Zad 3.

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Zadanie 3. Skonstruuj rozwiązanie następujących zagadnień metodą rozdzielania zmiennych:

a) $u_x = u_y$ dla $x, y \in \mathbb{R}$, $u(0, y) = e^y + e^{-2y}$;

b) $u_t = u_{xx} + u$ dla $x \in (0, 1)$, $t > 0$ oraz $u(x, 0) = \sin \pi x$, $u(0, t) = u(1, t) = 0$.

a) $u_x = u_y \qquad x, y \in \mathbb{R} \qquad u(0,y) = e^y + e^{-2y}$

$$u(x,y) = X(x)Y(y)$$

$$u_x = X'(x)Y(y) \qquad u_y = X(x)Y'(y)$$

$$\frac{X'(x)}{X(x)} = \frac{Y'(y)}{Y(y)} = \lambda$$

$$X(x) = c_1 e^{\lambda x} \qquad Y(y) = c_2 e^{\lambda y}$$

$$u(x,y) = c_1 c_2 e^{\lambda(x+y)} = c e^{\lambda(x+y)}$$

$$u(0,y) = c e^{\lambda y} = e^y + e^{-2y}$$

b) $u_t = u_{xx} + u \qquad u(x,0) = \sin \pi x$

$u(0,t) = u(1,t) = 0$

$$u(x,t) = X(x)T(t) \qquad u_t = X(x)T'(t)$$

$$u_{xx} = X''(x)T(t)$$

$$X(x)T'(t) = X''(x)T(t) + X(x)T(t)$$

$$\frac{T'(t)}{T(t)} = \frac{X''(x) + X(x)}{X(x)} = \lambda$$

$$T(t) = c_1 e^{\lambda t}$$

$$\frac{X''(x) + X(x)}{X(x)} = \lambda \to X''(x) = (\lambda - 1)X(x)$$

$$X(x) = c_2 \sin(\sqrt{\lambda - 1}\,x) + c_3 \cos(\sqrt{\lambda - 1}\,x)$$

$$u(x,t) = e^{\lambda t}\left[\alpha_1 \sin(\sqrt{\lambda - 1}\,x) + \alpha_2 \cos(\sqrt{\lambda - 1}\,x)\right]$$

$$u(0,t) = e^{\lambda t} \cdot \alpha_2 \rightsquigarrow \alpha_2 = 0$$

[illegible]

$$u(0,t) = e^{\lambda t} \cdot \alpha_2 \rightsquigarrow \alpha_2 = 0$$

$$u(1,t) = e^{\lambda t} \cdot \alpha_1 \sin\left(\sqrt{\lambda - 1}\right) \rightsquigarrow \sqrt{\lambda - 1} = k\pi, \quad k \in \mathbb{Z}$$

$$\lambda - 1 = k^2 \pi^2$$

$$\lambda = k^2 \pi^2 + 1$$

$$u_k(x,t) = e^{(k^2\pi^2+1)t} \cdot \alpha_k \sin(k\pi x)$$

$$\rightsquigarrow u(x,t) = \sum_{k \in \mathbb{Z}} e^{(k^2\pi^2+1)t} \alpha_k \sin(k\pi x)$$

$$u(x,0) = u_0 = \sin \pi x$$