CO 331 Course Notes

Coding Theory

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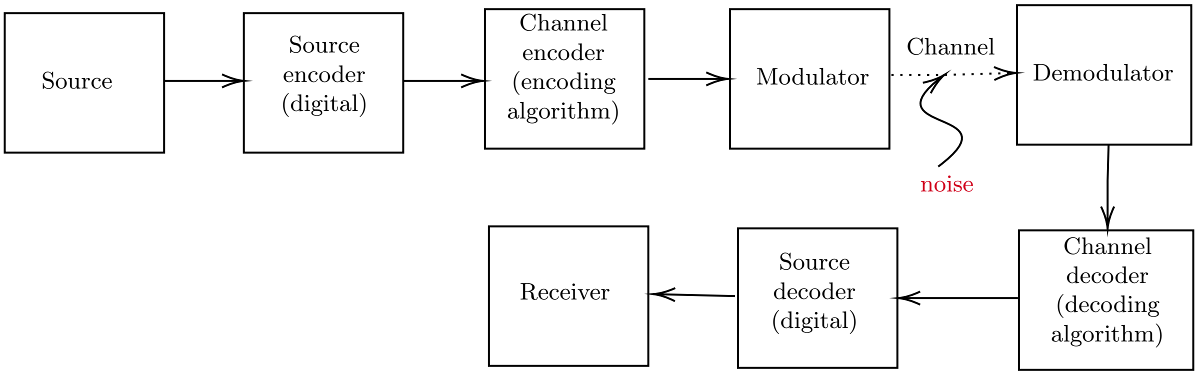
0 Introduction

Coding theory is about clever ways of adding redundancy to messages to allow (efficient) error detection and error correction.

Our basic communications model which we will use throughout the course is as follows.



Below, we give some examples of codes.

Parity code.

- **Encoding algorithm:** Add a 0 bit to the (binary) message m if the number of 1's in m is even; else add a 1 bit.
- **Decoding algorithm:** If the number of 1's in a received message r is even, then accept r; else declare that an error has occurred.

We note that if no errors occur during transmission of a message r, then the decoding algorithm will correctly accept r. Indeed, if the number of bits flipped is odd, then the parity of r is odd, and so the channel decoder will correctly declare that at least one error has occurred. On the other hand, if the number of bits flipped is even, then the parity of r is even, and hence the decoder correctly accepts r.

Replication code. For error correction, we are using "nearest neighbour decoding".

Source messages	Codeword	Number of errors per codeword that can always be detected	Number of errors per codeword that can always be corrected	Information rate
0 1	0 1	0	0	1
0 1	00 11	1	0	1/2
0 1	000 111	2	1	1/3
0 1	0000 1111	3	1	1/4
0 1	00000 11111	4	2	1/5

We see that we can always increase the error detecting capability and also the error correcting capability by increasing the number of times a bit is replicated. However, the price to pay for this is a decreasing information rate. This is a waste of potentially expensive bandwidth.

Goals of coding theory. We wish to design codes such that

(i) the error-correcting capability is high;

(ii) the information rate is high; and

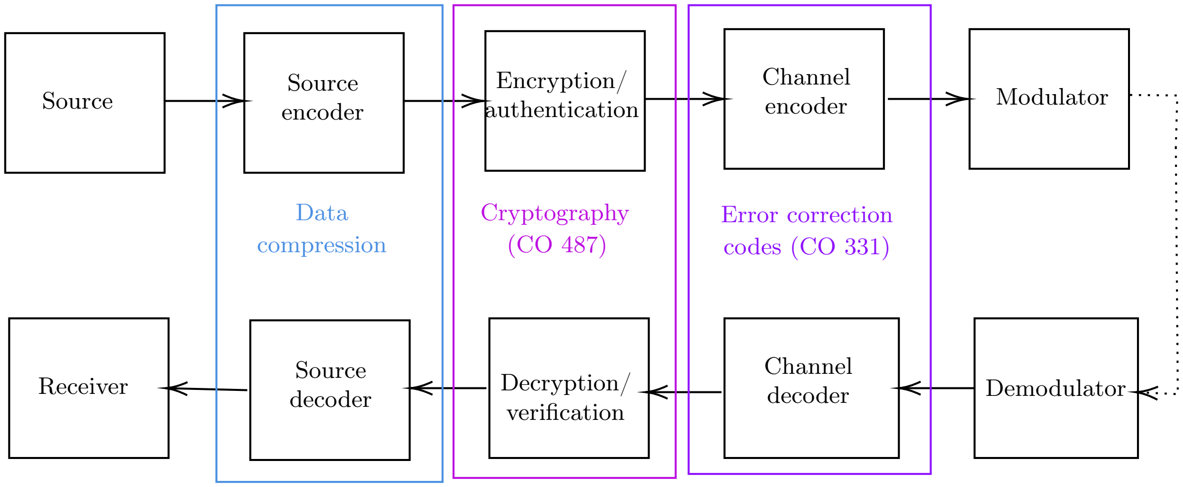
(iii) encoding and decoding can be done efficiently.

Course overview. The course will deal with algebraic methods of designing good (block) codes. The focus is on error correction, not error detection. These codes are used in wireless communications, space probes, CD/DVD players, storage, QR codes, and more.

Not covered in the course. Modern families of codes such as Turbo codes, LDPC codes, and Raptor codes due to lack of time, and the math used is not as elegant as the ones covered in the course.

The big picture. Coding theory in its broadest sense deals with techniques for the **efficient**, **secure**, and **reliable** transmission of data over communication channels that may be subject to **non-malicious errors** (noise) and **adversarial intrusion**. The latter includes passive intrusion (eavesdropping) and active intrusion (injection, deletion, modification).

Efficiency is associated with data compression, security is associated with cryptography, and reliability is associated with error correcting codes. They fit into our basic communications as shown in the diagram below.



1 Fundamentals

1.1 Basic definitions and concepts

Definition 1.1.1.

- An **alphabet** is a finite set of $q \geq 2$ symbols.
- A **word** (also known as a **vector** or **tuple**) is a finite sequence of symbols from A.
- A **code** C over A is a set of words of size at least 2.
- A **codeword** is a word in the code C.
- A **block code** is a code in which all codewords have the same length.
- A **block code of length n containing M codewords over A** is a subset $C \subseteq A^n$ with $|C| = M$. We call C an **$[n, M]$-code** over A.

Example 1.1.2. The code $C = \{00000, 11100, 00111, 10101\}$ is a $[5, 4]$-code over $\{0, 1\}$. The following is an encoding of messages (an injective function).

Message		Codewords
00	$\longrightarrow$	00000
01	$\longrightarrow$	11100
10	$\longrightarrow$	00111
11	$\longrightarrow$	10101

The channel encoder only transmits codewords. However, what is received may not be a codeword. For instance, suppose $r = 11000$ is received in Example 1.1.2. What should the channel decoder do?

Before we can answer this question, we should make some assumptions about the channel which we will use throughout the course. These will help determine whether a decoding strategy is good or not.

Assumptions about the channel.

(1) The channel transmits only symbols from A ("hard decision decoding").

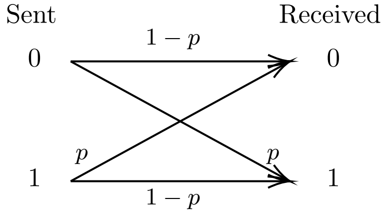
(2) No symbols are lost, added, or interchanged during transmission. (In practice, this may happen, but this can be accounted for using various techniques which we will not cover in this course.)

(3) The channel is a **q-symmetric channel**. That is, if $A = \{a_1, \ldots, a_q\}$, X_i denotes the i-th symbol sent, and Y_i denotes the i-th symbol received, then for all $i \geq 1$ and $1 \leq j, k \leq q$, we have

$$P(Y_i = a_j \mid X_i = a_k) = \begin{cases} 1 - p & \text{if } j = k \\ \frac{p}{q-1} & \text{if } j \neq k. \end{cases}$$

We call p the **symbol error probability** of the channel (we have $0 \leq p \leq 1$).

Definition 1.1.3. A 2-symmetric channel is called a **binary symmetric channel (BSC)**.



Remark 1.1.4. Consider a BSC with symbol error probability p.

(1) If $p = 0$, then the channel is **perfect**.

(2) If $p = 1/2$, then the channel is **useless**.

(3) If $1/2 < p \leq 1$, then flipping all received bits converts the channel to a BSC with $0 \leq p < 1/2$.

Based on these observations, we can assume without loss of generality that $0 < p < 1/2$ for a BSC.

Exercise 1.1.5. For a q-symmetric channel, show that one can take $0 < p < \frac{q-1}{q}$ without loss of generality. (Hint: First consider the case $q = 3$.)

Definition 1.1.6. The **information rate** (or **rate**) R of an $[n, M]$-code C over A is $R = (\log_q M)/n$.

Note that $0 \leq R \leq 1$. Ideally, R should be close to 1.

Example 1.1.7. If C encodes messages that are the k-tuples over A (so that $M = |A^k| = q^k$), then $R = k/n$.

Example 1.1.8. The rate of the binary code $C = \{00000, 11100, 00111, 10101\}$ in Example 1.2 is $R = 2/5$.

Definition 1.1.9. The **Hamming distance** (or **distance**) between two n-tuples over A is the number of coordinate positions in which they differ.

Theorem 1.1.10. The Hamming distance is a metric over A^n. That is, if d is the Hamming distance, then for all $x, y, z \in A^n$, we have

(1) $d(x, y) \geq 0$, with $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$; and

(3) $d(x, y) + d(y, z) \geq d(x, z)$.

Proof. Exercise. □

Definition 1.1.11. The **Hamming distance** (or **distance**) of an $[n, M]$-code C is

$$d(C) := \min\{d(x, y) : x, y \in C,\, x \neq y\}.$$

Example 1.1.12. The distance of $C = \{00000, 11100, 00111, 10101\}$ is $d(C) = 2$. (This can be verified by checking all $\binom{4}{2}$ pairs of codewords.)

1.2 Decoding strategies

Error detection. If C is used for error detection only, the strategy is as follows: a received word $r \in A^n$ is accepted if and only if $r \in C$.

Error correction. Let C be an $[n, M]$-code over A with distance d. Suppose $c \in C$ is transmitted and $r \in A^n$ is received. The (channel) decoder must decide one of the following:

(i) No errors have occurred; accept r.

(ii) Errors have occurred; correct (decode) r to a codeword $c \in C$.

(iii) Errors have occurred; no correction is possible.

Nearest neighbour decoding.

(i) **Incomplete maximum likelihood decoding (IMLD).** If there is a unique codeword $c \in C$ such that $d(r, c)$ is minimal, then correct r to c. If no such c exists, then report that errors have occurred, but correction is not possible (ask for retransmission, or disregard information).

(ii) **Complete maximum likelihood decoding (CMLD).** The same as IMLD, except that if there are two or more $c \in C$ for which $d(r, c)$ is minimal, correct r to an arbitrary one of these.

To see that IMLD is a reasonable strategy, we prove the following theorem.

Theorem 1.2.1. IMLD chooses the codeword c where the conditional probability $P(r \mid c) = P(r \text{ is received} \mid c \text{ is sent})$ is largest.

Proof. Suppose $c_1, c_2 \in C$ are codewords with $d_1 = d(c_1, r)$ and $d_2 = d(c_2, r)$. Without loss of generality, suppose that $d_1 > d_2$. Observe that $P(r \mid c_1) = (1-p)^{n-d_1}(\frac{p}{q-1})^{d_1}$ and $P(r \mid c_2) = (1-p)^{n-d_2}(\frac{p}{q-1})^{d_2}$, so we obtain

$$\frac{P(r \mid c_1)}{P(r \mid c_2)} = (1-p)^{d_2-d_1}\Big(\frac{p}{q-1}\Big)^{d_1-d_2} = \Big(\frac{p}{(1-p)(q-1)}\Big)^{d_1-d_2}.$$

Moreover, notice that

$$\frac{p}{(1-p)(q-1)} < 1 \iff p < (1-p)(q-1) \iff p < q - pq - 1 + p \iff pq < q-1 \iff p < \frac{q-1}{q}.$$

But the final inequality is an assumption we made about the channel, so $\frac{p}{(1-p)(q-1)} < 1$. Thus, $\frac{P(r|c_1)}{P(r|c_2)} < 1$ as well, so $P(r \mid c_1) < P(r \mid c_2)$, and the result follows. □

Minimum error probability decoding (MED). An **ideal strategy** would be to correct r to a codeword $c \in C$ for which $P(c \mid r) = P(c \text{ is sent} \mid r \text{ is received})$ is largest. This is MED.

We note that IMLD/CMLD is not the same as MED, which we will see in the following example.

Example 1.2.2. Consider $C = \{c_1 = 000, c_2 = 111\}$. Suppose that $P(c_1) = 0.1$ and $P(c_2) = 0.9$. Suppose that $p = 1/4$ (for a BSC). Now, suppose that $r = 100$ is the received word. Recall Bayes' Theorem, which states that if A and B are events and $P(B) \neq 0$, then

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}.$$

Then, we obtain

$$\begin{aligned} P(c_1 \mid r) &= \frac{P(r \mid c_1)P(c_1)}{P(r)} = \frac{p(1-p)^2 \cdot 0.1}{P(r)} = \frac{9}{640} \cdot \frac{1}{P(r)} \\ P(c_2 \mid r) &= \frac{P(r \mid c_2)P(c_2)}{P(r)} = \frac{(1-p)p^2 \cdot 0.9}{P(r)} = \frac{27}{640} \cdot \frac{1}{P(r)}. \end{aligned}$$

In particular, MED decodes r to c_2, but IMLD decodes r to c_1.

IMLD versus MED.

(i) IMLD maximizes $P(r \mid c)$, while MED maximizes $P(c \mid r)$. Generally, IMLD is not an ideal strategy, as it is not the same as MED which is an ideal strategy.

(ii) MED has the drawback that the decoding algorithm depends on the probability distribution of source messages. As such, it is not usable in practice.

(iii) If all source messages are equally likely, then CMLD and MED are equivalent. Indeed, we have

$$P(c_i \mid r) = \frac{P(r \mid c_i)P(c_i)}{P(r)} = P(r \mid c_i) \cdot \frac{1}{M \cdot P(r)},$$

and $M \cdot P(r)$ does not depend on c_i.

(iv) In practice, IMLD (or CMLD) is used. In this course, we will be using IMLD/CMLD.

1.3 Error correcting and detecting capabilities

First, we focus on detection only. Recall that our strategy was as follows: if r is received, then accept r if and only if $r \in C$.

Definition 1.3.1. A code C is an e**-error detecting code** if the decoder always makes the correct decision if e or fewer errors per codeword are introduced by the channel.

Example 1.3.2. Consider $C = \{000, 111\}$. Then C is a 2-error detecting code, but C is not a 3-error code.

Theorem 1.3.3. A code C of distance d is a $(d-1)$-error detecting code, but is not a d-error detecting code.

Proof. Suppose $c \in C$ is sent.

- If no errors occur, then c is received (and is accepted).
- Let r be the received word and suppose that $1 \leq d(r, c) \leq d - 1$; that is, there is at least 1 error and at most $d - 1$ errors. Then $r \notin C$ by the definition of Hamming distance. Thus, r is rejected.
- Since $d(C) = d$, there exist codewords $c_1, c_2 \in C$ with $d(c_1, c_2) = d$. If c_1 is sent and c_2 is received, then c_2 is accepted; the d errors go undetected. □

Now, we focus on correction. Our strategy here is IMLD/CMLD.

Definition 1.3.4. A code C is an e**-error correcting code** if the decoder always makes the correct decision if e or fewer errors per codeword are introduced by the channel.

Example 1.3.5. Again, consider $C = \{000, 111\}$. Then C is a 1-error correcting code, but C is not a 2-error correcting code.

Theorem 1.3.6. A code C of distance d is an e-error correcting code, where $e = \lfloor \frac{d-1}{2} \rfloor$.

Proof. Suppose that $c \in C$ is sent, at most $(d - 1)/2$ errors are introduced, and r is received. Then $d(r, c) \leq (d - 1)/2$. On the other hand, if c' is any other codeword, then

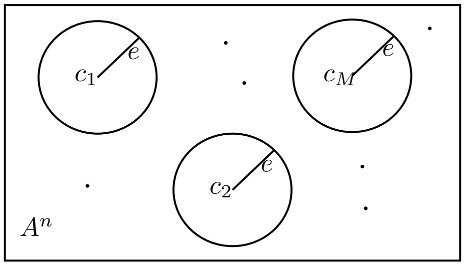
$$\begin{aligned} d(r, c') &\geq d(c, c') - d(r, c) && \text{(triangle inequality)} \\ &\geq d - (d-1)/2 && \text{(since } d(C) = d\text{)} \\ &= (d+1)/2 \\ &> (d-1)/2 \\ &\geq d(r, c). \end{aligned}$$

Hence, c is the unique codeword at minimum distance from r, and the decoder concludes that c was sent. □

Exercise 1.3.7. Suppose $d(C) = d$, and let $e = \lfloor \frac{d-1}{2} \rfloor$. Show that C is not an $(e+1)$-error correcting code.

Sphere packing. A natural question to ask is: given A, n, M, d, does there exist an $[n, M]$-code C over A of distance at least d? This can be phrased as an equivalent sphere packing problem: can we place M spheres of radius $e = \lfloor \frac{d-1}{2} \rfloor$ in A^n so that no spheres overlap?

Consider $C = \{c_1, c_2, \ldots, c_M\}$ and $e = \lfloor \frac{d-1}{2} \rfloor$. Let S_c be the sphere of radius e centered at c; equivalently, this is the set of all words within distance e of c. We also proved that if $c_1, c_2 \in C$ with $c_1 \neq c_2$, then $S_{c_1} \cap S_{c_2} = \varnothing$.



Let $n = 128$, $q = 2$, and $M = 2^{64}$. Does there exist a binary $[n, M]$-code with $d \geq 22$? If so, can encoding and decoding be done efficiently?

We will construct such a code at the end of the course. The main tools used will be linear algebra (over finite fields) and abstract algebra (rings and fields).

2 Finite fields

2.1 Introduction to finite fields

Definition 2.1.1. A **(commutative) ring** $(R, +, \cdot)$ consists of a set R, together with binary operations $+, \cdot : R \times R \to R$ such that

(1) $a + (b + c) = (a + b) + c$ for all $a, b, c, \in R$;

(2) $a + b = b + a$ for all $a, b \in R$;

(3) there exists an element $0 \in R$ such that $a + 0 = a$ for all $a \in R$;

(4) for all $a \in R$, there exists an element $-a \in R$ such that $a + (-a) = 0$;

(5) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all $a, b, c \in R$;

(6) $a \cdot b = b \cdot a$ for all $a, b \in R$;

(7) there exists an element $1 \in R$ with $1 \neq 0$ such that $a \cdot 1 = a$ for all $a \in R$; and

(8) $a \cdot (b + c) = a \cdot b + a \cdot c$ for all $a, b, c \in R$.

We usually identify the ring $(R, +, \cdot)$ with just the set R.

Example 2.1.2. The sets $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ equipped with their usual operations are all commutative rings.

Definition 2.1.3. A field $(F, +, \cdot)$ is a commutative ring with the additional property that for all $a \in F \setminus \{0\}$, there exists an element $a^{-1} \in F$ such that $a \cdot a^{-1} = 1$.

Definition 2.1.4. A field $(F, +, \cdot)$ is a **finite field** if F is finite; otherwise, it is an **infinite field**. If F is a finite field, its **order** is given by $|F|$.

Example 2.1.5. Note that $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ are all fields; in particular, they are infinite fields. However, $\mathbb{Z}$ is not a field, as 2 does not have a multiplicative inverse in $\mathbb{Z}$.

We wish to answer the following questions:

(1) For what integers $n \geq 2$ does there exist a finite field of order n?

(2) How does one construct such a field; that is, what are the elements of the field, and how are the field operations performed?

Let $n \geq 2$ be an integer. Recall that $\mathbb{Z}_n$ consists of the set of equivalence classes of integers modulo n, so $\mathbb{Z}_n = \{[0], [1], [2], \ldots, [n-1]\}$ with addition and multiplication given by $[a] + [b] = [a + b]$ and $[a] \cdot [b] = [a \cdot b]$. More simply, we write $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$ (dropping the brackets) and perform addition and multiplication modulo n.

Example 2.1.6. For example, in $\mathbb{Z}_9 = \{0, 1, 2, \ldots, 8\}$, we have $3 + 7 = 1$ and $3 \cdot 7 = 3$, since $3 + 7 \equiv 1 \pmod 9$ and $3 \cdot 7 \equiv 3 \pmod 9$.

Remark 2.1.7. It can be verified that $\mathbb{Z}_n$ is a commutative ring (axioms (1) to (8) all hold).

Theorem 2.1.8. Let $n \geq 2$. Then $\mathbb{Z}_n$ is a field if and only if n is prime.

Proof. For the forward direction, suppose that n is prime. Suppose that $0 \neq a \in \mathbb{Z}_n$ so that $1 \leq a \leq n-1$. Since n is prime, we have $\gcd(a, n) = 1$, and hence there exist $s, t \in \mathbb{Z}$ such that $as + nt = 1$. Reducing both sides modulo n gives $as \equiv 1 \pmod n$. In particular, we have $a^{-1} = s$, and hence $\mathbb{Z}_n$ is a field.

For the converse, suppose that n is composite. That is, $n = ab$ for some $2 \leq a, b \leq n-1$. Suppose for a contradiction that a^{-1} exists, so that $ac \equiv 1 \pmod n$ for some $c \in \mathbb{Z}_n$. Then $abc \equiv a \pmod n$, and so $nc \equiv a \pmod n$. Thus, $b \equiv 0 \pmod n$, which implies that $n \mid b$. But this is absurd, since we assumed that $2 \leq b \leq n-1$. Therefore, $\mathbb{Z}_n$ is not a field. □

Observe that Theorem 2.1.8 establishes the existence of finite fields of order n for every prime n. What about finite fields of order n where n is composite?

Definition 2.1.9. Let F be a field. The **characteristic** of F, denoted $\operatorname{char}(F)$, is the smallest integer m such that

$$\underbrace{1+\cdots+1}_{m \text{ times}} = 0.$$

If no such m exists, we define $\operatorname{char}(F) = 0$.

Example 2.1.10. For prime p, the field $\mathbb{Z}_p$ has characteristic p. On the other hand, the fields $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ all have characteristic 0.

Theorem 2.1.11. Let F be a field. If $\operatorname{char}(F) = 0$, then F is an infinite field.

Proof. Suppose for a contradiction that for positive integers $a < b$, we have

$$\underbrace{1+\cdots+1}_{a \text{ times}} = \underbrace{1+\cdots+1}_{b \text{ times}}.$$

Then, we see that

$$\underbrace{1+\cdots+1}_{b \text{ times}} - \underbrace{1+\cdots+1}_{a \text{ times}} = \underbrace{1+\cdots+1}_{b-a \text{ times}} = 0,$$

which contradicts our assumption that $\operatorname{char}(F) = 0$. Thus, the elements $1, 1+1, 1+1+1, \ldots$ are all distinct, so F must be infinite. □

Theorem 2.1.12. Let F be a field with $\operatorname{char}(F) = m > 0$. Then m is prime.

Proof. Suppose that m is composite, say $m = ab$ where $2 \leq a, b \leq m-1$. Let $s = 1 + \cdots + 1$ (a times) and $t = 1 + \cdots + 1$ (b times), and note that $s, t \neq 0$. It follows that

$$s \cdot t = (\underbrace{1+\cdots+1}_{a \text{ times}}) \cdot (\underbrace{1+\cdots+1}_{b \text{ times}}) = \underbrace{1+\cdots+1}_{ab = m \text{ times}} = 0.$$

Thus, $s \cdot t \cdot t^{-1} = s \cdot 1 = s = 0$, a contradiction. Hence, m must be prime. □

Let F be a finite field of characteristic p. Consider the subset

$$E = \{0, 1, 1+1, 1+1+1, \ldots, \underbrace{1+\cdots+1}_{p-1 \text{ times}}\} \subseteq F.$$

The elements of E are distinct, and it can be verified that E is a field under the same operations as F. In particular, E is a **subfield** of F. Identifying the elements of E with the elements of $\mathbb{Z}_p$ in the natural way, we see that E is essentially the same field as $\mathbb{Z}_p$. That is, $\mathbb{Z}_p$ is a subfield of F.

Note that if F is a finite field of characteristic p, then we can view F as a vector space over $\mathbb{Z}_p$ (where the vectors are elements of F, and the scalars are elements of $\mathbb{Z}_p$).

Theorem 2.1.13. Let F be a finite field of characteristic p. Then the order of F is p^n for some integer $n \geq 1$.

Proof. Let $n \geq 1$ be the dimension of F viewed as a vector space over $\mathbb{Z}_p$. Let $\{\alpha_1, \ldots, \alpha_n\}$ be a basis for F over $\mathbb{Z}_p$. Then every element $\beta \in F$ can be written uniquely in the form

$$\beta = c_1\alpha_1 + \cdots + c_n\alpha_n,$$

where each $c_i \in \mathbb{Z}_p$. In particular, we have $F = \{\sum_{i=1}^{n} c_i\alpha_i : c_i \in \mathbb{Z}_p\}$, so $|F| = p^n$. □

As a consequence of Theorem 2.1.13, there do not exist finite fields of order q when q is not of the form p^n where p is prime and n is a positive integer. However, given a fixed prime p and positive integer n, do there exist finite fields F such that $|F| = p^n$?

2.2 Existence of finite fields

Definition 2.2.1. Let F be a field. We denote by $F[x]$ the set of all polynomials in x with coefficients from F. Addition and multiplications of polynomials in $F[x]$ is done in the usual way, with coefficient arithmetic done in F.

Example 2.2.2. In $\mathbb{Z}_5[x]$, we have

$$(3x^4 + 2x^3 + x + 4) + (x^5 + 2x^4 + x^2 + 2x + 3) = x^5 + 2x^3 + x^2 + 3x + 2.$$

Remark 2.2.3. It can be shown that $F[x]$ is an infinite commutative ring.

Proposition 2.2.4 (Division algorithm in $F[x]$). Let F be a field. Let $f, g \in F[x]$ with $g \neq 0$. Then there exist unique polynomials $s, r \in F[x]$ with $\deg(r) < \deg(g)$ such that $f = sg + r$. (By convention, we define $\deg(0) = -\infty$).

Example 2.2.5. Let $f(x) = 3x^4 + 2x^3 + 2x^2 + x + 1$, $g(x) = 2x^2 + 3x + 4 \in \mathbb{Z}_5[x]$. Then using the division algorithm in $\mathbb{Z}_5[x]$, we can find that

$$f(x) = (4x^2 + 3)g(x) + (2x + 4),$$

so $s(x) = 4x^2 + 3$ and $r(x) = 2x + 4$ are the unique polynomials $s, r \in \mathbb{Z}_5[x]$ with $\deg(r) < \deg(g)$ such that $f = sg + r$.

Definition 2.2.6. Let F be a field and let $f \in F[x]$ with $\deg(f) \geq 1$. Let $g, h \in F[x]$. Then g **is congruent to** h **modulo** f, written $g \equiv h \pmod{f}$, if $g - h = \ell f$ for some $\ell \in F[x]$; equivalently, $f \mid g - h$.

The relation $\equiv \pmod{f}$ is an equivalence relation on $F[x]$, and partitions $F[x]$ into equivalence classes

$$[g] = \{h \in F[x] : h \equiv g \pmod{f}\}.$$

We can define addition and multiplication in the natural way; that is, $[g] + [h] = [g + h]$ and $[g] \cdot [h] = [g \cdot h]$. The set of equivalence classes is denoted by $F[x]/(f)$, and it can be verified that $F[x]/(f)$ is a commutative ring under the above operations.

Suppose now that $\deg(f) = n \geq 1$, and let $g \in F[x]$. Then we can write $g = sf + r$ where $s, r \in F[x]$ and $\deg(r) < n$. Notice that $g \equiv r \pmod{f}$, so $[g] = [r]$. Moreover, if $r_1, r_2 \in F[x]$ with $r_1 \neq r_2$ and $\deg(r_1), \deg(r_2) < n$, then $f \nmid r_1 - r_2$, so $r_1 \not\equiv r_2 \pmod{f}$. Thus, $[r_1] \neq [r_2]$. Therefore, the polynomials in $F[x]$ of degree less than n are a **complete set of representatives** of the equivalence classes of $F[x]/(f)$.

Now, let $F = \mathbb{Z}_p$. Then $\mathbb{Z}_p[x]/(f) = \{[r] : r \in \mathbb{Z}_p[x], \deg(r) < n\}$. We can see that $|\mathbb{Z}_p[x]/(f)| = p^n$, so $\mathbb{Z}_p[x]/(f)$ is a finite commutative ring of order p^n. But when is $\mathbb{Z}_p[x]/(f)$ a field?

Definition 2.2.7. Let F be a field and let $f \in F[x]$ with $\deg(f) \geq 1$. Then f is **irreducible over** F if f cannot be written as $f = gh$ where $g, h \in F[x]$ with $\deg(g), \deg(h) \geq 1$. Otherwise, we say that f is **reducible over** F.

Example 2.2.8. Let $f(x) = x^2 + 1$.

- $f(x)$ is irreducible over $\mathbb{R}$ since it has no roots in $\mathbb{R}$.
- $f(x)$ is reducible over $\mathbb{C}$ since we have $f(x) = (x + i)(x - i)$.
- $f(x)$ is reducible over $\mathbb{Z}_2$ since we can write $f(x) = (x + 1)(x + 1)$.
- $f(x)$ is irreducible over $\mathbb{Z}_3$ since it has no roots in $\mathbb{Z}_3$.

Theorem 2.2.9. Let F be a field and let $f \in F[x]$ with $\deg(f) \geq 1$. Then $F[x]/(f)$ is a field if and only if f is irreducible over F.

Proof. Analogous to the proof of Theorem 2.1.8 (which states that $\mathbb{Z}_n$ is a field if and only if n is prime). □

We conclude that if $f \in \mathbb{Z}_p[x]$ is irreducible with $\deg(f) = n \geq 1$, then $\mathbb{Z}_p[x]/(f)$ is a finite field of order p^n and characteristic p. The elements of $\mathbb{Z}_p[x]/(f)$ are the polynomials in $\mathbb{Z}_p[x]$ of degree less than n.

Example 2.2.10. Let $p = 2$ and $n = 2$. Let $f(x) = x^2 + x + 1 \in \mathbb{Z}_2[x]$. Then $f(0) = f(1) = 1$, so f has no roots in $\mathbb{Z}_2$. Hence, f is irreducible over $\mathbb{Z}_2$, and thus $F = \mathbb{Z}_2[x]/(x^2 + x + 1)$ is a finite field of order $2^2 = 4$. The elements of F are $\{0, 1, x, x + 1\}$.

Example 2.2.11. Let $p = 2$ and $n = 3$. To obtain a finite field of order $2^3 = 8$, we need to find an irreducible polynomial of degree 3 over $\mathbb{Z}_2$. Indeed, consider $f(x) = x^3 + x + 1$. Since $f(0) = f(1) = 1$, we see that f has no roots in $\mathbb{Z}_2$, and hence is irreducible over $\mathbb{Z}_2$. Thus, $F = \mathbb{Z}_2[x]/(x^3 + x + 1)$ is a finite field of order 8. The elements of F are $\{0, 1, x, x + 1, x^2, x^2 + 1, x^2 + x, x^2 + x + 1\}$.

Remark 2.2.12. Note that in the above example, we also could have chosen $g(x) = x^3 + x^2 + 1$ as our irreducible polynomial of degree 3 over $\mathbb{Z}_2$. Then $F' = \mathbb{Z}_2[x]/(x^3 + x^2 + 1)$ is also a finite field of order 8 with elements $\{0, 1, x, x + 1, x^2, x^2 + 1, x^2 + x, x^2 + x + 1\}$.

We note that F and F' are *not* the same field. Notice that in F, we have $x \cdot x^2 = x + 1$, whereas in F', we have $x \cdot x^2 = x^2 + 1$. Nonetheless, F and F' are **isomorphic**; that is, there is a bijection $\phi : F \to F'$ such that $\phi(a + b) = \phi(a) + \phi(b)$ and $\phi(a \cdot b) = \phi(a) \cdot \phi(b)$ for all $a, b \in F$.

Finally, we list some useful facts.

- Let p be prime and n be a positive integer. Then there exists an irreducible polynomial of degree n over $\mathbb{Z}_p$.
- There exists a finite field of order q if and only if $q = p^n$ for some prime p and positive integer n.
- Any two fields of the same order are isomorphic.

Definition 2.2.13. We will denote *the* finite field of order q (up to isomorphism) by $\mathrm{GF}(q)$, and call it the **Galois field of order** q.

We summarize our results in the following diagram.

$$\begin{tikzcd}
\textbf{Infinite ring} & & \textbf{Finite ring} & & \textbf{Finite field} \\
\mathbb{Z} \arrow[rr, "\text{Let } n\geq 2", "a\equiv b \pmod{n}"'] & & \mathbb{Z}_n \arrow[rr, "n \text{ prime}"] & & \mathbb{Z}_n\text{, order } n \\
\mathbb{Z}_p[x] \arrow[rr, "\text{Let } f\in\mathbb{Z}_p[x],\, \deg(f)=n\geq 1", "g\equiv h \pmod{f}"'] & & \mathbb{Z}_p[x]/(f) \arrow[rr, "f \text{ irreducible}"] & & \mathbb{Z}_p[x]/(f)\text{, order } p^n
\end{tikzcd}$$

2.3 Properties of finite fields

Theorem 2.3.1 (Freshman's dream). Let F be a field of characteristic p, and let $\alpha, \beta \in F$. Then for all $m \geq 1$, we have

$$(\alpha + \beta)^{p^m} = \alpha^{p^m} + \beta^{p^m}.$$

Proof. We proceed by induction on m. For $m = 1$, it follows from the Binomial Theorem that

$$(\alpha + \beta)^p = \binom{p}{0}\alpha^p + \sum_{i=1}^{p-1} \alpha^i \beta^{p-i} + \binom{p}{p}\beta^p.$$

Observe that for all $1 \leq i \leq p - 1$, we have

$$\binom{p}{i} = \frac{p(p-1)(p-2)\cdots(p-i+1)}{i(i-1)\cdots(1)} \equiv 0 \pmod{p},$$

since $\binom{p}{i}$ is an integer and p divides the numerator but not the numerator. Thus, for each $1 \leq i \leq p-1$, we obtain

$$\binom{p}{i}\alpha^i\beta^{p-i} = \underbrace{\alpha^i\beta^{p-i} + \cdots + \alpha^i\beta^{p-i}}_{\binom{p}{i} \text{ times}} = \underbrace{(1 + \cdots + 1)}_{\binom{p}{i} \text{ times}}\alpha^i\beta^{p-i} = 0.$$

Hence, $(\alpha+\beta)^p = \alpha^p + \beta^p$. We leave the inductive case as an exercise. □

Definition 2.3.2. The **multiplicative group of** GF(q) is the set GF$(q)^* = \text{GF}(q) \setminus \{0\}$.

The following theorem is a generalization of Fermat's Little Theorem for GF(q).

Theorem 2.3.3. If $\alpha \in \text{GF}(q)^*$, then $\alpha^{q-1} = 1$.

Proof. Let us denote the distinct elements of GF$(q)^*$ by $\alpha_1, \ldots, \alpha_{q-1}$. Consider the non-zero elements $\alpha\alpha_1, \ldots, \alpha\alpha_{q-1}$. Note that these elements are also distinct, because if $\alpha\alpha_i = \alpha\alpha_j$ for some $i \neq j$, then this would imply that $\alpha_i = \alpha^{-1}(\alpha\alpha_i) = \alpha^{-1}(\alpha\alpha_j) = \alpha_j$, a contradiction. Hence, we have $\{\alpha\alpha_1, \ldots, \alpha\alpha_{q-1}\} = \{\alpha_1, \ldots, \alpha_{q-1}\}$, and so

$$(\alpha\alpha_1)(\alpha\alpha_2)\cdots(\alpha\alpha_{q-1}) = \alpha_1\alpha_2\cdots\alpha_{q-1}.$$

Cancelling out terms on both sides, it follows that $\alpha^{q-1} = 1$. □

Corollary 2.3.4. If $\alpha \in \text{GF}(q)$, then $\alpha^q = \alpha$.

Proof. If $\alpha \in \text{GF}(q)^*$, then by Theorem 2.3.3, we have $\alpha^q = \alpha\alpha^{q-1} = \alpha \cdot 1 = \alpha$. Otherwise, we have $\alpha = 0$, and it is clear that $0^q = 0$. □

Definition 2.3.5. Let $\alpha \in \text{GF}(q)^*$. The **order** of α, denoted ord(α), is the smallest positive integer t such that $\alpha^t = 1$.

Remark 2.3.6. There is only one element in GF(q) of order 1, namely the element 1.

Theorem 2.3.7. Let $\alpha \in \text{GF}(q)^*$ with ord$(\alpha) = t$. Then $\alpha^s = 1$ if and only if $t \mid s$.

Proof. Suppose $s \in \mathbb{Z}$. Then long division of s by t yields $s = \ell t + r$ for some $0 \leq r < t$. We then see that

$$\alpha^s = \alpha^{\ell t + r} = (\alpha^t)^\ell \cdot \alpha^r = \alpha^r.$$

Note that $\alpha^s = 1$ if and only $r = 0$, and $r = 0$ if and only if $t \mid s$, so the result follows. □

Corollary 2.3.8. If $\alpha \in \text{GF}(q)^*$, then ord$(\alpha) \mid q-1$.

Proof. Apply Theorem 2.3.3 and Theorem 2.3.7. □

Example 2.3.9. Consider the field GF$(2^3) = \mathbb{Z}_2[x]/(x^3+x+1)$. The order of x^2+1 is 7, since the order cannot be 1 (by Remark 2.32) and must divide $2^3 - 1 = 7$.

Example 2.3.10. Note that x^4+x+1 is irreducible over $\mathbb{Z}_2$ (check this). Consider the field GF$(2^4) = \mathbb{Z}_2[x]/(x^4+x+1)$. Observe that $x^1 = x$, $x^3 = x^3$, and $x^5 = x^2 + x$, none of which are equal to 1, so ord(x) cannot be equal to 1, 3, or 5. Since ord$(x) \mid 15$, it must be that ord$(x) = 15$.

Remark 2.3.11. If $\alpha \in \text{GF}(q)^*$ with ord$(\alpha) = t$, then the elements $\alpha^0, \alpha^1, \alpha^2, \ldots, \alpha^{t-1}$ are distinct. In particular, if ord$(\alpha) = q-1$, then GF$(q)^* = \{\alpha^0, \alpha^1, \alpha^2, \ldots, \alpha^{q-2}\}$.

Definition 2.3.12. A **generator** of GF$(q)^*$ is an element of order $q-1$.

Example 2.3.13. As seen in Example 2.3.10, x is a generator of GF$(2^4) = \mathbb{Z}_2[x]/(x^4+x+1)$ since ord$(x) = 15$.

Note that every finite field GF(q) has a generator, but we will not prove this fact here.

3 Linear codes

3.1 Introduction to linear codes

We first introduce some notation. We will denote $F = \mathrm{GF}(q)$, and define $V_n(F) := F^n$. Note that $V_n(F)$ is an n-dimensional vector space over F with $|V_n(F)| = q^n$.

Definition 3.1.1. A **linear** (n, k)**-code over** F is a k-dimensional subspace of $V_n(F)$.

Let C be an (n, k)-code over F, and let $\{v_1, \ldots, v_k\}$ be an ordered basis for C. We list some properties of C.

Number of codewords. The elements of C are precisely

$$c_1v_1 + \cdots + c_kv_k,$$

where each $c_i \in F$. Thus, $|C| = M = q^k$.

Information rate. The rate of C is $R = (\log_q M)/n = (\log_q q^k)/n = k/n$.

Weight. Consider the following definition and theorem.

Definition 3.1.2. The **Hamming weight** $w(v)$ of a vector $v \in F_n(F)$ is the number of non-zero coordinates in v. The **Hamming weight** of a linear code C is given by

$$w(C) := \{w(c) : c \in C, \, c \neq 0\}.$$

Theorem 3.1.3. If C is a linear code, then $w(C) = d(C)$.

Proof. We have

$$\begin{aligned} d(C) &= \min\{d(x, y) : x, y \in C, \, x \neq y\} \\ &= \min\{w(x - y) : x, y \in C, \, x \neq y\} \text{ (since } d(x, y) = w(x - y)) \\ &= \min\{w(c) : c \in C, \, c \neq 0\} \text{ (since } C \text{ is linear, } x - y \in C) \\ &= w(C). \end{aligned}$$

□

Encoding. Since there are q^k codewords, there are also q^k source messages. We shall assume that the source messages are the elements of F^k. Then a convienient and natural bijection (known as an **encoding rule**) between F^k and C is given by

$$m = (m_1, \ldots, m_k) \mapsto c = m_1v_1 + \cdots + m_kv_k.$$

Generator matrix. This gives us a convenient way of representing C.

Definition 3.1.4. A **generator matrix** G for an (n, k)-code C is a $k \times n$ matrix whose rows from a basis for C; that is, with the ordered basis $\{v_1, \ldots, v_k\}$ for C, we have

$$G = \begin{bmatrix} v_1 \\ \vdots \\ v_k \end{bmatrix}_{k \times n}.$$

Note that the encoding rule above is more compactly stated as $c = mG$.

Example 3.1.5. Consider the binary $(5, 3)$-code

$$C = \langle c_1 = 10011, c_2 = 01001, c_3 = 00110 \rangle,$$

where c_1, c_2, c_3 are linearly independent over GF(2). A generator matrix for C is

$$G = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

The encoding rule (with respect to the ordered basis $\{c_1, c_2, c_3\}$) is $c = mG$. More explicitly, we have

m		c
000	→	00000
001	→	00110
010	→	01001
011	→	01111
100	→	10011
101	→	10101
110	→	11010
111	→	11100

Moreover, we clearly have $M = |C| = 2^3 = 8$, $R = 3/5$, and $d(C) = w(C) = 2$.

Definition 3.1.6. Let C be an (n, k)-code over F. A generator matrix G for C of the form

$$G = \big[I_k \mid A\big]_{k \times n}$$

is said to be in **standard form**. If C has a generator matrix in standard form, then C is a **systematic code**.

Example 3.1.7. The code $C = \langle 100011, 001001, 000110 \rangle$ is a non-systematic $(6, 3)$-code. On the other hand, $C' = \langle 100011, 001001, 010010 \rangle$ is a systematic code, since the generator matrix

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

for C' is in standard form.

Definition 3.1.8. Two codes C and C' over F are **equivalent** if C' can be obtained from C by choosing a permutation of the coordinate positions $\{1, 2, \ldots, n\}$, and then consistently rearranging every codeword of C according to this permutation.

Example 3.1.9. The codes C and C' from Example 3.1.7 are equivalent; in particular, the second and fourth coordinates were swapped.

The following facts are easy to prove.

(1) If C is linear and C' is equivalent to C, then C' is linear.

(2) Equivalent codes have the same length, dimension, and distance.

(3) Every linear code is equivalent to a systematic code.

3.2 Dual codes and parity-check matrices

Definition 3.2.1. Let $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in V_n(F)$. The **inner product** of x and y is defined to be

$$x \cdot y = \sum_{i=1}^{n} x_i y_i \in F.$$

EXAMPLE 3.2.2. Suppose that $x = (2, 0, 1, 1)$, $y = (2, 0, 2, 1) \in V_4(\mathbb{Z}_3)$. Then $x \cdot y = 1$.

Observe that for all $x, y, z \in V_n(F)$ and $\lambda \in F$, we have

(i) $x \cdot y = y \cdot x$;

(ii) $x \cdot (y + z) = x \cdot y + x \cdot z$; and

(iii) $(\lambda x) \cdot y = \lambda(x \cdot y)$.

REMARK 3.2.3. Note that $x \cdot x = 0$ does not necessarily imply that $x = 0$. Indeed, consider $x = 111100 \in V_6(\mathbb{Z}_2)$. Then $x \cdot x = 0$, but $x \neq 0$. More generally, if $x \in V_n(\mathbb{Z}_2)$, then $x \cdot x = 0$ if and only if $w(x)$ is even.

DEFINITION 3.2.4. Two vectors $x, y \in V_n(F)$ are said to be **orthogonal** if $x \cdot y = 0$.

We now take a look at the notion of the dual code of a linear code.

DEFINITION 3.2.5. Let C be an (n, k)-code over F. The **dual code** (or **orthogonal code**) of C is the code

$$C^{\perp} := \{x \in V_n(F) : x \cdot y \text{ for all } y \in C\}.$$

THEOREM 3.2.6. If C is an (n, k)-code over F, then $C^{\perp}$ is an $(n, n - k)$-code over F.

PROOF. Let G be a generator matrix of C, and denote the rows of G by $v_1, \ldots, v_k$.

CLAIM. If $x \in V_n(F)$, then $x \in C^{\perp}$ if and only if $v_1 \cdot x = v_2 \cdot x = \cdots = v_k \cdot x = 0$.

PROOF OF CLAIM. The forward direction is clear, since $v_1, \ldots, v_k \in C$. For the converse, suppose $v \in C$. Then we can write $v = \lambda_1 v_1 + \cdots + \lambda_k v_k$ where each $\lambda_i \in F$. We see that

$$v \cdot x = (\lambda_1 v_1 + \cdots + \lambda_k v_k) \cdot x = \lambda(v_1 \cdot x) + \cdots + \lambda_k(v_k \cdot x) = 0,$$

and since $v \in C$ was arbitrary, we have $x \in C^{\perp}$. ■

By the claim, we see that $C^{\perp} = \{x \in V_n(F) : Gx^T = 0\}$ is the null space of G. Since G has rank k, it follows that $C^{\perp}$ is a subspace of $V_n(F)$ of dimension $n - k$. □

THEOREM 3.2.7. If C is a linear code, then $(C^{\perp})^{\perp} = C$.

PROOF. Let C be an (n, k)-code. By Theorem 3.2.6, $C^{\perp}$ is an $(n, n - k)$-code, and applying Theorem 3.2.6 once more, we have that $(C^{\perp})^{\perp}$ is an (n, k)-code. Moreover, if $x \in C$, then for all $y \in C^{\perp}$, we have $x \cdot y = 0$, so $C \subseteq (C^{\perp})^{\perp}$. Since $\dim(C) = \dim((C^{\perp})^{\perp}) = k$, it follows that $C = (C^{\perp})^{\perp}$. □

DEFINITION 3.2.8. If C is a linear code, then a generator matrix H for $C^{\perp}$ is called a **parity-check matrix** for C.

REMARK 3.2.9. Let C be an (n, k)-code.

(1) Since $C^{\perp}$ is an $(n, n - k)$-code, a parity-check matrix for C is an $(n - k) \times n$ matrix.

(2) Since C has many generator matrices, we see that C also has many parity-check matrices.

THEOREM 3.2.10. Let C be an (n, k)-code with generator matrix $[I_k \mid A]$, where A is a $k \times (n - k)$ matrix. Then $H = [-A^T \mid I_{n-k}]$ is a generator matrix for $C^{\perp}$.

PROOF. Since $\operatorname{rank}(H) = n - k$, we see that H is a generator matrix for an $(n, n - k)$-code $\overline{C}$. Also, observe that

$$GH^T = \begin{bmatrix} I_k \mid A \end{bmatrix} \left[\begin{array}{c} -A \\ \hline I_{n-k} \end{array}\right] = -A + A = 0.$$

Thus, $\overline{C} \subseteq C^{\perp}$. Since $\dim(\overline{C}) = \dim(C^{\perp})$, we have $\overline{C} = C^{\perp}$. Hence, H is a generator matrix for $C^{\perp}$. □

Example 3.2.11. Consider the $(5, 2)$-code C over $\mathbb{Z}_3$ with generator matrix

$$G = \begin{bmatrix} 2 & 0 & 2 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

Let us find a parity-check matrix for C. First, we need to find a generator matrix for C in standard form. Indeed, performing some row operations, we obtain

$$G \xrightarrow{R_1 \leftarrow 2R_1} \begin{bmatrix} 1 & 0 & 1 & 2 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - R_1} \begin{bmatrix} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 1 & 1 \end{bmatrix}.$$

By Theorem 3.2.10, it follows that

$$H = \begin{bmatrix} 2 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 1 \end{bmatrix}$$

is a parity-check matrix for C.

3.3 Distance of a linear code

Looking just at a parity-check matrix of a linear code C, we can say something about the distance of C.

Theorem 3.3.1. Let H be a parity-check matrix for an (n, k)-code C over F. Let $s \geq 2$. Then $d(C) \geq s$ if and only if every $s - 1$ columns of H are linearly independent over F.

Proof. Let the columns of H be $h_1, \ldots, h_n$.

($\Leftarrow$) Suppose that $d(C) \leq s - 1$. Let $c = (c_1, \ldots, c_n) \in C$ be a codeword with $1 \leq w(c) \leq s - 1$. Without loss of generality, suppose that $c_j = 0$ for all $s \leq j \leq n$. Since $Hc^T = 0$, we have

$$c_1 h_1 + c_2 h_2 + \cdots + c_{s-1} h_{s-1} + c_s h_s + \cdots + c_n h_n = 0,$$

and hence $c_1 h_1 + \cdots + c_{s-1} h_{s-1} = 0$. Since at least one of $c_1, \ldots, c_{s-1}$ is non-zero, the $s - 1$ columns $h_1, \ldots, h_{s-1}$ are linearly dependent over F.

($\Rightarrow$) Suppose there is a set of $s - 1$ columns of H which is linearly dependent over F. Without loss of generality, suppose these columns are $h_1, \ldots, h_{s-1}$. Then, there exist scalars $\lambda_1, \ldots, \lambda_{s-1} \in F$, not all zero, such that

$$\lambda_1 h_1 + \lambda_2 h_2 + \cdots + \lambda_{s-1} h_{s-1} = 0.$$

Let $c = (\lambda_1, \ldots, \lambda_{s-1}, 0, \ldots, 0) \in V_n(F)$. Then $c \in C$ since $Hc^T = \lambda_1 h_1 + \cdots + \lambda_{s-1} h_{s-1} = 0$. But $1 \leq w(c) \leq s - 1$, so $1 \leq d(C) \leq s - 1$. □

As a direct consequence of Theorem 3.3.1, we have the following corollary.

Corollary 3.3.2. Let H be a parity-check matrix for a linear code C over F. Then $d(C)$ is the smallest number of columns of H that are linearly dependent over F.

Example 3.3.3. Recall Example 3.2.11, where we had a code C over $\mathbb{Z}_3$ with parity-check matrix

$$H = \begin{bmatrix} 2 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 1 \end{bmatrix}.$$

- No single column of H is linearly dependent over $\mathbb{Z}_3$ (as there is no zero column), so $d(C) \geq 2$.
- No two columns of H are linearly dependent over $\mathbb{Z}_3$ (there are no scalar multiples of each other), so $d(C) \geq 3$.

- There exist three columns of H which are linearly dependent over $\mathbb{Z}_3$. Indeed, we have

$$\begin{bmatrix}2\\1\\0\end{bmatrix} - 2\begin{bmatrix}1\\0\\0\end{bmatrix} - \begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix},$$

  so $d(C) \leq 3$.

Thus, it follows that $d(C) = 3$.

Example 3.3.4. Let H be a parity-check matrix for a binary linear code C.

(1) We have $d(C) = 1$ if and only if H has a zero column.

(2) We have $d(C) = 2$ if and only if H has no zero column, and two columns of H are the same.

(3) We have $d(C) = 3$ if and only if the columns of H are non-zero and distinct, and some column is the sum of two other columns.

Example 3.3.5. By our observations in Example 3.3.4, we can construct a binary $(7, 4, 3)$-code over C; that is, a binary $(7, 4)$-code with distance 3. First, we construct a parity-check matrix for C. Consider

$$H = \left[\begin{array}{ccc|cccc} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{array}\right]$$

which has no zero column, and all columns are distinct. Moreover, we can see that the fourth column is the sum of the first two columns, so $d(C) = 3$. Then, a generator matrix for C is given by

$$G = \left[\begin{array}{ccc|cccc} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right].$$

The code C is simply the set of linear combinations of the rows of G.

3.4 Hamming codes

We now focus on Hamming codes, which is an infinite family of single-error correcting codes discovered by Richard Hamming in 1950.

Definition 3.4.1. A **Hamming code of order r over $F = \mathrm{GF}(q)$** is an (n, k)-code over F with $n = (q^r - 1)/(q - 1)$, $k = n - r$, and parity-check matrix H_r being an $r \times n$ matrix whose columns are non-zero and no two of whose columns are scalar multiples of each other.

Definition 3.4.2. The binary $(7, 4, 3)$-code C in Example 3.3.5 is a Hamming code of order 3 over $\mathbb{Z}_2$.

Example 3.4.3. A parity-check matrix for a Hamming code of order 3 over $\mathbb{Z}_3$ is given by

$$H_3 = \left[\begin{array}{ccc|ccc|c|ccc|ccc} 1 & 0 & 0 & 1 & 1 & 0 & 1 & 2 & 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 2 & 1 & 2 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 2 & 1 & 2 & 1 & 1 \end{array}\right].$$

Note that we have $n = 13$, $k = 10$, and $d = 3$.

Remark 3.4.4.

(1) If $v \in V_r(F)$ with $v \neq 0$, then exactly one scalar multiple of v is a column of H_r (giving $n = (q^r - 1)/(q - 1)$ columns in total).

(2) The matrix H_r has rank r, since among its columns are scalar multiples of the unit vectors. Hence, a Hamming code of order r over $\mathrm{GF}(q)$ does indeed have dimension $k = n - r$.

(3) A Hamming code of order r over $\mathrm{GF}(q)$ has distance 3 (by design), so it is a single-error correcting code by Theorem 1.3.6.

For the remainder of this section, let H be a parity-check matrix of an (n, k, d)-code C over F with $d \geq 3$.

Definition 3.4.5. Suppose that $c \in C$ is sent, and $r \in V_n(F)$ is received. The **error vector** is given by $e = r - c$ (or equivalently, $r = c + e$).

Example 3.4.6. Over $\mathbb{Z}_3$, if $c = 100100$ is sent and $r = 120101$ is received, then $e = 020000$.

With the notion of an error vector, we can make the following observations.

(1) We have $Hr^T = H(c + e)^T = Hc^T + He^T = He^T$.

(2) If $e = 0$, then $He^T = 0$. (The converse is not true.)

(3) If $w(e) = 1$, say $e = (0, \ldots, 0, \alpha, 0, \ldots, 0)$ where α is at the i-th position, then $He^T = \alpha h_i$ where h_i is the i-th column of H which is non-zero by design. (The converse is not true.)

This suggests the following decoding algorithm.

Algorithm 3.4.7. We are given a parity-check matrix H and a received word r.

(1) Compute $s = Hr^T$.

(2) If $s = 0$, then accept r as the transmitted word (in this case, we have $e = 0$).

(3) If $s \neq 0$, then compare s with the columns of H. If $s = \alpha h_i$ for some $\alpha \in F$ and column h_i of H, then set $e = (0, \ldots, 0, \alpha, 0, \ldots, 0)$ with α at the i-th position, and decode r to $c = r - e$.

(4) Otherwise, report that more than one error has occurred.

If $w(e) = 0$ or $w(e) = 1$, then this decoding algorithm is guaranteed to make the correct decision.

Example 3.4.8. Consider the $(7, 4, 3)$-binary Hamming code with parity-check matrix

$$H = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Suppose that $r = 0111110$ is received. We see that

$$s = Hr^T = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},$$

which is the sixth column of H. Hence, we set $e = 0000010$ and decode r to $c = r - e = 0111100$. To check that this is correct, we can verify that $Hc^T = 0$.

3.5 Perfect codes

Definition 3.5.1. Let C be an $[n, M]$-code of distance d over A, with $|A| = q$ and $e = \lfloor (d-1)/2 \rfloor$. Then C is **perfect** if each $x \in A^n$ is contained in the sphere of radius e centered at some $c \in C$. Equivalently, C is perfect if it attains the sphere packing bound; that is, we have

$$M \cdot \sum_{i=0}^{e} \binom{n}{i} (q-1)^i = q^n.$$

For fixed q, n, and d, a perfect code has maximum possible M. In other words, a perfect code has maximum possible information rate $R = (\log_q M)/n$ for fixed q, n, and d.

Example 3.5.2. The trivial code $C = A^n$ is perfect with distance $d = 1$.

Example 3.5.3. Let n be odd. Then $C = \{000\cdots 0, 111\cdots 1\} \subseteq \{0,1\}^n$ is a perfect binary code with distance $d = n$. Indeed, let $e = (n-1)/2$ and observe that

$$\begin{aligned} M \cdot \sum_{i=0}^{e} \binom{n}{i}(q-1)^i &= 2\left[\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{e}\right] \\ &= \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{e} + \binom{n}{e+1} + \cdots + \binom{n}{n-1} + \binom{n}{n} \\ &= (1+1)^n = 2^n. \end{aligned}$$

Exercise 3.5.4. Prove that every perfect code has odd distance.

Exercise 3.5.5. Show that IMLD is equivalent to CMLD for a perfect code.

Example 3.5.6. All Hamming codes of order r over $\mathrm{GF}(q)$ are perfect. To see this, note that we have $n = (q^r - 1)/(q-1)$, $k = n - r$, $d = 3$, and $e = 1$, so we obtain

$$\begin{aligned} M \cdot \sum_{i=0}^{e} \binom{n}{i}(q-1)^i &= q^k \left[\binom{n}{0}(q-1)^0 + \binom{n}{1}(q-1)^1\right] \\ &= q^{n-r}\left[1 + \frac{q^r - 1}{q-1}(q-1)\right] = q^n. \end{aligned}$$

Theorem 3.5.7 (Tietäväinen, 1973). The only perfect codes are

(1) trivial codes $V_n(\mathrm{GF}(q))$;

(2) binary replication codes of odd length;

(3) Hamming codes, and all codes with the same $[n, M, d]$ parameters as them;

(4) the $(23, 12, 7)$-binary Golay code (see Section 4.1), and all codes equivalent to it;

(5) the $(11, 6, 5)$-ternary Golay code, and all codes equivalent to it.

A generator matrix for (5) above is given by

$$G = \left[\begin{array}{cccccc|ccccc} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 2 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 2 & 1 & 0 \end{array}\right].$$

In this course, we will not be studying this code.

3.6 Syndrome decoding

Let C be an (n, k)-code over $F = \mathrm{GF}(q)$ with parity-check matrix H.

Definition 3.6.1. Let $x, y \in V_n(F)$. We write $x \equiv y \pmod{C}$ if $x - y \in C$.

Remark 3.6.2.

(1) It is easily checked that $\equiv \pmod{C}$ is an equivalence relation.

(2) The set of equivalence classes partitions $V_n(F)$.

(3) The equivalence class containing $x \in V_n(F)$ is called a **coset** of C. This class is given by

$$C + x = \{y \in V_n(F) : y \equiv x \pmod{C}\} = \{c + x : c \in C\}.$$

Example 3.6.3. Consider a $(5, 2)$-binary code C with generator matrix

$$G = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

The cosets of C are

$$\begin{aligned} C = C + 00000 &= \{00000, 10111, 01110, 11001\} = C + 10111 = C + 01110 = C + 11001, \\ C + 10000 &= \{10000, 00111, 11110, 01001\} = C + 00111 = C + 11110 = C + 01001, \\ C + 01000 &= \{01000, 11111, 00110, 10001\}, \\ C + 00100 &= \{00100, 10011, 01010, 11101\}, \\ C + 00010 &= \{00010, 10101, 01100, 11011\}, \\ C + 00001 &= \{00001, 10110, 01111, 11000\}, \\ C + 10100 &= \{10100, 00011, 11010, 01101\}, \\ C + 10010 &= \{10010, 00101, 11100, 01011\}. \end{aligned}$$

Remark 3.6.4. We can generalize the previous example for an arbitrary code C.

(1) We see that $C + 0 = C$ is always a coset.

(2) If $y \in C + x$, then $C + y = C + x$.

(3) All cosets of C have the same size, namely q^k.

(4) The number of cosets of C is $q^n/q^k = q^{n-k}$.

We now turn to the main definition of this lecture.

Definition 3.6.5. Let H be a parity-check matrix for an (n, k)-code C over F. For each $x \in V_n(F)$, the **syndrome** of x (with respect to H) is $s = Hx^T$.

Note that $s = Hx^T$ is a vector of length $n - k$, and that every codeword has syndrome 0.

Theorem 3.6.6. Let $x, y \in V_n(F)$. Then $x \equiv y \pmod{C}$ if and only if $Hx^T = Hy^T$. In particular, cosets are characterized by their syndromes.

Proof. We have

$$x \equiv y \pmod{C} \iff x - y \in C \iff H(x - y)^T = 0 \iff Hx^T = Hy^T. \qquad \square$$

Recall that when $c \in C$ is sent and $r \in V_n(F)$ is received, the error vector is $e = r - c$. Since $r - e = c$, we have $r \equiv e \pmod{C}$. Thus, r and e are in the same coset of C.

A natural decoding strategy is as follows: given $r \in V_n(F)$, find a vector e of smallest weight that has the same syndrome as r; namely, $He^T = Hr^T$.

- **CMLD:** Decode r to $c = r - e$.
- **IMLD:** If e is unique, decode r to $c = r - e$; otherwise, reject r.

Given a parity-check matrix H and $r \in V_n(F)$, how do we efficiently find a vector e of smallest weight with the same syndrome as r?

Unfortunately, this problem is known to be **NP**-hard, which strongly suggests that no (general) efficient algorithm exists. In particular, if any **NP**-hard problem can be solved efficiently, then all problems in **NP** can also be solved efficiently (so that $\mathbf{P} = \mathbf{NP}$).

Despite the above discussion, we will nonetheless give an algorithm for syndrome decoding.

- **Setup.** For each coset of C, select an arbitrary vector of smallest weight in that coset, and call it the **coset leader** of that coset. Store a table of cosets and their syndromes, which we call the **syndrome table**. Note that the syndrome table consists of q^{n-k} rows as there are q^{n-k} cosets of C.
- **Decoding algorithm.** This follows the CMLD design principle. Given $r \in V_n(F)$, compute its syndrome $s = Hr^T$. Let the corresponding coset leader be e. Then decode r to $c = r - e$.

Note that the decoding algorithm is guaranteed to make the correct decision if the error vector is a coset leader; otherwise, it is guaranteed to make the wrong decision.

The following theorem helps us find coset leaders quickly.

THEOREM 3.6.7. Let C be an (n, k)-code over F with distance d. Let $x \in V_n(F)$ be a vector of weight $\leq \lfloor (d-1)/2 \rfloor$. Then x is a coset leader.

PROOF. Suppose that y is in the same coset as x, with $y \neq x$ and $w(y) \leq w(x) \leq \lfloor (d-1)/2 \rfloor$. Then $x \equiv y \pmod{C}$, so $x - y \in C$ and $x - y \neq 0$. But

$$w(x-y) = w(x+(-y)) \leq w(x) + w(-y) = w(x) + w(y) \leq \left\lfloor \frac{d-1}{2} \right\rfloor + \left\lfloor \frac{d-1}{2} \right\rfloor \leq d-1.$$

This contradicts the fact that $d(C) = d$, so no such y exists. Hence, x is a coset leader. □

EXAMPLE 3.6.8. Consider the $(5, 2)$-binary code C from Example 3.6.3 with generator matrix

$$G = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

Then, we have a parity-check matrix

$$H = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

for C, and note that $d(C) = 3$. The syndrome table is given below.

Coset leader	Syndrome
00000	000
10000	111
01000	110
00100	100
00010	010
00001	001
10100	011
10010	101

By Theorem 3.6.7, the first six coset leaders are actually guaranteed to be coset leaders as they have weight $\leq \lfloor (3-1)/2 \rfloor = 1$. The last two coset leaders were chosen arbitrarily from the set of vectors of weight 2.

We give an example of decoding. Suppose that the vector $r = 11011$ is received. Then we compute its syndrome $s = Hr^T = 010$. The corresponding coset leader is $e = 00010$, so we decode r to $c = r - e = 11001$.

Remark 3.6.9. Syndrome decoding is not efficient in general as the syndrome table is exponentially large. For an (n, k)-binary code, the syndrome table has size

$$2^{n-k}(n + (n - k)) = 2^{n-k}(2n - k)$$

bits, where 2^{n-k} is the number of cosets, n corresponds to the length of a coset leader, and $n - k$ is the length of the syndrome.

Actually, we can decrease the table size slightly to $2^{n-k}n$ bits, since the table can be sorted by syndrome, and so the syndromes do not need to be stored. However, this is still of exponential size.

4 Golay codes

4.1 The Golay code C_{24}

Consider the matrix

$$\hat{B} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ & & & & & \vdots & & & & & \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}_{12\times 11}$$

whose first row consists of all 1's, and for the following rows, each row is the left cyclic shift of the previous row. Then, let

$$\hat{G} = \left[\ I_{12} \mid \hat{B}\ \right]_{12\times 23}.$$

It turns out that $\hat{G}$ is a generator matrix for a $(23, 12)$-binary code called the **(binary) Golay code** C_{23}. We will prove later that $d(C_{23}) = 7$.

Assuming that $d(C_{23}) = 7$, we can prove that C_{23} is a perfect code. Letting $e = \lfloor (7-1)/2 \rfloor = 3$, we have

$$M \cdot \sum_{i=0}^{e} \binom{n}{i} (q-1)^i = 2^{12} \left[\binom{23}{0} + \binom{23}{1} + \binom{23}{2} + \binom{23}{3} \right] = 2^{23}.$$

The code C_{24} is an extension of C_{23} with generator matrix

$$G = \left[\ I_{12} \mid B\ \right]_{12\times 24},$$

where we set $j = [0, 1, 1, \ldots, 1]^T$ and

$$B = \left[\ j \mid \hat{B}\ \right]_{12\times 12} = \left[\begin{array}{c|ccccccccccc} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ \vdots & & & & & & \vdots & & & & & \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]_{12\times 12}.$$

We list some properties of C_{24}.

(1) C_{24} is a $(24, 12)$-binary code.

(2) It can be checked that $GG^T = 0$. Hence, we get $C_{24} \subseteq C_{24}^{\perp}$, so C_{24} is a **self-orthogonal code**. Since $\dim(C_{24}) = \dim(C_{24}^{\perp}) = 12$, we have $C_{24} = C_{24}^{\perp}$, so C_{24} is in fact a **self-dual code**.

(3) It can also be checked that B is symmetric; that is, $B^T = B$.

(4) A parity-check matrix for C_{24} is given by

$$H = \left[\ -B^T \mid I_{12}\ \right] = \left[\ B \mid I_{12}\ \right].$$

(5) Since $C_{24} = C_{24}^{\perp}$, we see that H is also a generator matrix for C_{24}.

THEOREM 4.1.1. C_{24} has distance 8.

PROOF. Denote the rows of G by $r_1, r_2, \ldots, r_{12}$. Note that $w(r_1) = 12$ and $w(r_i) = 8$ for all $2 \leq i \leq 12$. Hence, $4 \mid w(r_i)$ for all $1 \leq i \leq 12$. Now, since $r_i \cdot r_j = 0$ for all $1 \leq i < j \leq 12$ (since $GG^T = 0$), the number of coordinates of r_i and r_j that are both 1 must be even. Therefore, $4 \mid w(r_i + r_j)$, so every codeword has weight divisible by 4. Hence, we either have $d(C_{24}) = 4$ or $d(C_{24}) = 8$.

Now, we show that no codeword has weight 4, which will imply that $d(C_{24}) = 8$.

- Each row of G has weight ≥ 8.
- Adding any 2 rows of G, it can be checked that $w(r_1 + r_j) = 8$ for all $2 \leq j \leq 12$, and $w(r_i + r_j) = 8$ for all $2 \leq i < j \leq 12$.
- Adding any 3 rows of G, we have $c = r_i + r_j + r_k$ where $1 \leq i < j < k \leq 12$. Let $c = (x, y)$ where x and y each have length 12. Suppose that $w(c) = 4$. Since $w(x) = 3$, we have $w(y) = 1$. Since

  $$H = \left[\begin{array}{c|c} B & I_{12} \end{array}\right]$$

  is also a generator matrix for C_{24}, we see that c must be a single row of H. But this is impossible since each row of H has weight 8 or 12, so $w(c) \neq 4$.
- Adding any 4 rows of G, we have $c = r_i + r_j + r_k + r_\ell$ where $1 \leq i < j < k < \ell < 12$. Again, let $c = (x, y)$ where x and y each have length 12. Suppose that $w(c) = 4$. Then $w(x) = 4$ and $w(y) = 0$, but H does not any such vector in its row space. Thus, $w(c) \neq 4$.
- Finally, if c is the sum of ≥ 5 rows of G and $c = (x, y)$ where x and y each have length 12, then $w(x) \geq 5$, and hence $w(c) \neq 4$. □

Corollary 4.1.2. C_{23} has distance 7.

Proof. We obtain C_{24} by adding a single bit to each codeword in C_{23}, so C_{23} must have distance 7 or 8. But C_{23} has at least one codeword of weight 7 (such as the second row of $\hat{G}$), so $d(C_{23}) = 7$. □

4.2 A decoding algorithm for C_{24}

Recall that C_{24} has distance 8, and hence is a 3-error correcting code. Moreover, we saw that $G = [I_{12} \mid B]$ and $H = [B \mid I_{12}]$ are both generator matrices and parity-check matrices for C_{24}.

Decoding strategy. Compute a syndrome s of the received word r. Find a vector e of weight ≤ 3 that has the same syndrome s. If such a vector exists, then decode r to $c = r - e$; otherwise, reject r.

Correctness. If the error vector has weight ≤ 3, then the decoder always makes the correct decision. If the error vector has weight > 3, the decoder will either reject r or decode r to a codeword different from the transmitted one (which is an incorrect decision).

Decoding algorithm. Let $r = (x, y)$ and $e = (e_1, e_2)$, where x, y, e_1, e_2 all have length 12. Then there are 5 cases (which are not mutually exclusive) in the event that $w(e) \leq 3$.

(A) $w(e_1) = 0$ and $w(e_2) = 0$.

(B) $1 \leq w(e_1) \leq 3$ and $w(e_2) = 0$.

(C) $1 \leq w(e_1) \leq 2$ and $w(e_2) = 1$.

(D) $w(e_1) = 0$ and $1 \leq w(e_2) \leq 3$.

(E) $w(e_1) = 1$ and $1 \leq w(e_2) \leq 2$.

We now consider what happens in each of these cases.

(1) Compute the syndrome $s_1 = [I_{12} \mid B] r^T$. If $s_1 = 0$, then accept r and stop. **[Case A]**

(2) Note that $s_1 = [I_{12} \mid B] r^T = [I_{12} \mid B] e^T = e_1^T + B e_2^T$. If we are in case B, then $1 \leq w(s_1) \leq 3$.

If $w(s_1) \leq 3$, then correct x in the positions corresponding to the 1's in s_1, and stop. **[Case B]**

(3) We have $s_1 = e_1^T + Be_2^T$. If we are case C, then s_2 is equal to a column of B with one or two bits flipped (depending on which bits of e_1 have 1's).

Compare s_1 with the columns (or rows) of B. If any column of B, say column i, differs in exactly one (say j) or two (say j and k) coordinate positions from s_1, then decode $r = (x, y)$ by

- correcting x in position j or positions j and k; and
- correcting y in position i.

Then stop. **[Case C]**

(4) Compute the syndrome $s_2 = [B \mid I_{12}]r^T = [B \mid I_{12}]e^T = Be_1^T + e_2^T$. If $w(s_2) \leq 3$, then correct y in the positions corresponding to the 1's in s_2, and stop. **[Case D]**

(5) Analogous to step (3); then stop. **[Case E]**

(6) Since $w(e) \geq 4$, we reject r.

We summarize our results below.

Algorithm 4.2.1 (Decoding algorithm for C_{24}). Suppose that $r = (x, y)$ is received.

(1) **[Case A]** Compute $s_1 = [I_{12} \mid B]r^T$. If $s_1 = 0$, then accept r and stop.

(2) **[Case B]** If $w(s_1) \leq 3$, then set $e = (s_1^T, 0)$. Decode r to $c = r - e$ and stop.

(3) **[Case C]** Compare s_1 to the rows of B. If any row, say row i, differs in exactly one position (say j) or two positions (say j and k), then

- correct x in position j or positions j and k; and
- correct y in position i.

Then stop.

(4) **[Case D]** Compute $s_2 = [B \mid I_{12}]r^T$. If $w(s_2) \leq 3$, then set $e = (0, s_2^T)$. Decode r to $c = r - e$, then stop.

(5) **[Case E]** Compare s_2 to the rows of B. If any row, say row i, differs in exactly one position (say j) or two positions (say j and k), then

- correct x in position i; and
- correct y in position j or positions j and k.

Then stop.

(6) Reject r.

Example 4.2.2. Decode r = (1000 1000 0000 1001 0001 1101).

Solution. Compute $s_1 = [I_{12} \mid B]r^T$ = (0**1**00 **1**000 0000)^T. Since $w(s_1) \leq 3$, we set $e = (s_1^T, 0)$, and decode r to $c = r - e$ = (1**1**00 **0**000 0000 1001 0001 1101).

Example 4.2.3. Decode r = (1000 0010 0000 1000 1101 0010).

Solution. Compute $s_1 = [I_{12} \mid B]r^T$ = (1011 1**11**0 1011)^T. Then $w(s_1) > 3$. We see that s_1 differs in positions 6 and 7 from row 4 of B. Hence, we set e = (0000 0**11**0 0000 000**1** 0000 0000) and decode r to $c = r - e$ = (1000 0**10**0 0000 100**1** 1101 0010).

Remark 4.2.4.

(i) The above decoding algorithm only needs to store B, which has size 144 bits. In contrast, a syndrome table for C_{24} has size $2^{12} \times 24 = 98304$ bits.

(ii) Decoding is efficient and simple, which is good for hardware implementation.

(iii) Fun fact: C_{24} was used in the Voyager space mission to transmit photos of Jupiter and Saturn back to Earth.

In the remainder of this lecture, we determine if C_{24} is better than simpler codes such as the replication codes or the Hamming codes.

First, we will denote

- p to be the symbol error probability;
- $C = \{c_1, \dots, c_M\}$;
- w_i to be the probability that the decoding algorithm makes an incorrect decision if c_i is sent;
- $P_C = \frac{1}{M}\sum_{i=1}^{M} w_i = w_i$ to be the error probability of C; and
- $1 - P_C$ to be the probability that r is decoded correctly, called the **reliability** of C.

For C_{24}, we have

$$1 - P_{C_{24}} = (1-p)^{24} + \binom{24}{1} p(1-p)^{23} + \binom{24}{2} p^2(1-p)^{22} + \binom{24}{3} p^3(1-p)^{21}.$$

If no channel encoding is used, then the probability that a word is transmitted with no error is $(1-p)^{12}$.

Suppose now that the triplication code T is used to encode 12-bit messages. Then

$$1 - P_T = \left[(1-p)^3 + 3p(1-p)^2\right]^{12}.$$

Suppose a $(15, 11)$-binary Hamming code H is used to encode 11-bit messages. Then

$$1 - P_H = (1-p)^{15} + 15p(1-p)^{14}.$$

We summarize these results in the following table.

p	$(1-p)^{12}$	$1 - P_T$	$1 - P_H$	$1 - P_{C_{24}}$
0.1	0.282429	0.711206	0.549043	0.785738
0.01	0.886385	0.996430	0.990370	0.999909
0.001	0.988066	0.999964	0.999896	0.9999999895
Rate	1	$1/3 \approx 0.33$	$11/15 \approx 0.73$	$1/2 \approx 0.5$

We see that C_{24} is much better than the triplication code T, as it has higher rate and reliability. On the other hand, C_{24} has lower rate than the $(15, 11)$-binary Hamming code H, but this is compensated with a much higher reliability.

5 Cyclic codes

5.1 Introduction to cyclic codes

Definition 5.1.1 (Cyclic subspace). A subspace S of $V_n(F)$ is **cyclic** if $(a_0, a_1, \ldots, a_{n-1}) \in S$ implies that $(a_{n-1}, a_0, a_1, \ldots, a_{n-2}) \in S$. In other words, a cyclic subspace is a linear subspace that is closed under the operation of taking right cyclic shifts.

Definition 5.1.2 (Cyclic code). A **cyclic code** is a cyclic subspace of $V_n(F)$.

In this section, we will find an algebraic characterization of cyclic subspaces of $V_n(F)$ as ideals of the polynomial ring $R = F[x]/(x^n - 1)$. Indeed, let $R = F[x]/(x^n - 1)$ where $F = \mathrm{GF}(q)$. We see that R is a commutative ring, but it is not a field as the polynomial $x^n - 1$ is irreducible with $x - 1$ as a factor.

We have a bijection between $V_n(F)$ and R given by

$$V_n(F) \ni a = (a_0, a_1, a_2, \ldots, a_{n-1}) \leftrightarrow a(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1} \in R,$$

and this bijection preserves vector addition and scalar multiplication. We can use this bijection to define a natural multiplication on $V_n(F)$.

Definition 5.1.3 (Multiplication on $V_n(F)$). Let $a, b \in V_n(F)$. Then $a \cdot b = c \in V_n(F)$, where the vector c corresponds to the polynomial $c(x) = a(x) \cdot b(x) \pmod{x^n - 1}$.

Why did we choose $x^n - 1$ as the modulus? Let $a = (a_0, a_1, a_2, \ldots, a_{n-1}) \in V_n(F)$, and let $a(x)$ be the associated polynomial in R. Then

$$\begin{aligned} x \cdot a(x) &= a_0x + a_1x^2 + a_2x^3 + \cdots + a_{n-2}x^{n-1} + a_{n-1}x^n \\ &\equiv a_{n-1} + a_0x + a_1x^2 + a_2x^3 + \cdots + a_{n-2}x^{n-1} \pmod{x^n - 1}, \end{aligned}$$

and we can identify this polynomial with the vector $(a_{n-1}, a_0, a_1, \ldots, a_{n-2}) \in V_n(F)$. In particular, multiplication by x of a polynomial in R corresponds to a right cyclic shift of the associated vector in $V_n(F)$.

Definition 5.1.4 (Ideal). Let R be a commutative ring. A nonempty subset I of R is an **ideal** of R if

(i) $a, b \in I$ implies that $a + b \in I$; and

(ii) $a \in I$ and $b \in R$ implies that $a \cdot b \in I$.

Example 5.1.5. Let R be a ring. Then $\{0\}$ and R are trivial ideals of R.

Theorem 5.1.6 (Algebraic characterization of cyclic subspaces of $V_n(F)$). Let S be a non-empty subset of $V_n(F)$. Let I be the set of polynomials in $R = F[x]/(x^n - 1)$ corresponding to the elements of S. Then S is a cyclic subspace of $V_n(F)$ if and only if I is an ideal of R.

Proof. Suppose that S is a cyclic subspace of $V_n(F)$. Since S is nonempty and closed under addition, so is I. Now, let $a(x) \in I$ and $b(x) = b_0 + b_1x + \cdots + b_{n-1}x^{n-1} \in R$. Since S is a cyclic subspace, it is closed under right cyclic shifts, so $xa(x) \in I$. Moreover, we have $x^ia(x) \in I$ for all $0 \leq i \leq n-1$. Then, S is closed under scalar multiplication, so we have $b_ix^ia(x) \in I$ for all $0 \leq i \leq n - 1$. Finally, since I is closed under addition, it follows that $\sum_{i=0}^{n-1} b_ix^ia(x) = a(x)b(x) \in I$. Thus, I is an ideal of R.

Conversely, suppose that I is an ideal of R. Since I is closed under scalar multiplication, so is S. Since I is closed under multiplication by constant polynomials (namely, the elements of F), we see that S is closed under scalar multiplication. Hence, S is a subspace of $V_n(F)$. Next, since I is closed under multiplication by x, it follows that S is closed under right cyclic shifts, completing the proof. □

As a consequence of this theorem, in order to study cyclic subspaces of $V_n(F)$, it suffices to study ideals of the ring $R = F[x]/(x^n - 1)$.

5.2 Ideals of the ring $R = F[x]/(x^n - 1)$

Definition 5.2.1. Let R be a ring.

- Let $g \in R$ and let $\langle g \rangle = \{g \cdot r : r \in R\}$. Then $\langle g \rangle$ is an ideal of R, called the **ideal generated by** g.
- An ideal I of R is said to be **principal** if $I = \langle g \rangle$ for some $g \in I$ (that is, it is generated by a single element from the ideal).
- A ring R is a **principal ideal ring** if every ideal of R is principal.

Theorem 5.2.2. The ring $R = F[x]/(x^n - 1)$ is a principal ideal ring.

Proof. Let I be an ideal of R. If $I = \{0\}$, then $I = \langle 0 \rangle$. Suppose now that $I \neq \{0\}$. Let $g(x)$ be a non-zero polynomial of smallest degree in I. We claim that $I = \langle g \rangle$. Indeed, let $h(x) \in I$. Then we can write $h(x) = \ell(x)g(x) + r(x)$ where $\ell(x), r(x) \in F[x]$ with $\deg(r) < \deg(g)$. Since $h(x), \ell(x)g(x) \in I$ and I is an ideal, we have that $h(x) - \ell(x)g(x) = r(x) \in I$. But $\deg(r) < \deg(g)$ and we assumed that $g(x)$ is a non-zero polynomial of smallest degree, so it follows that $r(x) = 0$. Hence, we obtain $h(x) = \ell(x)g(x)$, so $I = \langle g \rangle$, and thus R is a principal ideal ring. □

Remark 5.2.3. In the proof of the previous theorem, we can take $g(x)$ to be monic (that is, $g(x)$ has leading coefficient 1) without loss of generality. This is because if $g(x) = \sum_{i=0}^{t} g_i x^i$ where $g_t \neq 0$, then the polynomial $g_t^{-1} g(x)$ is monic with the same degree as $g(x)$.

Definition 5.2.4 (Generator polynomial). Let I be an ideal of $R = F[x]/(x^n - 1)$.

- If $I = \{0\}$, then $x^n - 1$ is the **generator polynomial** of I.
- If $I \neq \{0\}$, then a monic polynomial of smallest degree in I is called a **generator polynomial** of I.

The following theorem asserts that if I is a non-zero ideal of $R = F[x]/(x^n - 1)$, then there is a unique generator polynomial of I, in which case we can call it *the* generator polynomial of I.

Theorem 5.2.5. Let I be a non-zero ideal of $R = F[x]/(x^n - 1)$.

(1) There is a unique monic polynomial of $g(x)$ of smallest degree in I which generates I.

(2) If $g(x)$ is the generator polynomial of I, then $g(x) \mid x^n - 1$ in $F[x]$.

Proof.

(1) Let $g(x)$ and $h(x)$ be monic polynomials of (the same) smallest degree in I. Then $g(x) - h(x) \in I$. But $\deg(g - h) < \deg(g)$, so $g(x) - h(x) = 0$, and hence $g(x) = h(x)$. Thus, a generator polynomial of I is unique.

(2) Write $x^n - 1 = \ell(x)g(x) + r(x)$ where $\ell(x), r(x) \in F[x]$ with $\deg(r) < \deg(g)$. Then $r(x) = -\ell(x)g(x) + (x^n - 1) \equiv -\ell(x)g(x) \pmod{x^n - 1}$. We have $r(x) \in I = \langle g \rangle$, and since $\deg(r) < \deg(g)$, we must have $r(x) = 0$. Therefore, we obtain $g(x) \mid x^n - 1$. □

Theorem 5.2.6. Suppose that $h(x)$ is a monic divisor of $x^n - 1$ in $F[x]$. Then $h(x)$ is the generator polynomial of $\langle h(x) \rangle$.

Proof. If $h(x) = x^n - 1$, then $I = \{0\}$. Suppose now that $h(x) \neq x^n - 1$ so that $I \neq \{0\}$. Let $g(x)$ be the monic polynomial of smallest degree in I. Since $h(x)$ generates I, we can write $g(x) \equiv a(x)h(x) \pmod{x^n - 1}$ for some $a(x) \in F[x]$ with $\deg(a) < n$. Then, we have $g(x) = a(x)h(x) + \ell(x)(x^n - 1)$ for some $\ell(x) \in F[x]$. Since $h(x) \mid x^n - 1$ (by part (2) of the previous theorem), we have $h(x) \mid g(x)$, so $\deg(h) \leq \deg(g)$. But we also have $\deg(g) \leq \deg(h)$, and hence $\deg(g) = \deg(h)$. Finally, since $g(x)$ and $h(x)$ are both monic, we must have $g(x) = h(x)$. Hence, $h(x)$ is the generator polynomial of $\langle h(x) \rangle$. □

Corollary 5.2.7. There is a bijection between cyclic subspaces of $V_n(F)$ and monic divisors of $x^n - 1$.

Proof. By Theorem 5.2.6, there is a bijection between the ideals of $R = F[x]/(x^n - 1)$ and the monic divisors of $x^n - 1$. □

Remark 5.2.8. Let $x^n - 1 = p_1(x)^{e_1} p_2(x)^{e_2} \cdots p_t(x)^{e_t}$ be the complete factorization of $x^n - 1$ over F, where $p_1, \ldots, p_t$ are monic irreducible polynomials and each $e_i \geq 1$. Then the set of all monic divisors of $x^n - 1$ is

$$\{p_1(x)^{f_1} p_2(x)^{f_2} \cdots p_t(x)^{f_t} : 0 \leq f_i \leq e_i \text{ for all } 1 \leq i \leq t\}.$$

Hence, the number of monic divisors of $x^n - 1$ over F is $\prod_{i=1}^{t}(e_i + 1) = (e_1 + 1)(e_2 + 1) \cdots (e_t + 1)$.

Example 5.2.9. Find all cyclic subspaces of $V_3(\mathbb{Z}_2)$.

Solution. The complete factorization of $x^3 - 1$ over $\mathbb{Z}_2$ is $x^3 - 1 = (1+x)(1+x+x^2)$, so the monic divisors of $x^3 - 1$ are

$$\begin{aligned}
g_1(x) &= 1, \\
g_2(x) &= 1 + x, \\
g_3(x) &= 1 + x + x^2, \\
g_4(x) &= (1 + x)(1 + x + x^2) = x^3 - 1.
\end{aligned}$$

Hence, $V_3(\mathbb{Z}_2)$ has 4 cyclic subspaces. They are given by

$$\begin{aligned}
S_1 &= \langle g_1(x) \rangle = \{000, 001, 010, 011, 100, 101, 110, 111\} = V_3(\mathbb{Z}_2), \\
S_2 &= \langle g_2(x) \rangle = \{000, 110, 011, 101\}, \\
S_3 &= \langle g_3(x) \rangle = \{000, 111\}, \\
S_4 &= \langle g_4(x) \rangle = \{000\}.
\end{aligned}$$

Observe that $\dim(S_i) + \deg(g_i) = 3$ for all $1 \leq i \leq 4$.

5.3 Dimension and generator matrix of a cyclic code

We generalize the observation that we made in Example 5.2.9.

Theorem 5.3.1. Let $g(x)$ be a monic divisor of $x^n - 1$ over F, where $F = \mathrm{GF}(q)$. Suppose that $\deg(g) = n - k$. Then the cyclic subspace S of $V_n(F)$ generated by $g(x)$ has dimension k. In particular, we have $\dim(S) + \deg(g) = n$.

Proof. Recall that $\langle g(x) \rangle = \{a(x)g(x) \pmod{x^n - 1} : a(x) \in F[x], \deg(a) < n\}$. We claim that

$$\langle g(x) \rangle = \{b(x)g(x) : b(x) \in F[x], \deg(b) < k\}.$$

To see this, let $h(x) = a(x)g(x) \pmod{x^n - 1}$ for some $a(x) \in F[x]$ with $\deg(a) < n$. THen we can write $a(x)g(x) = h(x) + \ell(x)(x^n - 1)$ for some $\ell(x) \in F[x]$. Now, we have $g(x) \mid h(x)$, so we have $h(x) = b(x)g(x)$ for some $b(x) \in F[x]$ with $\deg(b) < k$, proving the claim.

Finally, since there are q^k polynomials of degree $< k$ in $F[x]$, it follows that $\langle g(x) \rangle$ has size q^k by our claim. Hence, S has dimension k. □

Example 5.3.2. We construct a $(7, 4)$-cyclic code over $\mathbb{Z}_2$. Due to Theorem 5.3.1, we need to find a monic divisor of $x^7 - 1$ over $\mathbb{Z}_2$ of degree 3. The factorization of $x^7 - 1$ over $\mathbb{Z}_2$ is

$$x^7 - 1 = (1 + x)(1 + x + x^3)(1 + x^2 + x^3).$$

Choosing $g(x) = 1 + x^2 + x^3$, we see that $C = \langle g(x) \rangle$ is a $(7, 4)$-cyclic code over $\mathbb{Z}_2$.

We now find a generator matrix for C. In particular, we need to find a basis; that is, 4 linearly independent vectors in C. We can choose $g(x)$, $xg(x)$, $x^2g(x)$, and $x^3g(x)$ to get

$$G = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

Suppose now that we want to encode the message $m = 1001$. Then we have

$$c = mG = \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Equivalently, we can write

$$\begin{aligned} c(x) &= m(x)g(x) \\ &= (1 + x^3)(1 + x^2 + x^3) \\ &= 1 + x^2 + x^5 + x^6, \end{aligned}$$

which has associated vector $c = 1010011$.

Generalizing our results from the previous example, we obtain the following theorem.

THEOREM 5.3.3. Let $g(x)$ be the generator polynomial of an (n, k)-cyclic code C over F (so that $g(x)$ is a monic divisor of $x^n - 1$ over F of degree $n - k$). Then a (non-systematic) generator matrix for C is given by

$$G = \begin{bmatrix} g(x) \\ xg(x) \\ x^2g(x) \\ \vdots \\ x^{k-1}g(x) \end{bmatrix}_{k \times n}.$$

Encoding. The source messages are the polynomials in $F[x]$ of degree $< k$. If $m(x) = m_0 + m_1x + \cdots + m_{k-1}x^{k-1}$, then the encoding of m with respect to G is

$$c = mG = \begin{bmatrix} m_0 & m_1 & \cdots & m_{k-1} \end{bmatrix} \begin{bmatrix} g(x) \\ xg(x) \\ x^2g(x) \\ \vdots \\ x^{k-1}g(x) \end{bmatrix} = m_0g(x) + m_1xg(x) + \cdots + m_{k-1}x^{k-1}g(x),$$

so $c(x) = m(x)g(x)$. Note that no reduction by $x^n - 1$ is needed.

5.4 Dual code of a cyclic code

For the entirety of this section, we let C be an (n, k)-cyclic code over F with generator polynomial $g(x)$. Note that $g(x)$ is of the form

$$g(x) = g_0 + g_1x + \cdots + g_{n-k}x^{n-k} + g_{n-k+1}x^{n-k+1} + \cdots + g_{n-1}x^{n-1}$$

where $g_0 \neq 0$, $g_{n-k} = 1$, and $\sum_{i=n-k+1}^{n-1} g_ix^i = 0$.

Definition 5.4.1. The **parity-check polynomial** is given by $h(x) = (x^n - 1)/g(x)$.

We see that $h(x)$ is in the form

$$h(x) = h_0 + h_1(x) + \cdots + h_k x^k + h_{k+1}x^{k+1} + \cdots + h_{n-1}x^{n-1}$$

where $h_0 \neq 0$, $h_k = 1$, and $\sum_{i=k+1}^{n-1} h_i x^i = 0$.

Let $a(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} = g(x)h(x)$. Note that $a(x) = 0 \pmod{x^n - 1}$. Equating the coefficients of x^i for each $0 \leq i \leq n-1$ gives

$$0 = a_i = g_0 h_i + g_1 h_{i-1} + \cdots + g_i h_0 + g_{i+1}h_{n-1} + g_{i+2}h_{n-2} + \cdots + g_{n-1}h_{i+1}.$$

Thus, the vector $g = (g_0, g_1, \ldots, g_{n-1})$ is orthogonal to the vector $\bar{h} = (h_{n-1}, h_{n-2}, \ldots, h_1, h_0)$ (by taking $i = n-1$ in the above expression), as well as all its cyclic shifts. It follows that all cyclic shifts of g are orthogonal to all cyclic shifts of $\bar{h}$.

Recall that a generator matrix for C is

$$G = \begin{bmatrix} g_0 & g_1 & \cdots & g_{n-k-1} & g_{n-k} & 0 & \cdots & 0 \\ 0 & g_0 & g_1 & \cdots & g_{n-k-1} & g_{n-k} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & g_0 & g_1 & \cdots & g_{n-k-1} & g_{n-k} \end{bmatrix}_{k \times n}.$$

We define the matrix

$$H = \begin{bmatrix} h_k & h_{k-1} & \cdots & h_1 & h_0 & 0 & \cdots & 0 \\ 0 & h_k & h_{k-1} & \cdots & h_1 & h_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & h_k & h_{k-1} & \cdots & h_1 & h_0 \end{bmatrix}_{(n-k) \times n}.$$

By our above observation, we have $GH^T = 0$. Thus, if C' is the code generated by H, then $C' \subseteq C^\perp$. But $\operatorname{rank}(H) = n - k$ (since $h_k = 1$), so $\dim(C') = n - k = \dim(C^\perp)$. Hence, we obtain $C' = C^\perp$, so H is a (non-systematic) parity-check matrix for C.

Definition 5.4.2. Let $h(x) = h_0 + h_1 x + \cdots + h_k x^k$ be a polynomial of degree k (so that $h_k \neq 0$). The **reciprocal polynomial** of $h(x)$ is defined by

$$h_R(x) = x^k h(1/x) = h_k + h_{k-1}x + \cdots + h_0 x^k.$$

If $h_0 \neq 0$, we define $h^*(x) = h_0^{-1} h_R(x)$, which is monic polynomial of degree k.

Theorem 5.4.3. Let C be an (n, k)-cyclic code over F with generator polynomial $g(x)$. Suppose that $h(x) = (x^n - 1)/g(x)$ is the parity-check polynomial. Then $C^\perp$ is a cyclic code with generator polynomial $h^*(x) = h_0^{-1} h_R(x)$.

Proof. We have $g(x)h(x) = x^n - 1$, so $g(1/x)h(1/x) = (1/x)^n - 1$. Multiplying both sides by x^n gives

$$x^{n-k} g(1/x) \cdot x^k h(1/x) = -(x^n - 1).$$

Hence, we have $g_R(x)h_R(x) = -(x^n - 1)$, so $h_R(x) \mid x^n - 1$. In particular, $h^*(x)$ is a monic divisor of $x^n - 1$. From our previous discussion, we saw that the $(n, n-k)$-cyclic code generated by $h_R(x)$ (and thus $h^*(x)$ as well) is $C^\perp$. □

5.5 Computing syndromes for cyclic codes

Let C be an (n,k)-cyclic code over F with generator polynomial $g(x)$. We will find a nice parity-check matrix for C, by which we mean that computing syndromes with respect to this parity-check matrix will have a nice interpretation as polynomial division.

First, we find a generator matrix for C in the form $[R \mid I_k]$. For each $0 \le i \le k-1$, long division gives $x^{n-k+i} = \ell_i(x)g(x) + r_i(x)$ where $\deg(r_i) < n-k$ and $\deg(\ell_i) < k$. Then $x^{n-k+i} - r_i(x) = \ell_i(x)g(x) \in C$. Therefore, a generator matrix for C is given by

$$G = \begin{bmatrix} -r_0(x) + x^{n-k} \\ -r_1(x) + x^{n-k+1} \\ \vdots \\ -r_{k-1}(x) + x^{n-1} \end{bmatrix}_{k\times n} = \left[\begin{array}{c|c} -x^{n-k} \pmod{g(x)} & \\ -x^{n-k+1} \pmod{g(x)} & \\ \vdots & I_k \\ -x^{n-1} \pmod{g(x)} & \end{array}\right]_{k\times n} = [R \mid I_k].$$

Note that we have $\operatorname{rank}(G) = k$. Now, a (systematic) parity-check matrix for C is given by $H = [I_{n-k} \mid -R^T]$. Observe that

$$H^T = \left[\begin{array}{c} I_{n-k} \\ \hline -R \end{array}\right],$$

so the rows of H^T (equivalently, the columns of H) are $x^0 \pmod{g(x)}$, $x^1 \pmod{g(x)}, \dots, x^{n-1} \pmod{g(x)}$.

Theorem 5.5.1 (Computing syndromes). The syndrome of $r \in V_n(F)$ with respect to the above parity-check matrix H is $s \in V_{n-k}(F)$, where $s(x) = r(x) \pmod{g(x)}$.

Proof. Let $r = (r_0, r_1, \dots, r_{n-1}) \in V_n(F)$. Then the syndrome of r is $s = Hr^T$, and we have

$$\begin{aligned} s(x) &= [r_0x^0 \pmod{g(x)}] + [r_1x^1 \pmod{g(x)}] + \cdots + [r_{n-1}x^{n-1} \pmod{g(x)}] \\ &= r_0 + r_1x + \cdots + r_{n-1}x^{n-1} \pmod{g(x)} \\ &= r(x) \pmod{g(x)}. \end{aligned}$$

□

Example 5.5.2. Consider the $(15,9)$-binary cyclic code C with generator polynomial $g(x) = 1 + x + x^2 + x^3 + x^6$. Compute the syndrome of $r = 1100\,1000\,1110\,000$.

Solution. We have $r(x) = 1 + x + x^4 + x^8 + x^9 + x^{10}$. Then polynomial division gives

$$r(x) = (x^4 + x^3 + x^2 + x)g(x) + (x^5 + x^4 + x^3 + 1),$$

so we obtain $s(x) = x^5 + x^4 + x^3 + 1$, and hence $s = 100111$.

The syndromes of a vector and its cyclic shifts are closely related.

Theorem 5.5.3. Let $r(x)$ be a polynomial with syndrome polynomial $s(x) = s_0 + s_1x + \cdots + s_{n-k-1}x^{n-k-1}$.

(i) If $s_{n-k-1} = 0$, then the syndrome of $xr(x)$ is $xs(x)$.

(ii) If $s_{n-k-1} \neq 0$, then the syndrome of $xr(x)$ is $xs(x) - s_{n-k-1}g(x)$.

Note that these syndromes are not always cyclic shifts of each other.

Proof. Since $r(x)$ has syndrome $s(x)$, we have $r(x) = \ell(x)g(x) + s(x)$ for some $\ell(x) \in F[x]$ by Theorem 5.5.1. It follows that $xr(x) = x\ell(x)g(x) + xs(x)$.

(i) If $s_{n-k-1} = 0$, then $\deg(xs(x)) < n-k$, so we see that $xs(x)$ is the (unique) remainder when $xr(x)$ is divided by $g(x)$.

(ii) Suppose that $s_{n-k-1} \neq 0$. Then we can write

$$\begin{aligned} xr(x) &= x\ell(x)g(x) + xs(x) - s_{n-k-1}g(x) + s_{n-k-1}g(x) \\ &= [x\ell(x) + s_{n-k-1}]g(x) + \underbrace{[xs(x) - s_{n-k-1}g(x)]}_{\bar{s}(x)}. \end{aligned}$$

Observe that $\deg(\bar{s}) < n-k$, so $\bar{s}(x)$ is the (unique) remainder when $xr(x)$ is divided by $g(x)$. □

Therefore, given the syndrome s of r, we can easily compute the syndromes of the cyclic shifts of r.

Example 5.5.4. Recall Example 5.5.2, where we had a $(15,9)$-binary cyclic code C with generator polynomial $g(x) = 1+x+x^2+x^3+x^6 = 1111001$. We found that the syndrome of $r = 1100\,1000\,1110\,000$ is $s = 100111$. For each $0 \le i \le 14$, let $s_i(x)$ be the syndrome polynomial of $x^i r(x)$. We compute some of the syndromes below by applying Theorem 5.5.3.

i	$s_i(x)$
0	100111
1	101111
2	101011
3	101011
4	101000
5	010100
6	001010
7	000101
$\vdots$	$\vdots$

We show how $s_1 = 101111$ was computed. First, note that we have $s = 100111$, and in this case, we have $s_{15-9-1} = s_5 = 1 \neq 0$. Therefore, Theorem 5.5.3 says that the syndrome of $xr(x)$ is

$$xs(x) - s_{n-k-1}g(x) = 0100111 - 1111001 = 1011110,$$

which we can represent with the polynomial $1+x^2+x^3+x^4+x^5$. The syndromes of the other cyclic shifts of r can be found similarly.

5.6 Burst error correcting

So far in this course, we have assumed that errors occur randomly and independently of each other. However, in practice, errors tend to occur in bursts; that is, many bits close to each other might be affected. In the next section, we will see that cyclic codes are good at correcting burst errors. First, we will define cyclic burst errors more formally.

Definition 5.6.1. Let $e \in V_n(F)$. The **cyclic burst length** of e is the length of the shortest cyclic block of e that contains all the non-zero components of e.

Example 5.6.2. The cyclic burst length of $e = 0110100010$ is 7, because a cyclic block beginning with the second last bit of e up to the fifth bit of e contains all the non-zero entries of e, and furthermore, no shorter cyclic block of e contains all of its non-zero components.

Definition 5.6.3. A linear code C is a **t-cyclic burst error correcting code** if all cyclic burst errors of length $\le t$ are in different cosets of C; that is, they have different syndromes. The largest such t is the **cyclic burst error correcting capability** of C.

Example 5.6.4. Recall that $g(x) = 1+x+x^2+x^3+x^6$ generates a $(15,9)$-binary cyclic code C. In fact, C is a 3-cyclic burst error correcting code. To check this, we can verify that all cyclic burst errors of length ≤ 3 have different syndromes (which is extremely tedious to do by hand, and is best done with a computer program). In the following table, we give some of the syndromes and their integer representations; there are $1+15+15+15+15 = 61$ cyclic burst errors of length ≤ 3 in total.

Cyclic burst error	Syndrome	Integer representation
0	000000	0
x^0	100000	32
x^1	010000	16
x^2	001000	8
$\vdots$	$\vdots$	$\vdots$
x^{14}	111001	57
$1+x$	110000	48
$x(1+x)$	011000	24
$\vdots$	$\vdots$	$\vdots$
$x^{14}(1+x)$	011001	25
$1+x+x^2$	111000	56
$x(1+x+x^2)$	011100	28
$\vdots$	$\vdots$	$\vdots$
$x^{14}(1+x+x^2)$	001001	9
$1+x^2$	101000	40
$x(1+x^2)$	010100	20
$\vdots$	$\vdots$	$\vdots$
$x^{14}(1+x^2)$	101001	41

Example 5.6.5. The polynomial $g(x) = 1 + x^4 + x^6 + x^7 + x^8$ generates a $(15, 7)$-binary cyclic code C. It can be checked that C is a 4-cyclic burst error correcting code.

The following theorem gives us bounds on the cyclic burst error correcting capability of a code.

Theorem 5.6.6. Let C be an (n, k, d)-code over GF(q). Let t be the cyclic burst error correcting capability of C. Then $\lfloor (d-1)/2 \rfloor \le t \le n-k$.

Proof. First, we prove the lower bound. Recall that the vectors of weight $\le \lfloor (d-1)/2 \rfloor$ all lie in different cosets of C. In particular, all cyclic burst errors of length $\le \lfloor (d-1)/2 \rfloor$ lie in different cosets of C, so $\lfloor (d-1)/2 \rfloor \le t$.

Now, we prove the upper bound. Since C is a t-cyclic burst error correcting code, no two cyclic burst errors of length $\le t$ lie in the same coset of C. Namely, two vectors in which all the non-zero components are in the first t coordinate positions can lie in the same coset of C. Since there are q^t such vectors and q^{n-k} cosets of C, we must have $q^t \le q^{n-k}$, and hence $t \le n-k$. □

We can get an improved upper bound on the cyclic burst error correcting capability of a code.

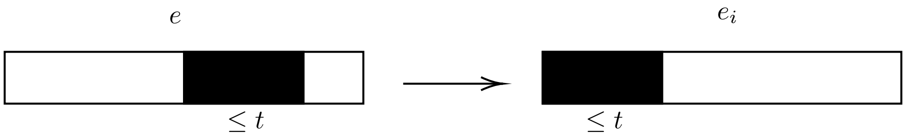
Exercise 5.6.7. Let C be an (n, k)-code over GF(q) with cyclic burst error correcting capability t. Prove that $t \le \lfloor (n-k)/2 \rfloor$.

5.7 Decoding algorithm for burst error correcting codes

Let C be an (n, k)-cyclic code over F with generator polynomial $g(x)$ and cyclic burst error correcting capability t (and hence $t \le n-k$).

Recall that $H = [I_{n-k} \mid -R^T]$ is a parity-check matrix for C, where the columns of H are given by $x^i \pmod{g(x)}$ for $0 \le i \le n-1$. Moreover, the syndrome of $r(x)$ is $s(x) = r(x) \pmod{g(x)}$.

Idea of the decoding algorithm. Suppose that the error vector e is a cyclic burst of length $\le t$. Then, some cyclic shift of e, say $e_i = x^i e$, has all of its non-zero entries in the first $n-k$ coordinate positions.



Then $s_i = He_i^T$ is a vector of length $n-k$ and has (non-cyclic) burst length $\leq t$. It follows that $e_i = (s_i, 0)$ and hence $e = x^{n-i}e_i$.

Let $r = c + e$ so that $x^i r - x^i e = x^i c$. Note that C is a cyclic code, so $x^i c$ is also a codeword, and hence $x^i r$ and $x^i e$ have the same syndrome. Therefore, in order to compute the syndromes of the cyclic shifts of the error vectors $s_i = He_i^T$, it suffices to compute the syndromes of $r_i = x^i r$ for all $0 \leq i \leq n-1$.

Algorithm 5.7.1 (Error-trapping decoding algorithm for cyclic burst error correcting codes). Let $r(x)$ be the received word, and let $s_i(x)$ denote the syndrome of $x^i r(x)$ for each $0 \leq i \leq n-1$.

```
for $i = 0$ to $n-1$ do
    Compute $s_i(x)$
    if $s_i$ has (non-cyclic) burst length $\leq t$ then
        Let $e(x) = x^{n-i}(s_i, 0)$
        Decode $r(x)$ to $c(x) = r(x) - e(x)$
    end if
end for
Reject $r$
```

If the error vector is in fact a cyclic burst of length $\leq t$, then the algorithm will make the correct decision.

Example 5.7.2. Recall that the $(15,9)$-binary cyclic code with generator polynomial $g(x) = 1+x+x^2+x^3+x^6$ is a 3-cyclic burst error correcting code. Decode the received word $r = 1110\,1110\,1100\,000$.

Solution. We begin by computing the syndromes of r and its cyclic shifts.

i	$s_i(x)$
0	110011
1	100101
2	101110
3	010111
4	110111
5	100111
6	101111
7	101011
8	101001
9	101000

Since s_9 has burst length ≤ 3, we let $e(x) = x^{15-9}s_9(x) = x^6(1+x^2) = 0000\,0010\,1000\,000$, and we decode r to $c = r - e = 1110\,1100\,0100\,000$.

Interleaving. Suppose we want to increase the cyclic burst error correcting capability of a code. Let C be an (n,k)-code with cyclic burst error correcting capability t. Suppose that

$$\begin{aligned} c_1 &= (c_{11}, c_{12}, \dots, c_{1n}) \in C, \\ c_2 &= (c_{21}, c_{22}, \dots, c_{2n}) \in C, \\ &\vdots \\ c_s &= (c_{s1}, c_{s2}, \dots, c_{sn}) \in C. \end{aligned}$$

Instead of transmitting $c_1, c_2, \ldots, c_s$ in that order, we transmit

$$c^* = (c_{11}, c_{21}, \ldots, c_{s1}, c_{12}, c_{22}, \ldots, c_{s2}, \ldots, c_{1n}, c_{2n}, \ldots, c_{sn}),$$

which are the columns of the above array. Then any cyclic burst of length $\leq st$ in c^* results in a cyclic burst of length $\leq t$ in each of the original codewords $c_1, c_2, \ldots, c_s$, which can then be corrected with the error-trapping algorithm. This is called **interleaving to a depth of** s.

Theorem 5.7.3 (Interleaving codes). *Let C be an (n, k)-code over F with cyclic burst error correcting capability t. Let C^* be the code obtained by interleaving C to a depth of s.*

(1) *C^* is an (ns, ks)-code over F with cyclic burst error correcting capability ts.*

(2) *If C is cyclic with generator polynomial $g(x)$, then C^* is cyclic with generator polynomial $g(x^s)$.*

Proof. We leave this as an exercise. For (1), show that C^* is linear with length ns, dimension ks, and cyclic burst error correcting capability ts. For (2), prove that $g(x^s)$ is a monic divisor of $x^{ns} - 1$ of degree $ns - ks$, and that $g(x^s) \mid c(x)$ for all $c \in C^*$. □

6 BCH codes

In this section, we will be studying a special class of cyclic codes called BCH codes. Before we get to that, we will need some more field theory.

6.1 Minimal polynomials

Recall that if $F = \mathrm{GF}(p^m)$ is a finite field of characteristic p, then $\mathbb{Z}_p$ is a subfield of F, and we can view F as an m-dimensional vector space over $\mathbb{Z}_p$.

More generally, for any prime power q, we have that $\mathrm{GF}(q)$ is a subfield of $\mathrm{GF}(q^m)$, and we can view $\mathrm{GF}(q^m)$ has an m-dimensional vector space over $\mathrm{GF}(q)$. We call $\mathrm{GF}(q^m)$ an **extension field** of $\mathrm{GF}(q)$.

Definition 6.1.1. Let $\alpha \in \mathrm{GF}(q^m)$. The **minimal polynomial of** α **over** $\mathrm{GF}(q)$, denoted by $m_\alpha(y)$, is the monic polynomial of smallest degree in $\mathrm{GF}(q)[y]$ that has α as a root.

Remark 6.1.2.

(1) If $m(y) \in \mathrm{GF}(q)[y]$ is a non-zero polynomial with $m(\alpha) = 0$ and c is the leading coefficient of $m(y)$, then $\hat{m}(y) = c^{-1}m(y)$ is a monic polynomial in $\mathrm{GF}(q)[y]$ with $\hat{m}(\alpha) = 0$. More generally, multiplying a polynomial by a non-zero constant does not change the roots of the polynomial.

(2) We have $m_0(y) = y$.

(3) Suppose that $\alpha \neq 0$ and let t be the order of α. Recall that $t \mid q^m - 1$. Then α is a root of the polynomial $y^t - 1 \in \mathrm{GF}(q)[y]$. Hence, there always exists a monic polynomial of smallest degree in $\mathrm{GF}(q)[y]$ having α as a root.

Example 6.1.3. The minimal polynomials over $\mathrm{GF}(2)$ of the elements in $\mathrm{GF}(2^2) = \mathbb{Z}_2[x]/(x^2 + x + 1) = \{0, 1, x, x + 1\}$ are

$$\begin{aligned} m_0(y) &= y, \\ m_1(y) &= y + 1, \\ m_x(y) &= y^2 + y + 1, \\ m_{x+1}(y) &= y^2 + y + 1. \end{aligned}$$

Theorem 6.1.4. Let $\alpha \in \mathrm{GF}(q^m)$.

(a) The minimal polynomial $m_\alpha(y)$ of α over $\mathrm{GF}(q)$ is unique.

(b) The minimal polynomial $m_\alpha(y)$ is irreducible over $\mathrm{GF}(q)$.

(c) We have $\deg(m_\alpha) \leq m$.

(d) If $f(y) \in \mathrm{GF}(q)[y]$, then $f(\alpha) = 0$ if and only if $m_\alpha(y) \mid f(y)$.

Proof.

(a) Suppose that $m_1(y), m_2(y) \in \mathrm{GF}(q)[y]$ are two monic polynomials of (the same) smallest degree with $m_1(\alpha) = m_2(\alpha) = 0$. Consider $r(y) = m_1(y) - m_2(y)$. Then $r(\alpha) = m_1(\alpha) - m_2(\alpha) = 0 - 0 = 0$. But $\deg(r) < \deg(m_1)$ as $m_1(y)$ and $m_2(y)$ are both monic, so we must have $r(y) = 0$. Therefore, we obtain $m_1(y) = m_2(y)$.

(b) Suppose towards a contradiction that $m_\alpha(y)$ is reducible over $\mathrm{GF}(q)$. Then we can write $m_\alpha(y) = s(y)t(y)$ where $s(y), t(y) \in \mathrm{GF}(q)[y]$ with $\deg(s), \deg(t) \leq \deg(m_\alpha)$. Then $m_\alpha(\alpha) = s(\alpha)t(\alpha) = 0$, so we either have $s(\alpha) = 0$ or $t(\alpha) = 0$. In either case, this contradicts the minimality of $\deg(m_\alpha)$, so $m_\alpha(y)$ must be irreducible over $\mathrm{GF}(q)$.

(c) Recall that $\mathrm{GF}(q^m)$ is an m-dimensional vector space over $\mathrm{GF}(q)$. Hence, the elements $1, \alpha, \alpha^2, \ldots, \alpha^m$ are linearly dependent over $\mathrm{GF}(q)$. In particular, we can write

$$a_0 + a_1\alpha + a_2\alpha^2 + \cdots + a_m\alpha^m = 0$$

for some $a_0, a_1, \ldots, a_m \in \mathrm{GF}(q)$ which are not all zero. Then α is a root of the non-zero polynomial

$$m(y) = a_0 + a_1 y + \cdots + a_m y^m \in \mathrm{GF}(q)[y],$$

so it follows that $\deg(m_\alpha) \leq m$.

(d) Let $f(y) \in \mathrm{GF}(q)[y]$. By the division algorithm for polynomials, we can write $f(y) = \ell(y)m_\alpha(y) + r(y)$ where $\ell(y), r(y) \in \mathrm{GF}(q)[y]$ and $\deg(r) < \deg(m_\alpha)$. Now, we have $f(\alpha) = \ell(\alpha)m_\alpha(\alpha) + r(\alpha) = r(\alpha)$, so we have

$$\begin{aligned} f(\alpha) = 0 &\iff r(\alpha) = 0 \\ &\iff r(y) = 0 \text{ (since } \deg(r) < \deg(m_\alpha)) \\ &\iff m_\alpha(y) \mid f(y). \end{aligned}$$

□

6.2 Computing minimal polynomials

We will show that the roots of the minimal polynomial $m_\alpha(y)$ of α over $\mathrm{GF}(q)$ are precisely the "conjugates" of α over $\mathrm{GF}(q)$. We first prove the following result.

Theorem 6.2.1. Let $\alpha \in \mathrm{GF}(q^m)$. Then $\alpha \in \mathrm{GF}(q)$ if and only if $\alpha^q = \alpha$.

Proof. Since $\beta^q = \beta$ for all $\beta \in \mathrm{GF}(q)$, the elements of $\mathrm{GF}(q)$ are all roots of the polynomial $y^q - y \in \mathrm{GF}(q^m)[y]$. Since this polynomial has degree q, these q elements are in fact the only roots. Thus, we have $\alpha \in \mathrm{GF}(q)$ if and only if $\alpha^q = \alpha$. □

Definition 6.2.2. Let $\alpha \in \mathrm{GF}(q^m)$. Let t be the smallest positive integer such that $\alpha^{q^t} = \alpha$ (and note that $t \leq m$). Then the **set of conjugates with respect to** $\mathrm{GF}(q)$ is

$$C(\alpha) := \{\alpha, \alpha^q, \alpha^{q^2}, \ldots, \alpha^{q^{t-1}}\}.$$

Note that the elements of $C(\alpha)$ are distinct.

Theorem 6.2.3. Let $\alpha \in \mathrm{GF}(q^m)$. The minimal polynomial of α over $\mathrm{GF}(q)$ is given by

$$m(y) = \prod_{\beta \in C(\alpha)} (y - \beta) = (y - \alpha)(y - \alpha^q)(y - \alpha^{q^2}) \cdots (y - \alpha^{q^{t-1}}).$$

Proof. Clearly, $m(y)$ is monic and has α as a root. Now, write $m(y) = \sum_{i=0}^{t} m_i y^i \in \mathrm{GF}(q^m)[y]$. In order to show that $m(y) \in \mathrm{GF}(q)[y]$, we need to verify that $m_i \in \mathrm{GF}(q)$ for all $0 \leq i \leq t$. Indeed, note that

$$m(y)^q = \prod_{\beta \in C(\alpha)} (y - \beta)^q = \prod_{\beta \in C(\alpha)} (y^q - \beta^q) = \prod_{\beta \in C(\alpha)} (y^q - \beta) = m(y^q) = \sum_{i=0}^{t} m_i y^{iq}, \tag{1}$$

where the third equality follows from the fact that $C(\alpha) = \{\beta^q : \beta \in C(\alpha)\}$. Moreover, we have

$$m(y)^q = \left(\sum_{i=0}^{t} m_i y^i\right)^q = \sum_{i=0}^{t} m_i^q y^{iq}. \tag{2}$$

Comparing the coefficients of the y^{iq} in (1) and (2) gives $m_i = m_i^q$ for all $0 \le i \le t$. By Theorem 6.2.1, we get $m_i \in \mathrm{GF}(q)$ for all $0 \le i \le t$, so $m(y) \in \mathrm{GF}(q)[y]$. Finally, suppose that $f(y) \in \mathrm{GF}(q)[y]$ is a non-zero polynomial such that $f(\alpha) = 0$. Write $f(y) = \sum_{i=0}^{d} f_i y^i$. Then

$$f(\alpha^q) = \sum_{i=0}^{d} f_i \alpha^{iq} = \left(\sum_{i=0}^{d} f_i \alpha^i \right)^q = f(\alpha)^q = 0,$$

so $\alpha, \alpha^q, \dots, \alpha^{q^{t-1}}$ are all roots of $f(y)$, which implies that $\deg(f) \ge t = \deg(m)$. Hence, $m(y)$ is the monic polynomial of smallest degree in $\mathrm{GF}(q)[y]$ that has α as a root. □

Example 6.2.4. Let $\mathrm{GF}(2^4) = \mathbb{Z}_2[x]/(x^4 + x + 1)$. Find the minimal polynomial of $\beta = x^2 + x^3$ over $\mathbb{Z}_2$.

Solution. When doing computations by hand, it is useful to have a generator α of $\mathrm{GF}(2^4)^*$ and a table of powers of α. It turns out that $\alpha = x$ is a generator of $\mathrm{GF}(2^4)^*$ with $\beta = \alpha^6$; see the GF(16) handout for the rest of the powers of α. Then we get $C(\beta) = \{\alpha^6, \alpha^{12}, \alpha^9, \alpha^3\}$ where $\alpha^9 = \alpha^{24}$ and $\alpha^3 = \alpha^{48}$. By Theorem 6.2.3, it follows that

$$\begin{aligned} m_\beta(y) &= (y - \alpha^3)(y - \alpha^6)(y - \alpha^9)(y - \alpha^{12}) \\ &= [y^2 - (\alpha^3 + \alpha^6)y + \alpha^9][y^2 - (\alpha^9 + \alpha^{12})y + \alpha^{21}] \\ &= (y^2 + \alpha^2 y + \alpha^9)(y^2 + \alpha^8 y + \alpha^6) \\ &= y^4 + (\alpha^2 + \alpha^8)y^3 + (\alpha^9 + \alpha^{10} + \alpha^6)y^2 + (\alpha^8 + \alpha^2)y + 1 \\ &= y^4 + y^3 + y^2 + y + 1 \end{aligned}$$

is the minimal polynomial of β over $\mathbb{Z}_2$.

6.3 Factoring $x^n - 1$ over $\mathrm{GF}(q)$ [Part 1]

Our goal is to describe the complete factorization of $x^n - 1$ over $\mathrm{GF}(q)$. In doing so, we will see how generator polynomials $g(x)$ can be selected so that we have a useful lower bound on the distance of the cyclic code generated by $g(x)$. Such codes are known as **BCH codes**.

Let p be the characteristic of $\mathrm{GF}(q)$. If $\gcd(n, q) \ne 1$, then we can write $n = n'p^\ell$ where $\ell \ge 1$ and $\gcd(n', p) = 1$. Then we have

$$x^n - 1 = (x^{n'} - 1)^{p^\ell}.$$

In particular, we may assume without loss of generality that $\gcd(n, q) = 1$.

We now introduce some notation.

- Let m be the smallest positive integer such that $q^m \equiv 1 \pmod{n}$.
- Let α be a generator of $\mathrm{GF}(q^m)^*$.
- Let $\beta = \alpha^{(q^m - 1)/n}$.

Note that $\beta \in \mathrm{GF}(q^m)$ with $\mathrm{ord}(\beta) = n$, and the elements $1, \beta, \beta^2, \dots, \beta^{n-1}$ are distinct. Moreover, for all $0 \le i \le n - 1$, we have

$$(\beta^i)^n = (\beta^n)^i = 1^i = 1.$$

Hence, the elements $1, \beta, \beta^2, \dots, \beta^{n-1}$ are roots of $x^n - 1$, and they are in fact the only roots as $x^n - 1$ has degree n. Therefore, we see that

$$x^n - 1 = (x - 1)(x - \beta)(x - \beta^2) \cdots (x - \beta^{n-1})$$

is the complete factorization of $x^n - 1$ over $\mathrm{GF}(q^m)$. However, we are seeking the complete factorization of $x^n - 1$ over $\mathrm{GF}(q)$, not $\mathrm{GF}(q^m)$.

Consider β^i where $0 \leq i \leq n-1$. Since β^i is a root of $x^n - 1$, we have $m_{\beta^i}(x) \mid x^n - 1$. Furthermore, the roots of $m_{\beta^i}(x)$ are

$$C(\beta^i) = \{\beta^i, \beta^{iq}, \beta^{iq^2}, \ldots, \beta^{iq^{t-1}}\}$$

where t is the smallest positive integer such that $iq^t \equiv i \pmod{n}$.

The above discussion motivates the following definition.

Definition 6.3.1. Suppose that $\gcd(n, q) = 1$, and let $0 \leq i \leq n-1$. The **cyclotomic coset of q modulo n containing** i is the set

$$C_i = \{i, iq \pmod{n}, iq^2 \pmod{n}, \ldots, iq^{t-1} \pmod{n}\},$$

where t is the smallest positive integer such that $iq^t \equiv i \pmod{n}$. Moreover, we define

$$C = \{C_i : 0 \leq i \leq n-1\}$$

to be the **set of cyclotomic cosets of q modulo** n.

Example 6.3.2. The cyclotomic cosets of $q = 2$ modulo $n = 15$ are

$$\begin{aligned}
&C_0 = \{0\},\\
&C_1 = \{1, 2, 4, 8\} = C_2 = C_4 = C_8,\\
&C_2 = \{3, 6, 12, 9\} = C_6 = C_{12} = C_9,\\
&C_5 = \{5, 10\} = C_{10},\\
&C_7 = \{7, 14, 13, 11\} = C_{14} = C_{13} = C_{11}.
\end{aligned}$$

Therefore, we have $C = \{C_0, C_1, C_3, C_5, C_7\}$.

Note that if $j \in C_i$, then $C_j = C_i$, as the above example shows. Moreover, observe that

$$m_{\beta^i}(x) = (x - \beta^i)(x - \beta^{iq})(x - \beta^{iq^2}) \cdots (x - \beta^{iq^{t-1}}) = \prod_{j \in C_i} (x - \beta^j)$$

is an irreducible factor of $x^n - 1$ over $\mathrm{GF}(q)$ of degree $|C_i|$. This leads to our main theorem.

Theorem 6.3.3. Suppose that $\gcd(n, q) = 1$.

(1) The number of monic irreducible factors of $x^n - 1$ over $\mathrm{GF}(q)$ is equal to the number of (distinct) cyclotomic cosets of q modulo n.

(2) The number of monic irreducible factors of $x^n - 1$ over $\mathrm{GF}(q)$ of degree d is equal to the number of (distinct) cyclotomic cosets of q modulo n of size d.

6.4 Factoring $x^n - 1$ over $\mathrm{GF}(q)$ [Part 2]

Theorem 6.4.1. Suppose that $\gcd(n, q) = 1$. Let m be the smallest positive integer such that $q^m \equiv 1 \pmod{n}$, and let $\beta \in \mathrm{GF}(q^m)$ be an element of order n. Then the monic irreducible factors of $x^n - 1$ over $\mathrm{GF}(q)$ are $\{m_{\beta^i}(x) : 0 \leq i \leq n-1\}$ where

$$m_{\beta^i}(x) = \prod_{j \in C_i} (x - \beta^j).$$

Note that if $j \in C_i$, then $m_{\beta^j}(x) = m_{\beta^i}(x)$.

Example 6.4.2. Factor $x^{15}-1$ over GF(2).

Solution. In Example 6.3.2, we saw the cyclotomic cosets of $q=2$ modulo $n=15$. In particular, we know that $x^{15}-1$ has five irreducible factors over GF(2); one factor has degree 1, one factor has degree 2, and three factors have degree 4. Let us find these factors.

The smallest positive integer m such that $2^m \equiv 1 \pmod{15}$ is $m=4$. We need an element $\beta \in \mathrm{GF}(q^m)$ of order 15. From Example 6.2.4, we can take $\beta=\alpha$, where α is a generator of $\mathrm{GF}(2^4)=\mathbb{Z}_2[\alpha]/(\alpha^4+\alpha+1)$. Then, we obtain

$$\begin{aligned} m_{\beta^0}(x) &= 1+x, \\ m_{\beta^1}(x) &= 1+x+x^4, \\ m_{\beta^3}(x) &= 1+x+x^2+x^3+x^4, \\ m_{\beta^5}(x) &= 1+x+x^2, \\ m_{\beta^7}(x) &= 1+x^3+x^4. \end{aligned}$$

These can be computed in a similar fashion to Example 6.2.4; in fact, $m_{\beta^3}(x)$ was obtained from there. It follows that the complete factorization of $x^{15}-1$ over GF(2) is

$$x^{15}-1=(1+x)(1+x+x^2)(1+x+x^4)(1+x^3+x^4)(1+x+x^2+x^3+x^4).$$

Example 6.4.3. Determine the number of cyclic subspaces of $V_{90}(\mathbb{Z}_3)$.

Solution. First, observe that $x^{90}-1=(x^{10}-1)^9$. To determine the factorization pattern of $x^{10}-1$ over $\mathbb{Z}_3$, we find the cyclotomic cosets of $q=3$ modulo $n=10$. Indeed, we have

$$\begin{aligned} C_0 &= \{0\}, \\ C_1 &= \{1,3,9,7\}, \\ C_2 &= \{2,4,8,6\}, \\ C_5 &= \{5\}. \end{aligned}$$

Hence, it follows that $x^{90}-1=[f_0(x)\cdot f_1(x)\cdot f_2(x)\cdot f_5(x)]^9$ where $f_0(x),f_1(x),f_2(x),f_5(x)\in\mathbb{Z}_3[x]$ are irreducible over $\mathbb{Z}_3$ with $\deg(f_0)=\deg(f_5)=1$ and $\deg(f_2)=\deg(f_3)=4$. In particular, we have $f_i(x)=m_{\beta^i}(x)$ where $\beta\in\mathrm{GF}(3^4)$ has order 10. Thus, the number of factors of $x^{90}-1$ over $\mathbb{Z}_3$, which is equal to the number of cyclic subspaces of $V_{90}(\mathbb{Z}_3)$, is

$$(9+1)(9+1)(9+1)(9+1)=10^4=10000.$$

6.5 BCH codes

BCH codes were discovered in 1960 by R. C. Bose and D. Ray-Chaudhuri, and independently in 1959 by A. Hocquenghem. They are cyclic codes that are constructed in such a way that a useful lower bound on their distance is known.

Setup. Suppose that $\gcd(n,q)=1$.

- Let m be the smallest positive integer such that $q^m\equiv 1 \pmod{n}$.
- Let α be a generator of $\mathrm{GF}(q^m)^*$, and let $\beta=\alpha^{(q^m-1)/n}$. Recall that $\mathrm{ord}(\beta)=n$.
- Let $m_{\beta^i}(x)$ denote the minimal polynomial of β^i over $\mathrm{GF}(q)$ for all $0\le i\le n-1$, and recall that $m_{\beta^i}(x)\mid x^n-1$.
- For all $i\ge n$, we will set $m_{\beta^i}(x)=m_{\beta^j}(x)$ where $j\equiv i \pmod{n}$ with $0\le j\le n-1$, since $\beta^i=\beta^j$.

Definition 6.5.1. A **BCH code** C over $\mathrm{GF}(q)$ of block length n and designed distance δ is a cyclic code generated by the polynomial

$$g(x) = \mathrm{lcm}\{m_{\beta^i}(x) : a \le i \le a + \delta - 2\}$$

for some integer a.

Remark 6.5.2.

(1) Since $m_{\beta^i}(x) \mid x^n - 1$ for all $a \le i \le a + \delta - 2$, it follows that $g(x) \mid x^n - 1$. Moreover, $g(x)$ is monic, so it is indeed the generator polynomial of a cyclic code of length n over $\mathrm{GF}(q)$.

(2) The $\delta - 1$ consecutive powers of β, namely $\beta^a, \beta^{a+1}, \dots, \beta^{a+\delta-2}$, are roots of $g(x)$.

(3) The **BCH bound** is given by $d(C) \ge \delta$, which we will prove in the following section.

Example 6.5.3. Let $q = 3$ and $n = 13$. Then $m = 13$ is the smallest positive integer such that $3^m \equiv 1 \pmod{13}$. Consider $\mathrm{GF}(3^3) = \mathbb{Z}_3[\alpha]/(\alpha^3 + 2\alpha^2 + 1)$. Then α is a generator of $\mathrm{GF}(3^3)^*$ (see the GF(27) handout). Moreover, $\beta = \alpha^2$ is an element of order 13. The cyclotomic cosets of $q = 3$ modulo $n = 13$ are

$$\begin{aligned} C_0 &= \{0\}, \\ C_1 &= \{1, 3, 9\}, \\ C_2 &= \{2, 6, 5\}, \\ C_4 &= \{4, 12, 10\}, \\ C_7 &= \{7, 8, 11\}. \end{aligned}$$

It follows that

$$\begin{aligned} m_{\beta^0}(x) &= x + 2, \\ m_{\beta^1}(x) &= x^3 + 2x^2 + 2x + 2, \\ m_{\beta^2}(x) &= x^3 + 2x + 2, \\ m_{\beta^4}(x) &= x^3 + x^2 + x + 2, \\ m_{\beta^7}(x) &= x^3 + 2x + 1. \end{aligned}$$

Now, let

$$g(x) = m_{\beta^0}(x) \cdot m_{\beta^1}(x) \cdot m_{\beta^2}(x) = 2 + 2x + x^4 + 2x^5 + x^6 + x^7.$$

The roots of $g(x)$ are $\beta^0, \beta^1, \beta^3, \beta^9, \beta^2, \beta^6, \beta^5$. Since $\beta^0, \beta^1, \beta^2, \beta^3$ are among these roots, then by taking $a = 0$, we see that $g(x)$ generates a $(13, 6)$-BCH code over GF(3) with designed distance $\delta = 5$. By the BCH bound, the distance of this code is at least 5.

Exercise 6.5.4. Show that the polynomial $g(x) = m_{\beta^0}(x) \cdot m_{\beta^4}(x) \cdot m_{\beta^7}(x)$ generates a $(13, 6)$-BCH code over GF(3) of distance at least 5.

6.6 BCH bound

At the end of Section 1, we asked if there exists a block code with parameters $q = 2$, $n = 128$, $M = 2^{64}$, and $d \ge 22$. We can equivalently pose this as a sphere packing problem; namely, can we place $M = 2^{64}$ spheres of radius $e = \lfloor (d-1)/2 \rfloor \ge 10$ in $V_{128}(\mathbb{Z}_2)$ so that no two spheres overlap?

The answer is yes! We will describe an **extended BCH code** with these parameters. First, we will describe a BCH code with parameters $q = 2$, $n = 127$, $k = 64$, and $\delta = 21$.

The smallest integer m such that $2^m \equiv 1 \pmod{127}$ is $m = 7$. The first few cyclotomic cosets of 2 modulo 127 are given by

$$\begin{aligned}
C_0 &= \{0\},\\
C_1 &= \{\mathbf{1}, \mathbf{2}, \mathbf{4}, \mathbf{8}, \mathbf{16}, 32, 64\},\\
C_3 &= \{\mathbf{3}, \mathbf{6}, \mathbf{12}, 24, 48, 96, 65\},\\
C_5 &= \{\mathbf{5}, \mathbf{10}, \mathbf{20}, 40, 80, 33, 66\},\\
C_7 &= \{\mathbf{7}, \mathbf{14}, 28, 56, 112, 97, 67\},\\
C_9 &= \{\mathbf{9}, \mathbf{18}, 36, 72, \mathbf{17}, 34, 68\},\\
C_{11} &= \{\mathbf{11}, 22, 44, 88, 49, 98, 69\},\\
C_{13} &= \{\mathbf{13}, 26, 52, 104, 81, 35, 70\},\\
C_{15} &= \{\mathbf{15}, 30, 60, 120, 113, 99, 71\},\\
C_{19} &= \{\mathbf{19}, 38, 76, 25, 50, 100, 73\}.
\end{aligned}$$

Let β be an element of order 127 in $\mathrm{GF}(2^7)^*$. Then the polynomial

$$g(x) = m_{\beta^1}(x) \cdot m_{\beta^3}(x) \cdot m_{\beta^5}(x) \cdot m_{\beta^7}(x) \cdot m_{\beta^9}(x) \cdot m_{\beta^{11}}(x) \cdot m_{\beta^{13}}(x) \cdot m_{\beta^{15}}(x) \cdot m_{\beta^{19}}(x)$$

is a degree 63 divisor of $x^{127} - 1$ over $\mathrm{GF}(2)$. Moreover, β^i is a root of $g(x)$ for all $1 \le i \le 20$, so $g(x)$ generates a $(127, 64)$-binary BCH code C with designed distance $\delta = 21$. Finally, the extended code of C, obtained by adding a parity bit to each codeword in C, is a $(128, 64)$-binary code with distance ≥ 22.

Definition 6.6.1. A **Vandermonde matrix** over a field F is a matrix of the form

$$A(x_1, x_2, \ldots, x_t) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_t \\ x_1^2 & x_2^2 & \cdots & x_t^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{t-1} & x_2^{t-1} & \cdots & x_t^{t-1} \end{bmatrix}_{t \times t}$$

where $x_1, x_2, \ldots, x_t \in F$.

Theorem 6.6.2. A Vandermonde matrix $A(x_1, x_2, \ldots, x_t)$ over F is non-singular if and only if the elements $x_1, x_2, \ldots, x_t \in F$ are distinct.

Proof. Perform the row operations

$$\begin{gathered}
R_t \leftarrow R_t - x_1 R_{t-1}\\
R_{t-1} \leftarrow R_{t-1} - x_1 R_{t-2}\\
\vdots\\
R_2 \leftarrow R_2 - x_1 R_1
\end{gathered}$$

to A to obtain the matrix

$$A_1 = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & x_2 - x_1 & x_3 - x_1 & \cdots & x_t - x_1 \\ 0 & x_2^2 - x_1 x_1 & x_3^2 - x_1 x_3 & \cdots & x_t^2 - x_1 x_t \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & x_2^{t-1} - x_1 x_2^{t-1} & x_3^{t-1} - x_1 x_3^{t-2} & \cdots & x_t^{t-1} - x_1 x_t^{t-2} \end{bmatrix}.$$

Now, we can compute $\det(A_1)$ by expanding along the first column to get

$$\det(A) = \det(A_1) = (x_2 - x_1)(x_3 - x_1)\cdots(x_t - x_1)\cdot \det \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_2 & x_3 & \cdots & x_t \\ x_2^2 & x_3^2 & \cdots & x_t^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_2^{t-2} & x_3^{t-2} & \cdots & x_t^{t-2} \end{bmatrix}.$$

By induction, it follows that

$$\det(A) = \prod_{1\le i<j\le t} (x_j - x_i).$$

In particular, we have $\det(A) \neq 0$ if and only if the elements $x_1, x_2, \ldots, x_t \in F$ are distinct. □

THEOREM 6.6.3 (BCH bound). Let C be an (n,k)-BCH code over $\mathrm{GF}(q)$ with designed distance δ. Then $d(C) \geq \delta$.

PROOF. Let $g(x)$ be the generator polynomial for C. For simplicity, suppose $a = 1$ so that $\beta, \beta^2, \ldots, \beta^{\delta-1}$ are the roots of $g(x)$ where $\beta \in \mathrm{GF}(q^m)$ is an element of order n and m is the smallest positive integer such that $q^m \equiv 1 \pmod{n}$. That is, we have $g(x) = \mathrm{lcm}\{m_{\beta^i}(x) : 1 \le i \le \delta - 1\}$. Now, let $r \in V_n(\mathrm{GF}(q))$. Then

$$\begin{aligned} r \in C &\iff g(x) \mid r(x) \\ &\iff m_{\beta^i}(x) \mid r(x) \text{ for all } 1 \le i \le \delta - 1 \\ &\iff r(\beta^i) = 0 \text{ for all } 1 \le i \le \delta - 1. \end{aligned}$$

Now, let

$$H_1 = \begin{bmatrix} 1 & \beta & \beta^2 & \cdots & \beta^{n-1} \\ 1 & \beta^2 & (\beta^2)^2 & \cdots & (\beta^2)^{n-1} \\ 1 & \beta^3 & (\beta^3)^2 & \cdots & (\beta^3)^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \beta^{\delta-1} & (\beta^{\delta-1})^2 & \cdots & (\beta^{\delta-1})^{n-1} \end{bmatrix}_{(\delta-1)\times n}.$$

Note that $r \in C$ if and only if $H_1 r^T = 0$. Moreover, no $t = \delta - 1$ columns of H_1 are linearly dependent over $\mathrm{GF}(q^m)$. Indeed, we have

$$\begin{aligned} \det \begin{bmatrix} \beta^{i_1} & \beta^{i_2} & \cdots & \beta^{i_t} \\ (\beta^2)^{i_1} & (\beta^2)^{i_2} & \cdots & (\beta^2)^{i_t} \\ \vdots & \vdots & \ddots & \vdots \\ (\beta^{\delta-1})^{i_1} & (\beta^{\delta-1})^{i_2} & \cdots & (\beta^{\delta-1})^{i_t} \end{bmatrix}_{t\times t} &= \beta^{i_1}\beta^{i_2}\cdots\beta^{i_t}\cdot\det \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \beta^{i_1} & \beta^{i_2} & \cdots & \beta^{i_t} \\ \vdots & \vdots & \ddots & \vdots \\ (\beta^{i_1})^{\delta-2} & (\beta^{i_2})^{\delta-2} & \cdots & (\beta^{i_t})^{\delta-2} \end{bmatrix} \\ &= \prod_{j=1}^{t} \beta^{i_j}\cdot \det(A(\beta^{i_1}, \beta^{i_2}, \ldots, \beta^{i_t})) \neq 0, \end{aligned}$$

where the inequality follows from Theorem 6.6.2 as the elements $\beta^{i_1}, \beta^{i_2}, \ldots, \beta^{i_t}$ are distinct. Since $\mathrm{GF}(q) \subseteq \mathrm{GF}(q^m)$, we also have that no $\delta - 1$ columns of H_1 are linearly dependent over $\mathrm{GF}(q)$.

Now, if $c \in C$ is a non-zero codeword with $w(c) < \delta$, then $H_1 c^T = 0$ implies that we can write 0 as a non-trivial linear combination of $\delta - 1$ or fewer columns of H_1, contradicting what we just proved. Thus, every non-zero codeword in C has weight $\geq \delta$, so $d(C) \geq \delta$. □

6.7 BCH decoding [Part 1]

Over the years, many efficient algorithms have been designed for decoding BCH codes. For now, we will present a decoding algorithm for a specific BCH code called C_{15}. This decoding algorithm captures the essential ideas for a decoding algorithm for general BCH codes.

First, we define the BCH code C_{15}. Let $\mathrm{GF}(2^4) = \mathbb{Z}_2[\alpha]/(\alpha^4 + \alpha + 1)$. Then α is a generator of $\mathrm{GF}(2^4)^*$, and $\beta = \alpha$ has order 15. Define the polynomial

$$g(x) = m_\beta(x) \cdot m_{\beta^3}(x) = (x^4 + x + 1)(x^4 + x^3 + x^2 + x + 1) = 1 + x^4 + x^6 + x^7 + x^8.$$

The roots of $g(x)$ include $\beta, \beta^2, \beta^3, \beta^4$, so $g(x)$ generates a $(15, 7)$-BCH code over $\mathbb{Z}_2$ with designed distance $\delta = 5$, which we call C_{15}. In fact, since $g(x)$ is a codeword of weight 5, we have $d(C_{15}) = 5$. Note that C_{15} is a 2-error correcting code.

Parity-check matrix. We now find a parity-check matrix for C_{15}. Let $r \in V_{15}(\mathbb{Z}_2)$. Then we have

$$\begin{aligned} r \in C_{15} &\iff g(x) \mid r(x) \\ &\iff m_\beta(x) \mid r(x) \text{ and } m_{\beta^3}(x) \mid r(x) \\ &\iff r(\beta) = 0 \text{ and } r(\beta^3) = 0. \end{aligned}$$

Thus, a parity-check matrix for C_{15} is given by

$$H = \begin{bmatrix} \beta^0 & \beta^1 & \beta^2 & \cdots & \beta^{14} \\ (\beta^3)^0 & (\beta^3)^1 & (\beta^3)^2 & \cdots & (\beta^3)^{14} \end{bmatrix}_{8\times 15}.$$

Note that H is a 2×15 matrix over $\mathrm{GF}(2^4)$, and we can convert it into an 8×15 matrix over $\mathbb{Z}_2$ by replacing each element in the matrix by its vector representation over $\mathbb{Z}_2$.

Syndromes. The syndrome of $r \in V_{15}(\mathbb{Z}_2)$ is

$$Hr^T = \begin{bmatrix} r(\beta) \\ r(\beta^3) \end{bmatrix} =: \begin{bmatrix} s_1 \\ s_3 \end{bmatrix}.$$

In this case, we do not require the parity-check matrix H to compute syndromes; we only have to compute $s_1 = r(\beta)$ and $s_3 = r(\beta^3)$.

Decoding strategy. If there is an error vector e of weight ≤ 2 that has the same syndrome (s_1, s_3) as r, then we decode r to $c = r - e$. Otherwise, we reject r.

6.8 BCH decoding [Part 2]

We now present a decoding algorithm for C_{15} with justification.

(1) Let $r \in V_{15}(\mathbb{Z}_2)$ be the received word.

(2) Compute $s_1 = r(\beta)$ and $s_3 = r(\beta^3)$.

(3) If $s_1 = s_3 = 0$, then accept r and stop.

(4) Suppose that $e(x) = x^i$; that is, exactly one error has occurred in position i, where $0 \leq i \leq 14$. Then $s_1 = r(\beta) = e(\beta) = \beta^i$ and $s_3 = r(\beta^3) = e(\beta^3) = \beta^{3i}$ so that $s_3 = s_1^3$.

If $s_1^3 = s_3$, then correct r in position i where $s_1 = \beta^i$, then stop.

(5) If $r(\beta^3) = e(\beta^3) \neq 0$, we have $e(x) \neq 0$. Moreover, if $s_1 = r(\beta) = 0$, then $e(\beta) = 0$ so that $m_\beta(x) \mid e(x)$. Hence, we have $w(e) \geq 3$, since the BCH code generated by $m_\beta(x)$ has designed distance $\delta \geq 3$.

If $s_1 = 0$ and $s_3 \neq 0$, then reject r and stop.

(6) If exactly two errors have occurred, say in positions i and j where $0 \leq i \neq j \leq 14$, then $e(x) = x^i + x^j$. Then we have $s_1 = r(\beta) = e(\beta) = \beta^i + \beta^j$ and

$$\begin{aligned} s_3 = r(\beta^3) = e(\beta^3) &= \beta^{3i} + \beta^{3j} \\ &= (\beta^i + \beta^j)(\beta^{2i} + \beta^{i+j} + \beta^{2j}) \\ &= (\beta^i + \beta^j)((\beta^i + \beta^j)^2 + \beta^{i+j}) \\ &= s_1(s_1^2 + \beta^{i+j}). \end{aligned}$$

In particular, we have $\beta^i + \beta^j = s_3/s_1 + s_1^2$, so β^i and β^j are both roots of the polynomial $z^2 + (\beta^i + \beta^j)z + \beta^{i+j} = z^2 + s_1 z + (s_3/s_1 + s_1^2)$.

Form the **error locator polynomial**

$$\sigma(z) = z^2 + s_1 z + \left(\frac{s_3}{s_1} + s_1^2\right)$$

and find its roots, if any, in $\mathrm{GF}(2^4)$. If there are two roots of $\sigma(z)$ given by β^i and β^j, then correct r in positions i and j, and stop.

(7) Reject r.

We summarize our algorithm below.

Algorithm 6.8.1.

(1) Let $r \in V_{15}(\mathbb{Z}_2)$ be the received word.

(2) Compute $s_1 = r(\beta)$ and $s_3 = r(\beta^3)$.

(3) If $s_1 = s_3 = 0$, then accept r and stop.

(4) If $s_1^3 = s_3$, then correct r in position i where $s_1 = \beta^i$, then stop.

(5) If $s_1 = 0$ and $s_3 \neq 0$, then reject r and stop.

(6) Form the **error locator polynomial**

$$\sigma(z) = z^2 + s_1 z + \left(\frac{s_3}{s_1} + s_1^2\right)$$

and find its roots in $\mathrm{GF}(2^4)$. If there are two distinct roots of $\sigma(z)$ given by β^i and β^j, then correct r in positions i and j, and stop.

(7) Reject r.

This algorithm is guaranteed to make the correct decision if $w(e) \leq 2$.

Example 6.8.2. Let $r = 10001\,00110\,00000 = 1 + x^4 + x^7 + x^8$ be the received word. Then we have

$$\begin{aligned} s_1 &= r(\beta) = 1 + \beta^4 + \beta^7 + \beta^8 = \beta + \beta^{11} = \beta^6, \\ s_3 &= r(\beta^3) = 1 + \beta^{12} + \beta^6 + \beta^9 + \beta^3. \end{aligned}$$

Observe that $s_1^3 = (\beta^6)^3 = \beta^{18} = \beta^3 = s_3$, so one error has occurred in position 6. Thus, we decode r to $c = 10001\,0\mathbf{1}110\,00000$.

Example 6.8.3. Let $r = 00111\,01110\,00000 = x^2 + x^3 + x^4 + x^6 + x^7 + x^8$ be the received word. We have

$$\begin{aligned} s_1 &= r(\beta) = \beta^2 + \beta^3 + \beta^4 + \beta^6 + \beta^7 + \beta^8 = \beta^{13}, \\ s_3 &= r(\beta^3) = \beta^6 + \beta^9 + \beta^{12} + \beta^3 + \beta^6 + \beta^9 = \beta^{10}. \end{aligned}$$

Notice that $s_1^3 = \beta^{39} = \beta^9 \neq s_3$. Now, the error locator polynomial is given by

$$\sigma(z) = z^2 + s_1 z + \left(\frac{s_3}{s_1} + s_1^2\right) = z^2 + \beta^{13} z + (\beta^{12} + \beta^{11}) = z^2 + \beta^{13} z + 1.$$

Suppose that the roots of $\sigma(z)$ are β^i and β^j. Then $\beta^i \cdot \beta^j = \beta^{i+j} = 1 = \beta^0$, so we see that $i + j \equiv 0 \pmod{15}$. Therefore, we check if $\beta^i + \beta^j = \beta^{13}$ for the pairs

$$(i, j) \in \{(1, 14), (2, 13), (3, 12), (4, 11), (5, 10), (6, 9), (7, 8)\}.$$

It turns out that $\beta^4 + \beta^{11} = \beta^{13}$, so two errors have occurred at positions 4 and 11. Finally, we decode r to $c = 0011\mathbf{0}\,01110\,0\mathbf{1}000$.

The general case. Suppose that C is an (n, k)-BCH code over $\mathbb{Z}_2$ with designed distance δ. Suppose the generator polynomial of C is

$$g(x) = \text{lcm}\{m_{\beta^i}(x) : 1 \leq i \leq \delta - 1\},$$

where $\beta \in \text{GF}(2^m)$ has order n. Then we have $d(C) \geq \delta$. Let $t = \lfloor(\delta - 1)/2\rfloor$.

Suppose that $c \in C$ is transmitted where $w(e) \leq t$, and r is received. Compute $s_i = r(\beta^i)$ for all $1 \leq i \leq \delta - 1$, and form the **syndrome polynomial**

$$S(z) = s_1 + s_2 z + s_3 z^2 + \cdots + s_{\delta-1} z^{\delta-2}.$$

From the syndrome polynomial $S(z)$, the error locator polynomial $\sigma(z)$ can be efficiently computed. The roots of $\sigma(z)$ are given by β^{-j}, where j denotes an error position.

Moreover, this algorithm generalizes to BCH codes over $\text{GF}(q)$.

7 Reed-Solomon codes

Reed-Solomon codes were invented by Irving Reed and Gustave Solomon in 1960.

Definition 7.1. A **Reed-Solomon (RS) code** is a BCH code of length n over $\mathrm{GF}(q)$ such that $n \mid q-1$.

Example 7.2. Let $q = 2^4$ and consider $\mathrm{GF}(2^4) = \mathbb{Z}_2[\alpha]/(\alpha^4+\alpha+1)$. Recall that α is a generator of $\mathrm{GF}(2^4)^*$. Let $\beta = \alpha^3$, and note that $\mathrm{ord}(\beta) = 5$ (so that $n = 5$). Let

$$\begin{aligned} g(x) &= \mathrm{lcm}\{m_\beta(x), m_{\beta^2}(x), m_{\beta^3}(x)\} \\ &= (x-\beta)(x-\beta^2)(x-\beta^3) \\ &= x^3+\alpha^{11}x^2+\alpha^2x+\alpha^3. \end{aligned}$$

Then $g(x)$ generates a $(5,2)$-RS code C over $\mathrm{GF}(2^4)$ with designed distance $\delta = 4$. In fact, we have $d(C) = 4$ since $g(x)$ is a codeword of weight 4. A generator matrix for C is given by

$$G = \begin{bmatrix} \alpha^3 & \alpha^2 & \alpha^{11} & 1 & 0 \\ 0 & \alpha^3 & \alpha^2 & \alpha^{11} & 1 \end{bmatrix}_{2\times 5}.$$

Consider the code C' obtained by C by replacing each symbol in the codewords of C by their binary vector representation. For instance, we set

$$(\alpha^3, \alpha^2, \alpha^{11}, 1, 0) \leftrightarrow (0001, 0010, 0111, 1000, 0000).$$

It is not difficult to see that C' is closed under addition and scalar multiplication over $\mathbb{Z}_2$. Thus, C' is a $(20,8)$-binary code.

More generally, suppose that $n \mid q-1$, and let $\beta \in \mathrm{GF}(q)$ be an element of order n. Then $m_{\beta^i}(x) = x - \beta^i$ for all $0 \le i \le n-1$. Hence, an RS code C of length n over $\mathrm{GF}(q)$ with designed distance δ is a BCH code over $\mathrm{GF}(q)$ with generator polynomial

$$g(x) = (x-\beta^a)(x-\beta^{a+1})(x-\beta^{a+2})\cdots(x-\beta^{a+\delta-2})$$

for some positive integer a. Since $\deg(g) = \delta - 1$, we have $w(g) \le \delta$, so $d(C) \le \delta$. On the other hand, the BCH bound gives $d(C) \ge \delta$, so $d(C) = \delta$.

Moreover, since $\dim(C) = k = n - \deg(g) = n - \delta + 1$, we have $k = n - d + 1$, and rearranging gives $d = n - k + 1$. Recall that $d \le n - k + 1$ for any (n,k,d)-code. Thus, RS codes are optimal in the sense that they achieve maximum distance among all (n,k)-codes over $\mathrm{GF}(q)$ for any fixed n, k, and q.

We note that RS codes have good cyclic burst error correcting capability. Indeed, let C be an RS code of length n over $\mathrm{GF}(2^4)$ with designed distance δ. Consider any codeword $c = (c_1, \ldots, c_n) \in C$, and note that each $c_i \in \mathrm{GF}(2^r)$. Let $e = \lfloor (d-1)/2 \rfloor = \lfloor (n-k)/2 \rfloor$.

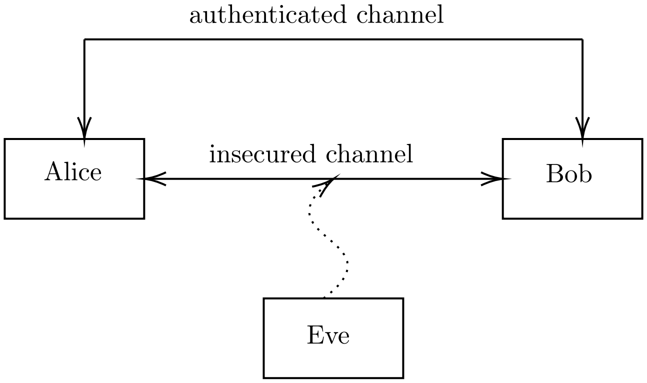
By identifying each c_i as a binary vector of length r, we can view c as a binary vector of length nr. Now, if c is transmitted and a cyclic burst error of length $\le 1 + (e-1)r$ is introduced, then at most e $\mathrm{GF}(2^r)$-symbols of c are received incorrectly. Thus, the received word can be decoded correctly.

Theorem 7.3. Let C be an (n,k)-RS code over $\mathrm{GF}(2^r)$. Let C' be the code obtained by replacing each symbol in the codewords of C by their r-bit binary representations. Then C' is an (nr, kr)-binary code with cyclic burst error correcting capability $t = 1 + (\lfloor (n-k)/2 \rfloor - 1)r$.

Example 7.4. Consider $\mathrm{GF}(2^8) = \mathbb{Z}_2[\alpha]/(\alpha^8+\alpha^4+\alpha^3+\alpha^2+1)$. Then $\beta = \alpha$ has order 255 (so that $q = 256$ and $n = 255$). Let $g(x) = \prod_{i=1}^{24}(x-\beta^i)$. Then $g(x)$ is the generator polynomial for a $(255, 231, 25)$-RS code C with error correcting capability $e = 12$. The corresponding code C' is then a $(2040, 1848)$-binary code with cyclic burst error correcting capability $t = 89$. This code C, and others derived from it, have been widely used in practice, such as in DVDs, CDs, and QR codes.

8 Code-based cryptography

Public-key encryption. The goal of public-key encryption is confidentiality when communicating over an insecured channel. The main feature of public-key encryption is that the two communicating parties do not share any secrets; they only share public information which has been authenticated.



Recall the basic RSA encryption scheme, which was covered in MATH 135.

- **Key generation.** Alice does the following:
  1. Randomlly select two large primes p and q.
  2. Compute $n = pq$ and $\Phi(n) = (p-1)(q-1)$.
  3. Select an arbitrary e such that $1 < e < \Phi(n)$ and $\gcd(e, \Phi(n)) = 1$.
  4. Compute $d = e^{-1} \pmod{\Phi(n)}$.
  5. Alice's public key is (n, e); her private key is d.
- **Encryption.** To encrypt the message to Alice, Bob does the following:
  1. Obtain an authentic copy of Alice's public key (n, e).
  2. Represent the message m as an integer from the interval $[0, n-1]$.
  3. Compute $c = m^e \pmod{n}$.
  4. Send c to Alice over the insecured channel.
- **Decryption.** To decrypt c, Alice simply computes $m = c^d \pmod{n}$.

The threat of quantum computers. The security of the RSA encryption scheme is based on the hardness of factoring n. However, it has been known since 1994 that factoring n is easy on a quantum computer. Elliptic curve cryptography, a widely used alternative to RSA, can also be broken easily by quantum computers. Fortunately, we are still far away from being able to build large-scale quantum computers. Nonetheless, it seems prudent to develop public-key encryption schemes that resist attacks even by quantum computers.

McEliece public-key encryption scheme (1978). The security of the McEliece public-key encryption scheme is based on the fact that decoding a random (binary) linear code is **NP**-hard. We give the basic idea in the next paragraph, and a high-level description of the encryption scheme following that.

We first select a code C for which an efficient decoding algorithm is known. We disguise C to get a "random looking" code $\hat{C}$. The code $\hat{C}$ is the public key, and the "disguising factor" is the private key. To encrypt the message, encode m to obtain a codeword $\hat{c} \in \hat{C}$, add a random error e to $\hat{c}$ to get $\hat{r}$, and send $\hat{r}$. Finally, to decrypt the message, use the "disguising factor" to convert the decoding problem into one for C, and use the decoding algorithm for C to recover e and m.

- **Key generation.** Alice does the following:
  1. Select a $k \times n$ generator matrix G for a t-error correcting binary Goppa code C.
  2. Select a random $k \times k$ invertible matrix S.
  3. Select a random $n \times n$ permutation matrix P.
  4. Compute $\hat{G} = SGP$, which will be a $k \times n$ matrix of rank k.
  5. Alice's public key is $(\hat{G}, t)$; her private key is (G, S, P).

  It is conjectured that $\hat{G}$ is indistinguishable from a random $k \times n$ binary matrix of rank k.

- **Encryption.** To encrypt a message for Alice, Bob does the following:
  1. Obtain an authentic copy of Alice's public key $(\hat{G}, t)$.
  2. Represent the message as a binary vector m of length k.
  3. Select a random binary vector $e \in V_n(\mathbb{Z}_2)$ of weight t.
  4. Compute $\hat{r} = m\hat{G} + e$, and send $\hat{r}$ to Alice over the insecured channel.

- **Decryption.** To decrypt $\hat{r}$, Alice does the following:
  1. Compute $r = \hat{r}P^{-1}$. We note that

     $$r = \hat{r}P^{-1} = m\hat{G}P^{-1} + eP^{-1} = (mSGP)P^{-1} + eP^{-1} = (mS)G + eP^{-1},$$

     and since P is a permutation matrix, it follows that eP^{-1} has weight t.
  2. Use the decoding algorithm for C to recover $m' = mS$.
  3. Compute $m = m'S^{-1}$.

  The security is based on the hardness of decoding $\hat{C}$, which is the code generated by $\hat{G}$.

To implement the McEliece public-key encryption scheme, some suggested parameters are $n = 4096$, $k = 3496$, and $t = 50$. Encryption is known to be very fast, and decryption is relatively fast. Moreover, this encryption scheme appears to resist quantum attacks.

Proposals that replace Goppa codes with Reed-Solomon codes, LDPC codes, and convolutional codes have all been broken. One secure alternative is to use quasi-cyclic MDPC codes.