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Solving the general Sylvester discrete-time periodic matrix equations via the gradient based iterative method

Masoud Hajarian

Department of Mathematics, Faculty of Mathematical Sciences, Shahid Beheshti University, General Campus, Evin, Tehran 19839, Iran

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ABSTRACT

The present work proposes a gradient based iterative method to find the solutions of the general Sylvester discrete-time periodic matrix equations

$$\sum_{j=1}^{m}\Big(A_{ij}X_iB_{ij}+C_{ij}X_{i+1}D_{ij}+E_{ij}Y_iF_{ij}+G_{ij}Y_{i+1}H_{ij}\Big)=M_i,$$
$$i=1,2,\ldots.$$

It is proven that the proposed iterative method can obtain the solutions of the periodic matrix equations for any initial matrices. Finally a numerical example is included to demonstrate the validity and applicability of the iterative method.

1. Introduction

This article considers the general Sylvester discrete-time periodic (GSDTP) matrix equations

$$\sum_{j=1}^{m}\Big(A_{ij}X_iB_{ij}+C_{ij}X_{i+1}D_{ij}+E_{ij}Y_iF_{ij}+G_{ij}Y_{i+1}H_{ij}\Big)=M_i,\quad i=1,2,\ldots,\tag{1.1}$$

where the coefficient matrices $A_{ij},C_{ij},E_{ij},G_{ij}\in\mathbb{R}^{p\times n}$, $B_{ij},D_{ij},F_{ij},H_{ij}\in\mathbb{R}^{n\times q}$, $M_i\in\mathbb{R}^{p\times q}$, and the solutions $X_i,Y_i\in\mathbb{R}^{n\times n}$ are periodic with period ω, i.e., $A_{i+\omega,j}=A_{i,j}$, $B_{i+\omega,j}=B_{i,j}$, $C_{i+\omega,j}=C_{i,j}$, $D_{i+\omega,j}=D_{i,j}$, $E_{i+\omega,j}=E_{i,j}$, $F_{i+\omega,j}=F_{i,j}$, $G_{i+\omega,j}=G_{i,j}$, $H_{i+\omega,j}=H_{i,j}$, $M_{i+\omega}=M_i$, $X_{i+\omega}=X_i$ and $Y_{i+\omega}=Y_i$ for $j=1,2,\ldots,m$ and $i=1,2,\ldots$. Solving the periodic matrix equations is of interest in linear periodic system theory, especially in the areas of optimal control, prediction and stability [1–7]. For example, the discrete-time periodic Lyapunov (DPL) matrix equations

$$A_kX_kA_k^T-X_{k+1}=-B_kB_k^T,\tag{1.2}$$

E-mail addresses: m_hajarian@sbu.ac.ir, mhajarian@aut.ac.ir, masoudhajarian@gmail.com.

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and

$$A_k^T X_{k+1} A_k - X_k = -Q_k, \tag{1.3}$$

appear in the semi-global stabilization problem of discrete-time linear periodic (DLP) systems

$$x(k+1) = A_k x(k) + B_k u_k,$$

with the periodic matrices [3]. The GSDTP matrix equations (1.1) are general and contain various linear discrete-time periodic matrix equations such as the DPL matrix equations. So far some numerical methods had been developed to solve the periodic matrix equations. In [8], the Bartels–Stewart and Hessenberg–Schur algorithms were extended for periodic Lyapunov and Sylvester equations. Varga introduced efficient numerically reliable algorithms based on the periodic Schur decomposition for the solution of periodic Lyapunov matrix equations [9]. In [10], Kressner proposed new variants of the squared Smith iteration and Krylov subspace based methods for the approximate solution of discrete-time periodic Lyapunov equations. Recently Benner et al. discussed the numerical solution of large-scale sparse projected periodic discrete-time Lyapunov equations in lifted form which arise in model reduction of periodic descriptor systems. [11].

In this article, a gradient based iterative method is proposed to solve the GSDTP matrix equations (1.1). The solutions of the GSDTP matrix equations have not been dealt with yet.

The outline of this article is as follows. In Section 2, first we introduce a gradient based iterative method for solving (1.1) and second it is shown that the introduced iterative method converges to the solutions for any initial matrices. We report a numerical example to show the effectiveness of the introduced iterative method in Section 3.

2. Main results

In this section, first we give the necessary and sufficient conditions for the existence of solutions of the GSDTP matrix equations (1.1). Then a gradient based iterative method is presented to solve (1.1). At the end of this section, we show that presented method converges to the exact solutions for any initial matrices.

We can equivalently transform the GSDTP matrix equations (1.1) to the following general Sylvester matrix equation

$$\sum_{j=1}^{m} \Big(\mathcal{A}_j \mathcal{X} \mathcal{B}_j + \mathcal{C}_j \mathcal{X} \mathcal{D}_j + \mathcal{E}_j \mathcal{Y} \mathcal{F}_j + \mathcal{G}_j \mathcal{Y} \mathcal{H}_j \Big) = \mathcal{M}, \tag{2.1}$$

where

$$\mathcal{A}_j = \begin{pmatrix} 0 & \cdots & 0 & A_{1,j} \\ A_{2,j} & & & 0 \\ & \ddots & & \vdots \\ 0 & & A_{\omega,j} & 0 \end{pmatrix}, \qquad \mathcal{B}_j = \begin{pmatrix} 0 & B_{2,j} & & 0 \\ \vdots & & \ddots & \\ 0 & & & B_{\omega,j} \\ B_{1,j} & 0 & \cdots & 0 \end{pmatrix},$$

$$\mathcal{E}_j = \begin{pmatrix} 0 & \cdots & 0 & E_{1,j} \\ E_{2,j} & & & 0 \\ & \ddots & & \vdots \\ 0 & & E_{\omega,j} & 0 \end{pmatrix},$$

$$\mathcal{F}_j = \begin{pmatrix} 0 & F_{2,j} & & 0 \\ \vdots & & \ddots & \\ 0 & & & F_{\omega,j} \\ F_{1,j} & 0 & \cdots & 0 \end{pmatrix}, \qquad \mathcal{C}_j = \operatorname{diag}\Big(C_{1,j}, C_{2,j}, \ldots, C_{\omega,j}\Big),$$

$$\mathcal{D}_j = \operatorname{diag}\left(D_{1,j}, D_{2,j}, \ldots, D_{\omega,j}\right),$$
$$\mathcal{G}_j = \operatorname{diag}\left(G_{1,j}, G_{2,j}, \ldots, G_{\omega,j}\right), \qquad \mathcal{H}_j = \operatorname{diag}\left(H_{1,j}, H_{2,j}, \ldots, H_{\omega,j}\right), \qquad \mathcal{M} = \operatorname{diag}\left(M_1, M_2, \ldots, M_\omega\right),$$
$$\mathcal{X} = \operatorname{diag}\left(X_2, X_3, \ldots, X_\omega, X_1\right), \qquad \mathcal{Y} = \operatorname{diag}\left(Y_2, Y_3, \ldots, Y_\omega, Y_1\right), \quad j = 1, 2, \ldots, m.$$

By applying the Kronecker product and vectorization operator, the general Sylvester matrix equations (2.1) can be transformed into the following linear system

$$\underbrace{\begin{pmatrix} \sum_{j=1}^{m}(\mathcal{B}_j^T \otimes \mathcal{A}_j + \mathcal{D}_j^T \otimes \mathcal{C}_j) & \sum_{j=1}^{m}(\mathcal{F}_j^T \otimes \mathcal{E}_j + \mathcal{H}_j^T \otimes \mathcal{G}_j) \end{pmatrix}}_{S} \underbrace{\begin{pmatrix} \operatorname{vec}(\mathcal{X}) \\ \operatorname{vec}(\mathcal{Y}) \end{pmatrix}}_{x} = \underbrace{\operatorname{vec}(\mathcal{M})}_{d}. \tag{2.2}$$

Noting that (2.2), we can obtain the necessary and sufficient conditions for the existence of solutions of the GSDTP matrix equations (1.1) by the following lemma.

Lemma 1. *The GSDTP matrix equations (1.1) have a unique solution group $(X_1, Y_1, \ldots, X_\omega, Y_\omega)$ if and only if $\operatorname{rank}((S, d)) = \operatorname{rank}(S)$ and S has a full column rank; in this case, the solution group $(X_1, Y_1, \ldots, X_\omega, Y_\omega)$ of (1.1) can be expressed as follows*

$$x = \begin{pmatrix} \operatorname{vec}(\mathcal{X}) \\ \operatorname{vec}(\mathcal{Y}) \end{pmatrix} = (S^T S)^{-1} S^T d = (S^T S)^{-1} S^T \operatorname{vec}(\mathcal{M}),$$

and the homogeneous GSDTP matrix equations

$$\sum_{j=1}^{m}\left(A_{ij}X_iB_{ij} + C_{ij}X_{i+1}D_{ij} + E_{ij}Y_iF_{ij} + G_{ij}Y_{i+1}H_{ij}\right) = 0, \quad i = 1, 2, \ldots, \tag{2.3}$$

have a unique solution group $(X_1, Y_1, \ldots, X_\omega, Y_\omega) = 0$.

Obviously we can obtain the solutions of the GSDTP matrix equations (1.1) by solving the general Sylvester matrix equations (2.1) or the linear system (2.2). For example, the iterative methods proposed in [12–18] can be used to solve (2.1) and (2.2). But the size of (2.1) and (2.2) is large. Hence solving (2.1) and (2.2) needs excessive computer time and memory space. To overcome the complications, we directly obtain a gradient based iterative method to solve the GSDTP matrix equations (1.1). One of the famous iterative method for solving the linear system $Ax = b$ is the gradient based iterative algorithm [19,20] as follows:

$$x^{(k+1)} = x^{(k)} + \mu A^T(b - Ax^{(k)}), \quad 0 < \mu \le \frac{2}{\|A\|^2}. \tag{2.4}$$

Recently the gradient based iterative methods were developed to solve several linear matrix equations [15,21,22]. For example, Ding and Chen proposed the gradient based iterative methods for solving (coupled) Sylvester matrix equations [19,20,23]. Zhou et al. constructed the gradient based iterative algorithm to approximate the solutions to the coupled linear matrix equations [24]. The gradient based iterative method can be constructed for solving (1.1) as follows.

Algorithm 1.

Step 1. *Choose the initial matrices $X_i(1), Y_i(1) \in \mathbb{R}^{n\times n}$ for $i = 1, 2, \ldots, \omega$ and a parameter $\delta > 0$;*

Step 2. *Set $X_{\omega+1}(1) = X_1(1)$, $Y_{\omega+1}(1) = Y_1(1)$, $C_{0,j} = C_{\omega,j}$, $D_{0,j} = D_{\omega,j}$, $G_{0,j} = G_{\omega,j}$ and $H_{0,j} = H_{\omega,j}$ for $j = 1, 2, \ldots, m$;*

Step 3. *Compute*

$$R_i(1) = M_i - \sum_{j=1}^{m}\Big(A_{ij}X_i(1)B_{ij} + C_{ij}X_{i+1}(1)D_{ij} + E_{ij}Y_i(1)F_{ij} + G_{ij}Y_{i+1}(1)H_{ij}\Big) \quad \textit{for } i = 1, 2, \ldots, \omega,$$

and set $R_0(1) = R_\omega(1)$*;*

Step 4. *For* $k = 1, 2, \ldots,$ *compute*

$$\begin{aligned} X_i(k+1) &= X_i(k) + \delta\sum_{j=1}^{m}\Big(A_{ij}^T R_i(k) B_{ij}^T + C_{i-1,j}^T R_{i-1}(k) D_{i-1,j}^T\Big), \quad \textit{for } i = 1, 2, \ldots, \omega, \\ Y_i(k+1) &= Y_i(k) + \delta\sum_{j=1}^{m}\Big(E_{ij}^T R_i(k) F_{ij}^T + G_{i-1,j}^T R_{i-1}(k) H_{i-1,j}^T\Big), \quad \textit{for } i = 1, 2, \ldots, \omega, \\ X_{\omega+1}(k+1) &= X_1(k+1), \qquad Y_{\omega+1}(k+1) = Y_1(k+1), \\ R_i(k+1) &= M_i - \sum_{j=1}^{m}\Big(A_{ij}X_i(k+1)B_{ij} + C_{ij}X_{i+1}(k+1)D_{ij} \\ &\quad + E_{ij}Y_i(k+1)F_{ij} + G_{ij}Y_{i+1}(k+1)H_{ij}\Big), \quad \textit{for } i = 1, 2, \ldots, \omega, \\ R_0(k+1) &= R_\omega(k+1). \end{aligned}$$

The convergence of the above algorithm is given in the following theorem.

Theorem 1. *Suppose that the GSDTP matrix equations (1.1) have a unique solution group* $(X_1^*, Y_1^*, \ldots, X_\omega^*, Y_\omega^*)$*. If the parameter* δ *satisfies the inequality*

$$0 < \delta < \frac{2}{\sum_{i=1}^{\omega}\sum_{j=1}^{n}\Big(\big\|A_{ij}B_{ij}\big\|^2 + \big\|C_{ij}D_{ij}\big\|^2 + \big\|E_{ij}F_{ij}\big\|^2 + \big\|G_{ij}H_{ij}\big\|^2\Big)}, \tag{2.5}$$

then for any initial matrix group $(X_1(1), Y_1(1), \ldots, X_\omega(1), Y_\omega(1))$*, the iterative solution group* $(X_1(k), Y_1(k), \ldots, X_\omega(k), Y_\omega(k))$ *generated by Algorithm 1 converges to the* $(X_1^*, Y_1^*, \ldots, X_\omega^*, Y_\omega^*)$*, that is*

$$\lim_{k\to\infty} X_i(k) = X_i^*, \qquad \lim_{k\to\infty} Y_i(k) = Y_i^* \quad \textit{for } i = 1, 2, \ldots, \omega. \tag{2.6}$$

Proof. To prove this theorem on the convergence, first let

$$\widehat{X}_i(k) = X_i(k) - X_i^*, \qquad \widehat{Y}_i(k) = Y_i(k) - Y_i^*, \quad \text{for } i = 1, 2, \ldots, \omega \tag{2.7}$$

denote the error matrices in the kth iteration of Algorithm 1. By using the error matrices we have

$$R_i(k) = -\sum_{j=1}^{m}\Big(A_{ij}\widehat{X}_i(k)B_{ij} + C_{ij}\widehat{X}_{i+1}(k)D_{ij} + E_{ij}\widehat{Y}_i(k)F_{ij} + G_{ij}\widehat{Y}_{i+1}(k)H_{ij}\Big) \quad \text{for } i = 1, 2, \ldots, \omega. \tag{2.8}$$

By substituting the above the error matrices into the sequences $\{X_i(k)\}$ and $\{Y_i(k)\}$, we can obtain

$$\begin{aligned} \widehat{X}_i(k+1) \;=\; & \widehat{X}_i(k) - \delta\sum_{k=1}^{m}\Bigg[A_{ik}^T\sum_{j=1}^{m}\Big(A_{ij}\widehat{X}_i(k)B_{ij} + C_{ij}\widehat{X}_{i+1}(k)D_{ij} + E_{ij}\widehat{Y}_i(k)F_{ij} + G_{ij}\widehat{Y}_{i+1}(k)H_{ij}\Big)B_{ik}^T \\ & + C_{i-1,k}^T\sum_{j=1}^{m}\Big(A_{i-1,j}\widehat{X}_{i-1}(k)B_{i-1,j} + C_{i-1,j}\widehat{X}_i(k)D_{i-1,j} \\ & + E_{i-1,j}\widehat{Y}_{i-1}(k)F_{i-1,j} + G_{i-1,j}\widehat{Y}_i(k)H_{i-1,j}\Big)D_{i-1,k}^T\Bigg], \end{aligned}$$

$$\begin{aligned}
\widehat{Y}_i(k+1) \;=\; & \widehat{Y}_i(k) - \delta \sum_{k=1}^{m} \Bigg[E_{ik}^T \sum_{j=1}^{m} \Big(A_{ij}\widehat{X}_i(k)B_{ij} + C_{ij}\widehat{X}_{i+1}(k)D_{ij} + E_{ij}\widehat{Y}_i(k)F_{ij} + G_{ij}\widehat{Y}_{i+1}(k)H_{ij} \Big) F_{ik}^T \\
& + G_{i-1,k}^T \sum_{j=1}^{m} \Big(A_{i-1,j}\widehat{X}_{i-1}(k)B_{i-1,j} + C_{i-1,j}\widehat{X}_i(k)D_{i-1,j} \\
& + E_{i-1,j}\widehat{Y}_{i-1}(k)F_{i-1,j} + G_{i-1,j}\widehat{Y}_i(k)H_{i-1,j} \Big) H_{i-1,k}^T \Bigg],
\end{aligned}$$

for $i = 1, 2, \ldots, \omega$. Now for $i = 1, 2, \ldots, \omega$, we can get

$$\begin{aligned}
\|\widehat{X}_i(k+1)\|^2 &= \mathrm{tr}\Big(\widehat{X}_i(k+1)^T \widehat{X}_i(k+1) \Big) \\
&= \|\widehat{X}_i(k)\|^2 - 2\delta \mathrm{tr}\Bigg(\widehat{X}_i(k)^T \sum_{k=1}^{m} \Bigg[A_{ik}^T \sum_{j=1}^{m} \Big(A_{ij}\widehat{X}_i(k)B_{ij} + C_{ij}\widehat{X}_{i+1}(k)D_{ij} \\
&\quad + E_{ij}\widehat{Y}_i(k)F_{ij} + G_{ij}\widehat{Y}_{i+1}(k)H_{ij} \Big) B_{ik}^T \\
&\quad + C_{i-1,k}^T \sum_{j=1}^{m} \Big(A_{i-1,j}\widehat{X}_{i-1}(k)B_{i-1,j} + C_{i-1,j}\widehat{X}_i(k)D_{i-1,j} \\
&\quad + E_{i-1,j}\widehat{Y}_{i-1}(k)F_{i-1,j} + G_{i-1,j}\widehat{Y}_i(k)H_{i-1,j} \Big) D_{i-1,k}^T \Bigg] \Bigg) \\
&\quad + \delta^2 \Bigg\| \sum_{k=1}^{m} \Bigg[A_{ik}^T \sum_{j=1}^{m} \Big(A_{ij}\widehat{X}_i(k)B_{ij} + C_{ij}\widehat{X}_{i+1}(k)D_{ij} + E_{ij}\widehat{Y}_i(k)F_{ij} + G_{ij}\widehat{Y}_{i+1}(k)H_{ij} \Big) B_{ik}^T \\
&\quad + C_{i-1,k}^T \sum_{j=1}^{m} \Big(A_{i-1,j}\widehat{X}_{i-1}(k)B_{i-1,j} + C_{i-1,j}\widehat{X}_i(k)D_{i-1,j} \\
&\quad + E_{i-1,j}\widehat{Y}_{i-1}(k)F_{i-1,j} + G_{i-1,j}\widehat{Y}_i(k)H_{i-1,j} \Big) D_{i-1,k}^T \Bigg] \Bigg\|^2 \\
&= \|\widehat{X}_i(k)\|^2 - 2\delta \mathrm{tr}\Bigg(\sum_{k=1}^{m} \Bigg[A_{ik}\widehat{X}_i(k)B_{ik} \sum_{j=1}^{m} \Big(A_{ij}\widehat{X}_i(k)B_{ij} + C_{ij}\widehat{X}_{i+1}(k)D_{ij} \\
&\quad + E_{ij}\widehat{Y}_i(k)F_{ij} + G_{ij}\widehat{Y}_{i+1}(k)H_{ij} \Big)^T \\
&\quad + C_{i-1,k}\widehat{X}_i(k)D_{i-1,k} \sum_{j=1}^{m} \Big(A_{i-1,j}\widehat{X}_{i-1}(k)B_{i-1,j} + C_{i-1,j}\widehat{X}_i(k)D_{i-1,j} \\
&\quad + E_{i-1,j}\widehat{Y}_{i-1}(k)F_{i-1,j} + G_{i-1,j}\widehat{Y}_i(k)H_{i-1,j} \Big)^T \Bigg] \Bigg) \\
&\quad + \delta^2 \Bigg\| \sum_{k=1}^{m} \Bigg[A_{ik}^T \sum_{j=1}^{m} \Big(A_{ij}\widehat{X}_i(k)B_{ij} + C_{ij}\widehat{X}_{i+1}(k)D_{ij} + E_{ij}\widehat{Y}_i(k)F_{ij} + G_{ij}\widehat{Y}_{i+1}(k)H_{ij} \Big) B_{ik}^T \\
&\quad + C_{i-1,k}^T \sum_{j=1}^{m} \Big(A_{i-1,j}\widehat{X}_{i-1}(k)B_{i-1,j} + C_{i-1,j}\widehat{X}_i(k)D_{i-1,j} \\
&\quad + E_{i-1,j}\widehat{Y}_{i-1}(k)F_{i-1,j} + G_{i-1,j}\widehat{Y}_i(k)H_{i-1,j} \Big) D_{i-1,k}^T \Bigg] \Bigg\|^2 \\
&= \|\widehat{X}_i(k)\|^2 - 2\delta \mathrm{tr}\Bigg(\sum_{k=1}^{m} \Bigg[A_{ik}\widehat{X}_i(k)B_{ik} \sum_{j=1}^{m} \Big(A_{ij}\widehat{X}_i(k)B_{ij} + C_{ij}\widehat{X}_{i+1}(k)D_{ij}
\end{aligned}$$

$$+ E_{ij}\widehat{Y}_i(k)F_{ij} + G_{ij}\widehat{Y}_{i+1}(k)H_{ij}\Big)^T$$
$$+ C_{i,k}\widehat{X}_{i+1}(k)D_{i,k}\sum_{j=1}^{m}\Big(A_{ij}\widehat{X}_i(k)B_{ij} + C_{ij}\widehat{X}_{i+1}(k)D_{ij} + E_{ij}\widehat{Y}_i(k)F_{ij} + G_{ij}\widehat{Y}_{i+1}(k)H_{ij}\Big)^T\Bigg]\Bigg)$$
$$+ \delta^2\Bigg\|\sum_{k=1}^{m}\Bigg[A_{ik}^T\sum_{j=1}^{m}\Big(A_{ij}\widehat{X}_i(k)B_{ij} + C_{ij}\widehat{X}_{i+1}(k)D_{ij} + E_{ij}\widehat{Y}_i(k)F_{ij} + G_{ij}\widehat{Y}_{i+1}(k)H_{ij}\Big)B_{ik}^T$$
$$+ C_{i,k}^T\sum_{j=1}^{m}\Big(A_{ij}\widehat{X}_i(k)B_{ij} + C_{ij}\widehat{X}_{i+1}(k)D_{ij} + E_{ij}\widehat{Y}_i(k)F_{ij} + G_{ij}\widehat{Y}_{i+1}(k)H_{ij}\Big)D_{i,k}^T\Bigg]\Bigg\|^2. \quad (2.9)$$

In a similar way, we can obtain

$$\|\widehat{Y}_i(k+1)\|^2 = \|\widehat{Y}_i(k)\|^2 - 2\delta\mathrm{tr}\Bigg(\sum_{k=1}^{m}\Bigg[E_{ik}\widehat{Y}_i(k)F_{ik}\sum_{j=1}^{m}\Big(A_{ij}\widehat{X}_i(k)B_{ij} + C_{ij}\widehat{X}_{i+1}(k)D_{ij}$$
$$+ E_{ij}\widehat{Y}_i(k)F_{ij} + G_{ij}\widehat{Y}_{i+1}(k)H_{ij}\Big)^T$$
$$+ G_{i,k}\widehat{Y}_{i+1}(k)H_{i,k}\sum_{j=1}^{m}\Big(A_{ij}\widehat{X}_i(k)B_{ij} + C_{ij}\widehat{X}_{i+1}(k)D_{ij} + E_{ij}\widehat{Y}_i(k)F_{ij} + G_{ij}\widehat{Y}_{i+1}(k)H_{ij}\Big)^T\Bigg]\Bigg)$$
$$+ \delta^2\Bigg\|\sum_{k=1}^{m}\Bigg[E_{ik}^T\sum_{j=1}^{m}\Big(A_{ij}\widehat{X}_i(k)B_{ij} + C_{ij}\widehat{X}_{i+1}(k)D_{ij} + E_{ij}\widehat{Y}_i(k)F_{ij} + G_{ij}\widehat{Y}_{i+1}(k)H_{ij}\Big)F_{ik}^T$$
$$+ G_{i,k}^T\sum_{j=1}^{m}\Big(A_{ij}\widehat{X}_i(k)B_{ij} + C_{ij}\widehat{X}_{i+1}(k)D_{ij} + E_{ij}\widehat{Y}_i(k)F_{ij} + G_{ij}\widehat{Y}_{i+1}(k)H_{ij}\Big)H_{i,k}^T\Bigg]\Bigg\|^2. \quad (2.10)$$

Now we define the nonnegative definite functions $Z(k)$ by

$$Z(k) = \sum_{i=1}^{\omega}(\|\widehat{X}_i(k)\|^2 + \|\widehat{Y}_i(k)\|^2).$$

It can be obtained

$$Z(k+1) = \sum_{i=1}^{\omega}(\|\widehat{X}_i(k+1)\|^2 + \|\widehat{Y}_i(k+1)\|^2) = \sum_{i=1}^{\omega}\Bigg\{\|\widehat{X}_i(k)\|^2 + \|\widehat{Y}_i(k)\|^2$$
$$- 2\delta\mathrm{tr}\Bigg(\sum_{k=1}^{m}\Big(A_{ik}\widehat{X}_i(k)B_{ik} + C_{i,k}\widehat{X}_{i+1}(k)D_{i,k} + E_{ij}\widehat{Y}_i(k)F_{ij} + G_{i,k}\widehat{Y}_{i+1}(k)H_{i,k}\Big)$$
$$\times\sum_{j=1}^{m}\Big(A_{ij}\widehat{X}_i(k)B_{ij} + C_{ij}\widehat{X}_{i+1}(k)D_{ij} + E_{ij}\widehat{Y}_i(k)F_{ij} + G_{ij}\widehat{Y}_{i+1}(k)H_{ij}\Big)^T\Bigg)$$
$$+ \delta^2\Bigg\{\Bigg\|\sum_{k=1}^{m}\Bigg[A_{ik}^T\sum_{j=1}^{m}\Big(A_{ij}\widehat{X}_i(k)B_{ij} + C_{ij}\widehat{X}_{i+1}(k)D_{ij} + E_{ij}\widehat{Y}_i(k)F_{ij} + G_{ij}\widehat{Y}_{i+1}(k)H_{ij}\Big)B_{ik}^T$$
$$+ C_{i,k}^T\sum_{j=1}^{m}\Big(A_{ij}\widehat{X}_i(k)B_{ij} + C_{ij}\widehat{X}_{i+1}(k)D_{ij} + E_{ij}\widehat{Y}_i(k)F_{ij} + G_{ij}\widehat{Y}_{i+1}(k)H_{ij}\Big)D_{i,k}^T\Bigg]\Bigg\|^2$$
$$+ \Bigg\|\sum_{k=1}^{m}\Bigg[E_{ik}^T\sum_{j=1}^{m}\Big(A_{ij}\widehat{X}_i(k)B_{ij} + C_{ij}\widehat{X}_{i+1}(k)D_{ij} + E_{ij}\widehat{Y}_i(k)F_{ij} + G_{ij}\widehat{Y}_{i+1}(k)H_{ij}\Big)F_{ik}^T$$
$$+ G_{i,k}^T\sum_{j=1}^{m}\Big(A_{ij}\widehat{X}_i(k)B_{ij} + C_{ij}\widehat{X}_{i+1}(k)D_{ij} + E_{ij}\widehat{Y}_i(k)F_{ij} + G_{ij}\widehat{Y}_{i+1}(k)H_{ij}\Big)H_{i,k}^T\Bigg]\Bigg\|^2\Bigg\}\Bigg\}$$

$$\leq Z(k) - 2\delta \sum_{i=1}^{\omega} \Bigg\{ \Big\| \sum_{k=1}^{m} \Big(A_{ik}\widehat{X}_i(k)B_{ik} + C_{i,k}\widehat{X}_{i+1}(k)D_{i,k} + E_{ij}\widehat{Y}_i(k)F_{ij} + G_{i,k}\widehat{Y}_{i+1}(k)H_{i,k} \Big) \Big\|^2$$

$$+ \delta^2 \Bigg\{ \sum_{k=1}^{m} \Big[\|A_{ik}B_{ik}\|^2 + \|C_{i,k}D_{i,k}\|^2 + \|E_{ik}F_{ik}\|^2 + \|G_{i,k}H_{i,k}\|^2 \Big]$$

$$+ \Big\| \sum_{j=1}^{m} \Big(A_{ij}\widehat{X}_i(k)B_{ij} + C_{ij}\widehat{X}_{i+1}(k)D_{ij} + E_{ij}\widehat{Y}_i(k)F_{ij} + G_{ij}\widehat{Y}_{i+1}(k)H_{ij} \Big) \Big\|^2 \Bigg\} \Bigg\}$$

$$\leq Z(k) - \delta \left[2 - \delta \sum_{i=1}^{\omega} \sum_{k=1}^{m} \Big(\|A_{ik}B_{ik}\|^2 + \|C_{i,k}D_{i,k}\|^2 + \|E_{ik}F_{ik}\|^2 + \|G_{i,k}H_{i,k}\|^2 \Big) \right]$$

$$\times \sum_{i=1}^{\omega} \Big\| \sum_{j=1}^{m} \Big(A_{ij}\widehat{X}_i(k)B_{ij} + C_{ij}\widehat{X}_{i+1}(k)D_{ij} + E_{ij}\widehat{Y}_i(k)F_{ij} + G_{ij}\widehat{Y}_{i+1}(k)H_{ij} \Big) \Big\|^2$$

$$\leq Z(1) - \delta \left[2 - \delta \sum_{i=1}^{\omega} \sum_{k=1}^{m} \Big(\|A_{ik}B_{ik}\|^2 + \|C_{i,k}D_{i,k}\|^2 + \|E_{ik}F_{ik}\|^2 + \|G_{i,k}H_{i,k}\|^2 \Big) \right]$$

$$\times \sum_{l=1}^{k} \sum_{i=1}^{\omega} \Big\| \sum_{j=1}^{m} \Big(A_{ij}\widehat{X}_i(l)B_{ij} + C_{ij}\widehat{X}_{i+1}(l)D_{ij} + E_{ij}\widehat{Y}_i(l)F_{ij} + G_{ij}\widehat{Y}_{i+1}(l)H_{ij} \Big) \Big\|^2. \tag{2.11}$$

By considering (2.5) and (2.11), we conclude that

$$\sum_{l=1}^{\infty} \sum_{i=1}^{\omega} \Big\| \sum_{j=1}^{m} \Big(A_{ij}\widehat{X}_i(l)B_{ij} + C_{ij}\widehat{X}_{i+1}(l)D_{ij} + E_{ij}\widehat{Y}_i(l)F_{ij} + G_{ij}\widehat{Y}_{i+1}(l)H_{ij} \Big) \Big\|^2 < \infty. \tag{2.12}$$

It follows from the necessary condition of the series convergence that

$$\lim_{l\to\infty} \sum_{i=1}^{\omega} \Big\| \sum_{j=1}^{m} \Big(A_{ij}\widehat{X}_i(l)B_{ij} + C_{ij}\widehat{X}_{i+1}(l)D_{ij} + E_{ij}\widehat{Y}_i(l)F_{ij} + G_{ij}\widehat{Y}_{i+1}(l)H_{ij} \Big) \Big\|^2 = 0. \tag{2.13}$$

Hence we can obtain

$$\lim_{l\to\infty} \sum_{j=1}^{m} \Big(A_{ij}\widehat{X}_i(l)B_{ij} + C_{ij}\widehat{X}_{i+1}(l)D_{ij} + E_{ij}\widehat{Y}_i(l)F_{ij} + G_{ij}\widehat{Y}_{i+1}(l)H_{ij} \Big) = 0, \quad i = 1, 2, \ldots, \omega. \tag{2.14}$$

Now according to Lemma 1, we can write

$$\lim_{l\to\infty} \widehat{X}_i(l) = 0, \quad \lim_{l\to\infty} \widehat{Y}_i(l) = 0, \quad \text{for } i = 1, 2, \ldots, \omega. \tag{2.15}$$

The proof is finished. □

3. Numerical examples

In this section we consider the discrete-time periodic matrix equation

$$A_i X_i + X_{i+1} B_i = C_i, \quad i = 1, 2, 3,$$

where

$$A_1 = \begin{pmatrix} 2.6756 & 0.3840 & 0.6085 & 0.0576 & 0.0841 \\ 0 & 2.4508 & 0.0158 & 0.3676 & 0.4544 \\ 0 & 0 & 2.2324 & 0.6315 & 0.4418 \\ 0 & 0 & 0 & 2.0784 & 0.3533 \\ 0 & 0 & 0 & 0 & 2.9943 \end{pmatrix},$$

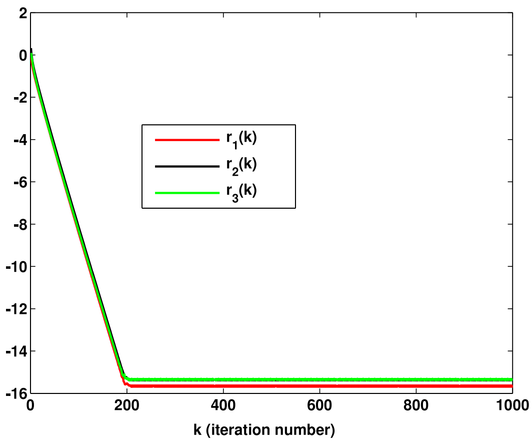

Fig. 1. The residuals.

$$A_2 = \begin{pmatrix} -3.2475 & 0.5915 & 0 & 0 & 0 \\ 0.3400 & -3.7362 & 0.2644 & 0 & 0 \\ 0.3142 & 0.0381 & -3.2519 & 0.6649 & 0 \\ 0.3651 & 0.4586 & 0.8729 & -2.7797 & 0.8903 \\ 0.3932 & 0.8699 & 0.2379 & 0.0099 & -2.9985 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} -7.4617 & 0.9200 & 0.1939 & 0.5488 & 0.6273 \\ 0.0099 & -6.6666 & 0.9048 & 0.9316 & 0.6991 \\ 0.4199 & 0.3678 & -7.2374 & 0.3352 & 0.3972 \\ 0.75370.6208 & 0.6318 & -6.4845 & 0.4136 & \\ 0.7939 & 0.7313 & 0.2344 & 0.3919 & -6.4036 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 9.1529 & 0.7621 & 0.6154 & 0.4057 & 0.0579 \\ 0.2311 & 9.2033 & 0.7919 & 0.9355 & 0.3529 \\ 0.6068 & 0.0185 & 9.4470 & 0.9169 & 0.8132 \\ 0.4860 & 0.8214 & 0.7382 & 8.7898 & 0.0099 \\ 0.8913 & 0.4447 & 0.1763 & 0.8936 & 8.3285 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} 8.8962 & 0.6979 & 0 & 0 & 0 \\ 0.6822 & 9.3352 & 0.8998 & 0 & 0 \\ 0.3028 & 0.8600 & 9.0740 & 0.2897 & 0 \\ 0.5417 & 0.8537 & 0.6449 & 8.5403 & 0.5681 \\ 0.1509 & 0.5936 & 0.8180 & 0.5341 & 9.3587 \end{pmatrix},$$

$$B_3 = \begin{pmatrix} 8.5536 & 0.2259 & 0 & 0 & 0 \\ 0.4235 & 8.2233 & 0.3798 & 0 & 0 \\ 0.5155 & 0.7604 & 8.0513 & 0.0592 & 0 \\ 0.3340 & 0.5298 & 0.6808 & 8.2317 & 0.0150 \\ 0.4329 & 0.6405 & 0.4611 & 0.0503 & 8.3431 \end{pmatrix},$$

and $C_1 = C_2 = C_3 = \mathrm{rand}(5)$. By applying Algorithm 1 with the initial matrices $X_1(1) = X_2(1) = X_3(1) = 0$ and parameter $\delta = 0.006$, we obtain results presented in Fig. 1 where

$$r_i(k) = \log_{10} \|C_i - A_i X_i(k) - X_{i+1}(k) B_i\|, \quad \text{for } i = 1, 2, 3.$$

From Fig. 1, it is clear that $r_1(k), r_2(k), r_3(k)$ are becoming smaller and smaller and go to zero as k increases. This indicates that the proposed algorithm is effective.

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References

[1] S. Bittanti, Deterministic and stochastic linear periodic systems, in: S. Bittanti (Ed.), Time Series and Linear Systems, Springer-Verlag, Berlin, 1986, pp. 141–182.
[2] S. Bittanti, P. Colaneri, Analysis of discrete-time linear periodic systems, Control Dyn. Syst. 78 (1996) 313–339.
[3] B. Zhou, G.R. Duan, Z. Lin, A parametric periodic Lyapunov equation with application in semi-global stabilization of discrete-time periodic systems subject to actuator saturation, Automatica 47 (2011) 316–325.
[4] B. Zhou, G.R. Duan, Periodic Lyapunov equation based approaches to the stabilization of continuous-time periodic linear systems, IEEE Trans. Automat. Control 57 (2012) 2139–2146.
[5] P. Benner, M.S. Hossain, T. Stykel, Low rank iterative methods of periodic projected Lyapunov equations and their application in model reduction of periodic descriptor systems, Chemnitz Sci. Comput. (2011). Preprints 11-01.
[6] E.K.W. Chu, H.Y. Fan, W.W. Lin, Projected generalized discrete-time periodic Lyapunov equations and balanced realization of periodic descriptor systems, SIAM J. Matrix Anal. Appl. 29 (2007) 982–1006.
[7] T. Stykel, Low-rank iterative methods for projected generalized Lyapunov equations, Electron. Trans. Numer. Anal. 30 (2008) 187–202.
[8] R. Byers, N. Rhee, Cyclic Schur and Hessenberg Schur numerical methods for solving periodic Lyapunov and Sylvester equations, Technical Report, Dept. of Mathematics, Univ. of Missouri at Kansas City, 1995.
[9] A. Varga, Periodic Lyapunov equations: some applications and new algorithms, Internat. J. Control 67 (1997) 69–87.
[10] D. Kressner, Large periodic Lyapunov equations: Algorithms and applications, in: Proc. of ECC03, Cambridge, UK, 2003.
[11] P. Benner, M.S. Hossain, T. Stykel, Low-rank iterative methods for periodic projected Lyapunov equations and their application in model reduction of periodic descriptor systems, Numer. Algorithms 67 (2014) 669–690.
[12] M. Hajarian, Matrix form of the CGS method for solving general coupled matrix equations, Appl. Math. Lett. 34 (2014) 37–42.
[13] S.K. Li, T.Z. Huang, LSQR iterative method for generalized coupled Sylvester matrix equations, Appl. Math. Model. 36 (2012) 3545–3554.
[14] M. Dehghan, M. Hajarian, Analysis of an iterative algorithm to solve the generalized coupled Sylvester matrix equations, Appl. Math. Model. 35 (2011) 3285–3300.
[15] M. Hajarian, Recent developments in iterative algorithms for solving linear matrix equations, in: I. Kyrchei (Ed.), Advances in Linear Algebra Research, Nova Sci. Publ., New York, 2015, pp. 239–286.
[16] M. Hajarian, Extending LSQR methods to solve the generalized Sylvester-transpose and periodic Sylvester matrix equations, Math. Methods Appl. Sci. 37 (2014) 2017–2028.
[17] M. Hajarian, Matrix algorithms for solving the generalized coupled Sylvester and periodic coupled matrix equations, Trans. Inst. Meas. Control 38 (2015) 963–970.
[18] M. Dehghan, M. Hajarian, Iterative algorithms for the generalized centro-symmetric and central anti-symmetric solutions of general coupled matrix, Eng. Comput. 29 (2012) 528–560.
[19] F. Ding, T. Chen, Iterative least squares solutions of coupled Sylvester matrix equations, Systems Control Lett. 54 (2005) 95–107.
[20] F. Ding, T. Chen, On iterative solutions of general coupled matrix equations, SIAM J. Control Optim. 44 (2006) 2269–2284.
[21] B. Zhou, J. Lam, G.R. Duan, Gradient-based maximal convergence rate iterative method for solving linear matrix equations, Int. J. Comput. Math. 87 (2010) 515–527.
[22] B. Zhou, G.R. Duan, Z.Y. Li, Gradient based iterative algorithm for solving coupled matrix equations, Systems Control Lett. 58 (2009) 327–333.
[23] F. Ding, T. Chen, Gradient based iterative algorithms for solving a class of matrix equations, IEEE Trans. Automat. Control 50 (2005) 1216–1221.
[24] B. Zhou, Z.Y. Li, G.R. Duan, Y. Wang, Weighted least squares solutions to general coupled Sylvester matrix equations, J. Comput. Appl. Math. 224 (2009) 759–776.