Row Echelon Form (REF) and Reduced Row Echelon Form (RREF)

1. Row Echelon Form (REF)

A matrix is in **Row Echelon Form (REF)** if:

1. **Leading Entries:** In each row, the first nonzero number (called the *leading entry*) is to the right of the leading entry in the row above it.
2. **Zero Rows:** Any rows that consist entirely of zeros are at the bottom of the matrix.
3. **Nonzero Leading Entries:** The leading entry in a row must be nonzero (unless the row is all zeros).

Example: The matrix:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is in **Row Echelon Form** because:

- The leading entry of each row is to the right of the one in the row above.
- All zero rows are at the bottom.

2. Reduced Row Echelon Form (RREF)

A matrix is in **Reduced Row Echelon Form (RREF)** if:

1. **Row Echelon Form:** The matrix satisfies all the conditions of Row Echelon Form.
2. **Leading Entry Equals 1:** Every leading entry in each row is exactly 1.
3. **Zeros Above Leading Entries:** Each leading entry is the only nonzero number in its column (all other elements in the column are zero).

Example: The matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is in **Reduced Row Echelon Form** because:

- It is in Row Echelon Form.
- Each leading entry is 1.
- Each column containing a leading entry has all other entries as 0.

Key Differences Between REF and RREF

Feature	**Row Echelon Form (REF)**	**Reduced Row Echelon Form (RREF)**
Leading Entries	Nonzero, no specific value required	Must be 1
Zero Rows	At the bottom	At the bottom
Other Column Entries	Not necessarily zero	Must be zero (except the leading entry)
Conversion to Form	Achieved using row operations	Requires additional operations after REF

Applications

- **REF:** Commonly used in the initial steps of solving linear systems through Gaussian elimination.
- **RREF:** Used for obtaining the final, simplified solution in systems of linear equations through Gauss-Jordan elimination.

Both forms help in determining properties like rank, consistency, and solutions of a matrix.

Matrix Inversion

The **inverse of a matrix** is defined only for square matrices (matrices with the same number of rows and columns) that are nonsingular (determinant is nonzero). For a square matrix A, its inverse is denoted as A^{-1}, and it satisfies the property:

$$A \cdot A^{-1} = A^{-1} \cdot A = I,$$

where I is the identity matrix.

Steps to Find the Inverse of a Matrix

Given a square matrix A, follow these steps to compute A^{-1}:

1. Check the Determinant

First, compute the determinant of A, denoted as $\det(A)$. If $\det(A) = 0$, the matrix is singular, and its inverse does not exist.

2. Find the Adjugate Matrix

The adjugate (or adjoint) of a matrix A, denoted as $\text{adj}(A)$, is calculated as follows:

- Compute the cofactor matrix of A, which involves:
    1. Calculating the minor for each element of A, where the minor is the determinant of the submatrix obtained by removing the row and column of the element.

2. Applying the cofactor sign pattern:

$$C_{ij} = (-1)^{i+j} \det(M_{ij}),$$

where M_{ij} is the minor matrix for the element in the i-th row and j-th column.

- Transpose the cofactor matrix to obtain $\mathrm{adj}(A)$.

3. Calculate the Inverse

The inverse of the matrix A is given by:

$$A^{-1} = \frac{1}{\det(A)} \mathrm{adj}(A).$$

Example

Consider the matrix:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

Step 1: Compute the Determinant

$$\det(A) = (1)(4) - (2)(3) = 4 - 6 = -2.$$

Step 2: Find the Adjugate The cofactor matrix is:

$$\mathrm{Cofactor}(A) = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}.$$

Taking the transpose:

$$\mathrm{adj}(A) = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}.$$

Step 3: Calculate the Inverse

$$A^{-1} = \frac{1}{\det(A)} \mathrm{adj}(A) = \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix}.$$

Verification

To verify, compute $A \cdot A^{-1}$:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus, the calculation is correct.

The Kernel of a Matrix

The **kernel** (also known as the null space) of a matrix A is the set of all vectors x such that:

$$A \cdot x = 0,$$

where A is an $m \times n$ matrix, x is an $n \times 1$ column vector, and 0 is the $m \times 1$ zero vector.

Definition

The kernel of a matrix A, denoted as $\ker(A)$, is given by:

$$\ker(A) = \{x \in R^n \mid A \cdot x = 0\}.$$

In other words, the kernel consists of all solutions to the homogeneous system of linear equations defined by $A \cdot x = 0$.

Steps to Find the Kernel

To compute the kernel of a matrix A:

1. **Set up the equation:** Write $A \cdot x = 0$, where x is a vector of variables.
2. **Row reduce:** Perform Gaussian elimination or row-reduction on the augmented matrix $[A \mid 0]$ to simplify the system.
3. **Express solutions:** Identify the free variables and write the solution in parametric form.
4. **Write the kernel:** The kernel is the span of the solution vectors.

Example

Consider the matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}.$$

Step 1: Set up the equation Let $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. Then:

$$A \cdot x = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1x_1 + 2x_2 + 3x_3 \\ 4x_1 + 5x_2 + 6x_3 \end{bmatrix}.$$

Set this equal to the zero vector:

$$\begin{bmatrix} 1x_1 + 2x_2 + 3x_3 \\ 4x_1 + 5x_2 + 6x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Step 2: Row reduce the augmented matrix The augmented matrix is:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \end{array}\right].$$

Perform Gaussian elimination:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & 0 \end{array}\right].$$

Simplify further:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right].$$

Step 3: Express the solutions From the row-reduced matrix, the system of equations is:

$$x_1 + 2x_2 + 3x_3 = 0,$$

$$x_2 + 2x_3 = 0.$$

Solve for the free variables:

$$x_2 = -2x_3, \quad x_1 = -2x_2 - 3x_3 = 4x_3 - 3x_3 = x_3.$$

Step 4: Write the kernel Let $x_3 = t$, where t is a free parameter. Then:

$$x_1 = t, \quad x_2 = -2t, \quad x_3 = t.$$

The solution is:

$$x = t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

Thus, the kernel of A is:

$$\ker(A) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}.$$

Properties of the Kernel

- The kernel of A is a subspace of R^n.
- The dimension of the kernel is called the **nullity** of A.
- The **Rank-Nullity Theorem** states:

  $$\operatorname{rank}(A) + \operatorname{nullity}(A) = n,$$

  where n is the number of columns of A.