Homework 2: Vectors, Subspaces, and Norms

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Problem 2.1 Summary and Context of Vectors/Subspaces/Norms

This week we covered vectors, subspaces, and norms. These are fundamental concepts in linear algebra, and therefore, they are foundational concepts in DSP. Vector spaces make up a framework for how we work in linear algebra, so it is very important that we define rules and develop an understanding of them. We also talked about norms, which are a measurement of distance. I've studied these in an undergraduate data science course where we used these norms to determine the distance between points in order to cluster them. Understanding the norms, how they work, and what they say about two points is very important in data science and scientific modeling. Using different norms can yield very different clustrering results.

This week's lectures laid the groundwork to introduce hilbert spaces, which are a very powerful tool that allow us to estimate signals using other functions, much like the fourier transform. However, to use this tool, one must understand vector spaces and bases.

Problem 2.2 Consider vector $V = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}\}$ in $\mathbb{R}^{2\times 2}$, prove 1, 2, 3, and 4.

$$\mathbf{v_1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \mathbf{v_3} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \mathbf{v_4} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

1. The span of $V = \mathbb{R}^{2\times 2}$

Vector $\mathbf{x} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ can be expressed as a linear combination of V

$\mathbf{x} = a_1\mathbf{v_1} + a_2\mathbf{v_2} + a_3\mathbf{v_3} + a_4\mathbf{v_4}$

$\mathbf{x} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ can represent every vector in $\mathbb{R}^{2\times 2}$

$\therefore V$ spans $\mathbb{R}^{2\times 2}$

2. V is linearly independent in $\mathbb{R}^{2\times 2}$

V is linearly independent if $\sum_{n=1}^{4} a_n\mathbf{v_n} = \mathbf{0}$ only when $A = \{a_1, a_2, a_3, a_4\} = \mathbf{0}$

$\mathbf{x} = a_1\mathbf{v_1} + a_2\mathbf{v_2} + a_3\mathbf{v_3} + a_4\mathbf{v_4}$ $\mathbf{x} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ can only equal $\mathbf{0}$ when $A = \mathbf{0}$

$\therefore V$ is linearly independent in $\mathbb{R}^{2\times 2}$

3. V is a basis for $\mathbb{R}^{2\times 2}$

V spans $\mathbb{R}^{2\times 2}$ (see 1) and V is linearly independent in $\mathbb{R}^{2\times 2}$ (see 2)

$\therefore V$ is a basis for $\mathbb{R}^{2\times 2}$

Problem 2.3 Let $C = [-1, 1]$, $C_e = \{f \in C : f \text{ is even}\}$, $C_o = \{f \in C : f \text{ is out}\}$

a) Prove that C_o and C_e are both subspaces of C

C_o closed under vector addition in C? Let $f(x), g(x) \in C_o$

$$h(x) = f(x) + g(x)$$
$$h(-x) = f(-x) + g(-x)$$
$$h(-x) = -f(x) - g(x)$$
$$h(-x) = -h(x)$$
$$h(x) \in C_o$$
$$\therefore C_o \text{ is closed over vector addition}$$

C_o closed under scalar multiplication in C? Let $f(x) \in C_o$, $a \in \mathbb{R}$

$$h(x) = af(x)$$
$$h(-x) = af(-x)$$
$$h(-x) = -af(x)$$
$$h(-x) = -h(x)$$
$$h(x) \in C_o$$
$$\therefore C_o \text{ is closed over scalar multiplication}$$

C_o contains the zero vector in C? Let $f(x) \in C_e$

$$z(x) = 0$$
$$f(x) + z(x) = f(x) + 0 = f(x)$$
$$z(x) \text{ is the zero vector}$$
$$z(-x) = 0$$
$$-z(x) = 0$$
$$z(-x) = -z(x)$$
$$\therefore C_o \text{ contains the zero vector}$$

C_o contains the zero vector and is closed under vector addition and scalar multiplication, so C_o is a subspace of C

C_e closed under vector addition in C? Let $f(x), g(x) \in C_o$

$$h(x) = f(x) + g(x)$$
$$h(-x) = f(-x) + g(-x)$$
$$h(-x) = f(x) + g(x)$$
$$h(-x) = h(x)$$
$$h(x) \in C_e$$
$$\therefore C_e \text{ is closed over vector addition}$$

C_e closed under scalar multiplication in C? Let $f(x) \in C_e$, $a \in \mathbb{R}$

$$h(x) = af(x)$$
$$h(-x) = af(-x)$$
$$h(-x) = af(x)$$
$$h(-x) = h(x)$$
$$h(x) \in C_e$$
$$\therefore C_e \text{ is closed over scalar multiplication}$$

C_e contains the zero vector? Let $f(x) \in C_e$

$$z(x) = 0$$
$$f(x) + z(x) = f(x) + 0 = f(x)$$
$$z(x) \text{ is the zero vector}$$
$$z(-x) = 0$$
$$z(x) = 0$$
$$z(-x) = z(x)$$
$$\therefore C_e \text{ contains the zero vector}$$

C_e contains the zero vector and is closed under vector addition and scalar multiplication, so C_e is a subspace of C

b) Prove that C_o and C_e are orthogonal subspaces

$$\langle x, y \rangle = \int_{-1}^{1} f(x)g(x)dx$$

$$\int_{-1}^{1} f(x)g(x)dx = \int_{-1}^{0} f(x)g(x)dx + \int_{0}^{1} f(x)g(x)dx$$

$$f(-x) = f(x), \quad g(-x) = g(x)$$

$$\int_{-1}^{0} f(x)g(x)dx = -\int_{0}^{1} f(x)g(x)dx$$

$$\int_{-1}^{1} f(x)g(x)dx = \int_{0}^{1} f(x)g(x)dx - \int_{0}^{1} f(x)g(x)dx = 0$$

$\boxed{\langle x, y \rangle = 0 \therefore C_o \text{ and } C_e \text{ are orthogonal subspaces}}$

Problem 2.4 Let V is all real number numbers greater than 0

a) Is V under $\boxplus$ and $\boxdot$ a vector space? Let $x \boxplus y = xy + 1$ for all $x, y \in V$, and $a \boxdot x = a^2x$ for all $x, y \in V$

$$x \boxplus y = xy + 1$$
$$y \boxplus x = yx + 1$$
$$x \boxplus y = y \boxplus x \quad \text{(Commutative Satisfied)}$$

$$x \boxplus (y \boxplus z) = xyz + x + 1$$
$$(x \boxplus y) \boxplus z = xyz + z + 1$$
$$x \boxplus (y \boxplus z) \neq (x \boxplus y) \boxplus z \quad \text{(Associative Not Satisfied)}$$

$$x \boxplus z = x$$
$$z = \mathbf{0} = 1 + \frac{1}{x} \quad \text{(V Has No Unique Zero Vector)}$$
$$(\therefore V \text{ Has No Vector } -x \text{ That Satisfies } x \boxplus (-x) = \mathbf{0})$$

$$a \boxdot (x \boxplus y) = a^2xy + a^2$$
$$(a \boxdot x) \boxplus (a \boxdot y) = a^4xy + 1$$
$$a \boxdot (x \boxplus y) \neq (a \boxdot x) \boxplus (a \boxdot y) \quad \text{(Distributive Not Satisfied)}$$

$$(ab) \boxdot x = (ab)2x = a^2b^2x$$
$$a(b \boxdot x) = a^2b^2x$$
$$(ab) \boxdot x = a(b \boxdot x) \quad \text{(Associative Satisfied)}$$

$$1 \boxdot x = 1^2x$$
$$1 \boxdot x = x \quad \text{(Multiplicitive Satisfied)}$$

$$0 \boxdot x \neq \mathbf{0}, \quad \mathbf{0} \notin V \quad \text{(Additive Not Satisfied)}$$

$$a \boxdot x \boxplus b \boxdot y \quad x, y \in V \quad \forall a, b \in \mathbb{R}$$
$$a \boxdot x \boxplus b \boxdot y = (a^2x)(a^2y) + 1 = a^4xy + 1$$
$$a^4xy + 1 \in \mathbb{R} \quad V \text{ is closed under vector addition and scalar multiplication}$$

V is not a vector space under $\boxplus$ and $\boxdot$.

V's $\boxplus$ opperator does not satisfy the Associative Identity, has no zero vector $\mathbf{0}$ that $x + \mathbf{0} = x \forall x \in S$, and has no vector $-x$ such that $x + (-x) = \mathbf{0}$

V's $\boxdot$ opperator does not satisfy the Distributive and Additive Identities

b) Is V under $\boxplus$ and $\boxdot$ a vector space? Let $x \boxplus y = xy$ for all $x, y \in V$, and $a \boxdot x = x^a$ for all $x, y \in V$

$$x \boxplus y = xy$$
$$y \boxplus x = yx$$
$$x \boxplus y = y \boxplus x \quad \text{(Commutative Satisfied)}$$

$$x \boxplus (y \boxplus z) = x(yz) = xyz$$
$$(x \boxplus y) \boxplus z = (xy)z = xyz$$
$$x \boxplus (y \boxplus z) = (x \boxplus y) \boxplus z \quad \text{(Associative Satisfied)}$$

$$x \boxplus z = x$$
$$z = 1 = \mathbf{0} \quad \text{(V Has A Unique Zero Vector)}$$

$$-x = \frac{1}{x}$$
$$x \boxplus (-x) = \mathbf{0}$$
$$x(1/x) = 1 = \mathbf{0}$$
$$V \text{ Has A Vector } -x \text{ That Satisfies } x \boxplus (-x) = \mathbf{0}$$

$$a \boxdot (x \boxplus y) = (xy)^a = x^a y^a$$
$$(a \boxdot x) \boxplus (a \boxdot y) = x^a y^a$$
$$a \boxdot (x \boxplus y) = (a \boxdot x) \boxplus (a \boxdot y) \quad \text{(Distributive Satisfied)}$$

$$(ab) \boxdot x = x^{ab}$$
$$a(b \boxdot x) = (x^b)^a = x^{ab}$$
$$(ab) \boxdot x = a(b \boxdot x) \quad \text{(Associative Satisfied)}$$

$$1 \boxdot x = x^1$$

$$1 \boxdot x = x \qquad \text{(Multiplicitive Satisfied)}$$

$$0 \boxdot x = x^0 = 1 = \mathbf{0}$$

$$0 \boxdot x = \mathbf{0}, \quad \mathbf{0} \in V \qquad \text{(Additive Satisfied)}$$

$$a \boxdot x \boxplus b \boxdot y \quad x, y \in V \quad \forall a, b \in \mathbb{R}$$

$$a \boxdot x \boxplus b \boxdot y = (x^a)(y^b)$$

$$x^a y^b \in V \quad \therefore V \text{ is closed under vector addition and scalar multiplication}$$

$$\boxed{V \text{ is a vector space under } \boxplus \text{ and } \boxdot.}$$

Problem 2.5 Prove the "reverse triangle inequality" holds in a normed vector space

Start by expanding $\|x-y\|^2$

$$\|x-y\|^2 = \langle x-y, x-y\rangle$$
$$\|x-y\|^2 = \langle x, x-y\rangle - \langle y, x-y\rangle$$
$$\|x-y\|^2 = \langle x, x\rangle - \langle x, y\rangle - \langle y, x\rangle + \langle y, y\rangle$$
$$\|x-y\|^2 = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2$$

Check if "reverse" triangle inequality is true:

$$\|x\| - \|y\| \leq \|x-y\|$$
$$(\|x\| - \|y\|)^2 \leq \|x-y\|^2$$
$$\|x\|^2 + \|y\|^2 - 2\|x\|\,\|y\| \leq \|x\|^2 + \|y\|^2 - 2\langle x, y\rangle$$

Given the Cauchy-Schwarz Inequality ($|\langle x, y\rangle| \leq \|x\|\,\|y\|$):

$\|x\| - \|y\| \leq \|x-y\|$ with equality iff x and y are colinear.

Problem 2.6 Let $\|\dot{\|}_p$ be the ℓ_p norm for vectors in $\mathbb{R}^N$ Given: $(a_1 + a_2)^\alpha \geq a_1^\alpha + a_2^\alpha$ for $\alpha \geq 1$

a) Prove $\|x\|_p \leq \|x\|_1$ for all $x \in \mathbb{R}$, $p \geq 1$

$$\|x\|_p = (\sum_{n=1}^{N} |x_n|^p)^{1/p}$$

$$\|x\|_1 = (\sum_{n=1}^{N} |x_n|^1)^1$$

$$(\sum_{n=1}^{N} |x_n|^p)^{1/p} \leq (\sum_{n=1}^{N} |x_n|^1)^1$$

$$\sum_{n=1}^{N} |x_n|^p \leq (\sum_{n=1}^{N} |x_n|)^p$$

$$(x_1^p + x_2^p + ... + x_n^p) \leq (x_1 + x_2 + ... + x_n)^p$$

$$\boxed{(a_1 + a_2)^\alpha \geq a_1^\alpha + a_2^\alpha \text{ for } \alpha \geq 1 \therefore \|x\|_p \leq \|x\|_1 \text{ for all } x \in \mathbb{R},\ p \geq 1}$$

b) Prove that $\|x\|_p \leq \|x\|_q$ for all $x \in \mathbb{R}$, $1 \leq q \leq p \leq \infty$

Let β be a real number greater than 1 where $p = \beta q$

$$\|x\|_p = (\sum_{n=1}^{N} |x_n|^{\beta q})^{\frac{1}{\beta q}}$$

$$\|x\|_q = (\sum_{n=1}^{N} |x_n|^{q})^{\frac{1}{q}}$$

$$(\sum_{n=1}^{N} |x_n|^{\beta q})^{\frac{1}{\beta q}} \leq (\sum_{n=1}^{N} |x_n|^{q})^{\frac{1}{q}}$$

$$\sum_{n=1}^{N} |x_n|^{\beta q} \leq (\sum_{n=1}^{N} |x_n|^{q})^{\beta}$$

$$(x_1^p + x_2^p + ... + x_n^p) \leq (x_1 + x_2 + ... + x_n)^p$$

$$\boxed{(a_1 + a_2)^\alpha \geq a_1^\alpha + a_2^\alpha \text{ for } \alpha \geq 1 \therefore \|x\|_p \leq \|x\|_n \text{ for all for all } x \in \mathbb{R},\ 1 \leq q \leq p \leq \infty}$$

Problem 2.7 Visualizing $\mathbb{R}^2$ with the unit ball

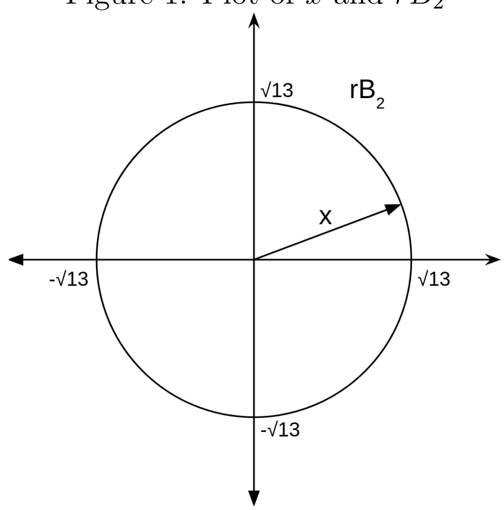
a) Find $r = \|x\|_p$ for $p = 1, 2, \infty$ and sketch x and rB

$$\boxed{\|x\|_1 = 3 + 2 = 5}$$

not a valid norm

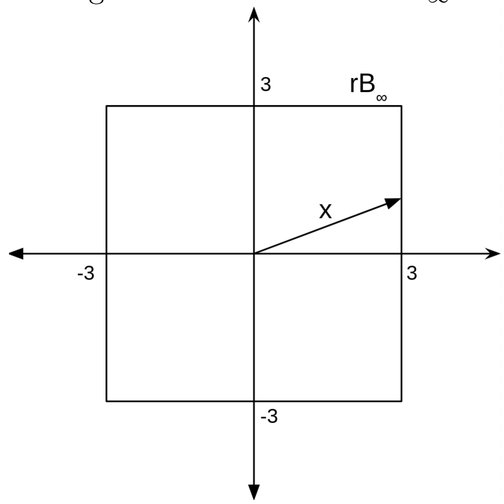
$$\boxed{\|x\|_2 = \sqrt{3^2 + 2^2} = \sqrt{13}}$$

Figure 1: Plot of x and rB_2



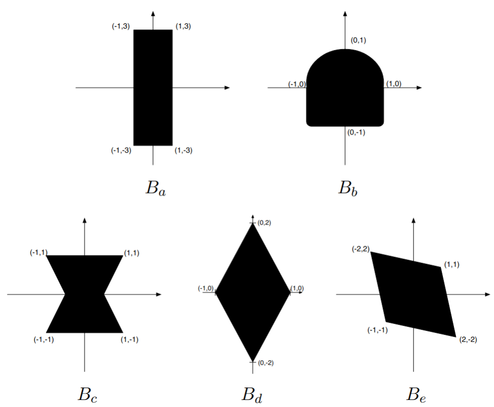
$$\boxed{\|x\|_\infty = \max(3, 2) = 3}$$

Figure 2: Plot of x and rB_∞



b) Consider the following shapes and determine which $\| \ \|_{B_j}$ is a valid norm.

Figure 3: Plots of B_p



B_a is a valid norm.

B_b is not a valid norm. It violates the homogeneity property of a valid norm.

$$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad a = -1$$

$$\|ax\| \neq |a| \, \|x\|$$

$$\| - x\|\| \neq \|x\|$$

$$\frac{1}{\sqrt{2}} \neq 1 \quad \text{The shape is not symetric}$$

B_c is not a valid norm: It violates the triangle inequality.

$$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad y = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\|x\| = \|y\| = 1$$

$$\|x + y\| > 2$$

$x + y = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ and due to B_c's concavity, $B_c < 1$ where $y = 0$ and $x > 0$, so B_c must be scaled by more than 2 to so that $x \in rB_p$.

$$\|x\| + \|y\| < \|x + y\|$$

B_d is a valid norm.

B_e is a valid norm:

$\boxed{\text{$j = a, d, e$ are valid norms. $j = b$ is not a valid norm because it violates the homogeneity property. $j = c$ is not a valid norm because it violates the triangle inequality.}}$

I'm sorry, but this took way too much effort in LaTeX. I'll still use it when convenient, but I can't do this for every assignment. Apologies.