Sundial Generator: basic concept

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1 Introduction

Sun position in cartesian coordinates in the reference system ($\mathbf{S}$, $\mathbf{E}$, $\mathbf{Z}$) is $\mathbf{s} = (s_S, s_E, s_Z)$ are [Spr07]:

$$\begin{aligned}\mathbf{s} = &-(\sin\delta\cos\phi - \cos\delta\sin\phi\cos\omega)\mathbf{S}\\ &-\cos\delta\sin\omega\mathbf{E}\\ &+(\cos\delta\cos\phi\cos\omega + \sin\delta\sin\phi)\mathbf{Z}\end{aligned} \tag{1}$$

where δ is the declination angle of the sun, ϕ is the latitude of the considered location and ω is the local time express as:

$$\omega = 15^\circ(\text{hour} - 12) \tag{2}$$

In the same reference system, we can define the normal of the wall $\mathbf{n}$, where $\mathbf{n} = (n_S, n_E, n_Z)$, via angle α and angle β as depicted in figure below. In particular, angle α is the angle between $\mathbf{n}$ and $\mathbf{Z}$ (i.e., $\alpha = \arccos\mathbf{n}\cdot\mathbf{Z}$), while angle β is the angle between $\mathbf{S}$ and the projections of $\mathbf{n}$ onto the plane specified by the two vectors $\mathbf{S}$ and $\mathbf{E}$, that can be defined as $\mathbf{n_{SE}}$.

Now we can express $\mathbf{s}$ in the reference system of the wall. To do so, we can apply two rotations. The first rotation is around the $\mathbf{Z}$ to align $\mathbf{S}$ with $\mathbf{n_{SE}}$. The second rotation is around$\mathbf{E}$ in order to align $\mathbf{Z}$ with $\mathbf{n}$.

Rotation arount $\mathbf{Z}$ is:

$$R_Z(\theta) = \begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix} \tag{3}$$

Rotation aound $\mathbf{E}$ is:

$$R_E(\theta) = \begin{bmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{bmatrix} \tag{4}$$

The vector $\mathbf{s}$ in the basis of the reference system of the wall is renamed as $\mathbf{s}'$ and will be:

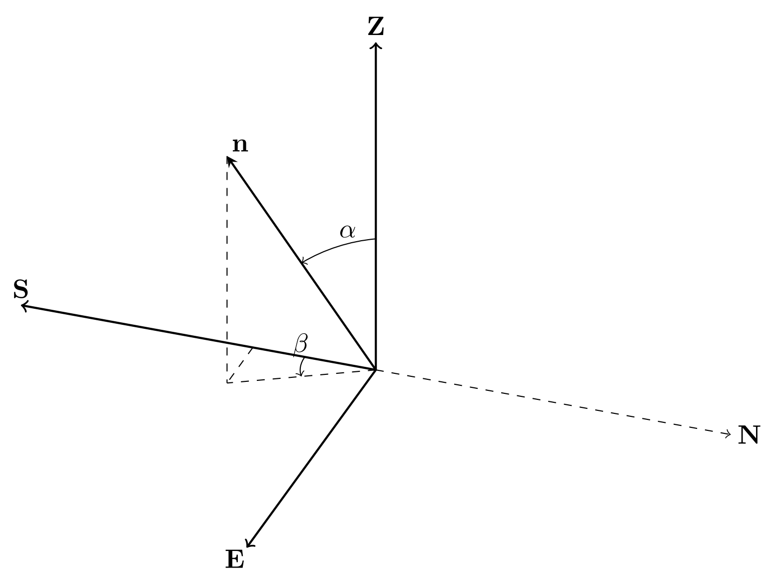
$$\mathbf{s}' = R_E^{-1}(\alpha)[R_Z^{-1}(\beta)\mathbf{s}^T] \tag{5}$$

Esplicitly:

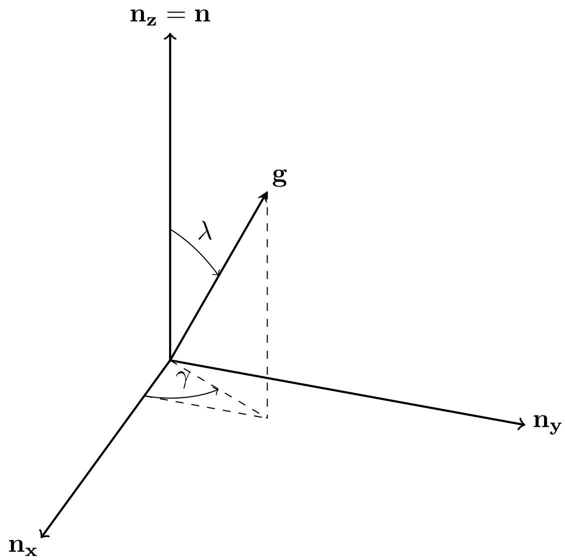
$$\begin{aligned}
\mathbf{s}' = \; & \Big[-\big((\sin\delta\cos\phi - \sin\phi\cos\delta\cos\omega)\cos\beta + \sin\beta\sin\omega\cos\delta\big)\cos\alpha \\
& -(\sin\delta\sin\phi + \cos\delta\cos\omega\cos\phi)\sin\alpha\Big]\,\mathbf{n_x} \\
& + \Big[(\sin\delta\cos\phi - \sin\phi\cos\delta\cos\omega)\sin\beta - \sin\omega\cos\beta\cos\delta\Big]\,\mathbf{n_y} \\
& + \Big[-\big((\sin\delta\cos\phi - \sin\phi\cos\delta\cos\omega)\cos\beta + \sin\beta\sin\omega\cos\delta\big)\sin\alpha \\
& +(\sin\delta\sin\phi + \cos\delta\cos\omega\cos\phi)\cos\alpha\Big]\,\mathbf{n_z}
\end{aligned} \tag{6}$$

We can rewrite this equation using the notation of $\mathbf{s}$ as simply:

$$\begin{aligned}
\mathbf{s}' = \; & \Big[-\big(s_S\cos\beta + s_E\sin\beta\big)\cos\alpha \\
& -s_Z\sin\alpha\Big]\,\mathbf{n_x} \\
& + \Big[s_S\sin\beta - s_E\cos\beta\Big]\,\mathbf{n_y} \\
& + \Big[-\big(s_S\cos\beta + s_E\sin\beta\big)\sin\alpha \\
& +s_Z\cos\alpha\Big]\,\mathbf{n_z}
\end{aligned} \tag{7}$$



Now that we have $\mathbf{s}'$ we can define the gnomon vector $\mathbf{g}$ angles as depicted in figure below. In details, the angle λ in the angle between the wall normal $\mathbf{n}$, or $\mathbf{n_z}$ and the gnomon $\mathbf{g}$, while γ is the angle between $\mathbf{n_x}$ and the projection of the gnomon onto the plane $\mathbf{n_x}\,\mathbf{n_y}$.



Now we can express the vector $\mathbf{s}$ in parametric form passing from the extreme point of $\mathbf{g} = h(g_x, g_y, g_z) = (\sin\lambda\cos\gamma, \sin\lambda\sin\gamma, \cos\lambda)$, where h is the gnomon length. The intersection os $\mathbf{s}$, when $\mathbf{s}\cdot\mathbf{g} > 0$ is actually the shadow of the gnomon projected onto the plane.

References

[Spr07] Alistair B. Sproul. Derivation of the solar geometric relationships using vector analysis. *Renewable Energy 2007-jun vol. 32 iss. 7*, 32, jun 2007.