Zad 3

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M6.3. $\boxed{\text{1 punkt}}$ Określmy wielomian $H_{2n+1} \in \Pi_n$ za pomocą wzoru

$$H_{2n+1}(x) = \sum_{k=0}^{n} f(x_k)\, h_k(x) + \sum_{k=0}^{n} f'(x_k)\, \bar{h}_k(x),$$

gdzie węzły $x_0, \dots, x_n$ są parami różne, ponadto

$$\left.\begin{aligned} h_k(x) &:= [1 - 2(x - x_k)\lambda'_k(x_k)]\lambda_k^2(x), \\ \bar{h}_k(x) &:= (x - x_k)\lambda_k^2(x), \\ \lambda_k(x) &:= \frac{p_{n+1}(x)}{(x - x_k)p'_{n+1}(x_k)}, \end{aligned}\right\} \quad (0 \leqslant k \leqslant n)$$

oraz $p_{n+1}(x) = (x - x_0)(x - x_1)\cdots(x - x_n)$. Wykazać, że H_{2n+1} spełnia warunki

(2) $$H_{2n+1}(x_i) = f(x_i), \quad H'_{2n+1}(x_i) = f'(x_i) \qquad (0 \leqslant i \leqslant n).$$

$$H_{2n+1}(x) = \sum_{k=0}^{n} f(x_k) h_k(x) + \sum_{k=0}^{n} f'(x_k) \bar{h}_k(x)$$

$$H'_{2n+1}(x) = \sum_{k=0}^{n} f(x_k) h'_k(x) + \sum_{k=0}^{n} f'(x_k) \bar{h}'_k(x)$$

chcemy, żeby

1. $\bar{h}_k(x_i) = 0$
2. $h_k(x_i) = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}$ dla $i \in \{0, \dots, n\}$

3. $h'_k(x_i) = 0$
4. $\bar{h}'_k(x_i) = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}$

pomocniczo

$$h_k(x) := [1 - 2(x - x_k)\lambda'_k(x_k)]\lambda_k^2(x),$$
$$\bar{h}_k(x) := (x - x_k)\lambda_k^2(x),$$
$$\lambda_k(x) := \frac{p_{n+1}(x)}{(x - x_k)p'_{n+1}(x_k)},$$

$$h'_k(x) = 2\lambda_k(x)\lambda'_k(x)\left[1 - 2(x - x_k)\lambda'_k(x_k)\right] - 2\lambda_k^2(x)\lambda'_k(x_k)$$

$$-2\lambda_k^2(x)\lambda_k'(x_k)$$

$$\bar{h}_k'(x) = 2\lambda_k(x)\lambda_k'(x)(x-x_k) + \lambda_k^2(x)$$

$$\lambda_k'(x) = \frac{p_{n+1}'(x)(x-x_k)p_{n+1}'(x_k) - p_{n+1}(x)\,p_{n+1}'(x_k)}{(x-x_k)^2(p_{n+1}'(x_k))^2} =$$

$$= \frac{p_{n+1}'(x)(x-x_k) - p_{n+1}(x)}{(x-x_k)\,p_{n+1}'(x_k)}$$

ad. 1

$$\bar{h}_k(x_i) = \underset{0 \text{ dla } i=k}{\underset{\shortparallel}{(x_i - x_k)}}\ \underset{0 \text{ dla } i\neq k}{\underset{\shortparallel}{\lambda_k^2(x_i)}} = 0 \qquad i = 0, \ldots, n \qquad \boxed{\checkmark}$$

ad. 2

$h_k(x) := [1 - 2(x - x_k)\lambda'_k(x_k)]\lambda_k^2(x),$

$i=k$

$$h_k(x_k) = [1-2(x_n-x_n)\lambda_k'(x_n)]\lambda_k^2(x_n) =$$

$$= \lambda_k^2(x_n) = \frac{\prod_{j\neq k}(x-x_j)}{p_{n+1}'(x_k)} = \frac{\prod_{j\neq k}(x-x_j)}{\prod_{j\neq k}(x-x_j)} = 1$$

$i\neq k$

$$h_k(x_i) = [1-2(x_i-x_n)\lambda_k'(x_n)]\underset{0}{\underset{\shortparallel}{\lambda_k^2(x_i)}} = 0 \qquad \boxed{\checkmark}$$

ad. 3

$$(x - x_k)p_{n+1}(x_k)$$

$$h_k'(x) = 2\ell_k(x)\ell_k'(x)\left[1 - 2(x - x_k)\ell_k'(x_k)\right] - 2\ell_k^2(x)\ell_k'(x_k)$$

$$\frac{p_{n+1}'(x)(x - x_k) - p_{n+1}(x)}{(x - x_k)\,p_{n+1}'(x_k)} = \ell_k'(x) = \frac{p_{n+1}'(x) - \prod_{j \neq k}(x - x_j)}{p_{n+1}'(x_n)}$$

$$h_k'(x_i) = 2\ell_k(x_i)\ell_k'(x_i)\left[1 - 2(x_i - x_n)\ell_k'(x_k)\right] - 2\ell_k^2(x_i)\ell_k'(x_n)$$

$i \neq k$ wtedy $\ell_k(x_i) = 0$ i $\bar{h}_k(x_i) = 0$

$i = k$ wtedy

$$h_k'(x_i) = 2\ell_k(x_n)\ell_k'(x_n)\Big[\underbrace{1 - \ell_n(x_n)}_{\overset{\shortparallel}{0}}\Big] = 0 \quad \boxed{\checkmark}$$

(liczone wyżej)

ad. 4

$$\bar{h}_k'(x) = 2\ell_k(x)\ell_k'(x)(x - x_k) + \ell_k^2(x)$$

$$\bar{h}_k'(x_n) = 2\ell_k(x_n)\ell_n'(x_n)(x_n - x_n) + \ell_n^2(x_n) = 0 + 1^2 = 1$$

$$\bar{h}_n'(x_i) = 2\ell_n(x_i)\ell_n'(x_i)(x_i - x_n) + \ell_n^2(x_i) =$$

$i \neq k$ $$= 0 + 0 = 0 \quad \boxed{\checkmark}$$

$\square$