# Useful Results

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Below are some results I proved in my Introduction to Real Analysis Course. I found some of them useful and even though some may seem obvious I wanted to do the formal proof for myself. Also, I wanted a place to put these results so I could come back and find them pretty easily when needed.

**Lemma 1.** *If $a \in \mathbb{R}$, $\varepsilon \in \mathbb{R}$, and $\varepsilon > 0$, then $a < a + \varepsilon$.*

*Proof.* Let $a, \varepsilon \in \mathbb{R}$ and suppose $\varepsilon > 0$. For the sake of contradiction assume $a \geq a + \varepsilon$. Then $a - (a + \varepsilon) \geq 0$, which implies that $-\varepsilon \geq 0$ and so $\varepsilon \leq 0$. However, $\varepsilon > 0$ by assumption and so we have that $\varepsilon > 0$ and $\varepsilon \leq 0$. This is a contradiction because for every real number $x$ we have $x < 0$, $x = 0$, or $x > 0$. Therefore our assumption that $a \geq a + \varepsilon$ must be false and so we can conclude $a < a + \varepsilon$. □

**Lemma 2.** *If $a, b \in \mathbb{R}$ and $a < b$, then $a < a + \dfrac{|b-a|}{2} < b$.*

*Proof.* Let $a, b \in \mathbb{R}$ and suppose $a < b$. Since $\dfrac{|b-a|}{2} > 0$ then by Lemma 1 $a < a + \dfrac{|b-a|}{2}$.

Now we will show that $a + \dfrac{|b-a|}{2} < b$ by considering two cases: $(b - a) < 0$ or $(b - a) > 0$. If $(b - a) < 0$ then $|b - a| = a - b$. By assumption $a < b$ and so

$$\begin{aligned} 3a < 3b &\Rightarrow 3a - b < 2b \\ &\Rightarrow 2a + (a - b) < 2b \\ &\Rightarrow 2a + |b - a| < 2b \\ &\Rightarrow a + \frac{|b-a|}{2} < b. \end{aligned}$$

In the case that $(b - a) > 0$ then $|b - a| = b - a$. By assumption $a < b$ and so

$$\begin{aligned} 2a - a < 2b - b &\Rightarrow 2a - a + b < 2b \\ &\Rightarrow 2a + (b - a) < 2b \\ &\Rightarrow 2a + |b - a| < 2b \\ &\Rightarrow a + \frac{|b-a|}{2} < b. \end{aligned}$$

Since the cases are exhaustive then we can conclude that $a + \frac{|b-a|}{2} < b$.

Now we have shown that $a < a + \frac{|b-a|}{2}$ and $a + \frac{|b-a|}{2} < b$. Therefore $a < a + \frac{|b-a|}{2} < b$. $\square$