On the Motion of Paired Vortices with a Common Axis.

By A. E. H. LOVE. Read and received March 8th, 1894.

1. The investigation in this paper was undertaken with the view of throwing some further light on a problem in the motion of vortex-rings which was first considered by Helmholtz in his original memoir on vortex-motion.* He found that two vortex-rings having the same axis and circulations in the same direction travel in the same direction parallel to the axis; "the foremost widens and travels more slowly, the pursuer shrinks and travels faster, till finally, if their velocities are not too different, it overtakes the first and penetrates it. Then the same game goes on in the opposite order, so that the rings pass through each other alternately."† It is extremely difficult to obtain a more detailed account of the motion here described. We are ignorant of the condition that the motion may be periodic, and we can only make guesses at the length of the period when the unknown condition is satisfied. Yet in applications of the vortex-atom theory to problems of radiation and chemical combination, it is conceivable that this period and the type of motion may play an important part. I propose here to imitate some of the circumstances of the problem by considering the case where there are present in an infinite fluid two pairs of cylindrical vortices of indefinitely small section, the circulations about the two vortices of each pair being equal and of opposite sign, the circulations about the four vortices being equal in absolute magnitude, and the line of symmetry for one pair coinciding with that for the other. A single pair of this kind moves parallel to the axis of symmetry with constant velocity. Two pairs with circulations in the proper directions influence each other's motions in a manner analogous to that exhibited by thin rings. I find a condition that the motion may be periodic, the length of the period, and the form of the curve described by one vortex of one pair relative to the homologous vortex of the other pair.

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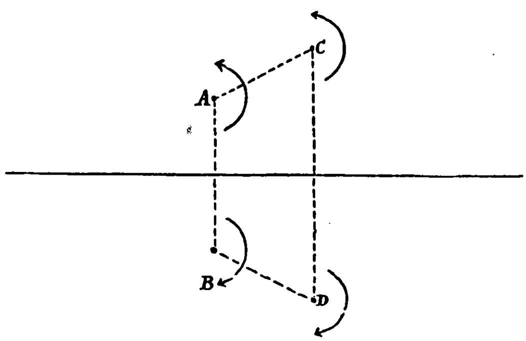
* "Ueber Integrale der hydrodynamischen Gleichungen, welche den Wirbelbewegungen entsprechen," *Crelle-Borchardt*, LV., 1858.

† Tait's translation of Helmholtz's memoir, *Phil. Mag.* (4), XXXIII., 1867.

2. Let the axis of symmetry be chosen as the axis of x, and let the coordinates of the vortices at time t be

$$(x_0, y_0),\ (x_0, -y_0),\ (x_1, y_1),\ (x_1, -y_1);$$

then the two with suffix 0 form a pair, and the two with suffix 1 also form a pair.



The figure shows a possible configuration of the system in which A, B are taken to be the pair with suffix 0.

Let m be the absolute value of the strength of either vortex (*i.e.*, half the circulation in any circuit surrounding it once, and not surrounding either of the others), and suppose the strengths of the four A, B, C, D to be m, $-m$, m, $-m$; then the stream function ψ which gives the motion at any point (x, y) is

$$\psi = \frac{m}{2\pi}\log\frac{(x-x_0)^2+(y+y_0)^2}{(x-x_0)^2+(y-y_0)^2} + \frac{m}{2\pi}\log\frac{(x-x_1)^2+(y+y_1)^2}{(x-x_1)^2+(y-y_1)^2} \quad \text{.....(1).}$$

From this expression we may form the components of velocity of the two vortices by leaving out the terms in $\partial\psi/\partial y$, and $-\partial\psi/\partial x$ which become infinite at (x_0, y_0) or (x_1, y_1).

We thus find

$$\left.\begin{aligned}\frac{dx_1}{dt} &= -\frac{m}{\pi}\,\frac{y_1-y_0}{(x_1-x_0)^2+(y_1-y_0)^2} + \frac{m}{\pi}\,\frac{y_1+y_0}{(x_1-x_0)^2+(y_1+y_0)^2} + \frac{m}{2\pi y_1} \\ \frac{dy_1}{dt} &= \frac{m}{\pi}\,\frac{x_1-x_0}{(x_1-x_0)^2+(y_1-y_0)^2} - \frac{m}{\pi}\,\frac{x_1-x_0}{(x_1-x_0)^2+(y_1+y_0)^2}\end{aligned}\right\} \quad \text{............(2),}$$

and dx_0/dt and dy_0/dt can be obtained from these by interchanging the suffixes 0, 1. The differential equations for x_0, y_0, x_1, y_1 as functions of t can clearly be put in the form

$$\frac{dx_0}{\frac{\partial\chi}{\partial y_0}} = \frac{dy_0}{-\frac{\partial\chi}{\partial x_0}} = \frac{dx_1}{\frac{\partial\chi}{\partial y_1}} = \frac{dy_1}{-\frac{\partial\chi}{\partial x_1}} = dt \quad\text{.................(3),}$$

where $$\chi = \frac{m}{2\pi}\log y_1 y_0 \frac{(x_1-x_0)^2+(y_1+y_0)^2}{(x_1-x_0)^2+(y_1-y_0)^2} \quad\text{.................(4).}$$

We find at once two integrals of the equations (3) in the form

$$\left.\begin{aligned} y_1+y_0 &= \text{const.} = 2c \\ y_1 y_0 \frac{(x_1-x_0)^2+(y_1+y_0)^2}{(x_1-x_0)^2+(y_1-y_0)^2} &= \text{const.} = a^2 \end{aligned}\right\}\text{...............(5).}$$

The first of these shows that the middle point of the line joining the two vortices A, C moves parallel to the axis x and at a distance c from it. It is easy to see by multiplying this equation by m that it represents the condition of constancy of momentum parallel to x of the fluid in the half-plane y positive, for this momentum is equal to that generated by impulsive pressure proportional to m per unit area applied at all points of half a barrier between the vortices A, B and at all points of half a barrier between C and D. The second of equations (5) is really $\chi = \text{const.}$, and it may in like manner be interpreted as representing the constancy of the energy of the fluid motion in the same half-plane. For, according to Helmholtz's formula, the energy of a plane vortex-motion, in terms of the density ρ, the spin ζ, and the stream function ψ, is

$$\rho\iint \psi\,\zeta\,dx\,dy,$$

the integral extending to all the points where ζ is different from zero. For isolated vortex-filaments this expression contains an infinite constant, and the finite part of it divided by ρ is the sum of the strengths of the vortices each multiplied by the part of ψ which is finite at the point occupied by the vortex.

3. The two equations (5) are sufficient to determine the paths of the vortices A and C relative to each other, for, if $2r$ is the distance between A and C at any time, and θ the angle the line AC makes with the axis x, they give rise to the polar equation

$$(r^2\cos^2\theta+c^2)(c^2-r^2\sin^2\theta) = a^2r^2 \quad\text{.................(6),}$$

which represents the curve described by either vortex relative to the middle point of the line joining them. In (x, y) coordinates with origin at this middle point the same equation is

$$x^2y^2+(a^2-c^2)\,x^2+(a^2+c^2)\,y^2-c^4=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(7).$$

The condition that the motion may be periodic is the condition that this equation represents a closed curve. This is the case if

$$a^2>c^2,$$

as is seen by writing (7) in the form

$$(x^2+a^2+c^2)(y^2+a^2-c^2)=a^4.$$

When this condition is satisfied one of the values of r^2 given by equation (6) is positive for every value of θ. This value of r^2 is

$$c^2\operatorname{cosec}^2\theta-\tfrac{1}{2}(a^2+c^2)\sec^2\theta\operatorname{cosec}^2\theta\left\{1-\sqrt{(1-\kappa^2\cos^2\theta)}\right\},$$

where
$$\kappa^2=4a^2c^2/(a^2+c^2)^2,$$

and the square root is to be taken positive. On evaluating the indeterminate, which occurs when $\theta=0$, we find

$$r^2=c^4/(a^2-c^2),$$

as is given directly by (6). This value becomes infinite when $a^2=c^2$, and for values of a^2/c^2 which are less than unity the curve has no real points on the line $\theta=0$, but there exists a pair of parallel asymptotes, $y^2=c^2-a^2$, between which there are no real points on the curve. In every case there is a positive value of r^2 for $\theta=\frac{1}{2}\pi$, so that every system of two pairs such as we are considering will either at some future time have its four vortices A, B, C, D in a straight line perpendicular to the axis x, or may be regarded as having been at some past time in this state. We may therefore interpret the constants a and c, more particularly in terms of the distances between the vortices in a pair at the instant when each vortex of one pair is passing between the corresponding vortex of the other pair and the axis of symmetry, or, as it may be otherwise expressed, when one pair is passing through the other pair.

Let r_0 and r_1 be the distances of A and C from the axis x when A, B, C, D are in a straight line; then we have

$$r_0+r_1=2c,$$

$$r_1r_0\left(\frac{r_1+r_0}{r_1-r_0}\right)^2=a^2,$$

and the condition $a^2 > c^2$ is

$$6r_0r_1 - r_0^2 - r_1 > 0.$$

Taking $r_1 > r_0$, this requires that $r_1 : r_0 < 3+2\sqrt{2}$. Thus the motion is periodic, if, at the instant when one pair passes through the other, the ratio of the breadths of the pairs is less than $3+2\sqrt{2}$. When the ratio has this precise value the smaller pair shoots ahead of the larger and widens, while the larger contracts, so that each is ultimately of the same breadth $2c$, and the distance between them is ultimately infinite. When the ratio in question is greater than $3+2\sqrt{2}$, the smaller shoots ahead and widens, and the latter falls behind and contracts, but in such a way that the former never attains so great a breadth as $2c$, nor the latter so small a breadth. When the ratio is less than $3+2\sqrt{2}$, the motion of the two pairs is similar to the motion described by Helmholtz for two rings on the same axis, and it is probable that there is for his case also a critical condition in which the rings, after one has passed through the other, ultimately separate to an infinite distance, and attain equal diameters.

4. The curves given by (7) when $a^2 > c^2$ always lie inside and near to the ellipses

$$\frac{x^2}{\dfrac{c^4}{a^2-c^2}} + \frac{y^2}{\dfrac{c^4}{a^2+c^2}} - 1 = 0,$$

and their longest and shortest diameters are the same as the major and minor axes of these ellipses. In the limiting case, when $a^2 = c^2$, it is clear that there are inflexions. In general it can be proved that the positions of the inflexions are given by the equation

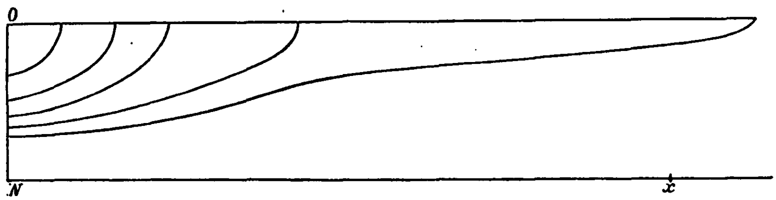
$$3y^4(a^2+c^2) - 2y^2c^4 + c^4(a^2-c^2) = 0.$$

All the roots are imaginary if $a^2 > \frac{1}{3}2\sqrt{3}\,c^2$, but, if $c^2 < a^2 < \frac{1}{3}2\sqrt{3}\,c^2$, they are all real. There are consequently no inflexions, unless the ratio of the major to the minor axis of the ellipse

$$x^2(a^2-c^2) + y^2(a^2+c^2) = c^4$$

exceeds $2+\sqrt{3}$, so that there are no inflexions unless the relative path is very elongated.

At the end of the paper will be found a table giving the principal axes of the ellipses which nearly coincide with the relative paths of the vortices for some simple values of the ratio $r_1 : r_0$, and a drawing showing the character of these curves is given here. The line Nx in the figure is the axis of symmetry of the two vortex-pairs, and the point O is the middle point of the line joining the two vortices on one side of the axis. The curves show quadrants of the relative paths of these two vortices for the values 2, 3, 4, 5, $\frac{17}{3}$ of the ratio $r_1 : r_0$. It is noteworthy how very nearly circular these curves are for values of this ratio up to 2.



5. We have next to consider the dependence of the relative situation of the vortices upon the time. For this purpose we form an equation connecting $d\theta/dt$ with θ. By means of this equation the time can be expressed as a function of θ, and, in particular, the period of the motion, when it is periodic, can be deduced.

The distance between the vortices A and C being $2r$, and the angle AC makes with the axis x being θ, we have

$$\frac{d\theta}{dt} = (x_1 - x_0)\frac{d(y_1-y_0)}{dt} - (y_1-y_0)\frac{d(x_1-x_0)}{dt}$$

$$= \frac{m}{2\pi}\left[\frac{(y_1-y_0)^2}{y_1 y_0} + 4 - 4\frac{(x_1-x_0)^2}{(x_1-x_0)^2+(y_1+y_0)^2}\right]$$

$$= \frac{2m}{\pi}\left[1 + \frac{r^2\sin^2\theta\,(r^2\cos^2\theta+c^2)}{a^2r^2} - \frac{r^2\cos^2\theta}{r^2\cos^2\theta+c^2}\right],$$

or

$$\frac{d\theta}{dt} = \frac{m}{2\pi}\,\frac{(r^2\cos^2\theta+c^2)^2\sin^2\theta+a^2c^2}{a^2r^2\,(r^2\cos^2\theta+c^2)} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8).$$

Now, from the equation of the relative path

$$(r^2\cos^2\theta+c^2)(r^2\sin^2\theta-c^2)+a^2r^2 = 0,$$

we find, for the positive value of r^2,

$$c^2\operatorname{cosec}^2\theta - \tfrac{1}{2}(a^2+c^2)\sec^2\theta\operatorname{cosec}^2\theta\left\{1-\sqrt{(1-\kappa^2\cos^2\theta)}\right\},$$

where

$$\kappa^2 = 4a^2c^2/(a^2+c^2)^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9),$$

are the square root is to be taken positive. From this, after some reductions, we find

$$\frac{d\theta}{dt}=\frac{m}{\pi a^2}\,\frac{\sin^2\theta\cos^2\theta\sqrt{(1-\kappa^2\cos^2\theta)}}{\sqrt{(1-\kappa^2\cos^2\theta)}-(\kappa'\cos^2\theta+\sin^2\theta)},$$

where $$\kappa'^2=1-\kappa^2,$$

so that $$\kappa'=(a^2-c^2)/(a^2+c^2).$$

The equation determining the time as a function of θ is

$$t=\frac{\pi a^2}{m}\left[\int\frac{d\theta}{\sin^2\theta\cos^2\theta}-\int\frac{\kappa'\cos^2\theta+\sin^2\theta}{\sin^2\theta\cos^2\theta}\,\frac{d\theta}{\sqrt{(1-\kappa^2\cos^2\theta)}}\right]\ldots(10).$$

To find the period τ we may take for the limits of θ the values 0 and $\frac{1}{2}\pi$, and multiply the right-hand side by 4, and then, writing $\frac{1}{2}\pi-\phi$ for θ, we find

$$\tau=\frac{4\pi a^2}{m}\left[\int_0^{\frac{1}{2}\pi}\frac{d\phi}{\sin^2\phi\cos^2\phi}-\int_0^{\frac{1}{2}\pi}\frac{\kappa'\sin^2\phi+\cos^2\phi}{\sin^2\phi\cos^2\phi}\,\frac{d\phi}{\sqrt{(1-\kappa^2\sin^2\phi)}}\right].$$

Introducing elliptic functions of modulus κ and argument u, such that
$$\phi=\operatorname{am}u,\quad \sin\phi=\operatorname{sn}u,\ \ldots,$$

we obtain, from (10),

$$\frac{mt}{\pi a^2}=\int\frac{d\phi}{\sin^2\phi}-\int\frac{du}{\operatorname{sn}^2u}+\int\frac{d\phi}{\cos^2\phi}-\kappa'\int\frac{du}{\operatorname{cn}^2u},$$

or $$\frac{mt}{\pi a^2}=\left[-\frac{\operatorname{cn}u}{\operatorname{sn}u}-u+E\operatorname{am}u+\frac{\operatorname{cn}u\operatorname{dn}u}{\operatorname{sn}u}\right]$$
$$+\left[\frac{\operatorname{sn}u}{\operatorname{cn}u}-\kappa'\left(u-\frac{1}{\kappa'^2}E\operatorname{am}u+\frac{\operatorname{sn}u\operatorname{dn}u}{\kappa'^2\operatorname{cn}u}\right)\right],$$

where $$E\operatorname{am}u=\int\operatorname{dn}^2u\,du;$$

and hence, observing that the terms that become infinite at the limits cancel one another, we deduce

$$\frac{m\tau}{\pi a^2}=4\,\frac{1+\kappa'}{\kappa'}\,(E-K\kappa'),$$

where K is the real quarter-period of the elliptic functions with modulus κ, and E is the complete elliptic integral of the second kind given by

$$E = \int_0^K \mathrm{dn}^2 u\, du.$$

It is easily verified that $E - K\kappa'$ is always positive and less than unity, for it vanishes when $\kappa = 0$, and becomes equal to 1 when $\kappa = 1$, and further

$$\frac{d}{d\kappa}(E - K\kappa') = \frac{1-\kappa'}{\kappa\kappa'}(K - E),$$

so that the function constantly increases when $1 > \kappa > 0$.

Since $$a^2/c^2 = (1+\kappa')/(1-\kappa'),$$

the period τ is thus proved to be the positive quantity

$$\tau = \frac{4\pi c^2}{m}\frac{(1+\kappa')^2}{\kappa'(1-\kappa')}(E - K\kappa') \quad \text{.................(11)},$$

and this becomes infinite when $\kappa' = 0$ or $a^2 = c^2$, in accordance with the result of § 3, that when $a^2 = c^2$ the motion ceases to be periodic.

6. To get an idea of the way the period varies with the velocities and sizes of the vortex-pairs, we shall compare it with the time taken by a vortex-pair of strength m and breadth $2c$, when undisturbed, to travel a distance equal to its breadth. The velocity of such a pair is $m/2\pi c$, and consequently the time in question is $4\pi c^2/m$. Calling this τ', we have

$$\frac{\tau}{\tau'} = \frac{(1+\kappa')^2}{\kappa'(1-\kappa')}(E - K\kappa').$$

As in the discussion of the relative path, we can make an interpretation in terms of the radii (r_1 and r_0) of the pairs at the instant when one is passing through the other. When $\kappa = 0$ or $\kappa' = 1$ the ratio τ/τ' can be shown to vanish, and when $\kappa' = 0$ or $\kappa = 1$ it becomes infinite. The value for κ very small is small, and can be proved to be $\pi^2(r_1 - r_0)^2/m$, which is the same as the period of rotation round each other of two vortex-filaments of equal strength m at a distance $r_1 - r_0$. As κ increases, the quantity $E - K\kappa'$ increases very slowly, and the ratio τ/τ' does not become large until κ is very nearly unity. The arithmetical details of some particular cases are given in the table of § 8 below.

The period of the slowest oscillation of a thin ring of radius c is known to be $\frac{1}{3}\sqrt{3}\,\pi c/V$, where V is the velocity of translation.* The latter velocity depends on the ratio of the radius of the section to the radius of the aperture, and becomes logarithmically infinite when this ratio vanishes. For the application to radiation it is perhaps best to regard τ' as comparable with the period required by a particle of fluid circulating between the vortices of the pair of breadth $2c$ which is never quite close to either vortex of the pair or to the axis. The period τ, in which one pair goes through the other, is thus of the same order of magnitude as the period in which a particle of the fluid in irrotational motion circulates through either pair when the other is absent. From the nature of the case it is highly probable that a like statement holds for rings, *i.e.*, that the period of one ring going through another is comparable with the period in which a particle of fluid carried forward with the ring circulates through it, and we appear to be justified in concluding that the period of two thin rings passing through each other alternately is long compared with any period of oscillation of either ring about its circular form.

7. So far we have been concerned with the motion of either of two vortices on the same side of the axis relative to the middle point of the line joining them. For the motion of this middle point we need only remark that it describes a straight line $y=c$ parallel to the axis with a variable velocity. This velocity is given by the equation

$$\frac{d}{dt}\tfrac{1}{2}(x_1+x_0)=\frac{mc}{2\pi a^2}\,\frac{2c^2+r^2\cos 2\theta}{r^2},$$

which is easily deduced from equations (2). It is easy to prove that this has its greatest value when $\theta=\frac{1}{2}\pi$ or $\frac{3}{2}\pi$, and its least value when $\theta=0$ or π. These values are respectively

$$\frac{mc}{2\pi a^2}\,\frac{2a^2+3c^2}{c^2}\quad\text{and}\quad\frac{mc}{2\pi a^2}\,\frac{2a^2-c^2}{c^2},$$

or

$$\frac{m}{2\pi c}\,\frac{5-\kappa'}{1+\kappa'}\quad\text{and}\quad\frac{m}{2\pi c}\,\frac{1+3\kappa'}{1+\kappa'}.$$

8. A table is appended showing arithmetical details of some particular cases. These are arranged so as to give simple values of r_1/r_0 lying between the two extreme values 0 and $3+2\sqrt{2}$ for which the motion is periodic. The first column gives the value assumed

* J. J. Thomson, "On the Motion of Vortex Rings," p. 35.

for $r_1 : r_0$, and the second gives the corresponding value of κ'. The number $(5-\kappa')/(1+\kappa')$ in the third column is the ratio of the greatest velocity of the middle point of the line joining the two vortices in the same half-plane to the velocity of a vortex-pair of strength m and breadth $2c$, and the number $(1+3\kappa')/(1+\kappa')$ in the fourth column is the ratio of the least velocity of the same middle point to the velocity of the same vortex pair. The number $c/\sqrt{(a^2-c^2)}$ in the fifth column is the ratio of the greatest semi-diameter of the relative path to the distance of the middle point from the axis of symmetry, and the number $c/\sqrt{(a^2+c^2)}$ in the sixth column is the ratio of the least semi-diameter of the relative path to the same distance. The number τ/τ' in the last column is the ratio of the period of one pair going through the other to the time occupied by a pair of strength m and breadth $2c$ in moving over a distance equal to its breadth.

$\frac{r_1}{r_0}$	κ'	$\frac{5-\kappa'}{1+\kappa'}$	$\frac{1+3\kappa'}{1+\kappa'}$	$\frac{c}{\sqrt{(a^2-c^2)}}$	$\frac{c}{\sqrt{(a^2+c^2)}}$	$\frac{\tau}{\tau'}$
1	1	2	2	0	0	0
2	$\frac{7}{9}$	2·375	1·875	·378...	·333...	·402...
3	$\frac{1}{2}$	3	1·666...	·707...	·5	1·195...
4	$\frac{7}{25}$	3·6875	1·4375	1·082...	·6	2·701...
5	$\frac{1}{9}$	4·4	1·2	2	·666...	8·538...
$\frac{17}{3}$	$\frac{1}{50}$	4·882...	1·039...	4·949	·7	45·11...
$3+2\sqrt{2}$	0	5	1	∞	·707...	∞