Useful Results

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Below are some results I proved in my Introduction to Real Analysis Course. I found some of them useful and even though some may seem obvious I wanted to do the formal proof for myself. Also, I wanted a place to put these results so I could come back and find them pretty easily when needed.

Lemma 1. *If* $a \in \mathbb{R}$, $\varepsilon \in \mathbb{R}$, *and* $\varepsilon > 0$, *then* $a < a + \varepsilon$.

Proof. Let $a, \varepsilon \in \mathbb{R}$ and suppose $\varepsilon > 0$. For the sake of contradiction assume $a \geq a + \varepsilon$. Then $a - (a + \varepsilon) \geq 0$, which implies that $-\varepsilon \geq 0$ and so $\varepsilon \leq 0$. However, $\varepsilon > 0$ by assumption and so we have that $\varepsilon > 0$ and $\varepsilon \leq 0$. This is a contradiction because for every real number x we have $x < 0$, $x = 0$, or $x > 0$. Therefore our assumption that $a \geq a + \varepsilon$ must be false and so we can conclude $a < a + \varepsilon$. $\square$

Lemma 2. *If* $a, b \in \mathbb{R}$ *and* $a < b$, *then* $a < a + \frac{|b-a|}{2} < b$.

Proof. Let $a, b \in \mathbb{R}$ and suppose $a < b$. Since $\frac{|b-a|}{2} > 0$ then by Lemma 1 $a < a + \frac{|b-a|}{2}$. $\square$