example, consider the harmonic series.

Let $x_k=1/k$, $k\geq 1$. Clearly, $\lim_{k\to\infty}x_k=0$. Let $f(x)=1/x$, $x>0$. On the interval $[k,k+1]$,
$$\frac{1}{k}\geq f(x),$$
so by the Comparison Theorem for integrals,
$$\begin{aligned}s_N=\sum_{k=1}^{N}x_k &\geq \sum_{k=1}^{N}\int_k^{k+1}\frac{1}{x}\,\mathrm{d}x\\ &=\int_1^{N+1}\frac{1}{x}\,\mathrm{d}x=\ln(N+1).\end{aligned}$$
Since $\lim_{N\to\infty}\ln(N+1)=\infty$, $\sum_{k=1}^{\infty}1/k$ diverges to ∞.

Note also that for $k \geq 1$,

$$0 < \frac{1}{k+1} \leq f(x), \;\; x \in [k, k+1],$$

so by the Comparison Theorem,

$$\sum_{k=1}^{N} \frac{1}{k+1} \leq \sum_{k=1}^{N} \int_{k}^{k+1} \frac{1}{x}\,\mathrm{d}x = \ln(N+1).$$

That is,

$$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N+1} \leq \ln(N+1),$$

and

$$\ln(N+1) \leq 1 + \frac{1}{2} + \cdots + \frac{1}{N} \leq \left(1 - \frac{1}{N+1}\right) + \ln(N+1).$$

The harmonic series $\sum_{k=1}^{\infty} 1/k$ "grows" like $\ln(x)$.

Proposition 5.6. Let $(x_k)_{k=1}^{\infty} \in \mathbb{R}^{\mathbb{N}}$ and suppose that $\gamma = \lim_{k\to\infty} x_k$ exists. Then

$$\sum_{k=1}^{\infty} (x_k - x_{k+1}) = x_1 - \gamma.$$

Proof. For each $N \geq 1$,

$$s_N := \sum_{k=1}^{N} (x_k - x_{k+1}) = x_1 - x_{N+1}.$$

But $\gamma = \lim_{k\to\infty} x_k = \lim_{N\to\infty} x_{N+1}$, and so

$$\begin{aligned}\sum_{k=1}^{\infty} (x_k - x_{k+1}) &= \lim_{N\to\infty} s_N \\ &= \lim_{N\to\infty} x_1 - x_{N+1} \\ &= x_1 - \gamma. \;\square\end{aligned}$$

Theorem 5.7 (The Cauchy Criterion for Series)**.** Let $(x_k)_{k=1}^{\infty} \in \mathbb{R}^{\mathbb{N}}$. The following are equivalent:

(a) $\sum_{k=1}^{\infty} x_k$ converges.

(b) For all $\varepsilon > 0$ there exists $N_0 \geq 1$ such that $m \geq n \geq N_0$ implies that

$$\left| \sum_{k=n+1}^{m} x_k \right| < \varepsilon.$$

Proof. Note that $\sum_{k=1}^{\infty} x_k$ converges if and only if the sequence $(s_N)_{N=1}^{\infty}$ converges, where $s_N = \sum_{k=1}^{N} x_k$, $N \geq 1$, if and only if (by the Cauchy Criterion for sequences) for all $\varepsilon > 0$, there exists $N_0 \geq 1$ such that $m \geq n \geq N_0$ implies

$$\left| \sum_{k=n+1}^{m} x_k \right| = \left| \sum_{k=1}^{m} x_k - \sum_{k=1}^{n} x_k \right| = |s_m - s_n| < \varepsilon. \;\square$$

Corollary 5.8. Let $(x_k)_{k=1}^{\infty} \in \mathbb{R}^{\mathbb{N}}$. The following are equivalent:

(a) $\sum_{k=1}^{\infty} x_k$ converges.

(b) For all $\varepsilon > 0$ there exists $N_1 \geq 1$ such that for all $n \geq N_1$, we have

$$\left| \sum_{k=n+1}^{\infty} x_k \right| < \varepsilon.$$

Proof.

(a) $\implies$ (b). Suppose that $\sum_{k=1}^{\infty} x_k$ converges, say $\sum_{k=1}^{\infty} x_k = \beta \in \mathbb{R}$. Let $s_N = \sum_{k=1}^{N} x_k$, $N \geq 1$, so that $\lim_{N\to\infty} s_N = \beta$.

Let $\varepsilon > 0$, and choose $N_1 \geq 1$ such that $n \geq N_1$ implies that

$$|s_n - \beta| < \varepsilon.$$

Note that

$$\begin{aligned}
\beta - s_n &= \sum_{k=1}^{\infty} x_k - \sum_{k=1}^{n} x_k \\
&= \lim_{N\to\infty} \sum_{k=1}^{N} x_k - \sum_{k=1}^{n} x_k \\
&= \lim_{N\to\infty} \sum_{k=n+1}^{N} x_k \\
&= \sum_{k=n+1}^{\infty} x_k.
\end{aligned}$$

Hence $n \geq N_1$ implies that

$$|\beta - s_n| = \left| \sum_{k=n+1}^{\infty} x_k \right| < \varepsilon.$$

(b) $\implies$ (a). We will apply the Cauchy Criterion for series.

Let $\varepsilon > 0$. Choose $N_1 \geq 1$ such that for all $n \geq N_1$, we have

$$\left| \sum_{k=n+1}^{\infty} x_k \right| < \frac{\varepsilon}{2}.$$

Suppose that $m \geq n \geq N_1$, so that

$$\left| \sum_{k=m+1}^{\infty} x_k \right| < \frac{\varepsilon}{2} \text{ and } \left| \sum_{k=n+1}^{\infty} x_k \right| < \frac{\varepsilon}{2}.$$

Consider

$$\begin{aligned}\left|\sum_{k=n+1}^{m} x_k\right| &= \left|\sum_{k=m+1}^{\infty} x_k - \sum_{k=n+1}^{\infty} x_k\right| \\ &\leq \left|\sum_{k=m+1}^{\infty} x_k\right| + \left|\sum_{k=n+1}^{\infty} x_k\right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

By the Cauchy Criterion for series, $\sum_{k=1}^{\infty} x_k$ converges. $\square$

Proposition 5.9. Let $(x_k)_{k=1}^{\infty}$, $(y_k)_{k=1}^{\infty} \in \mathbb{R}^{\mathbb{N}}$ and $\alpha \in \mathbb{R}$.

If $\sum_{k=1}^{\infty} x_k$ converges to $\beta_x \in \mathbb{R}$ and $\sum_{k=1}^{\infty} y_k$ converges to $\beta_y \in \mathbb{R}$, then

(a) $\sum_{k=1}^{\infty}(x_k + y_k)$ converges to $\beta_x + \beta_y$.

(b) $\sum_{k=1}^{\infty}(\alpha x_k) = \alpha\left(\sum_{k=1}^{\infty} x_k\right) = \alpha\beta_x$.

Proof.

(a) Let $s_N = \sum_{k=1}^{N} x_k$, $t_N = \sum_{k=1}^{N} y_k$, $N \geq 1$, so that $\lim_{N\to\infty} s_N = \beta_x$ and $\lim_{N\to\infty} t_N = \beta_y$. Recall that

$$\begin{aligned}\beta_x + \beta_y &= \lim_{N\to\infty} s_N + \lim_{N\to\infty} t_N \\ &= \lim_{N\to\infty} (s_N + t_N)\end{aligned}$$

since both limits exist.

But for all $N \geq 1$,

$$\begin{aligned}s_N + t_N &= \sum_{k=1}^{N} x_k + \sum_{k=1}^{N} y_k \\ &= \sum_{k=1}^{N}(x_k + y_k)\end{aligned}$$

Hence $\beta_x + \beta_y = \lim_{N\to\infty} \sum_{k=1}^{N}(x_k + y_k) = \sum_{k=1}^{\infty}(x_k + y_k)$.

(b) Left as an exercise. $\square$

Caveat 5.10. The converse of Proposition 5.9 is false.

For example, let $x_k = 1$, $y_k = -1$ for $k \geq 1$. Then $\sum_{k=1}^{\infty}(x_k + y_k) = \sum_{k=1}^{\infty}(0) = 0$, but $\sum_{k=1}^{\infty} x_k$ and $\sum_{k=1}^{\infty} y_k$ diverge to $+\infty$ and $-\infty$, respectively.

5.11. It is worth noting that the convergence of a series $\sum_{k=1}^{\infty} x_k$ has nothing to do with <u>finitely</u> many terms in the series.

Suppose that $(x_k)_{k=1}^{\infty} \in \mathbb{R}^{\mathbb{N}}$ is a sequence and that (P) is a property that each x_k may or may ot have.

We say that (P) holds for large k (for $(x_k)_{k=1}^\infty$) to mean that there exists $N \geq 1$ such that $k \geq N$ implies that x_k has property (P).

We can also extend this to properties of the terms in pairs of sequences.

If (P) is a property between pairs of real numbers (that may or may not hold), we say that (P) holds for large k relative to $(x_k)_{k=1}^\infty$ and $(y_k)_{k=1}^\infty$ if there exists $N \geq 1$ such that $k \geq N$ implies that the pair (x_k, y_k) has property (P).

As another example, say (P) is the property that $x \leq y$. Given $(x_k)_{k=1}^\infty, (y_k)_{k=1}^\infty \in \mathbb{R}^\mathbb{N}$, we say that (P) holds for large k if there exists $N \geq 1$ such that $x_k \leq y_k$ for all $k \geq N$.

Suppose that $x_k = y_k$ for large k. We claim that $\sum_{k=1}^\infty x_k$ converges if and only if $\sum_{k=1}^\infty y_k$ converges.

Let $s_N = \sum_{k=1}^N x_k$ and $t_N = \sum_{k=1}^N y_k$.

Let $\beta_x = \lim_{N\to\infty} s_N$ and $\beta_y = \lim_{N\to\infty} t_N$.

Choose $N_0 \geq 1$ such that $k \geq N_0$ implies that $x_k = y_k$. Then, with $N \geq N_0$,

$$\begin{aligned}
|s_N - t_N| &= \left|\sum_{k=1}^N x_k - \sum_{k=1}^N y_k\right| \\
&= \left|\left(\sum_{k=1}^{N_0-1} x_k - \sum_{k=1}^{N_0-1} y_k\right) + \underbrace{\left(\sum_{k=N_0}^{N} x_k - \sum_{k=N_0}^{N} y_k\right)}_{0}\right| \\
&= \underbrace{\left|\sum_{k=1}^{N_0-1} (x_k - y_k)\right|}_{\text{constant}}
\end{aligned}$$

But since $|s_N - t_N|$ is constant for all $N \geq N_0$, $(s_N)_{N=1}^\infty$ converges if and only if $(t_N)_{N=1}^\infty$ converges.

Theorem 5.12. Let $(x_k)_{k=1}^\infty$ with $x_k \geq 0$ for all $k \geq 1$. The following are equivalent:

(a) $\sum_{k=1}^\infty x_k$ converges.

(b) $(s_N)_{N_1}^\infty$ is bounded above.

Proof.

(a) $\implies$ (b). Let $s_N = \sum_{k=1}^N x_k$. Since $\sum_{k=1}^\infty x_k$ converges, $\lim_{N\to\infty} s_N$ exists. Hence $(s_N)_{N=1}^\infty$ is bounded.

(b) $\implies$ (a). Suppose that $(s_N)_{N_1}^\infty$ is bounded (hence bounded above). Note that for all $N \geq 1$,

$$\begin{aligned}
s_{N+1} - s_N &= \sum_{k=1}^{N+1} x_k - \sum_{k=1}^{N} x_k \\
&= x_{N+1} \\
&\geq 0.
\end{aligned}$$

Thus $(s_N)_{N=1}^{\infty}$ is increasing.

Hence $\sum_{k=1}^{\infty} x_k = \lim_{N\to\infty} s_N = \sup_{N\geq 1} s_N < \infty$, so $\sum_{k=1}^{\infty} x_k$ converges. $\square$

5.13 (The Integral Test)**.** Suppose that $f : [1,\infty) \to [0,\infty)$ is a monotone decreasing function. The following are equivalent:

(a) $\sum_{k=1}^{\infty} f(k)$ converges (i.e. $\sum_{k=1}^{\infty} f(k) < \infty$)

(b) $\int_1^{\infty} f$ converges (i.e. $\int_1^{\infty} f < \infty$)

Proof. Observe that since f is monotone decreasing on $[1,\infty)$, then if $[a,b] \subseteq [1,\infty)$ is any closed, bounded interval in $[1,\infty)$, then f is monotone decreasing on $[a,b]$, hence integrable on $[a,b]$. That is, f is locally integrable on $[1,\infty)$.

Let

$$\begin{aligned} g(x) &= \begin{cases} f(k+1) & x \in (k, k+1] \\ f(2) & x = 1 \end{cases} \\ h(x) &= f(k) \quad x \in [k, k+1) \end{aligned}$$

for $1 \leq k$.

Note that each of g, h are piecewise constant on $[1,\infty)$, and so they are locally integrable on $[1,\infty)$.

Moreover, $g(x) \leq f(x) \leq h(x)$ for all $x \in [1,\infty)$, so by the Comparison Theorem for Integrals, for each $N \geq 1$,

$$f(2) + \cdots + f(N) = \int_1^N g \leq \int_1^N f \leq \int_1^N h = f(1) + \cdots + f(N-1).$$

That is,

$$\sum_{k=2}^{N} f(k) \leq \int_1^N f \leq \sum_{k=1}^{N-1} f(k)$$

for all integers $N \geq 2$.

(a) $\implies$ (b). Suppose $\sum_{k=1}^{\infty} f(k) < \infty$. Then, since $0 \leq f(j)$ for all $j \geq 1$,

$$\int_2^N f \leq \sum_{k=1}^{N-1} f(k) \leq \sum_{k=1}^{\infty} f(k) =: \beta < \infty$$

Observe that if

$$F(x) = \int_1^x f(t)\,dt$$

then F is increasing on $[1, \infty)$ (since $f \geq 0$ on $[1, \infty)$). As such,

$$\begin{aligned}
\int_1^\infty f(t)\,dt &= \lim_{x\to\infty} \int_1^x f(t)\,dt \\
&= \sup_{x\geq 1} \int_1^x f(t)\,dt \\
&= \sup_{N\geq 2, N\in\mathbb{N}} \int_1^N f(t)\,dt \\
&\leq \beta = \sum_{k=1}^\infty f(k) < \infty
\end{aligned}$$

Thus $\int_1^\infty f(t)\,dt < \infty$.

(b) $\implies$ (a). Suppose $\int_1^\infty f(t)\,dt < \infty$. From above, for all $N \geq 2$, $N \in \mathbb{N}$,

$$\begin{aligned}
s_N := \sum_{k=1}^N f(k) &= f(1) + \sum_{k=2}^N f(k) \\
&\leq f(1) + \int_1^N f(t)\,dt \\
&\leq \underbrace{f(1) + \int_1^\infty f(t)\,dt}_{\text{constant}} < \infty
\end{aligned}$$

Thus $(s_N)_{N=1}^\infty$ is an increasing sequence and $(s_N)_{N=1}^\infty$ is bounded above, and so

$$\sum_{k=1}^\infty f(k) = \lim_{N\to\infty} s_N = \sup_{N\geq 1} s_N \leq f(1) + \int_1^\infty f(t)\,dt < \infty. \ \square$$

Corollary 5.14 (The p-Test for Series)**.** The series $\sum_{k=1}^\infty 1/k^p$ converges if and only if $p > 1$.

Proof. Exercise. $\square$

5.15 (The Comparison Test)**.** Suppose that $0 \leq a_k \leq b_k$ for large k. If $\sum_{k=1}^\infty b_k < \infty$, then $\sum_{k=1}^\infty a_k < \infty$.

Hence if $\sum_{k=1}^\infty a_k$ diverges, so does $\sum_{k=1}^\infty b_k$.

Proof. We are only interested in whether the series converges or diverges, so without loss of generality, assume that $0 \leq a_k \leq b_k$ holds for all $k \geq 1$.

For each $N \geq 1$,

$$s_N := \sum_{k=1}^N a_k \leq \sum_{k=1}^N b_k \leq \sum_{k=1}^\infty b_k =: \beta < \infty$$

But $a_k \geq 0$ for all $k \geq 1$, so $(s_N)_{N=1}^\infty$ is an increasing sequence bounded above by $\beta = \sum_{k=1}^\infty b_k$.

Hence

$$\sum_{k=1}^\infty a_k := \lim_{N\to\infty} s_N = \sup_{N\geq 1} s_N \leq \beta < \infty. \ \square$$

Example 5.16.

(a) Consider $\sum_{k=1}^{\infty} \frac{k^3 \log^2 k}{e^k}$. We show that this converges.

Note that if $a_k := \frac{k^3 \log^2 k}{e^k}$, then

$$0 \leq a_k \leq b_k := \frac{k^3 \cdot k^2}{e^k} = \frac{k^5}{e^k}.$$

By the Comparison Test for Series, it suffices to show that $\sum_{k=1}^{\infty} b_k < \infty$.

Let us write $h(x) = \frac{x^5}{e^{x/2}}$. Then $b_k = h(x) \cdot \frac{1}{(\sqrt{e})^k}$, $k \geq 1$. Then

$$\begin{aligned}
\lim_{x\to\infty} h(x) &= \lim_{x\to\infty} \frac{x^5}{e^{x/2}} \\
&\overset{\text{L'HR}}{=} \lim_{x\to\infty} \frac{5x^4}{(1/2)e^{x/2}} \\
&\overset{\text{L'HR}}{=} \lim_{x\to\infty} \frac{20x^3}{(1/2)^2 e^{x/2}} \\
&\overset{\text{L'HR}}{=} \lim_{x\to\infty} \frac{60x^2}{(1/2)^3 e^{x/2}} \\
&\overset{\text{L'HR}}{=} \lim_{x\to\infty} \frac{120x}{(1/2)^4 e^{x/2}} \\
&\overset{\text{L'HR}}{=} \lim_{x\to\infty} \frac{120}{(1/2)^5 e^{x/2}} = 0
\end{aligned}$$

hence $\lim_{x\to\infty} h(x)$ exists, and it follows that $\lim_{x\to\infty} \frac{x^5}{e^{x/2}} = 0$.

Now fix $N_0 \geq 1$ so that $x \geq N_0$ implies that $\frac{x^5}{e^{x/2}} < 1$. If $k \geq N_0$, $k \in \mathbb{Z}$, then

$$\begin{aligned}
b_k = \frac{k^5}{e^k} &= \frac{k^5}{e^{k/2}} \frac{1}{e^{k/2}} \\
&< 1 \cdot \frac{1}{e^{k/2}} = \left(\frac{1}{\sqrt{e}}\right)^k
\end{aligned}$$

Hence for large k,

$$b_k \leq \frac{1}{(\sqrt{e})^k}$$

and $\sum_{k=1}^{\infty} \frac{1}{(\sqrt{e})^k} < \infty$ since it is a geometric series with $1/\sqrt{e} < 1$.

By the Comparison Test for series,

$$\sum_{k=1}^{\infty} b_k < \infty$$

and applying the Comparison Test again,

$$\sum_{k=1}^{\infty} a_k < \infty$$

That is,

$$\sum_{k=1}^{\infty} \frac{k^3 \log^2 k}{e^k} < \infty.$$

(b) Consider $\sum_{k=1}^{\infty} \frac{1}{k^{\log k}}$. Note that for $k \geq 9$, $\log k \geq 2$, so

$$\frac{1}{k^{\log k}} \leq \frac{1}{k^2}$$

By the p-test, $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges. Then by the Comparison Test for series, $\sum_{k=1}^{\infty} \frac{1}{k^{\log k}}$ converges as well.

5.17 (The Limit Comparison Test)**.** Suppose that $0 \leq a_k \leq b_k$ for large k and that $L := \lim_{k\to\infty} a_k/b_k$ exists in $[0,\infty) \cup \{\infty\}$.

(a) If $0 < L < \infty$, then $\sum_{k=1}^{\infty} a_k$ converges if and only if $\sum_{k=1}^{\infty} b_k$ converges.

(b) If $L = 0$ and $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.

(c) If $L = \infty$ and $\sum_{k=1}^{\infty} b_k = \infty$, then $\sum_{k=1}^{\infty} a_k = \infty$.

Proof.

(a) With $0 < L < \infty$, choose $N_1 \geq 1$ such that $k \geq N_1$ implies that $\frac{1}{2} \leq \frac{a_k}{b_k} \leq 2L$. That is, $\frac{1}{2}b_k \leq a_k \leq (2L)b_k$.

Now if $\sum_{k=1}^{\infty} b_k < \infty$, then $\sum_{k=1}^{\infty}(2L)b_k < \infty$. By the Comparison Test, $\sum_{k=1}^{\infty} a_k < \infty$.

If $\sum_{k=1}^{\infty} b_k = \infty$, then $\sum_{k=1}^{\infty} \frac{1}{2}b_k = \infty$. By the Comparison Test, $0 \leq \frac{1}{2}b_k \leq a_k$ for large k implies that $\sum_{k=1}^{\infty} a_k = \infty$.

(b) If $L = 0$, then we can choose $N_2 \geq 1$ such that $k \geq N_2$ implies that $a_k, b_k \geq 0$ and $a_k/b_k < 1$. That is, $a_k < b_k$. By the Comparison Test, $\sum_{k=1}^{\infty} b_k < \infty$ implies that $\sum_{k=1}^{\infty} a_k < \infty$.

(c) Similar to (b). $\square$

Example 5.18. Suppose $(a_k)_{k=1}^{\infty} \in \mathbb{R}^{\mathbb{N}}$ and $\lim_{k\to\infty} |a_k| = 0$. Observe that

$$\lim_{k\to\infty} \frac{\sin|a_k|}{|a_k|} = 1,$$

so by the Limit Comparison Test, $\sum_{k=1}^{\infty} |a_k| < \infty$ if and only if $\sum_{k=1}^{\infty} \sin|a_k| < \infty$.

We now turn our attention to series where infinitely many terms may be negative. The strongest notion of convergence is the following:

Definition 5.19. Let $\sum_{k=1}^{\infty} x_k$ be an infinite series. We say that the series **converges absolutely** if $\sum_{k=1}^{\infty} |x_k|$ converges.

If $\sum_{k=1}^{\infty} x_k$ converges but is **not** absolutely convergent, then we say that $\sum_{k=1}^{\infty} x_k$ converges conditionally.

Example 5.20. Examples of absolute and conditional convergence:

(a) If $x_k \geq 0$ for large k and $\sum_{k=1}^{\infty} x_k$ converges, then $\sum_{k=1}^{\infty} x_k$ converges absolutely.

(b) We have seen that $\sum_{k=1}^{\infty} 1/k^2$ converges, so if $x_k = (-1)^k/k^2$, then $\sum_{k=1}^{\infty} x_k$ converges absolutely.

(c) We have seen that $\sum_{k=1}^{\infty} 1/k$ diverges, while the alternating harmonic series $\sum_{k=1}^{\infty}(-1)^k/k$ converges. Thus, the alternating harmonic series converges conditionally.

From the Cauchy Criterion for series, we get the following:

Proposition 5.21. Let $\sum_{k=1}^{\infty} x_k$ be a series. The following are equivalent:

(a) $\sum_{k=1}^{\infty} x_k$ converges absolutely.

(b) For all $\varepsilon > 0$ there exists $N > 0$ such that $m > n \geq N$ implies $\sum_{k=n+1}^{m} |x_k| < \varepsilon$.

Proposition 5.22. If the series $\sum_{k=1}^{\infty} x_k$ converges absolutely, then $\sum_{k=1}^{\infty} x_k$ converges.

Proof. We shall apply the Cauchy Criterion for series. Suppose $\sum_{k=1}^{\infty} x_k$ converges absolutely and let $\varepsilon > 0$. Choose $N > 0$ such that $m > n \geq N$ implies $\sum_{k=n+1}^{m} |x_k| < \varepsilon$.

Then $m > n \geq N$ implies

$$\left|\sum_{k=n+1}^{m} x_k\right| \leq \sum_{k=n+1}^{m} |x_k| < \varepsilon$$

so that by the Cauchy Criterion, $\sum_{k=1}^{\infty} x_k$ converges. □

Recall the following definition from MATH 147:

Definition 5.23. Given a sequence $(x_k)_{k=1}^{\infty} \in \mathbb{R}^{\mathbb{N}}$, the **limit supremum** of $(x_k)_{k=1}^{\infty}$ is

$$\limsup_{n\to\infty} x_n := \lim_{n\to\infty}\left(\sup_{k\geq n} x_k\right) \in \mathbb{R} \cup \{\pm\infty\}$$

while the **limit infimum** of $(x_k)_{k=1}^{\infty}$ is

$$\liminf_{n\to\infty} x_n := \lim_{n\to\infty}\left(\inf_{k\geq n} x_n\right) \in \mathbb{R} \cup \{\pm\infty\}$$

Recall also that with $\beta \in \mathbb{R}$,

(a) if $\limsup_{k\to\infty} x_k < \beta < \infty$, then $x_k < \beta$ for large k.

(b) if $\limsup_{k\to\infty} x_k > \beta$, then there exist $n_1 < n_2 < \dots$ so that $x_{n_j} > \beta$, $j \geq 1$.

(c) if $\lim_{k\to\infty} x_k = \beta$, then $\limsup_{k\to\infty} x_k = \liminf_{k\to\infty} x_k = \beta$.

Theorem 5.24 (The Root Test)**.** Let $(x_k)_{k=1}^{\infty} \in \mathbb{R}^{\mathbb{N}}$ and let $\rho := \limsup_{k\to\infty} |x_k|^{1/k}$.

(a) if $\rho < 1$, then $\sum_{k=1}^{\infty} x_k$ converges absolutely.

(b) if $\rho > 1$, then $\sum_{k=1}^{\infty} x_k$ diverges.

Proof.

(a) Let $\beta = \frac{1+\rho}{2}$, so that $\rho < \beta < 1$.

Then $\limsup_{k\to\infty} |x_k|^{1/k} < \beta$, so from above, we see that $|x_k|^{1/k} < \beta$; that is, $|x_k| < \beta^k$ for large k. Since $\beta < 1$, $\sum_{k=1}^{\infty} \beta^k$ converges. By the Comparison Test for series, $\sum_{k=1}^{\infty} |x_k|$ converges, or $\sum_{k=1}^{\infty} x_k$ converges absolutely.

(b) Let $\gamma = \frac{1+\rho}{2}$ so that $1 < \gamma < \rho$.

Then $\limsup_{k\to\infty} |x_k|^{1/k} > \gamma$, so from above, we can find $n_1 < n_2 < \dots$ so that $|x_k|^{1/k} > \gamma > 1$. Thus $|x_k| > 1$, and by the Divergence Test, $\sum_{k=1}^{\infty} x_k$ diverges. $\square$

5.25. The Root Test gives no information when $\rho = 1$. Note that $\sum_{k=1}^{\infty} 1/k$ diverges, while $\sum_{k=1}^{\infty} 1/k^2$ converges by the p-Test.

Let $x_k = 1/k$, $y_k = 1/k^2$. Then $|x_k|^{1/k} = \left(\frac{1}{k}\right)^{1/k} = \frac{1}{k^{1/k}} = k^{-1/k}$.

To find $\lim_{k\to\infty} k^{-1/k}$, we can consider $\log(x^{-1/x}) = -\frac{1}{x}\log x$. Then

$$\lim_{x\to\infty} \log(x^{-1/x}) = \lim_{x\to\infty} -\frac{\log x}{x} \overset{\text{L'HR}}{=} \lim_{x\to\infty} \frac{-1/x}{1} = 0$$

so $\lim_{k\to\infty} k^{-1/k} = e^0 = 1$.

Hence $\lim_{k\to\infty} |x_k|^{1/k} = 1$, and $\sum_{k=1}^{\infty} x_k$ diverges.

On the other hand, $y_k = x_k^2$, so

$$\begin{aligned}
\lim_{k\to\infty} |y_k|^{1/k} &= \lim_{k\to\infty} \left(|x_k|^{1/k}\right)^2 \\
&= \left(\lim_{k\to\infty} |x_k|^{1/k}\right)^2 \\
&= 1^2 = 1
\end{aligned}$$

while $\sum_{k=1}^{\infty} y_k$ converges.

Theorem 5.26 (The Ratio Test)**.** Let $(x_k)_{k=1}^{\infty} \in \mathbb{R}^{\mathbb{N}}$ and assume that $x_k \neq 0$ for large k. If

$$\rho := \lim_{k\to\infty} \left|\frac{x_{k+1}}{x_k}\right|$$

exists in $\mathbb{R} \cup \{\infty\}$, then we have the following:

(i) If $\rho < 1$, then $\sum_{k=1}^{\infty} x_k$ converges absolutely.

(ii) If $\rho > 1$, then $\sum_{k=1}^{\infty} x_k$ diverges.

Proof.

(i) Suppose that $\rho < 1$. If $\beta \in (\rho, 1)$, each term will eventually have $\left|\frac{x_{k+1}}{x_k}\right| < \beta$. Pick N so that $n \geq N$ implies $\left|\frac{x_{n+1}}{x_n}\right| < \beta$. In particular,

$$\left|\frac{x_{N+1}}{x_N}\right| < \beta \implies |x_{n+1}| < \beta |x_n|$$

Also,

$$\left|\frac{x_{N+2}}{x_{N+1}}\right| < \beta \implies |x_{N+2}| < \beta |x_{N+1}| < \beta^2 |x_N|$$

By induction, we have that

$$|x_{N+i}| < \beta^i |x_N|$$

Hence

$$\sum_{k=N+1}^{\infty} |x_k| = \sum_{k=1}^{\infty} |x_{N+k}| < \sum_{k=1}^{\infty} \beta^k |x_N| = |x_N| \sum_{k=1}^{\infty} \beta_k < \infty$$

since $\sum_{k=1}^{\infty} \beta^k$ is a geometric series and $\beta < 1$. Therefore, $\sum_{k=1}^{\infty} x_k$ converges absolutely.

(ii) Suppose $\rho > 1$. Choose $\beta \in (1, \rho)$ and note that for all $n \geq N$, it must be that $\left|\frac{x_{n+1}}{x_n}\right| > \beta$. From this, we can show that for all i,

$$|x_{N+i}| > \beta^i |x_N|$$

Hence

$$|x_{N+i}| > \underbrace{\beta^i}_{>1} |x_N| > |x_N| > 0$$

Thus the terms do not go to 0. Hence by the Divergence Test, $\sum_{k=1}^{\infty} x_k$ diverges. □

Exercise. Use the Ratio Test to determine if $\sum_{k=1}^{\infty} \frac{k^5}{e^k}$ converges.

5.27. If $\rho = 1$ in the Ratio Test, we obtain no information. For example, $\sum_{k=1}^{\infty} 1/k$ diverges, yet

$$\rho = \lim_{k\to\infty} \frac{1/(k+1)}{1/k} = \lim_{k\to\infty} \frac{k}{k+1} = 1$$

On the other hand, $\sum_{k=1}^{\infty} 1/k^2$ converges, and

$$\rho = \lim_{k\to\infty} \frac{1/(k+1)^2}{1/k^2} = \lim_{k\to\infty} \frac{k^2}{(k+1)^2} = 1$$

Definition 5.28. Let $(x_k)_{k=1}^{\infty} \in \mathbb{R}^{\mathbb{N}}$. A series $(y_k)_{k=1}^{\infty}$ is said to be a **rearrangement** of $\sum_{k=1}^{\infty} x_k$ if there is a bijective map $\sigma : \mathbb{N} \to \mathbb{N}$ such that $y_k = x_{\sigma(k)}$ for all k.

Note that on A1Q6, we proved that if $\sum_{k=1}^{\infty} x_k$ is a convergent series of positive terms, then any rearrangement $\sum_{k=1}^{\infty} y_k$ of this series is also convergent, and

$$\sum_{k=1}^{\infty} x_k = \sum_{k=1}^{\infty} y_k$$

Proposition 5.29. If $\sum_{k=1}^{\infty} x_k$ is an absolutely convergent series and $\sum_{k=1}^{\infty} y_k$ is some rearrangement of $\sum_{k=1}^{\infty} x_k$, then $\sum_{k=1}^{\infty} y_k$ is absolutely convergent, and

$$\sum_{k=1}^{\infty} x_k = \sum_{k=1}^{\infty} y_k$$

Proof. Let $\sigma : \mathbb{N} \to \mathbb{N}$ be a bijective function, so that

$$y_k = x_{\sigma(k)}$$

Since $\sum_{k=1}^{\infty} x_k$ converges absolutely, by definition, we have that $\sum_{k=1}^{\infty} |x_k|$ converges.

By A1Q6, we have that $\sum_{k=1}^{\infty} |x_{\sigma(k)}|$ converges (sum of positive terms).

Therefore $\sum_{k=1}^{\infty} |y_k|$ converges, so $\sum_{k=1}^{\infty} y_k$ converges absolutely.

Suppose that $\beta = \sum_{k=1}^{\infty} y_k$. Let $\varepsilon > 0$, and choose $M \in \mathbb{N}$ such that

(a) $\displaystyle\left|\beta - \sum_{k=1}^{M} y_k\right| < \frac{\varepsilon}{2}$

(b) $\displaystyle\sum_{k=M+1}^{\infty} |y_k| < \frac{\varepsilon}{2}$

Finally, choose $N = \max\{\sigma(1), \sigma(2), \dots, \sigma(M)\}$ so that

$$\begin{aligned}\{y_1, \dots, y_M\} &= \{x_{\sigma(1)}, \dots, x_{\sigma(M)}\} \\ &\subseteq \{x_1, \dots, x_N\}\end{aligned}$$

We claim that when $n \geq N$, we have

$$\left|\beta - \sum_{k=1}^{n} x_k\right| < \varepsilon$$

Indeed, for $n \geq N$, if we define $A_n = \{1, \dots, n\} \setminus \{\sigma(1), \dots, \sigma(M)\}$,

$$\begin{aligned}\left|\beta - \sum_{k=1}^{n} x_k\right| &= \left|\beta - \sum_{k=1}^{M} x_{\sigma(k)} - \sum_{k\in A_n} x_k\right| \\ &\leq \left|\beta - \sum_{k=1}^{M} y_k\right| + \sum_{k\in A_n} |x_k|\end{aligned}$$

Note that if $k \in A_n$, then $x_k = y_j = x_{\sigma(j)}$ for some $j \geq M+1$. So

$$\begin{aligned}\left|\beta - \sum_{k=1}^{n} x_k\right| &< \frac{\varepsilon}{2} + \sum_{j=M+1}^{\infty} |x_{\sigma(j)}| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

Therefore,

$$\sum_{k=1}^{\infty} x_k = \beta = \sum_{k=1}^{\infty} y_k \ \square$$

Notation 5.30. Given $x \in \mathbb{R}$, define

$$\begin{aligned}x^+ &= \max(x, 0) \\ x^- &= \max(-x, 0)\end{aligned}$$

We can find the following properties:

1. $x^+, x^- \geq 0$
2. $x^+ + x^- = |x|$
3. $x^+ - x^- = x$

From 2 and 3, we can deduce

4. $x^+ = \dfrac{x + |x|}{2}$
5. $x^- = \dfrac{-x + |x|}{2}$

Proposition 5.31. Let $(x_k)_{k=1}^{\infty} \in \mathbb{R}^{\mathbb{N}}$.

(a) If $\sum_{k=1}^{\infty} x_k$ converges absolutely, then so do $\sum_{k=1}^{\infty} x_k^+$ and $\sum_{k=1}^{\infty} x_k^-$. Moreover, we have

$$\sum_{k=1}^{\infty} |x_k| = \sum_{k=1}^{\infty} x_k^+ + \sum_{k=1}^{\infty} x_k^-$$
$$\sum_{k=1}^{\infty} x_k = \sum_{k=1}^{\infty} x_k^+ - \sum_{k=1}^{\infty} x_k^-$$

(b) If $\sum_{k=1}^{\infty} x_k$ converges conditionally, then $\sum_{k=1}^{\infty} x_k^+$ and $\sum_{k=1}^{\infty} x_k^-$ diverge to ∞.

Proof.

(a) Suppose $\sum_{k=1}^{\infty} x_k$ converges absolutely. Since $x_k^+, x_k^- \leq |x_k|$, we have from the Comparison Test that $\sum_{k=1}^{\infty} x_k^+$ and $\sum_{k=1}^{\infty} x_k^-$ converge (absolutely).

Moreover, since $|x_k| = x_k^+ + x_k^-$, we have

$$\begin{aligned}\sum_{k=1}^{\infty} |x_k| &= \sum_{k=1}^{\infty} (x_k^+ + x_k^-) \\ &= \sum_{k=1}^{\infty} x_k^+ + \sum_{k=1}^{\infty} x_k^-\end{aligned}$$

Likewise,

$$\begin{aligned}\sum_{k=1}^{\infty} x_k &= \sum_{k=1}^{\infty} (x_k^+ - x_k^-) \\ &= \sum_{k=1}^{\infty} x_k^+ - \sum_{k=1}^{\infty} x_k^-\end{aligned}$$

(b) Suppose that $\sum_{k=1}^{\infty} x_k$ converges conditionally, and to the contrary, also suppose that $\sum_{k=1}^{\infty} x_k^+$ converges. We have

$$\begin{aligned}\sum_{k=1}^{\infty} x_k^- &= \sum_{k=1}^{\infty}(x_k^+ - x_k)\\ &= \sum_{k=1}^{\infty} x_k^+ - \sum_{k=1}^{\infty} x_k\end{aligned}$$

Since these series both converge, $\sum_{k=1}^{\infty} x_k^-$ also converges. This means

$$\begin{aligned}\sum_{k=1}^{\infty} |x_k| &= \sum_{k=1}^{\infty}(x_k^+ + x_k^-)\\ &= \sum_{k=1}^{\infty} x_k^+ + \sum_{k=1}^{\infty} x_k^-\end{aligned}$$

converges, which is a contradiction. □

Theorem 5.32 (Riemann Rearrangement Theorem)**.** Suppose that $\sum_{k=1}^{\infty} x_k$ converges conditionally, and let β be any number in $\mathbb{R} \cup \{\pm\infty\}$.

Then there exists a rearrangement $\sum_{k=1}^{\infty} y_k$ of $\sum_{k=1}^{\infty} x_k$ such that $\sum_{k=1}^{\infty} y_k$ converges to β.

Proof. Let $\beta \in \mathbb{R}$ and let

$$\begin{aligned}A &= \{n \in \mathbb{N} \mid x_n = x_n^+\} && \text{(positive terms)}\\ B &= \{n \in \mathbb{N} \mid n \notin A\} && \text{(negative terms)}\end{aligned}$$

Then we can write

$$\begin{aligned}A &= \{k_1 < k_2 < k_3 < \dots\}\\ B &= \{\ell_1 < \ell_2 < \ell_3 < \dots\}\end{aligned}$$

and we have $\mathbb{N} = A \cup B$.

For each $n \in \mathbb{N}$, define

$$\begin{aligned}s_n &= x_{k_1} + x_{k_2} + \dots + x_{k_n}\\ t_n &= x_{\ell_1} + x_{\ell_2} + \dots + x_{\ell_n}\end{aligned}$$

We have that

$$\begin{aligned}\lim_{n\to\infty} s_n &= \infty\\ \lim_{n\to\infty} t_n &= -\infty\end{aligned}$$

This is because

$$\begin{aligned}s_n &= x_{k_1}^+ + \dots + x_{k_n}^+\\ t_n &= -x_{\ell_1}^- - \dots - x_{\ell_n}^-\end{aligned}$$

and these sums diverge by Proposition 5.31(b).

Let $N_1 = \min\{n \in \mathbb{N} \mid s_n > \beta\}$. We have $s_{N_1} - \beta \leq x_{k_{N_1}} = x^+_{k_{N_1}}$.

Let $M_1 = \min\{m \in \mathbb{N} \mid s_{N_1} + t_m < \beta\}$. So $\beta - (s_{N_1} + t_{M_1}) \leq |x_{\ell_{M_1}}| = x^-_{\ell_{M_1}}$.

Choose $N_2 = \min\{n > N_1 \mid s_n + t_{M_1} > \beta\}$. So $(s_{N_2} + t_{M_1}) - \beta \leq x_{k_{N_2}} = x^+_{k_{N_2}}$.

Define $M_2 = \min\{m > M_1 \mid s_{N_2} + t_m < \beta\}$. So $\beta - (s_{N_2} + t_{M_2}) < |x_{\ell_{M_2}}| = x^-_{\ell_{M_2}}$.

If $N_1, M_1, \ldots, N_r, M_r$ are defined, pick

$$N_{r+1} = \min\{n > N_r \mid s_n + t_{M_r} > \beta\}$$
$$M_{r+1} = \min\{m > M_r \mid s_{N_{r+1}} + t_m < \beta\}$$

As before, we have

$$(s_{N_{r+1}} + t_{M_r}) - \beta < x_{k_{N_{r+1}}} = x^+_{k_{N_{r+1}}} \tag{*}$$
$$\beta - (s_{N_{r+1}} - t_{M_{r+1}}) < x^-_{\ell_{M_{r+1}}} \tag{**}$$

Now let the rearrangement be

$$x_{k_1}, \ldots, x_{k_N}, x_{\ell_1}, \ldots, x_{\ell_{M_1}}, x_{k_{N_1+1}}, \ldots, x_{k_{N_2}}, x_{\ell_{M_1+1}}, \ldots, x_{\ell_{M_2}}, \ldots$$

and let $(w_n)_{n=1}^\infty$ be the sequence of partial sums. We claim that $w_n \to \beta$. Consider the following cases:

Case 1. Suppose n is such that

$$w_n = s_{N_p} + t_{M_p} + x_{k_{N_{p+1}}} + x_{k_{N_{p+2}}} + \cdots + x_{k_{N_{p+r}}}$$

for some p and some r (i.e. w_n ends with a positive term).

We have
$$s_{N_p} + t_{M_p} \leq w_n, \beta \leq s_{N_{p+1}} + t_{M_p}$$
Since $(s_{N_{p+1}} + t_{M_p}) - \beta \leq x^+_{k_{N_{p+1}}}$ by (*), and

$$\beta - (s_{N_p} + t_{M_p}) \leq x^-_{\ell_{M_p}}$$

by (**). Now
$$|w_n - \beta| \leq x^+_{k_{N_{p+1}}} + x^-_{\ell_{M_p}}$$
As $n \to \infty$, we have that $p \to \infty$. Hence $x^+_{k_{N_{p+1}}}, x^-_{\ell_{M_p}}$ become very small (since $\sum_{k=1}^\infty x_k$ converges). Therefore, $|w_n - \beta| \to 0$.

Case 2. The case where w_n ends with a negative term is left as an exercise.

Thus $\lim_{n\to\infty} w_n = \beta$. That is, $\sum_{k=1}^\infty x_{\sigma(k)} = \beta$. □

Lemma 5.33 (Abel's Formula)**.** Let $(a_k)_{k=1}^\infty \in \mathbb{R}^\mathbb{N}$ and define

$$\alpha(n, m) = \sum_{k=n}^m a_k$$

for each m, n with $m > n$. Then

$$\sum_{k=n}^{m} a_k b_k = \alpha(n,m)b_m + \sum_{k=n}^{m-1} \alpha(n,k)(b_k - b_{k+1})$$

Proof. Note that

(i) $\alpha(n,k) - \alpha(n,k-1) = a_k$

(ii) $\alpha(n,n) = a_n$

Now we have

$$\begin{aligned}
\sum_{k=n}^{m} a_k b_k &= a_n b_n + \sum_{k=n+1}^{m} a_k b_k \\
&= a_n b_n + \sum_{k=n+1}^{m} \big(\alpha(n,k) - \alpha(n,k-1)\big)\, b_k && \text{(by (i))} \\
&= a_n b_n + \sum_{k=n+1}^{m} \alpha(n,k) b_k - \sum_{k=n+1}^{m} \alpha(n,k-1) b_k \\
&= a_n b_n + \sum_{k=n+1}^{m} \alpha(n,k) b_k - \sum_{k=n}^{m-1} \alpha(n,k) b_{k+1} \\
&= a_n b_n + \alpha(n,m) b_m + \sum_{k=n+1}^{m-1} \alpha(n,k) b_k - \alpha(n,n) b_{n+1} - \sum_{k=n+1}^{m-1} \alpha(n,k) b_{k+1} \\
&= \underbrace{a_n b_n - \alpha(n,n) b_{n+1}}_{a_n b_n - a_n b_{n+1}} + \alpha(n,m) b_m + \sum_{k=n+1}^{m-1} \alpha(n,k)(b_k - b_{k+1}) \\
&= \alpha(n,m) b_m + \sum_{k=n}^{m-1} \alpha(n,k)(b_k - b_{k+1}). \ \square
\end{aligned}$$

5.34 (Dirichlet's Test)**.** Let $(a_k)_{k=1}^{\infty}, (b_k)_{k=1}^{\infty}$ be sequences of real numbers and define

$$s_N = \sum_{k=1}^{N} a_k$$

If

(i) $(s_N)_{N=1}^{\infty}$ are bounded

(ii) $(b_k)_{k=1}^{\infty}$ decreases monotonically to 0

then $\sum_{k=1}^{\infty} a_k b_k$ converges.

Proof. From (i), there exists $M > 0$ such that $|s_N| < M/2$ for all N. Then for $m > n > 1$, we have

$$|\alpha(n,m)| = \left|\sum_{k=n}^{m} a_k\right| = |s_m - s_{n-1}| \leq |s_m| + |s_{n-1}| \leq M/2 + M/2 = M$$

Hence $|\alpha(n,m)| \leq M$.

From (ii), there exists N_0 such that $b_n < \varepsilon/M$ for all $n \geq N_0$. We will show that for all $m > n \geq N_0$,

$$\left|\sum_{k=n}^{m} a_k b_k\right| < \varepsilon$$

This is sufficient by the Cauchy Criterion.

By Abel's Formula,

$$\begin{aligned}
\left|\sum_{k=n}^{m} a_k b_k\right| &= \left|\alpha(n,m)b_m + \sum_{k=n}^{m-1} \alpha(n,k)(b_k - b_{k+1})\right| \\
&\leq |\alpha(n,m)|b_m + \sum_{k=n}^{m-1} |\alpha(n,k)||b_k - b_{k+1}| \\
&\leq M b_m + \sum_{k=n}^{m-1} M(b_k - b_{k+1}) \\
&= M\left(b_m + \sum_{k=n}^{m-1}(b_k - b_{k+1})\right) \\
&= M(b_m + b_n - b_m) \\
&= M b_n < \varepsilon. \ \square
\end{aligned}$$

Corollary 5.35 (Alternating Series Test)**.** If $(b_k)_{k=1}^{\infty}$ is a sequence in $[0,\infty)^{\mathbb{N}}$ and this sequence decreases to 0 monotonically, then $\sum_{k=1}^{\infty}(-1)^k b_k$ converges.

Proof. Let $a_k = (-1)^k$. The partial sums are bounded. By Dirichlet's Test, this converges. $\square$

Example 5.36. Some examples of the Alternating Series Test:

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$

Since $b_k = 1/k$ decreases monotonically to 0, this converges by the Alternating Series Test.

(b) $\displaystyle\sum_{k=2}^{\infty} \frac{(-1)^k}{\ln k}$

Again, this converges by the Alternating Series Test.

Example 5.37. Let $(b_k)_{k=1}^{\infty}$ be a sequence in $[0,\infty)^{\mathbb{N}}$ and suppose that $(b_k)_{k=1}^{\infty}$ decreases monotonically to 0. Then $\sum_{k=1}^{\infty} b_k \sin(kx)$ converges for all $x \in \mathbb{R}$.

Proof. Since each function

$$\phi_k(x) = \sin(kx)$$

is 2π-periodic, we need only consider $x \in [0, 2\pi]$.

Also, if $x = 0$ or $x = 2\pi$, then $\sin(kx) = 0$ for all k, which implies convergence. Thus we can consider $x \in (0, 2\pi)$.

Fix some $x \in (0, 2\pi)$. Since $(b_k)_{k=1}^{\infty}$ decreases to 0, we can prove the result by showing that the partial sums

$$D_N(x) = \sum_{k=1}^{N} \sin(kx)$$

are bounded, and apply Dirichlet's Test.

To show this, recall that

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \tag{1}$$
$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \tag{2}$$

Subtracting (1) from (2) yields

$$2\sin\alpha\sin\beta = \cos(\alpha - \beta) - \cos(\alpha + \beta)$$

We have

$$\begin{aligned}
2\sin(x/2)D_N(x) &= \sum_{k=1}^{N} 2\sin(x/2)\sin(kx) \\
&= \sum_{k=1}^{N} \cos(x/2 - kx) - \cos(x/2 + kx) \\
&= \sum_{k=1}^{N} \cos\big((k - 1/2)x\big) - \cos\big((k + 1/2)x\big) \\
&= \cos(x/2) - \cos\big((N + 1/2)x\big)
\end{aligned}$$

Hence

$$\begin{aligned}
|D_N(x)| &= \left| \frac{\cos(x/2) - \cos\big((N+1/2)x\big)}{\underbrace{2\sin(x/2)}_{x\in(0,2\pi),\text{ so }\neq 0}} \right| \\
&\leq \frac{1+1}{2|\sin(x/2)|} = \frac{1}{\underbrace{\sin(x/2)}_{\text{constant}}} < \infty. \ \square
\end{aligned}$$

Theorem 5.38. Let $(b_k)_{k=1}^{\infty} \in [0, \infty)^{\mathbb{N}}$ such that $(b_k)_{k=1}^{\infty}$ is monotone decreasing and $\lim_{k\to\infty} b_k = 0$.

Let $\beta := \sum_{k=1}^{\infty}(-1)^k b_k$ and $t_N = \sum_{k=1}^{N}(-1)^k b_k$ denote the N-th partial sum of the series.

Then $|\beta - t_N| < b_{N+1}$.

Proof. Observe that

$$\begin{aligned}\beta - t_N &= \sum_{k=1}^{\infty}(-1)^k b_k - \sum_{k=1}^{N}(-1)^k b_k \\ &= \sum_{k=N+1}^{\infty}(-1)^k b_k \\ &= \lim_{m\to\infty}\sum_{k=N+1}^{m}(-1)^k b_k\end{aligned}$$

Let $r_m = \sum_{k=N+1}^{m}(-1)^k b_k$, $m \geq N+1$. Note that $\lim_{m\to\infty} r_m = \beta - t_N$, so $\beta - t_N = \lim_{m\to\infty} r_{2m}$. Consider two cases.

Case 1. N is odd.

Then we have

$$\begin{aligned}r_{2m} &= \sum_{k=N+1}^{2m}(-1)^k b_k \\ &= b_{N+1}\underbrace{-b_{N+2}+b_{N+3}}_{\leq 0}\underbrace{-b_{N+4}+b_{N+5}}_{\leq 0}+\cdots+b_{2m-2}\underbrace{-b_{2m-1}+b_{2m}}_{\leq 0} \\ &\leq b_{N+1}\end{aligned}$$

for all $m \geq N$ so $\beta - t_N = \lim_{m\to\infty} r_{2m} \leq b_{N+1}$.

On the other hand,

$$\begin{aligned}r_{2m} &= \underbrace{b_{N+1}-b_{N+2}}_{\geq 0}+\underbrace{b_{N+3}-b_{N+4}}_{\geq 0}+\cdots+\underbrace{b_{2m-2}-b_{2m-1}}_{\geq 0}+b_{2m} \\ &\geq b_{2m} \geq 0\end{aligned}$$

for all $m \geq N$. Thus $\beta - t_N = \lim_{m\to\infty} r_{2m} \geq 0$.

That is, $0 \leq \beta - t_N \leq b_{N+1}$.

Case 2. N is even.

For $m > N$, we have

$$\begin{aligned}0 \geq r_{2m+1} &= -b_{N+1}+\underbrace{b_{N+2}-b_{N+3}}_{\geq 0}+\cdots+\underbrace{b_{2m}-b_{2m+1}}_{\geq 0} \\ &\geq -b_{N+1}\end{aligned}$$

and so $\beta - t_N = \lim_{m\to\infty} r_{2m+1} \in [-b_{N+1}, 0]$. That is,

$$|\beta - t_N| \leq b_{N+1}. \ \square$$

Example 5.39. Let $\beta := \sum_{k=1}^{\infty}\dfrac{(-1)^{k+1}}{k}$. Estimate β to within 0.05.

Solution. Let $b_k = 1/k$, $k \geq 1$. Then $(b_k)_{k=1}^{\infty}$ is monotone decreasing with $\lim_{k\to\infty} b_k = 0$.

Then by Theorem 5.38, it suffices to find $N \geq 1$ such that $b_{N+1} < 0.05$; that is, $\frac{1}{N+1} < 0.05$. Clearly, $N = 20$ works, so we have

$$\begin{aligned}\beta \approx t_{20} &= \sum_{k=1}^{20} \frac{(-1)^{k+1}}{k} \\ &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{20} \\ &\approx 0.6678.\end{aligned}$$

5.40. Recall the following estimate from the Integral Test (Theorem 5.13).

Suppose that $f : [1, \infty) \to [0, \infty)$ is monotone decreasing. Then

$$f(2) + \cdots + f(N) \leq \int_1^N f(x)\,dx \leq f(1) + \cdots + f(N-1)$$

Hence

$$-f(1) \leq \int_1^N f(x)\,dx - \sum_{k=1}^{N} f(k) \leq -f(N)$$

or equivalently,

$$f(N) \leq \sum_{k=1}^{N} f(k) - \int_1^N f(x)\,dx \leq f(1)$$

A similar argument used to obtain this estimate also shows that

$$f(N) \leq \sum_{k=m}^{N} f(k) - \int_m^N f(x)\,dx \leq f(m)$$

for all integers $1 \leq m \leq N$.

6 Series of Functions

Definition 6.1. Let $\emptyset \neq E \subseteq \mathbb{R}$ and suppose that $f_n : E \to \mathbb{R}$ is a function for each $n \geq 1$. Let $f : E \to \mathbb{R}$ be a function as well.

We say that $(f_n)_{n=1}^{\infty}$ **converges pointwise** to f on E if for each $x \in E$,

$$f(x) = \lim_{n\to\infty} f_n(x)$$

We also write $f_n \xrightarrow{\text{pointwise}} f$ as $n \to \infty$.

Example 6.2.

(a) Let $f_n(x) = x^n$, $x \in [0,1]$, $n \geq 1$. Note that if $x \in [0,1)$, then

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = 0$$

Let $f(x) = 0$, $x \in [0,1)$. If $x = 1$, then

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} f_n(1) = \lim_{n\to\infty} 1^n = 1$$

Let $f(1) := 1$. Then $(f_n)_{n=1}^{\infty} \xrightarrow{\text{pointwise}} f$ as $n \to \infty$.

Note that each f_n is continuous on $[0,1]$, but f is not.

(b) Recall that $\mathbb{Q} \cap [0,1]$ is denumerable.

Let $(q_n)_{n=1}^{\infty}$ be an enumeration of $\mathbb{Q} \cap [0,1]$. For each $n \geq 1$, consider the function

$$f_n(x) = \begin{cases} 1 & x \in \{q_1, \ldots, q_n\} \\ 0 & \text{otherwise} \end{cases}$$

Note that the zero function $z(x) = 0$, $x \in [0,1]$ is Riemann integrable, and that $f_n = z$ except at finitely many points, namely $\{q_1, \ldots, q_n\}$. Hence $f_n \in \mathcal{R}[0,1]$, $n \geq 1$.

Now if $x \in [0,1] \setminus \mathbb{Q}$, then $f_n(x) = 0$ for all $n \geq 1$, so

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} 0 = 0$$

Define $f(x) = 0$, $x \in [0,1] \setminus \mathbb{Q}$. If $x \in \mathbb{Q} \cap [0,1]$, then there exists $N \geq 1$ such that $x = q_N$. For all $n \geq N$, $f_n(x) = 1$, so

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} 1 = 1$$

Define $f(x) = 1$, $x \in \mathbb{Q} \cap [0,1]$. Then $(f_n)_{n=1}^{\infty} \xrightarrow{\text{pointwise}} f$ on $[0,1]$. That is,

$$f = \aleph_{\mathbb{Q}\cap[0,1]} = \begin{cases} 1 & x \in \mathbb{Q} \cap [0,1] \\ 0 & \text{otherwise} \end{cases}$$

But as we have seen earlier, $f \notin \mathcal{R}[0,1]$.

Example 6.3. Differentiability and integrability of piecewise convergent functions:

(a) Let $f_n(x) = x^n/n$, $x \in [0,1]$, $n \geq 1$. Note that for all $x \in [0,1]$, we have that

$$0 = \lim_{n\to\infty} \frac{x^n}{n} = \lim_{n\to\infty} f_n(x)$$

so if $f(x) = 0$, $x \in [0,1]$ is the zero function, then $(f_n)_{n=1}^{\infty} \xrightarrow{\text{pointwise}} f$ on $[0,1]$.
Now each f_n is differentiable on $[0,1]$ and f is differentiable on $[0,1]$, but

$$1 = \lim_{n\to\infty} 1^{n-1} = \lim_{n\to\infty} f_n'(1)$$

(since $f_n'(x) = \dfrac{nx^{n-1}}{n} = x^{n-1}$) while $f'(1) = 0$.
Thus $(f_n)_{n=1}^{\infty}$ does not converge pointwise to f'.

(b) Let f_n be the piecewise linear function given by

$$\begin{aligned} f_n : [0,1] &\to \mathbb{R} \\ x &\mapsto \begin{cases} n^2 x & x \in [0, 1/n] \\ n - n^2(x - 1/n) & x \in [1/n, 2/n] \\ 0 & x \in [2/n, 1] \end{cases} \end{aligned}$$

Let $f \equiv 0$ on $[0,1]$. Then $f(0) = 0 = f_n(0)$ for all $n \geq 1$, so $f(0) = \lim_{n\to\infty} f_n(0)$.
If $0 < x \leq 1$, there exists $N \geq 1$ such that $2/N < x$. If $n \geq N$, $f_n(x) = 0$, and so

$$f(x) = 0 = \lim_{n\to\infty} f_n(x)$$

Thus $(f_n)_{n=1}^{\infty} \xrightarrow{\text{pointwise}} f$ on $[0,1]$.
Note that for each $f_n \in \mathcal{R}[0,1]$, since each f_n is continuous on $[0,1]$. Similarly, $f \in \mathcal{R}[0,1]$, but

$$\int_0^1 f = 0 \neq 1 = \lim_{n\to\infty} \int_0^1 f_n$$

The problem with pointwise convergence isthat it is not "strong enough" to behave nicely.

Definition 6.4. Let $\emptyset \neq E \subseteq \mathbb{R}$ be a set and $f_n : E \to \mathbb{R}$, $n \geq 1$ and $f : E \to \mathbb{R}$ be functions.

We say that $(f_n)_{n=1}^{\infty}$ **converges uniformly** to f on E if for all $\varepsilon > 0$, there exists $N \geq 1$ such that $n \geq N$ implies

$$|f_n(x) - f(x)| < \varepsilon$$

for all $x \in E$.

Remark 6.5.

(a) Evidently, the difference between uniform and pointwise convergence is that in the case of **uniform convergence**, we must be able to choose N that is independent of x.

(b) Here, we introduce some notation. Given that $\emptyset \neq E \subseteq \mathbb{R}$ and $g : E \to \mathbb{R}$ is a function, we write

$$\|g\|_E := \sup_{x\in E} |g(x)| \in [0,\infty) \cup \{\infty\}$$

Given this notation, we may express the notion of uniform convergence by saying that

$$(f_n)_{n=1}^{\infty} \xrightarrow{\text{uniformly}} f \iff \big(\forall \varepsilon > 0\, \exists N \geq 1 \text{ such that } n \geq N \implies \|f_n - f\|_E < \varepsilon\big)$$

(c) It should be clear that if $(f_n)_{n=1}^{\infty} \xrightarrow{\text{uniformly}} f$ on E, then $(f_n)_{n=1}^{\infty} \xrightarrow{\text{pointwise}} f$ on E. The converse is false.

Example 6.6.

(a) Consider the functions for each $n \geq 1$,

$$\begin{aligned} f_n : [0,1) &\to \mathbb{R} \\ x &\mapsto x^n \end{aligned} \qquad\qquad \begin{aligned} f : [0,1) &\to \mathbb{R} \\ x &\mapsto 0 \end{aligned}$$

For any $x \in [0,1)$,

$$f(x) = 0 = \lim_{n\to\infty} x^n = \lim_{n\to\infty} f_n(x)$$

so that $(f_n)_{n=1}^{\infty} \xrightarrow{\text{pointwise}} f$.

Note however, that for all $n \geq 1$,

$$\begin{aligned} \|f_n - f\|_{[0,1)} &= \sup_{x\in[0,1)} |f_n(x) - f(x)| \\ &= \sup_{x\in[0,1)} |x^n - 0| \\ &= 1 \end{aligned}$$

Thus $(f_n)_{n=1}^{\infty}$ does not converge uniformly to f.

(b) Suppose $0 < \delta < 1$. Consider the functions for each $n \geq 1$,

$$\begin{aligned} f_n : [0,\delta] &\to \mathbb{R} \\ x &\mapsto x^n \end{aligned} \qquad\qquad \begin{aligned} f : [0,\delta] &\to \mathbb{R} \\ x &\mapsto 0 \end{aligned}$$

Let $\varepsilon > 0$ Since $0 < \delta < 1$, we know that $\lim_{n\to\infty} \delta^n = 0$, and so choose $N \geq 0$ such that $n \geq N$ implies that $\delta^n < \varepsilon$. Thus for $n \geq N$,

$$\begin{aligned} \|f_n - f\|_{[0,\delta]} &= \sup_{x\in[0,\delta]} |f_n(x) - f(x)| \\ &= \sup_{x\in[0,\delta]} |x^n - 0| \\ &= \delta^n \leq \delta^N < \varepsilon \end{aligned}$$

Thus $(f_n)_{n=1}^{\infty} \xrightarrow{\text{uniformly}} f$ on $[0,\delta]$ for all $0 < \delta < 1$, but $(f_n)_{n=1}^{\infty}$ does not converge uniformly to f on $[0,1)$.

(c) Recall the functions $(f_n)_{n=1}^{\infty}$ from Example 6.3. We saw that $(f_n)_{n=1}^{\infty} \xrightarrow{\text{pointwise}} f$ where $f \equiv 0$ on $[0,1]$.

Note that for any $n \geq 1$,

$$\begin{aligned} \|f_n - f\|_{[0,1]} &= \sup_{x\in[0,1]} |f_n(x) - f(x)| \\ &\geq \big|f_n(1/n) - f(1/n)\big| \\ &= |n - 0| = n \end{aligned}$$

hence $\|f_n - f\|_E \nrightarrow 0$. That is, $(f_n)_{n=1}^{\infty}$ does not converge uniformly to f.

Theorem 6.7. Let $\emptyset \neq E \subseteq \mathbb{R}$ and $f_n : E \to \mathbb{R}$, $n \geq 1$ and $f : E \to \mathbb{R}$ be functions. Suppose that $(f_n)_{n=1}^{\infty} \xrightarrow{\text{uniformly}} f$ on E.

(a) If $x_0 \in E$ and each f_n is continuous at x_0, then f is continuous at x_0.

(b) If each f_n is continuous on E, then f is continuous on E.

Remark. Let $\emptyset \neq E \subseteq \mathbb{R}$ be a set and $g : E \to \mathbb{R}$ be a function. Let $x_0 \in E$. We say that g is **continuous** at x_0 if for all $\varepsilon > 0$ there exists $\delta > 0$ such that $\underline{x \in E}$ and $|x - x_0| < \delta$ implies that $|g(x) - g(x_0)| < \varepsilon$.

Proof.

(a) Let $\varepsilon > 0$. Choose $N \geq 1$ such that $n \geq N$ implies

$$\|f_n - f\|_E < \varepsilon/3$$

Observe that f_N is continuous at x_0 by hypothesis. Thus there exists $\delta > 0$ such that if $x \in E$ and $|x - x_0| < \delta$ then

$$|f_N(x) - f_N(x_0)| < \varepsilon/3$$

Then for $x \in E$ and $|x - x_0| < \delta$,

$$\begin{aligned} |f(x) - f(x_0)| &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon \end{aligned}$$

Hence f is continuous at x_0.

(b) This follows immediately from (a). $\square$

Theorem 6.8. Let $a < b \in \mathbb{R}$ and suppose $f_n, f : [a,b] \to \mathbb{R}$ are functions with $(f_n)_{n=1}^{\infty} \xrightarrow{\text{uniformly}} f$ on $[a,b]$.

(a) If $f_n \in \mathcal{R}[a,b]$ for all $n \geq 1$, then $f \in \mathcal{R}[a,b]$ and

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$$

(b) If $F_n(x) = \int_a^x f_n(t)\,dt$, $x \in [a,b]$, $n \geq 1$ and $F(x) = \int_a^x f(t)\,dt$, $x \in [a,b]$, then $(F_n)_{n=1}^{\infty} \xrightarrow{\text{uniformly}} F$ on $[a,b]$.

Proof.

Step 1. Show that f is bounded.

Let $\varepsilon > 0$. Choose $N_0 \geq 1$ such that $n \geq N_0$ implies

$$\|f_n - f\|_{[a,b]} < \varepsilon$$

Now $f_N \in \mathcal{R}[a,b]$ by hypothesis, hence bounded by $\|f_N\|_{[a,b]}$. Then for all $x \in [a,b]$,

$$|f(x)| \leq |f_N(x)| + \varepsilon \leq \|f_N\|_{[a,b]} + \varepsilon < \infty$$

so f is bounded.

Step 2. Show that $f \in \mathcal{R}[a,b]$.

To show this, we shall apply the Cauchy Criterion.

Let $\varepsilon > 0$. Choose $N \geq 1$ such that $n \geq N$ implies that

$$\|f_n - f\|_{[a,b]} < \frac{\varepsilon}{3(b-a)}$$

Observe that $f_N \in \mathcal{R}[a,b]$, and so by the Cauchy Criterion, there exists a partition $P \in \mathcal{P}[a,b]$ such that

$$U(f_N, P) - L(f_N, P) < \varepsilon/3$$

Note also that if we define $M = b - a$, $\|f_N - f\|_{[a,b]} < \dfrac{\varepsilon}{3(b-a)}$ implies that

$$\begin{aligned}
|U(f - f_N, P)| &= \left|\sum_{k=1}^{M} M_k(f - f_N, P)(p_k - p_{k-1})\right| \\
&\leq \sum_{k=1}^{M} |M_k(f - f_N, P)|(p_k - p_{k-1}) \\
&< \sum_{k=1}^{M} \frac{\varepsilon}{3(b-a)}(p_k - p_{k-1}) \\
&= \frac{\varepsilon}{3(b-a)}(b-a) < \varepsilon/3
\end{aligned}$$

and similarly,

$$|L(f - f_N), P)| < \varepsilon/3$$

Then,

$$\begin{aligned}
U(f,P) - L(f,P) &\leq \big(U(f - f_N, P)\big) + \big(U(f_N, P) - L(f_N, P)\big) - \big(L(f - f_N, P)\big) \\
&< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon
\end{aligned}$$

By the Cauchy Criterion, $f \in \mathcal{R}[a,b]$.

Step 3. Show that $(F_n)_{n=1}^{\infty} \xrightarrow{\text{uniformly}} F$ on $[a,b]$.

In particular, the function $F(x) = \int_a^x f(t)\,dt$, $x \in [a,b]$, exists. Note that if $n \geq N$ and $x \in [a,b]$, then

$$\begin{aligned} |F(x) - F_n(x)| &= \left| \int_a^x f(t) - f_n(t)\,dt \right| \\ &\leq \int_a^x |f(t) - f_n(t)|\,dt \\ &\leq \int_a^x \|f - f_n\|_{[a,b]}\,dt \\ &< \int_a^x \frac{\varepsilon}{3(b-a)}\,dt \\ &= \frac{\varepsilon}{3(b-a)}(x-a) \\ &\leq \frac{\varepsilon}{3(b-a)}(b-a) < \varepsilon \end{aligned}$$

Hence $(F_n)_{n=1}^\infty \xrightarrow{\text{uniformly}} F$ on $[a,b]$. In particular,

$$F(b) = \lim_{n\to\infty} F_n(b)$$

That is,

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n. \ \square$$

Proposition 6.9 (Cauchy Criterion for Uniform Convergence)**.** Let $\emptyset \neq E \subseteq \mathbb{R}$ and $f_n : E \to \mathbb{R}$ be functions for each $n \geq 1$. The following are equivalent:

(a) $(f_n)_{n=1}^\infty$ converges uniformly to some function $f : E \to \mathbb{R}$.

(b) For all $\varepsilon > 0$, there exists $N \geq 1$ such that $m \geq n \geq N$ implies

$$\|f_n - f_m\|_E < \varepsilon.$$

Proof.

(a) $\implies$ (b). Let $\varepsilon > 0$ and choose $N \geq 1$ such that $n \geq N$ implies

$$\|f_n - f\|_E < \varepsilon/2$$

Then $m \geq n \geq N$ implies that

$$\begin{aligned} \|f_n - f_m\|_E &\leq \|f_n - f\|_E + \|f - f_m\|_E \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned}$$

(b) $\implies$ (a). We check that there exists $f : E \to \mathbb{R}$ such that $(f_n)_{n=1}^\infty \xrightarrow{\text{pointwise}} f$.

But for any $x \in E$,

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_E$$

so if $\varepsilon > 0$ and we choose $N \geq 1$ such that $m \geq n \geq N$ implies that

$$\|f_m - f_n\|_E < \varepsilon/2$$

then $m \geq n \geq N$ implies that

$$|f_n(x) - f_m(x)| < \varepsilon/2$$

By the Cauchy Criterion for sequences of real numbers, $f(x) = \lim_{n\to\infty} f_n(x)$ exists.

Now we show that $(f_n)_{n=1}^{\infty} \xrightarrow{\text{uniformly}} f$. For $n \geq N$,

$$\begin{aligned} |f_n(x) - f(x)| &= \lim_{m\to\infty} |f_n(x) - f_m(x)| \\ &\leq \lim_{m\to\infty} \|f_n - f_m\|_E \\ &\leq \varepsilon/2 < \varepsilon \end{aligned}$$

That is, $(f_n)_{n=1}^{\infty} \xrightarrow{\text{uniformly}} f$ on E. $\square$

Theorem 6.10. Let $a < b \in \mathbb{R}$ and suppose that $x_0 \in (a,b)$. Let $f_n : (a,b) \to \mathbb{R}$, $n \geq 1$, be a sequence of functions satisfying

(a) $\big(f_n(x_0)\big)_{n=1}^{\infty}$ converges to some $y_0 \in \mathbb{R}$

(b) Each f_n is differentiable on (a,b), $n \geq 1$

(c) $(f_n')_{n=1}^{\infty}$ converges uniformly on (a,b)

Then $(f_n)_{n=1}^{\infty}$ converges uniformly to some function $f : (a,b) \to \mathbb{R}$ and f is differentiable on (a,b) with $f'(x) = \lim_{n\to\infty} f_n'(x)$ for all $x \in [a,b]$.

Proof. For each $\beta \in (a,b)$, consider the sequence of functions

$$g_{\beta,n}(x) = \begin{cases} \dfrac{f_n(x) - f_n(\beta)}{x - \beta} & x \neq \beta \\ f_n'(\beta) & x = \beta \end{cases}$$

Note that $g_{\beta,n}$ is continuous on (a,b) for all $n \geq 1$, $\beta \in (a,b)$.

Step 1. For each fixed $\beta \in (a,b)$, the sequence $(g_{\beta,n})_{n=1}^{\infty}$ converges uniformly on (a,b).

We will use the Cauchy Criterion. Let $\varepsilon > 0$. By (c), we can find $N \geq 1$ such that $m \geq n \geq N$ implies that

$$\left\|f_n' - f_m'\right\|_{(a,b)} < \varepsilon$$

Let $x \in (a,b)$.

Case 1. $x = \beta$.

Then for $m \geq n \geq N$,

$$\begin{aligned} |g_{\beta,n}(x) - g_{\beta,m}(x)| &= |g_{\beta,n}(\beta) - g_{\beta,n}(\beta)| \\ &= |f_n'(\beta) - f_m'(\beta)| \\ &\leq \left\|f_n' - f_m'\right\|_{(a,b)} \\ &< \varepsilon \end{aligned}$$

Case 2. $x \neq \beta$.

For $m \geq n \geq N$, we have

$$\begin{aligned} |g_{\beta,n}(x) - g_{\beta,m}(x)| &= \left| \frac{\big(f_n(x) - f_n(\beta)\big) - \big(f_m(x) - f_m(\beta)\big)}{x - \beta} \right| \\ &= \left| \frac{(f_n - f_m)(x) - (f_n - f_m)(\beta)}{x - \beta} \right| \\ &= |(f_n - f_m)'(c)| && \text{(for some } c \in (x, \beta) \text{ by MVT)} \\ &\leq \left\| f_n' - f_m' \right\|_{(a,b)} \\ &< \varepsilon \end{aligned}$$

Hence $(g_{\beta,n})_{n=1}^{\infty}$ converges uniformly by the Cauchy Criterion.

Step 2. $(f_n)_{n=1}^{\infty}$ converges uniformly on (a, b).

Note that for each $\beta \in (a, b)$,

$$f_n(x) = (x - \beta) g_{\beta,n}(x) + f_n(\beta),\ n \geq 1,\ x \in (a, b)$$

In particular, if $\beta = x_0$,

$$f_n(x) = (x - x_0) g_{x_0,n}(x) + \underbrace{f_n(x_0)}_{\text{converging to } y_0 \text{ by (a)}}$$

so for $m \geq n \geq N$ (as in Step 1), $x \in (a, b)$,

$$\begin{aligned} |f_n(x) - f_m(x)| &= \Big| (x - x_0) \left(g_{x_0,n}(x) - g_{x_0,m}(x) \right) + \left(f_n(x_0) - f_m(x_0) \right) \Big| \\ &\leq |x - x_0| |g_{x_0,n}(x) - g_{x_0,m}(x)| + |f_n(x_0) - f_m(x_0)| \\ &< (b - a)\varepsilon + |f_n(x_0) - f_m(x_0)| \end{aligned}$$

But the fact that $\big(f_n(x_0)\big)_{n=1}^{\infty}$ converges implies that we can find $N_1 \geq N$ such that $m \geq n \geq N_1$ implies

$$|f_n(x_0) - f_m(x_0)| < \varepsilon$$

Then $m \geq n \geq N_1$ implies

$$|f_n(x) - f_m(x)| \leq (b - a)\varepsilon + \varepsilon = \big(1 + (b - a)\big)\, \varepsilon$$

Hence $(f_n)_{n=1}^{\infty}$ converges uniformly (to some $f : (a, b) \to \mathbb{R}$).

Step 3. $f'(x) = \lim_{n\to\infty} f_n'(x)$ for all $x \in (a, b)$.

Recall that each $g_{\beta,n}$ is continuous on (a, b). Since g_β is a uniform limit of the sequence $(g_{\beta,n})_{n=1}^{\infty}$, it follows that g_β is also continuous. Now,

$$f_n'(\beta) = g_{\beta,n}(\beta)$$

and thus

$$\lim_{n\to\infty} f_n'(\beta) = \lim_{n\to\infty} g_{\beta,n}(\beta) = g_\beta(\beta)$$

Also, for $x \neq \beta$,

$$\frac{f(x) - f(\beta)}{x - \beta} = \lim_{n\to\infty} \frac{f_n(x) - f_n(\beta)}{x - \beta} = \lim_{n\to\infty} g_{\beta,n}(x) = g_\beta(x)$$

Since g_β is continuous,

$$f'(\beta) = \lim_{x\to\beta} \frac{f(x) - f(\beta)}{x - \beta} = \lim_{x\to\beta} g_\beta(x) = g_\beta(\beta) = \lim_{n\to\infty} f_n'(\beta)$$

for all $\beta \in (a, b)$. $\square$

Definition 6.11. Let $\emptyset \neq E \subseteq \mathbb{R}$ be a set and $f_k : E \to \mathbb{R}$ be a function for all $k \geq 1$.

A **series** $\sum_{k=1}^\infty f_k$ is a sequence $(s_N)_{N=1}^\infty$, where $s_N := \sum_{k=1}^N f_k$. We say that the series $\sum_{k=1}^\infty f_k$

(a) **converges pointwise** providing that $(s_N)_{N=1}^\infty$ converges pointwise on E.

(b) **converges uniformly** if $(s_N)_{N=1}^\infty$ converges uniformly on E.

(c) **converges absolutely pointwise** if for each $x \in E$, $\sum_{k=1}^\infty |f_k(x)|$ converges.

Example 6.12.

(a) Let $E = [0, 1)$ and $f_k(x) = x^k$, $x \in [0, 1)$, $k \geq 0$ (where $f_0(x) := 1$, $x \in [0, 1)$).
Note that for each $x \in [0, 1)$,

$$s_N(x) := \sum_{k=0}^N f_k(x) = \sum_{k=0}^N x^k = \frac{1 - x^{N+1}}{1 - x}$$

and

$$\sum_{k=1}^\infty f_k(x) = \lim_{N\to\infty} s_N(x) = \lim_{N\to\infty} \frac{1 - x^{N+1}}{1 - x} = \frac{1}{1 - x}$$

Let $f(x) = \frac{1}{1-x}$, $x \in [0, 1)$. From above, $\sum_{k=1}^\infty f_k(x)$ converges pointwise to f.
Note that if $N \geq 1$ however,

$$\begin{aligned} \|s_N - f\|_E &= \|s_N - f\|_{[0,1)} \\ &= \sup_{x\in[0,1)} |s_N(x) - f(x)| \\ &= \sup_{x\in[0,1)} \left| \frac{1 - x^{N+1}}{1 - x} - \frac{1}{1 - x} \right| \\ &= \sup_{x\in[0,1)} \frac{x^{N+1}}{1 - x} = \infty \end{aligned}$$

Hence $(s_N)_{N=1}^\infty$ does not converge uniformly to f. That is, $\sum_{k=1}^\infty f_k$ does not converge uniformly to f.

(b) Let $E = [0,1]$, $f_k(x) = x^k/k^2$, $x \in [0,1]$.

Recall that $\sum_{k=1}^{\infty} 1/k^2$ converges by the p-Test.

Let $\varepsilon > 0$. Then we can find $N \geq 1$ such that $m \geq n \geq N$ implies

$$\left|\sum_{k=n+1}^{m} \frac{1}{k^2}\right| = \sum_{k=n+1}^{m} \frac{1}{k^2} < \varepsilon$$

It follows for the same N that if $m \geq n \geq N$, then for all $x \in E = [0,1]$,

$$\begin{aligned} \|s_m - s_n\|_E &= \|s_m - s_n\|_{[0,1]} \\ &= \sup_{x\in[0,1]} |s_m(x) - s_n(x)| \\ &= \sup_{x\in[0,1]} \left|\sum_{k=1}^{m} \frac{x^k}{k^2} - \sum_{k=1}^{n} \frac{x^k}{k^2}\right| \\ &= \sup_{x\in[0,1]} \left|\sum_{k=n+1}^{m} \frac{x^k}{k^2}\right| \\ &\leq \sup_{x\in[0,1]} \sum_{k=n+1}^{m} \left|\frac{x^k}{k^2}\right| \\ &< \sum_{k=n+1}^{m} \frac{1}{k^2} < \varepsilon \end{aligned}$$

By the Cauchy Criterion, $(s_n)_{n=1}^{\infty}$ converges uniformly on $[0,1]$. That is, $\sum_{k=1}^{\infty} f_k$ converges uniformly on $[0,1]$ (and absolutely pointwise).

(c) Let $E = [0,1]$ and $f_k(x) = (-1)^k/k$, $x \in [0,1]$, $k \geq 1$.

Note that for each $x \in [0,1]$, $\sum_{k=1}^{\infty} f_k(x) = \sum_{k=1}^{\infty}(-1)^k/k$ converges.

Exercise. In fact, the series converges uniformly on $[0,1]$, since each function f_k is constant.

However, this series does not converge absolutely pointwise, since for any $x_0 \in [0,1]$, $\sum_{k=1}^{\infty} |f_k(x_0)| = \sum_{k=1}^{\infty} 1/k$ diverges.

Theorem 6.13. Let $\emptyset \neq E \subseteq \mathbb{R}$ and let $f_k : E \to \mathbb{R}$ be a function for each $k \geq 1$.

(a) If there exists a point $x_0 \in E$ such that each f_k is continuous and x_0 and if $\sum_{k=1}^{\infty} f_k$ converges uniformly to $f : E \to \mathbb{R}$, then f is continuous at x_0.

(b) If $E = [a,b]$ and each $f_k \in \mathcal{R}[a,b]$, $k \geq 1$, and if $\sum_{k=1}^{\infty} f_k$ converges uniformly to $f : [a,b] \to \mathbb{R}$, then $f \in \mathcal{R}[a,b]$ and

$$\int_a^b f = \sum_{k=1}^{\infty} \int_a^b f_k$$

(c) If $a < b \in \mathbb{R}$ and $E = (a,b)$, $f_k : (a,b) \to \mathbb{R}$ is differentiable for all $k \geq 1$, if there exists $x_0 \in (a,b)$ such that $\sum_{k=1}^{\infty} f_k(x_0)$ converges, and if $\sum_{k=1}^{\infty} f_k'$ converges uniformly on $E = (a,b)$, then $\sum_{k=1}^{\infty} f_k$ converges uniformly on $E = (a,b)$ and

$$\left(\sum_{k=1}^{\infty} f_k\right)' = \sum_{k=1}^{\infty} f_k'$$

Proof. Exercise. □

Theorem 6.14 (Weierstrass M-Test)**.** Let $\emptyset \neq E \subseteq \mathbb{R}$ and suppose that $f_k : E \to \mathbb{R}$ is a function, $k \geq 1$. Suppose also that for each $k \geq 1$, we can find $M_k > 0$ such that

(i) $\|f_k\|_E = \sup_{x \in E} |f_k(x)| \leq M_k$, $k \geq 1$

(ii) $\sum_{k=1}^{\infty} M_k < \infty$

Then f_k converges uniformly on E, and it also converges absolutely pointwise.

Proof. Let $\varepsilon > 0$. Since $\sum_{k=1}^{\infty} M_k$ converges, we can find $N \geq 1$ such that $m \geq n \geq N$ implies

$$\sum_{k=n+1}^{m} M_k < \varepsilon$$

Hence, with $s_n(x) = \sum_{k=1}^{n} f_k(x)$, $x \in E$, $n \geq 1$, $m \geq n \geq N$ implies that for all $x \in E$,

$$\begin{aligned} |s_m(x) - s_n(x)| &= \left| \sum_{k=1}^{m} f_k(x) - \sum_{k=1}^{n} f_k(x) \right| \\ &= \left| \sum_{k=n+1}^{m} f_k(x) \right| \\ &\leq \sum_{k=n+1}^{m} |f_k(x)| \qquad (*) \\ &\leq \sum_{k=n+1}^{m} M_k \\ &< \varepsilon \end{aligned}$$

Hence $(s_n)_{n=1}^{\infty}$ converges uniformly on E. That is, $\sum_{k=1}^{\infty} f_k$ converges uniformly on E. In fact, (*) shows that it converges absolutely pointwise. □

Theorem 6.15 (Dirichlet Test for Series of Functions)**.** Suppose that $\emptyset \neq E \subseteq \mathbb{R}$, $f_k, g_k : E \to \mathbb{R}$ are functions. Let $s_N = \sum_{k=1}^{N} f_k$, $N \geq 1$.

Also suppose that

(a) $\sup_{N \geq 1} \|s_N\|_E < \infty$

(b) $(g_k)_{k=1}^{\infty}$ is monotone decreasing

(c) $\lim_{k \to \infty} \|g_k\|_E = 0$

Then $\sum_{k=1}^{\infty} f_k g_k$ converges uniformly on E.

Proof. We will appeal to the Cauchy Criterion. First, for $m \geq n \geq 1$, set

$$F(n, m) = \sum_{k=n}^{m} f_k = s_m - s_n$$

and note that

$$\big\|F(n,m)\big\|_E = \|s_m - s_n\|_E \leq \|s_m\|_E + \|s_n\|_E \leq M + M = 2M$$

Let $\varepsilon > 0$. Note that by hypothesis (a), there exists $N \geq 1$ such that $k \geq N$ implies that

$$\|g_k\|_E < \frac{\varepsilon}{8M}$$

By Abel's Formula, for all $x \in E$, $m \geq n \geq N$,

$$\sum_{k=n}^{m} f_k(x)g_k(x) = F(n,m)(x)g_m(x) + \sum_{k=n}^{m-1} F(n,k)(x)\big(g_{k+1}(x) - g_k(x)\big)$$

and so

$$\begin{aligned}
\left|\sum_{k=n}^{m} f_k(x)g_k(x)\right| &\leq |F(n,m)(x)||g_m(x)| + \sum_{k=n}^{m-1} |F(n,k)(x)|\big(g_k(x) - g_{k+1}(x)\big) \\
&< 2M\left(\frac{\varepsilon}{8M}\right) + \sum_{k=n}^{m-1}(2M)\big(g_k(x) - g_{k+1}(x)\big) \\
&\leq \frac{\varepsilon}{4} + 2M\sum_{k=n}^{m-1}\big(g_k(x) - g_{k+1}(x)\big) \\
&= \frac{\varepsilon}{4} + 2M\big(g_n(x) - g_m(x)\big) \\
&< \frac{\varepsilon}{4} + 2M\big(|g_n(x)| + |g_m(x)|\big) \\
&< \frac{\varepsilon}{4} + 2M\left(\frac{\varepsilon}{8M} + \frac{\varepsilon}{8M}\right) \\
&= \frac{\varepsilon}{4} + \frac{\varepsilon}{2} < \varepsilon
\end{aligned}$$

By the Cauchy Criterion, $\sum_{k=1}^{\infty} f_k g_k$ converges uniformly on E. $\square$

Example 6.16. Let $(a_k)_{k=0}^{\infty} \in [0,\infty)^{\mathbb{N}}$ and suppose that

(i) $(a_k)_{k=0}^{\infty}$ is monotone decreasing

(ii) $\lim_{k\to\infty} a_k = 0$

Define $g_k(x) = a_k$, $x \in (0,2\pi)$ and $f_k(x) = \cos(kx)$, $x \in (0,2\pi)$, $k \geq 0$.

Let $0 < r_0 < s_0 < 2\pi$. We claim that $\sum_{k=0}^{\infty} a_k \cos(kx)$ converges uniformly on $[r_0,s_0] \subseteq (0,2\pi)$.

By Dirichlet's Test for Series of Functions, it suffices to prove that with $D_N = \sum_{k=0}^{N}\cos(kx)$. We have that $\sup_{N\geq 1}\|D_N\|_{[r_0,s_0]} < \infty$.

Recall that

$$\begin{aligned}
\sin(a+b) &= \sin a\cos b + \sin b\cos a \\
\sin(a-b) &= \sin a\cos b - \sin b\cos a
\end{aligned}$$

so that

$$2\sin b\cos a = \sin(a+b) - \sin(a-b)$$

Consider $b = x/2$ and $a = kx$ to get

$$2\sin(x/2)\cos(kx) = \sin\big((k+1/2)x\big) - \sin\big((k-1/2)x\big)$$

Thus

$$\begin{aligned}
2\sin(x/2)D_N &= \sum_{k=0}^{N} 2\sin(x/2)\cos(kx) \\
&= \sum_{k=0}^{N} \sin\big((k+1/2)x\big) - \sin\big((k-1/2)x\big) \\
&= \sin\big((N+1/2)x\big) - \sin(-x/2) \\
&= \sin\big((N+1/2)x\big) + \sin(x/2)
\end{aligned}$$

and so

$$\begin{aligned}
|2\sin(x/2)D_N(x)| &\le |\sin\big((N+1/2)x\big)| + |\sin(x/2)| \\
&\le 1 + 1 = 2
\end{aligned}$$

But for $0 < r_0 < s_0 < 2\pi$, $\sin(x/2)$ is bounded below on $[r_0, s_0]$ (by $\gamma_0 := \min\big(\sin(r_0/2), \sin(s_0/2)\big)$). Thus for all $x \in [r_0, s_0]$,

$$2\gamma_0|D_N(x)| \le 2$$

That is, for all $x \in [r_0, s_0]$, $N \ge 1$,

$$D_N(x) \le 1/\gamma_0$$

Hence $\|D_N\|_{[r_0,s_0]} \le 1/\gamma_0$ for all $N \ge 1$.

By Dirichlet's Test (Theorem 6.15),

$$\sum_{k=1}^{\infty} a_k\cos(kx) = \sum_{k=0}^{\infty} f_k g_k$$

converges uniformly on $[r_0, s_0]$.

6.17. Recall that we had a result that said if $a < b \in \mathbb{R}$, $(f_k)_{k=1}^{\infty}$ is a sequence of functions on (a, b) such that

(a) $\sum_{k=1}^{\infty} f_k(x_0)$ converges for some $x_0 \in (a, b)$

(b) f_k is differentiable on (a, b) for all $k \ge 1$

(c) $\sum_{k=1}^{\infty} f_k'$ is uniformly convergent on (a, b)

then $\sum_{k=1}^{\infty} f_k$ is uniformly convergent on (a, b), and

$$\left(\sum_{k=1}^{\infty} f_k\right)' = \sum_{k=1}^{\infty} f_k'$$

Let us verify that if we only assume that each f_k is differentiable and that $\sum_{k=1}^{\infty} f_k$ converges uniformly, then the conclusion fails.

Consider $a_k = 1/k$, $k \geq 1$, and $f_k(x) = \cos(kx)$, $k \geq 1$. By Example 6.16, $\sum_{k=1}^{\infty} \frac{1}{k}\cos(kx)$ converges uniformly on each $[r_0, s_0] \subseteq (0, 2\pi)$.

In particular, taking $r_0 = \pi/4$ and $s_0 = 3\pi/4$ gives that $\sum_{k=1}^{\infty} \frac{1}{k}\cos(kx)$ converges uniformly on $[\pi/4, 3\pi/4]$.

But for each $k \geq 1$,

$$f_k'(x) = \left(\frac{1}{k}\cos(kx)\right)' = -\sin(kx)$$

Moreover, for any $x \in [\pi/4, 3\pi/4]$, $\big(\sin(kx)\big)_{k=1}^{\infty}$ does not converge to 0, and thus $\sum_{k=1}^{\infty} f_k'(x)$ does not converge for any $x \in [\pi/4, 3\pi/4]$.

Example 6.18 (The Riemann Zeta Function)**.** Recall that $\sum_{n=1}^{\infty} 1/n^p$ converges if and only if $p > 1$ from the p-Test.

Thinking of p as a variable and renaming it "x", we may define a function

$$\begin{aligned} \zeta : (1, \infty) &\to \mathbb{R} \\ x &\mapsto \sum_{n=1}^{\infty} \frac{1}{n^x} \end{aligned}$$

We check if ζ is continuous on $(1, \infty)$.

Consider, for each $n \geq 1$, the function

$$\begin{aligned} f_n : (1, \infty) &\to \mathbb{R} \\ x &\mapsto \frac{1}{n^x} \end{aligned}$$

so that $\zeta = \sum_{n=1}^{\infty} f_n$ converges pointwise on $(1, \infty)$. For each $m \geq 1$, set

$$\zeta_m = \sum_{k=1}^{m} f_k$$

so that ζ_m is the m-th partial sum of the series $\sum_{k=1}^{\infty} f_k$.

Note that each ζ_m is continuous as each f_k is.

If $(\zeta_m)_{m=1}^{\infty}$ converges uniformly to ζ on $(1, \infty)$, then ζ is continuous on $(1, \infty)$ by a previous reuslt. We will now show that this fails.

Let $x_0 \in (1, \infty)$ be arbitrary and consider the function

$$\begin{aligned} \mu_{x_0} : [1, \infty) &\to [0, \infty) \\ t &\mapsto \frac{1}{t^{x_0}} \end{aligned}$$

Then μ_{x_0} is monotone decreasing on $[1, \infty)$. From 5.40, we know that if $g : [1, \infty) \to [0, \infty)$ is monotone decreasing and $1 \leq m < N$, then

$$g(N) = \sum_{k=m}^{N} g(k) - \int_m^N g(t)\, dt$$

Hence if $m < N - 1$, then

$$g(N) \leq \sum_{k=n+1}^{N} g(k) - \int_{m+1}^{N} g(t)\,dt$$

Applying this with $g = \mu_{x_0}$,

$$0 < \frac{1}{N^{x_0}} = \mu_{x_0}(N) \leq \sum_{k=m+1}^{N} \frac{1}{k^{x_0}} - \int_{m+1}^{N} \frac{1}{t^{x_0}}\,dt$$

Thus,

$$\begin{aligned}\int_{m+1}^{N} \frac{1}{t^{x_0}}\,dt &\leq \sum_{k=m+1}^{N} \frac{1}{k^{x_0}} \\ &\leq \sum_{k=m+1}^{\infty} \frac{1}{k^{x_0}} \\ &= \big|\zeta(x_0) - \zeta_m(x_0)\big|\end{aligned}$$

and

$$\int_{m+1}^{N} t^{-x_0}\,dt = \frac{t^{-x_0+1}}{-x_0+1}\bigg|_{t=m+1}^{t=N} = \frac{1}{1-x_0}\left(\frac{1}{N^{x_0-1}} - \frac{1}{(m+1)^{x_0-1}}\right)$$

so that

$$\begin{aligned}\int_{m+1}^{\infty} \frac{1}{t^{x_0}}\,dt &= \lim_{N\to\infty} \int_{m+1}^{N} \frac{1}{t^{x_0}}\,dt \\ &= \lim_{N\to\infty} \frac{1}{1-x_0}\left(\frac{1}{N^{x_0-1}} - \frac{1}{(m+1)^{x_0-1}}\right) \\ &= \frac{1}{1-x_0}\left(-\frac{1}{(m+1)^{x_0-1}}\right) \\ &= \frac{1}{x_0-1}\left(\frac{1}{(m+1)^{x_0-1}}\right) \\ &\leq |\zeta(x_0) - \zeta_m(x_0)|\end{aligned}$$

It follows that

$$\begin{aligned}\|\zeta - \zeta_m\|_{(1,\infty)} &= \sup_{x_0\in(1,\infty)} |\zeta(x_0) - \zeta_m(x_0)| \\ &\geq \sup_{x_0\in(1,\infty)} \frac{1}{x_0-1}\frac{1}{(m+1)^{x_0-1}} \\ &= \infty\end{aligned}$$

Note, however, that if we fix $1 < \alpha_0$, then

$$\|f_n\|_{[\alpha_0,\infty)} = \sup_{x\in[\alpha_0,\infty)} \left|\frac{1}{n^x}\right| \leq \frac{1}{n^{\alpha_0}}$$

so if we set $M_n := 1/n^{\alpha_0}$, then

(i) $\|f_n\|_{[\alpha_0,\infty)} \leq M_n$ for all $n \geq 1$

(ii) $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} 1/n^{\alpha_0} < \infty$ by the p-Test.

By the Weierstrass M-Test, $\sum_{n=1}^{\infty} f_n$ converges uniformly on $[\alpha_0, \infty)$, whence $\zeta = \sum_{n=1}^{\infty} f_n$ is continuous on $[\alpha_0, \infty)$. Since $\alpha_0 > 1$ was arbitrary, ζ is continuous on $\bigcup_{\alpha_0 > 1}[\alpha_0, \infty) = (1, \infty)$.

7 Power Series and Analyticity

7.1. We now consider a special class of series known as **power series**. They are series of the form

$$S(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k$$

where $a_k \in \mathbb{R}$ for all $k \geq 0$ and $x_0 \in \mathbb{R}$ is a fixed number.

Note. There is a *convention* that $S(x_0) = a_0$ (i.e. $(x_0 - x_0)^0 = 1$). This is not a definition of 0^0. It is only shorthand notation.

Example 7.2. Consider the power series

$$S(x) = \sum_{k=0}^{\infty} k^k x^k = 1 + \sum_{k=1}^{\infty} k^k x^k$$

By the Root Test, letting

$$\begin{aligned} \rho &= \limsup_{k \geq 1} |k^k x^k|^{1/k} \\ &= \limsup_{k \geq 1} |kx| \\ &= \begin{cases} \infty & x \neq 0 \\ 0 & x = 0 \end{cases} \end{aligned}$$

so that $S(x)$ converges absolutely if $x = 0$ and diverges otherwise.

Definition 7.3. Let $S(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k$ be a power series. The extended real number $R \in [0, \infty) \cup \{\infty\}$ is called the **radius of convergence** of S if $S(x)$ converges absolutely for $|x - x_0| < R$ and $S(x)$ diverges for $|x - x_0| > R$.

Example 7.4. Examples of radius of convergence:

(a) Let $S(x) = \sum_{k=0}^{\infty} k^k x^k$. The radius of convergence of S is $R = 0$.

(b) Let $S(x) = \sum_{k=0}^{\infty} x^k$. Then

$$\rho_x = \limsup_{k \geq 1} |x^k|^{1/k} = \limsup_{k \geq 1} |x| = |x|$$

and so by the Root Test, $R = 1$.

(c) Let $S(x) = 1 + \sum_{k=1}^{\infty} x^k / k^k$ and let

$$\begin{aligned} \rho_x &= \limsup_{k \geq 1} \left| \frac{x^k}{k^k} \right|^{1/k} \\ &= \limsup_{k \geq 1} \left| \frac{x}{k} \right| \\ &= 0 \end{aligned}$$

By the Root Test, $S(x)$ converges absolutely for all $x \in \mathbb{R}$. That is, $R = \infty$.

Theorem 7.5. Let $S(x) = \sum_{k=0}^{\infty} a_k(x - x_0)^k$ be a power series. Set $\rho = \limsup_{k\geq 1} |a_k|^{1/k}$. The radius of convergence of S is

$$R := \begin{cases} 0 & \text{if } \rho = \infty \\ 1/\rho & \text{if } 0 < \rho < \infty \\ \infty & \text{if } \rho = 0 \end{cases}$$

Furthermore,

(a) $S(x)$ converges absolutely for all $x \in (x_0 - R, x_0 + R)$.

Note. $R = \infty$ implies that $(x_0 - R, x_0 + R) = \mathbb{R}$.

(b) $S(x)$ diverges if $|x - x_0| > R$.

(c) if $[a, b] \subseteq (x_0 - R, x_0 + R)$, then S converges uniformly on $[a, b]$.

Proof. We first prove (a) and (b) together. For each $x \in \mathbb{R}$, set

$$\begin{aligned} r_x &= \limsup_{k\geq 1} |a_k(x - x_0)^k|^{1/k} \\ &= \left(\limsup_{k\geq 1} |a_k|^{1/k} \right) |x - x_0| \\ &= \rho|x - x_0| \end{aligned}$$

By the Root Test, $S(x)$ converges absolutely if $r_x < 1$ and $S(x)$ diverges if $r_x > 1$. But,

- if $\rho = 0$, then $r_x = \rho|x - x_0| = 0$ for all $x \in \mathbb{R}$, so $S(x)$ converges absolutely for all $x \in \mathbb{R}$; that is, $R = \infty$.
- if $\rho = \infty$, then for all $x \neq x_0$, $r_x = \rho|x - x_0| = \infty > 1$, so $S(x)$ diverges. Hence $R = 0$.
- if $0 < \rho < \infty$, then
$$r_x = \rho|x - x_0| < 1 \iff |x - x_0| < 1/\rho$$
and so $S(x)$ converges absolutely if $|x - x_0| < 1/\rho$ and diverges if $|x - x_0| > 1/\rho$. Thus $R = 1/\rho$.

Now we prove (c). There exists $0 < \gamma < R$ such that $[a, b] \subseteq (x_0 - \gamma, x_0 + \gamma)$. Note that $\gamma < R$ implies that $x_0 + \gamma \in (x_0 - R, x_0 + R)$, and so $S(x_0 + \gamma)$ converges absolutely. However,

$$\begin{aligned} S(x_0 + \gamma) &= \sum_{k=0}^{\infty} a_k \left((x_0 + \gamma) - x_0 \right)^k \\ &= \sum_{k=0}^{\infty} a_k \gamma^k \end{aligned}$$

Since this converges absolutely, we have that $\sum_{k=0}^{\infty} |a_k|\gamma^k$ converges.

For each $k \geq 0$, set $M_k := |a_k|\gamma^k$ so that $\sum_{k=0}^{\infty} M_k < \infty$. Note that if $x \in [a,b]$, then for $f_k(x) = a_k(x-x_0)^k$, we have

$$\begin{aligned} \|f_k\|_{[a,b]} &= \sup_{x\in[a,b]} |f_k(x)| \\ &= \sup_{x\in[a,b]} |a_k(x-x_0)^k| \\ &\leq \sup_{x\in[a,b]} |a_k|\gamma^k = M_k \end{aligned}$$

for all $k \geq 0$. Applying the Weierstrass M-Test, $S = \sum_{k=0}^{\infty} f_k$ converges uniformly on $[a,b]$. □

Theorem 7.6. Let $S(x) = \sum_{k=0}^{\infty} a_k(x-x_0)^k$ be a power series. Suppose that $R := \lim_{k\to\infty} |a_k/a_{k+1}|$ exists in $[0,\infty) \cup \{\infty\}$.

Then R is the radius of convergence of S.

Proof. Apply the Ratio Test to $\sum_{k=0}^{\infty} b_k$, where $b_k = a_k(x-x_0)^k$. □

Definition 7.7. Let $S(x) = \sum_{k=0}^{\infty} a_k(x-x_0)^k$ be a power series with radius of convergence R.

The **interval of convergence** of S is the largest interval in $\mathbb{R}$ where S converges.

Thus, the interval of convergence of S is

- $\mathbb{R}$ if $R = \infty$
- $\{x_0\}$ if $R = 0$
- one of the following if $0 < R < \infty$:
  - (i) $(x_0 - R, x_0 + R)$
  - (ii) $(x_0 - R, x_0 + R]$
  - (iii) $[x_0 - R, x_0 + R)$
  - (iv) $[x_0 - R, x_0 + R]$

Example 7.8. Let $S(x) = \sum_{k=1}^{\infty} \frac{x^k}{\sqrt{k}}$. By the Ratio Test, note that if $a_k = \frac{1}{\sqrt{k}}$,

$$R = \lim_{k\to\infty} \frac{|a_k|}{|a_{k+1}|} = \lim_{k\to\infty} \frac{1/\sqrt{k}}{1/\sqrt{k+1}} = \lim_{k\to\infty} \frac{\sqrt{k+1}}{\sqrt{k}} = \lim_{k\to\infty} \sqrt{1+\frac{1}{k}} = 1$$

so if J is the interval of convergence of S, then

$$(-1,1) \subseteq J \subseteq [-1,1]$$

It remains to check the endpoints -1 and 1.

But $S(1) = \sum_{k=1}^{\infty} 1/\sqrt{k} = \sum_{k=1}^{\infty} 1/k^{1/2}$, which diverges by the p-Test. That is, $1 \notin J$.

However, $S(-1) = \sum_{k=1}^{\infty} (-1)^k/\sqrt{k}$ converges by the Alternating Series Test. That is, $-1 \in J$, and thus $J = [-1,1)$.

Example 7.9.

(a) Let $S(x) = \sum_{k=1}^{\infty} x^k/k^2$. Applying the Ratio Test with $a_k = 1/k^2$ gives

$$R := \lim_{k\to\infty} \frac{|1/k^2|}{|1/(k+1)^2|} = \lim_{k\to\infty} \frac{(k+1)^2}{k^2} = \lim_{k\to\infty}(1+1/k)^2 = 1$$

so if J is the interval of convergence of S, then

$$(-1,1) \subseteq J \subseteq [-1,1]$$

Checking the endpoints, we have that

- $S(1) = \sum_{k=1}^{\infty} 1/k^2$ converges by the p-Test, so $1 \in J$.
- $S(-1) = \sum_{k=1}^{\infty}(-1)^k/k^2$ converges absolutely, hence it converges, so $-1 \in J$.

Thus $J = [-1,1]$.

(b) Let $S(x) = \sum_{k=1}^{\infty} x^k$. Then S is a geometric series, so $S(x)$ converges if and only if $|x-0| < 1$. That is, $x \in (-1,1)$. In this case, $J = (-1,1)$.

Theorem 7.10. Let $S(x) = \sum_{k=0}^{\infty} a_k(x-x_0)^k$ be a power series with radius of convergence $R > 0$.

Then S is continuous on $(x_0 - R, x_0 + R)$ and if $R = \infty$, then S is continuous on $\mathbb{R}$.

Proof. Let $y_0 \in (x_0 - R, x_0 + R)$. Then we can choose $0 < \gamma < R$ such that $y_0 \in [x_0 - \gamma, x_0 + \gamma]$. By Theorem 7.5, we know that the series $S(x) = \sum_{k=0}^{\infty} a_k(x-x_0)^k$ converges uniformly on $[x_0 - \gamma, x_0 + \gamma]$.

But each $f_k(x) := a_k(x-x_0)^k$ is continuous on $[x_0-\gamma, x_0+\gamma]$, and thus S is continuous on $[x_0-\gamma, x_0+\gamma]$. In particular, since $y_0 \in [x_0-\gamma, x_0+\gamma]$, S is continuous at y_0. $\square$

Theorem 7.11 (Abel's Theorem)**.** Suppose that $S(x) = \sum_{k=0}^{\infty} a_k(x-x_0)^k$ is a power series and that J (respectively R) is its interval of convergence (respectively radius of convergence). Suppose that $[\alpha,\beta] \subseteq J$ for some $\alpha < \beta$.

Then S converges uniformly on $[\alpha,\beta]$, and so it is continuous on $[\alpha,\beta]$.

Proof. Note that $\alpha < \beta$ and $[\alpha,\beta] \subseteq J$ implies that $R \neq 0$. If $R = \infty$, we can simply apply Theorem 7.5. So we can consider the case where $0 < R < \infty$. In fact, if $[\alpha,\beta] \subseteq (x_0 - R, x_0 + R)$, then once again we can apply Theorem 7.5. So we only need consider the cases $\alpha = x_0 - R$, $\beta = x_0 + R$, or both.

We deal with the case where $\beta = x_0 + R$ and $\alpha = x_0$, and leave the remaining cases as an exercise.

Thus we consider $[\alpha,\beta] = [x_0, x_0+R]$. Let $y \in [x_0, x_0+R]$, and consider

$$b_k = a_k R^k,\ k \geq 0$$
$$c_k\,\big(= c_k(y)\big) = \frac{(y-x_0)^k}{R^k},\ k \geq 0$$

so that

$$S(y) = \sum_{k=0}^{\infty} b_k c_k(y)$$

Note that

- for fixed y, $(c_k)_{k=0}^{\infty}$ is monotone decreasing
- by hypothesis, $S(x_0+R) = \sum_{k=0}^{\infty} a_k R^k = \sum_{k=0}^{\infty} b_k$ converges

Let $\varepsilon > 0$. By the Cauchy Criterion, there exists $N \geq 1$ such that $m > n \geq N$ implies that

$$\left|\sum_{k=n}^{m} b_k\right| < \varepsilon$$

Applying Abel's Formula,

$$\begin{aligned}
\left|\sum_{k=n}^{m} a_k (y-x_0)^k\right| &= \left|\sum_{k=n}^{m} b_k c_k\right| \\
&= \left|c_m \left(\sum_{k=n}^{m} b_k\right) + \sum_{k=n}^{m-1} (c_k - c_{k+1}) \left(\sum_{j=n}^{k} b_j\right)\right| \\
&\leq |c_m| \left|\sum_{k=n}^{m} b_k\right| + \sum_{k=n}^{m-1} |c_k - c_{k+1}| \varepsilon \\
&< c_m \varepsilon + \varepsilon \sum_{k=n}^{m-1} (c_k - c_{k+1}) \\
&= \varepsilon(c_m + c_n - c_m) \\
&= \varepsilon c_n \\
&\leq c_1 \varepsilon \\
&= c_1(y)\varepsilon
\end{aligned}$$

Note however, for all $y \in [x_0, x_0+R]$,

$$c_1(y) = \frac{y-x_0}{R} \leq \frac{(x_0+R)-x_0}{R} = 1$$

and so for all $y \in [x_0, x_0+R]$,

$$\left|\sum_{k=n}^{m} a_k (y-x_0)^k\right| < 1 \cdot \varepsilon = \varepsilon$$

It follows that S converges uniformly on $[x_0, x_0+R]$ by the Cauchy Criterion.

But with $f_k(x) := a_k(x-x_0)^k$, $k \geq 0$, each f_k is continuous on $[x_0, x_0+R]$ (since it is a polynomial) and so S must be continuous on $[x_0, x_0+R]$. $\square$

Remark 7.12. Even in the case where $S(x) = \sum_{k=0}^{\infty} a_k(x-x_0)^k$ has radius of convergence $0 < R < \infty$ and $S(x_0+R)$ converges, there is no guarantee that $S(x_0+R)$ converges absolutely. This is similar for $S(x_0-R)$.

For example, recall that with $S(x) = \sum_{k=1}^{\infty} x^k/\sqrt{k}$, the interval of convergence J of S was $J = [-1, 1)$. Indeed, although the series converges uniformly on $[-1, \beta]$ for all $-1 < \beta < 1$ (by Abel's Theorem), $S(-1) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^{1/2}}$ converges, but $\sum_{k=1}^{\infty} \left|\frac{(-1)^k}{k^{1/2}}\right|$ diverges, so $S(-1)$ does **not** converge absolutely.

7.13. Note that in general, if a function $F : (a, b) \to \mathbb{R}$ is differentiable on (a, b), there is no reason why $(F') : (a, b) \to \mathbb{R}$ should be differentiable.

For example, let $f(x) = |x|$, $x \in (-1, 1)$ and $F(x) = \int_0^x f(t)\,dt$, $x \in (-1, 1)$, then f being continuous on $(-1, 1)$ implies that F is differentiable on $(-1, 1)$ with $F'(x) = f(x)$, $x \in (-1, 1)$. However, F' is not differentiable at $x = 0$.

Lemma 7.14. Let $(a_k)_{k=0}^{\infty} \in \mathbb{R}^{\mathbb{N}\cup\{0\}}$. Then

$$\limsup_{k\geq 0} |a_k|^{1/k} = \limsup_{k\geq 0} |ka_k|^{1/k}$$

Proof. Let $\varepsilon > 0$. Note that $\lim_{n\to\infty} n^{1/n} = 1$ and thus, there exists $N \geq 1$ such that $n \geq N$ implies $1 < n^{1/n} < 1 + \varepsilon$. Hence,

$$\begin{aligned} \sup_{k\geq N} |ka_k|^{1/k} &= \sup_{k\geq N} |k|^{1/k}|a_k|^{1/k} \\ &\leq \sup_{k\geq N} (1+\varepsilon)|a_k|^{1/k} \\ &= (1+\varepsilon) \sup_{k\geq N} |a_k|^{1/k} \end{aligned}$$

Exercise. It follows that

$$\limsup_{k\geq 1} |ka_k|^{1/k} \leq (1+\varepsilon) \limsup_{k\geq 1} |a_k|^{1/k}$$

But $\varepsilon > 0$ was arbitrary, so

$$\limsup_{k\geq 1} |ka_k|^{1/k} \leq \limsup_{k\geq 1} |a_k|^{1/k}$$

On the other hand,

$$|a_k|^{1/k} \leq |ka_k|^{1/k}$$

for all $k \geq 1$, so

$$\limsup_{k\geq 0} |a_k|^{1/k} \leq \limsup_{k\geq 0} |ka_k|^{1/k}$$

Hence

$$\limsup_{k\geq 0} |a_k|^{1/k} = \limsup_{k\geq 0} |ka_k|^{1/k}. \ \square$$

Theorem 7.15. Let $S(x) = \sum_{k=0}^{\infty} a_k(x - x_0)^k$ be a power series with radius of convergence $R > 0$.

Then S is differentiable on $(x_0 - R, x_0 + R)$ and $S'(x) = \sum_{k=1}^{\infty} ka_k(x - x_0)^{k-1}$, $x \in (x_0 - R, x_0 + R)$. In other words, S is term by term differentiable.

Proof. Let $J_0 := (x_0 - R, x_0 + R)$. Consider $T(x) := \sum_{k=1}^{\infty} ka_k(x - x_0)^k$. Now define

$$\begin{aligned} \rho_S &:= \limsup_{k\geq 0} |a_k|^{1/k} \\ \rho_T &:= \limsup_{k\geq 0} |ka_k|^{1/k} \end{aligned}$$

If we denote the radius of convergence of T by R_T, then

$$R = \frac{1}{\rho_S} = \frac{1}{\rho_T} = R_T$$

and so T also converges absolutely on J_0, and if $[a,b] \subseteq J_0$, then T converges uniformly on $[a,b]$ by Abel's Theorem (Theorem 7.11).

Consider

$$S^*(x) = \begin{cases} \dfrac{T(x)}{x-x_0} & x \in J_0 \setminus \{x_0\} \\ a_1 & x = x_0 \end{cases}$$

Let $f_k(x) = a_k(x-x_0)^k$, $k \geq 0$. Then $\sum_{k=0}^{\infty} f_k(x_0) = a_0$ converges.

- If $[a,b] \subseteq J_0$, then each f_k is differentiable on $[a,b]$.
- $S^*(x) = \sum_{k=1}^{\infty} f_k'(x)$ converges absolutely on J_0 and thus uniformly on $[a,b] \subseteq J_0$ by Abel's Theorem.

By Theorem 6.13,

$$S(x) = \sum_{k=0}^{\infty} f_k(x) = \sum_{k=0}^{\infty} a_k(x-x_0)^k$$

converges uniformly on $[a,b]$, and

$$S'(x) = \sum_{k=1}^{\infty} f_k'(x) = \sum_{k=1}^{\infty} ka_k(x-x_0)^{k-1}$$

for all $x \in [a,b]$. But if $y \in J_0 = (x_0 - R, x_0 + R)$, then we can find $x_0 - R < a < y < b < x_0 + R$.

From above, for every $x \in [a,b]$,

$$S'(x) = \sum_{k=1}^{\infty} ka_k(x-x_0)^{k-1}$$

In particular,

$$S'(y) = \sum_{k=1}^{\infty} ka_k(y-x_0)^{k-1}. \ \square$$

Corollary 7.16. Let $S(x) = \sum_{k=0}^{\infty} a_k(x-x_0)^k$ be a power series with radius of convergence $R > 0$. Then S is infinitely differentiable on $(x_0 - R, x_0 + R)$ and

$$S^{(m)}(x) = \sum_{k=m}^{\infty} \frac{k!}{(k-m)!} a_k(x-x_0)^{k-m},\ x \in (x_0 - R, x_0 + R),\ m \geq 1$$

Proof. Use induction with Theorem 7.15. $\square$

Theorem 7.17. Let $S(x) = \sum_{k=0}^{\infty} a_k(x-x_0)^k$ be a power series. Let $\alpha < \beta \in \mathbb{R}$.

(a) If S converges on $[\alpha, \beta]$, then S is integrable on $[\alpha, \beta]$, and

$$\int_\alpha^\beta S(x)\,dx = \sum_{k=0}^\infty a_k \int_\alpha^\beta (x-x_0)^k\,dx$$

(b) If S converges on $[\alpha, \beta]$ and if $B := \sum_{k=0}^\infty \frac{a_k}{k+1}(\beta - x_0)^{k+1}$ and $A := \sum_{k=0}^\infty \frac{a_k}{k+1}(\alpha - x_0)^{k+1}$ both converge, then S is improperly integrable on $[\alpha, \beta]$ and

$$\int_\alpha^\beta S(x)\,dx = \sum_{k=0}^\infty a_k \int_\alpha^\beta (x-x_0)^k\,dx$$

Proof.

(a) By Abel's Theorem, S converges uniformly on $[\alpha, \beta]$. But each $f_k(x) = a_k(x-x_0)^k$ is integrable on $[\alpha, \beta]$, so by Theorem 6.13, S is integrable on $[\alpha, \beta]$, and

$$\begin{aligned}\int_\alpha^\beta S(x)\,dx &= \sum_{k=0}^\infty \int_\alpha^\beta a_k(x-x_0)^k\,dx \\ &= \sum_{k=0}^\infty a_k \int_\alpha^\beta (x-x_0)^k\,dx\end{aligned}$$

(b) Consider the function

$$g(t) = \int_\alpha^t S(x)\,dx,\ t \in [\alpha, \beta)$$

By (a), $g(t)$ is defined for all such t. Now,

$$\begin{aligned}g(t) = \int_\alpha^t S(x)\,dx &= \sum_{k=0}^\infty a_k \int_\alpha^t (x-x_0)^k\,dx \\ &= \sum_{k=0}^\infty \frac{a_k}{k+1}\left[(t-x_0)^{k+1} - (\alpha - x_0)^{k+1}\right] \\ &= \sum_{k=0}^\infty \frac{a_k}{k+1}\left[(t-x_0)^{k+1}\right] - A\end{aligned}$$

Note furthermore, that our hypothesis that B exists implies that $g(\beta) = B - A$ exists.

Hence g is a power series that converges on $[\alpha, \beta]$. By Abel's Theorem, g converges uniformly on $[\alpha, \beta]$. Since each term in the series representation of g is continuous, it follows that g itself is continuous on $[\alpha, \beta]$, so that

$$\begin{aligned}\sum_{k=0}^\infty \frac{a_k}{k+1}\int_\alpha^\beta (x-x_0)^k\,dx = B - A& \\ &= g(\beta) - g(\alpha) \\ &= \lim_{t\to\beta^-} g(t) - g(\alpha) \\ &= \lim_{t\to\beta^-} \int_\alpha^t S(x)\,dx - \int_\alpha^\alpha S(x)\,dx \\ &= \lim_{t\to\beta^-} \int_\alpha^t S(x)\,dt =: \int_\alpha^\beta S(x)\,dx. \ \square\end{aligned}$$

Theorem 7.18. Suppose that

$$S(x) = \sum_{k=0}^{\infty} a_k x^k$$
$$T(x) = \sum_{k=0}^{\infty} b_k x^k$$

both converge on $(-R, R)$.

Define $c_k = \sum_{j=0}^{k} a_j b_{k-j}$, $k \geq 0$. Then $\sum_{k=0}^{\infty} c_k x^k$ converges on $(-R, R)$, and

$$S(x)T(x) = \sum_{k=0}^{\infty} c_k x^k,\ x \in (-R, R)$$

Proof. For each $N \geq 1$, consider the partial sums

$$f_N(x) = \sum_{k=0}^{N} a_k x^k$$
$$g_N(x) = \sum_{k=0}^{N} b_k x^k$$
$$h_N(x) = \sum_{k=0}^{N} c_k x^k$$

for $x \in (-R, R)$. Then,

$$\begin{aligned} h_N(x) &= \sum_{k=0}^{N} \sum_{j=0}^{k} a_j b_{k-j} x^j x^{k-j} \\ &= \sum_{j=0}^{N} a_j x^j \left(\sum_{k=j}^{N} b_{k-j} x^{k-j} \right) \\ &= \sum_{j=0}^{N} a_j x^j \left(\sum_{k=0}^{N-j} b_k x^k \right) \\ &= \sum_{j=0}^{N} a_j x^j g_{N-j}(x) \\ &= g(x) f_N(x) + \sum_{j=0}^{N} a_j x^j \left(g_{N-j}(x) - g(x) \right) \end{aligned}$$

If we can show that for every $x \in (-R, R)$,

$$\lim_{N \to \infty} \sum_{j=0}^{N} a_j x^j \left(g_{N-j}(x) - g(x) \right) = 0$$

then since
$$\lim_{N\to\infty} g(x)f_N(x) = g(x)f(x),$$
it follows that $\lim_{N\to\infty} h_N(x) = g(x)f(x)$ exists. That is,
$$h(x) := \lim_{N\to\infty} g(x)f(x).$$

Let $\varepsilon > 0$ and fix $x_0 \in (-R, R)$.

Then $f(x_0) = \sum_{k=0}^{\infty} a_k x_0^k$ converges absolutely. Moreover, $\lim_{N\to\infty} g_N(x_0)$ exists. Thus we can find $M > 0$ such that $\sum_{k=0}^{\infty} |a_k x_0^k| < M$ and $|g_{N-j}(x) - g(x)| \leq M$ for all $N > j > 0$.

Next, choose $N_0 \geq 1$ such that $m \geq N_0$ implies $|g_m(x_0) - g(x_0)| < \frac{\varepsilon}{2M}$ and $\sum_{j=N_0+1}^{\infty} |a_j x_0^j| < \frac{\varepsilon}{2M}$.

Let $N > 2N_0$. Then
$$\begin{aligned}
\left|\sum_{j=0}^{N} a_j x^j \left(g_{N-j}(x) - g(x)\right)\right| &\leq \left|\sum_{j=0}^{N_0} a_j x^j \left(g_{N-j}(x) - g(x)\right)\right| + \left|\sum_{j=N_0+1}^{N} a_j x^j \left(g_{N-j}(x) - g(x)\right)\right| \\
&< \left(\sum_{j=0}^{N_0} |a_j x^j|\right)\left(\frac{\varepsilon}{2M}\right) + \left(\sum_{j=N_0+1}^{N} |a_j x^j|\right) M \\
&< M \cdot \frac{\varepsilon}{2M} + \frac{\varepsilon}{2M} \cdot M \\
&= \varepsilon
\end{aligned}$$

That is,
$$\lim_{N\to\infty} \left|\sum_{j=0}^{N} a_j x^j \left(g_{N-j}(x) - g(x)\right)\right| = 0$$
and we are done. □

Example 7.19. Let $S(x) = \sum_{k=0}^{\infty} x^k/k!$. Note that by the Ratio Test,
$$\lim_{k\to\infty} \left|\frac{1/k!}{1/(k+1)!}\right| = \lim_{k\to\infty} \frac{(k+1)!}{k!} = \lim_{k\to\infty} k+1 = \infty$$
exists in $[0, \infty) \cup \{\infty\}$ and so the radius of convergence of S is $R = \infty$. That is, S converges (absolutely) on $\mathbb{R}$. Moreover, by Theorem 7.15, S is differentiable on $\mathbb{R}$ and
$$\begin{aligned}
S'(x) &= \sum_{k=0}^{\infty} \frac{1}{k!}(x^k)' \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} k x^{k-1} \\
&= \sum_{k=1}^{\infty} \frac{1}{(k-1)!} x^{k-1} \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} x^k = S(x)
\end{aligned}$$

for all $x \in \mathbb{R}$. Now consider

$$Q(x) = \frac{S(x)}{e^x},\ x \in \mathbb{R}$$

Then

$$Q'(x) = \frac{S'(x)e^x - S(x)(e^x)'}{(e^x)^2} = \frac{S(x)e^x - S(x)e^x}{e^{2x}} = 0$$

for all $x \in \mathbb{R}$. Hence Q is a constant function, say $Q(x) = \alpha$ for all $x \in \mathbb{R}$. That is, for every $x \in \mathbb{R}$,

$$S(x) = \alpha e^x$$

But $\alpha = \alpha e^0 = S(0) = 1$. Thus $S(x) = e^x$, $x \in \mathbb{R}$.

Observe that we can use this series as the definition of the exponential function. That is, for all $x \in \mathbb{R}$,

$$e^x := \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Example 7.20. Consider $S(x) = \sum_{k=0}^{\infty}(-1)^k x^k$. Let

$$\rho = \limsup_{k \geq 1} |(-1)^k|^{1/k} = 1$$

Thus the radius R of convergence of S is $R = 1/\rho = 1$, and S converges absolutely on $(-1, 1)$.

For each $n \geq 1$, set $S_N(x) = \sum_{k=0}^{N}(-1)^k x^k$ and note that $S(x) = \lim_{N\to\infty} S_N(x)$. Now

$$\begin{aligned}(1+x)S_N(x) &= (1+x)(1 - x + x^2 - x^3 + \cdots + (-1)^N x^N) \\ &= 1 + (-1)^N x^{N+1}\end{aligned}$$

Hence

$$\begin{aligned}(1+x)S(x) &= \lim_{N\to\infty}(1+x)S_N(x) \\ &= \lim_{N\to\infty} 1 + (-1)^N x^{N+1} \\ &= 1 - 0 && \text{(since } |x| < 1\text{)} \\ &= 1\end{aligned}$$

and so $S(x) = \frac{1}{1+x}$, $x \in (-1, 1)$.

Suppose $x_0 \in (-1, 1)$. Then S converges uniformly on $[0, x_0]$ (if $x_0 \geq 0$, otherwise on $[x_0, 0]$ if $x_0 < 0$) and so

$$\begin{aligned}\int_0^{x_0} S(t)\,dt &= \sum_{k=0}^{\infty}(-1)^k \int_0^{x_0} t^k\,dt \\ &= \sum_{k=0}^{\infty}(-1)^k \left(\frac{t^{k+1}}{k+1}\right)\Bigg|_{t=0}^{t=x_0} \\ &= \sum_{k=0}^{\infty}\frac{(-1)^k}{k+1}x_0^{k+1} \\ &= x_0 - \frac{x_0^2}{2} + \frac{x_0^3}{3} - \frac{x_0^4}{4} + \dots\end{aligned}$$

But $S(t) = \frac{1}{1+t}$, $t \in [0, x_0]$, so

$$\begin{aligned}\int_0^{x_0} S(t)\,dt &= \int_0^{x_0} \frac{1}{1+t}\,dt \\ &= \ln(1+t)\Big|_{t=0}^{t=x_0} \\ &= \ln(1+x_0)\end{aligned}$$

It follows that for all $x_0 \in (-1,1)$,

$$\ln(1+x_0) = \int_0^{x_0} S(t)\,dt = x_0 - \frac{x_0^2}{2} + \frac{x_0^3}{3} - \frac{x_0^4}{4} + \dots$$

Example 7.21. Let $T(x) = \sum_{k=0}^{\infty}(-1)^k x^{2k}$.

Exercise. By the Root Test, the radius R of convergence of T is 1. Hence, T converges (absolutely) on $(-1,1)$.

Note that for $x \in (-1,1)$,

$$T(x) = S(x^2) = \frac{1}{1+x^2}$$

where S is the series from Example 7.20.

As in the previous example, if $x_0 \in (-1,1)$, then T converges uniformly on $[0, x_0]$ (or $[x_0, 0]$ if $x_0 < 0$) and thus,

$$\begin{aligned}\int_0^{x_0} T(t)\,dt &= \sum_{k=0}^{\infty}(-1)^k \int_0^{x_0} t^{2k}\,dt \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} t^{2k+1}\Big|_{t=0}^{t=x_0} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x_0^{2k+1} \\ &= x_0 - \frac{x_0^3}{3} + \frac{x_0^5}{5} - \frac{x_0^7}{7} + \dots\end{aligned}$$

But $T(t) = \frac{1}{1+t^2}$, $t \in [0, \infty]$, so

$$\begin{aligned}\int_0^{x_0} T(t)\,dt &= \int_0^{x_0} \frac{1}{1+t^2}\,dt \\ &= \arctan t\Big|_{t=0}^{t=x_0} \\ &= \arctan x_0\end{aligned}$$

It follows that for all $x_0 \in (-1,1)$,

$$\arctan x_0 = \int_0^{x_0} T(t)\,dt = x_0 - \frac{x_0^3}{3} + \frac{x_0^5}{5} - \frac{x_0^7}{7} + \dots$$

Definition 7.22. Let $a < b \in \mathbb{R}$ and $f : (a, b) \to \mathbb{R}$ be a function. We say that f is **real analytic** on (a, b) if for all $x_0 \in (a, b)$, there exists $\delta > 0$ and a sequence $(a_k)_{k=0}^{\infty}$ of real numbers such that

$$f(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k$$

for all $x \in (x_0 - \delta, x_0 + \delta)$.

Example 7.23. Recall we defined the exponential function as

$$E(x) = e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

and that this converges on $\mathbb{R}$.

Let $x_0 \in \mathbb{R}$ and consider the series

$$H_{x_0}(x) := \sum_{k=0}^{\infty} \frac{e^{x_0}}{k!}(x - x_0)^k$$

Using the Ratio Test,

$$\lim_{k\to\infty} \left| \frac{e^{x_0}/k!}{e^{x_0}/(k+1)!} \right| = \lim_{k\to\infty} \frac{(k+1)!}{k!} = \lim_{k\to\infty} k + 1 = \infty$$

and so the radius of convergence of H_{x_0} is $R = \infty$. Moreover, by Theorem 7.15, H_{x_0} is differentiable on $\mathbb{R}$, and for all $x \in \mathbb{R}$,

$$\begin{aligned} H'_{x_0}(x) &= \sum_{k=0}^{\infty} \left(\frac{e^{x_0}}{k!}(x - x_0)^k \right)' \\ &= \sum_{k=0}^{\infty} \frac{e^{x_0}}{k!} k (x - x_0)^{k-1} \\ &= \sum_{k=1}^{\infty} \frac{e^{x_0}}{(k-1)!}(x - x_0)^{k-1} = H_{x_0}(x) \end{aligned}$$

Now consider

$$Q(x) = \frac{H_{x_0}(x)}{E(x)},\ x \in \mathbb{R}$$

so that

$$\begin{aligned} Q'(x) &= \frac{H'_{x_0}(x)E(x) - H_{x_0}(x)E'(x)}{\big(E(x)\big)^2} \\ &= \frac{H_{x_0}(x)E(x) - H_{x_0}(x)E(x)}{\big(E(x)\big)^2} \\ &= 0 \end{aligned}$$

for all $x \in \mathbb{R}$. Hence Q is a constant function, say for every $x \in \mathbb{R}$,

$$Q(x) = \alpha$$

That is, for all $x \in \mathbb{R}$,

$$H_{x_0}(x) = \alpha E(x) = \alpha e^x$$

However,

$$e^{x_0} = H_{x_0}(x_0) = \alpha e^{x_0}$$

so $\alpha = 1$, and

$$H_{x_0}(x) = E(x)$$

for all $x \in \mathbb{R}$. It follows that

$$E(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$$

is real analytic on $\mathbb{R}$.

7.24. In the previous example, we did not explain where the coefficients

$$a_k := \frac{e^{x_0}}{k!},\ k \geq 0$$

for H_{x_0} came from.

Recall from Corollary 7.16 that if $S(x) = \sum_{k=0}^{\infty} a_k(x-x_0)^k$ is a power series, then

$$S^{(m)}(x) = \sum_{k=m}^{\infty} \frac{k!}{(k-m)!}(x-x_0)^{k-m}.$$

But then

$$S^{(m)}(x_0) = \frac{m!}{0!}a_m = m!a_m$$

and so

$$a_m = \frac{S^{(m)}(x_0)}{m!}$$

Proposition 7.25. Suppose that $S(x) = \sum_{k=0}^{\infty} a_k(x-x_0)^k$ is a power series with radius of convergence $R > 0$. Then

$$a_n = \frac{S^{(n)}(x_0)}{n!},\ n \geq 0$$

where $S^{(0)} := S$.

Proof. By Corollary 7.16, since S converges on $(x_0 - R, x_0 + R)$, S is infinitely differentiable on $(x_0 - R, x_0 + R)$ and

$$S^{(n)}(x) = \sum_{k=0}^{\infty} \frac{k!}{(k-n)!} a_k (x-x_0)^{k-n}$$

for $x \in (x_0 - R, x_0 + R)$. Hence for all $n \geq 0$,

$$S^{(n)}(x_0) = \frac{n!}{0!}a_n = n!a_n$$

That is,

$$a_n = \frac{S^{(n)}(x_0)}{n!}. \ \square$$

Definition 7.26. Suppose $(a,b) \subseteq \mathbb{R}$ and $f \in \zeta^\infty(a,b)$. Let $x_0 \in (a,b)$. We define the **Taylor series** for f at x_0 to be

$$T_{f,x_0}(x) := \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

If $x_0 = 0$, we also refer to this as the **MacLaurin series** for f.

Remark 7.27. Notes about Taylor series:

(a) By Proposition 7.25, if $S(x) = \sum_{k=0}^{\infty} a_k(x-x_0)^k$ is a power series with positive radius of convergence R, then

$$T_{S,x_0}(x) = S(x),\ x \in (x_0 - R, x_0 + R)$$

If $f \in \zeta^\infty(a,b)$, then it is unclear that $T_{f,x_0}(x) = f(x)$. In fact, we will see that this is generally false.

(b) Note that if f is real analytic on (a,b) and $x_0 \in (a,b)$, then there exists $\delta > 0$ such that

$$f(x) = \sum_{k=0}^{\infty} a_k(x-x_0)^k,\ x \in (x_0 - \delta, x_0 + \delta)$$

By Proposition 7.25,

$$a_n = \frac{f^{(n)}(x_0)}{n!},\ n \geq 0$$

so $f(x) = T_{f,x_0}(x)$, $x \in (x_0 - \delta, x_0 + \delta)$.

Example 7.28. Consider the function

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \end{aligned}$$

Exercises.

(i) Show that $f^{(k)}(0) = 0$ for all $k \geq 0$.

(ii) Show that f is infinitely differentiable on $\mathbb{R}$.

It follows that

$$\begin{aligned} T_{f,0}(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}(x-0)^k \\ &= \sum_{k=0}^{\infty} \frac{0}{k!} x^k \\ &= 0 \end{aligned}$$

which has radius of convergence $R_{T_{f,0}} = \infty$. But $x \neq 0$ implies that $f(x) \neq 0 = T_{f,0}(x)$.

In general, the Taylor series for f does **not** converge to f.

Example 7.29. Recall that in Example 7.23, we showed that if $f(x) = e^x$, $x \in \mathbb{R}$, then f is real analytic on $\mathbb{R}$. So for all $x_0 \in \mathbb{R}$,

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

But $f(x) = e^x$ implies $f^{(k)}(x) = e^x$, $k \geq 0$, $x \in \mathbb{R}$, so for all $x \in \mathbb{R}$,

$$f(x) = \sum_{k=0}^{\infty} \frac{e^{x_0}}{k!}(x - x_0)^k$$

This explains where the (Taylor) coefficients came from in Example 7.23.

Definition 7.30. Let $f \in \zeta^{\infty}(a, b)$ and $x_0 \in (a, b)$. The **remainder term** of order N of the Taylor series for f at x_0 is

$$R_N(x) := f(x) - \sum_{k=0}^{N-1} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

7.31. Recall Taylor's formula from MATH 147. If $f \in \zeta^{\infty}(a, b)$ and $x_0 \in (a, b)$, then given $x \in (a, b)$, there exists c between x_0 and x such that

$$R_N(x) = \frac{f^{(N)}(c)}{N!}(x - x_0)^N$$

Theorem 7.32. Let $(a, b) \subseteq \mathbb{R}$ and let $f \in \zeta^{\infty}(a, b)$. Suppose there exists $M > 0$ such that $\left\| f^{(n)} \right\|_{(a,b)} \leq M^n$ for all $n \geq 1$.

Then f is analytic on (a, b). Thus for each $x_0 \in (a, b)$,

$$f(x) = T_{f,x_0}(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

Proof. We consider two cases.

Case 1. $a, b \in \mathbb{R}$ with $a < b$.

In order for $f(x) = T_{f,x_0}(x)$, we need $\lim_{N\to\infty} R_N(x) = 0$. But by Taylor's formula, there exists c between x and x_0 such that

$$R_N(x) = \frac{f^{(N)}(c)}{N!}(x - x_0)^N$$

But then

$$|R_N(x)| \leq \frac{M^N}{N!}(b - a)^N = \frac{D^N}{N!}$$

where $D = M(b - a)$ is a constant. Since $\lim_{N\to\infty} D^N/N! = 0$, we have that $\lim_{N\to\infty} R_N(x) = 0$. That is, $f(x) = T_{f,x_0}(x)$.

Case 2. $a = -\infty$, $b = \infty$, or both.

Let $x_0 \in (a, b)$. Choose $a < a_0 < x_0 < b_0 < b$ with $a_0, b_0 \in \mathbb{R}$. By Case 1, f is real analytic on (a_0, b_0), so there exists $\delta > 0$ such that

$$f(x) = T_{f,x_0}(x),\ x \in (x_0 - \delta, x_0 + \delta). \ \square$$

Example 7.33. Some real analytic functions:

(a) Let $f(x) = \cos x$, $x \in \mathbb{R}$. Then $f^{(k)}(x) \in \{-\sin x, -\cos x, \sin x, \cos x\}$ and hence

$$\left\| f^{(k)} \right\|_{\mathbb{R}} \leq 1 \leq 1^k$$

for all $k \geq 1$. By Theorem 7.32, f is real analytic on $\mathbb{R}$, and

$$f(x) = T_{f,x_0}(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

When $x_0 = 0$, we have

$$\begin{aligned} f(x) = \cos x &= \frac{\cos 0}{0!} + \frac{(-\sin 0)}{1!}x + \frac{(-\cos 0)}{2!}x^2 + \frac{\sin 0}{3!}x^3 + \dots \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \end{aligned}$$

(b) Similarly, if $g(x) = \sin x$, then

$$g(x) = \sum_{k=0}^{\infty} \frac{g^{(k)}(0)}{k!}x^k = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

Theorem 7.34. Let $a_{kj} \in \mathbb{R}$, $1 \leq j, k$ and suppose that $\alpha_k := \sum_{j=1}^{\infty} |a_{kj}| < \infty$ for all $k \geq 1$. Suppose furthermore that $\sum_{k=1}^{\infty} a_k < \infty$.

Then $\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} a_{kj}$ exists, and

$$\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} a_{kj} = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{kj}$$

Proof. Let $E = \{1/n\}_{n=1}^{\infty} \cup \{0\}$, and for each $k \geq 1$, define $f_k : E \to \mathbb{R}$ by

$$f_k(1/N) = \sum_{j=1}^{N} a_{kj}$$

$$f_k(0) = \sum_{j=1}^{\infty} a_{kj}$$

Note that $\sum_{j=1}^{\infty} a_{kj}$ is absolutely summable by hypothesis, thus it is summable. That is, $f_k(0)$ exists for all $k \geq 1$.

Also note that for $1 \leq k$ fixed,

$$f_k(0) = \sum_{j=1}^{\infty} a_{kj} = \lim_{N \to \infty} \sum_{j=1}^{N} a_{kj} = \lim_{N \to \infty} f_k(1/N)$$

and so f_k is continuous at 0.

From above, f_k is continuous on E. Moreover,

$$\begin{aligned}\|f_k\|_E &= \sup\left\{|f_k(x)| : x \in E\right\} \\ &= \max\left(\sup_{N\geq 1} |f(1/N)|, |f_k(0)|\right) \\ &= \max\left(\sup_{N\geq 1}\left|\sum_{j=1}^{N} a_{kj}\right|, \left|\sum_{j=1}^{\infty} a_{kj}\right|\right) \\ &\leq \sum_{j=1}^{\infty} |a_{kj}| = \alpha_k < \infty\end{aligned}$$

by hypothesis. In fact,

$$\sum_{k=1}^{\infty} \|f_k\|_E \leq \sum_{k=1}^{\infty} \alpha_k < \infty$$

Applying the Weierstrass M-Test, we find that $\sum_{k=1}^{\infty} f_k$ converges uniformly on E to a <u>continuous</u> function $f : E \to \mathbb{R}$.

Thus f is continuous at $x = 0$; that is,

$$f(0) = \lim_{N\to\infty} f(1/N) \tag{*}$$

But $f(0) = \sum_{k=1}^{\infty} f_k(0) = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} a_{kj}$, while

$$\begin{aligned}\lim_{N\to\infty} f(1/N) &= \lim_{N\to\infty} \sum_{k=1}^{\infty} f_k(1/N) \\ &= \lim_{N\to\infty} \sum_{k=1}^{\infty}\sum_{j=1}^{N} a_{kj} \\ &= \lim_{N\to\infty}\lim_{M\to\infty} \sum_{k=1}^{M}\sum_{j=1}^{N} a_{kj} \\ &= \lim_{N\to\infty}\lim_{M\to\infty} \sum_{j=1}^{N}\sum_{k=1}^{M} a_{kj} \\ &= \lim_{N\to\infty} \sum_{j=1}^{N}\lim_{M\to\infty}\sum_{k=1}^{M} a_{kj} \\ &= \lim_{N\to\infty} \sum_{j=1}^{N}\sum_{k=1}^{\infty} a_{kj} \\ &= \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{kj}\end{aligned}$$

By (*),

$$\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} a_{kj} = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{kj}. \ \square$$

Theorem 7.35. Let $S(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k$ be a power series and suppose that S has radius of convergence $R > 0$.

Then S is real analytic on $(x_0 - R, x_0 + R)$.

Proof. By setting $w = x - x_0$, we may assume without loss of generality that $x_0 = 0$. Hence

$$S(x) = \sum_{k=0}^{\infty} a_k x^k$$

is converging on $(-R, R)$.

Let $y_0 \in (-R, R)$. Choose $\delta > 0$ such that $(y_0 - \delta, y_0 + \delta) \subseteq (-R, R)$.

Observe that if $x \in (y_0 - \delta, y_0 + \delta)$, then

$$z := |x - y_0| + |y_0| < R \qquad (**)$$

Now,

$$\begin{aligned}
S(x) &= \sum_{k=0}^{\infty} a_k x^k \\
&= \sum_{k=0}^{\infty} a_k \left(y_0 + (x - y_0)\right)^k \\
&= \sum_{k=0}^{\infty} a_k \left[\binom{k}{0} y_0^k + \binom{k}{1} y_0^{k-1}(x - y_0) + \cdots + \binom{k}{k}(x - y_0)^k\right] \\
&= \sum_{k=0}^{\infty} a_k \left[\sum_{j=0}^{k} \binom{k}{j} y_0^{k-j}(x - y_0)^j\right]
\end{aligned}$$

By (**), if $z \in (y_0 - \delta, y_0 + \delta)$, then $z = |y_0| + |x - y_0| < R$. So S converges absolutely at z. Hence,

$$\sum_{k=0}^{\infty} |a_k| \left(|y_0| + |x - y_0|\right)^k < \infty$$

That is,

$$\sum_{k=0}^{\infty} |a_k| \sum_{j=0}^{k} \binom{k}{j} |y_0|^{k-j} |x - y_0|^j = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \underbrace{|a_k| \binom{k}{j} |y_0|^{k-j} |x - y_0|^j}_{v_{kj} = a_k \binom{k}{j} y_0^{k-j} (x - y_0)^j,\, j \le k} < \infty$$

By Theorem 7.34,

$$\begin{aligned}
S(x) &= \sum_{k=0}^{\infty} \sum_{j=0}^{k} a_k \binom{k}{j} y_0^{k-j} (x - y_0)^j \\
&= \sum_{j=0}^{\infty} \left[\sum_{k=j}^{\infty} a_k \frac{k!}{(k-j)!} y_0^{k-j}\right] \frac{(x - y_0)^j}{j!}
\end{aligned}$$

Note that

$$
\begin{aligned}
S(y_0) &= \sum_{k=0}^{\infty} a_k y_0^k \\
S'(y_0) &= \sum_{k=1}^{\infty} k a_k y_0^{k-1} \\
S^{(2)}(y_0) &= \sum_{k=2}^{\infty} k(k-1) a_k y_0^{k-2} \\
S^{(j)}(y_0) &= \sum_{k=j}^{\infty} k(k-1)\dots(k-j+1) a_k y_0^{k-j}
\end{aligned}
$$

Thus

$$S(x) = \sum_{j=0}^{\infty} \frac{S^{(j)}(y_0)}{j!}(x-y)^j$$

which is the Taylor series for S at y_0. This completes the proof. $\square$

Example 7.36. Recall from Example 7.21 that

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

for $|x| < 1$. But

$$S(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

is a power series which converges in $(-1, 1)$.

By Theorem 7.35, $\arctan x$ is real analytic in $(-1, 1)$.

8 An Interesting Application

Example 8.1. We shall now produce an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$ which is nowhere differentiable.

To begin, let

$$\begin{aligned} g_0 : [-1,1] &\to \mathbb{R} \\ x &\mapsto |x| \end{aligned}$$

We extend g_0 to a continuous function $g : \mathbb{R} \to \mathbb{R}$ with period 2. That is, for all $y \in \mathbb{R}$,

$$\begin{aligned} g(y+2) &= g(y) \\ g(y) &= g_0(y),\ |x| \le 1 \end{aligned}$$

For each $n \ge 0$, set

$$f_n(x) = \left(3/4\right)^n g(4^n x),\ x \in \mathbb{R}$$

Note that $f_0 = g$ and that each f_n is continuous on $\mathbb{R}$, $n \ge 0$. Moreover,

$$\|g\|_{\mathbb{R}} := \sup_{x \in \mathbb{R}} |g(x)| = \sup_{x \in [-1,1]} |g(x)| = 1$$

and thus

$$\|f_n\|_{\mathbb{R}} = \sup_{x \in \mathbb{R}} |f_n(x)| = \left(3/4\right)^n$$

In fact, since

$$\sum_{n=0}^{\infty} \left(3/4\right)^n = \frac{1}{1-3/4} = 4 < \infty$$

it follows from the Weierstrass M-Test that $\sum_{n=0}^{\infty} f_n$ converges uniformly on $\mathbb{R}$ to a continuous function f.

We claim that f is nowhere differentiable. First, observe that

(I) if $x < y$ and $(x,y) \cap \mathbb{Z} = \emptyset$, then

$$|g(y) - g(x)| = |y - x|$$

(II) if $x < y$ and $(x,y) \cap \mathbb{Z} \neq \emptyset$, then

$$|g(y) - g(x)| \le |y - x|$$

Let $x_0 \in \mathbb{R}$. We shall prove that f is not differentiable at x_0.

Indeed, if $f'(x_0) \in \mathbb{R}$ exists, then

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

But we shall find a sequence $(t_m)_{m=1}^{\infty}$ such that

(i) $\lim_{m\to\infty} t_m = 0$

(ii) $\lim_{m\to\infty} \left| \dfrac{f(x_0+t_m)-f(x_0)}{t_m} \right| = \infty$

Fix $m \geq 1$. Consider the interval $J_m := (4^m x_0 - 1/2, 4^m x_0 + 1/2)$. Note that $J_m \cap \mathbb{Z}$ either has 0 elements or 1 element.

We define t_m such that

$$4^m(x_0+t_m) = \begin{cases} 4^m x_0 + 1/2 & \text{if } (4^m x_0, 4^m x_0 + 1/2) \cap \mathbb{Z} = \emptyset \\ 4^m x_0 - 1/2 & \text{otherwise} \end{cases}$$

Thus,

$$t_m = \begin{cases} \frac{1}{2\cdot 4^m} & \text{if } (4^m x_0, 4^m x_0 + 1/2) \cap \mathbb{Z} = \emptyset \\ \frac{-1}{2\cdot 4^m} & \text{if } (4^m x_0, 4^m x_0 + 1/2) \cap \mathbb{Z} \neq \emptyset \end{cases}$$

It follows that there does not exist an integer between $4^m x_0$ and $4^m(x_0+t_m)$.

Now by condition (I),

$$\begin{aligned} \left| g\left(4^m(x_0+t_m)\right) - g(4^m x_0) \right| &= \left|4^m(x_0+t_m) - 4^m x_0\right| \\ &= 4^m |t_m| \end{aligned}$$

Thus,

$$\frac{f(x_0+t_m)-f(x_0)}{t_m} = \sum_{n=0}^{\infty} \frac{f_n(x_0+t_m)-f_n(x_0)}{t_m}$$

First, suppose $n > m$. Then

$$\begin{aligned} f_n(x_0+t_m) - f_n(x_0) &= (3/4)^n \left[g\left(4^n(x_0+t_m)\right) - g(4^n x_0) \right] \\ &= (3/4)^n \left[g(4^n x_0 + 4^n t_m) - g(4^n x_0) \right] \end{aligned}$$

But $|4^n t_m| = 4^n \cdot \dfrac{1}{2\cdot 4^m} = \dfrac{4^{n-m}}{2} = 2k$ for some $k \in \mathbb{N}$.

Since g has period 2,

$$\begin{aligned} g(4^n x_0 + 4^n t_m) &= g(4^n x_0 + 2k) \\ &= g(4^n x_0) \end{aligned}$$

and so

$$f_n(x_0+t_m) - f_n(x_0) = 0$$

Thus,

$$\begin{aligned} \frac{\sum_{n=0}^{\infty} f(x_0+t_m) - f_n(x_0)}{t_m} &= \sum_{n=0}^{m} \frac{f(x_0+t_m)-f(x_0)}{t_m} \\ &= \sum_{n=0}^{m-1} \frac{f_n(x_0+t_m)-f_n(x_0)}{t_m} + \frac{f_m(x_0+t_m)-f_m(x_0)}{t_m} \end{aligned}$$

Now,

$$\begin{aligned}\left|\frac{f_m(x_0+t_m)-f_m(x_0)}{t_m}\right| &= \left|\frac{(3/4)^m\left[g\left(4^m(x_0+t_m)\right)-g(4^m x_0)\right]}{t_m}\right| \\ &= \left(\frac{3}{4}\right)^m \frac{1}{|t_m|}|4^m(x_0+t_m)-4^m x_0| \\ &= 3^m\end{aligned}$$

while if $n<m$, then

$$\begin{aligned}\frac{|f_n(x_0+t_m)-f_n(x_0)|}{|t_m|} &= \left(\frac{3}{4}\right)^n \frac{1}{|t_m|}\left|g\left(4^m(x_0+t_m)\right)-g(4^m x_0)\right| \\ &\le \left(\frac{3}{4}\right)^n \frac{1}{|t_m|}|4^n x_0+4^n t_m-4^n x_0| \\ &\le 3^n\end{aligned}$$

and so

$$\begin{aligned}\left|\frac{f(x_0+t_m)-f(x_0)}{t_m}\right| &\ge \left|\frac{f_m(x_0+t_m)-f_m(x_0)}{t_m}\right| - \left|\sum_{n=0}^{m-1}\frac{f_n(x_0+t_m)-f_n(x_0)}{t_m}\right| \\ &\ge 3^m-\sum_{n=0}^{m-1}\frac{|f_n(x_0+t_m)-f_n(x_0)|}{|t_m|} \\ &\ge 3^m-\sum_{n=0}^{m-1}3^n \\ &\ge 3^m-\left(\frac{3^m-1}{2}\right) \\ &> \frac{3^m}{2}\end{aligned}$$

Thus,

$$\lim_{m\to\infty}\left|\frac{f(x_0+t_m)-f(x_0)}{t_m}\right| = \infty$$

That is, $f'(x_0)$ does not exist. Since $x_0\in\mathbb{R}$ was arbitrary, f is nowhere differentiable. $\square$