Simulation of LLN, CLT and Gambler's Ruin

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Introduction

During long period of human practical activities, people have discovered that even if some random event may appear in one experiment or not, when there are a large number of experiments, we can find the pattern that the frequency will be around a fixed number. What's more some random factors that determine the errors will follow the normal distribution pattern. In this project, I will simulate the related laws(law of large number and central limit law) to provide an intuition of these facts and simulate an application of LLN(law of large number), which is called "Gambler's Ruin".

Content of LLN and CLT

Definition of Law of Large Number (LLN):

if $X_1, X_2, ..., X_n, ...,$ *are a sequence of random variables Let* :

$$S_n = \frac{X_1+X_2+...+X_n}{n}$$

if there exist a constant sequence $a_1, a_2, .., a_n, ..$ *such that for all* $\epsilon > 0 \Rightarrow$

$$\lim_{n\to\infty} P(\mid S_n - a_n \mid) = 1$$

then we say X_n *follows law of large number.*

DefinitionofTheCentrallimitTheorem(CLT)

Let (X_i) *be independent identical distributed(i.i.d.) random variables with* $\mathbf{E}X = \mu$, $Var(X_i) = \sigma^2$. *Let* $S_n = \sum_{i=1}^{n}(X_i)$, *Then*

$$\lim_{n\to\infty} \frac{1}{\sigma\sqrt{n}}(S_n - \mu) = N(0,1)$$

weakly.

Brief Introduction of Gambler's Ruin

Gambler's Ruin:

This is a interesting theory of gamble supported by probability theory.

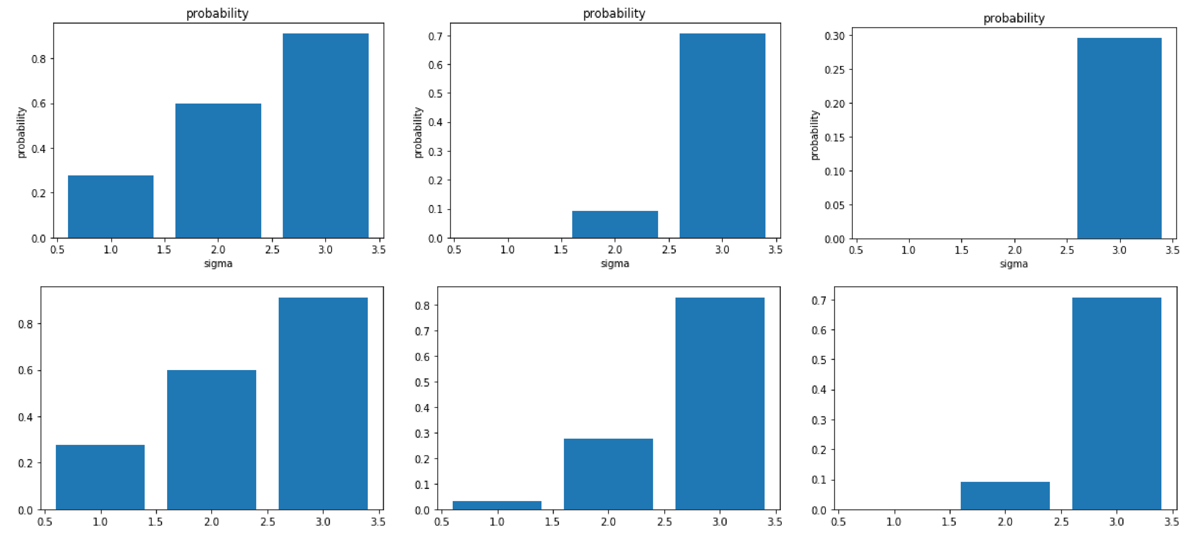
It states that when in a "fair" game, any gamblers with finite wagers, if he/she continuously gamble without stopping, he/she will run out all of their wagers.

Where 'fair' means that at each bet, the probability of win or lose the game is equal.

This is not an intuitional result, because we usually think that we will remain around the adjacent of our original wagers. But we can run a simulation of this theory and by LLN, we will get the result of this theory.

Simulation of Law of Large Number

Here are some random plots:



In the first row we simulate the probability measure of $P(\mid \frac{1}{n}\sum_{i=1}^{n}(Y_i) - \mu \mid > \epsilon)$ where Y_i are independent random variable taking from [0,1] and uniformly distributed and μ taking value of 1(P value equals 0.5).

From left to right is for n = 10, 100, 1000, X-axis is the value of ϵ Y-axis is the value of probability.

from the graph we can see that as n increasing for the same value of ϵ the value of probability of $P(\mid \sum_{i=1}^{n}(Y_i) - \mu \mid > \epsilon)$ is decreasing. Which implies that the average value of Y_i tends to the expectation of it.

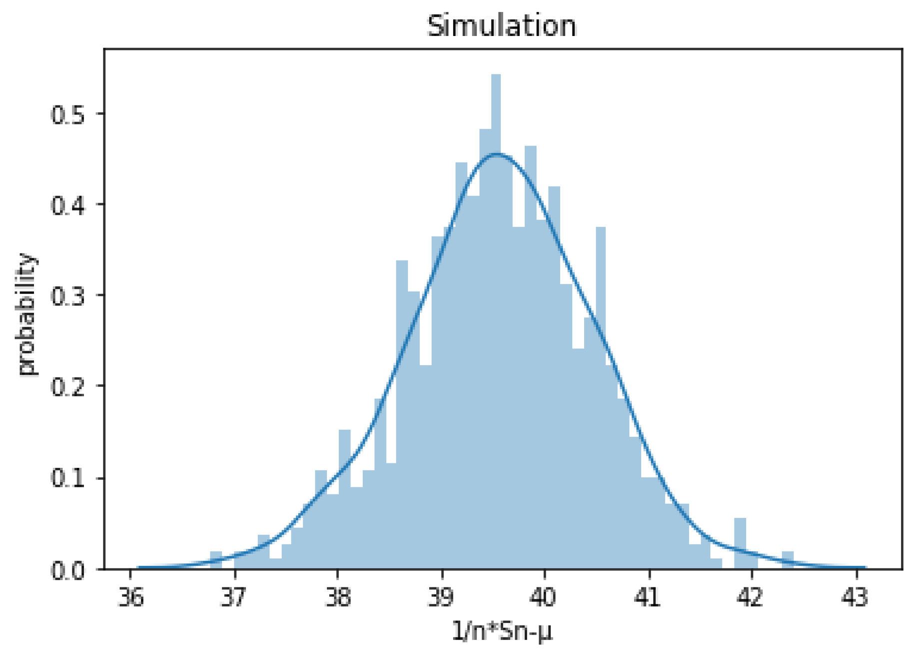
In the second row we simulate the probability measure of $P(\mid \sum_{i=1}^{n}(X_i) - \mu \mid > \epsilon)$ where X_i is defined as

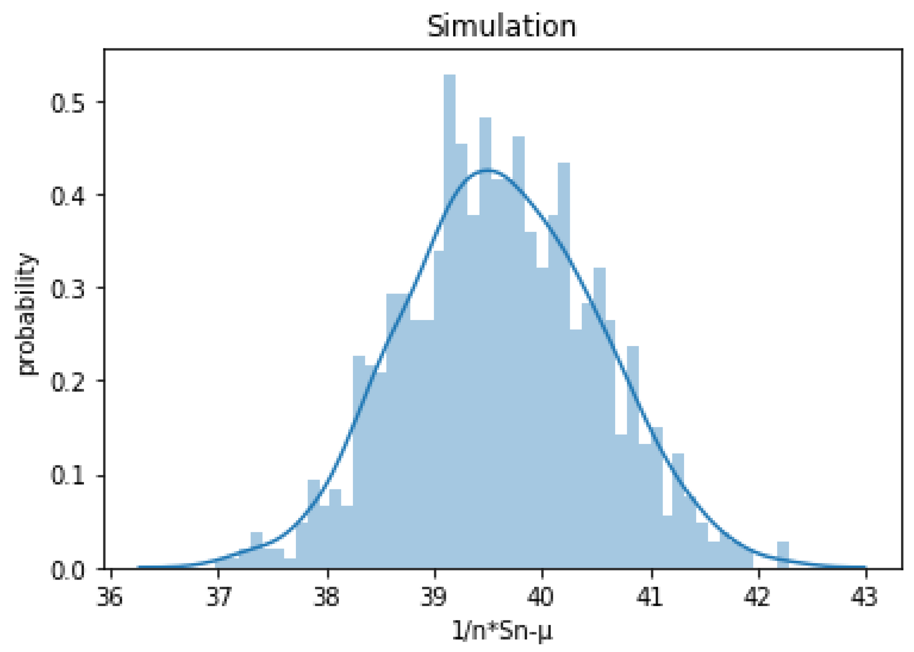
$$X_i = \begin{cases} 1 & \text{if } Y_i(\omega) < p \\ -1 & \text{otherwise} \end{cases}$$

The axis are the same as the previous one. And we can see that it shows the same trend that as n increasing the probability for the same σ value. Therefore we can conclude that when n tends to infinity, the frequency of the event is equal to the probability of the random event.

Simulation of the Central Limit Theorem

Here are some random plots:





Left-hand-side two graphs show the simulation of $\frac{1}{\sqrt{N}}S_n - \mu$ where S_n is the defined as $\sum_{i=1}^{n}(X_i)$. The x-axis is the value of $\frac{1}{\sqrt{N}}S_n - \mu$ and y-axis is the probability in that region.

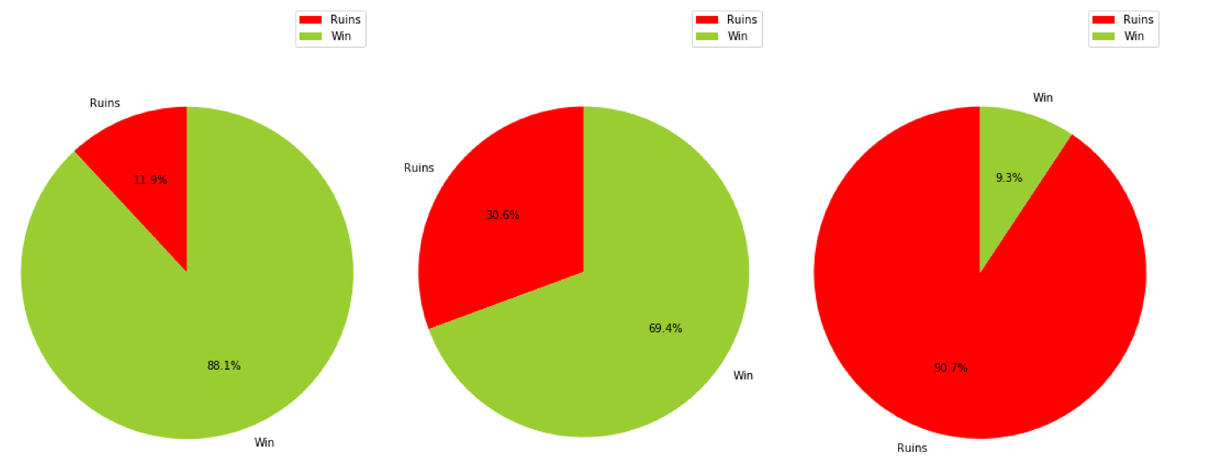
In these simulation we take p value as 0.7 and n to be 1000 in order to see the pattern of simulation.

Although the definition of CLT uses the $\frac{1}{\sigma\sqrt{n}}(S_n - \mu)$ to get normal distribution, we can get this by multiplying and adding some constant, which means that it should look like a stretched normal distribution.

By observing the result of simulation we find that it shows symmetry pattern and reach peak near the mean value of simulation. Therefore we can conclude that when X_i follows Bernouli distribution, the probability for each S_n will similar to the normal distribution.

Simulation of Gambler's Ruin

Here are some random plots:



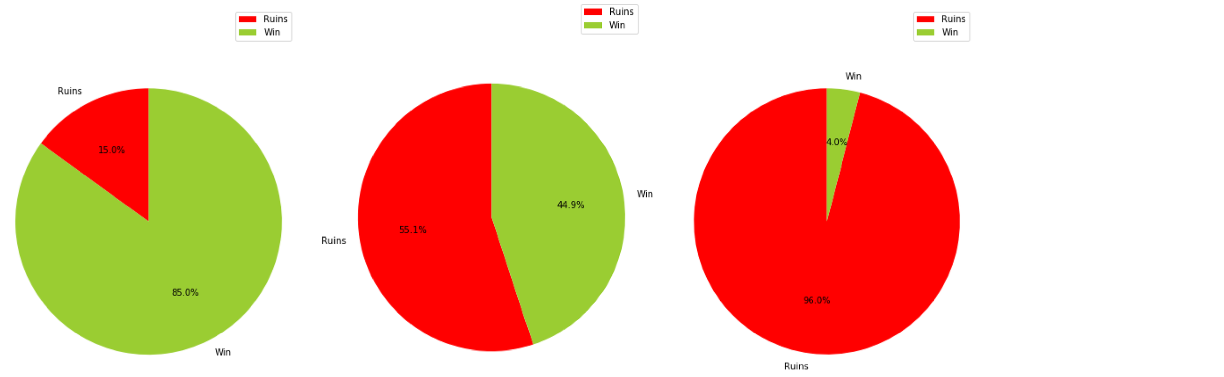
From left to right is the proportion of gambler who lost their money within 4000, 10000, 800000 gambles starting with 100 wagers.

By observing that as n increasing the proportion of gambler's who finally run out all of their money is also increasing.

By using LLN we can conclude that when the times of gamble is large enough even it is a fair gamble, gamblers will lost all their wagers.

Further Gambler's Ruin

Here are some random plots:



We can deduce further by reduce the value of p (the probability of winning one gamble).

Therefore, this time I reduce p to 0.33, which I think is quiet near to the real probability for a gambler to win one time.

From left to right, the times of gamble is 250, 300, 400 respectively and we can see the gamblers who does not go bankrupt are reducing quickly.

Conclusions

Overall, LLN and CLT are really useful theorem in real life to analysis many events like the fairness of a game and error analysis.

References

[1] Xianping Li. Elementary Probability version 3 Fudan university Press, 2010.

[2] Xue-Mei Li. Law of large numbers and Central Limit Theorem. Imperial College London MR1, 2020.