Zad 13.

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13. Niech $A(x)$ będzie funkcją tworzącą ciągu a_n. Wylicz funkcje tworzące ciągów:

(a) $b_n = na_n$

(b) $c_n = a_n/n, c_0 = 0$

(c) $s_n = a_0 + a_1 + a_2 + \cdots + a_n$

(d) $d_n = \begin{cases} a_n & \text{gdy} \quad n = 2k \\ 0 & \text{gdy} \quad n = 2k+1 \end{cases}$

Wsk.: W (c) skorzystaj ze wzoru na iloczyn szeregów potęgowych dla $A(x)$ i $1 + x + x^2 + \cdots$

(a) $b_n = na_n$

$$B(x) = \sum_{n=0}^{\infty} n a_n x^n = x \sum_{n=0}^{\infty} a_n \cdot n x^{n-1} = x \frac{d}{dx} \sum_{n=0}^{\infty} a_n x^n = x A'(x)$$

(b) $C(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=1}^{\infty} c_n x^n = \sum_{n=1}^{\infty} \frac{1}{n} \cdot a_n x^n \qquad C'(x) = \sum_{n=1}^{\infty} a_n x^{n-1}$

$$C(x) = \int C'(x)dx = \int \Big(\sum_{n=1}^{\infty} a_n x^{n-1}\Big) dx = \int \Big(\frac{1}{x} \sum_{n=1}^{\infty} a_n x^n\Big) dx \underset{(*)}{=} \int \frac{A(x) - A(0)}{x} dx$$

(*) $A(0) = a_0 \qquad A(x) - a_0 = \sum_{n=1}^{\infty} a_n x^n$

(c) $s_n = a_0 + a_1 + \ldots + a_n$

$$\Big(\sum_{n=0}^{\infty} a_n x^n\Big)\Big(\sum_{n=0}^{\infty} x^n\Big) = \sum_{n=0}^{\infty} \Big(\sum_{k=0}^{n} a_k\Big) x^n = \sum_{n=0}^{\infty} (a_0 + \ldots + a_n) x^n = S(x)$$

$$\Big(\sum_{n=0}^{\infty} a_n x^n\Big)\Big(\sum_{n=0}^{\infty} x^n\Big) = A(x) \cdot \frac{1}{1-x} = \frac{A(x)}{1-x} = S(x)$$

(d) $d_n = \begin{cases} a_n & ; \quad n = 2k \\ 0 & ; \quad n = 2k+1 \end{cases}$

$$D(x) = \sum_{n=0}^{\infty} d_n x^n = \sum_{n=0}^{\infty} a_{2n} x^{2n} = a_0 x^0 + a_2 x^2 + a_4 x^4 + \ldots =$$

$$= a_0 x^0 + \underbrace{a_1 x^1}_{=0} + a_2 x^2 + \underbrace{a_3 x^3}_{=0} + \ldots =$$

$$= \Big[(a_0 x^0 + a_1 x^1 + a_2 x^2 + \ldots) + (a_0 x^0 - a_1 x^1 + a_2 x^2 - a_3 x^3 + \ldots)\Big] \cdot \frac{1}{2} =$$

$$= \Big[\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} a_n (-x)^n\Big] \cdot \frac{1}{2} = \frac{A(x) + A(-x)}{2}$$