CO 353 Course Notes

Computational Discrete Optimization

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1 Shortest Paths

1.1 Preliminaries on Graphs

An **(undirected) graph** G is a pair (V, E), where E is a set of unordered pairs of elements in V. The elements of V are called **vertices** or **nodes**; the elements of E are called **edges**.

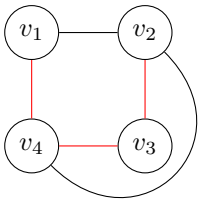
Let $u, v \in V$ and let $e = uv \in E$ be an edge.

- We say that e is **incident** to u and v.
- The vertices u and v are said to be **adjacent**.
- We call u and v the **endpoints** of e.

By default, we assume that there are no parallel edges (i.e. two edges $e = uv$ and $e' = u'v'$ in E with $\{u, v\} = \{u', v'\}$) and no loops (i.e. an edge $e = uv \in E$ with $u = v$).

For distinct $u, v \in V$, a u, v**-path** is a sequence of vertices $w_1, \ldots, w_k$ such that $w_1 = u$, $w_k = v$, and $w_i w_{i+1} \in E$ for all $i = 1, \ldots, k-1$.

For example, consider the following graph $G = (V, E)$ with vertices $V = \{v_1, v_2, v_3, v_4\}$ and edges $E = \{v_1v_2, v_1v_4, v_2v_3, v_2v_4, v_3v_4\}$.



The lines in red form a v_1, v_2-path, namely v_1, v_4, v_3, v_2. Another v_1, v_2-path can be obtained by simply traversing the edge v_1v_2.

A **cycle** in G is a sequence of vertices $w_1, \ldots, w_{k+1}$ such that $w_i w_{i+1} \in E$ for all $i = 1, \ldots, k$, the vertices $w_1, \ldots, w_k$ are all distinct, and $w_1 = w_{k+1}$.

Finally, a graph G is **connected** if for any pair of distinct vertices $u, v \in V$, there exists a u, v-path in G.

1.2 Shortest Paths Problem

Given a *directed* graph $G = (V, E)$ with edge lengths $\ell_e \geq 0$ for each $e \in E$ and a distinguished start vertex $s \in V$, we wish to find shortest paths from s to every other vertex in V. Note that when we work with directed graphs, we will denote the directed edges with (v_1, v_2) as opposed to v_1v_2 in the case of undirected graphs, where the order of the vertices did not matter.

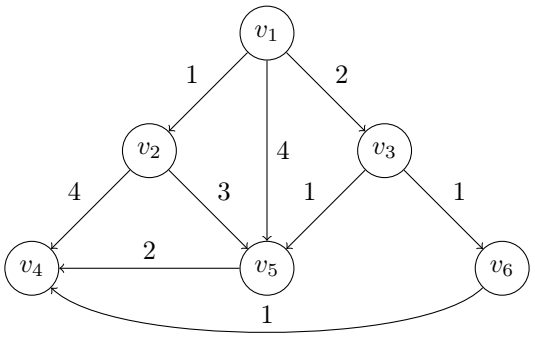
The **length** of a path P given by the sequence $w_1, \ldots, w_k$ is given by

$$\ell(P) := \sum_{i=1}^{k-1} \ell_{(w_i, w_{i+1})} = \sum_{e \in P} \ell_e,$$

where the second sum makes sense because there are no parallel edges. Then the **shortest-path distance** from s to a vertex $u \in V$ is defined to be

$$d(u) := \min_{s,\, u\text{-paths } P} \ell(P).$$

For example, we can consider the following instance of an undirected graph with given edge lengths and starting vertex $s = v_1$.



In this case, we have $d(v_2) = 1$, since the only possible path from v_1 to v_2 is by taking the edge (v_1, v_2). There are multiple paths from v_1 to v_5; the shortest one is v_1, v_3, v_5 giving $d(v_5) = 3$.

Note that we always set $d(s) = 0$. We now make some observations:

(i) If $(u, v) \in E$, then $d(v) \leq d(u) + \ell_{(u,v)}$, since such an s, v-path is always an option.

(ii) For every $v \in V$ distinct from s, there exists $w \in V$ such that $d(v) = d(w) + \ell_{(w,v)}$ and $(w, v) \in E$. This can be seen by chopping off the last edge from a shortest path from s to v.

1.3 Dijkstra's Algorithm

In 1959, Dijkstra came up with the following algorithm to solve the shortest paths problem. The main idea is to maintain a set $A \subseteq V$ of "explored" nodes; that is, a set of nodes for which we already know the shortest-path distances. We'll also maintain labels $d'(v)$ for $v \in V \setminus A$ with upper bounds on the shortest-path distances from s.

Input. A directed graph $G = (V, E)$, edge lengths $\ell_e \geq 0$ for all $e \in E$, and a start vertex $v \in V$.

Output. For all $v \in V$, the length $d(v)$ for the shortest-path from s to v.

Step 1. **(Initialization.)** Set $A \leftarrow \{s\}$, $d(s) \leftarrow 0$, and $d'(v) \leftarrow \infty$ for all $v \in V \setminus A$.

Step 2. While $A \neq V$:

Step 2.1. **(Push down the upper bounds.)** For each $v \in V \setminus A$, compute

$$d'(v) \leftarrow \min \left\{ d'(v), \min_{\substack{u \in A \\ (u,v) \in E}} \{d(u) + \ell_{(u,v)}\} \right\}.$$

Step 2.2. **(Add a new vertex.)** Set $w \leftarrow \arg\min_{v \in V \setminus A} d'(v)$, $A \leftarrow A \cup \{w\}$, and $d(w) \leftarrow d'(w)$.

Suppose that for each vertex $w \in V$, we keep track of the node u determining its upper bound $d'(w)$. That is, the node u is such that $(u, w) \in E$ and $d'(w) = d(u) + \ell_{(u,w)}$. Then at the end of the algorithm, a shortest path from s to w can be obtained as a shortest path from s to u adjoined with the edge $(u, w) \in E$. Moreover, these edges selected by Dijkstra's algorithm form an arborescence, which is a nice graph structure that we'll discuss more later.

Next, let's prove the correctness of Dijkstra's algorithm. In particular, we need to show that for every $v \in V$, the distance from s to v is computed correctly. We'll assume that the graph is connected; that is, for every $v \in V$, there is an s, v-path in G. (Note that the algorithm won't terminate otherwise, but it can be adjusted to deal with this.)

Proof of correctness of Dijkstra's algorithm.

We proceed by induction on $|A|$, and show that at each point in time, $d(v)$ is computed correctly for all $v \in A$. The case where $|A| = 1$ is clear because at the start of the algorithm, we initialize $A = \{s\}$ with $d(s) = 0$, which is correct.

Assume that $d(v)$ is computed correctly for every $v \in A$ when that $|A| = k$. Suppose that we are adding a new vertex w to A in Step 2.2 of the algorithm. Consider the vertex $u \in A$ such that $(u, w) \in E$ and

$$d'(w) = d(u) + \ell_{(u,w)}.$$

Specifically, this is the vertex u determining the upper bound $d'(w)$ which we discussed in the paragraph following the description of the algorithm.

For the sake of contradiction, assume that the distance from s to w is not $d'(w)$. Let P_u be a shortest path from s to u, and let P' be a shortest path from s to w. Then by our assumption, we know that

$$\ell(P') < \ell(P_u) + \ell_{(u,w)} = d'(w).$$

Now, let $x, y \in V$ be such that $(x, y) \in E$ lies on the shortest path P' from s to w, with $x \in A$ and $y \in V \setminus A$. (This exists because at some point, the path must exit A to get from s to w.) Then we obtain

$$d'(y) \leq d(x) + \ell_{(x,y)} \leq \ell(P') < \ell(P_u) + \ell_{(u,w)} = d'(w),$$

where the first inequality is because of how $d'(y)$ is computed in Step 2.1, and the second inequality is because the shortest path from x to y adjoined with the edge (x, y) is part of the path P', noting that $\ell_e \geq 0$ for all $e \in E$. But this contradicts our choice of $w = \arg\min_{v \in V \setminus A}\{d'(v)\}$ in Step 2.2 since $y \in V \setminus A$ but $d'(y) < d'(w)$. □

The **running time** of an algorithm is the number of elementary operations that the algorithm performs as a function of the input size. The **input size** is the number of bits needed to specify the input.

- For example, in order to specify an integer $n \geq 0$ in binary, we require about $\log_2(n)$ bits.
- To specify a graph $G = (V, E)$ with integral edge lengths $\ell_e \geq 0$ for each $e \in E$, we require approximately $|V| + |E| + \sum_{e \in E} \log_2(\ell_e)$ bits. Note that we can specify an edge with two pointers to the endpoints.

We will need big-O notation to describe running time, because we are interested in the asymptotic behaviour of algorithms. For two functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ and $g : \mathbb{R}^+ \to \mathbb{R}^+$, we say that $f(n) = O(g(n))$ if there exist constants $n_0 \in \mathbb{N}$ and $c \geq 0$ such that $f(n) \leq c \cdot g(n)$ for all $n \geq n_0$. For example, we have $2n^2 + 1 = O(n^2)$ and $\log(n) = O(n)$. An algorithm is then considered **efficient** if its running time is bounded above by a polynomial function of the input size.

Let's consider the running time of Dijkstra's algorithm by looking at a naive implementation. For ease of notation, we will write $|V| = n$ and $|E| = m$. Note that $G = (V, E)$ is connected, so $n - 1 \leq m \leq n^2$.

- Step 1 can be performed using $O(n)$ operations since we are only assigning values for each vertex.
- In Step 2, there are at most n iterations.
  - Step 2.1 can be performed using $O(m)$ operations because each edge will participate in at most two comparisons throughout the entire iteration.
  - Step 2.2 takes $O(n)$ operations in order to determine the vertex of minimum upper bound.

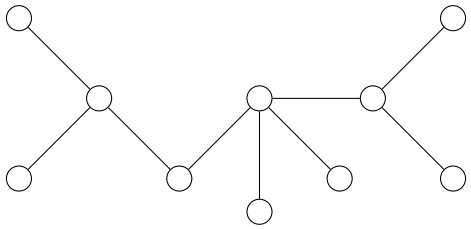
Therefore, the running time of the naive implementation is $O(n + mn + n^2) = O(mn)$, which is polynomial in the input size.

We note that there are better implementations of Dijkstra's algorithm than the naive one that we have just stated. For Step 2.1, we can use m Decrease-Key calls and for Step 2.2, we can use n Extract-Min calls. In particular, by using Fibonacci heaps, Decrease-Key has running time $O(1)$ and Extract-Min has running time $O(\log n)$, which brings the total running time down to $O(m + n \log n)$.

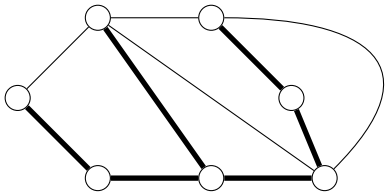
2 Minimum Spanning Trees

2.1 Trees

A **tree** is a connected acyclic graph; that is, a connected graph containing no cycles.



Given a graph $G = (V, E)$, a **spanning tree** of G is a graph $T = (V, F)$ such that $F \subseteq E$ and T is a tree. We illustrate an example of a graph G with a subtree T using bold edges below.



In an introductory graph theory course, such as MATH 239, it is shown that every tree on n vertices has $n - 1$ edges. The following theorem then gives us a useful characterization of trees.

Theorem 2.1: Fundamental Theorem of Trees

Let $T = (V, F)$ be a graph. The following are equivalent:

(i) T is a tree.

(ii) T is connected and $|F| = |V| - 1$.

(iii) T is acyclic and $|F| = |V| - 1$.

In particular, if we know that two of the conditions hold, then the third one is guaranteed.

2.2 Minimum Spanning Trees

Given a connected graph $G = (V, E)$ and edge costs c_e for each $e \in E$, our goal is to find a spanning tree T of minimum cost

$$c(T) := \sum_{e \in T} c_e.$$

First, we'll set some notation. For a vertex $v \in V$, we define $\delta(v)$ to be the set of edges in E incident to v. More generally, given a subset of vertices $S \subseteq V$, the **cut induced by** S is defined to be the set

$$\delta(S) := \{uv \in E : u \in S, v \notin S\}.$$

The following theorem will be extremely important for finding a minimum spanning tree.

Theorem 2.2: Cut Property

Suppose that the costs c_e for $e \in E$ are distinct. Let $S \subseteq V$ be such that $S \neq \varnothing$ and $S \neq V$, and let

$$e = \arg\min_{f \in \delta(S)} c_f.$$

Then every minimum spanning tree contains the edge e.

Proof of Theorem 2.2.

We proceed by contradiction. Let $S \subseteq V$ be such that $S \neq \varnothing$ and $S \neq V$, and let $e = \arg\min_{f \in S} c_f$. Suppose that there is a minimum spanning tree $T = (V, F)$ such that $e \notin F$.

Consider the graph $(V, F \cup \{e\})$. Note that $|F \cup \{e\}| = |V|$ and this graph is connected, so it cannot be a tree by Theorem 2.1. In particular, it must contain a cycle C.

Next, note that $|C \cap \delta(S)|$ must be even because for any edge leaving the cut $\delta(S)$, there must be another edge coming back into the cut. Moreover, since $e \in C \cap \delta(S)$, we have $C \cap \delta(S) \neq \varnothing$. This implies that $|C \cap \delta(S)| \geq 2$, so there is another edge $e' \neq e$ with $e' \in C \cap \delta(S)$.

Consider now the graph $T' = (V, (F \cup \{e\}) \setminus \{e'\})$. Then $|(F \cup \{e\}) \setminus \{e'\}| = |F| = |V| - 1$ and T' is connected because if a path between two vertices used the edge e', then we could go along the edges in $C \setminus \{e'\}$ instead. So by Theorem 2.1, T' is also a spanning tree. Finally, observe that

$$c(T) - c(T') = c_{e'} - c_e > 0$$

because we have $e = \arg\min_{f \in S} c_f$ and $e' \neq e$ with $e' \in \delta(S)$, as well as the assumption that the edge costs c_e for $e \in E$ were distinct. But T' has lower cost than T, contradicting our assumption that T was a minimum spanning tree. □

The following property relating cycles to minimum spanning trees can also be proved similarly to Theorem 2.2. The main idea is to try to create shortcuts using the edges in the cycle.

Theorem 2.3: Cycle Property

Let $G = (V, E)$ be connected with distinct edge costs c_e for $e \in E$. Let C be a cycle in G and let

$$e = \arg\max_{f \in C} c_f.$$

Then no minimum spanning tree contains the edge e.

2.3 Prim's Algorithm

Prim's algorithm takes the idea of the cut property (Theorem 2.2) and uses it to compute a minimum spanning tree starting from an arbitrary vertex $s \in V$. At each iteration of the algorithm, we will keep track of a set of a partial tree construction T and vertices $A \subseteq V$ that are connected to s in T.

Input. A connected graph $G = (V, E)$ and edge costs c_e for all $e \in E$.

Output. The edges T of a minimum spanning tree of G.

Step 1. **(Initialization.)** Let $s \in V$ be an arbitrary vertex, then set $A \leftarrow \{s\}$ and $T \leftarrow \varnothing$.

Step 2. While $A \neq V$:

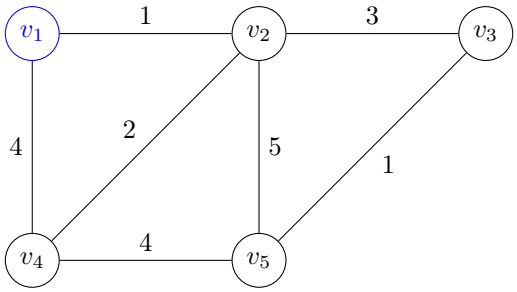
Step 2.1. **(Pick an edge for the minimum spanning tree.)** Set

$$e \leftarrow \arg\min_{f \in \delta(A)} c_f,$$

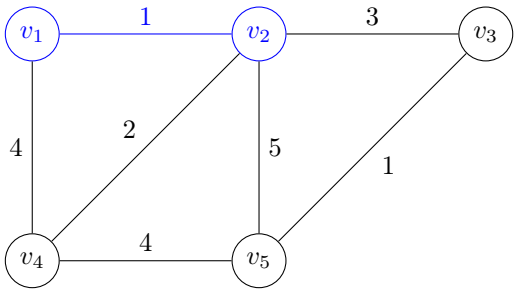
where $e = uv$ with $u \in A$ and $v \notin A$.

Step 2.2. **(Update values.)** Set $A \leftarrow A \cup \{v\}$ and $T \leftarrow T \cup \{e\}$.

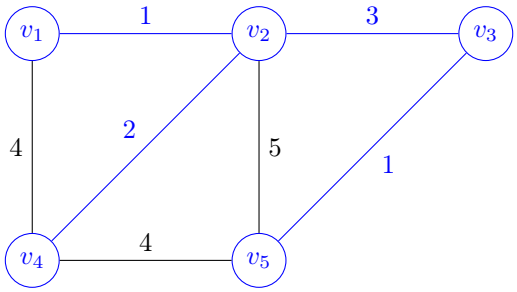
Let's run Prim's algorithm on a simple example. Consider the following graph $G = (V, E)$, and suppose that at Step 1, we pick v_1 to be the initial vertex.



Then at the first iteration of Step 2.1, we have $\delta(A) = \{v_1v_2, v_1v_4\}$, so we pick $e = v_1v_2$ because it has minimum cost. At Step 2.2, we set $A = \{v_1, v_2\}$ and $T = \{v_1v_2\}$.



The edge v_2v_4 is of minimum cost in the cut induced by A, so we set $A = \{v_1, v_2, v_4\}$ and $T = \{v_1v_2, v_2v_4\}$ at the next iteration. Continuing in this fashion, we obtain the following minimum spanning tree.



Note that Prim's algorithm may seen similar to Dijkstra's algorithm which also yields a spanning tree in the process of computing shortest paths. However, these algorithms fundamentally solve different problems, and there are examples where the resulting spanning trees do not coincide. Moreover, Dijkstra's algorithm takes a directed graph as input, whereas Prim's algorithm takes an undirected graph.

Next, let's prove that Prim's algorithm always gives us a minimum spanning tree. For simplicity, we will assume that all edge costs c_e for $e \in E$ are distinct.

Proof of correctness of Prim's algorithm.

At every iteration of Step 2, we add one edge to T, and we run through $|V| - 1$ iterations. Therefore, T has exactly $|V| - 1$ edges. Moreover, an invariant of Prim's algorithm is that all vertices in A remain connected to s, so the final output is also connected. Therefore, we indeed obtain a spanning tree by Theorem 2.1. Finally, by the cut property (Theorem 2.2) and our assumption that all the edge costs are distinct, T contains only the edges that are in every minimum spanning tree, so T itself is a minimum spanning tree. □

As a consequence, we also have the following useful result.

COROLLARY 2.4

Let $G = (V, E)$ be a connected graph with distinct edge costs c_e for $e \in E$. Then G has a unique minimum spanning tree.

As usual, let's consider the running time, denoting $|V| = n$ and $|E| = m$. For an efficient implementation, we keep track of a key $d'(v)$ for each $v \in V \setminus A$. Once a vertex w is added to A in Step 2.2, we update the keys via $d'(v) \leftarrow \min\{d'(v), c_{wv}\}$ for each $v \in V \setminus A$.

Note that Step 1 takes $O(1)$ time, and there are $n - 1$ iterations of Step 2. Implementing Prim's algorithm using priority queues, we note that each iteration of Step 2.1 involves one EXTRACT-MIN call, and going through all iterations of Step 2.2 takes at most m DECREASE-KEY calls in total. We recall that by using Fibonacci heaps, DECREASE-KEY is $O(1)$ and EXTRACT-MIN is $O(\log n)$, so we have a total running time of $O(n \log n + m)$.

2.4 Kruskal's Algorithm

Kruskal's algorithm is another greedy algorithm that finds a minimum spanning tree. It first sorts the edges by ascending edge costs and continually adds an edge to a partial tree construction T as long as no cycle is induced by that edge. The algorithm we give stops once we go through every edge, but note that it is enough to stop once the number of edges we've added hits $|V| - 1$.

Input. A connected graph $G = (V, E)$ and edge costs c_e for all $e \in E$. (We write $m = |E|$ and $n = |V|$.)

Output. The edges T of a minimum spanning tree of G.

Step 1. **(Initialization.)** Sort the edges such that $c_{e_1} \leq c_{e_2} \leq \cdots \leq c_{e_m}$. Set $T \leftarrow \varnothing$ and $j \leftarrow 1$.

Step 2. While $j \neq m + 1$:

(Add an edge if it does not form a cycle.) If $T \cup \{e_j\}$ is acyclic, set $T \leftarrow T \cup \{e_j\}$.

(Go to the next edge.) Regardless if we add an edge or not, increment $j \leftarrow j + 1$.

To prove the correctness of Kruskal's algorithm, we'll again use the cut property (Theorem 2.2). As with Prim's algorithm, the proof will consist of two parts: proving that the result T is a spanning tree, and showing that it is of minimum cost. We will also assume that the edge costs c_e are distinct.

We use a couple of facts that will be repeated many times throughout the course. These concern equivalent notions to the definitions we have.

(1) A graph $G = (V, E)$ is disconnected if and only if there exists a nontrivial cut $\delta(S)$ for some $\varnothing \subsetneq S \subsetneq V$ such that $\delta(S) \cap E = \varnothing$.

(2) Let $G = (V, E)$ be a graph and let $u, v \in V$ be distinct vertices. Then G has no u, v-path if and only if there exists $S \subseteq V$ such that $u \in S$, $v \notin S$, and $\delta(S) \cap E = \varnothing$.

Proof of correctness of Kruskal's algorithm.

Let T denote the output set consisting of edges. We know that T is acyclic by construction because Step 2 of the algorithm rules out all edges that create a cycle. We will show that (V, T) is connected by way of contradiction, and thus it is a spanning tree by Theorem 2.1.

Suppose not, so there is a nontrivial cut $\delta(S)$ for some $\varnothing \subsetneq S \subsetneq V$ such that $\delta(S) \cap T = \varnothing$ by (1) above. Since G is connected, we have $\delta(S) \neq \varnothing$, so there exists some edge $e \in \delta(S)$. Let's look at the moment where e was considered in the algorithm. Since e was rejected, it must be that $T \cup \{e\}$ contains a cycle C with $e \in C$. But $|C \cap \delta(S)|$ is even and $e \in C \cap \delta(S)$, so $|C \cap \delta(S)| \geq 2$. We then have $|(C \setminus \{e\}) \cap \delta(S)| \geq 1$. But $C \setminus \{e\} \subseteq T$, which contradicts the fact that $\delta(S) \cap T = \varnothing$.

Now, we show that we have a minimum spanning tree. Consider the moment an arbitrary edge $e = uv$ is added to T; let T' be the set of edges in T before the addition of e to T. Then T' has no u, v-path (else a u, v-path adjoined with e would form a cycle). Hence, by (2), there exists $S \subseteq V$ such that $u \in S$, $v \notin S$, and $\delta(S) \cap T' = \varnothing$. Due to the way that the edges are sorted in Step 1, we have $e = \arg\min_{f \in \delta(S)} c_f$. It follows from the cut property (Theorem 2.2) that including e is the correct decision. □

To implement Kruskal's algorithm, first observe that Step 1 takes $O(m \log m)$ time to sort m numbers. For Step 2, we can make use of the UnionFind data structure, which helps maintain a list of connected components. It has the following methods:

- MakeUnionFind(V) creates a UnionFind data structure for V and takes $O(n)$ time.
- Find(v) determines the name of the connected component where v lies. This takes $O(\log n)$.
- Union(A, B) merges two connected components and takes $O(1)$ via a change of pointer.

We initialize the data structure once with MakeUnionFind. We use Find in every iteration of Step 2 to check that Find(u) $\neq$ Find(v) for the edge $e = uv$. In this case, u and v are in different components and we can join them with Union; otherwise we don't have to do anything. There are m iterations of Step 2, so Kruskal's algorithm takes $O(m \log n) = O(m \log m)$ time (since $n - 1 \leq m \leq n^2$ when G is connected).

2.5 Maximum Spacing Clustering

Given a set $U = \{p_1, \ldots, p_n\}$ of n objects and a distance $d(p_i, p_j) = d(p_j, p_i) \geq 0$ between points, we wish to find a k**-clustering** (a partition $C_1, \ldots, C_k$ of the set U) with maximum spacing

$$\max_{C_1,\ldots,C_k \text{ partition of } U} \left\{ \min_{1 \leq i < j \leq k} \left\{ \min_{p \in C_i, q \in C_j} d(p, q) \right\} \right\}.$$

Note that k is fixed above, and the term inside the maximum is the **spacing** of the k-clustering $C_1, \ldots, C_k$.

We can solve this problem with an adaptation of Kruskal's algorithm, called the **single linkage clustering algorithm**. We can view it as being equivalent to working over the complete graph K_n on the points $p_1, \ldots, p_n$, and the edge costs corresponding to the distances. Here, we can stop early once we hit k clusters.

Input. A set $U = \{p_1, \ldots, p_n\}$ with distances $d(p_i, p_j) \geq 0$ between points, and a fixed integer k.

Output. A k-clustering of U with maximum spacing.

Step 1. **(Initialization.)** Start with n clusters, each containing its own object from $p_1, \ldots, p_n$.

Step 2. While #clusters $> k$:

(Merge clusters.) Merge the clusters C and C' that achieve

$$\min_{C \neq C' \text{ clusters}} \left\{ \min_{p \in C, q \in C'} d(p, q) \right\}.$$

3 Minimum Cost Arborescences

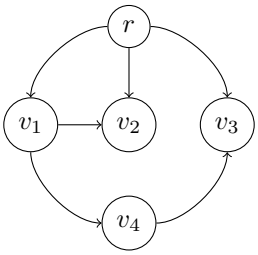
3.1 Arborescences and a Characterization

Let $G = (V, E)$ be a directed graph and let r be a distinguished node, which is commonly called a root. An **arborescence** with respect to r (or rooted at r) is a directed subgraph $T = (V, F)$ with $F \subseteq E$ such that

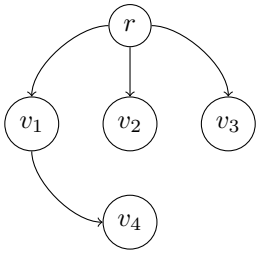
(i) undirected T (that is, T obtained from disregarding all directions) is a spanning tree; and

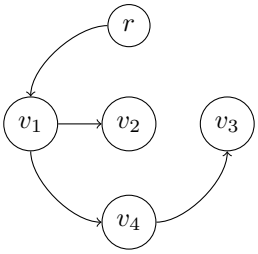
(ii) for every $v \in V$ with $v \neq r$, there is a directed path in T from r to v.

For example, consider the following graph $G = (V, E)$.

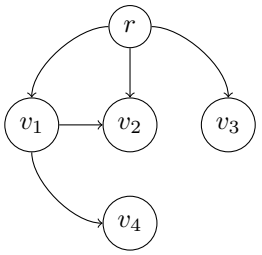


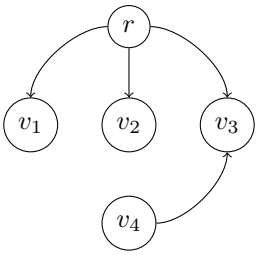
Then the following two subgraphs are arborescences rooted at r.





On the other hand, the following two subgraphs are not arborescences rooted at r: the first one is not a tree, and the second has no directed path from r to v_4.





The following theorem gives us a useful characterization of arborescences.

Theorem 3.1: Characterization of Arborescences

Let $G = (V, E)$ be a connected graph and let $T = (V, F)$ be a subgraph. Then T is an arborescence rooted at r if and only if both of the following conditions hold:

(1) every $v \in V$ with $v \neq r$ has exactly one incoming edge in T; and

(2) T has no directed cycles.

Proof of Theorem 3.1.

($\Rightarrow$) First, we check that condition (1) holds. Consider a vertex $v \in V$ with $v \neq r$. Since undirected T is a spanning tree, there is a unique (simple) path from r to v in undirected T. The last edge on this path is incoming for v. Hence, all other edges incident to v in undirected T should be outgoing edges for v (because the neighbours of v also have unique simple paths from v to them in undirected T). This proves (1). To see that condition (2) holds, note that undirected T is acyclic as it is a spanning tree, so it could not possibly have any directed cycles either.

($\Leftarrow$) Suppose that $V = (T, F)$ satisfies conditions (1) and (2). First, we show that for all $v \in V$ with $v \neq r$, there exists a directed path from r to v in T, which is condition (ii) of the definition of an arborescence. Let (v_1, v) be the unique edge incoming to v by condition (1), let (v_2, v_1) be the unique edge incoming to v_1, and so on. By contradiction, suppose that we cannot reach r by backtracking in this way. Then at some point, we must visit the same node at least twice because G is a finite graph. But this creates a directed cycle, contradicting condition (2). Thus, there is a directed path from r to v in T.

Now, we verify condition (i) that undirected T is a spanning tree. Note that we can get from r to any other vertex v, so undirected T is connected. Moreover, r has no incoming edges because if (v, r) is an incoming edge, then the directed path from r to v followed by (v, r) forms a directed cycle, contradicting condition (2). Since every edge is incoming for exactly one of its endpoints, the total number of edges in T is

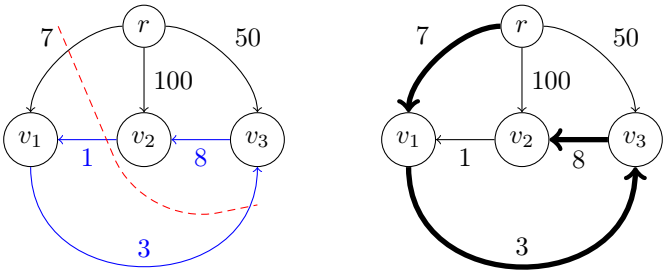
$$\sum_{u=r} 0 + \sum_{\substack{u \in V \\ u \neq r}} 1 = |V| - 1.$$

By the fundamental theorem of trees (Theorem 2.1), it follows that undirected T is a spanning tree. □

3.2 Minimum Cost Arborescences

Given a directed graph $G = (V, E)$, a distinguished node r, and edge costs $c_e \geq 0$ for each $e \in E$, our goal is to find an arborescence rooted at r so that the total edge cost is minimized.

Let's try to transfer our knowledge from the minimum spanning tree problem. Do the analogues of the cycle and cut properties hold for arborescences? It turns out we can find a counterexample to both in the same graph. Consider the following graph $G = (V, E)$ with associated edge costs.



The minimum cost arborescence is given to the right with cost $7 + 3 + 8 = 18$. Consider the cut (in red) and cycle (in blue) above. Then the cut property fails to hold because the edge (v_2, v_1) of cost 1 was not picked in a minimum cost arborescence, and the cycle property fails to hold because the edge (v_3, v_2) of maximum cost 8 in the cycle was picked in a minimum cost arborescence.

Can we instead consider the union of minimum cost incoming edges (vertex by vertex), excluding the root r? For the above example, if we start with v_1, we'd pick (v_2, v_1) as it is the minimum cost edge incoming to v_1. Then (v_1, v_3) is the minimum cost edge incoming to v_3 and (v_3, v_2) is the minimum cost edge incoming to v_2. This yields the same directed cycle highlighted in blue above!

So this strategy does not immediately work. But if the strategy did happen to give us an arborescence, then we are already done! All we need to do is tweak it slightly. The idea is to compute

$$y_v = \min_{(u,v)\in E} c_{(u,v)}$$

for all vertices $v \in V$ with $v \neq r$. In particular, any edge using this vertex needs to pay this cost anyways. Then for all $(u, v) \in E$, we can define new edge costs

$$c'_{(u,v)} = c_{(u,v)} - y_v.$$

We give the edge costs c'_e (in blue) and the values y_v (in purple) for the above example. Notice that some of the edges turn out to be free.