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2000 Eur. J. Phys. 21 269

(http://iopscience.iop.org/0143-0807/21/3/310)

Eur. J. Phys. **21** (2000) 269–273. Printed in the UK PII: S0143-0807(00)11774-X

Instability of vortex leapfrogging

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Received 10 February 2000

Abstract. A strange instability gives a new twist to a 100-year old problem in the theory of vortex motion. Computer simulation software is included in the online edition of this paper.

1. Introduction

The theory of vortex motion is one of the great successes of classical fluid mechanics, and finds application in such diverse areas as meteorology, aerodynamics and superfluid physics [1].

One of the most elementary problems involves a so-called *vortex pair*, where two vortices of equal strength but opposite sign are placed close to one another (see figure 1). Each vortex is then swept along at the local flow velocity due to the other, and, as a consequence, the two vortices travel along together, with uniform speed. The closer the vortices, i.e. the smaller the breadth of the pair, the faster the speed of travel.

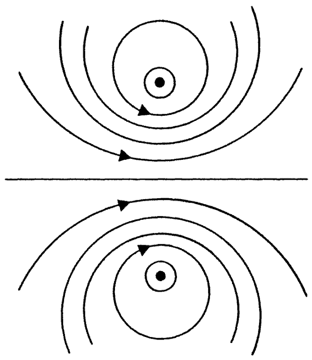

Figure 1. Flow due to a vortex pair. The pair, and the whole flow pattern, move to the right at uniform speed.

What happens, however, *if one such vortex pair chases another*? In fact, as a result of the mutual interaction, the pair in front becomes wider and slows down, while the one behind becomes more narrow and speeds up. In this way it can happen that the two vortex pairs exchange places, over and over again, by repeatedly passing through one another†, as illustrated in figure 2.

This 'leapfrogging' of two vortex pairs was first investigated theoretically by A E H Love (figure 3) who found that the nature of the vortex motion depends on the ratio of the breadths

† The corresponding phenomenon with circular symmetry is relatively well-known, and can be demonstrated with smoke rings [2, 3].

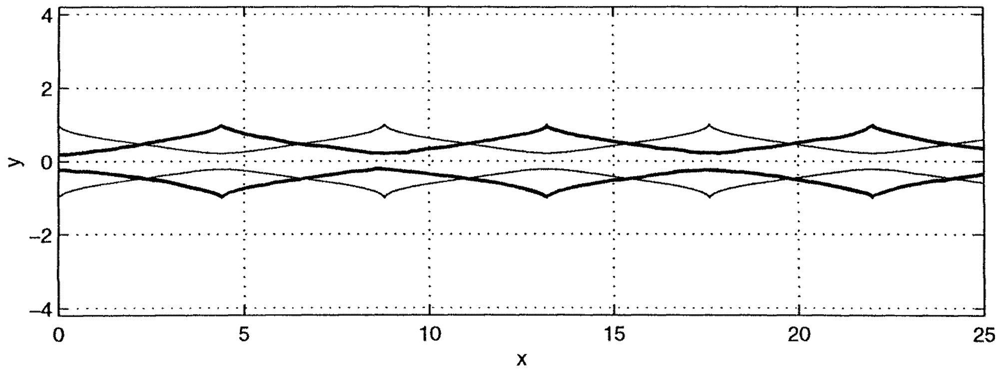

Figure 2. A modern computation showing leapfrogging trajectories when $\alpha = 0.220$. The pair of smaller breadth begins by travelling faster, but soon widens and slows down. The other pair, on the other hand, becomes more narrow, and speeds up, until the two pairs exchange places.

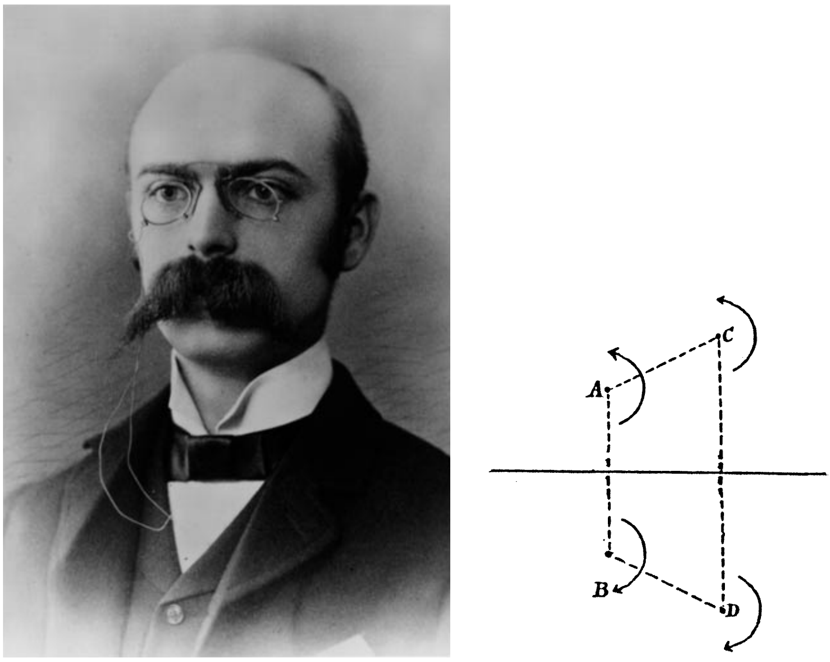

Figure 3. A E H Love (1863–1940) and the original sketch of two vortex pairs from his 1894 paper.

of the pairs at the instant when one pair passes through the other [4]. If we denote the ratio of the smaller breadth to the larger by α, then Love proved that leapfrogging was only possible if

$$\alpha > 3 - 2\sqrt{2}. \tag{1}$$

If α is less than this critical value, the smaller pair passes through so quickly that the other one is unable to catch up, and the two vortex pairs move steadily further apart as time goes on. Leapfrogging is a possible motion of the system, then, if $\alpha > 0.172$, but an obvious question remains: is such a motion *stable*, or will small disturbances to the system lead to a quite different kind of behaviour?

2. Is leapfrogging stable?

Following Love, we use the simplest vortex model consistent with classical fluid mechanics, namely a single point (or 'line') vortex with a circulatory flow about it, the speed of this flow being k/r. Here k is a constant, denoting the strength of the vortex, and r is the distance from its centre. The circulatory flow speed therefore diminishes with distance r.

Now, if we have a system of N such vortices, with strengths k_i, $i = 1, \ldots, N$, each vortex will move, at any given instant, at the local flow velocity due to all the others. This remarkable consequence of the equations of fluid motion, deduced by Helmholtz in 1858, implies that the positions (x_i, y_i) of the various vortices are governed by the following set of nonlinear differential equations:

$$\begin{aligned} \frac{\mathrm{d}x_i}{\mathrm{d}t} &= -\sum_{j=1, j\neq i}^{N} \frac{k_j(y_i - y_j)}{r_{ij}^2} \\ \frac{\mathrm{d}y_i}{\mathrm{d}t} &= \sum_{j=1, j\neq i}^{N} \frac{k_j(x_i - x_j)}{r_{ij}^2} \end{aligned} \tag{2}$$

where

$$r_{ij}^2 = (x_i - x_j)^2 + (y_i - y_j)^2. \tag{3}$$

Given the initial positions of the vortices, we integrate these equations numerically using a fourth-order Runge-Kutta method.

In the leapfrogging problem we have $N = 4$, and if the two vortex pairs are of equal strength then $k_1 = k_3 = 1$, say, while $k_2 = k_4 = -1$. For 'undisturbed' leapfrogging we start vortex 1 at $(0, 1)$, vortex 2 at $(0, -1)$, vortex 3 at $(0, \alpha)$ and vortex 4 at $(0, -\alpha)$, where $\alpha < 1$. When investigating the stability of the resulting motion to small disturbances we typically start vortex 1 at $(\epsilon, 1)$ instead.

The outcome of the numerical integrations depends crucially on the parameter α:

(i) $\alpha < 0.172$. At such low values of α the computational results confirm Love's result, that leapfrogging is simply not possible.

(ii) $0.172 < \alpha < 0.29$. In this regime leapfrogging is certainly possible, but it is unstable to small disturbances which are not symmetric about the original symmetry axis. In figure 4, for instance, the value of α is the same as in figure 2, but the uppermost vortex in the original state has been given a small perturbation $\epsilon = 0.000001$. Three leapfroggings take place, but after the last of these the four vortices have become so out of alignment at 'pass-through' that they suddenly re-group into two different vortex pairs, which then go their separate ways at more or less uniform speed.

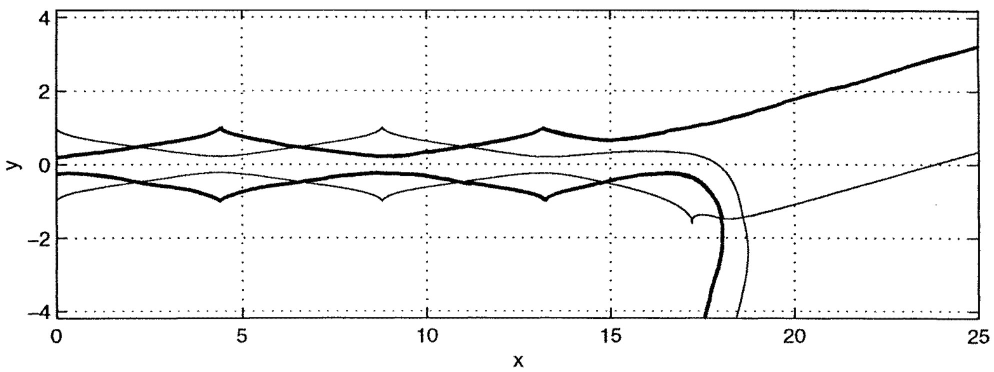

Figure 4. Instability of vortex leapfrogging, with $\alpha = 0.220$ and $\epsilon = 0.000001$.

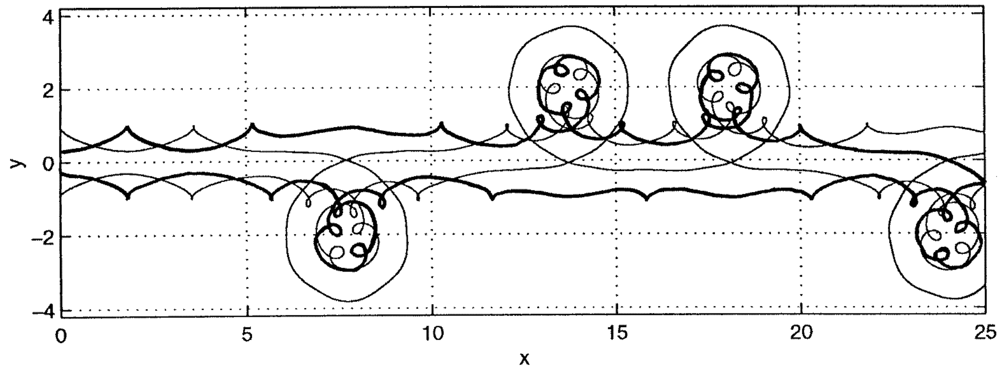

Figure 5. 'Walkabout' instability of vortex leapfrogging, with $\alpha = 0.310$ and $\epsilon = 0.007$.

(iii) $0.29 < \alpha < 0.382$. In this regime, leapfrogging is again unstable, though in a much more peculiar way. Instead of splitting into two separate pairs, the vortices typically continue in a disorderly leapfrogging motion which is punctuated by periods of 3-vortex 'walkabout' (see figure 5). What appears to happen is that two like-signed vortices occasionally get so close that they revolve rapidly around one another, and the combination of the two is then strong enough to sweep one of the other, oppositely-signed, vortices out of its usual path. The three then go off together on a roughly circular excursion, before meeting up again with the fourth, which has been continuing on a more or less straight path in the meantime. 'Walkabout' episodes of this kind take place on seemingly unpredictable sides of the original symmetry axis and at seemingly irregular intervals.

(iv) $\alpha > 0.382$. For these values of α, leapfrogging is stable to small disturbances, essentially because vortices of like sign are close enough to 'stick together' and avoid disruption by a vortex of opposite sign.

In summary, then, leapfrogging of identical vortex pairs is unstable in the range $0.172 < \alpha < 0.382$.

If the vortex pairs are *not* equal, so that one pair is weaker (smaller k) than the other, the range of unstable leapfrogging becomes more narrow and shifts to larger values of α. In the limit when one pair is so weak that it is just swept about passively by the other, the region of instability disappears altogether, and leapfrogging changes directly from being impossible if $\alpha < 0.479$ to being both possible and stable if $\alpha > 0.479$.

There can be no doubt, then, that the instability which we have described arises from a genuine *interaction* between the two vortex pairs involved.

3. Computer simulations

The software LEAPFROG, specially written as DOS programs to accompany this paper, is included in the online edition at `www.iop.org`, and can also be freely downloaded from the author's website at `www.jesus.ox.ac.uk/~dacheson`. It includes a computer animation showing the vortex leapfrogging. The 'walkabout' instability, in particular, is much more striking and comprehensible when seen in animated form.

LEAPFROG comes as a ready-to-run demonstration, but also allows the user to experiment by entering different values of the parameter α and different disturbances ϵ. Thus while figure 4 shows a reasonably typical outcome when $\alpha = 0.220$, it can easily happen that the vortices go their separate ways in their original pairings, rather than by 'exchange' . This is what happens, for instance, if ϵ is 0.00002 rather than the value 0.000001 in figure 4, and an added curiosity in that case is that one pair is eventually scattered almost directly 'backwards', i.e. almost parallel to the negative x-axis.

As a representation of real fluid motion our vortex model is highly idealized, of course, mainly because of the neglect of viscous effects. But the N-vortex problem may, more generally, have another, quite different, role to play, namely as an effective and stimulating way of introducing some of the ideas of modern nonlinear dynamics, including chaos. Too often, perhaps, these ideas are illustrated only by somewhat manufactured mathematical equations. Interacting point vortices, on the other hand, provide an easily visualized 'real' system, suitable perhaps for a student computing project, and loosely related to the rather more familiar gravitational N-body problem.

References

[1] Acheson D J 1990 *Elementary Fluid Dynamics* (Oxford: Oxford University Press)
[2] Yamada H and Matsui T 1978 *Phys. Fluids* **21** 292–4
[3] Riley N and Stevens D P 1993 *Fluid Dyn. Res.* **11** 235–44
[4] Love A E H 1894 *Proc. Lond. Math. Soc.* **25** 185–94