THE

LONDON, EDINBURGH, AND DUBLIN

PHILOSOPHICAL MAGAZINE

AND

JOURNAL OF SCIENCE.

SUPPLEMENT TO VOL. XXXIII. FOURTH SERIES.

LXIII. *On Integrals of the Hydrodynamical Equations, which express Vortex-motion.* By H. HELMHOLTZ*.

HITHERTO, in integrating the hydrodynamical equations, the assumption has been made that the components of the velocity of each element of the fluid in three directions at right angles to each other are the differential coefficients, with reference to the coordinates, of a definite function which we shall call the *velocity-potential*. Lagrange† no doubt has shown that this assumption is lawful if the motion of the fluid has been produced by, and continued under, the action of forces which have a potential; and also that the influence of moving solids which are in contact with the fluid does not affect the lawfulness of the assumption. And, since the greater number of natural forces which can be defined with mathematical strictness can be expressed as differential coefficients of a potential, by far the greater number of mathematically investigable cases of fluid-motion belong to the class in which a velocity-potential exists.

Yet Euler‡ has distinctly pointed out that there are cases of fluid-motion in which no velocity-potential exists,—for instance, the rotation of a fluid about an axis when every element has the same angular velocity. Among the forces which can produce such motions may be named magnetic attractions acting upon a fluid conducting electric currents, and particularly friction, whether among the elements of the fluid or against fixed bodies. The effect of fluid friction has not hitherto been mathematically defined; yet it is very great, and, except in the case of indefinitely small oscillations, produces most marked differences between theory and fact. The difficulty of defining this effect, and of finding expressions for its measurement, mainly con-

* From Crelle's *Journal*, vol. lv. (1858), kindly communicated by Professor Tait.

† *Mécanique Analytique* (Paris, 1815), vol. ii. p. 304.

‡ *Histoire de l'Académie des Sciences de Berlin* (1755), p. 292.

sisted in the fact that no idea had been formed of the species of motion which friction produces in fluids. Hence it appeared to me to be of importance to investigate the species of motion for which there is no velocity-potential.

The following investigation shows that when there is a velocity-potential the elements of the fluid have no rotation, but that there is at least a portion of the fluid elements in rotation when there is no velocity-potential.

By *vortex-lines* (*Wirbellinien*) I denote lines drawn through the fluid so as at every point to coincide with the instantaneous axis of rotation of the corresponding fluid element.

By *vortex-filaments* (*Wirbelfäden*) I denote portions of the fluid bounded by vortex-lines drawn through every point of the boundary of an infinitely small closed curve.

The investigation shows that, if all the forces which act on the fluid have a potential,—

1. No element of the fluid which was not originally in rotation is made to rotate.

2. The elements which at any time belong to one vortex-line, however they may be translated, remain on one vortex-line.

3. The product of the section and the angular velocity of an infinitely thin vortex-filament is constant throughout its whole length, and retains the same value during all displacements of the filament. Hence vortex-filaments must either be closed curves, or must have their ends in the bounding surface of the fluid.

This last theorem enables us to determine the angular velocity when the form of the vortex-filament at different times is given. Besides, there is given a solution of the problem of finding the velocities of the fluid elements at any instant, if at that instant the angular velocities are given: an arbitrary function, however, remains undetermined, and is to be applied to satisfy the boundary conditions.

This last example leads to a remarkable analogy between the vortex-motion of fluids and the electro-magnetic action of electric currents. If, for instance, in a simply-connected * (*einfach zusammenhängend*) space full of fluid there be a velocity-potential, the velocities of the fluid elements are equal to, and in the same direction as, the forces exerted on a magnetic particle in the interior of the space by a certain distribution of magnetic masses or electric currents on its surface. But, if vortex-fila-

* I use this expression in the sense in which Riemann (Crelle, vol. liv. p. 108) speaks of simply and complexly connected surfaces. An $n$ly connected space is thus one which can be cut through by $n-1$, but no more, surfaces, without being separated into two detached portions. In this sense a ring is a doubly-connected space. The cutting surfaces must be completely enclosed within the lines in which they cut the bounding surface of the space considered.

ments exist in such a space, the velocities of the fluid elements are represented by the forces exerted on a magnetic particle by closed electric currents which flow partly through the vortex-filaments in the interior of the fluid mass, partly on its surface, their intensity being proportional to the product of the section of the vortex-filament and the angular velocity.

I shall therefore frequently avail myself of the analogy of magnetic masses or electric currents, simply to give a briefer and more vivid representation of quantities which are just such functions of the coordinates as the attractive forces exerted by these masses or currents on a magnetic particle, or the corresponding potential functions.

By means of these theorems the various species of fluid-motion which are concealed in the yet unstudied integrals of the hydrodynamical equations can at least be represented, even although the complete integration is possible only in a few of the simplest cases—as when we have one or two straight or circular vortex-filaments in a mass of fluid which is either infinite in all directions or bounded in one direction by an infinite plane.

It can be shown that straight parallel vortex-filaments, in a fluid mass which is limited only by planes perpendicular to the filaments, revolve about their common centre of gravity, if we determine this point by employing the angular velocity as we would the density of a mass. The position of the centre of gravity remains unaltered. On the other hand, in the case of circular vortex-filaments which are all perpendicular to a common axis, the centre of gravity of their section moves on parallel to the axis.

§ 1.

At a point x, y, z in a liquid let p be the pressure, u, v, w the rectangular components of the velocity, X, Y, Z the components of external forces acting on unit of mass, and h the density (whose variations will be supposed indefinitely small), all at time t. Then we have the following known equations of motion for the interior particles of the fluid:—

$$\left.\begin{aligned} \mathrm{X}-\frac{1}{h}\cdot\frac{dp}{dx}&=\frac{du}{dt}+u\frac{du}{dx}+v\frac{du}{dy}+w\frac{du}{dz},\\ \mathrm{Y}-\frac{1}{h}\cdot\frac{dp}{dy}&=\frac{dv}{dt}+u\frac{dv}{dx}+v\frac{dv}{dy}+w\frac{dv}{dz},\\ \mathrm{Z}-\frac{1}{h}\cdot\frac{dp}{dz}&=\frac{dw}{dt}+u\frac{dw}{dx}+v\frac{dw}{dy}+w\frac{dw}{dz},\\ 0&=\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}. \end{aligned}\right\} \quad . \quad . \quad (1)$$

Hitherto, with scarcely an exception, no cases have been treated except those where not only have the forces X, Y, Z a potential V, so that

$$X=\frac{dV}{dx},\quad Y=\frac{dV}{dy},\quad Z=\frac{dV}{dz}, \quad . \quad . \quad . \quad (1\,a)$$

but also a velocity-potential ϕ can be found, giving

$$u=\frac{d\phi}{dx},\quad v=\frac{d\phi}{dy},\quad w=\frac{d\phi}{dz}. \quad . \quad . \quad . \quad . \quad (1\,b)$$

By these assumptions the problem is immensely simplified, since the first three of the equations (1) give a common integral from which p is to be found, ϕ having previously been determined so as to satisfy the fourth equation, which becomes in this case

$$\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}+\frac{d^2\phi}{dz^2}=0,$$

and coincides with the known differential equation for the potential of magnetic masses which are external to the space in which the equation holds good. And it is known that every such function ϕ which satisfies the above equation within a simply-connected* space, can be expressed as the potential of a definite distribution of magnetic matter on the bounding surface, as I have already mentioned in the introduction.

In order that the substitution (1 b) may be lawful, we must have

$$\frac{du}{dy}-\frac{dv}{dx}=0,\quad \frac{dv}{dz}-\frac{dw}{dy}=0,\quad \frac{dw}{dx}-\frac{du}{dz}=0. \quad . \quad (1\,c)$$

To understand the mechanical meaning of these last three conditions, we may consider the changes undergone by an indefinitely small volume of the fluid during the time dt as compounded of three separate motions. 1st. A *translation* of the whole in space. 2nd. An *expansion* or *contraction* of the whole parallel to three dilatation-axes, so that any rectangular parallelepiped whose edges are parallel to these axes may remain rectangular, while its edges alter their length but remain parallel to their original directions. 3rd. A *rotation* about some instantaneous axis, which, as we know, may be considered as the resultant of three rotations about the axes of coordinates.

If the conditions (1 c) are fulfilled at a point whose coordi-

* In complexly-connected spaces ϕ may have more values than one; and for multiple-valued functions which satisfy the above differential equation Green's fundamental theorem does not hold; and hence a great number of its consequences which Gauss and Green have deduced for magnetic potential functions also fail, since the latter, from their very nature, can have but single values.

nates are $\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{z}$, we may put at that point

$$u=\mathrm{A}, \quad \frac{du}{dx}=a, \quad \frac{dw}{dy}=\frac{dv}{dz}=\alpha,$$

$$v=\mathrm{B}, \quad \frac{dv}{dy}=b, \quad \frac{du}{dz}=\frac{dw}{dx}=\beta,$$

$$w=\mathrm{C}, \quad \frac{dw}{dz}=c, \quad \frac{dv}{dx}=\frac{du}{dy}=\gamma,$$

whence we have for a point x, y, z indefinitely near $\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{z}$,

$$\begin{aligned} u&=\mathrm{A}+a(x-\mathfrak{x})+\gamma(y-\mathfrak{y})+\beta(z-\mathfrak{z}),\\ v&=\mathrm{B}+\gamma(x-\mathfrak{x})+b(y-\mathfrak{y})+\alpha(z-\mathfrak{z}),\\ w&=\mathrm{C}+\beta(x-\mathfrak{x})+\alpha(y-\mathfrak{y})+c(z-\mathfrak{z}).\end{aligned}$$

But if we make

$$\begin{aligned}\phi=&\mathrm{A}(x-\mathfrak{x})+\mathrm{B}(y+\mathfrak{y})+\mathrm{C}(z-\mathfrak{z})\\ &+\alpha(y-\mathfrak{y})(z-\mathfrak{z})+\beta(x-\mathfrak{x})(z-\mathfrak{z})+\gamma(x-\mathfrak{x})(y-\mathfrak{y})\\ &+\tfrac{1}{2}a(x-\mathfrak{x})^2+\tfrac{1}{2}b(y-\mathfrak{y})^2+\tfrac{1}{2}c(z-\mathfrak{z})^2,\end{aligned}$$

then we have

$$u=\frac{d\phi}{dx}, \quad v=\frac{d\phi}{dy}, \quad w=\frac{d\phi}{dz}.$$

It is known that we can in one definite way, by change of axes to those of x_1, y_1, z_1 ($\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{z}$ being the origin), reduce the expression for ϕ to the form

$$\phi=\mathrm{A}_1x_1+\mathrm{B}_1y_1+\mathrm{C}_1z_1+\tfrac{1}{2}a_1x_1^2+\tfrac{1}{2}b_1y_1^2+\tfrac{1}{2}c\,z_1^2;$$

and the components of the velocity parallel to these new axes are

$$u_1=\mathrm{A}_1+a_1x_1, \quad v_1=\mathrm{B}_1+b_1y_1, \quad w_1=\mathrm{C}_1+c\,z_1.$$

The component u_1 is thus the same for all points for which x_1 has the same value; hence particles which at the beginning of the time dt are in a plane parallel to one of the coordinate planes will at the end of dt also lie in such a plane. Hence an indefinitely small parallelepiped with its edges parallel to the axes of x_1, y_1, z_1 will move parallel to itself in space, and suffer only dilatations or compressions (cases (1) and (2) above).

Let us return to the first system of axes, and suppose that in addition to the above motions there are, for the element of the fluid just considered, angular velocities ξ, η, ζ about axes through $\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{z}$ parallel to the coordinate axes. These give rise to the component velocities

$$\begin{array}{ccc} 0, & (z-\mathfrak{z})\xi, & -(y-\mathfrak{y})\xi, \\ -(z-\mathfrak{z})\eta, & 0, & (x-\mathfrak{x})\eta, \\ (y-\mathfrak{y})\zeta, & -(x-\mathfrak{x})\zeta, & 0; \end{array}$$

and the velocities of the element whose coordinates are x, y, z are now

$$\begin{aligned} u &= \mathrm{A}+a(x-\mathfrak{x})+(\gamma+\zeta)(y-\mathfrak{y})+(\beta-\eta)(z-\mathfrak{z}), \\ v &= \mathrm{B}+(\gamma-\zeta)(x-\mathfrak{x})+b(y-\mathfrak{y})+(\alpha+\xi)(z-\mathfrak{z}), \\ w &= \mathrm{C}+(\beta+\eta)(x-\mathfrak{x})+(\alpha-\xi)(y-\mathfrak{y})+c(z-\mathfrak{z}). \end{aligned}$$

From these, by differentiation,

$$\left.\begin{aligned} \frac{dv}{dz}-\frac{dw}{dy} &= 2\xi, \\ \frac{dw}{dx}-\frac{du}{dz} &= 2\eta, \\ \frac{du}{dy}-\frac{dv}{dx} &= 2\zeta. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The quantities on the left, which are zero by (1 c) when there is a velocity-potential, are now the doubles of the angular velocities of the fluid element about the coordinate axes. The existence of a velocity-potential, then, is inconsistent with the existence of a rotation of the fluid element.

As a further characteristic property of motion with a velocity-potential, we may adduce that no such motion can exist in a rigid and unmoved closed vessel full of fluid, whose interior is a simply-connected space S. For if n be the normal to the bounding surface drawn inwards, the velocity-component perpendicular to the surface is $\frac{d\phi}{dn}$, and must of course vanish. But Green* has shown that

$$\iiint\left[\left(\frac{d\phi}{dx}\right)^2+\left(\frac{d\phi}{dy}\right)^2+\left(\frac{d\phi}{dz}\right)^2\right]dx\,dy\,dz=-\int\phi\frac{d\phi}{dn}d\omega,$$

where the first integral extends to the whole space S, and the second to the whole bounding surface of S, an element of which is denoted by $d\omega$. As in this case $\frac{d\phi}{dn}$ is identically zero over the whole surface, the triple integral must vanish, and therefore throughout the whole space S we must have

$$\frac{d\phi}{dx}=0, \quad \frac{d\phi}{dy}=0, \quad \frac{d\phi}{dz}=0;$$

so that there cannot exist such a motion of the fluid.

* This, as before remarked, is not true for complexly-connected spaces.

Every motion of a fluid enclosed in a singly-connected space, when a velocity-potential exists, is therefore dependent on the motion of the free surface of the fluid. When the latter (*i. e.* $\frac{d\phi}{dn}$) is given, the whole motion of the enclosed fluid is completely determined. For suppose that two functions $\phi_{\prime}$ and $\phi_{\prime\prime}$ could satisfy the condition

$$\frac{d\phi}{dn}=\psi$$

at the surface of S, and also

$$\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}+\frac{d^2\phi}{dz^2}=0$$

through the whole interior of S. The function $\phi_{\prime}-\phi_{\prime\prime}$ satisfies the latter; but at the free surface $\frac{d(\phi_{\prime}-\phi_{\prime\prime})}{dn}=0$, and therefore, as before, we must have for the whole interior

$$\frac{d(\phi_{\prime}-\phi_{\prime\prime})}{dx}=0,\quad \frac{d(\phi_{\prime}-\phi_{\prime\prime})}{dy}=0,\quad \frac{d(\phi_{\prime}-\phi_{\prime\prime})}{dz}=0,$$

and the component velocities at every point in the fluid are therefore alike for both cases.

Thus it is only when there is no velocity-potential that some fluid elements can rotate, and that others can move round along a closed curve, in a simply-connected closed space. We may therefore call the motions which have no velocity-potential, generally, *vortex-motions*.

§ 2.

We must next determine the variations of the angular velocities ξ, η, ζ during the motion, when the only external forces are such as have a potential. Let us note once for all, that if ψ be a function of x, y, z, and t, and if it increase by $\delta\psi$ when these increase by δx, δy, δz, δt respectively, then

$$\delta\psi=\frac{d\psi}{dt}\,\delta t+\frac{d\psi}{dx}\,\delta x+\frac{d\psi}{dy}\,\delta y+\frac{d\psi}{dz}\,.\,\delta z.$$

If we now take the change of ψ in the time δt for the same elementary volume of fluid, we must give δx, δy, δz the values which they have for the moving element—that is,

$$\delta x=u\delta t,\quad \delta y=v\delta t,\quad \delta z=w\delta t\,;$$

and we obtain

$$\frac{\delta\psi}{\delta t}=\frac{d\psi}{dt}+u\frac{d\psi}{dx}+v\frac{d\psi}{dy}+w\frac{d\psi}{dz}.$$

We will employ the symbol $\frac{\delta\psi}{\delta t}$ in what follows only in the sense that $\frac{\delta\psi}{\delta t}dt$ is the change of ψ during dt for that element of the fluid whose coordinates at the commencement of dt were x, y, z.

If we eliminate p by differentiation from (1), and with the help of (2) introduce the new expressions, supposing (1 a) to be true for the forces X, Y, Z, we obtain the following equations:—

$$\left.\begin{aligned}\frac{\delta\xi}{\delta t}&=\xi\frac{du}{dx}+\eta\frac{du}{dy}+\zeta\frac{du}{dz},\\ \frac{\delta\eta}{\delta t}&=\xi\frac{dv}{dx}+\eta\frac{dv}{dy}+\zeta\frac{dv}{dz},\\ \frac{\delta\zeta}{\delta t}&=\xi\frac{dw}{dx}+\eta\frac{dw}{dy}+\zeta\frac{dw}{dz}\end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \quad (3)$$

or, which amounts to the same,

$$\left.\begin{aligned}\frac{\delta\xi}{\delta t}&=\xi\frac{du}{dx}+\eta\frac{dv}{dx}+\zeta\frac{dw}{dx},\\ \frac{\delta\eta}{\delta t}&=\xi\frac{du}{dy}+\eta\frac{dv}{dy}+\zeta\frac{dw}{dy},\\ \frac{\delta\zeta}{\delta t}&=\xi\frac{du}{dz}+\eta\frac{dv}{dz}+\zeta\frac{dw}{dz}.\end{aligned}\right\} \quad . \; . \; . \; . \; . \quad (3a)$$

If in a fluid element ξ, η, ζ are simultaneously equal to zero, we have also

$$\frac{\delta\xi}{\delta t}=\frac{\delta\eta}{\delta t}=\frac{\delta\zeta}{\delta t}=0.$$

Hence *those elements of the fluid which at any instant have no rotation, remain during the whole motion without rotation.*

We can apply the method of the parallelogram of forces to rotations. Since ξ, η, ζ are the angular velocities about rectangular axes, the angular velocity about the instantaneous axis is

$$q=\sqrt{\xi^2+\eta^2+\zeta^2},$$

and the direction-cosines of that axis are

$$\frac{\xi}{q},\ \frac{\eta}{q},\ \frac{\zeta}{q}.$$

If we now take in the direction of this instantaneous axis the indefinitely small portion $q\epsilon$, its projections on the axes are $\epsilon\xi$, $\epsilon\eta$, and $\epsilon\zeta$. While at x, y, z the components of the velocity are u, v, w, at the other end of $q\epsilon$ they are

$$u_1 = u + \epsilon\xi \frac{du}{dx} + \epsilon\eta \frac{du}{dy} + \epsilon\zeta \frac{du}{dz},$$

$$v_1 = v + \epsilon\xi \frac{dv}{dx} + \epsilon\eta \frac{dv}{dy} + \epsilon\zeta \frac{dv}{dz},$$

$$w_1 = w + \epsilon\xi \frac{dw}{dx} + \epsilon\eta \frac{dw}{dy} + \epsilon\zeta \frac{dw}{dz}.$$

At the end of the time dt, the projections of the distance between the two elements of fluid, which at the beginning of dt limited the line $q\epsilon$, have values which by the aid of (3) may be thus written:—

$$\epsilon\xi + (u_1 - u)dt = \epsilon\left(\xi + \frac{\delta\xi}{\delta t}\,dt\right),$$

$$\epsilon\eta + (v_1 - v)dt = \epsilon\left(\eta + \frac{\delta\eta}{\delta t}\,dt\right),$$

$$\epsilon\zeta + (w_1 - w)dt = \epsilon\left(\zeta + \frac{\delta\zeta}{\delta t}\,dt\right).$$

The left-hand sides of these equations give the projections of the new position of the joining line $q\epsilon$, the right-hand the projections of the new velocity of rotation, multiplied by the constant factor ϵ. It follows from these equations that the joining line between the elements, which at the commencement of dt limited the portion $q\epsilon$ of the instantaneous axis, also after the lapse of dt coincides with the altered axis of rotation.

If we call *vortex-line* a line whose direction coincides everywhere with the instantaneous axis of rotation of the there-situated element of fluid as above described, we can enunciate the above theorem in the following manner:—*Each vortex-line remains continually composed of the same elements of fluid,* and swims forward with them in the fluid.

The rectangular components of the angular velocity vary directly as the projections of the portion $q\epsilon$ of the axis of rotation; it follows from this that *the magnitude of the resultant angular velocity in a defined element varies in the same proportion as the distance between this and its neighbour along the axis of rotation.*

Conceive that vortex-lines are drawn through every point in the circumference of any indefinitely small surface; there will thus be set apart in the fluid a filament of indefinitely small section which we shall call *vortex-filament.* The volume of a portion of such a filament bounded by two given fluid elements, which (by the preceding propositions) remains filled by the same element of fluid, must in the motion remain constant, and its section must therefore vary inversely as its length. Hence the last theorem may be stated as follows:—*The product of the section and the an-*

gular velocity, in a portion of a vortex-filament containing the same element of fluid, remains constant during the motion of that element.

From equations (2) it follows directly that

$$\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}=0.$$

And, further, from this, that

$$\int\int\int\left(\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}\right)dx\,dy\,dz=0,$$

the integration being extended over any given portion S of the fluid mass. Integrating partially we have

$$\int\int\xi\,dy\,dz+\int\int\eta\,dx\,dz+\int\int\zeta\,dx\,dy=0,$$

where the integration extends to the whole surface of S. Calling $d\omega$ an element of this surface, and α, β, γ the three angles made with the axes by the normal to $d\omega$ drawn outwards, we have

$$dy\,dz=d\omega\cos\alpha,\quad dx\,dz=d\omega\cos\beta,\quad dx\,dy=d\omega\cos\gamma.$$

Hence

$$\int\int(\xi\cos\alpha+\eta\cos\beta+\zeta\cos\gamma)\,\delta\omega=0;$$

or if we call q the resultant angular velocity, and θ the angle between its axis and the normal,

$$\int\int q\cos\theta\,.\,d\omega=0,$$

the integration extending to the whole surface of S.

Now let S be a portion of a vortex-filament bounded by two indefinitely small planes $\omega_{\prime}$ and $\omega_{\prime\prime}$ perpendicular to the axis of the filament; $\cos\theta$ is equal to 1 at one of these, and -1 at the other, and equal to 0 for the rest of the surface of S; hence if $q_{\prime}$ and $q_{\prime\prime}$ be the angular velocities in $\omega_{\prime}$ and $\omega_{\prime\prime}$, the last equation reduces itself to

$$q_{\prime}\omega_{\prime}=q_{\prime\prime}\omega_{\prime\prime},$$

whence it follows that *the product of the velocity of rotation and the cross section is constant throughout the whole length of any one vortex-filament.* That it does not alter by the motion of the filament itself has been already proved.

It also follows from this that a vortex-filament can never end within a fluid, but must either return ring-shaped into itself within the fluid, or reach to the boundaries of the fluid, since, if a vortex-filament ended anywhere within a fluid, a closed surface could be constructed for which $\int q\cos\theta\,d\omega$ would not vanish.

§ 3.

If the motion of the vortex-filaments in a fluid can be determined, the preceding theorems enable us to determine ξ, η, ζ completely. We shall now consider the problem of finding u, v, w from ξ, η, ζ.

Thus, let there be given within a mass of fluid which includes the space S the values of ξ, η, ζ, which latter satisfy the condition

$$\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}=0; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2a)$$

u, v, and w must be found so as to satisfy within the whole space S the conditions

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)_4$$

$$\left.\begin{aligned}\frac{dv}{dz}-\frac{dw}{dy}&=2\xi,\\ \frac{dw}{dx}-\frac{du}{dz}&=2\eta,\\ \frac{du}{dy}-\frac{dv}{dx}&=2\zeta.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

We require also the necessary conditions for the bounding surface of S according to the particular problem.

According to the given values of ξ, η, ζ, we may have some vortex-filaments which are reentrant within the space S, and also some which reach the boundary of S and then break off. If the latter be the case, we can always continue these filaments along the surface of S or without it till they return into themselves, so that a greater space S_1 exists which contains only reentrant vortex-filaments. And at the whole of the surface of S_1 either ξ, η, ζ and their resultant q are each $=0$, or at all events

$$\xi\cos\alpha+\eta\cos\beta+\zeta\cos\gamma=q\cos\theta=0, \quad . \quad . \quad (2\,b)$$

where α, β, γ, θ have the same values as before. We find values of u, v, w which satisfy $(1)_4$ and (2) if we put

$$\left.\begin{aligned}u&=\frac{dP}{dx}+\frac{dN}{dy}-\frac{dM}{dz},\\ v&=\frac{dP}{dy}+\frac{dL}{dz}-\frac{dN}{dx},\\ w&=\frac{dP}{dz}+\frac{dM}{dx}-\frac{dL}{dy},\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and determine the functions, L, M, N, and P so as to satisfy within the space S_1 the conditions

$$\left.\begin{aligned}\frac{d^2L}{dx^2}+\frac{d^2L}{dy^2}+\frac{d^2L}{dz^2}&=2\xi,\\ \frac{d^2M}{dx^2}+\frac{d^2M}{dy^2}+\frac{d^2M}{dz^2}&=2\eta,\\ \frac{d^2N}{dx^2}+\frac{d^2N}{dy^2}+\frac{d^2N}{dz^2}&=2\zeta,\\ \frac{d^2P}{dx^2}+\frac{d^2P}{dy^2}+\frac{d^2P}{dz^2}&=0.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (5)$$

The method of integrating these equations is known. L, M, N are the potential functions of imaginary magnetic matter distributed through the space S_1 with the densities

$$-\frac{\xi}{2\pi}, \quad -\frac{\eta}{2\pi}, \quad -\frac{\zeta}{2\pi},$$

P the potential of masses external to the space S. If we denot by r the distance of a point a, b, c from x, y, z, and by ξ_a, η_a, ζ_a the values of ξ, η, ζ at that point, we have

$$\left.\begin{aligned}L&=-\frac{1}{2\pi}\iiint\frac{\xi_a}{r}\,da\,db\,dc,\\ M&=-\frac{1}{2\pi}\iiint\frac{\eta_a}{r}\,da\,db\,dc,\\ N&=-\frac{1}{2\pi}\iiint\frac{\zeta_a}{r}\,da\,db\,dc,\end{aligned}\right\} \quad . \quad . \quad . \quad (5a)$$

the integration extending to the whole space S_1, and

$$P=\iiint\frac{k}{r}\,da\,db\,dc,$$

where k is an arbitrary function of a, b, c; and the integration extends through all space exterior to S. The arbitrary function k must be taken so as to satisfy the conditions at the bounding surface, a problem whose difficulty resembles that of magnetic and electric distribution. That the values of u, v, and w in (4) satisfy the conditions in $(1)_4$ is proved by differentiation, with attention to the fourth of equations (5).

We also find by differentiation of (4), attending to the first three of equations (5), that

$$\begin{aligned}\frac{dv}{dz}-\frac{dw}{dy}&=2\xi-\frac{d}{dx}\left(\frac{dL}{dx}+\frac{dM}{dy}+\frac{dN}{dz}\right),\\ \frac{dw}{dx}-\frac{du}{dz}&=2\eta-\frac{d}{dy}\left(\frac{dL}{dx}+\frac{dM}{dy}+\frac{dN}{dz}\right),\\ \frac{du}{dy}-\frac{dv}{dx}&=2\zeta-\frac{d}{dz}\left(\frac{dL}{dx}+\frac{dM}{dy}+\frac{dN}{dz}\right).\end{aligned}$$

Equations (2) are thus satisfied if it can be shown that in the complete space S_1

$$\frac{dL}{dx}+\frac{dM}{dy}+\frac{dN}{dz}=0. \quad \ldots \ldots \ldots \quad (5\,b)$$

That this is the case is seen from equations (5 *a*),

$$\frac{dL}{dx}=+\frac{1}{2\pi}\iiint\frac{\xi_a(x-a)}{r^3}\,da\,db\,dc,$$

or by partial integration

$$\frac{dL}{dx}=\frac{1}{2\pi}\iint\frac{\xi_a}{r}\,db\,dc-\frac{1}{2\pi}\iiint\frac{1}{r}\cdot\frac{d\xi_a}{da}\,da\,db\,dc,$$

$$\frac{dM}{dy}=\frac{1}{2\pi}\iint\frac{\eta_a}{r}\,da\,dc-\frac{1}{2\pi}\iiint\frac{1}{r}\cdot\frac{d\eta_a}{db}\,da\,db\,dc,$$

$$\frac{dN}{dz}=\frac{1}{2\pi}\iint\frac{\zeta_a}{r}\,da\,db-\frac{1}{2\pi}\iiint\frac{1}{r}\cdot\frac{d\zeta_a}{dc}\,da\,db\,dc.$$

Adding these three equations, and again putting $d\omega$ for the surface-element of S, we have

$$\frac{dL}{dx}+\frac{dM}{dy}+\frac{dN}{dz}=\frac{1}{2\pi}\int(\xi_a\cos\alpha+\eta_a\cos\beta+\zeta_a\cos\gamma)d\omega$$

$$-\frac{1}{2\pi}\iiint\frac{1}{r}\left(\frac{d\xi_a}{da}+\frac{d\eta_a}{db}+\frac{d\zeta_a}{dc}\right)da\,db\,dc.$$

But throughout the entire space

$$\frac{d\xi_a}{da}+\frac{d\eta_a}{db}+\frac{d\zeta_a}{dc}=0. \quad \ldots \ldots \ldots \quad (2\,a)$$

And over the whole surface,

$$\xi_a\cos\alpha+\eta_a\cos\beta+\zeta_a\cos\gamma=0. \quad \ldots \quad (2\,b)$$

Both integrals therefore vanish, and equations (5 *b*) are satisfied as well as (2). (4) and (5) or (5 *a*) are thus integrals of $(1)_4$ and (2).

The analogy, mentioned in the introduction, between the distance-action of vortex-filaments and the electromagnetic action of current-conducting wires, which gives a very good means of exhibiting the form of vortex-motions, is deducible from these theorems.

If we put in (4) the values of L, M, N from (5 *a*), and denote by Δu, Δv, Δw the indefinitely small elements of u, v, w which in the integrals result from the element $da\,db\,dc$, also their resultant by Δp, we have

$$\Delta u = \frac{1}{2\pi} \cdot \frac{(y-b)\zeta_a - (z-c)\eta_a}{r^3}\, da\, db\, dc,$$

$$\Delta v = \frac{1}{2\pi} \cdot \frac{(z-c)\xi_a - (x-a)\zeta_a}{r^3}\, da\, db\, dc,$$

$$\Delta w = \frac{1}{2\pi} \cdot \frac{(x-a)\eta_a - (y-b)\xi_a}{r^3}\, da\, db\, dc;$$

and it follows from these that

$$\Delta u(x-a) + \Delta v(y-b) + \Delta w(z-c) = 0;$$

hence the resultant of Δu, Δv, Δw is at right angles to r. Further,

$$\xi_a \Delta u + \eta_a \Delta v + \zeta_a \Delta w = 0.$$

Hence this resultant is also at right angles to the resultant axis of rotation at a, b, c. Lastly,

$$\Delta p = \sqrt{(\Delta u)^2 + (\Delta v)^2 + (\Delta w)^2} = \frac{da\, db\, dc}{2\pi r^2}\, q \sin \nu,$$

where q is the resultant of ξ_a, η_a, ζ_a, and ν the angle it makes with r, which is found from

$$qr \cos \nu = (x-a)\xi_a + (y-b)\eta_a + (z-c)\zeta_a.$$

Each rotating element of fluid (a) *implies in each other element* (b) *of the same fluid mass a velocity whose direction is perpendicular to the plane through* (b) *and the axis of rotation of* (a). *The magnitude of this velocity is directly proportional to the volume of* (a), *its angular velocity, and the sine of the angle between the line* (a) (b) *and that axis of rotation, and inversely proportional to the square of the distance between* (a) *and* (b).

The same law holds for the force exerted by an element of an electric current at (a), parallel to its axis of rotation, on a particle of magnetism at (b).

The mathematical connexion of these phenomena consists in this—that in the fluid vortices, for any element of the fluid which has no rotation, a velocity-potential ϕ exists satisfying the equation

$$\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2} = 0;$$

and this holds everywhere but within the vortex-filaments. If we consider the latter as always reentrant either within or without the fluid, the space for which the above equation for ϕ holds is complexly connected, since it remains single if we conceive surfaces of separation through it, each of which is completely bounded by a vortex-filament. In such complexly connected spaces a function

ϕ which satisfies the above equation can have more than one value; and it must be so if it represent currents reentering, since the velocities of the fluid outside the vortex-filaments are proportional to the differential coefficients of ϕ, and therefore the motion of the fluid must correspond to ever increasing values of ϕ. If the current returns into itself, we come again to a point where it formerly was, and find there a second greater value of ϕ. Since this may occur indefinitely, there must be for every point of such a complexly-connected space an infinite number of distinct values of ϕ differing by equal quantities like those of $\tan^{-1}\frac{x}{y}$, which is such a many-valued function and satisfies the differential equation.

Such also is the case with the electromagnetic effects of a closed electric current. This acts at a distance just as a determinate arrangement of magnetic matter on a surface bounded by the conductor. Exterior to the current, therefore, the forces exerted on a particle of magnetism may be considered as the differential coefficients of a function V which satisfies the equation

$$\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}=0.$$

But in this case also the space in which this equation holds is complexly connected, and V has more than one value.

Thus in the vortex-motion of fluids, as in electromagnetic effects, the velocities or forces external to the vortex-filaments (or electric-current-penetrated space) depend upon potential functions *with more values than one,* which satisfy the general differential equation of magnetic potential functions; while within the vortex-filaments or the space traversed by electric currents, velocities and electromagnetic forces can be expressed (both in an analogous manner) by those functions which appear in the equations (4), (5), and (5 *a*). On the other hand, in simply streaming fluid-motion and magnetic forces we have to do with potential functions *with only one value,* just as in the cases of gravitation, electric attractions, and constant currents of heat and electricity.

The latter integrals of the hydrodynamical equations, in which a single-valued velocity-potential exists, we may call *integrals of the first class*; those, on the other hand, where there is rotation of some of the elements of the fluid, and in consequence a velocity-potential with more than one value in the non-rotating elements, *integrals of the second class.* It may occur that in the latter case only such portions of space are to be treated in the example as contain no rotating elements,—for instance, the motion of fluids in ring-shaped vessels, where a vortex-filament may be supposed to lie along the axis of the vessel, and where the problem belongs to those which can be solved by the assumption of a velocity-potential.

In the hydrodynamic integrals of the first class the velocities of the fluid elements are in the direction of, and proportional to, the forces which a determinate magnetic distribution outside the fluid would exert on a magnetic particle in the places of the elements.

In the hydrodynamic integrals of the second class the velocities of the fluid elements are in the direction of, and proportional to, the forces which would act on a particle of magnetism if closed electric currents passed through the vortex-filaments with a density proportional to the angular velocity in these filaments, combined with magnetic masses outside the fluid. The electric currents inside the fluid must move with their respective vortex-filaments and have constant intensity. The assumed distribution of magnetic matter outside or at the surface of the fluid must be taken so as to satisfy the conditions at the surface. Each magnetic mass can also, as we know, be replaced by electric currents. Thus, instead of using for the values of u, v, w the potential-function P of an external mass κ, we get quite as general a solution if we give ξ, η, and ζ outside of, or at the bounding surface of, the fluid any values such that only closed current-filaments exist; and then the integration in (5 a) must be extended to all space in which ξ, η, and ζ are different from zero.

§ 4.

In hydrodynamic integrals of the first class it is sufficient, as I have shown above, to know the motion of the surface. By this the whole motion in the interior is determined. In integrals of the second class, on the other hand, the motion of the vortex-filaments in the interior of the fluid must be found with reference to their mutual action, and with attention to the conditions at the surface, by which the problem becomes much more complicated. Even this problem can be solved in certain simple cases—namely, when rotation of the fluid elements takes place only in known surfaces or lines, and the form of these surfaces or lines remains unchanged during the motion.

The properties of surfaces bounded by an indefinitely thin sheet of rotating elements can be easily deduced from (5 a). If ξ, η, ζ differ from zero only in an indefinitely thin sheet, their potential functions L, M, N will, by known theorems, have equal values on both sides of the sheet; but their differential coefficients, taken in the direction of the normal to the sheet, will be different. Suppose the coordinate axes so placed that at the point of the vortex-sheet we are considering the axis of z is the normal to the sheet, and that of x the axis of rotation of the element, so that $\eta=\zeta=0$, the potentials M and N and their differential coefficients have the same value at both sides. Such is also the case with L, $\frac{dL}{dx}$, and $\frac{dL}{dy}$; but $\frac{dL}{dz}$ has two distinct values, whose difference is $2\xi\epsilon$, if ϵ denote the thickness of the sheet. Consequently equations (4) show that u and w have the same values on both sides of the vortex-surface, but the values of v differ by $2\xi\epsilon$. *Hence the values of that component of the velocity which is a tangent to the vortex-surface and at right angles to the vortex-lines differ on opposite sides of the surface.* Within the sheet of revolving elements we must take this component of the velocity as gradually and equably varying from one value to the other. For if ξ is here constant through the whole thickness of the shell, and α represent a proper fraction, v', v_1 the values of v at the sides, v_α the value in the shell at a distance $\alpha\epsilon$ from the first side, we saw that $v'-v_1=2\xi\epsilon$, while between was a sheet of thickness ϵ and angular velocity ξ. We have in the same way $v'-v_\alpha=2\xi\epsilon\alpha=\alpha(v'-v_1)$, which expresses the above result. As we must consider the revolving elements as being themselves moved, and the change of their distribution on the surface depends on their motion, we must assign as their mean velocity along the surface for the whole thickness of the sheet the arithmetical mean of v' and v_1.

Such a vortex-sheet will be produced if two separate moving masses of fluid come in contact. At the surface of contact the velocity perpendicular to this must be the same for both, but the tangential velocities will in general be different in the two. Thus the surface of contact will have the properties of a vortex-sheet.

Hence in general isolated vortex-filaments cannot be supposed indefinitely thin, since otherwise the velocities at opposite sides would be indefinitely great and in opposite directions, and the proper velocity of the filament would remain undetermined. To obtain, therefore, certain general conclusions about the motion of very fine filaments of any section, we must make use of the principle of the *conservation of vis viva.*

Before we proceed to treat of separate examples, we will first write the expression for the *vis viva* K of the moving mass of fluid,

$$K=\tfrac{1}{2}h\iiint(u^2+v^2+w^2)\,dx\,dy\,dz. \quad . \quad . \quad . \quad (6)$$

We now from equations (4) substitute in this integral

$$u^2=u\left(\frac{dP}{dx}+\frac{dN}{dy}-\frac{dM}{dz}\right),$$

$$v^2=v\left(\frac{dP}{dy}+\frac{dL}{dz}-\frac{dN}{dx}\right),$$

$$w^2=w\left(\frac{dP}{dz}+\frac{dM}{dx}-\frac{dL}{dy}\right),$$

and integrate partially, denoting by α, β, γ, and θ the angles which the inwardly directed normal of the element $d\omega$ of the fluid mass makes with the coordinate axes and with the resultant velocity q; we thus obtain, attending to equations (2) and $(1)_4$,

$$\left.\begin{aligned} \mathrm{K} = -\tfrac{1}{2}h\int d\omega\,[\mathrm{P}q\cos\theta + \mathrm{L}(v\cos\gamma - w\cos\beta) + \mathrm{M}(w\cos\alpha - u\cos\gamma) \\ + \mathrm{N}(u\cos\beta - v\cos\alpha)] - h\iiint(\mathrm{L}\xi + \mathrm{M}\eta + \mathrm{N}\zeta)\,dx\,dy\,dz. \end{aligned}\right\} \quad (6a)$$

The value of $\frac{d\mathrm{K}}{dt}$ is found from (1) if we multiply the first by u, the second by v, and the third by w, and add,

$$h\left(u\frac{du}{dt} + v\frac{dv}{dt} + w\frac{dw}{dt}\right) = -\left(u\frac{dp}{dx} + v\frac{dp}{dy} + w\frac{dp}{dz}\right)$$

$$+h\left(u\frac{d\mathrm{V}}{dx} + v\frac{d\mathrm{V}}{dy} + w\frac{d\mathrm{V}}{dz}\right) - \frac{h}{2}\left(u\frac{d(q^2)}{dx} + v\frac{d(q^2)}{dy} + w\frac{d(q^2)}{dz}\right).$$

If both sides be multiplied by $dx\,dy\,dz$ and integrated through the whole extent of the fluid, noticing that by $(1)_4$

$$\iiint\left(u\frac{d\psi}{dx} + v\frac{d\psi}{dy} + w\frac{d\psi}{dz}\right)dx\,dy\,dz = -\int\psi q\cos\theta\,d\omega,$$

if ψ denote in the interior of the fluid mass a continuous and single-valued function, we obtain

$$\frac{d\mathrm{K}}{dt} = \int d\omega(p - h\mathrm{U} + \tfrac{1}{2}hq^2)q\cos\theta. \quad . \quad . \quad . \quad (6b)$$

If the fluid mass be entirely enclosed in a rigid envelope, $q\cos\theta$ must be zero at every point of its surface. Hence $\frac{d\mathrm{K}}{dt} = 0$, or $\mathrm{K} =$ constant.

If we consider this rigid envelope as being at an infinite distance from the origin of coordinates, but the vortex-filaments at a finite distance, the potential functions L, M, N, whose masses ξ, η, ζ are each in sum equal to nothing, are, at an infinite distance R, proportional to R^{-2}, and their differential coefficients as R^{-3}; but the surface-element $d\omega$, if it always correspond to the same solid angle at the origin, is as R^2. The first integral in the expression for K (6 a), which is extended over the surface of the fluid mass, will vary as R^{-3}, and therefore vanish for an infinite value of R. The value of K thus becomes

$$\mathrm{K} = -h\iiint(\mathrm{L}\xi + \mathrm{M}\eta + \mathrm{N}\zeta)\,dx\,dy\,dz; \quad . \quad . \quad (6)c$$

and this value does not alter during the motion.

§ 5.

Straight parallel Vortex-filaments.

We shall first consider the case where only straight vortex-filaments, parallel to the axis of z, exist, whether in an indefinitely extended mass of fluid, or in a similar mass limited by two infinite planes perpendicular to the filaments, which comes to the same thing. All the motions are then confined to planes perpendicular to the axis of z, and are exactly the same in all such planes.

We put therefore

$$w=\frac{du}{dz}=\frac{dv}{dz}=\frac{dp}{dz}=\frac{d\mathrm{V}}{dz}=0.$$

Then equations (2) become

$$\xi=0,\quad \eta=0,\quad 2\zeta=\frac{du}{dy}-\frac{dv}{dx}.$$

(3) become

$$\frac{\delta\zeta}{\delta t}=0.$$

The vortex-filaments thus retain constant angular velocity, so that they also retain the same section.

Equations (4) become

$$u=\frac{d\mathrm{N}}{dy},\quad v=-\frac{d\mathrm{N}}{dx},$$

$$\frac{d^2\mathrm{N}}{dx^2}+\frac{d^2\mathrm{N}}{dy^2}=2\zeta.$$

By the remark at the end of § 3 we put $\mathrm{P}=0$. The equation of current lines is thus $\mathrm{N}=$ constant.

N is in this case the potential function of indefinitely long lines; and is infinitely great, but its differential coefficients are finite. Let a and b be the coordinates of a vortex-filament whose section is $da\,db$, we have

$$-v=\frac{d\mathrm{N}}{dx}=\frac{\zeta da\,db}{\pi}\cdot\frac{x-a}{r^2},$$

$$u=\frac{d\mathrm{N}}{dy}=\frac{\zeta da\,db}{\pi}\cdot\frac{y-b}{r^2}.$$

From this it follows that the resultant velocity q is perpendicular to r, which again is perpendicular to the vortex-filament, and that

$$q=\frac{\zeta da\,db}{\pi r}.$$

If we have a number of vortex-filaments whose coordinates are

x_1, y_1, x_2, y_2, &c. in a fluid mass indefinite in the directions of x and y, and denote the product of the section and the angular velocity in them by m_1, m_2, &c., then, forming the sums

$$U = m_1 u_1 + m_2 u_2 + m_3 u_3 \text{ \&c.},$$
$$V = m_1 v_1 + m_2 v_2 + m_3 v_3 \text{ \&c.},$$

these will be each equal to 0, since the portion of the sum V which arises from the effect of the second vortex-filament on the first, is destroyed by the effect of the first on the second. They are respectively

$$m_1 \cdot \frac{m_2}{\pi} \cdot \frac{x_1 - x_2}{r^2} \text{ and } m_2 \cdot \frac{m_1}{\pi} \cdot \frac{x_2 - x_1}{r^2},$$

and so for other pairs in each sum. But U is the velocity of the centre of gravity of the masses m_1, m_2, ... in the direction of x multiplied by the sum of these masses; so of V in the direction of y. Both velocities are thus zero, unless the sum of the masses be zero, in which case there is no centre of gravity. The centre of gravity of the vortex-filaments remains, therefore, stationary during their motions about one another; and since this is true for any distribution of vortex-filaments, it will also be true of isolated ones of indefinitely small section.

From this we derive the following consequences:—

1. If there be a single rectilinear vortex-filament of indefinitely small section in a fluid infinite in all directions perpendicular to it, the motion of an element of the fluid at finite distance from it depends only on the product ($\xi da\, db = m$) of the velocity of rotation and the section, not on the form of that section. The elements of the fluid revolve about it with tangential velocity $= \frac{m}{\pi r}$, where r is the distance from the centre of gravity of the filament. The position of the centre of gravity, the angular velocity, the area of the section, and therefore, of course, the magnitude m remain unaltered, even if the form of the indefinitely small section may alter.

2. If there be two rectilinear vortex-filaments of indefinitely small section in an unlimited fluid, each will cause the other to move in a direction perpendicular to the line joining them. Thus the length of this joining line will not be altered. They will thus turn about their common centre of gravity at constant distances from it. If the rotation be in the same direction for both (that is, of the same sign) their centre of gravity lies between them. If in opposite directions (that is, of different signs), their centre of gravity lies in the line joining them produced. And if, in addition, the product of the velocity and the section be the same for both, so that the centre of gravity is at an infi-

nite distance, they travel forwards with equal velocity, and in parallel directions perpendicular to the line joining them.

To this last case may also be referred that in which a vortex-filament of indefinitely small section moves near an infinite plane to which it is parallel. The condition at the limits (viz. that the fluid must move parallel to the plane) will be fulfilled if instead of the plane there be an infinite mass of fluid with another vortex-filament the image (with respect to the plane) of the first. From this it follows that the vortex-filament moves parallel to the plane in the direction in which the elements of the fluid between it and the plane move, and with one-fourth of the velocity which the elements at the foot of a perpendicular from the filament on the plane have.

With rectilinear vortex-filaments the assumption of an indefinitely small section leads to no inadmissible consequences, since each filament exerts upon itself no displacing action, and is only displaced by the action of other filaments which may be present. It is different with curved filaments.

§ 6. *Circular Vortex-filaments.*

Let there exist in an infinite mass of fluid only circular vortex-filaments whose planes are parallel to that of xy, and whose centres are in the axis of z, so that all is symmetrical about that axis. Let us change the coordinates by assuming

$$x=\chi\cos\epsilon, \qquad a=g\cos e,$$
$$y=\chi\sin\epsilon, \qquad b=g\sin e,$$
$$z=z, \qquad c=c.$$

The angular velocity σ is, by the above assumption, a function of χ and z or of g and c only, and the axis of rotation is perpendicular to χ (or g) and axis of z. The rectangular components of the angular velocity at the point g, e, c are, therefore,

$$\xi=-\sigma\sin e, \quad \eta=\sigma\cos e, \quad \zeta=0.$$

In equations (5 a) we have

$$r^2=(z-c)^2+\chi^2+g^2-2\chi g\cos(\epsilon-e),$$
$$\mathrm{L}=\frac{1}{2\pi}\iiint\frac{\sigma\sin e}{r}g\,dg\,de\,dc,$$
$$\mathrm{M}=-\frac{1}{2\pi}\iiint\frac{\sigma\cos e}{r}g\,dg\,de\,dc,$$
$$\mathrm{N}=0.$$

From the equations for L and M we obtain, multiplying by $\cos\epsilon$

and sin ϵ, and adding and subtracting,

$$\mathrm{L}\sin\epsilon-\mathrm{M}\cos\epsilon=-\frac{1}{2\pi}\iiint\frac{\sigma\cos(\epsilon-e)}{r}g\,dg\,d(\epsilon-e)\,dc,$$

$$\mathrm{L}\cos\epsilon+\mathrm{M}\sin\epsilon=+\frac{1}{2\pi}\iiint\frac{\sigma\sin(\epsilon-e)}{r}g\,dg\,d(\epsilon-e)\,dc.$$

In both integrals e and ϵ appear only in the form $(\epsilon-e)$; and this may therefore be taken as the variable in the integration. In the second integral the element for $\epsilon-e=\mathfrak{t}$ is destroyed by that for $\epsilon-e=2\pi-\mathfrak{t}$; it is therefore zero. If we put

$$\psi=\frac{1}{2\pi}\iiint\frac{\sigma\cos e\,g\,dg\,de\,dc}{\sqrt{(z-c)^2+\chi^2+g^2-2g\chi\cos e}}, \quad . \quad (7)$$

we have

$$\mathrm{M}\cos\epsilon-\mathrm{L}\sin\epsilon=\psi,$$

$$\mathrm{M}\sin\epsilon+\mathrm{L}\cos\epsilon=0;$$

or

$$\mathrm{L}=-\psi\sin\epsilon,\quad \mathrm{M}=\psi\cos\epsilon. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7\,a)$$

Calling τ the velocity in the direction of the radius χ, and noticing that on account of the symmetry about the axis of z there can be no velocity in the direction of the circumference, we have

$$u=\tau\cos\epsilon,\quad v=\tau\sin\epsilon;$$

and from equations (4)

$$u=-\frac{d\mathrm{M}}{dz},\quad v=\frac{d\mathrm{L}}{dz},\quad w=\frac{d\mathrm{M}}{dx}-\frac{d\mathrm{L}}{dy}.$$

From this

$$\tau=-\frac{d\psi}{dz},\quad w=\frac{d\psi}{d\chi}+\frac{\psi}{\chi},$$

or

$$\tau\chi=-\frac{d(\psi\chi)}{dz},\quad w\chi=\frac{d(\psi\chi)}{d\chi}. \quad . \quad . \quad . \quad . \quad . \quad (7\,b)$$

The equation of the current-lines is therefore

$$\psi\chi=\text{constant}.$$

If we perform approximately the integration for ψ for a vortex-filament of indefinitely small section, putting $\sigma dg\,dc=m_1$, and the corresponding part of $\psi=\psi_{m_1}$, we have

$$\psi_{m_1}=\frac{m_1}{\pi}\sqrt{\frac{g}{\chi}}\left\{\frac{2}{\kappa}(\mathrm{F}-\mathrm{E})-\kappa\mathrm{F}\right\},$$

where

$$\kappa^2=\frac{4g\chi}{(g+\chi)^2+(z-c)^2},$$

and F and E are the complete elliptic integrals of the first and second orders for the modulus κ.

If we put, for sake of brevity,

$$U=\frac{2}{\kappa}(F-E)-\kappa F,$$

where, therefore, U is a function of κ, we have

$$\tau\chi=\frac{m_1}{\pi}\sqrt{-\chi\frac{dU}{d\kappa}}\cdot\kappa\frac{z-c}{(g+\chi)^2+(z-c)^2}.$$

If there be a second vortex-filament m at the point χ, z, and we denote by τ_1 the velocity in the direction of g which it gives to m_1, we shall find this, if we put in the expression for τ instead of

$$\tau\ \chi\ g\ z\ c\ m_1,$$
$$\tau_1\ g\ \chi\ c\ z\ m.$$

By this process U and κ are not changed, and we have

$$m\tau\chi+m_1\tau_1 g=0. \quad \ldots \ldots \quad (8)$$

Let us next determine the value of the velocity parallel to the axis of z, which m_1, whose coordinates are g and c, produces, and we find

$$w\chi=\frac{1}{2}\frac{m_1}{\pi}\sqrt{\frac{g}{\chi}}U+\frac{m_1}{\pi}\sqrt{-g\chi\frac{dU}{d\kappa}}\cdot\frac{\kappa}{2\chi}\cdot\frac{(z-c)^2+g^2-\chi^2}{(g+\chi)^2+(z-c)^2}.$$

If we call w_1 the velocity parallel to z which the vortex-ring m, whose coordinates are z and χ, produces at the position of m_1, we require only to make again the same interchange of letters as before. Hence we find

$$2mw\chi^2+2m_1w_1g^2-m\tau\chi z-m_1\tau_1 gc=\frac{2mm_1}{\pi}\sqrt{g\pi}\,.\,U. \quad . \quad (8a)$$

Similar sums can be made for any assumed number of vortex-rings. Denote $\sigma dg\,dc$ in the nth ring by m_n and the components of the velocity it receives from the others by τ_n and w_n, omitting, however, for the time that which each ring can impart to itself. Call also its radius ρ_n, and λ its distance from a plane parallel to xy, which magnitudes, no doubt, correspond in direction to what we have called χ and z, but as belonging to a particular ring they are functions of the time, and not independent variables like χ and z. Finally, let the value of ψ, as far as it arises from the other vortex-rings, be ψ_n. We find from (8) and (8 a), by writing out and adding these equations for each pair of rings,

$$\Sigma(m_n\rho_n\tau_n)=0,$$
$$\Sigma(2m_nw_n\rho_n{}^2-m_n\tau_n\rho_n\lambda_n)=\Sigma(m_n\rho_n\psi_n).$$

As long as we consider a finite number of separate and indefinitely thin vortex-rings in these sums, we can only introduce in w, τ, and ψ the portions produced by the other rings. If, however, we suppose the space to be continuously filled with an infinite number of such, ψ is the potential function of a continuous mass, w and τ its differential coefficients; and we know that for such a function, as well as for its differential coefficients, the parts due to an indefinitely small portion indefinitely near the point for which the value is sought, are indefinitely small in comparison with those of finite masses at a finite distance*.

Let us change the sums into integrals: we may suppose the entire amount of their value at any point to be expressed by w, τ, and ψ, and put

$$w=\frac{d\lambda}{dt}, \quad \tau=\frac{d\rho}{dt}.$$

For this purpose we express m as $\sigma d\rho\, d\lambda$,

$$\iint \sigma\rho \frac{d\rho}{dt}\, d\rho\, d\lambda=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$2\iint \sigma\rho^2 \frac{d\lambda}{dt}\, d\rho\, d\lambda - \iint \sigma\rho\lambda \frac{d\rho}{dt}\, d\rho\, d\lambda = \iint \sigma\rho\psi\, d\rho\, d\lambda. \quad (9\,a)$$

Since the product $\sigma d\rho\, d\lambda$ is by § 2 constant with respect to the time, (9) can be integrated with respect to t, and we have

$$\tfrac{1}{2}\iint \sigma\rho^2 d\rho\, d\lambda=\text{const.}$$

Consider the space to be divided by a plane which passes through the axis of z, and therefore cuts all the vortex-rings; consider σ as the density of a slice of the mass, and call $\mathfrak{M}$ the entire mass in the slice made by the plane;

$$\mathfrak{M}=\iint \sigma d\rho\, d\lambda;$$

and if R^2 is the mean value of ρ^2 for all the elements of mass,

$$\iint \sigma\rho \,.\, \rho d\rho\, d\lambda=\mathfrak{M}R^2;$$

and since this integral and the value of $\mathfrak{M}$ remain unchanged during the lapse of time, R also remains unchanged during the motion.

Therefore if there exist in the unlimited fluid only one circular vortex-filament of indefinitely small section, its radius remains unaltered.

The magnitude of the *vis viva* is in our case, by (6 c),

* Compare Gauss in the *Resultate des magnetischen Vereins*, 1839, p. 7.

$$\begin{aligned} K &= -h\iiint(L\xi + M\eta)\,da\,db\,dc \\ &= -h\iiint\psi\sigma\rho d\rho\, d\lambda\, d\epsilon \\ &= -2\pi h\iint\psi\sigma\rho d\rho\, d\lambda. \end{aligned}$$

This is also constant as regards time.

Again, we remark that, as $\sigma d\rho\, d\lambda$ is constant with respect to t,

$$\frac{d}{dt}\iint\sigma\rho^2\lambda d\rho\, d\lambda = 2\iint\sigma\rho\lambda\frac{d\rho}{dt}\,d\rho\, d\lambda + \iint\sigma\rho^2\frac{d\lambda}{dt}\,d\lambda\, d\rho,$$

hence the equation (9 a), if we call l the value of λ for the centre of gravity of the section of the vortex-filament and multiply (9) by it and add, becomes

$$2\frac{d}{dt}\iint\sigma\rho^2\lambda d\rho\, d\lambda + 5\iint\sigma\rho(l-\lambda)\frac{d\rho}{dt}\,d\rho\, d\lambda = -\frac{K}{2\pi h}. \qquad (9\,b)$$

If the section of the vortex-filament is indefinitely small and ϵ an indefinitely small magnitude of the same order as $l-\lambda$ and the other linear dimensions of the section, but $\sigma d\rho\, d\lambda$ finite, then ψ and K are of the same order of indefinitely great quantities as $\log\epsilon$. For very small distances v from the vortex-ring we have

$$\begin{aligned} v &= \sqrt{(g-\chi)^2+(z-c)^2}, \\ \kappa^2 &= 1-\frac{v^2}{4g^2}, \\ \psi_{m_1} &= \frac{m_1}{\pi}\log\left(\frac{\sqrt{1-\kappa^2}}{4}\right) = \frac{m_1}{\pi}\log\frac{v}{8g}. \end{aligned}$$

In the value of K, ψ is multiplied by ρ or g. If g is finite and v of the same order as ϵ, K is of the order $\log\epsilon$. Only when g is indefinitely great and of the order $\frac{1}{\epsilon}$ does K become indefinitely great, of the order $\frac{1}{\epsilon}\log\epsilon$. Then the circle becomes a straight line. But, on the other hand, $\frac{d\rho}{dt}$, which is equal to $\frac{d\psi}{dz}$, becomes of the order $\frac{1}{\epsilon}$, the second integral therefore is finite, and for finite values of ρ is indefinitely small compared with K. In this case we may put in the first integral l instead of λ, and we find

$$2\frac{d}{dt}(\mathfrak{M}R^2 l) = -\frac{K}{2\pi h},$$

or

$$2\mathfrak{M}R^2 l = C - \frac{K}{2\pi h}t.$$

Since $\mathfrak{M}$ and R are constants, l must vary proportionally to the time. If $\mathfrak{M}$ is positive, the motion of the elements of fluid on the outer side of the ring is in the direction of z positive, on the inner side in the direction of z negative; K, h, and R are from their nature necessarily positive.

Hence *in a circular vortex-filament of very small section in an indefinitely extended fluid, the centre of gravity of the section has, from the commencement, an approximately constant and very great velocity parallel to the axis of the vortex-ring, and this is directed towards the side to which the fluid flows through the ring.* Indefinitely thin vortex-filaments of finite radius would have indefinitely great velocity of translation. But if the radius be indefinitely great, of the order $\frac{1}{\epsilon}$, then R^2 is indefinitely great compared with K, and l becomes constant. The vortex-filament, which has now become rectilinear, becomes stationary, as we have already proved for the case of such filaments.

We can now see generally how two ring-formed vortex-filaments having the same axis would mutually affect each other, since each, in addition to its proper motion, has that of its elements of fluid as produced by the other. If they have the same direction of rotation, they travel in the same direction; the foremost widens and travels more slowly, the pursuer shrinks and travels faster, till finally, if their velocities are not too different, it overtakes the first and penetrates it. Then the same game goes on in the opposite order, so that the rings pass through each other alternately.

If they have equal radii and equal and opposite angular velocities, they will approach each other and widen one another; so that finally, when they are very near each other, their velocity of approach becomes smaller and smaller, and their rate of widening faster and faster. If they are perfectly symmetrical, the velocity of fluid elements midway between them parallel to the axis is zero. Here, then, we might imagine a rigid plane to be inserted, which would not disturb the motion, and so obtain the case of a vortex-ring which encounters a fixed plane.

In addition it may be noticed that it is easy in nature to study these motions of circular vortex-rings, by drawing rapidly for a short space along the surface of a fluid a half-immersed circular disk, or the nearly semicircular point of a spoon, and quickly withdrawing it. There remain in the fluid half vortex-rings whose axis is in the free surface. The free surface forms a bounding plane of the fluid through the axis, and thus there is no essential change in the motion. These vortex-rings travel on, widen when they come to a wall, and are widened or contracted by other vortex-rings, exactly as we have deduced from theory.

The above version of one of the most important recent investigations in mathematical physics was made long ago for my own use, and does not pretend to be an exact translation. Professor Helmholtz has been kind enough to revise it; and it may therefore be accepted as representing the spirit of the original. A portion of the contents of the paper had been anticipated by Professor Stokes in various excellent papers in the Cambridge Philosophical Transactions; but the discovery of the nature and motions of vortex-filaments is entirely novel, and of great consequence. Sir W. Thomson has recently propounded a very singular speculation as to the ultimate nature of matter, mainly founded on the properties of the Helmholtz ring. I append an extract from a letter I have just received from him, which fills an important gap towards the end of Professor Helmholtz's work.—P. G. TAIT.

"Following as nearly as may be Helmholtz's notation, let g be the radius of the circular axis of a uniform vortex-ring, and a the radius of the section of its core (which will be approximately circular when a is small in comparison with g), the vortex motion being so instituted that there is no molecular rotation in any part of the fluid exterior to this core, and that in the core the angular velocity of the molecular rotation is approximately ω, or rigorously

$$\frac{\omega\chi}{g}$$

for any fluid particle at distance χ from the straight axis.

"I find that the velocity of translation is approximately equal to

$$\frac{\omega a^2}{2g}\left(\log\frac{8g}{a}-\frac{1}{4}\right),$$

(quantities of the same order as this multiplied by $\frac{a}{g}$ being neglected.)

"The velocity of the liquid at the surface of the core is approximately constant and equal to ωa. At the centre of the ring it is $\frac{\pi\omega a^2}{g}$.

"If these be denoted by Q and W respectively, and if T be the velocity of translation, we therefore have

$$T=\frac{a}{2g}\left(\log\frac{8g}{a}-\frac{1}{4}\right)Q,$$

$$=\frac{\log\frac{8g}{a}-\frac{1}{4}}{2\pi}W.$$

"Hence the velocity of translation is very large in comparison with the fluid velocity along the axis through the centre of the ring, when the section is so small that log $\frac{8g}{a}$ is large in comparison with 2π. But the velocity of translation is always small in comparison with the velocity of the fluid at the surface of the core, and the more so the smaller is the diameter of the section in comparison with the diameter of the ring.

"These results remove completely the difficulty which has hitherto been felt with reference to the translation of infinitely thin vortex-filaments. I have only succeeded in obtaining them since the communication of my mathematical paper (April 29, 1867) to the Royal Society of Edinburgh, but hope to be allowed to add a proof of them to that paper should it be accepted for the Transactions."

"May 17, 1867."

LXIV. *On the Action of Magnetism upon the Electric Discharge in highly Rarefied Gaseous Media*.* *By* Professor A. DE LA RIVE†.

IN the memoir which I recently published "On the Propagation of Electricity in Elastic Fluids," I reserved for a subsequent publication the investigation of the manner in which this propagation is modified by the action of magnetism. I demonstrated the existence of this action as early as 1849, by showing that a magnetic pole causes jets of electricity which escape from it radially to rotate. M. Plücker subsequently proved by several remarkable experiments that this action is general. The luminous veins which show themselves in rarefied gases traversed by the discharges of a Ruhmkorff's apparatus are in fact attracted and repelled in the same way as electric currents passing along metallic wires would be. In a word, this action is subject to the laws of electrodynamics, with the difference, however, that, all the parts of the moveable conductor being independent of each other instead of being connected together as they are in a rigid wire, they obey perfectly the forces by which they are solicited, and take up positions of equilibrium determined by these forces. It follows that each luminous vein assumes the form of a magnetic curve, the only condition under which equilibrium can be produced, since the action of the magnet upon an element of the current is then nothing, the direction of this action be-

* Translated from the *Archives des Sciences Physiques et Naturelles*, vol. xxvii. p. 289 (December 1866).

† This paper forms a continuation of the one which I published, on the propagation of electricity in highly rarefied elastic fluids, in the Number of the *Archives des Sciences Physiques et Naturelles* for July 1866 (vol. xxvi. p. 177). [A translation of the paper here referred to will be found at p. 241 of the present volume of the Philosophical Magazine.]