Otto-von-Guericke-Universität Magdeburg Sommersemester 2015
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Linear Systems and Matrix Equations – Bartels Stewart Algorithm.

We want to compute the solution X of the Lyapunov equation

$$AX + XA^T = W, \quad A \in \mathbb{R}^{n\times n}, \quad W = W^T \in \mathbb{R}^{n\times n}. \tag{1}$$

Algorithmus 1 Bartels-Stewart (complex)

Input: $A, W \in \mathbb{C}^{n\times n}, W = W^H$
Output: $X = X^H$ solving (1)
1: Compute $T = Q^H AQ$ with QR-algorithm
2: **if** $\operatorname{diag}(T) \cap \operatorname{diag}(-T) \neq \emptyset$ **then**
3: STOP (no unique solution)
4: **end if**
5: $\begin{bmatrix} W_1 & W_2 \\ W_2^H & W_3 \end{bmatrix} \leftarrow Q^H W Q$
6: $k = n - 1$
7: **while** $k > 1$ **do**
8: Solve $X_3 = \frac{\tilde{W}_3}{T_3+\overline{T}_3}$ with $T_3 = T_{kk}$
9: Solve

$$T_1 X_2 + X_2 \overline{T}_3^H = W_1$$

with $T_1 = T(1:k, 1:k), W_1 = W_1 - T_2 X_2^H - X_2 T_2^H$
10: $k = k - 1$
11: **end while**
12: Solve $T_1 X_1 + X_1 T_1^H = W_2 - T_2 X_2^H - X_2 T_2^H$ in $\mathbb{C}^{1\times 1}$
13: Back transformation $X \leftarrow QXQ^H$

The Bartels-Stewart algorithm requires approximately $32n^3$ elementary operations, consisting of $\approx 25n^3$ for the QR algorithm, and $\approx 3n^3$ for each of the steps 5 and 13. The whole backward substitution (**while**-loop) requires only n^3 flops. The QR Algorithmus as well as the backward substitution can be implemented numerically backwards stable, and, moreover, only unitary similarity transformations are used. Hence, the Bartels-Stewart algorithm can be considered as numerically backwards stable.
The Bartels-Stewart algorithm for solving Lyapunov and Sylvester equations is, e.g., in the Matlab Control Toolbox routine `lyap` implemented.

Literatur

[1] R. Bartels and G. Stewart, *Solution of the Matrix Equation $AX + XB = C$: Algorithm 432*, Comm. ACM, 15 (1972), pp. 820–826.

[2] D. Sorensen and Y. Zhou, P. Benner, P. Kürschner, and J. Saak, *Direct methods for matrix Sylvester and Lyapunov equations*, J. Appl. Math, 2003 (2003), pp. 277–303.