Zad 14.

poniedziałek, 5 grudnia 2022 23:50

14. Wylicz funkcje tworzące ciągów

(a) $a_n = n^2$ (b) $a_n = n^3$ (c) $a_n = \binom{n+k}{k}$

a) $a_n = n^2$

$$A(x) = \sum_{n=0}^{\infty} n^2 x^n = x \sum_{n=0}^{\infty} n \cdot n x^{n-1} =$$

$$= x \sum_{n=0}^{\infty} n \cdot \frac{d}{dx}(x^n) = x \frac{d}{dx} \sum_{n=0}^{\infty} n x^n =$$

$$= x \frac{d}{dx}\left[x \frac{d}{dx} \sum_{n=0}^{\infty} x^n\right] = x \frac{d}{dx}\left[x \left(\frac{1}{1-x}\right)'\right] =$$

$$= x \frac{d}{dx}\left[\frac{x}{(1-x)^2}\right] = x \cdot \frac{1+x}{(1-x)^3} = \frac{x(1+x)}{(1-x)^3}$$

b) $a_n = n^3$

$$A(x) = \sum_{n=0}^{\infty} n^3 x^n \overset{(a)}{=} x \frac{d}{dx}\left[\frac{x(1+x)}{(1-x)^3}\right]$$

$$\left[\frac{x(1+x)}{(1-x)^3}\right]' = \frac{[(1+x)+x](1-x)^3 + 3x(1+x)(1-x)^2}{(1-x)^6} =$$

$$= \frac{(1+x)(1-x) + x(1-x) + 3x(1+x)}{(1-x)^4} =$$

$$= \frac{1 - x^2 + x - x^2 + 3x + 3x^2}{(1-x)^4} = \frac{x^2 + 4x + 1}{(1-x)^4} =$$

zatem $A(x) = \dfrac{x(x^2+4x+1)}{(1-x)^4}$

c) $a_n = \binom{n+k}{k}$ k - ustalone, $k \geq 0$

$$A(x) = \sum \binom{n+k}{k} x^n = \sum \frac{(n+k)!}{\ } x^n =$$

$$A(x) = \sum_{n=0}^{\infty} \binom{n+k}{k} x^n = \sum_{n=0}^{\infty} \frac{(n+k)!}{k!\,n!} x^n =$$

$$= \sum_{n=0}^{\infty} \frac{1}{k!} \cdot \frac{n!(n+1)\cdots(n+k)}{n!} x^n = \frac{1}{k!} \sum_{n=0}^{\infty} (n+1)\cdots(n+k) x^n =$$

$$= \frac{1}{k!} \sum_{n=0}^{\infty} \frac{d^k}{dx^k} x^{n+k} = \frac{1}{k!} \cdot \frac{d^k}{dx^k} \sum_{n=0}^{\infty} x^{n+k} =$$

$$= \frac{1}{k!} \cdot \frac{d^k}{dx^k} \left(x^k \sum_{n=0}^{\infty} x^n \right) = \frac{1}{k!} \frac{d^k}{dx^k} \left(x^k \frac{1}{1-x} \right) =$$

no i tego się
chyba nie da ładnie
uprościć