CHAPTER 10
CIRCLES

- **Circle:** The closed plane figure consisting of all those points of the plane which are at a constant distance from a fixed point is called a circle.

  The fixed point is called the centre and the constant distance, the radius of the circle.

- **Terms related to a circle**
  - **Chord:** The line segment joining any two points of a circle is called a chord.
  - **Diameter:** A chord of a circle passing through the centre of the circle is called a diameter.
  - **Circumference:** The perimeter of a circle is called the circumference of the circle.
  - **Arc:** It is a part of a circle between two distinct points of the circle.

    The two points divide the circle into two parts usually unequal, the larger part is called major arc and the smaller part minor arc.

- **Theorems**

1. Equal chords of a circle subtend equal angles at the centre.
2. If the angles subtended by the chords of a circle at the centre are equal, then the chords are of equal length.
3. The perpendicular from the centre of a circle to a chord bisects the chord.
4. The line drawn through the centre of a circle to bisect a chord is perpendicular to the chord.
5. One and only one circle can be drawn passing through three given non-collinear points.

> Note:
>
> An infinite number of circles can be drawn through a point.
>
> An infinite number of circles can also be drawn passing through any two points in a plane.

6. Chords of equal lengths are equidistant from the centre.
7. Chords equidistant from the centre of a circle are equal in length.

8. The angle subtended by an arc of a circle at the centre is double the angle subtended by it at any point on the remaining part of the circle.

Corollary: Angle in a semi-circle is a right angle.

9. Angles in the same segment of a circle are equal.

10. If a line segment joining two points subtends equal angles at two other points lying on the same side of the line containing the line segment, the four points are concyclic.

An angle in a segment of a circle is the angle subtended at any point on the corresponding arc of the segment by the corresponding chord.

11. Opposite angles of a cyclic quadrilateral are supplementary.

12. If the opposite angles of a quadrilateral are supplementary, then the quadrilateral is cyclic.

A quadrilateral is cyclic if all its vertices lie on a circle.

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SOLUTIONS

EXERCISE 10

1. If two circles intersect at two points, prove that their centres lie on the perpendicular bisector of the common chord.

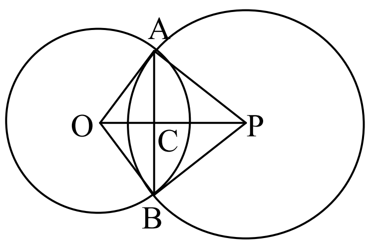
Solution:

Given: Two circles with centres O and P intersect each other at A and B. OP intersects AB at C.

To prove: O and P lies on the perpendicular bisector of AB.

i.e., AC = BC and OP ⊥ AB.

Construction: OA, OB, PA and PB are joined.



Proof: In ΔAOP and ΔBOP, we have

OA = OB [radii of a same circle]

PA = PB [radii of a same circle]

OP = OP [common side]

Then, ΔAOP ≅ ΔBOP [by SSS congruence]

∴ ∠AOP = ∠BOP i.e. ∠AOC = ∠BOC

In ΔAOC and ΔBOC, we have

OA = OB [radii of a same circle]

∠AOC = ∠BOC

OC = OC [common side]

Then, ΔAOC ≅ ΔBOC [by SSS congruence]

∴ AC = BC

And ∠OCA = ∠OCB

But $\angle OCA + \angle OCB = 180^0$ [being linear pair angles]

$\therefore \angle OCA = \angle OCB = \frac{1}{2} \times 180^0 = 90^0$

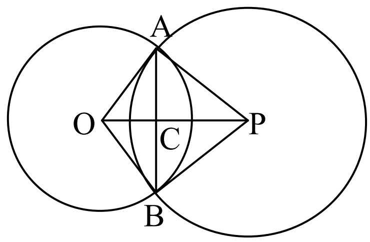
i.e. OP ⊥ AB.

Hence OP lies on the perpendicular bisector of AB.

2. Two circles of radii 5 cm and 3 cm intersect at two points and the distance between their centres is 4 cm. Find the length of the common chord.

Solution:

Two circles with centres O and P intersect at A and B such that OA = 3 cm, PA = 5 cm and OP = 4 cm. OP intersects AB at C.



We know, OP is the perpendicular bisector of AB.

Let OC = x cm. Then, PC = $(4 - x)$cm

In the right ΔAOC, we have

$AC^2 + OC^2 = OA^2$ [by Pythagoras Theorem]

$\Rightarrow AC^2 + x^2 = 3^2$

$\Rightarrow AC^2 = 9 - x^2$ ------------ (1)

In the right ΔAPC, we have

$$AC^2 + PC^2 = AP^2 \text{ [by Pythagoras Theorem]}$$

$$\Rightarrow AC^2 + (4-x)^2 = 5^2$$

$$\Rightarrow AC^2 + 16 - 8x + x^2 = 25$$

$$\Rightarrow AC^2 = 8x + 9 - x^2 \text{------------ (2)}$$

From (1) and (2), we have

$$8x + 9 - x^2 = 9 - x^2$$

$$\Rightarrow 8x = 0$$

$$\Rightarrow x = 0$$

Substituting $x = 0$ in (1), we have

$$AC^2 = 9$$

$$\Rightarrow AC^2 = 3^2$$

$$\Rightarrow AC = 3$$

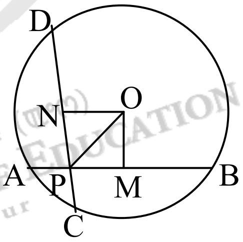
We know, $AB = 2AC = 2 \times 3 = 6$

∴ the required length of the common chord is 6 cm.

3. If two equal chords of a circle intersect within the circle, prove that the line joining the points of intersection to the centre makes equal angles with the chords.

Solution:

Given: Two equal chords AB and CD of a circle with centre O intersect at P.



To prove: ∠OPB = ∠OPD

Construction: OM ⊥ AB and ON ⊥ CD are drawn.

Proof: AB and CD are equal chords of the circle.

∴ OM = ON [∵ equal chords are equidistant from the centre]

In ΔOMP and ΔONP, we have

∠OMP = ∠ONP $(= 90^0)$

OP = OP [common side}

OM = ON

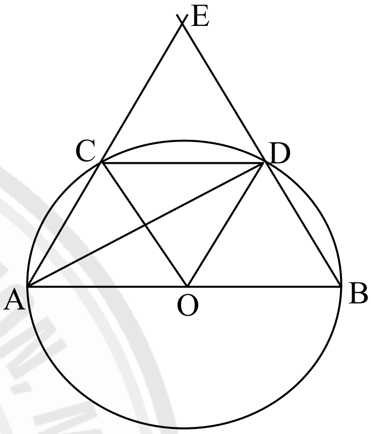
Then, ΔOMP ≅ ΔONP [by RHS congruence]

⇒ ∠OPM = ∠OPN

∴ ∠OPB = ∠OPD

4. AB is a diameter of a circle, CD is a chord equal to the radius of the circle. AC and BD when extended intersect at a point E. Prove that $\angle AEB = 60^0$.

Solution:



Given: AB is a diameter of a circle with centre O. CD is a chord equal to the radius of the circle. AC and BD when extended intersect at a point E.

To prove: $\angle AEB = 60^0$

Construction: OC, OD and AD are joined.

Proof : We have, OC = OD = CD (= radius)

Then, ΔCOD is an equilateral triangle.

$\therefore \angle COD = 60^0$

But $2\angle CAD = \angle COD$ [$\because$arc DC makes $\angle COD$ at the centre and $\angle CAD$ at a point on the circle]

$\Rightarrow \angle CAD = \frac{1}{2}\angle COD$

$\Rightarrow \angle EAD = \frac{1}{2} \times 60^0 = 30^0$

We know, AD $\perp$ BE [$\because \angle ADB = 90^0$, as angle in a semi-circle is a right angle]

$\therefore \angle ADE = 90^0$

Now, in the ΔADE, we have

$\angle AED + \angle ADE + \angle EAD = 180^0$ [by angle sum property of a triangle]

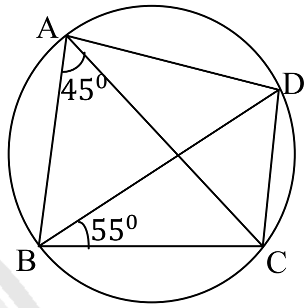
$\Rightarrow \angle AEB + 90^0 + 30^0 = 180^0$ [$\because \angle AED = \angle AEB$]

$\Rightarrow \angle AEB + 120^0 = 180^0$

$\therefore \angle AEB = 60^0$

5. ABCD is a cyclic quadrilateral in which AC and BD are its diagonals. If $\angle DBC = 55^0$ and $\angle BAC = 45^0$, find $\angle BCD$.

Solution:



In the cyclic quadrilateral ABCD, AC and BD are diagonals such that $\angle DBC = 55^0$ and $\angle BAC = 45^0$.

We know, $\angle BAC = \angle BDC$ [angles in a same segment of a circle]

$\therefore \angle BDC = 45^0$

Now, in ΔBCD, we have

$\angle BCD + \angle BDC + \angle DBC = 180^0$ [by angle sum property of a triangle]

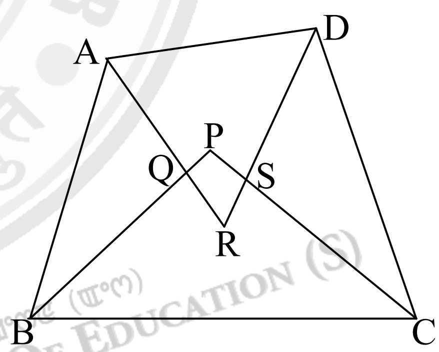
$\Rightarrow \angle BCD + 45^0 + 55^0 = 180^0$

$\Rightarrow \angle BCD + 100^0 = 180^0$

$\therefore \angle BCD = 80^0$

6. Prove that the quadrilateral formed (if possible) by the internal angle bisectors of any quadrilateral is cyclic.

Solution:



Given: PQRS is a quadrilateral formed by the internal angle bisectors of a quadrilateral ABCD.

To prove: PQRS is a cyclic quadrilateral.

Proof: In ΔBPC, we have

$\angle BPC + \angle PBC + \angle PCB = 180^0$ [angle sum property of a triangle]

$\Rightarrow \angle QPS + \frac{1}{2}\angle ABC + \frac{1}{2}\angle BCD = 180^0$

$\Rightarrow \angle QPS = 180^0 - \frac{1}{2}(\angle ABC + \angle BCD)$

Similarly, $\angle QRS = 180^0 - \frac{1}{2}(\angle BAD + \angle ADC)$

Now, $\angle QPS + \angle QRS = \left\{180^0 - \frac{1}{2}(\angle ABC + \angle BCD)\right\} + \left\{180^0 - \frac{1}{2}(\angle BAD + \angle ADC)\right\}$

$= \left\{360^0 - \frac{1}{2}(\angle ABC + \angle BCD + \angle BAD + \angle ADC)\right\}$

$= 360^0 - \frac{1}{2} \times 360^0$ [$\because$sum of four angles of a quadrilateral is 360^0]

$= 360^0 - 180^0$

$= 180^0$

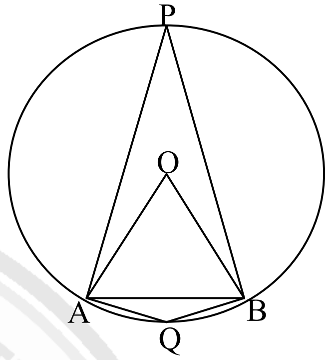
But $\angle QPS$ and $\angle QRS$ are opposite angles of the quadrilateral PQRS.

Hence the quadrilateral PQRS is cyclic.

7. A chord of a circle is equal to the radius of the circle. Find the angle subtended by the chord at a point on the minor arc and also at a point on the major arc.

Solution:

AB is a chord of a circle with centre O equal to the radius of the circle. P and Q are points on the major and minor arcs.



We have, OA = OB = AB (= radius)

Then, ΔAOB is an equilateral triangle.

$\therefore \angle AOB = 60^0$

We know,

$\angle AOB$ and $\angle APB$ are angles subtended by the arc AB at the centre O and at a point P on the remaining part of the circle.

$\therefore 2\angle APB = \angle AOB$

$\Rightarrow \angle APB = \frac{1}{2}\angle AOB = \frac{1}{2} \times 60^0 = 30^0$

$\angle AQB$ and $\angle APB$ are the opposite angles of a cyclic quadrilateral APBQ.

$\therefore \angle AQB + \angle APB = 180^0$

$\Rightarrow \angle AQB + 30^0 = 180^0$

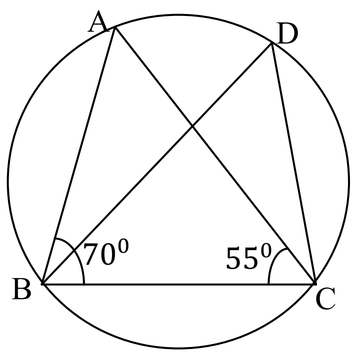
$\Rightarrow \angle AQB = 180^0 + 30^0 = 120^0$

$\therefore$ The angles subtended by the chord at the major arc is 30^0 and at the minor arc is 120^0.

8. In the adjoining figure, $\angle ABC = 70^0$, $\angle ACB = 55^0$. Find $\angle BDC$.

Solution:

In the ΔABC, we have



$\angle BAC + \angle ABC + \angle ACB = 180^0$ [by angle sum property of triangle]

$\Rightarrow \angle BAC + 70^0 + 55^0 = 180^0$

$\Rightarrow \angle BAC = 180^0 - 125^0$

$\Rightarrow \angle BAC = 55^0$

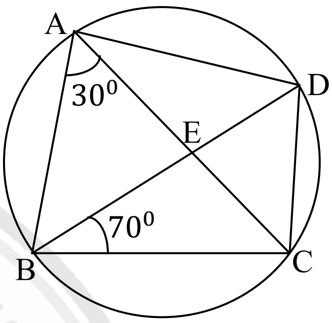
But $\angle BAC = \angle BDC$ [being angles in a same segment of a circle]

$\therefore \angle BDC = 55^0$

9. ABCD is a cyclic quadrilateral whose diagonals intersect at a point E. If $\angle DBC = 70^0$, $\angle BAC = 30^0$, find $\angle BCD$. Further, if AB = BC, find $\angle ECD$.

Solution:

ABCD is a cyclic quadrilateral whose diagonals intersect at a point E such that $\angle DBC = 70^0$, $\angle BAC = 30^0$.



We know, $\angle BAC = \angle BDC$ [being angles in the same segment]

$\therefore \angle BDC = 30^0$

In ΔBCD, by angle sum property of triangle, we have

$\angle BCD + \angle DBC + \angle BDC = 180^0$

$\Rightarrow \angle BCD + 70^0 + 30^0 = 180^0$

$\Rightarrow \angle BCD + 100^0 = 180^0$

$\therefore \angle BCD = 80^0$

In ΔABC, if AB = BC, then $\angle BCA = \angle BAC$ i.e. $\angle BCA = 30^0$

We know, $\angle ACD + \angle BCA = \angle BCD$

$\Rightarrow \angle ACD + 30^0 = 80^0$

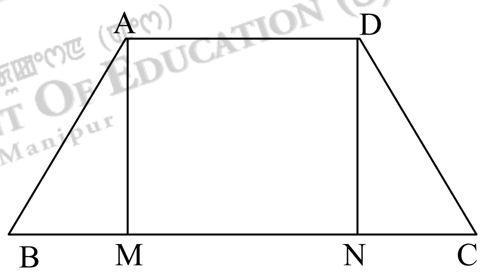
$\Rightarrow \angle ACD = 50^0$

i.e. $\angle ECD = 50^0$

10. If the non-parallel sides of a trapezium are equal, prove that it is cyclic.

Solution:

Given: ABCD is a trapezium in which AD ∥ BC and AB = DC.



To prove: ABCD is cyclic.

Construction: AM and DN are drawn perpendicular to BC.

Proof: In ΔABM and ΔDCN, we have

$\angle AMB = \angle DNC$ $(= 90^0)$

AB = DC (given)

AM = DN [$\because$ distance between two parallels are equal]

Then, $\Delta ABM \cong \Delta DCN$ [by RHS congruence]

$\therefore \angle ABM = \angle DCN$ i.e. $\angle ABC = \angle DCB$

We know, $\angle BAD + \angle ABC = 180^0$ [being interior angles on a same side of a transversal]

$\Rightarrow \angle BAD + \angle DCB = 180^0$

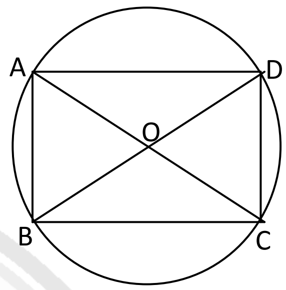
$\therefore$ Opposite angles of ABCD are supplementary.

Hence, ABCD is cyclic.

11. If diagonals of a cyclic quadrilateral are diameters of the circle through the vertices of a quadrilateral, prove that it is a rectangle.

Solution:

Given: ABCD is a cyclic quadrilateral in which diagonals AC and BD are diameters of the circle.



To prove: ABCD is a rectangle.

Proof: We have, AC and BD are diameters of the circle.

∴ ∠BAD, ∠ABC, ∠BCD and ∠ADC are angles in semi-circles.

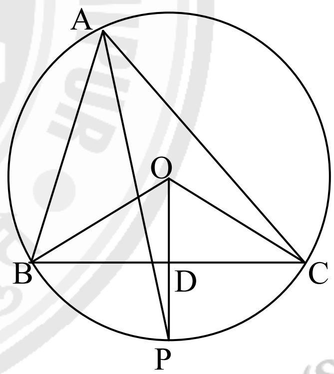
Then, $\angle BAD = \angle ABC = \angle BCD = \angle ADC = 90^0$

Hence ABCD is a rectangle.

12. In any triangle ABC, if angle bisectors of ∠A and perpendicular bisector of BC intersect, prove that they intersect on the circumcircle of the triangle ABC.

Solution:

Let us suppose that the circle with centre O is the circumcircle of ΔABC and OD is the perpendicular bisector of BC meeting the circle at P.



It is sufficient to prove that AP is the bisector of ∠A.

In ΔOBD and ΔOCD,

$\angle ODB = \angle ODC\ (= 90^0)$

OB = OC (radii of a circle)

OD = OD (common side)

∴ ΔOBD ≅ ΔOCD (by RHS congruence)

$\Rightarrow \angle BOD = \angle COD$

$\Rightarrow \angle BOP = \angle COP$

$\Rightarrow \frac{1}{2}\angle BOP = \frac{1}{2}\angle COP$

$\Rightarrow \angle BAP = \angle CAP$

∴ AP is the bisector of ∠A of ΔABC.

Hence, in any triangle ABC, if angle bisectors of ∠A and perpendicular bisector of BC intersect, they intersect on the circumcircle of the triangle ABC.

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