Zad 1.

środa, 19 kwietnia 2023 15:32

Zadanie 1. Wyprowadź wzór na n-tą iterację Picarda $y_n(x)$ i oblicz jej granicę gdy $n \to \infty$ dla podanych zagadnień Cauchy'ego:

a) $y' = -y \quad y(0) = 1,$ b) $y' = 2yt \quad y(0) = 1,$ c) $y' = -y^2 \quad y(0) = 0.$

a) $f(t,y) = -y(t)$

$$y_0 \equiv 1$$

$$y_1 = 1 + \int_0^t f(s, y_0(s))\, ds =$$

$$= 1 + \int_0^t -1\, ds = 1 - t$$

$$y_2 = 1 + \int_0^t f(s, y_1(s))\, ds =$$

$$= 1 + \int_0^t -1 + s\, ds = 1 - t + \frac{t^2}{2}$$

$$y_n = 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \dots \pm \frac{t^n}{n!}$$

$$y_n \xrightarrow[n\to\infty]{} e^{-t}$$

b) $f(t,y) = 2yt$

$$y_0 \equiv 1$$

$$y_1 = 1 + \int_0^t f(s, y_0(s))\, ds =$$

$$= 1 + \int_0^t 2y_0(s) s\, ds = 1 + \int_0^t 2s\, ds =$$

$$= 1 + s^2$$

$$y_2 = 1 + \int_0^t f(s, y_1(s))\, ds =$$

$$= 1 + \int_0^t 2y_1(s) s\, ds = 1 + \int_0^t 2s + 2s^3 =$$

$$= 1 + s^2 + \frac{s^4}{2}$$

$$= 1 + s^2 + \frac{s^4}{2}$$

$$y_3 = 1 + \int_0^t 2s\left(1 + s^2 + \frac{s^4}{2}\right) ds =$$

$$= 1 + \int_0^t 2s + 2s^3 + s^5 \, ds =$$

$$= 1 + s^2 + \frac{s^4}{2} + \frac{s^6}{6}$$

$$y_4 = 1 + \int_0^t 2s + 2s^3 + s^5 + \frac{1}{3}s^7 \, ds =$$

$$= 1 + t^2 + \frac{t^4}{2} + \frac{t^6}{6} + \frac{t^8}{24}$$

$$y_n(t) = \sum_{k=0}^{n} t^{2k} \cdot \frac{1}{k!} \xrightarrow[n \to \infty]{} e^{t^2}$$

c) c) $y' = -y^2 \quad y(0) = 0.$ $\qquad f(t, y) = -y^2$

$$y_0 \equiv 0$$

$$y_1 = \int_0^t -y_0^2 \, ds = 0$$

$$y_n = 0 \xrightarrow[n \to \infty]{} 0$$