CO 353 Course Notes

Computational Discrete Optimization

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1 Shortest Paths

1.1 Preliminaries on Graphs

An **(undirected) graph** G is a pair (V, E), where E is a set of unordered pairs of elements in V. The elements of V are called **vertices** or **nodes**; the elements of E are called **edges**.

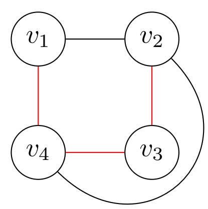
Let $u, v \in V$ and let $e = uv \in E$ be an edge.

- We say that e is **incident** to u and v.
- The vertices u and v are said to be **adjacent**.
- We call u and v the **endpoints** of e.

By default, we assume that there are no parallel edges (i.e. two edges $e = uv$ and $e' = u'v'$ in E with $\{u, v\} = \{u', v'\}$) and no loops (i.e. an edge $e = uv \in E$ with $u = v$).

For distinct $u, v \in V$, a u, v**-path** is a sequence of vertices $w_1, \dots, w_k$ such that $w_1 = u$, $w_k = v$, and $w_i w_{i+1} \in E$ for all $i = 1, \dots, k-1$.

For example, consider the following graph $G = (V, E)$ with vertices $V = \{v_1, v_2, v_3, v_4\}$ and edges $E = \{v_1v_2, v_1v_4, v_2v_3, v_2v_4, v_3v_4\}$.



The lines in red form a v_1, v_2-path, namely v_1, v_4, v_3, v_2. Another v_1, v_2-path can be obtained by simply traversing the edge v_1v_2.

A **cycle** in G is a sequence of vertices $w_1, \dots, w_{k+1}$ such that $w_i w_{i+1} \in E$ for all $i = 1, \dots, k$, the vertices $w_1, \dots, w_k$ are all distinct, and $w_1 = w_{k+1}$.

Finally, a graph G is **connected** if for any pair of distinct vertices $u, v \in V$, there exists a u, v-path in G.

1.2 Shortest Paths Problem

Given a *directed* graph $G = (V, E)$ with edge lengths $\ell_e \geq 0$ for each $e \in E$ and a distinguished start vertex $s \in V$, we wish to find shortest paths from s to every other vertex in V. Note that when we work with directed graphs, we will denote the directed edges with (v_1, v_2) as opposed to v_1v_2 in the case of undirected graphs, where the order of the vertices did not matter.

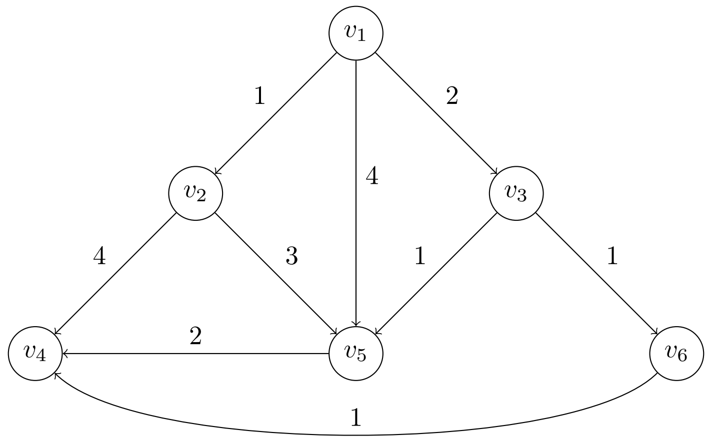
The **length** of a path P given by the sequence $w_1, \dots, w_k$ is given by

$$\ell(P) := \sum_{i=1}^{k-1} \ell_{(w_i, w_{i+1})} = \sum_{e \in P} \ell_e,$$

where the second sum makes sense because there are no parallel edges. Then the **shortest-path distance** from s to a vertex $u \in V$ is defined to be

$$d(u) := \min_{s,\, u\text{-paths } P} \ell(P).$$

For example, we can consider the following instance of an undirected graph with given edge lengths and starting vertex $s = v_1$.



In this case, we have $d(v_2) = 1$, since the only possible path from v_1 to v_2 is by taking the edge (v_1, v_2). There are multiple paths from v_1 to v_5; the shortest one is v_1, v_3, v_5 giving $d(v_5) = 3$.

Note that we always set $d(s) = 0$. We now make some observations:

(i) If $(u, v) \in E$, then $d(v) \leq d(u) + \ell_{(u,v)}$, since such an s, v-path is always an option.

(ii) For every $v \in V$ distinct from s, there exists $w \in V$ such that $d(v) = d(w) + \ell_{(w,v)}$ and $(w, v) \in E$. This can be seen by chopping off the last edge from a shortest path from s to v.

1.3 Dijkstra's Algorithm

In 1959, Dijkstra came up with the following algorithm to solve the shortest paths problem. The main idea is to maintain a set $A \subseteq V$ of "explored" nodes; that is, a set of nodes for which we already know the shortest-path distances. We'll also maintain labels $d'(v)$ for $v \in V \setminus A$ with upper bounds on the shortest-path distances from s.

Input. A directed graph $G = (V, E)$, edge lengths $\ell_e \geq 0$ for all $e \in E$, and a start vertex $v \in V$.

Output. For all $v \in V$, the length $d(v)$ for the shortest-path from s to v.

Step 1. **Initialization.** $A \leftarrow \{s\}$, $d(s) \leftarrow 0$, and $d'(v) \leftarrow \infty$ for all $v \in V \setminus A$.

Step 2. While $A \neq V$:

Step 2.1. **Push down the upper bounds.** For each $v \in V \setminus A$, compute

$$d'(v) \leftarrow \min \left\{ d'(v), \min_{\substack{u \in A \\ (u,v) \in E}} \{d(u) + \ell_{(u,v)}\} \right\}.$$

Step 2.2. **Add a new vertex.** Set $w \leftarrow \arg\min_{v \in V \setminus A} d'(v)$, $A \leftarrow A \cup \{w\}$, and $d(w) \leftarrow d'(w)$.

Suppose that for each vertex $w \in V$, we keep track of the node u determining its upper bound $d'(w)$. That is, the node u is such that $(u, w) \in E$ and $d'(w) = d(u) + \ell_{(u,w)}$. Then at the end of the algorithm, a shortest path from s to w can be obtained as a shortest path from s to u adjoined with the edge $(u, w) \in E$. Moreover, these edges selected by Dijkstra's algorithm form an arborescence, which is a nice graph structure that we'll discuss more later.

Next, let's prove the correctness of Dijkstra's algorithm. In particular, we need to show that for every $v \in V$, the distance from s to v is computed correctly. We'll assume that the graph is connected; that is, for every $v \in V$, there is an s,v-path in G. (Note that the algorithm won't terminate otherwise, but it can be adjusted to deal with this.)

Proof of correctness of Dijkstra's algorithm.

We proceed by induction on $|A|$, and show that at each point in time, $d(v)$ is computed correctly for all $v \in A$. The case where $|A| = 1$ is clear because at the start of the algorithm, we initialize $A = \{s\}$ with $d(s) = 0$, which is correct.

Assume that $d(v)$ is computed correctly for every $v \in A$ when that $|A| = k$. Suppose that we are adding a new vertex w to A in Step 2.2 of the algorithm. Consider the vertex $u \in A$ such that $(u, w) \in E$ and

$$d'(w) = d(u) + \ell_{(u,w)}.$$

Specifically, this is the vertex u determining the upper bound $d'(w)$ which we discussed in the paragraph following the description of the algorithm.

For the sake of contradiction, assume that the distance from s to w is not $d'(w)$. Let P_u be a shortest path from s to u, and let P' be a shortest path from s to w. Then by our assumption, we know that

$$\ell(P') < \ell(P_u) + \ell_{(u,w)} = d'(w).$$

Now, let $x, y \in V$ be such that $(x, y) \in E$ lies on the shortest path P' from s to w, with $x \in A$ and $y \in V \setminus A$. (This exists because at some point, the path must exit A to get from s to w.) Then we obtain

$$d'(y) \leq d(x) + \ell_{(x,y)} \leq \ell(P') < \ell(P_u) + \ell_{(u,w)} = d'(w),$$

where the first inequality is because of how $d'(y)$ is computed in Step 2.1, and the second inequality is because the shortest path from x to y adjoined with the edge (x, y) is part of the path P', noting that $\ell_e \geq 0$ for all $e \in E$. But this contradicts our choice of $w = \arg\min_{v \in V \setminus A}\{d'(v)\}$ in Step 2.2 since $y \in V \setminus A$ but $d'(y) < d'(w)$. $\square$