CHAPTER 7
TRIANGLES

Axioms and Theorems

1. **SAS Congruence:** Two triangles are congruent if any two sides and the included angle of one are equal to any two sides and the included angle of the other.
2. In an isosceles triangle, the angles opposite to the equal sides are equal.
3. **ASA Congruence:** Two triangles are congruent if any two angles and the included side of one are equal to any two angles and the included side of the other.
4. **AAS Congruence:** If any two angles and a non-included side of one triangle are equal to corresponding angles and side of another triangle, then the two triangles are congruent.
5. The sides opposite to equal angles of a triangle are equal.
6. **SSS Congruence:** If three sides of a triangle are equal respectively to the corresponding three sides of another triangle, then the two triangles are congruent.

SOLUTIONS

EXERCISE 7.1

1. ABC is an isosceles triangle in which AB = AC. If BA is produced to D so that AD = AB, prove that $\angle BCD = 90^0$.

Solution:

Given: In the isosceles ΔABC, AB = AC, BA is produced to D such that AB = AD

To prove: $\angle BCD = 90^0$

Proof: As AB = AC in ΔABC, $\angle ACB = \angle ABC$

As AC = AD in ΔADC, $\angle ADC = \angle ACD$

$\therefore \angle ACB + \angle ACD = \angle ABC + \angle ADC$

$\Rightarrow \angle BCD = \angle DBC + \angle BDC$

In Δ BCD,

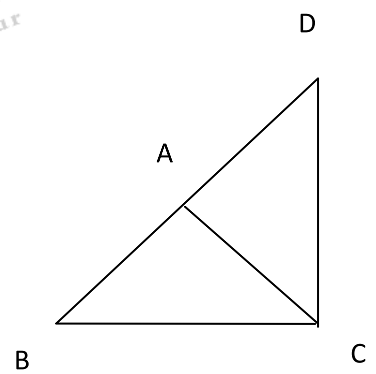
$\angle BCD + \angle DBC + \angle BDC = 180^0$

$\Rightarrow \angle BCD + \angle BCD = 180^0$

$\Rightarrow 2\angle BCD = 180^0$

$\Rightarrow \angle BCD = \frac{180^0}{2} = 90^0$

$\therefore \angle BCD = 90^0$

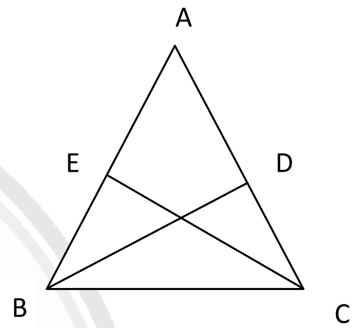


2. BD and CE are two altitudes of aΔABC such that BD = CE. Prove that ΔABC is isosceles.

Solution:

Given: BD and CE are altitudes of a ΔABC such that BD = CE

To prove: ΔABC is isosceles



Proof: In ΔDCB and ΔEBC,

BD = CE

BC = CB

$\angle$BDC = $\angle$CEB [90^0]

$\therefore$ DCB $\cong$ Δ EBC [RHS congruence]

$\Rightarrow \angle$ C = $\angle$B

In Δ ABC, $\angle$ C = $\angle$B

$\therefore$ AB = AC

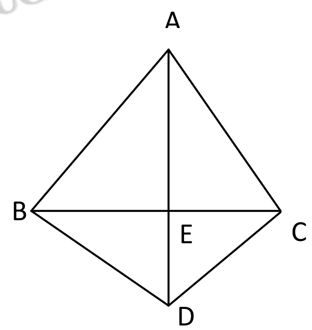
As two sides are equal in length, Δ ABC is isosceles.

3. If two isosceles triangles have a common base, prove that the line joining their vertices bisects the base at right angles.

Solution:

Given: Two isosceles triangles ABC and DBC have a common base BC. AD is joined intersecting BC at E.

To prove: BE = CE and AD $\perp$ BC.



Proof: In Δ ABD and Δ ACD,

AB = AC [being equal sides of an isosceles triangle]

DB = DC [being equal sides of an isosceles triangle]

AD = AD [common side]

$\therefore$ Δ ABD $\cong$ Δ ACD [SSS congruence]

$\Rightarrow$ $\angle$ BAD = $\angle$ CAD i.e. $\angle$ BAE = $\angle$ CAE

In Δ ABE and Δ ACE,

$AB = AC, \angle BAE = \angle CAE, AD = AD$

$\therefore \Delta ABE \cong \Delta ACE$ [SAS congruence]

$\Rightarrow BE = CE$

and, $\angle AEB = \angle AEC$

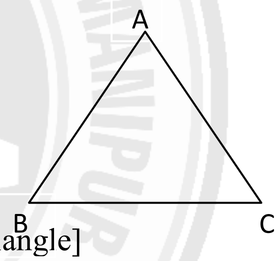
But, $\angle AEB + \angle AEC = 180^0$ [Linear pair angles]

$\therefore \angle AEB = \angle AEC = \frac{180^0}{2} = 90^0$

So, $AD \perp BC$

Hence $BE = CE$ and $AD \perp BC$

4. In a Δ ABC, $\angle A = 100^0$ and AB = AC. Find $\angle B$ and $\angle C$.



Solution: In Δ ABC, $\angle A = 100^0$ and AB = AC

$\therefore \angle C = \angle B$

But $\angle A + \angle B + \angle C = 180^0$ [by angle sum property of triangle]

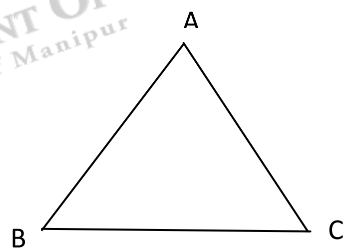
$\Rightarrow 100^0 + \angle B + \angle B = 180^0$

$\Rightarrow 2\angle B = 180^0 - 100^0$

$\Rightarrow \angle B = \frac{80^0}{2} = 40^0$

$\therefore \angle B = \angle C = 40^0$

5. In a Δ ABC, if AB = AC and $\angle B = 70^0$, Find $\angle A$.



Solution: In Δ ABC, $AB = AC, \angle B = 70^0$.

We have, $\angle C = \angle B$ [$\because$ AB = AC]

$\therefore \angle C = 70^0$

We know, $\angle A + \angle B + \angle C = 180^0$ [by angle sum property of triangle]

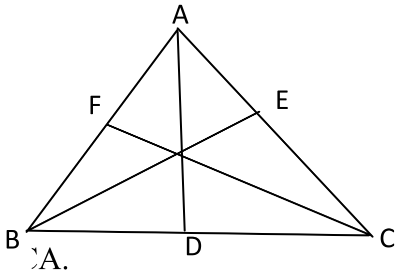
$\Rightarrow \angle A + 70^0 + 70^0 = 180^0$

$\Rightarrow \angle A + 140^0 = 180^0$

$\Rightarrow \angle A = 180^0 - 140^0 = 40^0$

$\therefore \angle A = 40^0$

6. Prove that the medians of an equilateral triangle are equal.



Solution:

Given: AD, BE, CF are medians of equilateral Δ ABC

To prove: AD = BE = CF

Proof: Since ABC is an equilateral triangle, AB = BC = CA.

$\therefore \angle A = \angle B = \angle C$

In Δ FBC and Δ ECB, we have

$BC = CB$

$\angle B = \angle C$

$BF = CE \quad \left[\because \frac{AB}{2} = \frac{AC}{2}\right]$

$\therefore \Delta\, FBC \cong \Delta ECB$ [SAS congruence]

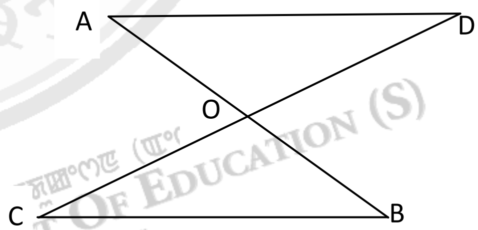
$\Rightarrow CF = BE$

Similarly, $\Delta\, EAB \cong \Delta DBA$

$\Rightarrow BE = AD$

So, AD = BE = CF

7. Two lines AB and CD intersect at O such that BC is equal and parallel to AD. Prove that the lines AB and CD bisect each other at O.



Solution:

Given: Two lines AB and CD intersect at O such that BC is equal and parallel to AD.

To prove: AB and CD bisect each other at O.

Proof: In ΔAOD and ΔBOC,

AD = BC [given]

$\angle ODA = \angle\, OCB$ [Alternate angles as AD ∥ BC]

$\angle OAD = \angle OBC$

$\therefore \Delta\, AOD \cong \Delta BOC$ [ASA congruence]

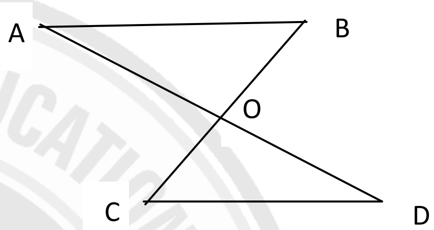
$\Rightarrow OA = OB$ and $OD = OC$

So, AB and CD bisect each other at O.

8. Line segment AB is parallel to another equal line segment CD. O is the mid-point of AD. Prove that (i) Δ AOB $\cong$ ΔDOC (ii) O is mid-point of BC.

Solution:

Given: AB is parallel to another equal line segment CD and O is the mid-point of AD.



To prove: (i) Δ AOB $\cong$ ΔDOC

(ii) O is mid-point of BC

Proof: In ΔAOB and ΔDOC

AB = CD [Given]

OA = OD [$\because$ O is the mid_point of AD]

$\angle OAB = \angle ODC$[$\because$ alternate angles]

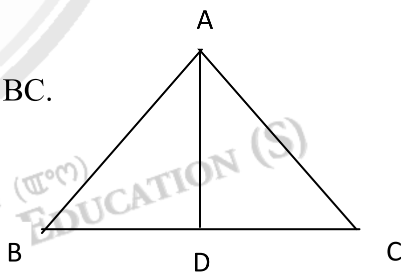
$\therefore$ Δ AOB $\cong$ ΔDOC [SAS]

$\Rightarrow OB = OC$

So, O is the mid-point of BC.

9. InΔABC, the bisector AD of $\angle$A is perpendicular to the side BC. Prove that Δ ABC is isosceles.

Solution:



Given: In Δ ABC, the bisector AD of $\angle$A is perpendicular to BC.

To prove: Δ ABC is isosceles.

Proof: As AD bisects $\angle A$, $\angle BAD = \angle CAD$

As AD $\perp$ BC, $\angle ADB = \angle ADC$ [90^0]

In ΔADB and ΔADC,

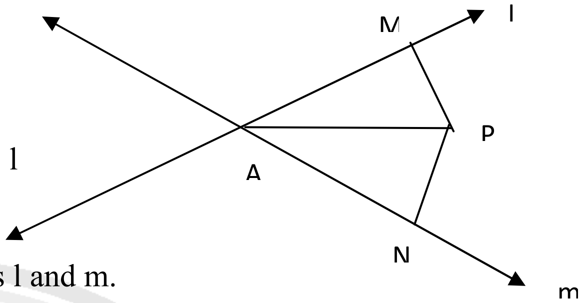
$\angle ADB = \angle ADC$, AD = AD, $\angle BAD = \angle CAD$

$\therefore$ Δ ADB $\cong$ ΔADC [ASA congruence]

$\Rightarrow AB = AC$

As two sides are equal in length, Δ ABC is isosceles.

10. P is a point equidistant from two lines l and m intersecting at a point A. Prove that the line AP bisects the angle between the lines.



Solution:

Given: P is a point equidistant from two lines l and m intersecting at a point A.

To prove: AP bisects the angle between the lines l and m.

Construction: We draw PM and PN perpendicular to l and m respectively.

Proof: As P is equidistant from l and m, PM = PN

In ΔPMA and ΔPNA,

$\angle PMA = \angle PNA\ [90^0]$

AP = AP

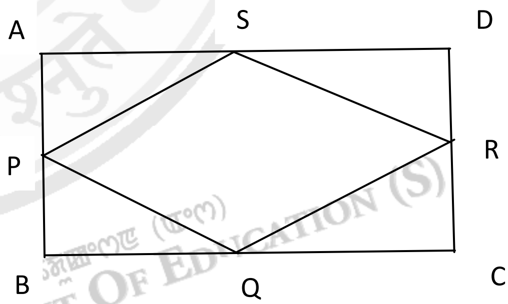
PM = PN

$\therefore \Delta\ PMA \cong \Delta PNA$ [RHS congruence]

$\Rightarrow \angle PAM = \angle PAN$

So, AP bisects the angle between l and m.

11. ABCD is a rectangle. P,Q,R and S are the mid points of AB, BC, CD and DA respectively. Prove that PQRS is rhombus.



Solution:

Given: P,Q,R,S are the mid-points of AB, BC, CD,DA of rectangle ABCD.

To prove: PQRS is a rhombus.

Proof: In ΔPAS and ΔPBQ.,

AP =BP [$\because$ P is the mid_point of AB]

$\angle PAS = \angle PBQ\ [90^0]$

AS=BQ $\left[\because \frac{AD}{2} = \frac{BC}{2}\right]$

$\therefore \Delta PAS \cong \Delta PBQ$ [SAS congruence]

$\Rightarrow PS = PQ$

Similarly, $\Delta QBP \cong \Delta QCR \Rightarrow PQ = QR$

And $\Delta RCQ \cong \Delta RDS \Rightarrow QR = RS$

$\therefore PS = PQ = QR = RS$

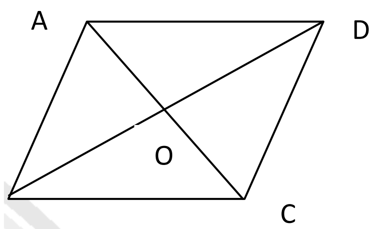
As all the four sides are equal in length, PQRS is a rhombus.

12. Prove that the diagonals of a rhombus bisect each other at right angles.

Solution:

Given: The diagonals AC and BD of a rhombus ABCD intersect at D.

To prove: OA=OC, OB=OD and AC ⊥ BD



Proof: In ΔAOD and ΔCOB

$AD = BC$ [sides of rhombus]

$\angle OAD = \angle OCB$ [alternate angles]

$\angle ODA = \angle OBC$

$\therefore \Delta AOD \cong \Delta COB$ [ASA]

$\Rightarrow OA = OC, OD = OB$

In ΔAOB and ΔCOB,

$OA = OC$

$OB = OB$

$AB = CB$

$\therefore \Delta AOB \cong \Delta COB$ [SSS]

$\Rightarrow \angle AOB = \angle COB$

But, $\angle AOB + \angle COB = 180^0$[linear pair]

$\Rightarrow \angle AOB + \angle AOB = 180°$

$\Rightarrow 2\angle AOB = 180°$

$$\Rightarrow \angle AOB = \frac{180°}{2} = 90°$$

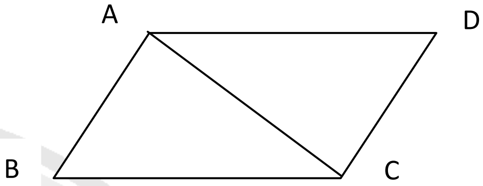
$\Rightarrow AC \perp BD$

$\therefore OA = OC, OB = OD$ and $AC \perp BD$.

13. ABCD is a parallelogram and AC is one of the diagonals. Prove that ΔABC ≅ ΔCDA.

Solution:

Given: AC is one of the diagonal of a parallelogram ABCD.



To prove: $\Delta ABC \cong \Delta CDA$

Proof: In ΔABC and ΔACD

$\angle ACB = \angle CAD$[alternate angles]

$AC = CA$

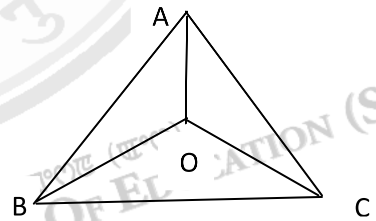
$\angle CAB = \angle ACD$

$\therefore \Delta ABC \cong \Delta CDA$ [ASA congruence]

14. AB and AC are equal sides of an isosceles ΔABC. If the bisectors of $\angle ABC$ and $\angle ACB$ intersect each other at O, prove that $\Delta AOB \cong \Delta AOC$.

Solution:

Given: In isosceles ΔABC, $AB = AC$. The bisectors of $\angle ABC$ and $\angle ACB$ intersect at O.



To prove: $\Delta AOB \cong \Delta AOC$

Proof: As AB=AC in ΔABC,

$\angle C = \angle B$

$\Rightarrow \frac{1}{2}\angle C = \frac{1}{2}\angle B$

$\Rightarrow \angle OCB = \angle OBC$

As $\angle OCB = \angle OBC$ in ΔOBC, $OB = OC$

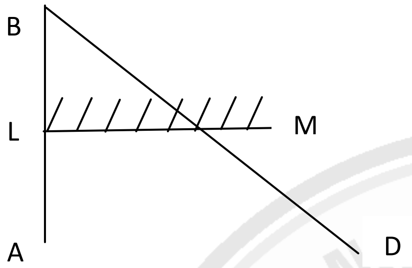
In ΔAOB and ΔAOC,

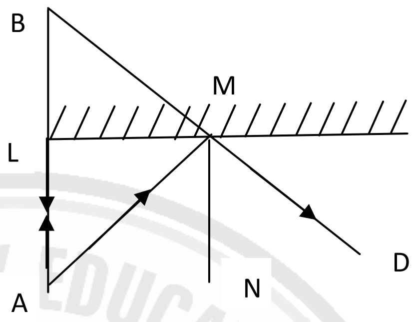
$OA = OA$, $AB = AC$, $OB = OC$

$\therefore \Delta AOB \cong \Delta AOC$ [SSS congruence]

15. The image of an object placed at a point A before a plane mirror LM is seen at the point B by an observer at D as in the adjoining figure. Prove that the image is as far behind the mirror as the object is in front of the mirror.

Solution:





A light ray coming perpendicularly from A to the mirror is reflected in the same way. Another ray AM incident at M is reflected as MD such that $\angle AMN = \angle DMN$ where MN is normal to the plane mirror.

$$AB \parallel MN$$

$$\therefore \angle DMN = \angle MBL \text{ [Corresponding angles]}$$

$$\text{and } \angle AMN = \angle MAL \text{ [alternate angles]}$$

$$\text{so, } \angle MAL = \angle MBL \ [\because \angle AMN = \angle DMN]$$

$$\text{In } \Delta MAL \text{ and } \Delta MBL$$

$$\angle MLA = \angle MLB \ [90°]$$

$$\angle MAL = \angle MBL$$

$$LM = LM$$

$$\therefore \Delta MAL \cong \Delta MBL \text{ [AAS]}$$

$$\Rightarrow AL = BL$$

Hence, the image is as far behind the mirror as the object is in front of the mirror.

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Theorem (RHS Congruence):

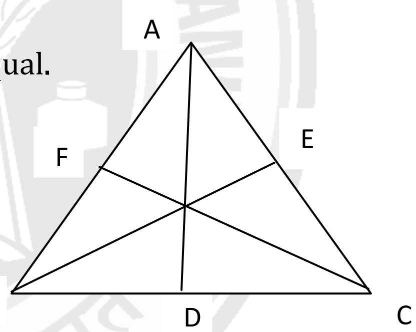
If the hypotenuse and a side of a right angled triangle are equal to the hypotenuse and a side of another right angled triangle, the two right triangles are congruent.

SOLUTIONS

EXERCISE 7.2

1. If the altitudes of a triangle are equal prove that the triangle is equilateral.

Solution:



Given: The altitudes AD, BE, CF of ΔABC are equal.

To prove: ΔABC is equilateral.

Proof: In ΔFBC and ΔECB, we have

$\angle BFC = \angle CEB$ [$= 90°$]

$BC = BC$

$CF = BE$

$\therefore \Delta FBC \cong \Delta ECB$ [RHS congruence]

$\Rightarrow \angle B = \angle C$ i.e. $AB = AC$

Similarly, $\Delta EAB \cong \Delta DBA$

$\Rightarrow \angle A = \angle B$ i.e. $AB = BC$

In ΔABC, $AB = BC = CA$

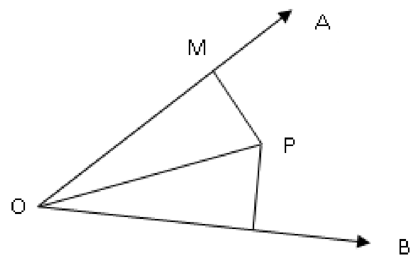
So, ΔABC is equilateral.

2. If perpendiculars from any point within an angle on its arms are congruent, prove that the point lies on the bisector of that angle.

Solution:

Given: From a point P within an angle $\angle AOB$, $PM \perp OA$ and $PN \perp OB$ such that $PM = PN$

To prove: P lies on the bisector of $\angle AOB$ i.e. $\angle AOP = \angle BOP$



Proof: In ΔMOP and ΔNOP,

$\angle PMO = \angle PNO[90°]$

$OP = OP$

$PM = PN$

$\therefore \Delta MOP \cong \Delta NOP[RHS]$

$\Rightarrow \angle MOP = \angle NOP$

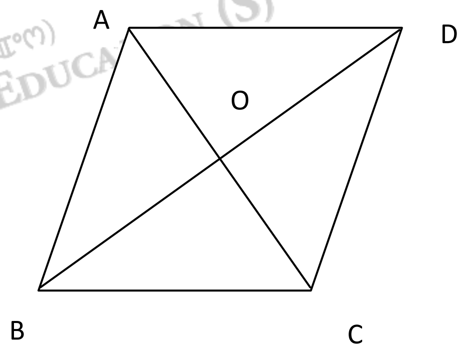
$\Rightarrow \angle AOP = \angle BOP$

$\therefore$ P lies on the bisector of $\angle AOB$.

3. The diagonals AC and BD of a rhombus ABCD bisect each other at right anles at O. Prove that (i) $\Delta AOB \cong \Delta COD$, (ii) $\Delta AOD \cong \Delta COB$.

Solution:

Given: The diagonals AC and BD of a rhombus ABCD bisect each other at right angles at O



To prove: (i)$\Delta AOB \cong \Delta COD$

(ii)$\Delta AOD \cong \Delta COB$

Proof: As AC and BD bisect each other at right angles at O,

$OA = OC, OB = OD$ and $AC \perp BD$.

(i) In ΔAOB and ΔCOD,

$OA = OC, OB = OD$ and $\angle AOB = \angle COD[= 90°]$

$\therefore \Delta AOB \cong \angle COD$[SAS congruence]

(ii)In ΔAOD and ΔCOB,

$OA = OC,$

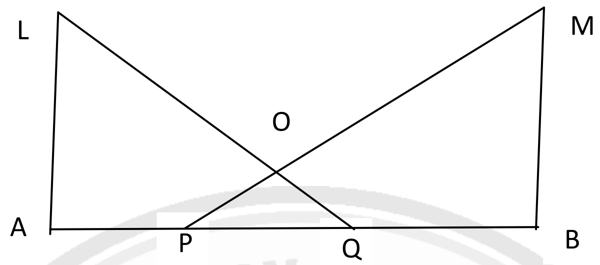
$OD = OB$

$\angle AOD = \angle COB\ [= 90°]$

$\therefore \Delta AOD \cong \Delta COB$ [SAS congruence]

4. AB is trisected at P and Q such that P is between A and Q and AL $\perp$ AB, BM $\perp$ AB. If LQ and MP intersect at O and LQ=MP, prove that Δ OPQ is isosceles.

Solution:



Given: AB is trisected at P and Q such that P is between A and Q. AL $\perp$ AB, BM $\perp$ AB. LQ and MP intersect at O and LQ = MP

To prove: ΔOPQ is isosceles

Proof: As AB is trisected at P and Q,

$AP = PQ = BQ$

$\therefore AP + PQ = BQ + PQ$

$\Rightarrow AQ = BP$

In ΔLAQ and ΔMBP,

$AQ = BP$

$LQ = MP$[Given]

$\angle LAQ = \angle MBP$ [$= 90°$]

$\therefore \Delta LAQ \cong \Delta MBP$

$\Rightarrow \angle LQA = \angle MPB$

$\Rightarrow OQP = \angle OPQ$

As $\angle OQP = \angle OPQ$ in Δ POQ,

$OP = OQ$

So, ΔOPQ is isosceles

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Inequality Relations in a triangle

Theorem: If two sides of triangle are unequal, the angle opposite to the longer side is larger.

Theorem: In a triangle, the side opposite to the larger angle is longer.

Theorem: The sum of any two sides of a triangle is greater than the third side.

SOLUTIONS

EXERCISE 7.3

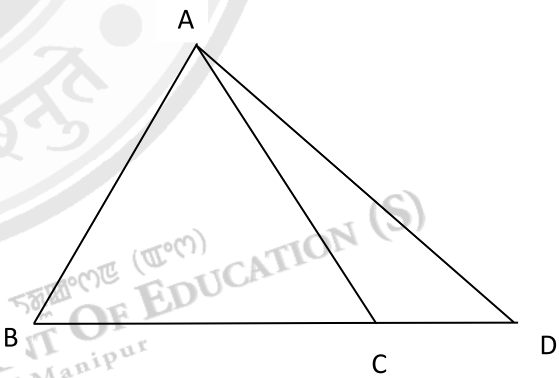
1. If D is any point on the base BC produced of an isosceles triangle ABC, prove that AD > AB

Solution:

Given: D is a point on the base BC produced of an isosceles $\Delta\, ABC$.

To prove: AD>AB

Proof: In the isosceles $\Delta\, ABC$,



$\mathrm{AB} = \mathrm{AC}$

$\therefore \angle\mathrm{ACB} = \angle\mathrm{ABC}$

As $\angle$ACB is exterior to ΔACD,

$\angle\mathrm{ACB} = \angle\mathrm{CAD} + \angle\mathrm{ADC}$

$\Rightarrow \angle\mathrm{ABC} = \angle\mathrm{CAD} + \angle\mathrm{ADB}$

$\Rightarrow \angle\mathrm{ABD} = \angle\mathrm{CAD} + \angle\mathrm{ADB}$

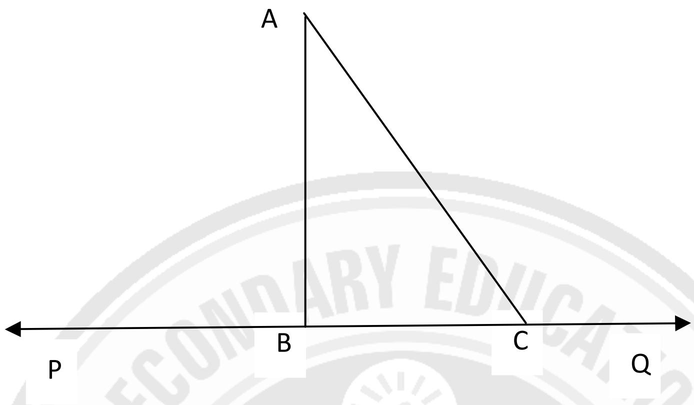
$\Rightarrow \angle\mathrm{ABD} > \angle ADB$

As $\angle\mathrm{ABD} > \angle ADB$ in ΔABD,

$\mathrm{AD} > AB$

2. Of all the line segments that can be drawn to a given line, from a point, not lying on it, prove that the perpendicular line segment is the shortest.

Solution:



Given: A is a point not lying on a line PQ. AB is perpendicular to PQ and AC is not perpendicular to PQ.

To prove: AB<AC

Proof: As $AB \perp PQ, \angle ABC = 90°$

In ΔABC,

$\angle ABC + \angle ACB + \angle BAC = 180°$

$\Rightarrow 90^0 + \angle ACB + \angle BAC = 180°$

$\Rightarrow \angle ACB + \angle BAC = 180° - 90^0 = 90^0$

$\therefore \angle ABC = \angle ACB + \angle BAC$

$\Rightarrow \angle ABC > \angle ACB$

As $\angle ABC > \angle ACB$ in $\Delta ABC, AC > AB$

$\therefore AB < AC$

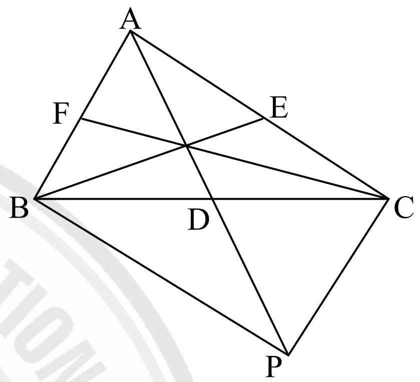
3. Prove that the perimeter of triangle is greater than the sum of its three medians.

Solution:

Given: AD, BE and CF are medians of ΔABC.

To prove: $AB + BC + CA = AD + BE + CF$

Construction: AD is produced to P such that AD = DP. BP and CP are joined.



Proof: Since AD = DP and BD = DC, ABPC is a parallelogram.

$\Rightarrow BP = AC$

In ΔABP, we have

$AB + BP > AP$

$\Rightarrow AB + CA > 2.AD$ ------------- (1)

Similarly, $AB + BC > 2.BE$ ------------- (2)

And $CA + BC > 2.CF$ ------------- (3)

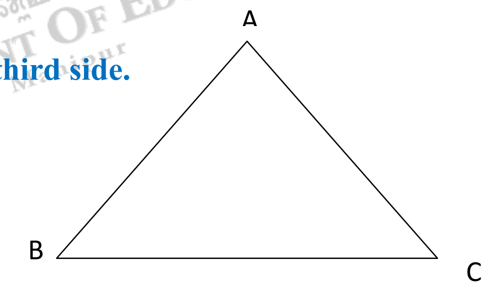
Adding (1), (2) and (3), we get

$2.AB + 2.BC + 2.CA > 2.AD + 2.BE + 2.CF$

$\Rightarrow 2(AB + BC + CA) > 2(AD + BE + CF)$

$\therefore AB + BC + CA > AD + BE + CF$

4. Show that the difference of any two sides is less than the third side.



Solution:

Given: ABC is a triangle.

To prove: $BC - AB < AC$

Proof: We know that the sum of any two sides of a triangle is greater than the third side.

$\therefore AB + AC > BC$

$\Rightarrow BC < AB + AC$

$\Rightarrow BC - AB < AC$

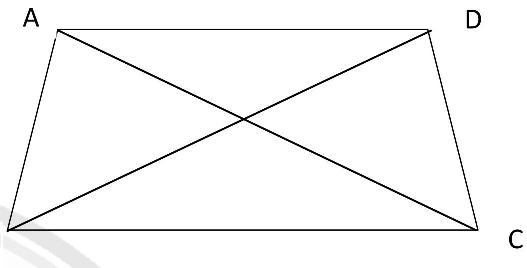
$\therefore BC - AB < AC$

5. Prove that in quadrilateral, the sum of sides is greater than the sum of its diagonals.

Solution:

Given: ABCD is a quadrilateral in which AC and BD are diagonals.

To prove: $AB + BC + CD + DA > AC + BD$



Proof: *In* ΔABC,

$AB + BC > AC$ ……………..(1)

In ΔBCD,

$BC + CD > BD$ ……………..(2)

In ΔADC,

$CD + DA > AC$ ……………..(3)

In ΔABD,

$AB + DA > BD$ ……………..(4)

Adding (1), (2), (3) and (4),

$AB+BC+BC+CD+CD+DA+AB+DA > AC+BD+AC+BD$

$\Rightarrow 2AB+2BC+2CD+2DA > 2AC+ 2BD$

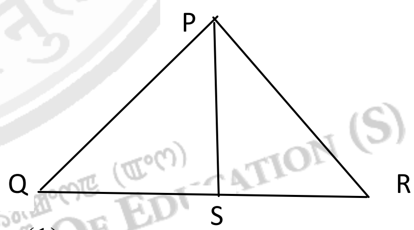
$\Rightarrow 2(AB+BC+CD+DA) > 2(AC + BD)$

$\Rightarrow AB+BC+CD+DA > AC+BD$

$\therefore AB+BC+CD+DA > AC+BD$

6. In Δ PQR, S is any point on the side QR. Show that $PQ + QR + PR > 2PS$.

Solution:



Given: S is any point on the side QR of Δ PQR.

To prove: $PQ + QR + PR > 2PS$

Proof: In Δ PQS,

$PQ + QS > PS$ …………………………(1)

In Δ PSR,

$PR + RS > PS$ …………………………. (2)

Adding (1) and (2),

$PQ + QS + PR + RS + > PS + PS$

$\Rightarrow$ $PQ + (QS + RS) + PR > 2PS$

$\Rightarrow$ $PQ + QR + PR > 2PS$

$\therefore PQ + QR + PR > 2PS$

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