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Week 1: Fundamental Concepts & Discrete Distributions

Definitions

- **Population:** The complete collection of all individuals or items under consideration in a statistical study.
- **Sample:** A subset of the population from which information is actually collected.

Parameters vs Statistics

- **Population Parameters** (constants, usually unknown):
 - $\mu \rightarrow$ population mean
 - $\sigma \rightarrow$ population standard deviation
 - **Sample Statistics** (random variables):
 - $\bar{x} \rightarrow$ sample mean
 - $s \rightarrow$ sample standard deviation
-

Measures of Centrality and Variation

Measures of Centrality (Centre)

1. **Mean (\bar{x}):** Arithmetic average

$$\bar{x} = \frac{\sum x_i}{n}$$

2. **Median:** Middle value when data is ordered.
3. **Mode:** Most frequently occurring value.

Measures of Variation (Spread)

- **Range:** Max – Min. Very sensitive to outliers.
- **Sample Variance (s^2):**

$$s^2 = \frac{\sum (x_i - \bar{x})^2}{n - 1}$$

- **Sample Standard Deviation (s):**

$$s = \sqrt{\frac{\sum (x_i - \bar{x})^2}{n - 1}}$$

Probability Theory Basics

Core Rules

- **Sample Space (Ω):** Set of all possible outcomes. $P(\Omega) = 1$
- **Empty Set (\emptyset):** Impossible event. $P(\emptyset) = 0$
- **Probability Bounds:** $0 \leq P(A) \leq 1$
- **Complement Rule:** $P(A^c) = 1 - P(A)$. Notation: A^c , \bar{A} , or A' .

Combining Events

- **Intersection (AND):** $A \cap B$
- **Union (OR):** $A \cup B$

Addition Rules

- **General Addition Rule:** $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
 - **Disjoint (Mutually Exclusive) Events:**
 - Cannot occur together
 - $A \cap B = \emptyset$
 - $P(A \cup B) = P(A) + P(B)$
-

Conditional Probability and Independence

Conditional Probability

Probability that B occurs given that A has occurred:

$$P(B | A) = \frac{P(A \cap B)}{P(A)}$$

Multiplication Law

$$P(A \cap B) = P(B | A) \times P(A)$$

Independence

Two events are independent if one does not affect the other:

- $P(B | A) = P(B)$, or
 - $P(A \cap B) = P(A) \times P(B)$
-

Advanced Probability Theorems

Bayes' Theorem

Used to reverse conditional probabilities:

$$P(A | B) = \frac{P(B | A) \times P(A)}{P(B)}$$

Law of Total Probability

If A_1, A_2, \dots, A_n partition the sample space:

$$P(B) = \sum_{i=1}^n P(B | A_i) \times P(A_i)$$

Discrete Random Variables

Definition

A **Random Variable** (X) is a numerical model for a measurement.

- **Discrete RV:** Takes a finite or countably infinite number of values.
- **Bernoulli RV:** Simplest discrete RV.
Takes value:

- 1 for success
- 0 for failure

Probability Mass Function (pmf)

$$f(x) = P(X = x)$$

Expected Value (Mean)

The long-run average or centre of gravity:

$$E(X) = \mu = \sum x \cdot P(X = x)$$

Example (Fair die):

$$E(X) = 3.5$$

Cumulative Distribution Function (CDF)

$$F(x) = P(X \leq x)$$

- For discrete RVs, the CDF has a **step shape**.
 - **At least rule:** $P(X \geq k) = 1 - P(X < k)$
-

Discrete Probability Distributions

Binomial Distribution

Used for the number of successes in n trials.

Assumptions (Always state in exams)

1. Fixed number of trials (n)
2. Constant probability of success (p)
3. Trials are independent

Model

$$P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x}$$

Parameters

- **Mean:** $\mu = np$
- **Standard Deviation:**

$$\sigma = \sqrt{np(1 - p)}$$

Poisson Distribution

Used for counting arrivals in a fixed interval of time or space.

Assumptions

1. Probability proportional to interval size
2. Probability of two or more arrivals in a very small interval is negligible
3. Non-overlapping intervals are independent

Model

$$P(X = x) = \frac{e^{-\alpha t} (\alpha t)^x}{x!}$$

- α = average rate per unit
- t = length of interval

Key Property (Very Exam Important)

$$Rate = \lambda = E(X) = Var(X) = \alpha t$$

Week 2: Continuous, Sampling & Hypothesis Testing

Continuous Random Variables

Definition

A continuous random variable can take values anywhere in a continuum, such as height, temperature, or sales.

- **Density Function ($f(x)$):**

A curve where the area under the curve between two points represents probability.

- **Total Area:**

The total area under $f(x)$ is always 1:

$$\int_{-\infty}^{\infty} f(x) dx = 1$$

Uniform Distribution

The simplest continuous distribution where probability is constant between a and b .

- **PDF:**

$$f(x) = \frac{1}{b-a}, \quad a \leq x \leq b$$

The Normal Distribution

Properties

- Defined by **Mean** (μ) and **Variance** (σ^2).
- Notation: $X \sim N(\mu, \sigma^2)$

Empirical Rule (68, 95, 99.7)

- 68% of data lies within $\mu \pm 1\sigma$
 - 95% of data lies within $\mu \pm 2\sigma$
 - 99.7% of data lies within $\mu \pm 3\sigma$
-

Sampling Distributions

Central Limit Theorem (CLT)

Regardless of the population distribution, if sample size n is large, the distribution of the sample mean \bar{X} is approximately normal.

- **Mean of \bar{X} :** $E(\bar{X}) = \mu$
- **Variance of \bar{X} :**

$$\text{Var}(\bar{X}) = \frac{\sigma^2}{n}$$

- **Standard Error:** $\frac{\sigma}{\sqrt{n}}$

- **Z Statistic:**

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$$

Hypothesis Testing Basics

Core Concepts

- **Null Hypothesis (H_0):**
Assumed true. Always contains equality ($=, \leq, \geq$).
- **Alternative Hypothesis (H_1):**
The claim we seek evidence for. Always contains inequality ($\neq, <, >$).

Errors

- **Type I Error (α):**
Rejecting H_0 when it is actually true.
- **Type II Error (β):**
Failing to reject H_0 when it is actually false.

The p-value

The probability of observing a result at least as extreme as the one obtained, assuming H_0 is true.

- **Decision Rule:**
Reject H_0 if

$$\text{p-value} < \alpha$$

Confidence Intervals

Definition

An interval constructed around \bar{x} where we are reasonably confident the true population mean μ lies.

- **Interpretation:**
In repeated sampling, 95% of such intervals would contain μ .

Formula (Known σ or Large n)

$$\bar{x} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

Example: Cola Cans

- $\bar{x} = 299.64$
- $n = 100$
- $\sigma = 1.2$

Resulting interval:

$$[299.40, 299.88]$$

Since 300 is not in the interval, reject H_0 .

Hypothesis Tests for Proportions

Used for categorical data.

Example: Thanos Snap

- $H_0 : p = 0.5$
- $H_1 : p \neq 0.5$
- Observed: 64 vanished out of 100

R Code:

```
prop.test(64, 100, p = 0.5)
```

- p-value = 0.0069
Reject H_0 .
-

One Sample t-Test

Used when population variance σ^2 is unknown.

- Uses Student's t distribution
- Degrees of freedom: $df = n - 1$

Assumptions

1. Data is numeric and continuous.
2. Data is normally distributed.

Normality Test: Shapiro-Wilk

- If p-value > 0.05 , assume normality.

Example: Corrib River Radiation

- $H_0 : \mu \geq 5$
- $H_1 : \mu < 5$

R Code:

```
t.test(corrib, mu = 5, alternative = "less")
```

- p-value = 0.002
Reject H_0 . Water is safe.
-

Comparing Two Means: Independent Samples

Used to compare two separate groups.

Steps

1. **Check Normality:**
Shapiro-Wilk test on both groups.
2. **Check Variances:**
 - Levene's Test (robust)
 - Bartlett's Test (requires normality)If p-value > 0.05, assume equal variances.
3. **Run t-Test:**
 - Welch Two Sample t-test (default in R)

R Code:

```
t.test(x, y, alternative = "less")
```

Comparing Two Means: Paired Samples

Used when observations are dependent or matched.

Logic

Performs a one sample t-test on the differences between paired observations.

Example: Diet Study

- $H_1 : \mu_{\text{diff}} > 0$

R Code:

```
t.test(before, after, paired = TRUE, alternative = "greater")
```

- p-value = 0.02
Reject H_0 . Diet worked.

Warning

Using an independent t-test on paired data is incorrect and can increase the chance of a Type II error.

Week 3: Enumerative Data Analysis and MLE

Enumerative Data Analysis (Chi-Squared)

Qualitative vs Quantitative

Previously we analysed **quantitative data** (height, weight, marks).

Now we analyse **qualitative (categorical) data**:

- Data consists of **counts / frequencies**
- Examples: Eye colour, Yes/No, Defective/Not defective

We compare **Observed vs Expected frequencies**.

The Chi-Squared Distribution (χ^2)

- Not symmetric
 - Right skewed
 - Range: $0 \rightarrow \infty$
 - Depends on **degrees of freedom (df)**
 - As df increases, it becomes more Normal shaped
 - Right tail area = significance level α
-

Chi-Squared Goodness-of-Fit Test

Purpose

Tests whether observed categorical data matches a claimed distribution.

Test Statistic

$$\chi^2 = \sum \frac{(O - E)^2}{E}$$

Where:

- O = Observed frequency
- E = Expected frequency
- $df = k - 1$

Large χ^2 means observed differs strongly from expected.

M&Ms Example

Claim (H_0):

30% Brown, 20% Yellow, 20% Red, 10% Orange, 10% Green, 10% Blue

Hypotheses:

- H_0 : Distribution matches claim
- H_1 : Distribution differs

Reject H_0 if $\chi^2_{calc} > \chi^2_{critical}$.

R Code

```
chocolate <- c(67, 36, 43, 24, 23, 7)
probs <- c(0.3, 0.2, 0.2, 0.1, 0.1, 0.1)
chisq.test(chocolate, p = probs)
```

Chi-Squared Test of Independence

Purpose

Tests whether two categorical variables are related.

Hypotheses

- H_0 : Variables are independent
 - H_1 : Variables are dependent
-

Expected Counts Formula

For contingency table:

$$E_{ij} = \frac{(\text{Row Total})(\text{Column Total})}{\text{Grand Total}}$$

Degrees of freedom:

$$df = (r - 1)(c - 1)$$

Assumptions

1. Categorical variables
2. Independent observations
3. Rule of 5:
 - At least 80% of expected counts ≥ 5
 - No expected count < 1

If violated, combine categories or use Fisher's test.

Fisher's Exact Test

Used for small samples.

Independent

```
wilcox.test(group_A, group_B, alternative = "two.sided")
```

Paired

```
wilcox.test(group_A, group_B, alternative = "two.sided", paired = TRUE)
```

Mann-Whitney U Test / Wilcoxon Test

Used for median

```
wilcox.test
```

Effect Size: Statistical vs Practical Significance

The Problem with Large Samples

- **Statistical significance** tells you if a difference exists.
 - **Practical importance** tells you if the difference matters.
 - With very large n , even tiny differences can produce small p values.
 - Example: A 2 second improvement may be statistically significant but practically useless.
-

The Solution: Effect Size Effect size measures the **magnitude** of a difference.

Chi-Squared Tests: Phi Coefficient For 2×2 tables:

$$\phi = \sqrt{\frac{\chi^2}{n}}$$

Guidelines:

- 0.1 small
 - 0.3 medium
 - 0.5 large
-

t Tests: Cohen's d Used when comparing two means.

$$d = \frac{\bar{x}_1 - \bar{x}_2}{s}$$

For independent samples, use the pooled standard deviation.

Guidelines:

- 0.2 small
- 0.5 medium
- 0.8+ large

Maximum Likelihood Estimation

The Core Idea

How do we find the “best” parameters such as μ or λ ?

MLE finds the parameter that makes your data most likely.

- **Fisher's Principle:** Choose parameter θ that makes the observed data most probable.
 - Goal: Find θ that maximizes $P(\text{data} \mid \theta)$ i.e. the Likelihood Function $L(\theta)$.
-

MLE Step by Step

Likelihood Function Write the probability of the entire dataset.

If observations are independent:

$$L(\theta) = \prod f(x_i \mid \theta)$$

Log-Likelihood Take the natural log:

$$\ell(\theta) = \sum \ln(f(x_i | \theta))$$

Why?

- Differentiating a product is messy
 - Differentiating a sum is easier
 - Logs turn products into sums
-

Differentiate Find derivative with respect to θ :

$$\frac{d\ell}{d\theta}$$

Solve Set derivative equal to 0 and solve for θ .

This gives the MLE estimate.

MLE Examples

Poisson Distribution (Horse Kicks)

- Data: Counts of deaths by horse kicks (von Bortkiewicz data)
- Model:

$$X \sim \text{Poisson}(\lambda)$$

MLE Result

$$\hat{\lambda}_{MLE} = \frac{1}{n} \sum x_i = \bar{x}$$

Takeaway: For Poisson, the MLE for λ is the **sample mean**.

Normal Distribution

We estimate two parameters: μ and σ^2 .

Estimating the Mean

$$\hat{\mu} = \bar{x}$$

Takeaway: MLE mean equals the sample mean.

Estimating the Variance

$$\hat{\sigma}_{MLE}^2 = \frac{1}{n} \sum (x_i - \bar{x})^2$$

Bias Issue

- MLE divides by $n \rightarrow$ biased (underestimates variance)
- Sample variance:

$$s^2 = \frac{1}{n-1} \sum (x_i - \bar{x})^2$$

Uses Bessel's correction and is unbiased.

Conclusion: For large n , difference is negligible.

R Implementation

For complex models, solve numerically.

Note: R minimizes functions, so use the **negative log-likelihood**.

```
library(stats4)

# 1. Define Negative Log-Likelihood
nloglik <- function(lambda) {
  return(-sum(dpois(data, lambda, log = TRUE)))
}

# 2. Run Optimizer
fit <- mle(nloglik, start = list(lambda = 1))
summary(fit)
```

Week 4

Complex MLE & The Need for Optimization

When Math Fails (The Gamma Distribution)

- The Gamma distribution models right-skewed data, for example insurance claims.

- It uses two parameters: α (shape) and β (scale).
 - **The Problem:** When you take the derivative of the Gamma log-likelihood and set it equal to 0, there is no simple closed-form solution. You cannot solve it by hand.
 - **The Solution:** Numerical optimization. We use a computer to find where the derivative is approximately zero, which corresponds to the peak of the likelihood.
-

Numerical Optimization Methods

When we cannot find the maximum likelihood mathematically, we use algorithms to walk uphill to the peak.

Gradient Ascent / Descent

- **How it works:** Finds the direction of the steepest slope, the gradient, and takes a step in that direction.
- **Pros:** Simple to implement; only needs first derivatives.
- **Cons:** Slow, linear convergence; choosing the right step size is tricky.

Newton's Method

- **How it works:** Uses curvature, the Hessian matrix of second derivatives, to fit a quadratic curve and jump straight to its maximum.
- **Pros:** Very fast, quadratic convergence; fewer, smarter steps.
- **Cons:** Fails if the Hessian matrix is not invertible or near saddle points; computationally expensive because it requires second derivatives.

BFGS (Quasi-Newton)

- **How it works:** Achieves Newton-like speed without computing second derivatives. It approximates the Hessian matrix using previous gradient information.
- **Pros:** Fast, robust, and requires no second derivatives. This is the default in R's `optim()` and `mle()`.

Nelder-Mead (Simplex)

- **How it works:** Uses no derivatives. It constructs a simplex, a geometric shape of points, that reflects and shrinks over the surface to find the peak.
 - **Pros:** Extremely robust; works on non-smooth functions and poor starting values.
 - **Cons:** Slow; struggles in high-dimensional problems with many parameters.
-

MLE Optimization in R

The Negative Log-Likelihood Trick

- R's optimization functions such as `optim()` and `nlm()` are designed to minimize, not maximize.
- To compute the Maximum Likelihood Estimate, we minimize the Negative Log-Likelihood.
- If $\ell(\theta)$ is the log-likelihood, we minimize $-\ell(\theta)$.

Using `log=TRUE`

- When computing likelihoods in R, always use `log=TRUE` inside density functions, for example `dgamma(x, shape, scale, log=TRUE)`.
- This computes the log-probability directly, which is more numerically stable than computing a very small probability and then taking its logarithm.

Optimization Pitfalls

- **Local Maxima:** The algorithm may converge to a smaller local peak instead of the global maximum.
 - **Solution:** Try multiple starting values. If all runs converge to the same point, you likely found the global maximum. If not, the likelihood may be multimodal.
 - **Check Convergence:** In R, `optim()$convergence == 0` indicates successful convergence. Any non-zero value indicates failure.
-

Why We Love MLE (Theoretical Properties)

Even when computed numerically, MLE has excellent theoretical properties.

1. **Consistency:** As sample size $n \rightarrow \infty$, $\hat{\theta} \rightarrow \theta$.
 2. **Equivariance:** If $\hat{\theta}$ is the MLE of θ , then $g(\hat{\theta})$ is the MLE of $g(\theta)$.
 3. **Asymptotic Normality:** For large samples, $\hat{\theta} \approx \mathcal{N}\left(\theta, \frac{1}{I(\theta)}\right)$, where $I(\theta)$ is the Fisher Information.
 4. **Asymptotic Efficiency:** For large samples, the MLE achieves the minimum possible variance among regular estimators.
-

The Likelihood Ratio Test (LRT)

Concept

Used to compare two nested models to determine whether additional parameters significantly improve model fit.

- H_0 (Restricted Model): Parameters are fixed, for example a fair coin with $p = 0.5$.

- H_1 (Unrestricted Model): Parameters are estimated using MLE, for example $p = \hat{p}$.

The Test Statistic

$$\Lambda = -2 [\ell(\hat{\theta}_0) - \ell(\hat{\theta})]$$

- $\ell(\hat{\theta})$: Log-likelihood of the unrestricted model.
- $\ell(\hat{\theta}_0)$: Log-likelihood of the restricted model.

The Distribution

- Under H_0 , $\Lambda \sim \chi^2_{df}$
- Degrees of freedom df equal the number of restrictions imposed under H_0 .

Profile Likelihood & Confidence Intervals

- Since the LRT statistic follows a χ^2 distribution asymptotically, we can invert the test to construct confidence intervals without assuming normality.
- In R, `confint(fit)` computes profile likelihood confidence intervals.

Complete Exam Quick Reference Table

Concept / Test	Formula or R Function	Use Case	Key Exam Notes
Sample Mean	$\bar{x} = \frac{\sum x_i}{n}$	Estimate μ	Centre of data
Sample Variance	$s^2 = \frac{\sum (x_i - \bar{x})^2}{n-1}$	Spread	Uses $n - 1$
Sample Std Dev	$s = \sqrt{\frac{\sum (x_i - \bar{x})^2}{n-1}}$	Spread in units	Root of variance
Addition Rule	$P(A \cup B) = P(A) + P(B) - P(A \cap B)$	Combine events	Avoid double counting
Complement Rule	$P(A^c) = 1 - P(A)$	At least one questions	Often simplifies
Conditional Prob	$P(B A) = \frac{P(A \cap B)}{P(A)}$	Given info	Order matters
Independence	$P(A \cap B) = P(A)P(B)$	Check independence	Only if unrelated
Bayes Theorem	$P(A B) = \frac{P(B A)P(A)}{P(B)}$	Reverse conditional	Common trap
Binomial Mean	$\mu = np$	Expected successes	Fixed n, p
Binomial SD	$\sigma = \sqrt{np(1-p)}$	Spread	Memorise
Poisson Mean	$\mu = \lambda$	Arrivals	Mean = variance
Uniform PDF	$f(x) = \frac{1}{b-a}$	Constant density	Area = probability
Z Statistic	$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$	Mean tests	Known σ

Concept / Test	Formula or R Function	Use Case	Key Exam Notes
Confidence Interval	$\bar{x} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$	Estimate mean	Check if μ_0 inside
Shapiro Wilk Test	<code>shapiro.test(x)</code>	Normality	H_0 : Normal
Levene Test	<code>leveneTest()</code>	Compare variances	Robust
Bartlett Test	<code>bartlett.test()</code>	Compare variances	Needs normality
One Sample t Test	<code>t.test(x, mu=...)</code>	Mean vs constant	Unknown σ
Independent t Test	<code>t.test(x, y)</code>	Two groups	Welch default
Paired t Test	<code>t.test(x, y, paired=TRUE)</code>	Before vs after	Uses differences
Proportion Test	<code>prop.test(x, n)</code>	Test proportion	Large samples
Chi Square Statistic	$\chi^2 = \sum \frac{(O-E)^2}{E}$	Categorical tests	Large = big difference
Goodness of Fit	<code>chisq.test(x, p=probs)</code>	Match distribution	$df = k - 1$
Independence Test	<code>chisq.test(matrix)</code>	Relationship test	$df = (r - 1)(c - 1)$
Fisher Exact Test	<code>fisher.test(matrix)</code>	Small samples	Use if counts < 5
Effect Size (Phi)	$\phi = \sqrt{\frac{\chi^2}{n}}$	Strength of association	0.1 small, 0.3 med, 0.5 large
Cohen's d	$d = \frac{\bar{x}_1 - \bar{x}_2}{s}$	Effect size for mean differences	0.2 small, 0.5 medium, 0.8 large.
Likelihood	$L(\theta) = \prod f(x_i \theta)$	Parameter estimation	Maximise
Log Likelihood	$\ell(\theta) = \log L(\theta)$	Simplify math	Turns product into sum
MLE Normal Mean	$\hat{\mu} = \bar{x}$	Estimate mean	Same as sample mean
MLE Normal Variance	$\hat{\sigma}^2 = \frac{1}{n} \sum (x_i - \bar{x})^2$	Estimate variance	Biased
Cohen d	$d = \frac{\bar{x}_1 - \bar{x}_2}{s}$	t test effect size	0.2 small, 0.5 med, 0.8 large
BFGS	<code>optim(method="BFGS")</code>	General-purpose MLE optimization	Fast, robust, no second derivatives required
Nelder-Mead	<code>optim(method="Nelder-Mead")</code>	Non-smooth likelihoods	Very robust but slower, weak in high dimensions
Negative Log-Likelihood	$-\sum \log f(x_i \theta)$	Convert maximization to minimization	R minimizes by default

Concept / Test	Formula or R Function	Use Case	Key Exam Notes
Convergence Check	<code>fit\$convergence == 0</code>	Verify optimizer success	0 indicates successful convergence
Equivariance (MLE)	If $\hat{\theta}$ is MLE, then $g(\hat{\theta})$ is MLE of $g(\theta)$	Transformations of parameters	Core theoretical property
LRT Statistic	$\Lambda = -2[\ell(\hat{\theta}_0) - \ell(\hat{\theta})]$	Compare nested models	Based on log-likelihood difference
LRT Distribution	$\Lambda \sim \chi^2_{df}$	Compute p-values	df equals number of restrictions
Profile Confidence Intervals	<code>confint(fit)</code>	Construct CIs via LRT	Does not rely on normal approximation