Extra Exercises from book

December 6, 2022

Below are some exercises from our book that were suggested to be done in addition to the weekly homework. The book is Introduction to Real Analysis 4^{th} edition by Bartle and Shebert. The section and exercise numbers refer to the book. I have worked almost all of these problems and am working to get them typed up.

Section 11.1

3. Write out the Induction argument in the proof of part (b) of the Open Set Properties 11.1.4.

Claim. *The intersection of any finite collection of open sets in $\mathbb{R}$ is open.*

Proof. Let $G_n := G_1 \cap G_2 \cap \ldots \cap G_n$ where $n \in \mathbb{N}$ and G_n is an open set for all $n \in \mathbb{N}$. To show that G is open we will use an induction argument.

Base Case: Letting $n = 1$ we have that $G = G_1$ and G_1 is an open set by assumption so that G is open in this case.

Induction Step: Suppose $n \geq 1$ and $\bigcap_{i=1}^{n}(G_i)$ is an open set. Then $\bigcap_{i=1}^{n+1}(G_i) = (G_{n+1}) \cap \left(\bigcap_{i=1}^{n} G_i\right)$. Both G_{n+1} and $\bigcap_{i=1}^{n}(G_i)$ are open by assumption. In the book it is proven that the intersection of two open sets are open and so we can conclude that $\bigcap_{i=1}^{n+1}(G_i)$ is open. □

5. Show that the set $\mathbb{N}$ of natural numbers is a closed set in $\mathbb{R}$.

Claim. *The set $\mathbb{N}$ of natural numbers is a closed set in $\mathbb{R}$.*

Proof. To show that $\mathbb{N}$ is a closed set in $\mathbb{R}$ we will show that $\mathbb{R} \setminus \mathbb{N}$ is open. Let $G_n := (n, n+1)$ for all $n \in \mathbb{N}$. Since G_n is an open interval for all $n \in \mathbb{N}$, then G_n is an open set for all $n \in \mathbb{N}$. If we show that $\mathbb{R} \setminus \mathbb{N} = \bigcup_{n \in \mathbb{N}} G_n$ and that $\bigcup_{n \in \mathbb{N}} G_n$ is open, then we can conclude the complement of $\mathbb{N}$ in $\mathbb{R}$ is open, which means $\mathbb{N}$ is closed. Now

$$
\begin{aligned}
x \in \mathbb{R} \setminus \mathbb{N} &\iff \exists k \in \{1, 2, 3, \ldots\} \text{ such that } x \in (k, k+1) \\
&\iff x \in G_k \text{ for some } k \in \mathbb{N} \\
&\iff x \in \bigcup_{k \in \mathbb{N}} G_k.
\end{aligned}
$$

□

Therefore $\mathbb{R} \setminus \mathbb{N} = \bigcup_{n \in \mathbb{N}} G_n$. Since the union of an arbitrary collection of open sets is open then $\bigcup_{n \in \mathbb{N}} G_n$ is open and therefore $\mathbb{R} \setminus \mathbb{N}$ is open. Thus the complement of $\mathbb{N}$ in $\mathbb{R}$ is open, so $\mathbb{N}$ is closed.