

STAT 430 - Notes

Matrix Analysis

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1 Matrix

An $m \times n$ matrix A can be written as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

2 Notation

- a_{ij} is the (i, j) component of A , so a_{ij} is in the i^{th} row and j^{th} column.
- each horizontal line of A is a **row** of A
- each vertical line of A is a **column** of A
- $m \times n$ is the **size** of A , i.e., A has m rows and n columns.

A shorthand notation for A is $A = [a_{ij}]$.

3 Other formulations of a matrix

1. **Column form** We can write A as $A = [A_{\bullet 1} \quad A_{\bullet 2} \quad \cdots \quad A_{\bullet n}]$ where $A_{\bullet k}$ represents the k^{th} column of A , i.e.,

$$A_{\bullet k} = \begin{bmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{mk} \end{bmatrix}$$

Remark. If A is an $m \times 1$ matrix then A is called an m -column vector or simply an m -vector.

2. **Row form** We can write A as

$$A = \begin{bmatrix} A_{1\bullet} \\ A_{2\bullet} \\ \vdots \\ A_{m\bullet} \end{bmatrix}$$

where $A_{\ell\bullet}$ represents the ℓ^{th} row of A , i.e., $A_{\ell\bullet} = [a_{\ell 1} \quad a_{\ell 2} \quad \cdots \quad a_{\ell n}]$.

4 Examples

1. Let A be an $m \times n$ matrix. If $m = n$, then A is a **square matrix**.
2. Given $A = [a_{ij}]$ such that $a_{ij} = 0, \forall i, j$. Then A is called the **zero matrix**, i.e., $A = 0$.

5 Equivalence of matrices

Given two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ we say that $A = B$ if

1. A and B are the same size,
2. $a_{ij} = b_{ij}$ for all i, j .

6 Basic Matrix Operations

1. **Addition:** Let A and B be $m \times n$ matrices. Then

$$A + B = [a_{ij} + b_{ij}].$$

2. **Scalar multiplication:** Let c be a scalar, i.e., c is either a real or complex number. Given $A = [a_{ij}]$ then

$$cA = [c \cdot a_{ij}].$$

6.0.1 Examples

Let $A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}_{3 \times 2}$ and $B = \begin{bmatrix} -1 & 1 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}_{3 \times 2}$.
Find $2A + B$.

Solution: First we find $2A$ as

$$2A = 2 \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 0 & 2 \\ 2 & 0 \end{bmatrix}.$$

Then we find $2A + B$ as

$$2A + B = \begin{bmatrix} 4 & 2 \\ 0 & 2 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} -1 & 1 \\ 0 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 0 & 2 \\ 2 & -1 \end{bmatrix}.$$

Definition. Let A be an $m \times n$ matrix. Then the **additive inverse** of A is defined as

$$-A = (-1)A = [-a_{ij}].$$

Definition. Let A and B be $m \times n$ matrices. Then the **difference** of A and B is defined as

$$\begin{aligned} A - B &= A + (-B) \\ &= [a_{ij}] + [-b_{ij}] \\ &= [a_{ij} - b_{ij}]. \end{aligned}$$

3. **Transpose** Let A be an $m \times n$ matrix and $A = [a_{ij}]$. Let A^T be the transpose of A where

$$[A^T]_{ij} = [A]_{ji} = a_{ji}.$$

Then A^T is an $m \times n$ matrix, where the k^{th} row is the k^{th} column of A .

Remark. For a square matrix, the transpose operation does not change the size of the matrix.

4. Conjugate and conjugate transpose

Let $z = a + bi$ be a complex number where $a, b \in \mathbb{R}$, and $i = \sqrt{-1}$ is the imaginary unit. We call a the real part of z , i.e., $a = \text{Re}(z)$ and b is the imaginary part of z , i.e., $b = \text{Im}(z)$.

The **conjugate** of z is

$$\bar{z} = \overline{a + bi} = a - bi.$$

Particularly, if z is a real number, i.e., $z = a$, then $\bar{z} = a$.

Suppose $A = [a_{ij}]$ is a complex matrix, i.e., a_{ij} is a complex number. Therefore,

$$\bar{A} = [\bar{a}_{ij}].$$

The **conjugate transpose** of A is

$$A^* = (\bar{A})^T.$$

Example. Let $A = \begin{bmatrix} 1 & 1+i \\ i & 3 \\ 0 & 2 \end{bmatrix}$.

Then to calculate the conjugate transpose we first calculate the conjugate of A as $\bar{A} = \begin{bmatrix} 1 & 1-i \\ -i & 3 \\ 0 & 2 \end{bmatrix}$. Then we calculate the conjugate transpose as

$$(\bar{A})^T = \begin{bmatrix} 1 & -i & 0 \\ 1-i & 3 & 2 \end{bmatrix} = A^*.$$

Remark. The conjugate and transpose operations are commutative. Thus $A^* = \overline{(A^T)}$.

7 Properties of matrix operations

I) Addition

- (a) Commutative Property $A + B = B + A$
- (b) Associative Property $(A + B) + C = A + (B + C)$.
- (c) $A + 0 = A$
- (d) Let $-A$ be the additive inverse of A . Then $A + (-A) = 0$.

II) Scalar multiplication

Let α and β be two scalars.

- (a) $(\alpha \cdot \beta)A = \alpha(\beta A) = \beta(\alpha A)$
- (b) $(\alpha + \beta)A = \alpha A + \beta A$.

To see this we have

$$\begin{aligned} (\alpha + \beta)A &= [(\alpha + \beta)a_{ij}] \\ &= [\alpha a_{ij} + \beta a_{ij}] \\ &= [\alpha a_{ij}] + [\beta a_{ij}] \\ &= \alpha A + \beta A. \end{aligned}$$

- (c) $1 \cdot A = A$

III) Transpose and conjugate transpose

- (a) $(A + B)^T = A^T + B^T$

- (b) $(cA)^T = c \cdot A^T$
- (c) $(A^T)^T = A$
- (d) $(A + B)^* = A^* + B^*$
- (e) $(c \cdot A)^* = \bar{c} \cdot A^*$ where \bar{c} is the conjugate of complex number c
- (f) $(A^*)^* = A$

Remark. Below we use the fact that given 2 complex numbers z and w , i.e. $z, w \in \mathbb{C}$, then

$$\overline{z \cdot w} = \bar{z} \cdot \bar{w}.$$

Lemma. Given a complex number c and a complex matrix A we have

$$(c \cdot A)^* = \bar{c} \cdot A^*.$$

Proof. Recall that $A^* = (\bar{A})^T$. Now using the remark above we can write

$$\begin{aligned} [\overline{c \cdot A}]_{ij} &= \overline{c \cdot a_{ij}} \\ &= \bar{c} \cdot \overline{a_{ij}} \\ &= \bar{c} \cdot [\bar{A}]_{ij}, \quad \forall i, j. \end{aligned}$$

Thus, $\overline{c \cdot A} = \bar{c} \cdot \bar{A}$. Therefore

$$\begin{aligned} (c \cdot A)^* &= (\overline{c \cdot A})^T \\ &= (\bar{c} \cdot \bar{A})^T \\ &= \bar{c} \cdot (\bar{A})^T \\ &= \bar{c} \cdot A^*. \end{aligned}$$

□

Remark. Let $A : m \times n$ matrix. If A and A^T have the same size, then $m = n$, i.e., A is a square matrix.

Definition. Let A be a square matrix, i.e., $A : n \times n = [a_{ij}]$.

1. If $A = A^T$, then A is **symmetric**, i.e., $a_{ij} = a_{ji}$, $\forall i, j$.
2. If $A = -A^T$, then A is **skew symmetric**, i.e., $a_{ij} = -a_{ji}$, $\forall i, j$.
3. If $A = A^*$, then A is **Hermitian**, i.e., $a_{ij} = \overline{a_{ji}}$, $\forall i, j$.

4. If $A = -A^*$, then A is **skew Hermitian**, i.e., $a_{ij} = -\overline{a_{ji}}$, $\forall i, j$.

Using the above definitions and properties we can prove the following statements:

1. Let A be a skew symmetric matrix. Then $a_{ii} = 0$ for all i and j .

Proof. Since A is skew symmetric and so $a_{ij} = -a_{ji}$ for all i and j . Letting $j = i$, we have $a_{ii} = -a_{ii}$ and after adding a_{ii} to both sides we have $2a_{ii} = 0$. Since $2 \neq 0$, then we must have $a_{ii} = 0$. \square

2. If A is skew Hermitian the a_{ii} are pure imaginary numbers.

Proof. Need to complete. \square

3. Let A be a square matrix. Then $A + A^T$ is symmetric.

Proof. Let $B = A + A^T$. We want to show $B = B^T$. So

$$\begin{aligned} B^T &= (A + A^T)^T \\ &= A^T + (A^T)^T \\ &= A^T + A \\ &= A + A^T \\ &= B. \end{aligned}$$

We have shown $B^T = B$ and therefore B is symmetric. \square

8 Diagonal entries and diagonal matrices

Definition. Let A be an $n \times n$ matrix and we can write A as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}_{n \times n}.$$

The entries a_{ii} are called the **diagonal** entries of A and the other entries are **non-diagonal** entries.

A **diagonal** matrix A is a matrix whose non-diagonal entries are all zero.

An **upper triangular** matrix is a square matrix whose $a_{ij} = 0$, where $i > j$.

A **lower triangular** matrix is a square matrix whose $a_{ij} = 0$, where $i < j$.

The following statements follow from the definition of matrix operations and the above definitions:

1. If A is upper triangular then A^T is lower triangular.
2. If A is lower triangular, then A^T is upper triangular.
3. A is lower and upper triangular if and only if A is diagonal.

Example. Let the $n \times n$ matrix I be defined as

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{n \times n}.$$

Then I is called the **Identity matrix**.

9 Matrix-vector product

Definition. Let A be an $m \times n$ matrix, i.e., $A = [A_{\bullet 1} \ A_{\bullet 2} \ \dots A_{\bullet n}]$, which is the column form of A . Let $b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ be an n -(column) vector. Then the **matrix-vector product** Ab is defined as

$$A \cdot b = (b_1 \cdot A_{\bullet 1}) + (b_2 \cdot A_{\bullet 2}) + \dots + (b_n \cdot A_{\bullet n}).$$

Note that by the definition of scalar multiplication, $b_i \cdot A_{\bullet k}$ and for $k = 1, 2, \dots, n$ and $i = 1, 2, \dots, m$ is an m -vector (with m the number of rows in A). Then by the definition of vector addition, $A \cdot b$ is also an m -vector or an $m \times 1$ matrix.

9.0.1 Properties of matrix-vector products

Let $A : m \times n$ matrix and let u, v : n -vector and let c be a scalar. Then

1. $A(u + v) = Au + Av$

Proof. Need to fill in. □

2. $A(c \cdot u) = c \cdot (A \cdot u)$

Proof. Need to fill in. □

Remark. Recall that the identity matrix I can be written as

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{n \times n}.$$

We can also write I as $I = [e_1 \ e_2 \ \cdots \ e_n]$ where e_k is the k^{th} column of I .

Example. Let $A : n \times n$ matrix. Then

$$A \cdot e_1 = A_{\bullet 1} \cdot 1 + A_{\bullet 2} \cdot 0 + \cdots + A_{\bullet n} \cdot 0 = A_{\bullet 1}$$

where $e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$. Likewise, $A \cdot e_2 = A_{\bullet 2}$, \dots , $A \cdot e_n = A_{\bullet n}$.

10 Matrix multiplication

Definition. Two matrices $A_{m \times n}$ and $B_{r \times p}$ are **conformable** if $n = r$, i.e., $B_{n \times p}$.

It can be shown that if A and B are two square matrices of the same size then they are conformable.

10.0.1 Matrix product

Given $A_{m \times n}$ and $B_{n \times p}$ where $B = [B_{\bullet 1} \ B_{\bullet 2} \ \cdots \ B_{\bullet p}]$ and each $B_{\bullet k}$ is an n -vector, then

$$A \cdot B = [A \cdot B_{\bullet 1} \ A \cdot B_{\bullet 2} \ \cdots \ A \cdot B_{\bullet p}].$$

By the definition of a matrix-vector product, $A \cdot B_{\bullet k}$ for $k = 1, 2, \dots, p$ is an m -vector, or an $m \times 1$ matrix. Therefore, $A \cdot B$ has m rows and p columns and so $A \cdot B_{m \times p}$ matrix.

Another definition of matrix product is:

$$[A \cdot B]_{ij} = A_{i \bullet} \cdot B_{\bullet j} = \sum_{k=1}^n a_{ik} \cdot b_{kj}.$$

10.0.2 Caution and examples

1. Even if $A \cdot B$ is well defined, $B \cdot A$ may not. For example, given $A : m \times n$ and $B : n \times p$, the product $A \cdot B$ is well defined, but if $m \neq p$ then $B \cdot A$ is not defined.
2. Let $A : m \times n$ and $B : n \times m$. In this case $A \cdot B$ and $B \cdot A$ are well defined. But, $A \cdot B : m \times m$ and $B \cdot A : n \times n$. If $m \neq n$, then $AB \neq BA$.

3. Let $A : n \times n$ and $B : n \times n$. In this case, $A \cdot B$ and $B \cdot A$ are both well defined and of the same size. However, $AB \neq BA$ in general. For example, let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}_{2 \times 2}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}_{2 \times 2}$. Then $AB = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and $BA = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. Therefore, $A \cdot B \neq B \cdot A$.

Definition. Let $A, B : n \times n$ matrices. If $A \cdot B = B \cdot A$, then we say that A and B **commute**, otherwise they do not commute.

4. (No cancellation law in general)

Consider $A = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix}$. Both A and B are non-zero, but $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Let C be the 2×2 zero matrix. Then $A \cdot C = 0$ and therefore $A \cdot B = A \cdot C$ but $B \neq C$.

11 Product of matrix and identity matrix

Recall that the $n \times n$ identity matrix

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{n \times n} = [e_1 \quad e_2 \quad \dots \quad e_n]$$

where e_i is the i th column of I_n .

Given an $m \times n$ matrix A ,

$$A \cdot I_n = A \cdot [e_1 \quad e_2 \quad \dots \quad e_n] = [Ae_1 \quad Ae_2 \quad \dots \quad Ae_n].$$

Since $A \cdot e_i = A_{\bullet i}$, the i th column of A , then

$$A \cdot I_n = [A_{\bullet 1} \quad A_{\bullet 2} \quad \dots \quad A_{\bullet n}] = A.$$

Similarly, let I_n be the $m \times n$ identity matrix. Then $I_n \cdot A = A$.

Example. Suppose $A : m \times n$ matrix, such that $A \cdot A$ is well defined.

Question: What can be said about m and n ?

Answer: Since $A \cdot A$ is well defined then A and A are conformable, then the number of columns in A must equal the number of rows in A . Thus, $m = n$.

Remark. For the next few sections let A be a $n \times n$ square matrix.

12 k^{th} power of a matrix

Since A is a square matrix then $A \cdot A$ is well defined and $A \cdot A = A^2$. Similarly,

$$\begin{aligned} A^3 &= A \cdot A \cdot A \\ &\vdots \\ A^k &= A \cdot A \cdots A \quad k \in \mathbb{N}. \end{aligned}$$

By convention, $A^0 = I_n$.

13 Properties of matrix products

1. $A \cdot (B + C) = A \cdot B + A \cdot C$
2. $(D + E) \cdot F = D \cdot F + E \cdot F$
3. $(A \cdot B) \cdot C = A \cdot (B \cdot C)$
4. Let α be a scalar, then $A(\alpha \cdot B) = \alpha \cdot (A \cdot B) = (\alpha \cdot A) \cdot B$
5. $(A \cdot B)^T = B^T \cdot A^T$, $(A \cdot B)^* = B^* \cdot A^*$

Proof. (Proof of 5) Consider the pure transpose case only. Let $A : m \times n$ and $B : n \times p$. First we show that $(A \cdot B)^T$ and $B^T \cdot A^T$ are the same size. Since $A \cdot B$ is $m \times p$ then by the definition of transpose, $(A \cdot B)^T$ is $p \times m$. Now, $B^T : p \times n$ and $A^T : n \times m$ so the number of columns in B^T is n and the number of columns in A^T is n , thus $B^T \cdot A^T$ is well defined and is a $p \times m$ matrix.

Next, we want to show that the corresponding entries of $(A \cdot B)^T$ and $B^T \cdot A^T$ are equal. Consider $i = 1, \dots, p$ and $j = 1, \dots, m$. Then,

$$\begin{aligned} [(B^T \cdot A^T)]_{ij} &= (B^T)_{i\bullet} \cdot (A^T)_{\bullet j} \\ &= \sum_{k=1}^n [B^T]_{ik} \cdot [A^T]_{kj} \\ &= \sum_{k=1}^n [B]_{ki} \cdot [A]_{jk} \\ &= \sum_{k=1}^n [A]_{jk} \cdot [B]_{ki}. \end{aligned}$$

Also,

$$\begin{aligned}
[(A \cdot B)^T]_{ij} &= [A \cdot B]_{ji} \\
&= A_{j\bullet} \cdot B_{\bullet i} \\
&= \sum_{k=1}^n [A]_{jk} \cdot [B]_{ki}.
\end{aligned}$$

Therefore, $[(AB)^T]_{ij} = [(B^T \cdot A^T)]_{ij}$. Thus $(A \cdot B)^T = B^T \cdot A^T$. □

Example. Let $A : m \times n$ matrix. Show that $A \cdot A^T$ and $A^T \cdot A$ are well defined and symmetric.

Proof. Since $A : m \times n$ then $A^T : n \times m$. Therefore $A \cdot A^T$ is well defined and is $m \times m$. Also, A^T has m columns and A has m rows and so $A^T \cdot A$ is well defined and is an $n \times n$ matrix.

Let $B = A \cdot A^T$, then

$$\begin{aligned}
B^T &= (A \cdot A^T)^T \\
&= (A^T)^T \cdot A^T \\
&= A \cdot A^T \\
&= B.
\end{aligned}$$

Thus, $B^T = B$, so B is symmetric. Likewise we can show that $A^T \cdot A$ is symmetric. □

Definition. Let A be an $n \times n$ matrix. The trace of A is the summation of the diagonal entries, i.e.,

$$trace(A) = \sum_{i=1}^n a_{ii},$$

where $A = [a_{ij}]$.

Example. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}_{2 \times 2}$. Then $trace(A) = 1 + (-4) = -3$.

Example. $trace(I_n) = n$

Lemma. Let $A : m \times n$ and $B : n \times m$ such that AB and BA are well defined. Note that AB and BA must be square matrices. Then $trace(A \cdot B) = trace(B \cdot A)$.

Proof.

$$\begin{aligned}
\text{trace}(A \cdot B) &= \sum_{i=1}^m [A \cdot B]_{ii} \\
&= \sum_{i=1}^m (A_{i\bullet} B_{\bullet i}) \\
&= \sum_{i=1}^m \left(\sum_{k=1}^n [A]_{ik} [B]_{ki} \right) \\
&= \sum_{k=1}^n \left(\sum_{i=1}^m [A]_{ik} [B]_{ki} \right) \\
&= \sum_{k=1}^n \left(\sum_{i=1}^m [B]_{ki} [A]_{ik} \right) \quad (\text{in parentheses is } B_{k\bullet} A_{\bullet k}) \\
&= \sum_{k=1}^n (B_{k\bullet} A_{\bullet k}) \\
&= \sum_{k=1}^n [B \cdot A]_{kk} \\
&= \text{trace}(B \cdot A).
\end{aligned}$$

□

Word of caution

- $\text{trace}(A \cdot B) \neq \text{trace}(A) \cdot \text{trace}(B)$. Counterexample: Let $A = B = I_n$. Then $\text{trace}(A) = \text{trace}(B) = n$, but $A \cdot B = I_n$ and so $\text{trace}(AB) = n$. Therefore, $\text{trace}(A \cdot B) = n$ and $\text{trace}(A) \cdot \text{trace}(B) = n^2$.

Cycling property of trace Let $A, B, C : n \times n$ matrices. It can be shown that $\text{trace}(A \cdot B \cdot C) = \text{trace}(C \cdot A \cdot B)$.

14 Partition of a matrix

Let $A : m \times n$ matrix.

$$A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array} \right]$$

where A_{ij} is the (i, j) -block of A .

Example.

$$A = \left[\begin{array}{cc|c} 0 & 1 & 1 \\ 1 & 0 & 1 \\ \hline -2 & 3 & 0 \end{array} \right]$$

In the above matrix, $A_{11} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $A_{12} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $A_{21} = \begin{bmatrix} -2 & 3 \end{bmatrix}$, and $A_{22} = \begin{bmatrix} 0 \end{bmatrix}$.

15 Block matrix multiplication

Example. Let $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ and $B = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \end{bmatrix}$. Assume that (A_{ik}, B_{kj}) is conformable and so $A_{ik} \cdot B_{kj}$ is well defined and so $A \cdot B$ is well defined. Then

$$A \cdot B = \begin{bmatrix} (A_{11}B_{11} + A_{12}B_{21}) & (A_{11}B_{12} + A_{12}B_{22}) & (A_{11}B_{13} + A_{12}B_{23}) \\ (A_{21}B_{11} + A_{22}B_{21}) & (A_{21}B_{12} + A_{22}B_{22}) & (A_{21}B_{13} + A_{22}B_{23}) \end{bmatrix}.$$

Example. Let $A = \begin{bmatrix} C & I \\ I & 0 \end{bmatrix}$ and $B = \begin{bmatrix} I & 0 \\ D & E \end{bmatrix}$. Then

$$AB = \begin{bmatrix} (CI + ID) & (C0 + IE) \\ (II + 0D) & (I0 + 0E) \end{bmatrix} = \begin{bmatrix} C + D & E \\ I & 0 \end{bmatrix}.$$

Example. The product of upper block triangular matrices is also block upper triangular. Let $A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$ and $B = \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{bmatrix}$. Then

$$AB = \begin{bmatrix} A_{11}B_{11} & (A_{11}B_{12} + A_{12}B_{22}) \\ 0 & A_{22}B_{22} \end{bmatrix}.$$

16 Linear functions

Let V and W be two sets (i.e., vector spaces) with the same field, like \mathbb{R} or \mathbb{C} , and two basic operations on V and W :

1. vector addition:

$$\begin{aligned} u + w &\in V & \forall u, v &\in V \\ x + y &\in W & \forall x, y &\in W \end{aligned}$$

2. scalar multiplication:

$$\begin{aligned} \alpha \cdot u &\in V & \forall u &\in V, \alpha \in \mathbb{R} \text{ or } \mathbb{C} \\ \alpha \cdot x &\in W & \forall x &\in W, \alpha \in \mathbb{R} \text{ or } \mathbb{C} \end{aligned}$$

Example. 1. $V = \mathbb{R}^n$: n-dimensional Euclidean space and $W = \mathbb{R}^m$ is m-dimensional Euclidean space.

2. $V = \mathbb{C}^n$: n-dimensional complex space and $W = \mathbb{C}^m$ is m-dimensional complex space.

3. $V = \mathbb{R}^{m \times n}$: the space of all $m \times n$ real matrices.

Definition. A function $f : V \rightarrow W$ is **linear** if:

1. $f(u + v) = f(u) + f(v)$, $\forall u, v \in V$. This is sometimes called the superposition property.
2. $f(\alpha \cdot u) = \alpha \cdot f(u)$, $\forall u \in V$ and scalar $\alpha \in \mathbb{R}$ or \mathbb{C} . This is sometimes called the scalar property.

Example. Given an $m \times n$ real matrix A , define the function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ as

$$f(x) = A \cdot x \text{ for } x \in \mathbb{R}^n \text{ where } x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n.$$

Claim. $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is linear.

Proof. By question 3 from homework 1, $f(u+v) = A(u+v) = Au + Av = f(u) + f(v)$ for all $u, v \in \mathbb{R}^n$. Also, for scalar α and $u \in \mathbb{R}^n$, $f(\alpha \cdot u) = A(\alpha \cdot u) = \alpha(A \cdot u) = \alpha f(u)$. \square

Example. Consider the trace function $trace : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ where $\mathbb{R}^{n \times n}$ is the space of all $n \times n$ matrices.

Show: $trace(\cdot)$ is linear.

Proof. First we show $trace(\cdot)$ satisfies the superposition property. Given $A = [a_{ij}]$ and $B = [b_{ij}]$ in $\mathbb{R}^{n \times n}$ then

$$\begin{aligned} trace(A + B) &= \sum_{i=1}^n [A + B]_{ii} \\ &= \sum_{i=1}^n [A]_{ii} + [B]_{ii} \\ &= \sum_{i=1}^n [A]_{ii} + \sum_{i=1}^n [B]_{ii} \\ &= trace(A) + trace(B). \end{aligned}$$

Now we show $trace(\cdot)$ satisfies the scalar property. Let $\alpha \in \mathbb{R}$ and $A = [a_{ij}] \in \mathbb{R}^{n \times n}$. Then

$$\begin{aligned} trace(\alpha A) &= \sum_{i=1}^n [\alpha \cdot A]_{ii} \\ &= \sum_{i=1}^n \alpha [A]_{ii} \\ &= \alpha \sum_{i=1}^n [A]_{ii} \\ &= \alpha \cdot trace(A). \end{aligned}$$

Therefore, $\text{trace}(\cdot)$ is linear. □

Proposition. A function $f : V \rightarrow W$ is linear if and only if

$$f(\alpha \cdot u + v) = \alpha \cdot f(u) + f(v), \quad \forall u, v \in V, \alpha \in \mathbb{R} \text{ or } \mathbb{C}.$$

Proof. (\Rightarrow) Suppose f is linear. Then for any $u, v \in V$ and scalar α ,

$$f(\alpha \cdot u + v) = f(\alpha \cdot u) + f(v) = \alpha \cdot f(u) + f(v)$$

where the first equality follows from the superposition property and the second from the scalar property of linear functions.

(\Leftarrow) For MATH 603 □

17 Matrix Inverse

Definition. A $n \times n$ matrix A is invertible (i.e., non-singular) if (and only if) there exists an $n \times n$ matrix B such that

$$A \cdot B = I_n \text{ and } B \cdot A = I_n.$$

Here B is called an inverse of A , denoted A^{-1} .

Claim. If A is invertible then its inverse is unique.

Proof. Since A is invertible, there exists a matrix B such that $A \cdot B = B \cdot A = I_n$. Now suppose there exists a matrix B' such that $A \cdot B' = B' \cdot A = I_n$. Then since $A \cdot B'$,

$$\begin{aligned} B &= B \cdot I_n = B \cdot (A \cdot B') \\ &= (B \cdot A) \cdot B' \\ &= I_n B' \\ &= B'. \end{aligned}$$

Thus $B = B'$. Therefore, the inverse of A is unique. □

Proposition. Let A be an $n \times n$ matrix. The following are equivalent:

- (0) A is invertible.
- (1) A is row equivalent to I_n .
- (2) The equation $Ax = 0$ has the solution $x = 0$ only.
- (3) The equation $Ax = b$ has a solution for any n -vector b .
- (4) $\text{rank}(A) = n$.
- (5) There exists an $n \times n$ matrix B such that $B \cdot A = I_n$.

(6) There exists an $n \times n$ matrix C such that $A \cdot C = I_n$.

Proof. In the following we assume as facts (2) and (3) to prove $5) \Rightarrow 0)$ and $6) \Rightarrow 0)$.

$5) \Rightarrow 0)$. By 5) there exists a matrix such that $B \cdot A = I_n$. It is sufficient to show that the equation $Ax = 0$ has the solution $x = 0$ only to prove that A is invertible.

Let u be a solution to $Ax = 0$, i.e., $Au = 0$. If we right multiply the latter equation by B we have $B(Au) = B \cdot 0 = 0$. Now consider

$$B(A \cdot u) = (B \cdot A)u = I_n \cdot u = u.$$

Thus $u = 0$. Therefore, 2) holds and since 2) implies 0) then 5) implies 0).

$6) \Rightarrow 0)$ By 6) there exists a matrix C such that $AC = I_n$.

Claim: For any n -vector b , $Ax = b$ has a solution.

Since there exists a matrix C such that $A \cdot C = I_n$ then given any n -vector b , $(A \cdot C) \cdot b = A \cdot (C \cdot b) = I_n \cdot b = b$. Thus $A \cdot (C \cdot b) = b$ implies that $x = C \cdot b$ is a solution to the equation $Ax = b$. Thus, 3) holds $\Rightarrow 0)$ holds.

□

18 Properties of an invertible matrix

Suppose A is invertible.

(1) A^{-1} is invertible, and $(A^{-1})^{-1}$.

Proof. $A \cdot A^{-1} = A^{-1} \cdot A = I_n$. Fill in details.

□

(2) A^T and A^* are invertible, and $(A^T)^{-1} = (A^{-1})^T$ and $(A^*)^{-1} = (A^{-1})^*$

Proof. Since A is invertible then

$$\begin{aligned} A \cdot A^{-1} = A^{-1} \cdot A = I_n &\Rightarrow (A \cdot A^{-1})^T = (A^{-1} \cdot A)^T = (I_n)^T = I_n \\ &\Rightarrow (A^{-1})^T \cdot A^T = A^T \cdot (A^{-1})^T = I_n \end{aligned}$$

Therefore A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.

□

(3) Let B be another invertible matrix. Then $A \cdot B$ is invertible, and its inverse $(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}$.

Proof.

$$\begin{aligned} A \cdot B \cdot (B^{-1} \cdot A^{-1}) &= A \cdot (B \cdot B^{-1}) \cdot A^{-1} \\ &= A \cdot A^{-1} \\ &= I_n. \end{aligned}$$

Therefore $(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}$. □

Corollary. This is an extension of 3 above. Let A_1, \dots, A_k be invertible matrices. Then $A_1 \cdot A_2 \cdots A_k$ is invertible, and $(A_1 \cdot A_2 \cdots A_k)^{-1} = A_k^{-1} \cdots A_2^{-1} \cdot A_1^{-1}$. Can use proof by induction on k .

19 Examples

Example. Let A be a square matrix such that $I - A$ is invertible. Show:

$$A \cdot (I - A)^{-1} = (I - A)^{-1} \cdot A \quad (\text{commutative}) .$$

Proof. Compute

$$\begin{aligned} A \cdot (I - A) &= A \cdot I - AA = A - A^2 \\ &= I \cdot A - A \cdot A \\ &= (I - A) \cdot A. \end{aligned}$$

Then

$$\begin{aligned} A \cdot (I - A)(I - A)^{-1} &= (I - A) \cdot A \cdot (I - A)^{-1} \Rightarrow A = (I - A) \cdot A \cdot (I - A)^{-1} \\ &\Rightarrow (I - A)^{-1} \cdot A = A \cdot (I - A)^{-1}. \end{aligned}$$

□

Example. We will use the following preliminary results in this example.

- (1) Given $x, y \in \mathbb{R}^n$ (i.e., $x, y \in \mathbb{R}^{n \times 1}$ are n -column vectors). The following is a scalar:

$$x^T \cdot y = (x^T \cdot y)^T = y^T \cdot x \quad (\text{standard inner product}) .$$

- (2) Given $x \in \mathbb{R}^n$ such that $x^T x = 0$, then $x = 0$.

Proof. Write this as $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$.

Then

$$\begin{aligned}
x^T \cdot x &= \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \\
&= x_1^2 + x_2^2 + \cdots + x_n^2 \\
&= 0.
\end{aligned}$$

Since $x^T x = 0$, then $x_1^2 + x_2^2 + \cdots + x_n^2 = 0$.

If $x_i \neq 0$ for some i then $x^T x > 0$. This shows that each $x_i = 0$ or $x = 0$. \square

(3) Show $x^T S x = 0, \forall x \in \mathbb{R}^n$.

Proof. For any $x \in \mathbb{R}^n$

$$\begin{aligned}
x^T S x &= x^T \cdot (Sx) \\
&= (S \cdot x)^T \cdot x \quad (\text{by preliminary result 1}) \\
&= (x^T \cdot S^T) \cdot x \\
&= (x^T \cdot -S) \cdot x \quad (S \text{ is skew symmetric}) \\
&= x^T \cdot (-1) \cdot S \cdot x \\
&= (-1) \cdot x^T S \cdot x.
\end{aligned}$$

Therefore,

$$\begin{aligned}
x^T S x &= -x^T S x \Rightarrow 2 \cdot x^T S x = 0 \\
&\Rightarrow x^T S x = 0.
\end{aligned}$$

\square

Let S be a real skew symmetric matrix, i.e., $S = -S^T$ or $S^T = -S$.

Show that $\forall \alpha \in \mathbb{R}, I + \alpha \cdot S$ is invertible.

Proof. Let $\alpha \in \mathbb{R}$ be arbitrary. To show $I + \alpha \cdot S$ is invertible it suffices to show that

$$(I + \alpha S) \cdot x = 0$$

has the solution $x = 0$ only.

Suppose \hat{x} is an arbitrary solution to $(I + \alpha S) \cdot x = 0$. Therefore,

$$\begin{aligned}
(I + \alpha S) \cdot \hat{x} &= 0 \Rightarrow \hat{x}^T (I + \alpha S) \hat{x} = \hat{x}^T \cdot 0 \\
&\Rightarrow \hat{x}^T \hat{x} + \alpha \hat{x}^T S \hat{x} = 0 \\
&\Rightarrow \hat{x}^T \hat{x} = 0 \\
&\Rightarrow \text{By 2), } \hat{x} = 0.
\end{aligned}$$

Therefore $(I + \alpha S) \cdot x = 0$ has the solution $x = 0$ only. Thus $I + \alpha \cdot S$ is invertible. \square

20 Vector Space

Definition. A **vector space** is a set V , along with a field F (which is \mathbb{R} or \mathbb{C}), and two basic operations

- (1) vector addition “+” and
- (2) scalar multiplication “ \cdot ”

which satisfy:

A1): V is closed under vector addition, i.e., $x, y \in V$ then $x + y \in V$.

A2): For any $x, y \in V$, $(x + y) + z = x + (y + z)$ (associative)

A3): For any $x, y \in V$, $x + y = y + x$ (commutative)

A4): There exists a zero vector in V , denoted by 0 , such that $x + 0 = x$ and $0 + x = x$, $\forall x \in V$.

A5): For any $x \in V$, there exists a vector in V , denoted by $-x$ such that $x + (-x) = 0$. Here $-x$ is additive inverse of x .

M1): V is closed under scalar multiplication, i.e., for any $x \in V$ and scalar α , $\alpha \cdot x \in V$.

M2): For any scalar α, β and $x \in V$,

$$(\alpha \cdot \beta) \cdot x = \alpha(\beta \cdot x).$$

M3): For any scalar α , and $x, y \in V$,

$$\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y.$$

M4): For any scalars α, β and $x \in V$

$$(\alpha + \beta)x = \alpha \cdot x + \beta \cdot x.$$

M5): When $\alpha = 1$, $\alpha \cdot x = x$, $\forall x \in V$.

Any $x \in V$ is called a **vector** in V .

Example. Vector Spaces

- (1) \mathbb{R}^n : the n -dimensional Euclidean space. Likewise, \mathbb{C}^n is also a vector space.
- (2) $\mathbb{R}^{m \times n}$: the set of all $m \times n$ real matrices. Likewise, $\mathbb{C}^{m \times n}$ is also a vector space.
- (2') The set of all $n \times n$ upper triangular real matrices is a vector space, which is a subspace of $\mathbb{R}^{n \times n}$.

- (3) The set of all real sequences

$$\begin{aligned}X + Y &= (x_1 + y_1, x_2 + y_2, \dots), \\ \alpha \cdot X &= (\alpha x_1, \alpha x_2, \dots).\end{aligned}$$

This set is a vector space.

- (3') The set of **convergent** real sequences is a vector space and is a subspace of the above vector space.

- (3'') The set of **bounded** real sequences is also a vector space.

- (4) The set of all real-valued continuous functions on \mathbb{R} (or $[a, b] \subseteq \mathbb{R}$).

- (5) The set of all **divergent** real sequences is **NOT** a vector space.

This is because:

- 1) It has no "zero vector"
- 2) The closure under scalar multiplication fails. If we multiply any element by scalar 0, then the sequence will be convergent.
- 3) The closure under vector addition fails. Add $-x$ to x and will be zero vector, which is convergent.

21 Vector Spaces: Important Facts

1. V has the unique zero vector 0 . (Related to A4)
2. For any $x \in V$, there exists the unique additive inverse $-x$, and $-x = (-1) \cdot x$.
3. For any $x \in V$, $0 \cdot x = 0 \in V$.
4. For any scalar α , $\alpha \cdot 0 = 0 \in V$.
5. A finite sum of vectors in V belongs to V . Can use induction on the number of vectors in sum to prove.

22 Subspace

Definition. Let $(V, \mathbb{F}, +, \cdot)$ be a vector space. If a subset S of V is a vector space (along with the same field \mathbb{F} and basic operations of $+$ and \cdot of V) then S is a **subspace** of V .

Example. Let $(V, \mathbb{F}, +, \cdot)$ be a vector space.

- V is a subspace of itself. So every vector space has at least one subspace.
- $S = \{0\}$, the singleton set of the zero vector in V is a subspace of V and is the zero or trivial subspace. Provide details of proof.

Remark. From the above example we can conclude that every vector space has at least 2 subspaces.

Proposition. Let S be a subset of a vector space $(V, \mathbb{F}, +, \cdot)$. Then S is a subspace of V if and only if

1. S is closed under “+”; (A1) and
2. S is closed under “ \cdot ” (M1).

Proof. (\Leftarrow) Suppose S is a subspace of V . Since S is a subspace of V , then S is a subspace and satisfies A1-A5 and M1-M5. Thus, A1 and M1 hold.

(\Rightarrow) Suppose the subset S satisfies A1 and M1.

We want to show: S satisfies A2-A5 and M2-M5.

Since $S \subseteq V$ and V is a vector space then A.2, A.3, and M.2-M.5 hold on V and therefore hold on S as well.

Then it suffices to show that S satisfies A.4 and A.5.

A.4 Pick an arbitrary $x \in S$. Then $0 \cdot x = 0$ for the scalar 0 and any vector x , and since S is closed under “ \cdot ”, then the zero vector is in S . Also, $0 + x = x + 0 = x$ for all $x \in S$.

A.5 For any $x \in S$, recall that its additive inverse $-x = (-1) \cdot x$ by Fact 2. Since S is closed under “ \cdot ”, then $(-1) \cdot x \in S \implies -x \in S$. Thus, A.5 holds.

Therefore, we have shown that S is a vector space and so S is a subspace of V . □

Example. Examples of sets that are subspaces.

(1) Let $V = \mathbb{R}^3$ and let $S = \left\{ \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} : x_1 \in \mathbb{R} \right\}$. Show that S is a subspace of \mathbb{R}^3 .

Proof. Let $x, y \in S$. Then

$$x = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ 0 \\ 0 \end{bmatrix}.$$

The sum of x and y is

$$x + y = \begin{bmatrix} x_1 + y_1 \\ 0 \\ 0 \end{bmatrix}.$$

Therefore, S is closed under “+”. Given any $\alpha \in \mathbb{R}$ and $x = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} \in S$,

$$\alpha \cdot x = \begin{bmatrix} \alpha \cdot x_1 \\ 0 \\ 0 \end{bmatrix} \in S.$$

Therefore, S is closed under scalar multiplication. Thus S is a subspace of V . \square

(2) Let $V = \mathbb{R}^{n \times n}$.

(i) Let S_1 be the set of all $n \times n$ diagonal matrices in $\mathbb{R}^{n \times n}$. Since S is closed under standard matrix addition and scalar multiplication then S is a subspace of V .

(ii) Let S_2 be the set of all symmetric matrices in $\mathbb{R}^{n \times n}$.

Let $A, B \in S_2$, and therefore $A = A^T$ and $B = B^T$. Now consider the transpose of $A + B$,

$$(A + B)^T = A^T + B^T = A + B,$$

which shows that $(A + B)$ is symmetric and therefore $(A + B) \in S_2$.

Now let $\alpha \in \mathbb{R}$ and $A \in S_2$ and consider their product

$$(\alpha A)^T = \alpha A^T = \alpha A,$$

which shows that αA is symmetric and therefore in S_2 . Since S_2 is closed under vector addition and scalar multiplication then S_2 is a subspace of V .

Example. Examples of sets that are **not** subspaces.

(1) Let $V = \mathbb{R}^2$ and $S = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : x_1 \leq 0, x_2 \geq 0 \right\}$. Let

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in S.$$

Therefore, $x_1 \leq 0, y_1 \leq 0$ and $x_2 \geq 0, y_2 \geq 0$. The sum of x and y is

$$x + y = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}$$

where $x_1 + y_1 \leq 0$ and $x_2 + y_2 \geq 0$. Thus $x + y \in S$ and so S is closed under vector addition.

Given an arbitrary $\alpha \in \mathbb{R}$ and $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in S$. For example, $x = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \in S$ and

$\alpha = -1$. Then the product $\alpha \cdot x = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \notin S$ and therefore S is **not** closed under vector multiplication. Thus, S is not a subspace of V .

- (2) Let $V = \mathbb{R}^{2 \times 2}$ and S is the subset of all singular (i.e., not invertible) matrices in $\mathbb{R}^{2 \times 2}$. Let $A \in S$ and $\alpha \in \mathbb{R}$. If $\alpha = 0$, then $\alpha \cdot A = 0$ is singular. If $\alpha \neq 0$, then $\alpha \cdot A$ is singular because otherwise $\alpha A \cdot (1/\alpha \cdot B) = I$ and so $A \cdot B = I$. This contradicts our assumption that A is not invertible. Therefore, αA is singular. Thus, S is closed under vector multiplication.

Claim: S is not closed under “+”.

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Both A and B are because if a matrix is in reduced row echelon form and any diagonal element is 0 then the matrix is singular. Thus $A, B \in S$. However, $A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, which is the identity matrix and thus is invertible or nonsingular. Therefore, $A + B \notin S$ and so S is not closed under “+”.

23 Linear Combination

Definition. Let $\{v_1, v_2, \dots, v_p\}$ be a finite set of vectors in a vector space V . Given scalars $\alpha_1, \alpha_2, \dots, \alpha_p$. The **linear combination** of v_1, v_2, \dots, v_p using $\alpha_1, \alpha_2, \dots, \alpha_p$ is

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_p v_p$$

where the α_i for $i = 1, 2, \dots, p$ are called the weights or coefficients.

Remark. • By definition, $\alpha \cdot x \in V$ so $\alpha_1 v_1 \in V, \dots, \alpha_p v_p \in V$.

• Any finite sum of vectors from V is in V .

Definition. Given a finite set $S = \{v_1, \dots, v_p\}$ in V , the set spanned by v_1, \dots, v_p (or S) is

$$\begin{aligned} \text{span}(S) &= \{\alpha_1 v_1 + \dots + \alpha_p v_p : \text{scalars } \alpha_i, i = 1, \dots, p\} \\ &= \text{the set of all linear combinations of } v_1, \dots, v_p. \end{aligned}$$

Clearly, $\text{span}(S) \subseteq V$, where S is a spanning set of $\text{span}(S)$.

Proposition. The set $\text{span}(S)$ is a subspace of V .

Proof. First we show that $\text{span}(S)$ is closed under “+”. Let x, y be two vectors in $\text{span}(S)$. Therefore, there exists scalars $\alpha_1, \dots, \alpha_p$ such that

$$x = \alpha_1 v_1 + \dots + \alpha_p v_p.$$

Similarly, there exists scalars β_1, \dots, β_p such that

$$y = \beta_1 v_1 + \dots + \beta_p v_p.$$

Then the sum of x and y is

$$\begin{aligned} x + y &= (\alpha_1 v_1 + \cdots + \alpha_p v_p) + (\beta_1 v_1 + \cdots + \beta_p v_p) \\ &= (\alpha_1 v_1 + \beta_1 v_1) + (\alpha_2 v_2 + \beta_2 v_2) + \cdots + (\alpha_p v_p + \beta_p v_p) \\ &= (\alpha_1 + \beta_1) v_1 + (\alpha_2 + \beta_2) v_2 + \cdots + (\alpha_p + \beta_p) v_p. \end{aligned}$$

Therefore, $x + y$ is a linear combination of v_1, \dots, v_p . Thus $x + y \in \text{span}(S)$ and so $\text{span}(S)$ is closed under “+”.

Now we show that $\text{span}(S)$ is closed under “.”. Let $x \in \text{span}(S)$ and γ be a scalar. Therefore there exists scalars $\alpha_1, \dots, \alpha_p$ such that

$$x = \alpha_1 v_1 + \cdots + \alpha_p v_p.$$

Then the product of γ and x is

$$\begin{aligned} \gamma \cdot x &= \gamma \cdot (\alpha_1 v_1 + \cdots + \alpha_p v_p) \\ &= (\gamma \cdot \alpha_1) v_1 + \cdots + (\gamma \cdot \alpha_p) v_p. \end{aligned}$$

Therefore, $\gamma \cdot x$ is a linear combination of v_1, \dots, v_p and so $\gamma \cdot x \in \text{span}(S)$. Thus $\text{span}(S)$ is closed under “.”. Thus, $\text{span}(S)$ is a subspace. \square

Here $\text{span}(S)$ is a subspace spanned by v_1, \dots, v_p (or S).

Proposition. Let $S = \{v_1, \dots, v_p\}$. Then $S \subseteq \text{span}(S)$.

Proof. For each v_i ,

$$v_i = 0 \cdot v_1 + \cdots + 0 \cdot v_{i-1} + \underline{1 \cdot v_i} + 0 \cdot v_{i+1} + \cdots + 0 \cdot v_p = 1 \cdot v_i.$$

Then $v_i \in \text{span}(S)$ and therefore $S \subseteq \text{span}(S)$. \square

Remark. In the previous proof, S is generally not a subspace while $\text{span}(S)$ is a subspace.

Proposition. $\text{span}(S)$ is the smallest subset containing S .

Proof. Need to fill in \square

Definition. Given a subspace W of V , if a finite set S spans W , i.e., $W = \text{span}(S)$, then W is **spanned by** S or S **spans** W .

Example. (1) Show that $\{e_1, e_2\}$ spans \mathbb{R}^2 or $\mathbb{R}^2 = \text{span}\{e_1, e_2\}$ where $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\text{and } e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Proof. It suffices to show that any $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is a linear combination of e_1 and e_2 .
Then

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2 \end{bmatrix}.$$

Therefore, $\{e_1, e_2\}$ spans \mathbb{R}^2 . □

(2) Show $\left\{e_1, e_2, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ spans \mathbb{R}^2 .

Proof. It suffices to show any $x \in \mathbb{R}^2$ is a linear combination of e_1 , e_2 and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

For any $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$,

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 0 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

□

Therefore, $\{e_1, e_2\}$ spans \mathbb{R}^2 .

(3) Does $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ spans \mathbb{R}^2 ?

Answer: Yes

For any $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$, we need to show that there exists $\alpha_1, \alpha_2 \in \mathbb{R}$ such that

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ 0 \end{bmatrix} + \begin{bmatrix} \alpha_2 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} \alpha_1 + \alpha_2 \\ \alpha_2 \end{bmatrix}.$$

This leads to the linear equation with unknowns α_1 and α_2 :

$$\begin{cases} \alpha_1 + \alpha_2 &= x_1 \\ \alpha_2 &= x_2 \end{cases}$$

which implies

$$\begin{cases} \alpha_2 &= x_2 \\ \alpha_1 &= x_1 - x_2 \end{cases}.$$

Therefore, $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ spans \mathbb{R}^2 .

(4) Show that $\{E^{11}, E^{12}, E^{21}, E^{22}\}$ spans $\mathbb{R}^{2 \times 2}$ where

$$E^{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E^{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E^{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \text{ and } E^{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

For any matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in \mathbb{R}^{2 \times 2}$,

$$A = a_{11} \cdot E^{11} + a_{12} \cdot E^{12} + a_{21} \cdot E^{21} + a_{22} \cdot E^{22}.$$

Let W be the subspace of all 2×2 symmetric matrices in $\mathbb{R}^{2 \times 2}$. Show

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$$

spans W .

All matrices in W is symmetric.

Proof. For any $A \in W$, $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ such that $a_{12} = a_{21}$. Then

$$\begin{aligned} \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} &= \begin{bmatrix} a_{11} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & a_{12} \\ a_{12} & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & a_{22} \end{bmatrix} \\ &= a_{11} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + a_{12} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + a_{22} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

Therefore, A is a linear combination of $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Thus, W is spanned by $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ or $W = \text{span} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$. \square

24 Set operations on subspaces

Let U and W be two subspaces of V .

(1) $U \cap W$ is a subspace of V .

Proof. Since U and W are subspaces of V , then $0 \in U$ and $0 \in W$ and therefore $U \cap W$ is nonempty. First we show that $U \cap W$ is closed under “+”. For any $x, y \in U \cap W$, $x, y \in U$ and $x, y \in W$. Then $x + y \in U$ and $x + y \in W$ by the closure property of the subspaces. Therefore, $U \cap W$ is closed under “+”.

Similarly, $U \cap W$ is closed under “.” and therefore $U \cap W$ is a subspace. \square

(2) The union U and W is **not** a subspace in general.

Consider the example where $U = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} \subseteq \mathbb{R}^2$, and $W = \text{span} \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \subseteq \mathbb{R}^2$. Therefore, $U \cup W = \text{span}(e_1) \cup \text{span}(e_2)$.

Claim: $U \cup W$ is not closed under “+”. This is because $e_1 \in U$, $e_2 \in W$ and therefore $e_1 + e_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin U \cup W$.

(3) The (algebraic) sum of U and W ,

$$U + W = \{u + w : u \in U \text{ and } w \in W\}.$$

Example: When $U = \text{span} \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ and $W = \text{span} \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$. Then $U + W \in \mathbb{R}^2$.

Proposition. Let U and W be two subspaces of V . Then the (algebraic) sum of $U + W$ is a subspace, where $U + W = \{u + w : u \in U, w \in W\}$.

Proof. First we show that $U + W$ is closed under “+”. Let $x, y \in U + W$. Then there exists $u \in U$ and $w \in W$ such that $x = u + w$. Also, there exists $u' \in U$ and $w' \in W$ such that $y = u' + w'$. Then

$$\begin{aligned} x + y &= (u + w) + (u' + w') \\ &= (u + u') + (w + w'). \end{aligned}$$

Now U is a subspace, and $u, u' \in U$, therefore $u + u' \in U$ because U is a subspace and closed under vector addition. Likewise $w + w' \in W$. Then $x + y \in U + W$.

Now we will show that $U + W$ is closed under “.”. Let $x \in U + W$ and α be an arbitrary scalar. Then $x = u + w$ for some $u \in U$ and $w \in W$. Then $\alpha \cdot x = \alpha(u + w) = \alpha u + \alpha w$ where $\alpha u \in U$ and $\alpha w \in W$ and therefore $\alpha \cdot x \in U + W$.

Thus $U + W$ is a subspace. \square

Remark. If S_U is a spanning set of U and S_W is a spanning set of W , then $S_W \cup S_U$ is a spanning set of $U + W$.

Sketch of the proof: Let $S_U = \{u_1, \dots, u_p\}$ and $S_W = \{w_1, \dots, w_r\}$. Consider an arbitrary $x \in U + W$. Then $x = u + w$ for some $u \in U$ and $w \in W$. Since $u \in U$ and S_U spans U , there exists scalars $\alpha_1, \dots, \alpha_p$ such that $u = \alpha_1 u_1 + \dots + \alpha_p u_p$. Likewise, there exists β_1, \dots, β_r such that $w = \beta_1 w_1 + \dots + \beta_r w_r$. Then

$$x = (\alpha_1 u_1 + \dots + \alpha_p u_p) + (\beta_1 w_1 + \dots + \beta_r w_r).$$

So x is a linear combination of u_1, \dots, u_p and w_1, \dots, w_r and so $x \in \text{span}(S_U \cup S_W)$. Therefore $U + W \subseteq \text{span}(S_U \cup S_W)$. Similarly, you can show that $\text{span}(S_U \cup S_W) \subseteq U + W$. Thus $U + W = \text{span}(S_U \cup S_W)$.

25 Range and null space

Definition. Let $f : V \rightarrow W$ be a linear function where V and W are vector spaces over the same field.

The **range** of f , denoted by $range(f)$, is

$$range(f) = \{f(x) : x \in V\} \subseteq W$$

which is the set of all images of x under f .

The **null space** of F , denoted by $Null(f)$ is

$$Null(f) = \{x \in V : f(x) = 0\} \subseteq V.$$

Proposition. Let $f : V \rightarrow W$ be a linear function. Then

- (1) The range of f is a subspace of W .
- (2) $Null(f)$ is a subspace of V .

Proof. (1) We need to show that $range(f)$ is closed under “+” and “.”. We will only show the former. Let $y, z \in range(f)$. Since $y \in range(f)$ then there exists a x such that $y = f(x)$. Likewise, there exists x' in V such that $z = f(x')$. Consider $y + z = f(x) + f(x')$. Since f is linear, by superposition property, $f(x) + f(x') = f(x + x')$. This shows $y + z = f(x + x')$. Since $x, x' \in V$ then $x + x' \in V$ and therefore $y + z \in range(f)$.

- (2) $Null(f)$ is closed under “+” and “.”. We will show the former. Let $x, x' \in Null(f)$. Then $f(x) = 0$ and $f(x') = 0$. Since f is linear function; $f(x + x') = f(x) + f(x') = 0 + 0 = 0$. Therefore, $x + x' \in Null(f)$ and so $Null(f)$ is closed under “+”. We can also show that $Null(f)$ is closed under “.”. Thus $Null(f)$ is a subspace. □

26 Matrix defined linear functions

Let $A \in \mathbb{R}^{m \times n}$. Consider the function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ given by $f(x) = A \cdot x$, $x \in \mathbb{R}^n$. It's shown that f is a linear function.

- Then the $range(f) = range(A) = r(A) = \{Ax : x \in \mathbb{R}^n\}$, which is a subspace of \mathbb{R}^m .
- The $Null(f) = Null(A) = N(A) = \{x \in \mathbb{R}^n : Ax = 0\}$, which is a subspace of \mathbb{R}^n .
- Similarly we have $R(A^T)$ which is a subspace of \mathbb{R}^n .

- Also, $N(A^T)$ is a subspace of \mathbb{R}^m .

The spaces $R(A)$, $N(A)$, $R(A^T)$, and $N(A^T)$ are the four **fundamental subspaces** associated with A .

27 Spanning Sets of $R(A)$

Write the matrix A in the column form:

$$A = [A_1 \quad A_2 \quad \cdots \quad A_n].$$

Therefore, for any $x \in \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$

$$Ax = x_1 \cdot A_{\bullet 1} + x_2 \cdot A_{\bullet 2} + \cdots + x_n \cdot A_{\bullet n},$$

which is a linear combination of $A_{\bullet 1}, \dots, A_{\bullet n}$. Thus, $R(A)$ is the set of all linear combinations of $A_{\bullet 1}, \dots, A_{\bullet n}$. Therefore, $R(A) = \text{span}\{A_{\bullet 1}, \dots, A_{\bullet n}\}$. Likewise, $R(A^T)$ is spanned by the columns of A^T .

28 Linear Independence

Proposition. Let $S = \{v_1, \dots, v_r, v_{r+1}\}$ span a subspace U . Suppose v_{r+1} is a linear combination of v_1, \dots, v_r , then $U = \text{span}\{v_1, \dots, v_r\}$.

Proof. Clearly, $U = \text{span}\{v_1, \dots, v_r, v_{r+1}\}$.

Claim 1: $\text{span}\{v_1, \dots, v_r\} \subseteq U$.

Consider an arbitrary vector $x \in \text{span}\{v_1, \dots, v_r\}$. Therefore, x is a linear combination of v_1, \dots, v_r so there exist scalars $\alpha_1, \dots, \alpha_r$ such that

$$\begin{aligned} x &= \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_r v_r \\ &= \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_r v_r + 0 \cdot v_{r+1}. \end{aligned}$$

So $x \in U$.

Claim: 2: $U \subseteq \text{span}\{v_1, \dots, v_r\}$.

Since v_{r+1} is a linear combination of v_1, \dots, v_r then there exists scalars β_1, \dots, β_r such that $v_{r+1} = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_r v_r$.

For any $x \in U$, x is a linear combination of $v_1, v_2, \dots, v_r, v_{r+1}$ so there exists scalars $\alpha_1, \alpha_2, \dots, \alpha_r, \alpha_{r+1}$ such that

$$\begin{aligned}
x &= \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_r v_r + \alpha_{r+1} v_{r+1} \\
&= \alpha_1 v_1 + \cdots + \alpha_r v_r + \alpha_{r+1} (\beta_1 v_1 + \cdots + \beta_r v_r) \\
&= (\alpha_1 + \alpha_{r+1} \cdot \beta_1) v_1 + \cdots + (\alpha_r + \alpha_{r+1} \cdot \beta_r) v_r.
\end{aligned}$$

Therefore, $x \in \text{span}\{v_1, \dots, v_r\}$.

Thus $U = \text{span}\{v_1, \dots, v_r\}$. □

Definition. A set $\{v_1, \dots, v_p\}$ in a vector space V is **linearly dependent** if there exist scalars $\alpha_1, \dots, \alpha_p$, not all zero, such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_p v_p = 0.$$

Otherwise, $\{v_1, \dots, v_p\}$ is **linearly independent**. That is, if $\alpha_1 v_1 + \cdots + \alpha_p v_p = 0$, then $\alpha_1 = \alpha_2 = \cdots = \alpha_p = 0$. (\Leftarrow is trivial).

Example.

- (1) $\{0\}$: linearly dependent because for any nonzero scalar α , $\alpha \cdot 0 = 0$.

Let $v \neq 0$, then $\{v\}$ is linearly independent.

Proof. Let α be a scalar such that $\alpha \cdot v = 0$. For sake of contradiction, assume $\alpha \neq 0$, which implies that $\frac{1}{\alpha}$ exists. Then $\frac{1}{\alpha}(\alpha \cdot v) = \frac{1}{\alpha} \cdot 0$, which implies that $v = 0$. However this contradicts the assumption that $v \neq 0$. So $\alpha = 0$, then $\{v\}$ is linearly independent. □

Summary: $\{v\}$ is linearly independent $\iff v \neq 0$.

- (2) $\{u, v\}$ is linearly independent if and only if one of u and v is a multiple of the other.

Proof. “if”: Suppose $u = \alpha \cdot v$ (WLOG) for some scalar α . Then $u - \alpha v = 0$ if and only if $u + (-\alpha)v = 1 \cdot u + (-\alpha) \cdot v = 0$. Since not every α is 0, then $\{u, v\}$ is linearly dependent.

Not every α is 0, therefore $\{u, v\}$ is linearly dependent.

“only if”: Suppose $\{u, v\}$ is linearly independent. Then there exists scalars α_1, α_2 not both zero such that

$$\alpha_1 u + \alpha_2 v = 0.$$

WLOG, assume $\alpha_1 \neq 0$, which implies $\frac{1}{\alpha_1}$ exists. Then multiplying the previous equation on both sides by $\frac{1}{\alpha_1}$ and rearranging we have

$$u = -\frac{\alpha_2}{\alpha_2}v.$$

Therefore, u is a multiple of v . Likewise, if $\alpha_2 \neq 0$, then v is a multiple of u . \square

Lemma. *A finite set $S = \{v_1, \dots, v_p\}$ is linearly dependent if and only if one of the vectors in S is a linear combination of the rest of the vectors in S .*

Proof. “if”: WLOG, assume that v_1 is a linear combination of v_2, \dots, v_p . Then there exists scalars such that $\alpha_2, \dots, \alpha_p$ such that

$$v_1 = \alpha_2 v_2 + \alpha_3 v_3 + \dots + \alpha_p v_p,$$

which implies that

$$(1)v_1 + (-\alpha_2)v_2 + \dots + (-\alpha_p)v_p = 0.$$

Since the above linear combination is equal to 0, but the first scalar is 1, then S is linearly dependent.

“only if”: Suppose S is linearly dependent. Then there exists scalars $\alpha_1, \dots, \alpha_p$, not all zero, such that

$$\alpha_1 v_1 + \dots + \alpha_p v_p = 0.$$

WLOG, assume that $\alpha_1 \neq 0$, which implies $\frac{1}{\alpha_1}$ exists. Then multiplying the previous equation by $\frac{1}{\alpha_1}$ and rearranging we have

$$v_1 = -\frac{\alpha_2}{\alpha_1}v_2 - \dots - \frac{\alpha_p}{\alpha_1}v_p.$$

Therefore, v_1 is a linear combination of v_2, \dots, v_p . \square

Proposition. *A finite set $S = \{v_1, \dots, v_p\}$ is linearly independent if and only if none of the vectors in S is a linear combination of the rest of the vectors in S .*

Proof. Contrapositive of the preceding lemma. \square

29 Minimal Spanning Set

Definition. A spanning set S of a vector space V (i.e., $V = \text{span}(S)$) is **minimal** if none of the vectors in S is a linear combination of the rest of the vectors in S .

Fact: A spanning set S is minimal if and only if it is linearly independent.

Lemma. *Let $S = \{v_1, \dots, v_p\}$ be a minimal spanning set for a vector space V . If we remove any vector from S , then the resulting set does **not** span V .*

Proof. WLOG, assume we remove v_1 from S such that the resulting set is $S' = \{v_2, \dots, v_p\}$.

Claim: S' does not span V .

From the given we know that $S = \{s_1, s_2, \dots, s_p\}$ spans V . Therefore, every $v_i \in S$ belongs to V , which implies $v_1 \in V$. On the other hand, S is minimal and therefore v_1 is not a linear combination of v_2, \dots, v_p . Thus, $v_1 \notin \text{span}(S')$ and so S' does not span V . □

Remark. In the above proof we found a vector $v_1 \in V$ that is not in $\text{span}(S')$ so $\text{span}(S') \neq V$.

30 Maximal linearly independent set

Definition. A linearly independent set S in a vector space V is maximal if for any $z \in V$, $S \cup \{z\}$ is linearly dependent.

Lemma. Let $S = \{v_1, \dots, v_p\}$ be a linearly independent set in V . Then for any given $z \in V$, $z \in \text{span}(S)$ if and only if $S \cup \{z\}$ is linearly dependent.

Proof. “Only if”: Suppose $z \in \text{span}(S)$. Therefore, z is a linear combination of v_1, \dots, v_p . Then $S \cup \{z\} = \{v_1, \dots, v_p, z\}$ where z is a linear combination of v_1, \dots, v_p . Thus $S \cup \{z\}$ is linearly dependent.

“If”: Suppose $S \cup \{x\}$ is linearly dependent, which means there exists scalars $\alpha_1, \dots, \alpha_p$ and β , not all zero, such that

$$\alpha_1 v_1 + \dots + \alpha_p v_p + \beta z = 0.$$

Claim: The scalar $\beta \neq 0$.

For the sake of contradiction that $\beta = 0$, which implies $\beta \cdot z = 0$. Then the above equation reduces to

$$\alpha_1 v_1 + \dots + \alpha_p v_p = 0.$$

Since $S = \{v_1, \dots, v_p\}$ is linearly independent, then $\alpha_1 = \alpha_2 = \dots = \alpha_p = 0$. (and $\beta = 0$). However, this is a contradiction. Therefore, $\beta \neq 0$.

Now since $\beta \neq 0$ then we can multiply the equation

$$\alpha_1 v_1 + \dots + \alpha_p v_p + \beta z = 0$$

through by β and then rearranging we have

$$z = -\frac{\alpha_1}{\beta} v_1 - \dots - \frac{\alpha_p}{\beta} v_p.$$

Therefore, $z \in \text{span}(S)$. □

Theorem 30.1.

- (1) A maximal linearly independent set S in V spans V , i.e., $V = \text{span}(S)$.
- (2) A linearly independent set is maximal if and only if it is a minimal spanning set.

Proof. (1) Suppose S is a **maximal** linearly independent set. Then for any $z \in V$, $S \cup \{z\}$ is linearly dependent. By the preceding lemma, $z \in \text{span}(S)$. Since $z \in \text{span}(S)$ for any $z \in V$, then $\text{span}(S) = V$ or S spans V .

- (2) “only if”: Let S be a maximal linear independent set. By 1), S spans V . Therefore, S is linearly independent and so S is a minimal spanning set.

“if”: Let S be a minimal spanning set. Then S is linearly independent and so S spans V . Thus, for any vector $z \in V$, $z \in \text{span}(S)$. By the preceding lemma, $S \cup \{z\}$ is linearly dependent. Therefore, S is a maximal linear independent set. \square

31 More on linearly independent sets

Fact: Let $S = \{v_1, \dots, v_p\}$ be a linearly independent set. Then any (non-empty) subset of S is linearly independent.

Proof. WLOG, suppose $S' = \{v_1, \dots, v_r\} \subseteq S$, where $r < p$. Now we want to show that S' is linearly independent.

Let $\alpha_1, \dots, \alpha_r$ be such that $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_r v_r = 0$. Then we can add in the v_{r+1} to v_p vectors to the previous linear combination by taking $\alpha_{r+1} = \dots = \alpha_p = 0$,

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_r v_r + 0 \cdot v_{r+1} + \dots + 0 \cdot v_p = 0.$$

Now since S is linearly independent then it must be the case that $\alpha_1 = 0, \alpha_2 = 0, \dots, \alpha_r = 0$ and therefore $\{v_1, \dots, v_r\}$ is linearly independent. \square

Let A be an $m \times n$ matrix whose column form is

$$A = [A_{\bullet 1} \quad A_{\bullet 2} \quad \dots \quad A_{\bullet n}]$$

where each column $A_{\bullet i}$ is in \mathbb{R}^m or \mathbb{C}^m . If

$$(A \cdot x = 0 \iff x = 0) \iff N(A) = \{0\}$$

then the columns of A are linearly independent.

Proposition. Let $\{v_1, v_2, \dots, v_r\}$ be a linearly independent set in \mathbb{R}^n and $P \in \mathbb{R}^{n \times n}$ be invertible. Then $\{P \cdot v_1, P \cdot v_2, \dots, P \cdot v_r\}$ is linearly independent.

Proof. Let $\alpha_1, \dots, \alpha_r$ be scalars in \mathbb{R} such that

$$\alpha_1 P \cdot v_1 + \alpha_2 P \cdot v_2 + \dots + \alpha_r P \cdot v_r = 0.$$

Then we can rearrange each of the terms and remove P from the left so that we have

$$P[\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_r v_r] = 0$$

and since P is invertible we can left multiply both sides by P^{-1} to obtain

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_r v_r = P^{-1} \dot{0} = 0.$$

Therefore, $\{Pv_1, \dots, Pv_r\}$ are linearly independent. □

Let A be a matrix in $\mathbb{R}^{m \times n}$ or $\mathbb{C}^{m \times n}$. Then the range of A , $R(A)$ is the **column space** of A , which is a subset of \mathbb{R}^m or \mathbb{C}^m .

We can write A in column form and use row reduction to get in an equivalent form $P \cdot U$

$$\begin{aligned} A &= [A_{\bullet 1} \quad A_{\bullet 2} \quad \cdots \quad A_{\bullet n}] \\ &\vdots \\ &= P \cdot U \end{aligned}$$

where P is an $m \times m$ invertible matrix and U is an $m \times n$ matrix that is the echelon form of A ,

$$U = [U_{\bullet 1} \quad U_{\bullet 2} \quad \cdots \quad U_{\bullet r} \quad U_{\bullet r+1} \quad \cdots \quad U_{\bullet n}],$$

where $U_{bullet 1}, \dots, U_{\bullet r}$ are the pivot columns and $U_{\bullet r+1}, \dots, U_{\bullet n}$ are the non-pivot columns.

Facts:

- (1) The pivot columns of U are linearly independent.
- (2) A non-pivot column of U is a linear combination of pivot columns of U .

We can write A as

$$A = [A_{\bullet 1} \quad \cdots \quad A_{\bullet r} \quad A_{\bullet r+1} \quad \cdots \quad A_{\bullet n}],$$

and then $PA_{\bullet 1}, \dots, PA_{\bullet r}$ are linearly independent because multiplying a set of linearly independent vectors by an invertible matrix preserves linear independence.

Therefore the pivot columns of A are linearly independent and each non-pivot column of A is a linear combination of the pivot columns of A . Therefore

$$R(A) = \text{span}\{A_{\bullet 1}, \dots, A_{\bullet n}\} = \text{span}\{A_{\bullet 1}, \dots, A_{\bullet r}\}$$

where $\text{span}\{A_{\bullet 1}, \dots, A_{\bullet r}\}$ is a minimal spanning set. So to find a minimal spanning set, find the pivot columns of A .

32 Basis and Dimension

Assumption: V is a vector space spanned by a finite set.

Definition. If a set $B \in V$ spans V and is linearly independent then B is a **basis** for V .

B is a basis $\iff V = \text{span}(B)$ and B is linearly independent.

Fact:

B is a basis $\iff B$ is a minimal spanning set
 $\iff B$ is maximal linearly independent

Example. (1) Let $V = \mathbb{R}^2$.

Consider $B = \{e_1, e_2\} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$. Clearly, B spans V , and is linearly independent. Therefore, B is a basis (i.e., it is the **standard** basis).

- Consider $B' = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$. In a previous example we have shown that B' spans \mathbb{R}^2 and none of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a multiple of the other. Therefore, B' is linearly independent and is a basis.

- (2) Let $V = \mathbb{R}^n$. Let $A \in \mathbb{R}^{n \times n}$ be an invertible matrix. Then $B = \{A_{\bullet 1}, \dots, A_{\bullet n}\}$ is a basis for V .

Proof. First we prove the spanning property. Let $y \in \mathbb{R}^n$ be arbitrary. Therefore, y is a linear combination of the columns of A and so there exists and $x \in \mathbb{R}^n$ such that $y = A \cdot x$. Since A is invertible then the equation $Ax = y$ has the unique solution given by $A^{-1} \cdot y$. Therefore, B spans \mathbb{R}^n .

Now we prove that B is linearly independent. To show B is linearly independent it suffices to show that $N(A) = 0$. Since A is invertible then $N(A) = \{0\}$, which implies that B is linearly independent. Thus B is a basis. \square

- (3) Let W be the subspace of all 2×2 upper triangular matrices in $\mathbb{R}^{2 \times 2}$. Consider

$$B = \{E^{11}, E^{12}, E^{22}\} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

We claim that B is a basis for W .

Proof. We first prove the spanning property. We can write every $A \in W$ as

$$A = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix} = a_{11} \cdot E^{11} + a_{12} \cdot E^{12} + a_{22} \cdot E^{22}.$$

Therefore, B spans W . Now to show linear independence, suppose $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ are such that $\alpha_1 E^{11} + \alpha_2 E^{12} + \alpha_3 E^{22} = 0$. Then if

$$\begin{bmatrix} \alpha_1 & \alpha_2 \\ 0 & \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

we must $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Therefore, B is linearly independent and so B is a basis. □

Lemma. Let $B = \{v_1, \dots, v_n\}$ be a basis for a vector space V . Then for any $x \in V$, there exist **unique** scalars $\alpha_1, \dots, \alpha_n$ such that

$$x = \alpha_1 v_1 + \dots + \alpha_n v_n.$$

Proof. Let x be an arbitrary vector in V . Since B is a basis then B spans V and so every vector in V is a linear combination of v_1, \dots, v_n . Then there exist scalars $\alpha_1, \dots, \alpha_n$ such that

$$x = \alpha_1 v_1 + \dots + \alpha_n v_n.$$

To show uniqueness of $\alpha_1, \dots, \alpha_n$ let β_1, \dots, β_n be scalars such that

$$x = \beta_1 v_1 + \dots + \beta_n v_n.$$

Now if we subtract $x = \alpha_1 v_1 + \dots + \alpha_n v_n$ and the previous equation we get

$$0 = (\alpha_1 - \beta_1)v_1 + \dots + (\alpha_n - \beta_n)v_n.$$

Since B is linearly independent then it must be the case that $\alpha_1 - \beta_1 = 0, \alpha_2 - \beta_2 = 0, \dots, \alpha_n - \beta_n = 0$. Therefore, $\beta_1 = \alpha_1, \beta_2 = \alpha_2, \dots, \beta_n = \alpha_n$. This shows the uniqueness of the α_i 's. □

Definition. Let $B = \{v_1, \dots, v_n\}$ be a basis for V . Then for each $x \in V$, $(\alpha_1, \dots, \alpha_n)$ is the coordinate of x with respect to the basis B denoted by $[x]_\beta$. Therefore

$$[x]_\beta = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \in \mathbb{R}^n.$$

Definition. The **dimension** of a vector space V is the number of vectors in a basis of V , and it is denoted by $\dim V$.

Example. (1) $\dim \mathbb{R}^n = n$.

(2) $\dim\{0\} = 0$.

Lemma. Let A be an $m \times n$ matrix. Suppose $n > m$. Then the columns of A are linearly dependent.

Theorem 32.1. Let B_1 and B_2 be two bases for a vector space V . Then $\#(B_1) = \#(B_2)$, where $\#(B_i)$ is the number of vectors in B_i for $i = 1, 2$.

Proof. Let $B_1 = \{v_1, \dots, v_p\}$ and $B_2 = \{u_1, \dots, u_r\}$ be two bases for V . Since B_1 is a basis for V then B_1 spans V . Also, each $u_i \in B_2$ is in V and therefore each u_i is a linear combination of the vectors in B_1 , v_1, \dots, v_p , i.e.,

$$u_i = \alpha_{i1}v_1 + \alpha_{i2}v_2 + \dots + \alpha_{ip}v_p$$

for $i = 1, \dots, r$.

Therefore, we can write B_2 as

$$B_2 = [u_1 \quad \dots \quad u_r] = [v_1 \quad \dots \quad v_p] \cdot \begin{bmatrix} \alpha_{11} & \alpha_{21} & \dots & \alpha_{r1} \\ \alpha_{12} & \alpha_{22} & \dots & \alpha_{r2} \\ \vdots & \vdots & & \vdots \\ \alpha_{1p} & \alpha_{2p} & & \alpha_{rp} \end{bmatrix}_{p \times r} = B_1 \cdot A$$

where A is a $p \times r$ matrix.

Claim: $r \leq p$.

Suppose otherwise; i.e., $r > p$. Then by the preceding lemma, the columns of A are linearly dependent. Then there exists a vector $w \neq 0$ such that $Aw = 0$. If we let multiply the above equation by w then we have

$$[u_1 \quad \dots \quad u_r] \cdot w = [v_1 \quad \dots \quad v_p] \cdot A \cdot w = 0.$$

Now let $w = \begin{bmatrix} w_1 \\ \vdots \\ w_r \end{bmatrix} \neq 0$. Then

$$w_1u_1 + w_2u_2 + \dots + w_ru_r = 0,$$

which implies that $\{u_1, \dots, u_r\}$ is linearly dependent. However, this contradicts our assumption that $B_2 = \{u_1, \dots, u_r\}$ is a basis. Thus it must be the case that

$$r \leq p \iff \#(B_2) \leq \#(B_1).$$

By interchanging the roles of B_1 and B_2 we obtain $p \leq r \iff \#(B_1) \leq \#(B_2)$. Thus, $\#(B_1) = \#(B_2)$. \square

Proposition. Let V be an n -dimensional vector space, i.e., $\dim V = n$. Then

- (1) A maximal linearly independent set in V has n vectors.
- (2) A minimal spanning set in V has n vectors.
- (3) A spanning set for V has at least n vectors.
- (4) A linearly independent set in V has at most n vectors.

Proof. (1) and (2): Recall that a maximal linearly independent set and a minimal spanning set are bases for V . Since $\dim V = n$ then each basis has exactly n vectors. Thus, a maximal linearly independent or a minimal spanning set contains n vectors.

(3) Let S be a spanning set for V . If S is minimal, then by 2) $\#(S) = n$. If S is not minimal, then it can be reduced to a minimal spanning set S' , where $n = \#(S') < \#(S)$. Thus, S has at least n vectors. \square

Proposition. Let V be an n -dimensional vector space, and W is a subspace of V .

- (1) $\dim W \leq \dim V$
- (2) If $\dim W = \dim V$, then $W = V$.

Proof. (1) Let B be a basis for the subspace W . Then B is linearly independent and $B \subseteq W \subseteq V$. Then

$$\dim W = \#(B) \leq n = \dim V.$$

Therefore, $\dim W \leq \dim V$.

- (2) Let $\dim W = \dim V$. Suppose for the sake of contradiction that $W \neq V$. Since $W \subseteq V$ and $W \neq V$ then there exists a vector $z \in V$ but $z \notin W$. Let B be a basis for W , i.e., $W = \text{span}(B)$. Since $z \notin W$, then $z \notin \text{span}(B)$. Since B is a basis then B is linearly independent and so $B \cup \{z\}$ is linearly independent. Therefore $\#(B \cup \{z\}) = \#(B) + 1$. Since $\dim V = \dim W = n$ and therefore $\#(B) = n$. But $\#(B \cup \{z\}) = n + 1$ and therefore $B \cup \{z\}$ is a linear independent set in V with $n + 1$ vectors. However by a previous proposition, this is a contradiction. Therefore, our initial assumption that $W \neq V$ must be false. Thus, $W = V$. \square

Theorem 32.2. Let U and W be two subspaces of a vector space V (of finite dimension). Then

$$\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W).$$

Proof. Not given in class. \square

Example. Let $V = \mathbb{R}^3$ and let $U = \text{span}\{e_1\}$ and $W = \text{span}\{e_1, e_2\}$. Since U is a subspace of W then $U \subseteq V$ and $U + W = W$ so $U \cap W = U$. Therefore, $\dim U = 1$ and $\dim W = 2$. Then $\dim(U + W) = \dim(W) = 2$. On the right we have $\dim U + \dim W - \dim(U \cap W) = 1 + 2 = 2$.

33 Dimension of four fundamental subspaces associated with an $m \times n$ matrix

Without loss of generality let $A \in \mathbb{R}^{m \times n}$. Then

- $R(A)$: range or the column space of A is a subspace of \mathbb{R}^m .
- $N(A)$: null space of A is a subspace of \mathbb{R}^n .
- $R(A^T)$: range or column space of A^T is a subspace of \mathbb{R}^n .
- $N(A^T)$: null space of A^T is a subspace of \mathbb{R}^m .

Definition. The dimension of $R(A)$ is called the **rank** of A , i.e., $\dim[R(A)] = \text{rank}(A)$.
The dimension of the null space $N(A)$ is the **nullity** of A .

Proposition. $\text{rank}(A) = 0$ if and only if $A = 0$, i.e., A is the zero matrix.

Proof. If $A = 0$ then $R(A) = \{0\}$, which implies $\dim[R(A)] = 0$. If $\text{rank}(A) = 0$, then $\dim[R(A)] = 0$, which implies $R(A) = \{0\}$ and so $A = 0$. \square

Let A be an $m \times n$ real matrix.

- (1) The pivot columns of A form a basis for $R(A)$.
- (2) Let r be the number of pivots in A . Then

$$\dim[R(A)] = r = \text{rank}(A).$$

- (3) $\dim[N(A)] =$ the number of non-pivot columns of $A = n - r$.

Theorem 33.1. (Rank-nullity theorem) Let A be an $m \times n$ matrix. Then $\text{rank}(A) + \dim[N(A)] = n$.

Fact:

- $\text{rank}(A) = \dim[R(A^T)] =$ number of pivots in $A = r$.
- $\text{rank}(A^T) + \dim[N(A^T)] = m$.

34 Rank of a matrix product

Lemma. (Special Case) Let $A : m \times n$ matrix, and $P : m \times m$ invertible. Then $\text{rank}(P \cdot A) = \text{rank}(A)$.

Proof. Need to show: $\dim(R(P \cdot A)) = \dim(R(A))$.

Let $A_{\bullet 1}, A_{\bullet 2}, \dots, A_{\bullet r}$ be all pivot columns of A such that $\{A_{\bullet 1}, A_{\bullet 2}, \dots, A_{\bullet r}\}$ is a basis for $R(A)$.

Claim: $\{PA_{\bullet 1}, PA_{\bullet 2}, \dots, PA_{\bullet r}\}$ is a basis for $R(P \cdot A)$.

- **Spanning Property:** Since $\{A_{\bullet 1}, A_{\bullet 2}, \dots, A_{\bullet r}\}$ is a basis for $R(A)$ then $\{A_{\bullet 1}, A_{\bullet 2}, \dots, A_{\bullet r}\}$ spans $R(A)$. Therefore, $\{PA_{\bullet 1}, PA_{\bullet 2}, \dots, PA_{\bullet r}\}$.
- **Linear Independence:** Let $\alpha_1, \alpha_2, \dots, \alpha_r$ be such that $\alpha_1 PA_{\bullet 1} + \dots + \alpha_r PA_{\bullet r} = 0$. Then $P(\alpha_1 A_{\bullet 1} + \dots + \alpha_r A_{\bullet r}) = 0$. Since P is invertible, we can right multiply both sides of the previous equation by P^{-1} and then we have $\alpha_1 A_{\bullet 1} + \dots + \alpha_r A_{\bullet r} = 0$. Since $\{A_{\bullet 1}, \dots, A_{\bullet r}\}$ is a basis, then the set is linear independent and so $\alpha_1 = 0, \alpha_2 = 0, \dots, \alpha_r = 0$. Thus, $\{PA_{\bullet 1}, \dots, PA_{\bullet r}\}$ is linearly independent and so $\{PA_{\bullet 1}, \dots, PA_{\bullet r}\}$ is a basis for $R(PA)$. Now since $\dim(R(A)) = r = \dim(R(PA))$ then $\text{rank}(A) = \text{rank}(P \cdot A)$.

□

Now we will prove a more general theorem, but first we need some remarks and a preliminary lemma. The statement of the general theorem is: Let A be an $m \times n$ matrix and B be a $n \times p$ matrix. Then $\text{rank}(A \cdot B) = \text{rank}(B) - \dim(N(A) \cap R(B))$.

Remark.

- (1) Since A is $m \times n$ then $N(A)$ is a subspace of \mathbb{R}^n . Since B is $n \times p$ then $R(B)$ is a subspace of \mathbb{R}^n . Therefore, $N(A) \cap R(B)$ is a subspace of \mathbb{R}^n .
- (2) For any $z \in (N(A) \cap R(B))$, $z \in N(A)$ so that $Az = 0$ and $z \in R(B)$ so that $z = B \cdot w$ for some w . Therefore, $Az = ABw = 0$ and so any $z \in N(A) \cap R(B)$ does not “contribute” to $\dim(R(A \cdot B))$.

Lemma. Let V be an n -dimensional vector space and W be a r -dimensional space of V with $r < n$. Suppose $\{v_1, \dots, v_r\}$ is a basis for W . Then there exists $v_{r+1}, v_{r+2}, \dots, v_n \in V$ such that $\{v_1, \dots, v_r, v_{r+1}, \dots, v_n\}$ is a basis for V .

Proof. Need to show $\{v_1, \dots, v_r, v_{r+1}, \dots, v_n\}$ is maximally linearly independent, which implies that $\{v_1, \dots, v_r, v_{r+1}, \dots, v_n\}$ is a basis. □

Theorem 34.1. Let A be an $m \times n$ matrix and B be a $n \times p$ matrix. Then $\text{rank}(A \cdot B) = \text{rank}(B) - \dim(N(A) \cap R(B))$.

Proof. Since $N(A) \cap R(B)$ is a subspace of \mathbb{R}^n . Let $\{v_1, \dots, v_r\}$ be a basis for $N(A) \cap R(B)$. Then $\dim(N(A) \cap R(B)) = r$.

Now consider 2 cases:

- (1) $r = n$. We know $N(A) \cap R(B)$ is a subspace of $N(A)$ and a subspace of $R(B)$. Since $r = n$, then $n = \dim(N(A) \cap R(B)) \leq \dim(R(B)) \leq n$, where the last inequality follows because $R(B)$ is a subspace of \mathbb{R}^n . Therefore, $\dim(R(B)) = n$ and so $\text{rank}(B) = n$. Also, since $r = n$, then $n = \dim(N(A) \cap R(B)) \leq \dim(N(A)) \leq n$. The last inequality holds again because $N(A)$ is a subspace of \mathbb{R}^n . Therefore, $\dim(N(A)) = n$ and by the rank-nullity theorem we have

$$\begin{aligned}
\text{rank}(A) &= n - \dim(N(A)) \\
&= n - n \\
&= 0.
\end{aligned}$$

Since $\text{rank}(A) = 0$, then $A = 0$. Now since $\text{rank}(B) = 0$ and $A = 0$ then $(A \cdot B) = 0$. Thus

$$\text{rank}(A \cdot B) = 0 = \text{rank}(B) - \dim(N(A) \cap R(B)).$$

- (2) $r < n$. In this case, we use the preliminary lemma to obtain $z_1, z_2, \dots, z_t \in R(B)$ such that $\{v_1, \dots, v_r, z_1, \dots, z_t\}$ is a basis for $R(B)$. Therefore $\text{rank}(B) = \dim(R(B)) = r+t$. Then $\{v_1, \dots, v_r, z_1, \dots, z_t\}$ spans $R(B)$ and so $\{Av_1, \dots, Av_r, Az_1, \dots, Az_t\}$ spans $R(B)$.

Recall that $\{v_1, v_r\}$ is a basis for $N(A) \cap R(B)$. Therefore, for each $v_i \in N(A)$, for all $i = 1, \dots, r$, we have $Av_i = 0$ for all $i = 1, \dots, r$. Now we can drop the zero vectors from $\{Av_1, \dots, Av_r, Az_1, \dots, Az_t\}$ so that we have $\{Az_1, \dots, Az_t\}$, which also spans $R(A \cdot B)$.

Claim: $\{Az_1, \dots, Az_t\}$ is linearly independent.

Let $\alpha_1, \dots, \alpha_t$ be scalars such that

$$\alpha_1 Az_1 + \dots + \alpha_t Az_t = A(\alpha_1 z_1 + \dots + \alpha_t z_t) = 0.$$

Now define $y = (\alpha_1 z_1 + \dots + \alpha_t z_t)$. From above we see that $y \in N(A)$ and $y \in R(B)$ and so $y \in N(A) \cap R(B)$. Since $\{v_1, \dots, v_r\}$ is a basis for $N(A) \cap R(B)$ and $y \in N(A) \cap R(B)$ then there exists scalars β_1, \dots, β_r such that

$$y = \beta_1 v_1 + \dots + \beta_r v_r$$

.

Now since $y = \alpha_1 z_1 + \dots + \alpha_t z_t$ and $y = \beta_1 v_1 + \dots + \beta_r v_r$ then

$$\alpha_1 z_1 + \dots + \alpha_t z_t = \beta_1 v_1 + \dots + \beta_r v_r$$

and then subtracting $\beta_1 v_1 + \dots + \beta_r v_r$ from both sides we have

$$\alpha_1 z_1 + \dots + \alpha_t z_t + (-\beta_1)v_1 + \dots + (-\beta_r)v_r = 0.$$

Since $\{v_1, \dots, v_r, z_1, \dots, z_t\}$ is a basis for $R(B)$ then $\{v_1, \dots, v_r, z_1, \dots, z_t\}$ is linearly independent. Therefore $\beta_1 = 0 = \dots = \beta_r = 0$ and $-\alpha_1 = 0 = \dots = -\alpha_t = 0$ so $\alpha_1 = 0 = \dots = \alpha_t = 0$. Thus, $\{Az_1, \dots, Az_t\}$ is linearly independent.

Then $\{Az_1, \dots, Az_t\}$ is a basis for $R(A \cdot B)$ and so $\dim(R(AB)) = t$ and then $\text{rank}(AB) = t$. Therefore, $\text{rank}(B) = \dim(R(B)) = r + t$ and $\dim(N(A) \cap R(B)) = r$ and so

$$\text{rank}(AB) = t = r + t - r = \text{rank}(B) - \dim(N(A) \cap R(B)).$$

□

Example.

- (1) Let $A : m \times n$ and $P : m \times n$ invertible matrix.
- (2) Rank of $A \cdot A^T$ and $A^T \cdot A$ where $A \in \mathbb{R}^{m \times n}$.

Theorem 34.2. $\text{rank}(A \cdot A^T) = \text{rank}(A)$ and $\text{rank}(A^T \cdot A) = \text{rank}(A)$.

Proof. Note that $\text{rank}(A^T \cdot A) = \text{rank}(A) - \dim(N(A^T) \cap R(A))$.

Claim: $N(A^T) \cap R(A) = \{0\}$.

Let $z \in N(A^T) \cap R(A)$. Then $z \in N(A^T)$ and $z \in R(A)$. Therefore, $A^T z = 0$ and $z = Ax$ for some x . Then $A^T z = A^T Ax = 0$. Then left multiply by x^T ,

$$\begin{aligned} \Rightarrow x^T A^T Ax &= x^T 0 \\ \Rightarrow (A \cdot x)^T Ax &= 0 \in \mathbb{R} \\ \Rightarrow z^T z &= 0 \in \mathbb{R} \\ \Rightarrow z &= 0. \end{aligned}$$

Therefore, $N(A^T) \cap R(A) = \{0\}$. Then $\text{rank}(A^T \cdot A) = \text{rank}(A)$.

To show $\text{rank}(A \cdot A^T) = \text{rank}(A)$, we see that

$$\begin{aligned} \text{rank}(A \cdot A^T) &= \text{rank}((A^T)^T A^T) \\ &= \text{rank}(A^T) \\ &= \text{rank}(A). \end{aligned}$$

□

Theorem 34.3. Let A be an $m \times n$ matrix. Then

- (1) $R(A^T \cdot A) = R(A^T)$ and $R(A \cdot A^T) = R(A)$.
- (2) $N(A^T \cdot A) = N(A)$ and $N(A \cdot A^T) = N(A^T)$.

Proof.

- (1) Prove $R(A^T \cdot A) = R(A)$ only. Note that $R(A \cdot B)$ is a subspace of $R(A)$ and therefore $R(A^T \cdot A)$ is a subspace of $R(A^T)$. Then it suffices to show that $\dim(R(A^T \cdot A)) = \dim(R(A^T)) \iff \text{rank}(A^T \cdot A) = \text{rank}(A^T)$. Since $\text{rank}(A^T \cdot A) = \text{rank}(A^T)$ and so $R(A^T \cdot A) = R(A^T)$.

- (2) Show $N(A^T \cdot A) = N(A)$ only.

Clearly, $N(A)$ is a subspace of $N(A^T \cdot A)$. This is because $x \in N(A) \iff Ax = 0 \Rightarrow A^T Ax = 0 \Rightarrow N(A^T \cdot A)$.

It suffices to show that

$$\dim(N(A)) = \dim(N(A^T \cdot A))$$

Recall from the rank-nullity theorem that $\text{rank}(A) + \dim(N(A)) = n$ and so $\dim(N(A)) = n - \text{rank}(A)$. Likewise, since $A^T A$ is an $n \times n$ matrix, then $\dim(A^T \cdot A) = n - \text{rank}(A^T \cdot A)$. Since $\text{rank}(A^T \cdot A) = \text{rank}(A)$ and $\dim(N(A)) = \dim(N(A^T \cdot A))$. Therefore, $N(A^T \cdot A) = N(A)$.

□

Remark. The following implications of the above theorems.

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.

- (1) Consider the linear equation

$$Ax = b.$$

In general, this equation may **not** have a solution.

- (2) Consider the **normal** equation

$$A^T \cdot Ax = A^T b.$$

Fact: $A^T \cdot Ax = A^T b$ always has a solution (or is always **consistent**).

Proof. Since $R(A^T A) = R(A^T)$ and $A^T b \in R(A^T)$ then $A^T b \in R(A^T A)$. Then there exists a $p \in \mathbb{R}^n$ such that $A^T A p = A^T b$. Therefore, p is a solution to $A^T \cdot A \cdot x = A^T \cdot b$. □

- (3) If $Ax = b$ has a solution or is consistent, then $Ax = b$ and $A^T Ax = A^T b$ have the same solution set.
- (4) Suppose $Ax = b$ has a **unique** solution. Then $A^T \cdot A$ is invertible.

Proof. Since $Ax = b$ has a unique solution. Therefore $N(A) = \{0\}$ because otherwise there exists a $0 \neq z \in N(A)$. Then $Ap = b$ and since A is linear then $A(p + z) = Ap + Az = b + 0 = b$. However, this is a contradiction. Therefore, $N(A) = \{0\}$.

Since $N(A^T \cdot A) = N(A)$ and $N(A) = \{0\}$ then $N(A^T A) = \{0\}$. Thus, $A^T A$ is invertible. Then the unique solution to $A^T Ax = A^T b$ is

$$x_* = (A^T A)^{-1} \cdot A^T \cdot b,$$

which is the unique solution to $Ax = b$. □

35 Application to the least squares problem

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. The least-squares (LS) problem is to minimize

$$\begin{aligned} F(x) &= \|Ax - b\|_2^2 \\ &= (Ax - b)^T (Ax - b) \\ &\geq 0. \end{aligned}$$

An optimal solution x_* to the LS problem satisfies:

$$\nabla F(x_*) = 0$$

where $\nabla F(x) = 2(A^T Ax - A^T b)$. Since x_* is such that $A^T Ax_* = A^T b$, i.e., x_* is solution to normal equation. Therefore, the LS problem always has a solution.

Claim: x_* satisfying the normal equation is an optimal solution to the LS problem.

Proof. Let $x = x_* + u$, where $u \in \mathbb{R}^n$ is arbitrary. Then

$$\begin{aligned} F(x) &= (Ax - b)^T (Ax - b) \\ &= (Ax_* - b + Au)^T (Ax_* - b + Au) \\ &= (Ax_* - b)^T (Ax_* - b) + 2(Ax_* - b)^T Au + (Au)^T (Au). \end{aligned}$$

Note that the first term on the RHS above is $F(x_*)$, the second term $2(Ax_* - b)^T Au$ goes to zero, and the final term is a sum of squares and so is ≥ 0 .

Note that

$$2[(Ax_* - b)^T A]u = 2[A^T (Ax_* - b)]^T \cdot u = 0$$

because term in brackets is equal to 0 because x_* satisfies the normal equation.

Therefore, $F(x) \geq F(x_*)$, for all $u \in \mathbb{R}^n$ or any $x \in \mathbb{R}^n$. Thus, x_* is an optimal solution to the LS problem. □

36 Coordinate of a vector

Definition. Let V be an n -dimensional vector space, and $B = \{v_1, \dots, v_n\}$ be a basis for V .

For each $x \in V$, let

$$[x]_B = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} \in \mathbb{R}^n \text{ or } \mathbb{C}^n$$

where $\alpha_1, \dots, \alpha_n$ satisfy $x = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$. Then $[x]_B$ is the **coordinate vector** for x .

Fact:

- (1) $x \neq 0$ if and only if $[x]_B \neq 0$.
- (2) $x = 0$ if and only $[x]_B = 0$.
- (3) For each $x \in V$, $[x]_B$ is unique.

Remark. Recall that a linear function $T : U \rightarrow V$ where U, V are finite dimensional vector spaces, satisfies

- (1) $T(x + y) = T(x) + T(y)$, for all $x, y \in U$
- (2) $T(\alpha x) = \alpha \cdot T(x)$, for all $x, y \in U$ and scalar α .

Then for any $x, y \in U$ and scalars α, β

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y).$$

Generalization

$$T\left(\sum_{i=1}^n \beta_i u_i\right) = \sum_{i=1}^n \beta_i \cdot T(u_i) \text{ for all } u_i \in V, i = 1, \dots, n.$$

Special Case:

If $T : V \rightarrow V$ is a linear function (i.e., $U = V$), then T is call a **linear operator**.

Example. Linear Operator

- (1) $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ given by $T(x) = A \cdot x$, where A is an $n \times n$ real matrix.
- (2) General linear operators: e.g., identity operator $I : V \rightarrow V$ given by $I(x) = x$ for all $x \in V$.
- (3) Some other operators are projection operators and reflection operators.

37 Coordinate Matrix

Let linear function $T : U \rightarrow V$ where U, V are finite dimensional vector spaces. Let $B = \{u_1, \dots, u_n\}$ be a basis for U and $B' = \{v_1, \dots, v_m\}$ be a basis for V .

Question: How to characterize the relation between $[x]_B$ and $[T(x)]_{B'}$?

We can use the **coordinate matrix** of T from B to B' .

For each $u_i \in B$, $T(u_i) \in V$, since $T(u_i) = \alpha_{1i}v_1 + \alpha_{2i}v_2 + \dots + \alpha_{mi}v_m$ then for weights $\alpha_{1i}, \dots, \alpha_{mi}$

$$T(u_i) = \begin{bmatrix} v_1 & \dots & v_m \end{bmatrix} \begin{bmatrix} \alpha_{1i} \\ \alpha_{2i} \\ \vdots \\ \alpha_{mi} \end{bmatrix}$$

Define the following $m \times n$ matrix,

$$[T]_{BB'} = [[T(u_1)]_{B'}, [T(u_2)]_{B'}, \dots, [T(u_n)]_{B'}] = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \vdots & \vdots & & \vdots \\ \alpha_{m1} & \alpha_{m2} & \dots & \alpha_{mn} \end{bmatrix} \in \mathbb{R}^{m \times n}.$$

Let $[T]_{BB'}$ denote this matrix, which is called the **coordinate matrix** of T from B to B' .

Proposition. $[T(x)]_{B'} = [T]_{BB'} \cdot [x]_B$ for all $x \in U$.

Proof. Let x be an arbitrary vector in U . Since $\{u_1, \dots, u_n\}$ is a basis for U then we can write x as a linear combination of the vectors B

$$x = \beta_1 u_1 + \dots + \beta_n u_n.$$

Then the coordinate vector of x with respect to B is

$$[x]_B = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix} \in \mathbb{R}^n.$$

Then we have

$$\begin{aligned}
T(x) &= T(\beta_1 u_1 + \cdots + \beta_n u_n) \\
&= \beta_1 T(u_1) + \cdots + \beta_n T(u_n) \\
&= [T(u_1) \quad T(u_2) \quad \cdots \quad T(u_n)] \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix} \\
&= [v_1 + \cdots + v_m] [T]_{BB'} \cdot [x]_B.
\end{aligned}$$

Therefore $[T(x)]_{B'} = [T]_{BB'} \cdot [x]_B$. □

Remark. For a linear operator $T : V \rightarrow V$, if B is a basis for V , the coordinate matrix of T from B to B is $[T]_{BB}$, which can be written as $([T]_{BB}) = [T]_B$. (Because same basis for domain and codomain.)

If V is an n -dimensional vector space, then $[T]_B$ is an $n \times n$ matrix for any basis B for V .

38 Change of basis

Question Let T be a linear operator and B and B' be two bases for V . What is the relation between $[T]_B$ and $[T]_{B'}$?

38.1 Change of basis

Let $B = \{u_1, \dots, u_n\}$ and $B' = \{v_1, \dots, v_n\}$ be two bases for an n -dimensional vector space V . Then the coordinate matrix of the identity operator $I : V \rightarrow V$ from B to B' is $[I]_{BB'}$.

Then by construction of $[I]_{BB'}$, we have

$$\begin{aligned}
[I]_{BB'} &= [[I(u_1)]_{B'} \quad [I(u_2)]_{B'} \quad \cdots \quad [I(u_n)]_{B'}] \\
&= [[u_1]_{B'} \quad [u_2]_{B'} \quad \cdots \quad [u_n]_{B'}]_{n \times n} \\
&= P.
\end{aligned}$$

Remark.

- (1) For any $x \in V$, $[x]_{B'} = [I]_{BB'} \cdot [x]_B = P \cdot [x]_B$.
- (2) $P = [I]_{BB'}$ is invertible.

Proof. Recall $[T(x)]_{B'} = [T]_{BB'} \cdot [x]_B$. Let $T = I$ the identity operator. Then $[x]_{B'} = [I]_{BB'} \cdot [x]_B$ for all $x \in V$.

Suppose P was not invertible. Then there exists $b \neq 0$ in \mathbb{R}^n such that $P \cdot b = 0$. Then there exists an $x \in V$ such that $[x]_B = b$. Since $b \neq 0$ then $x \neq 0$. Then $P \cdot b = [I]_{BB'} \cdot [x]_B = [x]_{B'}$ (by 1). Then $[x]_{B'} = 0$ and so $x = 0$. However this contradicts our assumption that if $b \neq 0$ then $x \neq 0$. Thus P is invertible. \square

Theorem 38.1. Let $P = [I]_{BB'}$, the change of basis matrix. Then

$$(1) [T]_B = P^{-1} \cdot [T]_{B'} \cdot P,$$

$$(2) [T]_{B'} = P \cdot [T]_B \cdot P^{-1}.$$

Proof. It suffices to show (1), which is equivalent to $P \cdot [T]_B = [T]_{B'} \cdot P$. To show $P \cdot [T]_B = [T]_{B'} \cdot P$,

$$\begin{aligned} P \cdot [T]_B &= P \begin{bmatrix} [T(u_1)]_B & \cdots & [T(u_n)]_B \end{bmatrix} \\ &= \begin{bmatrix} P[T(u_1)]_B & \cdots & P[T(u_n)]_B \end{bmatrix} \\ &= \begin{bmatrix} [T(u_1)]_{B'} & \cdots & [T(u_n)]_{B'} \end{bmatrix} \\ &= \begin{bmatrix} [T]_{B'} \cdot [u_1]_{B'} & \cdots & [T]_{B'} \cdot [u_n]_{B'} \end{bmatrix} \quad (1) \\ &= [T]_{B'} \begin{bmatrix} [u_1]_{B'} & \cdots & [u_n]_{B'} \end{bmatrix} \\ &= [T]_{B'} \cdot P. \end{aligned}$$

where (1) follows because $[T(x)]_{B'} = [T]_{B'} \cdot [x]_{B'}$.

Therefore, $P \cdot [T]_B = [T]_{B'} \cdot P$. \square

39 Similarity

Definition. Let A and B be two $n \times n$ matrices. If there exists an invertible matrix P such that $A = P^{-1}BP$ then A is **similar** to B , and we write $A \simeq B$.

39.1 Properties of similar relations

(1) Any $n \times n$ matrix is similar to itself. (By letting $P = I$).

(2) If A is similar to B , then B is similar to A .

Since A is similar to B then $A = P^{-1}BP$ and so $B = PAP^{-1} = (P^{-1})^{-1}AP^{-1}$.

(3) If $A \simeq B$ and $B \simeq C$ then $A \simeq C$.

(4) If A is similar to B then A^k is similar to B^k for any $k = 1, 2, \dots$

(5) If A is similar to B then $\text{rank}(A) = \text{rank}(B)$ and $\text{trace}(A) = \text{trace}(B)$. Also, $\det(A) = \det(B)$ and A and B have the same eigenvalues.

Proof. To show $\text{trace}(A) = \text{trace}(B)$. We have

$$\begin{aligned}\text{trace}(A) &= \text{trace}(P^{-1}BP) \\ &= \text{trace}((B \cdot P) \cdot P^{-1}) \\ &= \text{trace}(B \cdot I) \\ &= \text{trace}(B).\end{aligned}$$

□

Remark. In the above proof we used the fact that $\text{trace}(C \cdot D) = \text{trace}(D \cdot C)$, if C and D are square.

40 Inner products, norms, and orthogonality

40.1 Inner products on \mathbb{R}^n and \mathbb{C}^n

Let $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$ be two vectors in \mathbb{C}^n or \mathbb{R}^n .

If $x, y \in \mathbb{R}^n$, then the (standard) inner product is

$$x^T y = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n \in \mathbb{R}^n$$

If $x, y \in \mathbb{C}^n$ then the (standard) inner product of x and y is

$$x^* \cdot y = \begin{bmatrix} \overline{x_1} & \overline{x_2} & \cdots & \overline{x_n} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \overline{x_1} \cdot y_1 + \cdots + \overline{x_n} \cdot y_n \in \mathbb{C}^n$$

40.2 Properties of (standard) inner products on \mathbb{R}^n or \mathbb{C}^n

Let $x, y \in \mathbb{C}^n$. Then the following hold:

- (1) $y^* \cdot x = \overline{x^* \cdot y}$ (When $x, y \in \mathbb{R}^n$, then $x^T y = y^T x$)
- (2) For any $\alpha \in \mathbb{C}$, $x^*(\alpha \cdot y) = \alpha(x^* \cdot y)$, and $(\alpha \cdot x)^* \cdot y = \overline{\alpha}(x^* \cdot y)$.
- (3) For any $z \in \mathbb{C}^n$, $x^* \cdot (y + z) = x^* \cdot y + x^* \cdot z$.
- (4) $x^* x \geq 0$ and $x^* \cdot x = 0$ if and only $x = 0$. (Note that $x^* \cdot x \in \mathbb{R}$).

Proof. (1) We have that $y^* \cdot x = \overline{y_1} \cdot x_1 + \overline{y_2} \cdot x_2 + \cdots + \overline{y_n} \cdot x_n$ and

$$\begin{aligned} \overline{x^* \cdot y} &= \overline{\overline{x_1} \cdot y_1 + \cdots + \overline{x_n} \cdot y_n} \\ &= \overline{\overline{x_1} \cdot y_1} + \cdots + \overline{\overline{x_n} \cdot y_n} \\ &= \overline{\overline{x_1}} \cdot \overline{y_1} + \cdots + \overline{\overline{x_n}} \cdot \overline{y_n} \\ &= x_1 \cdot \overline{y_1} + \cdots + x_n \overline{y_n} \\ &= \overline{y_1} \cdot x_1 + \cdots + \overline{y_n} \cdot x_n. \end{aligned}$$

Therefore, $y^* \cdot x = x^* \cdot y$.

(2) Let $\alpha \in \mathbb{C}$. Then $x^* \cdot (\alpha \cdot y) = \alpha(x^* \cdot y)$ (by definition of $x^* \cdot y$).

To show $(\alpha x)^* \cdot y = \overline{\alpha}(x^* \cdot y)$ we have

$$\begin{aligned} (\alpha \cdot x)^* \cdot y &= \overline{y^* \cdot (\alpha x)} \\ &= \overline{\alpha \cdot (y^* \cdot x)} \\ &= \overline{\alpha} \cdot \overline{y^* \cdot x} \\ &= \overline{\alpha} \cdot \overline{x^* y} \\ &= \overline{\alpha} \cdot x^* y. \end{aligned}$$

(3) skipped

(4) Given $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{C}^n$, $x^* \cdot x = \overline{x_1} \cdot x_1 + \cdots + \overline{x_n} \cdot x_n$.

Since each $x_i \cdot \overline{x_i} \in \mathbb{R}$ and $x_i \cdot \overline{x_i} \geq 0$ for all $i = 1, \dots, n$ then $x_1 \cdot \overline{x_1} + \cdots + x_n \cdot \overline{x_n} = x^* \cdot x \geq 0$.

Clearly, if $x = 0$, then $x^* \cdot x = 0$. Conversely, if $x^* \cdot x = 0$, then since $x^* \cdot x = |x_1|^2 + |x_2|^2 + \cdots + |x_n|^2 = 0$ then $|x_i| = 0$ for all $i = 1, 2, \dots, n$. Therefore, $x_i = 0$ for all $i = 1, 2, \dots, n$ and so $x = 0$. □

Remark. Recall that if $z = a + bi \in \mathbb{C}$ where $a, b \in \mathbb{R}$ then $z \cdot \bar{z} = (a + bi)(a - bi) = a^2 + b^2 \in \mathbb{R}$ and $|z|^2 = a^2 + b^2 \geq 0$ where $|z| = \sqrt{a^2 + b^2}$.

41 Norms on \mathbb{R}^n and \mathbb{C}^n

Definition. Let $x \in \mathbb{C}^n$. Then it's (induced) norm is

$$||x|| = \sqrt{x^* x}$$

If $x \in \mathbb{R}^n$, then $||x|| = \sqrt{x^T \cdot x}$.

41.1 Properties of the induced norms

Let $x \in \mathbb{C}^n$.

(1) $\|x\| \geq 0$ and $\|x\| = 0 \iff x = 0$.

(2) For any $\alpha \in \mathbb{C}$, $\|\alpha \cdot x\| = |\alpha| \cdot \|x\|$ where $|\alpha|$ is the modulus of $\alpha \in \mathbb{C}$ for $\alpha = a + bi \in \mathbb{C}$.

If $x \in \mathbb{R}$, then $|\alpha|$ is the absolute value of α .

(3) Cauchy-Schwartz Inequality.

$$|x^* \cdot y| \leq \|x\| \cdot \|y\|, \quad \forall x, y \in \mathbb{C}^n$$

(4) Triangle Inequality for $\|\cdot\|$.

$$\|x + y\| \leq \|x\| + \|y\|.$$

Proof. (1) Since $x^* \cdot x \geq 0$ and $x^* \cdot x \in \mathbb{R}$ for all $x \in \mathbb{C}^n$ then $\|x\| = \sqrt{x^* \cdot x} \geq 0$.
Further,

$$\begin{aligned} \|x\| = 0 &\iff \sqrt{x^* \cdot x} = 0 \\ &\iff x^* \cdot x = 0 \\ &\iff x = 0. \end{aligned}$$

where the last \iff follows from properties of the inner product.

(2) Let $\alpha \in \mathbb{C}$. Then

$$\begin{aligned} \|\alpha \cdot x\|^2 &= (\alpha \cdot x)^* \cdot (\alpha \cdot x) \\ &= \alpha(\alpha x)^* \cdot (x) \\ &= \alpha \cdot \bar{\alpha}(x^* \cdot x) \\ &= |\alpha|^2 \cdot \|x\|^2. \end{aligned}$$

Therefore, taking square roots we have $\|\alpha \cdot x\| = |\alpha| \cdot \|x\|$.

Remark. Before proving the Cauchy-Schwartz inequality we need a couple of facts. Let $z = a + bi$ where $a, b \in \mathbb{R}$.

(i) $z \cdot \bar{z} = (a + bi)(a - bi) = a^2 + b^2 = |z|^2$.

(ii) $z + \bar{z} = (a + bi) + (a - bi) = 2a \leq 2|z|$, which implies that $z + \bar{z} \leq 2|z|$.

(3) Proof of Cauchy-Schwartz Inequality.

When $x = 0$, then $|x^*y| = 0$ and $\|x\| \cdot \|y\| = 0$. Then the C-S inequality holds.

Now consider the case when $x \neq 0$ and therefore $\|x\| > 0$.

Define α as

$$\alpha = \frac{x^* \cdot y}{\|x\|^2}$$

for a given $y \in \mathbb{C}$. Note that since $\|x\| \geq 0$ then α is well defined. Then

$$\begin{aligned} x^* \cdot (\alpha x - y) &= \alpha \cdot x^* \cdot x - x^* \cdot y \\ &= \alpha \cdot \|x\|^2 - x^* \cdot y \\ &= \frac{x^* \cdot y}{\|x\|^2} \|x\|^2 - x^* \cdot y \\ &= 0. \end{aligned}$$

Note that $\|\alpha \cdot x - y\|^2 \geq 0$ and

$$\begin{aligned} \|\alpha \cdot x - y\|^2 &= (\alpha \cdot x - y)^* \cdot (\alpha x - y) \\ &= [(\alpha x)^* - y^*] \cdot (\alpha x - y) \\ &= [\bar{\alpha} x^* - y^*] \cdot (\alpha x - y) \\ &= \bar{\alpha} \cdot x^* \cdot (\alpha x - y) - y^* \cdot (\alpha x - y) \\ &= y^* \cdot y - \alpha y^* \cdot x \end{aligned}$$

where the last equality follows because we know from earlier that $x^* \cdot (\alpha x - y) = 0$. Therefore

$$y^* \cdot y - \alpha \cdot y^* \cdot x = \|\alpha x - y\|^2 \geq 0$$

and so

$$\|y\|^2 \geq \alpha \cdot y^* \cdot x = \frac{x^* \cdot y}{\|x\|^2} y^* \cdot x.$$

Multiplying by $\|x\|^2$ on both sides the the previous inequality we have

$$\begin{aligned} \|x\|^2 \cdot \|y\|^2 &\geq (x^* \cdot y)(y^* \cdot x) \\ &= (x^* \cdot y)(x^* \cdot y) \\ &= (x^* \cdot y) \cdot \overline{x^* \cdot y} \\ &= \|z\|^2 \\ &= |x^* y|^2. \end{aligned}$$

Therefore, $\|x\|^2\|y\|^2 \geq |x^* \cdot y|^2$ and taking square roots on both sides we have

$$\|x\| \cdot \|y\| \geq |x^* \cdot y|.$$

(4) Proof of Triangle Inequality

$$\begin{aligned} \|x + y\|^2 &= (x + y)^*(x + y) \\ &= (x^* + y^*) \cdot (x + y) \\ &= (x^* \cdot x) + (x^* \cdot y) + (y^* \cdot x) + (y^* \cdot y) \\ &= \|x\|^2 + x^* \cdot y + x^* \cdot y + \|y\|^2 \\ &= \|x\|^2 + 2 \cdot \operatorname{Re}(x^* \cdot y) + \|y\|^2 \\ &\leq \|x\|^2 + 2 \cdot |x^* \cdot y| + \|y\|^2 \\ &\leq \|x\|^2 + 2\|x\| \cdot \|y\| + \|y\|^2 \\ &= (\|x\| + \|y\|)^2. \end{aligned}$$

Therefore, $\|x + y\|^2 \leq (\|x\| + \|y\|)^2$ and taking the square root of both sides we have

$$\|x + y\| \leq \|x\| + \|y\|.$$

□

42 General Inner Product Space

Definition. An **inner product** on a real or complex vector space V is a function

$$\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R} \text{ or } \mathbb{C}$$

that satisfies the following four properties:

- (1) $\langle x, x \rangle \geq 0$ for $x \in V$, and $\langle x, x \rangle = 0 \iff x = 0$.
- (2) $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$ for all $x, y, z \in V$.
- (3) $\langle x, \alpha \cdot y \rangle = \alpha \langle x, y \rangle$ for all $x, y \in V$ and scalar α in \mathbb{R} or \mathbb{C} .
- (4) $\langle y, x \rangle = \overline{\langle x, y \rangle}$ for all $x, y \in V$.

Then V , along with the inner product $\langle \cdot, \cdot \rangle$ is an **inner product space**. (IPS)

42.1 Implications of Inner Products

(i) It can be shown that

$$\langle x, \alpha_1 v_1 + \cdots + \alpha_p v_p \rangle = \alpha_1 \langle x, v_1 \rangle + \alpha_2 \langle x, v_2 \rangle + \cdots + \alpha_p \langle x, v_p \rangle.$$

(ii) The induced norm is

$$||x|| = \sqrt{\langle x, x \rangle}$$

for all $x \in V$.

It can be shown that:

- $||x|| \geq 0$, and $||x|| = 0 \iff x = 0$
- $||\alpha x|| = |\alpha| \cdot ||x||$ for all $x \in V$ and scalar α .
- $|\langle x, y \rangle| \leq ||x|| \cdot ||y||$ (Cauchy Schwartz)
- $||x + y|| \leq ||x|| + ||y||$ (Triangle Inequality)

(iii) $\langle x, 0 \rangle$ for all $x \in V$.

Proof. Note that

$$\begin{aligned} \langle x, 0 \rangle &= \langle x, y + (-y) \rangle \\ &= \langle x, y \rangle + \langle x, (-1) \cdot y \rangle \\ &= \langle x, y \rangle + (-1) \langle x, y \rangle \\ &= 0. \end{aligned}$$

□

Example. Inner Products

(1) Let $V = \mathbb{R}$ and let $A \in \mathbb{R}^{n \times n}$ be invertible. For any $x, y \in \mathbb{R}^n$

$$\langle x, y \rangle_A = (Ax)^T(Ay)$$

Note that if $A = I$, the identity matrix, then the above is the standard inner product.

Claim: $\langle \cdot, \cdot \rangle_A$ is an inner product on V .

Proof. It's easy to show $\langle \cdot, \cdot \rangle_A$ satisfies conditions 2) – 4). Only show condition 1).

Since $\langle x, x \rangle_A = (Ax)^T(Ax) = z^T z$ where $z = Ax \in \mathbb{R}^n$ and $z^T z \geq 0$, then $\langle x, x \rangle_A \geq 0$. Further, if $x = 0$, then $\langle x, x \rangle_A = 0$.

Conversely, if $\langle x, x \rangle_A = 0$, then $z^T z = 0$, where $z = A \cdot x$, which implies that $z = 0$ and so $Ax = 0$. Since A is invertible, then $x = 0$, then $\langle x, x \rangle_A = 0 \iff x = 0$. Therefore, $\langle \cdot, \cdot \rangle_A$ is an inner product. □

(2) Let $V = \mathbb{R}^{n \times n}$, the space of all real matrices of size $n \times n$. For any $A, B \in V$, let

$$\langle A, B \rangle = \text{trace}(A^T \cdot B).$$

.

Claim: $\langle \cdot, \cdot \rangle$ is an inner product.

Proof. Fill in. □

The induced norm is

$$\|A\|_F = \sqrt{\langle A, A \rangle} = \sqrt{\text{trace}(A^T \cdot A)}.$$

Here, $\|\cdot\|_F$ is the Frobenius norm for a matrix.

Definition. Let $\|\cdot\|$ be a norm on V . Then a vector $x \in V$ is a **unit vector** if $\|x\| = 1$.

Fact: A unit vector must be a non-zero vector.

42.2 Normalization of a Vector

Given any non-zero vector $z \in V$,

$$\frac{z}{\|z\|}$$

is a unit vector.

Proof.

$$\begin{aligned} \left\| \frac{z}{\|z\|} \right\| &= \left\| \frac{1}{\|z\|} \cdot z \right\| \\ &= |\alpha| \cdot \|z\| \\ &= \alpha \cdot \|z\| \\ &= \frac{1}{\|z\|} \cdot \|z\| \\ &= 1 \end{aligned}$$

where $\alpha = \frac{1}{\|z\|} > 0$. Therefore $\frac{z}{\|z\|}$ is a unit vector. □

43 Orthogonal Vectors

Definition. Let $(V, \langle \cdot, \cdot \rangle)$ be a real/complex inner product space. Given $x, y \in V$, x is **orthogonal** to y (i.e., $x \perp y$) if $\langle x, y \rangle = 0$.

Remark. Facts about Orthogonal Vectors

(1) If $x \perp y$, then $y \perp x$?

Claim: $\langle x, y \rangle = 0 \implies \langle y, x \rangle = 0$.

Proof. Since $\langle y, x \rangle = \overline{\langle x, y \rangle} = \bar{0} = 0$, then $y \perp x$. □

(2) Is the zero vector orthogonal to any vector in V ? Answer: Yes because $\langle x, 0 \rangle = 0$ for all $x \in V$. Thus, $0 \perp x$ for all $x \in V$.

Definition. Let $S = \{v_1, \dots, v_p\}$ be a finite set in an inner product space V . We say S is an **orthogonal set** if

$$\langle v_i, v_j \rangle = 0 \quad \forall i \neq j$$

or

$$v_i \perp v_j \quad \forall v_i, v_j \in S \text{ with } i \neq j.$$

If in addition, each $v_i \in S$ is a unit vector, then S is an **orthonormal set** (or simply an O.N. set).

Remark. Note that $\|v_i\| = 1 \iff \|v_i\|^2 = 1$. Therefore, S is orthonormal if and only if:

$$\langle v_i, v_j \rangle = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} = \delta_{ij}$$

where δ_{ij} is the delta function.

Example. Let $V = \mathbb{R}^3$ and $B = \{e_1, e_2, e_3\}$ be the standard basis for \mathbb{R}^3 . Show that B is an O.N. set.

Proof. For any i, j from $\{1, 2, 3\}$, we have

$$\langle e_i, e_j \rangle = e_i^T \cdot e_j = \begin{cases} 0, & i \neq j, \\ 1, & i = j \end{cases}.$$

Therefore, B is orthonormal. □

44 Properties of Orthogonal Sets

Lemma. *An orthogonal set of non-zero vectors is linearly independent.*

Proof. Let $\{u_1, \dots, u_p\}$ be an orthogonal set where each $u_i \neq 0$.

Suppose there are scalars $\alpha_1, \dots, \alpha_p$ such that

$$\alpha_1 u_1 + \dots + \alpha_p u_p = 0.$$

We want to show $\alpha_1 = 0, \alpha_2 = 0, \dots, \alpha_p = 0$.

Note that for each u_i $\langle u_i, 0 \rangle = 0$ and so

$$\langle u_i, \alpha_1 u_1 + \dots + \alpha_i u_i + \dots + \alpha_p u_p \rangle = 0.$$

The left hand side is:

$$\alpha_1 \langle u_i, u_1 \rangle + \dots + \alpha_i \langle u_i, u_i \rangle + \dots + \alpha_p \langle u_i, u_p \rangle.$$

Since $\{u_1, \dots, u_p\}$ is an orthogonal set then $\langle u_i, u_j \rangle = 0$ if $i \neq j$. Therefore, the left hand side equals $\alpha_i \langle u_i, u_i \rangle = \alpha_i \|u_i\|^2$. (Since $u_i \neq 0$ then $\langle u_i, u_i \rangle > 0$). Since $u_i \neq 0$ then $\|u_i\|^2 > 0$ and then since $\alpha_i \cdot \|u_i\|^2 = 0$ then $\alpha_i = 0$.

Therefore, $\alpha_1 = 0, \alpha_2 = 0, \dots, \alpha_p = 0$ and so $\{u_1, \dots, u_p\}$ is linearly independent. \square

Remark. In the above proof we can repeat the process for each i to show that $\alpha_i = 0$.

Corollary. An O.N. set is linearly independent.

Proof. This is because each vector in an O.N. set is a unit vector and must be non-zero. Then by the preceding lemma, an O.N. set is linearly independent. \square

45 Orthonormal Basis

Definition. Let B be a basis for a finite dimensional inner product space (IPS). If B is an orthonormal set, then B is an **orthonormal basis**.

Corollary. Let V be an n -dimensional IPS. An O.N. set of n vectors in V is an O.N. basis for V .

Proof. Let B be an O.N. set of n vectors. By the above corollary, B is linearly independent. If V is of $\dim n$ and B has n vectors then B is a maximally linearly independent set in V . Therefore, B is a basis for V and so B is an orthonormal basis for V . \square

Proposition. Let $B = \{u_1, \dots, u_n\}$ be an O.N. basis for V . Given a vector $x \in V$, x can be written as

$$x = \langle u_1, x \rangle u_1 + \dots + \langle u_n, x \rangle u_n$$

where $\langle u_i, x \rangle$ are the Fourier coefficients and writing x this way is a special case of the Fourier Expansion.

Remark. The coordinate vector of x with respect to basis B can be written as

$$[x]_B = \begin{bmatrix} \langle u_1, x \rangle \\ \langle u_2, x \rangle \\ \vdots \\ \langle u_n, x \rangle \end{bmatrix}.$$

Proof. Since $B = \{u_1, \dots, u_p\}$ is an O.N. basis for V then for any given vector $x \in V$, there exists unique scalars $\alpha_1, \dots, \alpha_n$ such that

$$x = \alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n.$$

For each u_i (i.e., fix i),

$$\begin{aligned} \langle u_i, x \rangle &= \langle u_i, \alpha_1 u_1 + \dots + \alpha_i u_i + \dots + \alpha_n u_n \rangle \\ &= \alpha_1 \langle u_i, u_1 \rangle + \dots + \alpha_i \langle u_i, u_i \rangle + \dots + \alpha_n \langle u_i, u_n \rangle. \end{aligned}$$

Since B is O.N. then

$$\langle u_i, u_j \rangle = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases}.$$

Therefore, $\langle u_i, x \rangle = \alpha_i$ for all $i = 1, \dots, n$. (Repeat the above process for each fixed i).

□

46 Idea for Gram-Schmidt Procedure/Process

Goal: Let $B = \{v_1, \dots, v_n\}$ be a basis for an n -dimensional IPS V . Construct an O.N. basis from B .

Remark. In general, there could be multiple O.N. bases and the Gram-Schmidt process gives one O.N. basis that depends on what basis you start from.

We will use the following preliminary result below:

Lemma. *Any subset of a linearly independent set is also linearly independent.*

Gram-Schmidt Process: To find an orthonormal basis for an n -dimensional IPS V from $B = \{v_1, \dots, v_n\}$, the Gram-Schmidt process uses the following process:

Step 1 Let $W_1 = \text{span}\{v_1\}$.

Clearly, $\{v_1\}$ spans W_1 , and is linearly independent because it is a subset of B . Since $\{v_1\}$ is a basis for W_1 then $\dim W_1 = 1$ and therefore W_1 is a one-dimensional subspace of V .

Step 2 Let $W_2 = \text{span}\{v_1, v_2\}$.

Likewise, $\{v_1, v_2\}$ is a basis for W_2 and therefore $\dim W_2 = 2$ and so W_2 is a two dimensional subspace of V . Also, W_1 is a one-dimensional subspace of W_2 (i.e., $W_1 \subseteq W_2$).

Steps 3, ..., n Similarly, we can define $W_2, W_3, \dots, W_k, \dots, W_n$, i.e., $W_k = \text{span}\{v_1, v_2, \dots, v_k\}$ is an k -dimensional subspace of V . Finally, $W_n = \text{span}\{v_1, \dots, v_n\} = V$.

Note that $W_1 \subseteq W_2 \subseteq W_3 \subseteq \dots \subseteq W_k \subseteq \dots \subseteq W_n = V$.

Remark. The idea of the GS process is to construct an O.N. basis for W_1 and use that to construct O.N. basis for W_2 , ... O.N. basis for $W_n = V$.

Definition. Consider the subspace W_k . Suppose $\{u_1, \dots, u_k\}$ is an O.N. basis for W_k . Consider an arbitrary vector $x \in V$ (not necessarily in W_k). We call

$$p = \langle u_1, x \rangle u_1 + \dots + \langle u_k, x \rangle u_k$$

the **orthogonal projection** of x onto W_k denoted by $\text{proj}_{\mathbf{W}_k} \mathbf{x}$.

Properties of $\text{proj}_{\mathbf{W}_k} \mathbf{x}$.

- (1) $\text{proj}_{\mathbf{W}_k} \mathbf{x} \in W_k$ because x is a linear combination of u_1, \dots, u_k , which are basis vectors for W_k .
- (2) $(x - \text{proj}_{\mathbf{W}_k} \mathbf{x})$ is orthogonal to each u_i , $i = 1, \dots, k$.

Proof. For each u_i , $i = 1, \dots, k$ compute

$$\langle u_i, x - \text{proj}_{\mathbf{W}_k} \mathbf{x} \rangle = \langle u_i, x \rangle - \langle u_i, p \rangle$$

where

$$\begin{aligned} \langle u_i, p \rangle &= \langle u_i, \alpha_1 u_1 + \dots + \alpha_k u_k \rangle \\ &= \alpha_i \langle u_i, u_i \rangle \\ &= \alpha_i \\ &= \langle u_i, x \rangle. \end{aligned}$$

with $\alpha_k = \langle u_i, x \rangle$ and $\langle u_i, u_i \rangle = 1$. Therefore, $\langle u_i, x - \text{proj}_{\mathbf{W}_k} \mathbf{x} \rangle = 0$ and so $x - \text{proj}_{\mathbf{W}_k} \mathbf{x} \perp u_i$ for all $i = 1, \dots, k$.

Therefore $x - \text{proj}_{\mathbf{W}_k} \mathbf{x}$ is orthogonal to any vector in W_k .

□

47 Gram-Schmidt Procedure/Process

Goal: Construct an O.N. basis from a basis $B = \{v_1, \dots, v_n\}$ for an n -dimensional IPS V .

47.1 Preliminary Results

We will use the following preliminary results:

(1)

$$\begin{aligned} W_1 &= \text{span}\{v_1\} \\ W_2 &= \text{span}\{v_1, v_2\} \\ &\vdots \\ W_k &= \text{span}\{v_1, \dots, v_k\} \\ &\vdots \\ W_n &= \text{span}\{v_1, \dots, v_n\} = V \end{aligned}$$

(2) Projection of $x \in V$ onto W_k , assuming $\{u_1, \dots, u_k\}$ is an O.N. basis for W_k is

$$\text{proj}_{W_k} \mathbf{x} = \langle u_1, x \rangle u_1 + \dots + \langle u_k, x \rangle u_k.$$

Then,

- (1) $\text{proj}_{W_k} \mathbf{x} \in W_k$
- (2) $(x - \text{proj}_{W_k} \mathbf{x}) \perp (u_i)$ for all $i = 1, \dots, k$.

47.2 Gram-Schmidt Procedure (Inductive Process)

Step 1 Local goal in this step is to construct an O.N. basis for W_1 . Since $W_1 = \text{span}\{v_1\}$ and $v_1 \neq 0$ then $\{v_1\}$ is a basis for W_1 . Using the normalization of v_1 we have

$$u_1 = \frac{v_1}{\|v_1\|} \in W_1$$

and u_1 is a unit vector. Therefore, $\{u_1\}$ is an orthonormal basis for W_1 .

Step 2 Local goal is to construct O.N. basis for $W_2 = \text{span}\{v_1, v_2\}$, where W_1 is a subspace of W_2 .

From step 1, $\{u_1\}$ is an O.N. basis for $W_1 \subseteq W_2$. To generate a vector in W_2 that is orthogonal to u_1 , we consider

$$\begin{aligned}\tilde{u}_2 &= v_2 - \text{proj}_{\mathbf{W}_1} \mathbf{v}_2 \\ &= v_2 - \langle v_2, u_1 \rangle \cdot u_1 \in W_1.\end{aligned}$$

Properties of \tilde{u}_2 :

- (1) $\tilde{u}_2 \in W_2$. Since \tilde{u}_2 is a linear combination of v_1 and v_2 then $\tilde{u}_2 \in W_2$.
- (2) $\tilde{u}_2 \neq 0$. This is because otherwise, $v_2 = \langle v_2, u_1 \rangle \cdot u_1 =$ a multiple of u_1 . (as $u_1 = \frac{v_1}{\|v_1\|}$).
- (3) $\tilde{u}_2 \perp u_1$. Since $\{u_1, \tilde{u}_2\}$ is an orthogonal set, and both $\tilde{u}_2 \neq 0$ and $u_1 \neq 0$.

Using the normalization of \tilde{u}_2 we have

$$u_2 = \frac{\tilde{u}_2}{\|\tilde{u}_2\|} = \frac{v_2 - \langle u_1, v_2 \rangle u_1}{\|v_2 - \langle u_1, v_2 \rangle u_1\|} \in W_2.$$

Since $\{u_1, u_2\} \subset W_2$, $u_1 \perp u_2$, and $\|u_1\| = \|u_2\| = 1$, then $\{u_1, u_2\}$ is an orthonormal set in W_2 and therefore linearly independent. Thus, $\{u_1, u_2\}$ is an O.N. basis W_2 where

$$u_1 = \frac{v_1}{\|v_1\|}, \quad u_2 = \frac{v_2 - \langle u_1, v_2 \rangle u_1}{\|v_2 - \langle u_1, v_2 \rangle u_1\|}.$$

Step $k + 1$: Where $1 \leq k \leq n - 1$. Local goal is to construct an O.N. basis for W_{k+1} .

Induction Hypothesis: W_k has an orthonormal basis given by $\{u_1, \dots, u_k\}$.

Now define

$$\begin{aligned}\tilde{u}_{k+1} &= v_{k+1} - \text{proj}_{\mathbf{W}_k} \mathbf{v}_{k+1} \\ &= v_{k+1} - (\langle u_1, v_{k+1} \rangle u_1 + \dots + \langle u_k, v_{k+1} \rangle u_k) \in W_{k+1}\end{aligned}$$

Properties of \tilde{u}_{k+1}

- (1) $\tilde{u}_{k+1} \in W_{k+1}$. This is because $v_{k+1} \in W_{k+1}$ and $\text{proj}_{\mathbf{W}_k} \mathbf{v}_{k+1} \in W_k \subset W_{k+1}$ such that $v_{k+1} - \text{proj}_{\mathbf{W}_k} \mathbf{v}_{k+1} = \tilde{u}_{k+1}$ and $v_{k+1} - \text{proj}_{\mathbf{W}_k} \mathbf{v}_{k+1} \in W_{k+1}$.
- (2) $\tilde{u}_{k+1} \neq 0$. Using a result from HW # 8.
- (3) $\tilde{u}_{k+1} \perp u_i$ for all $i = 1, \dots, k$

Using the normalization of \tilde{u}_{k+1} we have

$$u_{k+1} = \frac{\tilde{u}_{k+1}}{\|\tilde{u}_{k+1}\|} = \frac{v_{k+1} - (\langle u_1, v_{k+1} \rangle u_1 + \dots + \langle u_k, v_{k+1} \rangle u_k)}{\|v_{k+1} - (\langle u_1, v_{k+1} \rangle u_1 + \dots + \langle u_k, v_{k+1} \rangle u_k)\|}$$

Therefore, $\{u_1, \dots, u_k, u_{k+1}\}$ is an O.N. set in W_{k+1} , which are all orthogonal by assumption and 3) above. Since $\{u_1, \dots, u_k, u_{k+1}\}$ is an O.N. then it is linearly independent and therefore $\{u_1, \dots, u_k, u_{k+1}\}$ is an O.N. basis for W_{k+1} .

By the induction principle, we obtain an orthonormal basis $\{u_1, \dots, u_n\}$ for $W_n = V$.

47.3 Summary of Gram-Schmidt Procedure

$$(1) \quad u_1 = \frac{v_1}{\|v_1\|}$$

$$(2) \quad u_2 = \frac{\tilde{u}_2}{\|\tilde{u}_2\|} = \frac{v_2 - \langle u_1, v_2 \rangle u_1}{\|v_2 - \langle u_1, v_2 \rangle u_1\|}$$

$$(k) \quad u_k = \frac{v_k - \sum_{i=1}^{k-1} \langle u_i, v_k \rangle u_i}{\left\| v_k - \sum_{i=1}^{k-1} \langle u_i, v_k \rangle u_i \right\|}$$

$$(n) \quad u_n = \frac{v_n - \sum_{i=1}^{n-1} \langle u_i, v_n \rangle u_i}{\left\| v_n - \sum_{i=1}^{n-1} \langle u_i, v_n \rangle u_i \right\|}$$

Example. Let $V = \mathbb{R}^3$ and $B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$. Find an O.N. basis for \mathbb{R}^3 from B .

Solution:

$$u_1 = \frac{v_1}{\|v_1\|} = v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{because } \|v_1\| = 1.$$

$$\text{Then } v_2 - \langle u_1, v_2 \rangle \cdot u_1 = v_2 - u_1 = v_2 - v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

$$\text{Normalizing we have } u_2 = \frac{v_2 - \langle u_1, v_2 \rangle u_1}{\|v_2 - \langle u_1, v_2 \rangle u_1\|} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

$$\text{Then } v_3 - \langle u_1, v_3 \rangle u_1 - \langle u_2, v_3 \rangle u_2 = v_3 - u_1 - u_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

$$\text{Therefore, } u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Thus, $\{u_1, u_2, u_3\}$ is an O.N. basis for \mathbb{R}^3

48 Orthogonal and Unitary Matrices

Definition.

A matrix $P \in \mathbb{R}^{n \times n}$ is **orthogonal** if its columns form an O.N. basis for \mathbb{R}^n .

A matrix $U \in \mathbb{C}^{n \times n}$ is **unitary** if its columns form an O.N. basis for \mathbb{C}^n .

Proposition. *Let $P \in \mathbb{R}^{n \times n}$. Then P is orthogonal if and only if the columns of P , $\{p_1, p_2, \dots, p_n\}$, is an O.N. basis for \mathbb{R}^n .*

Proof. \Leftarrow Let $P = [p_1 \ p_2 \ \cdots \ p_n]$, where p_k is the k^{th} column of P . The condition that $\{p_1, p_2, \dots, p_n\}$, is an O.N. basis for \mathbb{R}^n is equivalent to

$$\langle p_i, p_j \rangle = p_i^T p_j = \begin{cases} 0, & \text{if } i \neq j, \\ 1, & \text{if } i = j \end{cases}.$$

Then if we multiply P^T by P we have

$$\begin{aligned} P^T \cdot P &= \begin{bmatrix} p_1^T \\ p_2^T \\ \vdots \\ p_n^T \end{bmatrix} [p_1 \ p_2 \ \cdots \ p_n] \\ &= \begin{bmatrix} p_1^T & p_1^T p_2 & \cdots & p_1^T p_n \\ p_2^T & p_2^T p_2 & \cdots & p_2^T p_n \\ \vdots & \vdots & \ddots & \vdots \\ p_n^T & p_n^T p_2 & \cdots & p_n^T p_n \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \end{aligned}$$

By definition, P is orthogonal if $P^T P = I_n$, thus if $\{p_1, p_2, \dots, p_n\}$, is an O.N. basis for \mathbb{R}^n , then P is orthogonal.

(\Rightarrow) Conversely, if $P^T P = I_n$ then

$$\langle p_i, p_j \rangle = p_i^T p_j = \begin{cases} 0, & \text{if } i \neq j, \\ 1, & \text{if } i = j. \end{cases}$$

Therefore, $\{p_1, p_2, \dots, p_n\}$ is an O.N. basis for \mathbb{R}^n and so P is orthogonal matrix.

In summary, P is an orthogonal matrix $\iff P^T \cdot P = I_n$. \square

Theorem 48.1. Let $P \in \mathbb{R}^{n \times n}$. Then P is an orthogonal matrix if and only if $\|P \cdot x\| = \|x\|$ for all $x \in \mathbb{R}^n$, where $\|x\| = \sqrt{x^T x}$ is the induced norm (or the 2-norm/Euclidean norm).

Here, P is an **isometry**. (This means it does not change the length of the norm).

Proof. "only if": Suppose P is orthogonal. Then $P^T P = I_n$. Then for a given $x \in \mathbb{R}^n$,

$$\begin{aligned} \|Px\|^2 &= (Px)^T (Px) \\ &= x^T \cdot (P^T \cdot P)x \\ &= x^T \cdot x \\ &= \|x\|^2. \end{aligned}$$

Then, taking square roots we have $\|Px\| = \|x\|$.

"If": Suppose $\|Px\| = \|x\|$ for all $x \in \mathbb{R}^n$. Then

$$\begin{aligned} \|Px\|^2 = \|x\|^2 &\iff (Px)^T (Px) = x^T x \\ &\iff x^T P^T \cdot Px - x^T Ix = 0 \\ &\iff x^T \cdot (P^T P - I) \cdot x = 0. \end{aligned}$$

In the above equality we cannot conclude that $A = 0$. To see this, suppose $x^T \cdot A \cdot x = 0$ for all $x \in \mathbb{R}^n$. Let $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Then A is skew symmetric (i.e., $A^T = -A$). Then for any $x \in \mathbb{R}^2$. Then

$$x^T A x = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix} = x_1 x_2 - x_1 x_2 = 0.$$

Now let $x = e_i$, $i = 1, \dots, n$ where e_i is the i^{th} standard basis vector of \mathbb{R}^n .

Then $Px = P \cdot e_i = p_i$, where p_i is the i^{th} column of P and therefore $\|Px\|^2 = \|p_i\|^2$ and $\|x\|^2 = \|e_i\|^2 = 1$. Since $\|p_i\|^2 = 1$ then $p_i^T p_i = 1$ for all $i = 1, \dots, n$.

Let $x = e_i + e_j$ where $i \neq j$. Then $e_i^T e_j = 0$ and $P \cdot x - P(e_i + e_j) = P e_i + P e_j = P_i + P_j$. Further

$$\begin{aligned} \|Px\|^2 &= (p_i + p_j)^T \cdot (p_i + p_j) \\ &= p_i^T p_i + p_i^T p_j + p_j^T p_i + p_j^T p_j \\ &= 2 + 2p_i^T p_j. \end{aligned}$$

Also,

$$\begin{aligned} \|x\|^2 &= (e_i + e_j)^T (e_i + e_j) \\ &= e_i^T e_i + e_i^T e_j + e_j^T e_i + e_j^T e_j \\ &= 2. \end{aligned}$$

Since $\|Px\|^2 = \|x\|^2$ then $2 + 2 \cdot P_i^T P_j = 2$ and subtracting 2 and dividing by 2 on both side we have $P_i^T P_j = 0$ for all $i \neq j$. Thus, P is an orthogonal matrix. \square

The above theorem also holds for unitary matrices.

Example. (Isometries)

(1) Let $P = I_n$, the identity matrix.

(2) Let $P = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}_{2 \times 2}$, where $\theta \in (-\pi, \pi]$ in \mathbb{R}^2 .

Then

$$P^T \cdot P = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2.$$

Thus, P is orthogonal.

Remark. In (2) above the matrix $P = P(\theta)$ is a rotation matrix and $P(\theta)$ is commutative in the sense that $P(\theta_1) \cdot P(\theta_2) = P(\theta_1 + \theta_2)$.

49 Global Properties of Orthogonal (or Unitary) Matrices

Let $O(n)$ be the set of all $n \times n$ orthogonal matrices. ($O(n) \subseteq \mathbb{R}^{n \times n}$)

Properties of $O(n)$ under the matrix product “.”

(1) Let $A, B \in O(n)$. **Claim:** $A \cdot B \in O(n)$.

Proof. Since $A, B \in O(n)$ then $A^T A = B^T B = I$. Then

$$(A \cdot B)^T \cdot (A \cdot B) = B^T \cdot A^T \cdot A \cdot B = I_n.$$

Therefore, $A \cdot B$ is orthogonal and so $A \cdot B \in O(n)$. Thus, $O(n)$ is closed under “.”. \square

(2) The matrix product “.” is associative, i.e., $(AB) \cdot C = A \cdot (BC)$.

- (3) The identity matrix $I_n \in O(n)$.
- (4) For any $A \in O(n)$, $A^{-1} \in O(n)$.

Proof. Since $A \in O(n)$ then $A^T \cdot A = I_n$ and so $A^{-1} = A^T$. Therefore,

$$(A^{-1})^T(A^{-1}) = (A^T)^T \cdot A^{-1} = A \cdot A^{-1} = I_n.$$

Therefore, $A^{-1} \in O(n)$. □

Remark. From the above properties, $(O(n), “\cdot”)$ is a group, more specifically a Lie group.

For any $P \in O(n)$, $P^T \cdot P = I_n$ and $\det(P^T \cdot P) = (\det P)^2$ and $\det I_n = 1$. Therefore, $\det P = \pm 1$.

The Special $O(n)$ group is the set $SO(n) = \{P \in O(n) : \det P = 1\}$. This is the Special Orthogonal Group of Order n . When $n = 2$, we get the space of all rotation matrices. $SU(n)$ is the special unitary group of order n .

50 Projection

Let V be a vector space, and X and Y be two subspaces of V .

Remark. Recall that $X + Y = \{x + y : x \in X \text{ and } y \in Y\}$ and $X \cap Y = \{v : v \in X \text{ and } v \in Y\}$ are subspaces of V and $Y \subseteq X + Y$ and $X \subseteq X + Y$.

If X and Y are finite dimensional, then $\dim(X + Y) = \dim(X) + \dim(Y) - \dim(X \cap Y)$

Definition. The subspaces X and Y are **complementary** subspaces of V if

- (1) $V = X + Y$
- (2) $X \cap Y = \{0\}$

In this case, V is the **direct sum** of X and Y , and it is written as

$$V = X \oplus Y.$$

Theorem 50.1. Let X and Y be subspaces of V with bases $B_x = \{u_1, \dots, u_p\}$ and $B_y = \{v_1, \dots, v_r\}$ for X and Y , respectively. Then the following are equivalent:

1. $V = X \oplus Y$
2. For any $v \in V$, there exists “unique” vectors $x \in X$ and $y \in Y$ such that $V = X + Y$.
3. $B_x \cap B_y = \emptyset$ and $B_x \cup B_y$ is a basis for V .

Proof. We will prove the following chain of implications: $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$.

(1 \Rightarrow 2) Since $V = X \oplus Y$, then $V = X + Y$, and $X \cap Y = \{0\}$. For an arbitrary vector $v \in V$, $v \in V = X + Y$ and therefore, there exists an $x \in X$ and a $y \in Y$ such that $v = x + y$. To prove uniqueness, suppose that there exists an $x' \in X$ and a $y' \in Y$ such that $x' + y' = v$. Since $x + y = v$, then $x + y = v = x' + y'$. Therefore, $x - x' = y' - y$. Since $x \in X$ and $x' \in X$ and X is a subspace, then $x - x' \in X$ (because X is closed under “+” and “.”). Likewise, $y - y' \in Y$ because $y \in Y$ and $y' \in Y$ and Y is a subspace. Since $x - x' = y - y'$ and $X \cap Y = \{0\}$, then $x - x' = y - y' = 0$, which implies that $x' = x$ and $y' = y$. This proves uniqueness.

(2 \Rightarrow 3) First we show that $B_X \cap B_Y = \emptyset$. Suppose not. Then there exists a $v \in B_X \cap B_Y$ such that $v \neq 0$ and $v \in B_X$ and $v \in B_Y$. Also, $v \in X \cap Y$. However, $v = v + 0 = 0 + v$ where $v \in X$ and $0 \in Y$ in the first inequality and $0 \in X$ and $v \in Y$ in the second equality. However, this contradicts that unique decomposition in (2).

Now we show that $B_X \cup B_Y$ is a basis for V . Since $B_X \cap B_Y = \emptyset$ then $B_X \cup B_Y = \{u_1, \dots, u_p, v_1, \dots, v_r\}$. To show that $B_X \cup B_Y$ spans V , we see that for all $v \in V$, $v = x + y$, where $x \in X$ and $y \in Y$. Since B_X is a basis for X then $x = \alpha_1 u_1 + \dots + \alpha_p u_p$ for some scalars α_i for $i = 1, \dots, p$. Likewise, $y = \beta_1 v_1 + \dots + \beta_r v_r$ for some scalars β_i , $i = 1, \dots, r$. Then

$$\begin{aligned} v &= x + y \\ &= \alpha_1 u_1 + \dots + \alpha_p u_p + \beta_1 v_1 + \dots + \beta_r v_r. \end{aligned}$$

Therefore $v \in \text{span}(B_X \cup B_Y)$ and since $v \in V$ was arbitrary, this shows that $V = \text{span}(B_X \cup B_Y)$. This proves the spanning property. To show that $B_X \cup B_Y$ is linearly independent, let α_i and β_i be such that

$$\alpha_1 u_1 + \dots + \alpha_p u_p + \beta_1 v_1 + \dots + \beta_r v_r = 0$$

where $(\alpha_1 u_1 + \dots + \alpha_p u_p) \in X$ and $(\beta_1 v_1 + \dots + \beta_r v_r) \in Y$. By part (2) we can write the zero vector in V as the unique composition $0 = 0 + 0$ where the first zero is in X and the second zero is in Y . Therefore $\alpha_1 u_1 + \dots + \alpha_p u_p = 0$ and $\beta_1 v_1 + \dots + \beta_r v_r = 0$. Since B_X and B_Y are linearly independent, then $\alpha_i = 0$ for all i and $\beta_j = 0$ for all j . Thus, $B_X \cup B_Y$ is linearly independent, which implies that $B_X \cup B_Y$ is a basis for V .

(3 \Rightarrow 1) Assume $B_X \cap B_Y = \emptyset$ and $B_X \cup B_Y$ is a basis for V . First we show that $V = X + Y$. Since $B_X \cup B_Y$ is a basis for V , then for any $v \in V$, we can write v as linear combination of $u_1, \dots, u_p, v_1, \dots, v_r$. Therefore

$$v = (\alpha_1 u_1 + \dots + \alpha_p u_p) + (\beta_1 v_1 + \dots + \beta_r v_r)$$

where $(\alpha_1 u_1 + \dots + \alpha_p u_p) \in X$ and $(\beta_1 v_1 + \dots + \beta_r v_r) \in Y$. Thus, $v \in X + Y$ and since $v \in V$ was arbitrary, then $V = X + Y$.

To show that $X \cap Y = \{0\}$, let $z \in X \cap Y$. Then $z \in X$ and $z \in Y$. Since $B_X = \{u_1, \dots, u_p\}$ is a basis for X , then $z = \alpha_1 u_1 + \dots + \alpha_p u_p$. Likewise, $z = \beta_1 v_1 + \dots + \beta_r v_r$. Then

$$\alpha_1 u_1 + \dots + \alpha_p u_p = \beta_1 v_1 + \dots + \beta_r v_r$$

and subtracting all terms from the right hand side we have

$$\alpha_1 u_1 + \dots + \alpha_p u_p - \beta_1 v_1 - \dots - \beta_r v_r = 0.$$

Since $B_X \cup B_Y$ is basis for V , then $B_X \cup B_Y$ is linearly independent. Therefore, since the above expression is a linear combination of vectors from $B_X \cup B_Y$, then $\alpha_i = 0$ for all i and $-\beta_j = 0$ for j , which implies that $\beta_j = 0$ for all j . This implies that $z = \alpha_1 u_1 + \dots + \alpha_p u_p = 0$ and so $X \cap Y = \{0\}$.

Thus, $V = X \oplus Y$.

□

Definition. Suppose $V = X \oplus Y$. Since for any $v \in V$ there exists a unique $x \in X$ and $y \in Y$ such that $v = x + y$, then $v \rightarrow x$ defines a function and $v \rightarrow y$ defines a function.

Here we call x the **projection of $v \in V$ onto X (along Y)** and we call y the **projection of $v \in V$ onto Y (along X)**.

Definition. A linear operator $P : V \rightarrow V$ is a **projector** if there exists complementary subspaces X and Y of V , such that $Pv = x$ for all $v \in V$, where $v = x + y$ for $x \in X$ and $y \in Y$.

Remark. In the above definition, $v = x + y$ is a key decomposition that is used in solving problems.

Remark. Facts about a projector:

- (1) For any $x \in X$, $P \cdot x = x$ because $x = x + 0$, where $x \in X$ and $0 \in Y$.
- (2) for any $y \in Y$, $P \cdot y = 0$ because $y = 0 + y$, where $0 \in X$ and $y \in Y$.
- (3) For any two linear operators R and Q , $R = Q$ if and only if $Rv = Qv$ for all $v \in V$. (if given any input, the output is the same).

Theorem 50.2. Let $P : V \rightarrow V$ be a projector matrix onto the subspace X along Y . Then

- (1) $P^2 (= P \circ P) = P$ (idempotent property)
- (2) $I - P$ is the projector onto Y along X .
- (3) $R(P) = \{v \in V : Pv = v\} (= S) = X$.

Proof. (1) We want to show that $P^2v = Pv$ for all $v \in V$. Since P is a projector matrix, then X and Y are complementary subspaces. By the above theorem, we can uniquely write any $v \in V$ as $v = x + y$, where $x \in X$ and $y \in Y$. Since $Pv = x \in X$, then $P^2v = P(Pv) = P \cdot x = x = Pv$. Therefore, since $P^2v = Pv$ for all $v \in V$, then $P^2 = P$.

(2) For any $v \in V$ we have the unique decomposition of v as $v = x + y$ where $x \in X$ and $y \in Y$, with $Pv = x$. Now

$$\begin{aligned}(I - P)v &= Iv - Pv \\ &= v - Pv \\ &= v - x \\ &= x + y - x \\ &= y.\end{aligned}$$

Therefore, for any $v \in V$, $(I - P)(v) = y$, which is the y part of the unique decomposition $v = x + y$. Thus, $I - P$ is the projector onto Y along X .

(3) First we show that $R(P) = S$. Clearly, for any $v \in S$, $Pv = v$ where $Pv \in R(P)$. This implies that $v \in R(P)$. Therefore, $S \subseteq R(P)$.

Conversely, let $z \in R(P)$. Then $z = P \cdot v$ for some $v \in V$ and so $Pz = P(Pv) = P^2(v)$. Now since $P^2 = P$ (idempotent property) then $P^2v = Pv = z$, which implies that $Pz = z$. Thus, $z \in S$. Since $S \subseteq R(P)$ and $R(P) \subseteq S$, then $R(P) = S$.

Now we show that $X = S$. First, $X \subseteq S$ because for any $x \in X$, $Px = x \in S$, which implies that $X \subseteq S$. To show that $S \subseteq X$, let $z \in S$. Then $Pz = z$. Since $Pv \in X$ for $v \in V$ then $Pz \in X$ and so $z \in X$, which implies that $S \subseteq X$. Since we have shown that $X \subseteq S$ and $S \subseteq X$, then $X = S$.

□

In fact, $N(P) = Y$, which we show in the homework.

51 Matrix representation of a projector P on \mathbb{R}^n

Let $V = \mathbb{R}^n$ and X, Y be complementary subspaces. Let P be the projector onto X along Y . Then $R(P) = X$ and $N(P) = Y$.

Let $B_X = \{x_1, \dots, x_r\}$ and $B_Y = \{y_1, \dots, y_{n-r}\}$. Since $B_X \cap B_Y = \emptyset$ and $B = B_X \cup B_Y = \{x_1, \dots, x_r, y_1, \dots, y_{n-r}\}$ is a basis for \mathbb{R}^n , then

$$[P]_B = \begin{bmatrix} [Px_1]_B & [Px_r]_B & [Py_1]_B & [Py_{n-r}]_B \end{bmatrix}$$

where $Px_i = x_i$ ($x_i \in X$) for all $i = 1, \dots, r$ and $Py_i = 0$ ($y_i \in Y$) for all $i = 1, \dots, n - r$. Then

$$[P]_B = \begin{bmatrix} [x_1]_B & \cdots & [x_r]_B & 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

where $[x_1]_B = e_1, \dots, [x_r]_B = e_r$. Let $\tilde{B} = \{e_1, e_2, \dots, e_n\}$ be the standard basis for \mathbb{R}^n . Then

$$[P]_{\tilde{B}} = [I]_{B\tilde{B}} \cdot [P]_B \cdot ([I]_{B\tilde{B}})^{-1}$$

where

$$\begin{aligned} [I]_{B\tilde{B}} &= \begin{bmatrix} [Ix_1]_{\tilde{B}} & \cdots & [Ix_r]_{\tilde{B}} & [Iy_1]_{\tilde{B}} & \cdots & [Iy_{n-r}]_{\tilde{B}} \end{bmatrix} \\ &= \begin{bmatrix} x_1 & \cdots & x_r & y_1 & \cdots & y_{n-r} \end{bmatrix}_{n \times n}. \end{aligned}$$

Therefore

$$\begin{aligned} [P]_{\tilde{B}} &= \begin{bmatrix} x_1 & \cdots & x_r & y_1 & \cdots & y_{n-r} \end{bmatrix} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \left(\begin{bmatrix} x_1 & \cdots & x_r & y_1 & \cdots & y_{n-r} \end{bmatrix}^{-1} \right) \\ &= \begin{bmatrix} B_X & 0 \end{bmatrix} \begin{bmatrix} B_X & B_Y \end{bmatrix}^{-1}. \end{aligned}$$

Therefore, the matrix representation of the projector P is

$$[P]_{\tilde{B}} = \begin{bmatrix} B_X & 0 \end{bmatrix} \cdot \begin{bmatrix} B_X & B_Y \end{bmatrix}^{-1}.$$

Example. Let $V = \mathbb{R}^3$. Let $X = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ and $Y = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

Since $B_X \cap B_Y = \emptyset$ and $B_X \cup B_Y$ is a basis for \mathbb{R}^3 then $V = X \oplus Y$.

To find the matrix representation of the projector P onto the X space,

$$\begin{aligned} [P]_{\tilde{B}} &= \begin{bmatrix} B_X & 0 \end{bmatrix} \cdot \begin{bmatrix} B_X & B_Y \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

Theorem 51.1. Let $P : V \rightarrow V$ be a linear operator. Then P is a projector if and only if $P^2 = P$, i.e., P , is idempotent.

Proof. (\rightarrow) If P is a projector then $P^2 = P$, from theorem above.

(\leftarrow) Suppose $P^2 = P$. We will show that $V = R(P) \oplus N(P)$, i.e., $R(P)$ and $N(P)$ are complementary subspaces of V .

First we show that $V = R(P) + N(P)$. For any $v \in V$, $v = Pv + (v - Pv) = Pv + (I - P)v$. Clearly, $Pv \in R(P)$. Since $P(I - P)v = Pv - P^2v$ and $P^2 = P$ then $P(I - P)v = 0$ and so $(I - P)v \in N(P)$. Therefore, $v \in R(P) + N(P)$. Since $v \in V$ was arbitrary this shows that $V = R(P) + N(P)$.

Now we show that $R(P) \cap N(P) = \{0\}$. Let $z \in R(P) \cap N(P)$. Then $z \in R(P)$ and $z \in N(P)$, which means $z = Pv$ for some $v \in V$ and $Pz = 0$. Therefore $0 = P \cdot z = P(P \cdot v) = P^2v$ and since P is idempotent then $Pv = z$. Thus, $z = 0$. Since z was arbitrary then this shows that $R(P) \cap N(P) = \{0\}$ and so $V = R(P) \oplus N(P)$.

Therefore, for any $v \in V$, v is uniquely decomposed into the sum

$$v = x + y$$

where $x \in R(P)$ and $y \in N(P)$. Therefore, P is a projector onto $R(P)$ along $N(P)$. \square

52 Orthogonal Complement

Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space. This is a general assumption in this section.

Definition. A vector $v \in V$ is orthogonal to a non-empty set S in V if it is orthogonal to each vector in S , i.e., $v \perp z$ for all $z \in S$. In this case, $v \perp S$.

Proposition. Let M be a subspace of V with a basis $B_M = \{u_1, \dots, u_p\}$. Then a vector $v \perp M$ if and only if $v \perp u_i$ for all $i = 1, \dots, p$.

Proof. (\Rightarrow) Suppose $v \perp M$. Then $v \perp z$ for all $z \in M$. Since $u_i \in M$ for all $i = 1, \dots, p$ then $v \perp u_i$ for all $i = 1, \dots, p$.

(\Leftarrow) Suppose $v \perp u_i$ for all $i = 1, \dots, p$. Since B_M is a basis for M , then for any $z \in M$ we can write as

$$z = \alpha_1 u_1 + \dots + \alpha_p u_p$$

for some scalars $\alpha_1, \dots, \alpha_p$. Now consider the inner product of v and z , which is

$$\begin{aligned} \langle v, z \rangle &= \langle v, \alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_p u_p \rangle \\ &= \alpha_1 \langle v, u_1 \rangle + \dots + \alpha_p \langle v, u_p \rangle \\ &= 0. \end{aligned}$$

Therefore, $v \perp z$ for all $z \in M$ and so $v \perp M$. \square

Definition. Given a non-empty set S in V , the orthogonal complement of S , denoted by S^\perp , is the collection of all vectors in V that are orthogonal to S , i.e.,

$$S^\perp = \{v \in V : v \perp S\} = \{v \in V : v \perp z, \forall z \in S\}$$

Proposition. For any non-empty set S , S^\perp is a subspace.

Proof. We proved in a homework exercise. \square

Theorem 52.1. Let V be a finite dimensional inner product space, and M be a subspace of V . Then

$$V = M \oplus M^\perp$$

i.e., M and its orthogonal complement M^\perp are complementary subspaces of V .

Proof. First we show that $M \cap M^\perp = \{0\}$. Let $z \in M \cap M^\perp$. Then $z \in M^\perp$ and $z \in M$. Since $z \in M^\perp$ then $z \perp u$ for all $u \in M$. Then since $u \in M$, $z \perp z$ and so $\langle z, z \rangle = 0$. Thus, $z = 0$. Since z was arbitrary, this shows that $M \cap M^\perp = \{0\}$.

Now we show that $V = M + M^\perp$. V is a finite dimensional inner product space. Therefore M and M^\perp are finite dimensional product spaces such that M has an orthonormal basis B_M , and M^\perp has an orthonormal basis B_{M^\perp} . Since $B \subseteq M$, $B_{M^\perp} \subseteq M^\perp$, and $M \cap M^\perp = \{0\}$, then $B_M \cap B_{M^\perp} = \emptyset$ and $B_M \cup B_{M^\perp}$ is an orthonormal set in M . Therefore, $B_M \cup B_{M^\perp}$ is linearly independent, which implies that $B_M \cup B_{M^\perp}$ spans $M + M^\perp$. Therefore, $B_M \cup B_{M^\perp}$ is a basis for $M + M^\perp$.

Claim: $B_M \cup B_{M^\perp}$ spans V .

Suppose not. Then there exists a $u \in V$ such that $u \notin \text{span}(B_M \cup B_{M^\perp}) = M + M^\perp$. Therefore, by the Gram-Schmidt procedure,

$$w = u - \text{proj}_{M+M^\perp} u \neq 0 \text{ and } w \perp M + M^\perp$$

where $\text{proj}_{M+M^\perp} u$ is the orthogonal projection of u onto $M + M^\perp$. Note that $M \subseteq M + M^\perp$ (because we can write $v = v + 0$ where $v \in M$ and $0 \in M^\perp$.) Since $w \perp M + M^\perp$ then $w \perp M$, which implies that $w \in M^\perp$. On the other hand, $M^\perp \subseteq M + M^\perp$. Therefore, $w \in M^\perp \subseteq M + M^\perp$ and so $w \in M + M^\perp$. Therefore, $w \perp w$ which implies that $w = 0$. However, this contradicts the fact above that $w \neq 0$. Therefore, $B_M \cup B_{M^\perp}$ spans V , i.e., $V = M + M^\perp$. Thus, $V = M \oplus M^\perp$. \square

Proposition. Let V be a finite dimensional inner product space and M is a subspace of V . Then, $(M^\perp)^\perp = M$.

Proof. First we prove that $M \subseteq (M^\perp)^\perp$. Let $v \in M$. Therefore, for any $w \in M^\perp$, $v \perp w$. Because $v \perp w$ for all $w \in M^\perp$, then $v \in (M^\perp)^\perp$, which implies that $M \subseteq (M^\perp)^\perp$.

Now we prove that $(M^\perp)^\perp \subseteq M$. Let $v \in (M^\perp)^\perp \subseteq V$. Since $V = M \oplus M^\perp$, then we can write v as $v = u + w$ where $u \in M$ and $w \in M^\perp$. Since $v \in (M^\perp)^\perp$ and $w \in M^\perp$, then $v \perp w$ or equivalently, $\langle w, v \rangle = 0$. Since $v = u + w$, then $\langle w, v \rangle = \langle w, u + w \rangle = \langle w, u \rangle + \langle w, w \rangle$. Because $u \in M$ and $w \in M^\perp$, then $w \perp u$ or $\langle w, u \rangle = 0$. Therefore, $0 = 0 + \langle w, w \rangle$, which implies that $\langle w, w \rangle = 0$ and so $w = 0$. Thus, $v = u \in M$. Since $v \in M$ then $(M^\perp)^\perp \subseteq M$. \square

53 Orthogonal Complements of Fundamental Subspaces of a Matrix

Let A be an $m \times m$ real matrix, i.e., $A \in \mathbb{R}^{m \times m}$. Then

- $R(A)$ and $N(A^T)$ are subspaces of \mathbb{R}^m
- $R(A^T)$ and $N(A)$ are subspaces of \mathbb{R}^n

Proposition. $[R(A)]^\perp = N(A^T)$ and $[R(A^T)]^\perp = N(A)$.

Proof.

$$\begin{aligned}
x \in [R(A)]^\perp &\iff x \perp z, \forall z \in R(A) \\
&\iff z^T x = 0, \forall z \in R(A) \\
&\iff (Ay)^T \cdot x = 0, \forall y \in \mathbb{R}^n \\
&\iff y^T \cdot (A^T \cdot x) = 0, \forall y \in \mathbb{R}^n \\
&\iff A^T x = 0 \\
&\iff x \in N(A^T).
\end{aligned}$$

Therefore, $[R(A)]^T = N(A^T)$. □

Remark. It follows that $\mathbb{R}^m = R(A) \oplus N(A^T)$ and $\mathbb{R}^n = R(A^T) \oplus N(A)$.

54 Normal Matrix

Definition. An $n \times n$ complex matrix A is normal if $A^*A = AA^*$.

A special case is when $A \in \mathbb{R}^{n \times n}$, then A is normal $\iff A^T A = A A^T$.

Example. (1) A is real symmetric/skewed symmetric.

(2) A is hermitian/skewed hermitian.

Proposition. Let $A \in \mathbb{R}^{n \times n}$ be a normal matrix. Then $[R(A)]^\perp = N(A)$ and $\mathbb{R}^n = R(A) \oplus N(A)$.

Proof. We use the fact that $N(A^T) = N(A \cdot A^T)$ and $N(A) = N(A^T A)$. Note that $[R(A)]^T = N(A^T) = N(A \cdot A^T) = N(A^T \cdot A)$. (Since A is normal.) Then $N(A^T A) = N(A)$. Then $[R(A)]^\perp = N(A)$. Thus, $\mathbb{R}^n = R(A) \oplus N(A)$. □

55 Orthogonal Projection

Let V be a finite-dimensional inner product space. Suppose $V = M \oplus M^\perp$, where M is a subspace of V and M^\perp is the orthogonal complement of M . Then for any $v \in V$, $v = m + n$, where $m \in M$, $n \in M^\perp$ (unique decomposition). This gives the linear operator $P_M : V \rightarrow V$, given by $P_M(v) = m \in M$. Here P_M is **orthogonal projector** onto M space (We showed this in HW#10, Question 4).

55.1 Orthogonal Projectors onto M

Let $B_M = \{u_1, \dots, u_p\}$ be an orthonormal basis for M and $B_{M^\perp} = \{w_1, \dots, w_r\}$ be an orthonormal basis for M^\perp . Then, $B_M \cap B_{M^\perp} = \emptyset$ and $B_M \cup B_{M^\perp}$ is an orthonormal basis for $V = M \oplus M^\perp$. Then let $B = B_M \cup B_{M^\perp} = \{u_1, \dots, u_p, w_1, \dots, w_r\}$. Then for all $v \in V$, $v = \langle v, u_1 \rangle u_1 + \dots + \langle v, u_p \rangle u_p + \langle v, w_1 \rangle w_1 + \dots + \langle v, w_r \rangle w_r$ where $\langle v, u_1 \rangle u_1 + \dots + \langle v, u_p \rangle u_p = u \in M$ and $\langle v, w_1 \rangle w_1 + \dots + \langle v, w_r \rangle w_r = w \in M^\perp$. Then the projection of v onto M is,

$$\text{proj}_M \mathbf{v} = u = \langle v, u_1 \rangle u_1 + \dots + \langle v, u_p \rangle u_p.$$

55.2 Matrix Representation of an Orthogonal Projector P_M on $V = \mathbb{R}^n$

Let $B_M = \{u_1, \dots, u_p\}$ be an arbitrary basis for M , and $B_{M^\perp} = \{w_1, \dots, w_{n-p}\}$ be an orthonormal basis for M^\perp . Then let the matrices S and Q be

$$S = [u_1, \dots, u_p] \text{ and } Q = [w_1, \dots, w_{n-p}].$$

Facts about S and Q :

- (1) The columns of S and Q are linearly independent. Then $Sx = 0$ implies that $x = 0$. Therefore, $S^\perp S$ is invertible because

$$S^\perp Sx = 0 \implies x^T S^T Sx = 0 \implies (Sx)^T (Sx) = 0 \implies Sx = 0 \implies x = 0$$

- (2) $Q^T Q = I_{n-p}$.

- (3)

$$S^T Q = \begin{bmatrix} u_1^T \\ \vdots \\ u_p^T \end{bmatrix} [w_1, \dots, w_{n-p}] = \begin{bmatrix} u_1^T w_1 & \dots & u_1^T w_{n-p} \\ \vdots & & \vdots \\ u_p^T w_1 & \dots & u_p^T w_{n-p} \end{bmatrix} = 0.$$

Since $\langle u_i, w_j \rangle = 0$ where $u_i \in M$ and $w_j \in M^\perp$.

Similarly, $Q^T S = 0$ because $Q^T S = (S^T Q)^T$.

Proposition. The matrix representation of P_M given by $W = S(S^T \cdot S)^{-1} S^T$, where $W \in \mathbb{R}^{n \times n}$ and $S = [u_1, \dots, u_p] \in \mathbb{R}^{n \times p}$,

Proof. Recall that $W = \begin{bmatrix} S & 0 \end{bmatrix} \cdot \begin{bmatrix} S & Q \end{bmatrix}^{-1}$, by results for general projection.

We claim that

$$\begin{bmatrix} S & Q \end{bmatrix}^{-1} = \begin{bmatrix} (S^T S)^{-1} S^T \\ Q^T \end{bmatrix}.$$

Since

$$\begin{bmatrix} (S^T S)^{-1} S^T \\ Q^T \end{bmatrix} \begin{bmatrix} S & Q \end{bmatrix} = \begin{bmatrix} (S^T S)^{-1} S^T S & (S^T S)^{-1} S^T Q \\ Q^T S & Q^T Q \end{bmatrix} = \begin{bmatrix} I_p & 0 \\ 0 & I_{n-p} \end{bmatrix}$$

Thus,

$$\begin{bmatrix} S & Q \end{bmatrix}^{-1} = \begin{bmatrix} (S^T S)^{-1} S^T \\ Q^T \end{bmatrix}.$$

Then,

$$W = \begin{bmatrix} S & 0 \end{bmatrix} \begin{bmatrix} (S^T S)^{-1} S^T \\ Q^T \end{bmatrix} = S(S^T S)^{-1} S^T.$$

□

Remark.

- (1) The matrix representation of P_M is independent of the basis for M .
- (2) $W = W^T$ (i.e., W is symmetric).

To see this, note that

$$\begin{aligned} W^T &= [S(S^T S)^{-1} S^T]^T = (S^T)^T \cdot [(S^T S)^{-1}]^T \cdot S^T \\ &= S \cdot [(S^T S)^T]^{-1} \cdot S^T \\ &= S \cdot [S^T S]^{-1} S^T \\ &= W \end{aligned}$$

Therefore, $W^T = W$ and so W is symmetric.

Theorem 55.1. Suppose $P : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a projector ($P^2 = P$). Let W be its matrix representation. Then P is an orthogonal projector if and only if $W = W^T$.

Proof. (\Rightarrow) We already covered.

(\Leftarrow) Suppose W is symmetric, i.e., $W = W^T$. Since P is a projector and W is the matrix representation, then $\mathbb{R}^n = R(P) \oplus N(P)$ and $R(P) = R(W)$, $N(P) = N(W)$. To show that P is orthogonal projector, we want to show $R(P) \perp N(P)$ (or equivalently $R(W) \perp N(W)$). Recall that $[R(W)]^\perp = N(W^T)$. Since W is symmetric then $W^T = W$ and so $[R(W)]^\perp = N(W)$. Then $\mathbb{R}^n = R(W) \oplus N(W)$ and $R(W) \perp N(W)$. Thus P is an orthogonal projector. □

Theorem 55.2. Let P_M be an orthogonal projection onto the subspace M . Then, for all $z \in V$, $P_M(z) \in M$ is the closest point in M to z , i.e., for all $u \in M$, $\|z - \text{proj}_M z\| \leq \|u - z\|$.

Proof. We will use the facts:

- (1) IF $x \perp y$, then $\|x + y\|^2 = \|x\|^2 + \|y\|^2$.
- (2) $(z - \text{proj}_{\mathbf{M}} \mathbf{z}) \perp M$, which implies $z - \text{proj}_{\mathbf{M}} \mathbf{z} \in M^\perp$.

For any $u \in M$, $\|u - z\|^2 = \|u - P_M z + P_M z - z\|^2$ where $u - P_M z \in M$ and $P_M z - z \in M^\perp$.

Thus, $\|u - z\|^2 = \|u - P_M z\|^2 + \|P_M z - z\|^2$. Therefore, $\|u - \text{proj}_{\mathbf{M}} \mathbf{z}\|^2 > 0$. Then $\|u - z\|^2 \geq \|\text{proj}_{\mathbf{M}} \mathbf{z} - z\|^2$ implies $\|u - z\| \geq \|\text{proj}_{\mathbf{M}} \mathbf{z} - z\| = \|z - \text{proj}_{\mathbf{M}} \mathbf{z}\|$. This implies that $\|u - z\| \geq \|z - \text{proj}_{\mathbf{M}} \mathbf{z}\|$ for all $u \in M$. Thus, $P_M(z)$ is the closest point in M to z . \square

56 Determinant

Definition. Let A be an $n \times n$ matrix, i.e., $A \in \mathbb{R}^{n \times n}$ or $A \in \mathbb{C}^{n \times n}$. Then the determinant of A , using co-factor expansion is an inductive process on n . When $n = 1$ then A is a scalar and $\det(A) = A$. Now consider the general case. Then $A = [a_{ij}]$ where a_{ij} is the (i, j) element of A . In this case, the cofactor expansion across the i^{th} row is

$$\det A = \sum_{j=1}^n (-1)^{i+j} \cdot a_{ij} \cdot \det A_{ij}$$

where A_{ij} is the submatrix of A after removing the i^{th} row and the j^{th} column of A . Therefore, A_{ij} is an $(n-1) \times (n-1)$ matrix. By the inductive hypothesis, $\det A_{ij}$ is known. Similarly, co-factor expansion down the j^{th} column is

$$\det A = \sum_{i=1}^n (-1)^{i+j} \cdot a_{ij} \cdot \det A_{ij}$$

where j is fixed and $i \in \{1, \dots, n\}$.

Example.

- (1) Let $n = 2$ and let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. Using co-factor expansion across the first row we get,

$$\det A = (-1)^{1+1} \cdot a_{11} \cdot \det(A_{11}) + (-1)^{1+2} \cdot a_{12} \cdot \det A_{12}$$

where $A_{11} = [a_{22}]$ and $A_{12} = [a_{21}]$. Then $\det A = a_{11} \cdot a_{22} - a_{12} \cdot a_{21}$.

In this case, $\det A$ has 2 terms, each of which is a product of 2 elements of A .

- (2) Let $n = 3$ and $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. Using co-factor expansion across the first row we have

$$\begin{aligned}
\det(A) &= a_{11} \cdot \det A_{11} - a_{12} \cdot \det A_{12} + a_{13} \cdot \det A_{13} \\
&= a_{11} \cdot \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \cdot \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \cdot \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \\
&= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22}).
\end{aligned}$$

In this case, $\det(A)$ has 6 terms, each of which is a product of 3 elements of A .

Remark. For an $n \times n$ matrix, $\det A$ contains $n!$ terms, each of which is a product of n elements of A .

Example. Let A be an $n \times n$ lower triangular matrix. Then

$$\begin{aligned}
\det A &= \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ \vdots & & \ddots & 0 \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \\
&= a_{11} \cdot \det A_{11} \\
&= a_{11} \cdot \det \begin{bmatrix} a_{22} & 0 & 0 & 0 \\ a_{32} & a_{33} & 0 & 0 \\ \vdots & & \ddots & 0 \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \\
&= a_{11}a_{22} \cdots a_{nn}.
\end{aligned}$$

Similarly, the determinant of an upper triangular matrix is the product of the diagonal entries.

57 Properties of Determinants

- (1) The determinant is linear in each row or column when the other column/row is fixed.
- (2) Determinant under elementary row operations. There are three elementary row operations:
 - (1) row scaling: If we multiply a row of A by a scalar α to get B , then the $\alpha \cdot \det A = \det B$.
 - (2) If we switch a row in A to get B then $\det B = -\det A$.
 - (3) If we replace a row in A to get B then $\det B = \det A$.
- (3)

Theorem 57.1. A square matrix A is invertible if and only if $\det A \neq 0$.

Proof. We will use the following facts in the proof:

- (a) An echelon form of A is an upper triangular matrix and each pivot is a diagonal entry of the echelon form.
- (b) A is invertible if and only if each column/row has a pivot.

Let U be an echelon form of A . Then A is invertible if and only if $\det U \neq 0$. To get U , one only needs row switching and row replacement,

$$A \sim U_1 \sim U_2 \sim \dots \sim U$$

where each \sim represents a row switch or row replacement. Therefore,

$$\det U = (-1)^r \cdot \det A$$

where r is the number of row switchings used in this process. Then

$$\det U \neq 0 \iff \det A \neq 0.$$

Thus, A is invertible if and only if $\det A \neq 0$. □

(4) $\det(A^T) = \det A$

(5) $\det(A \cdot B) = \det A \cdot \det B$

(6) $\det \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} = \det A \cdot \det B$

(7) Suppose A is invertible. Then $\det(A^{-1}) = \frac{1}{\det A}$.

Proof. Because $A^{-1} \cdot A = I$ then $\det(A^{-1} \cdot A) = \det I = 1$. Therefore, $\det A \cdot \det A^{-1} = 1$, which implies $\det(A^{-1}) = \frac{1}{\det A}$. □

(8) Suppose A and B are similar. Then $\det A = \det B$.

Proof. We did in homework. □

58 Eigenvalues and Eigenvectors

Let A be an $n \times n$ real/complex matrix.

Definition. A scalar λ (which is possibly complex number) is an **eigenvalue** of A if and only if there is a nonzero vector v such that

$$Av = \lambda \cdot v$$

where v is an **eigenvector** of A associated with the eigenvalue λ .

Here (λ, v) is called an eigen-pair.

Definition. The set of all distinct eigenvalues of A is the **spectrum** of A , denoted by $\sigma(A)$.

To determine all of the eigenvalues of A :

$$\begin{aligned}\lambda \text{ is an eigenvalue of } A &\iff Av = \lambda v \text{ for some } v \neq 0 \\ &\iff Av - \lambda v = 0 \text{ for some } v \neq 0 \\ &\iff (A - \lambda I)v = 0 \text{ for some } v \neq 0 \\ &\iff A - \lambda I \text{ is singular or noninvertible} \\ &\iff \det(A - \lambda I) = 0.\end{aligned}$$

Therefore, λ is an eigenvalue of A if and only if $\det(A - \lambda I) = 0$.

Let $p(\lambda) = \det(A - \lambda I)$, where $\lambda \in \mathbb{C}$ is a scalar variable.

It can be shown that $p(\lambda)$ is a polynomial function of degree n and $p(\lambda)$ is the **characteristic polynomial** of A . Also,

$$p(\lambda) = 0$$

is the characteristic equation of A .

This equation has n (possibly repeated) roots/solutions. Therefore, A has at most n distinct eigenvalues.

Remark. (a) Even if A is a real matrix, its eigenvalues maybe complex. As an example, consider the matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. This matrix is a skew symmetric matrix.

Then $p(\lambda) = \det(A - \lambda I) = 0$. Because

$$\begin{aligned}\det(A - \lambda I) &= \det \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} \\ &= (-\lambda)^2 + 1 \\ &= \lambda^2 + 1 \\ &= p(\lambda)\end{aligned}$$

then

$$\begin{aligned} p(\lambda) &\iff \lambda^2 + 1 = 0 \\ &\iff \lambda = \pm\sqrt{-1} \end{aligned}$$

where $i = \sqrt{-1}$.

- (b) Let A be an $n \times n$ hermitian matrix; i.e., $A^* = A$ (or $A^T = A$). Then any eigenvalue of A is a real number.

Proof. Let λ be an eigenvalue of A . Then there exists a vector $v \neq 0$ such that $Av = \lambda v$. Subtracting λv from both sides and taking the conjugate transpose then we have $(Av)^* = (\lambda v)^*$. By properties of the conjugate transpose, then $v^* \cdot A^* = \bar{\lambda} \cdot v^*$. Since A is hermitian, then $A^* = A$. Therefore $v^* A = \bar{\lambda} \cdot v^*$. Right multiplying both sides by v we have $v^*(Av) = \bar{\lambda} \cdot v^* \cdot v$ and then because λ is an eigenvalue of A we can replace Av with λv , $v^*(\lambda \cdot v) = \bar{\lambda} \cdot v^* \cdot v$. Then moving the scalar λ to the left side we have $\lambda \cdot (v^* \cdot v) = \bar{\lambda}(v^* \cdot v)$. Because v is an eigenvector we know $v \neq 0$ and so $v^* \cdot v = \langle v, v \rangle > 0$. Therefore, it must be the case that $\lambda = \bar{\lambda}$. Now let $\lambda = a + bi$. Then $\lambda = \bar{\lambda}$ implies that $a + bi = a - bi$ and so $bi = -bi$, and therefore, $b = -b$. Since $b \in \mathbb{R}$, then this means that $b = 0$. Thus, $\lambda = a \in \mathbb{R}$. \square

Corollary. If A is a real symmetric matrix, i.e., $A = A^T \in \mathbb{R}^{n \times n}$, then each eigenvalue of A is real.

- (c) If A is skew Hermitian, i.e., $A^* = -A$, then each eigenvalue of A is imaginary.

Proof. Use similar technique as above proof. \square

- (d) A is singular (i.e., non-invertible) if and only if A has a zero eigenvalue (i.e., $\lambda = 0$ is an eigenvalue of A).

Proof.

Suppose $\lambda = 0$ is an eigenvalue of A and so $\det(A - \lambda I) = 0 \iff \det A = 0$
 $\iff A$ is singular .

\square

Definition. Let $p(\lambda)$ be the characteristic polynomial of an $n \times n$ matrix A and let $\sigma(A) = \{\lambda_1, \dots, \lambda_S\}$ be the distinct eigenvalues of A . Therefore,

$$p(\lambda) = (-1)^n \cdot (\lambda - \lambda_1)^{a_1} \cdot (\lambda - \lambda_2)^{a_2} \cdots (\lambda - \lambda_S)^{a_S}$$

where a_i , $i = 1, \dots, S$, is the number of times that λ_i is repeated. Therefore, $1 \leq a_i \leq n$ and $a_1 + a_2 + \cdots + a_S = n$. Each a_i is the **algebraic multiplicity** of λ_i .

To determine the eigenvectors associated with an eigenvalue λ and matrix A .

Let $\lambda \in \mathbb{C}$ be an eigenvalue of an $n \times n$ matrix $A \iff \exists v \neq 0$ such that $(A - \lambda I)v = 0$
 $\iff \exists v \neq 0$ such that $v \in N(A - \lambda I)$.

Then $v \neq 0$ is an eigenvector associated with λ if and only if $v \neq 0$ and $v \in N(A - \lambda I)$.

Here $N(A - \lambda I)$ is called the **eigenspace** associated with λ , and $N(A - \lambda I)$ is a non-trivial subspace for any eigenvalue λ of A .

Remark.

- (1) $\dim(N(A - \lambda I)) \geq 1$ for all $\lambda \in \sigma(A)$ where $\sigma(A)$ is the spectrum of A , i.e., the set of all distinct eigenvalues of A .
- (2) $\dim(N(A - \lambda_i I)) \leq a_i$, where a_i is the algebraic multiplicity of λ_i . Therefore,

$$\sum_{i=1}^S \dim(N(A - \lambda_i I)) \leq \sum_{i=1}^S a_i = n$$

where S is the total number of distinct eigenvalues of A .

59 Diagonal Matrix and Diagonalization

Recall that an $n \times n$ matrix D is diagonal if

$$D = \begin{bmatrix} d_{11} & 0 & 0 & 0 \\ 0 & d_{22} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & d_{nn} \end{bmatrix}.$$

Some useful properties of diagonal matrices:

(1)

$$D^2 = D \cdot D = \begin{bmatrix} d_{11}^2 & 0 & 0 & 0 \\ 0 & d_{22}^2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & d_{nn}^2 \end{bmatrix}$$

and

$$D^k = D \cdot D \cdots D = \begin{bmatrix} d_{11}^k & 0 & 0 & 0 \\ 0 & d_{22}^k & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & d_{nn}^k \end{bmatrix}$$

for all $k = 1, 2, \dots$

(2) $D \cdot \tilde{D} = \tilde{D} \cdot D$ where D, \tilde{D} are diagonal.

Definition. An $n \times n$ matrix A is called **diagonalizable** if A is similar to a diagonal matrix D , i.e., there exists an invertible matrix P such that $A = P \cdot D \cdot P^{-1}$.

Theorem 59.1. An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors. Further, the diagonal entries of the diagonal matrix D are the eigenvalues of A .

Proof. (\Leftarrow) Suppose A has n linearly independent eigenvectors, p_1, p_2, \dots, p_n . Since each p_i is an eigenvector associated with an eigenvalue λ_i , then $Ap_i = \lambda_i \cdot p_i$.

Let the matrix $P = [p_1 \ p_2 \ \cdots \ p_n]$ and because $\{p_1, p_2, \dots, p_n\}$ are linearly independent, then P is invertible. Further,

$$\begin{aligned} A \cdot P &= A \cdot [p_1 \ p_2 \ \cdots \ p_n] \\ &= [Ap_1 \ Ap_2 \ \cdots \ Ap_n] \\ &= [\lambda_1 p_1 \ \lambda_2 p_2 \ \cdots \ \lambda_n p_n] \\ &= [p_1 \ p_2 \ \cdots \ p_n] \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_n \end{bmatrix} \\ &= P \cdot D. \end{aligned}$$

Therefore, $A \cdot P = P \cdot D$ where D is diagonal and P is invertible. Also, $A = P \cdot D \cdot P^{-1}$ and so A is similar to D and therefore A is diagonalizable.

(\Rightarrow) Suppose A is diagonalizable. Then $A = P \cdot D \cdot P^{-1}$ where

$$D = \begin{bmatrix} d_{11} & 0 & 0 & 0 \\ 0 & d_{22} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & d_{nn} \end{bmatrix}$$

and $P = [p_1 \ p_2 \ \cdots \ p_n]$, which is invertible. Therefore $\{p_1, p_2, \dots, p_n\}$ is linearly independent.

Also, $A \cdot P = P \cdot D$. Therefore,

$$A [p_1 \ p_2 \ \cdots \ p_n] = [p_1 \ p_2 \ \cdots \ p_n] \begin{bmatrix} d_{11} & 0 & 0 & 0 \\ 0 & d_{22} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & d_{nn} \end{bmatrix}$$

and so

$$[Ap_1 \ Ap_2 \ \cdots \ Ap_n] = [d_{11}p_1 \ d_{22}p_2 \ \cdots \ d_{nn}p_n]$$

because if two matrices are the same then the columns are the same.

Therefore, $Ap_i = d_{ii} \cdot p_i$ for all $i = 1, \dots, n$. Note that p_i cannot be zero because $\{p_1, p_2, \dots, p_n\}$ is linearly independent.

Therefore each d_{ii} is an eigenvalue of A , and p_i is the eigenvector associated with d_{ii} . Thus, A has n linearly independent eigenvectors. \square

60 Eigenbasis and Diagonalizable Matrices

Definition. Given an $n \times n$ matrix A , a basis $B = \{v_1, \dots, v_n\}$ is an **eigenbasis** for \mathbb{R}^n or \mathbb{C}^n if each v_i is an eigenvector.

Corollary. A is diagonalizable if and only if A has an eigenbasis.

Proposition. Let $S = \{v_1, \dots, v_p\}$ of eigenvectors of A , where each v_i is associated with a “distinct” eigenvalue λ_i . Then S is linearly independent.

Proof. First consider the case where $p > 1$. For sake of contradiction suppose S is linearly independent (where $p \geq 2$). Let $\{v_1, \dots, v_r\}$ be a minimal spanning set of $\text{span}(S)$, where $r < p$.

Therefore, $\{v_1, \dots, v_r\}$ is linearly independent and v_{r+1} is a linear combination of v_1, \dots, v_r , i.e.,

$$v_{r+1} = \alpha_1 v_1 + \dots + \alpha_r v_r$$

for some scalars $\alpha_1, \dots, \alpha_r$. Then multiplying both sides by A we have

$$Av_{r+1} = \alpha_1 Av_1 + \dots + \alpha_r Av_r.$$

Then we can rewrite the v_{r+1} as

$$\lambda_{r+1} v_{r+1} = \alpha_1 \lambda_1 v_1 + \dots + \alpha_r \lambda_r v_r.$$

Then plugging in $v_{r+1} = \alpha_1 v_1 + \dots + \alpha_r v_r$ in the above equation, we have

$$\lambda_{r+1} \cdot (\alpha_1 v_1 + \dots + \alpha_r v_r) = \alpha_1 \lambda_1 v_1 + \dots + \alpha_r \lambda_r v_r$$

and multiplying through on the right hand side by λ_{r+1} we have

$$\alpha_1 \lambda_{r+1} \cdot v_1 + \dots + \alpha_r \lambda_{r+1} v_r = \alpha_1 \lambda_1 v_1 + \dots + \alpha_r \lambda_r v_r$$

and moving all terms on the right to left side of the equation and we get

$$\alpha_1 (\lambda - \lambda_{r+1}) \cdot v_1 + \dots + \alpha_r (\lambda_r - \lambda_{r+1}) v_r = 0.$$

Since $\{v_1, \dots, v_r\}$ is linearly independent then $\alpha_1 (\lambda_1 - \lambda_{r+1}) = 0, \dots, \alpha_r (\lambda_r - \lambda_{r+1}) = 0$.

Then since $\lambda_i \neq \lambda_j$ for $i \neq j$ then $\alpha_1 = 0, \alpha_2 = 0, \dots, \alpha_r = 0$.

This shows that $v_{r+1} = \alpha_1 v_1 + \dots + \alpha_r v_r = 0$, which is a contradiction because v_{r+1} is an eigenvector and cannot be equal to 0.

Thus, S is linearly independent. \square

Remark. The following is an implication of the above proposition:

Suppose an $n \times n$ matrix A has n distinct eigenvalues $\lambda_1, \dots, \lambda_n$. In this case, each λ_j has its eigenvector v_i , where $v_i \neq 0$.

Then by the preceding proposition, $\{v_1, v_2, \dots, v_n\}$ is linearly independent. Then by a preceding theorem for necessary and sufficient condition for diagonalizability, A is diagonalizable.

Remark. Recall that $\sigma(A) = \{\lambda_1, \dots, \lambda_S\}$, i.e., the set of distinct eigenvalues of A .

For each $\lambda_i \in \sigma(A)$ its eigenspace is $N(A - \lambda_i I)$. Let B_i be a basis for the eigenspace $N(A - \lambda_i I)$.

Then for any $\lambda_i, \lambda_j \in \sigma(A) = \{\lambda_1, \dots, \lambda_S\}$ with $i \neq j$, $B_i \cap B_j = \emptyset$.

Proposition. In the above setting, where $\sigma(A) = \{\lambda_1, \dots, \lambda_S\}$, $B = B_1 \cup B_2 \cup \dots \cup B_S$ is linearly independent.

Proof. Let $G_k = B_1 \cup B_2 \cup \dots \cup B_k$, for $k = 1, \dots, S$. We use induction on k .

For the base case, assume $k = 1$. In this case, $G_1 = B_1$. Since B_1 is a basis for $N(A - \lambda_1 I)$ then B_1 is linearly independent. Thus, G_1 is linearly independent.

Now suppose that G_k is linearly independent and consider G_{k+1} . Note that $G_{k+1} = G_k \cup B_{k+1}$ where $G_k = \{v_1, \dots, v_p\}$ and $B_{k+1} = \{u_1, \dots, u_r\}$ is a basis for $N(A - \lambda_{k+1} I)$. Note that the v_i in G_k are eigenvectors associated with other eigenvalues.

Suppose scalars $\alpha_1, \dots, \alpha_r$ and β_1, \dots, β_p are such that

$$\alpha_1 u_1 + \dots + \alpha_r u_r + \beta_1 v_1 + \dots + \beta_p v_p = 0.$$

We want to show that $\alpha_1 = \dots = \alpha_r = \beta_1 = \dots = \beta_p = 0$.

From the above equation we have

$$(A - \lambda_{k+1} I) \left(\sum_{i=1}^r \alpha_i u_i + \sum_{j=1}^p \beta_j v_j \right) = 0.$$

Distributing the $(A - \lambda_{k+1} I)$ term we get,

$$(A - \lambda_{k+1} I) \sum_{i=1}^r \alpha_i u_i + \sum_{j=1}^p \beta_j (A - \lambda_{k+1} I) \cdot v_j = 0.$$

Since each $u_i \in N(A - \lambda_{k+1} I)$ then $\sum_{i=1}^r \alpha_i u_i \in N(A - \lambda_{k+1} I)$ because the null space is a subspace. Therefore $(A - \lambda_{k+1} I) \sum_{i=1}^r \alpha_i u_i = 0$. Further,

$$\begin{aligned} (A - \lambda_{k+1} I) v_j &= A v_j - \lambda_{k+1} v_j \\ &= A q_j v_j - \lambda_{k+1} v_j \\ &= (\lambda_{q_j} - \lambda_{k+1}) v_j \end{aligned}$$

where $(\lambda_{q_j} - \lambda_{k+1}) \neq 0$ because $\lambda_{q_j} \neq \lambda_{k+1}$.

Then since $\sum_{j=1}^p \beta_j (\lambda_{q_j} - \lambda_{k+1}) v_j = 0$ then $G_k = \{v_1, \dots, v_k\}$ are linearly independent. Therefore $\beta_j \cdot (\lambda_{q_j} - \lambda_{k+1}) = 0$ for all $j = 1, \dots, p$. This implies that $\beta_j = 0$ for $j = 1, \dots, p$.

This shows that $\alpha_1 u_1 + \cdots + \alpha_r u_r = 0$ and therefore $B_{k+1} = \{u_1, \dots, u_r\}$ is a basis for $N(A - \lambda_{k+1}I)$ and so $\{u_1, \dots, u_r\}$ is linearly independent. Thus, $\alpha_1 = \cdots = \alpha_r = 0$ and so $G_{k+1} = G_k \cup B_{k+1}$ is linearly independent. By induction principle, each G_k is linearly independent. Thus G_S is linearly independent. \square

Theorem 60.1. An $n \times n$ matrix A is diagonalizable if and only if for each $\lambda_i \in \sigma(A)$, its geometric multiplicity, i.e., $\dim(N(A - \lambda_i I))$, equals its algebraic multiplicity.

Proof. (\Leftarrow) Suppose $\dim[N(A - \lambda_j I)] = a_i$ for all $\lambda_i \in \sigma(A)$ where $\{\lambda_1, \dots, \lambda_S\}$. Since $\sum_{i=1}^S a_i = n$, then $\sum_{i=1}^S [N(A - \lambda_j I)] = n$.

Let B_i be a basis for each eigenspace $N(A - \lambda_i I)$, for all $\lambda_i \in \sigma(A)$. Therefore, $\#(B_i) = \dim[N(A - \lambda_i I)]$ for all $\lambda_i \in \sigma(A)$, where $\#(\cdot)$ is number of vectors in a finite set. Therefore, $\sum_{i=1}^S \#(B_i) = n$.

Recall that $B_i \cap B_j = \emptyset$ for all $i \neq j$. Therefore, $\#(B_1 \cup B_2 \cup \cdots \cup B_S) = n$, where $B = B_1 \cup B_2 \cup \cdots \cup B_S$.

Also, from the preceding proposition, we know $B = B_1 \cup B_2 \cup \cdots \cup B_S$ is linearly independent. Therefore, B is a set of n linearly independent eigenvectors and so A is diagonalizable.

(\Rightarrow) Suppose A is diagonalizable. Then there exists an invertible matrix P such that $A = PDP^{-1}$ where D is a diagonal matrix. Without loss of generality, D can be written as

$$D = \begin{bmatrix} \lambda_1 & & & & & & \\ & \ddots & & & & & \\ & & \lambda_1 & & & & \\ & & & \lambda_2 & & & \\ & & & & \ddots & & \\ & & & & & \lambda_2 & \\ & & & & & & \ddots & \\ & & & & & & & \lambda_S & \\ & & & & & & & & \ddots & \\ & & & & & & & & & \lambda_S \end{bmatrix}.$$

Here a_i is the algebraic multiplicity and $a_1 + a_2 + \cdots + a_S = n$. Consider the eigenvalue λ_1 . Then

$$\begin{aligned}
A - \lambda_1 I &= P \cdot D \cdot P^{-1} - P(\lambda_1 I) \cdot P^{-1} \\
&= P \cdot (D - \lambda_1 I) P^{-1} \\
&= P \cdot \begin{bmatrix} 0 & & & \\ & \ddots & & \\ & & 0 & \\ & & & \ddots \\ & & & & \ddots \end{bmatrix} \cdot P^{-1}
\end{aligned}$$

where all the first a_1 eigenvalues are set to zero. Then the number of pivots in previous matrix (all a_1 entries are **not** pivots but all others are). Therefore,

$$\text{rank}(A - \lambda_1 I) = n - a_1.$$

Since $A - \lambda_1 I$ is similar to $D - \lambda_1 I$, then

$$\begin{aligned}
\text{rank}(A - \lambda_1 I) &= \text{rank}(D - \lambda_1 I) \\
&= n - a_1.
\end{aligned}$$

From the rank-nullity theorem, we have

$$\begin{aligned}
\dim[N(A - \lambda_1 I)] &= n - \text{rank}(A - \lambda_1 I) \\
&= n - (n - a_1) \\
&= a_1.
\end{aligned}$$

Therefore, $\dim[N(A - \lambda_1 I)] = a_1$.

Likewise, for each $\lambda_2, \dots, \lambda_S \in \sigma(A)$ then $\dim[N(A - \lambda_i I)] = a_i$ for all $i = 2, \dots, S$. \square

Remark. The above proof shows that A is diagonalizable if and only if

$$B = B_1 \cup B_2 \cup \dots \cup B_S$$

has n linearly independent eigenvectors

$$\begin{aligned}
&\iff B \text{ is an eigenbasis for } \mathbb{R}^n \text{ or } \mathbb{C}^n \\
&\iff \mathbb{R}^n \text{ (or } \mathbb{C}^n) = N(A - \lambda_1 I) \oplus \dots \oplus N(A - \lambda_S I)
\end{aligned}$$

Then any vector can be written as a unique combination of vectors from

$$N(A - \lambda_1 I) \oplus N(A - \lambda_2 I) \oplus \dots \oplus N(A - \lambda_S I).$$

Example.

(1) Let $A = \begin{bmatrix} 4 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}_{3 \times 3}$

where A is lower triangular. Therefore, $A - \lambda I$ is also lower triangular. Then $\det(A - \lambda I) = (4 - \lambda)(3 - \lambda)^2$. Then the two distinct eigenvalues are $\lambda_1 = 4$ with algebraic multiplicity $a_1 = 1$ and $\lambda_2 = 3$ with algebraic multiplicity $a_2 = 2$.

Now we find the geometric multiplicity of both eigenvalues. Consider $\lambda_1 = 4$. Then

$$A - \lambda_1 I = \begin{bmatrix} 0 & 0 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

and so the $\text{rank}(A - \lambda_1 I) = 2$. Therefore, $\dim[N(A - \lambda_1 I)] = 3 - 2 = 1 = a_1$.

Now consider $\lambda_2 = 3$. Then

$$A - \lambda_2 I = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and so $\text{rank}(A - \lambda_2 I) = 1$. Therefore, $\dim[N(A - \lambda_2 I)] = 3 - 1 = 2 = a_2$.

Therefore, A is diagonalizable.

(2) Now consider

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

which is an upper triangular matrix. Therefore, A has two distinct eigenvalues $\lambda_1 = 4$ with algebraic multiplicity $a_1 = 1$ and $\lambda_2 = 3$ with algebraic multiplicity $a_2 = 2$.

For $\lambda_1 = 4$, $\dim[(N - \lambda_1 I)] = 1 = a_1$.

For $\lambda_2 = 3$

$$A - \lambda_2 I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore, $\text{rank}(A - \lambda_2 I) = 2$ and so $\dim[N(A - \lambda_2 I)] = 3 - 2 = 1 < a_2 = 2$.

Thus, A is **not** diagonalizable.

61 Unitary Diagonalization and Spectral Theorem

61.1 Facts about normal matrices

An $n \times n$ real/complex matrix A is **normal** if and only if $A \cdot A^* = A^* \cdot A$ (or $AA^T = A^T A$).

Facts about normal matrices:

- (1) $N(A^*) = N(AA^*) = N(A^*A) = N(A)$.
- (2) $R(A) \perp N(A)$, $\Rightarrow A$ represents an orthogonal projector. (if $A^2 = A$).
- (3) Let λ_i, λ_j be two distinct eigenvalues of A . Then $N(A - \lambda_i I) \perp N(A - \lambda_j I)$.

Theorem 61.1. (Unitary Diagonalization) Let A be an $n \times n$ complex matrix. Then $A = U \cdot D \cdot U^*$, where U is a unitary matrix (i.e., $U \cdot U^* = U^* \cdot U = I$) and D is diagonal, if and only if A is normal.

In other words, A has an orthonormal eigenbasis for \mathbb{C}^n if and only if A is a normal matrix.

Proof. We will only prove (\Rightarrow) here because (\Leftarrow) is more involved.

(\Rightarrow) Suppose D is diagonal matrix. Then

$$D \cdot D^* = D^* \cdot D.$$

This is because $D \cdot D^*$ and $D^* \cdot D$ are diagonal, and $(D \cdot D^*)_{ii} = d_i \cdot \bar{d}_i$ and $(D^* \cdot D)_{ii} = \bar{d}_i \cdot d_i$.

Assume $A = U \cdot D \cdot U^*$, where U is unitary and D is diagonal. Therefore,

$$\begin{aligned} A \cdot A^* &= (U \cdot D \cdot U^*) \cdot (U \cdot D \cdot U^*)^* \\ &= U \cdot D \cdot D^* \cdot U^* \end{aligned}$$

and

$$\begin{aligned} A^* \cdot A &= (U \cdot D \cdot U^*)^* \cdot (U \cdot D \cdot U^*) \\ &= U \cdot D^* \cdot D \cdot U^*. \end{aligned}$$

Therefore, $AA^* = A^*A$ and so A is normal. □

Remark. The following is a special case of the above theorem.

Let $A \in \mathbb{R}^{n \times n}$ be a real symmetric matrix, i.e., $A = A^T$. (This implies A is normal)
Then,

- (1) Each eigenvalue of A is real.
- (2) All eigenvectors are in \mathbb{R}^n .
- (3) By the past theorem, there exists an orthogonal matrix $P \in \mathbb{R}^{n \times n}$ (Therefore, $P \cdot P^T = P^T \cdot P = I$.) and a diagonal matrix D such that $A = P^T \cdot D \cdot P$. This is called the spectral decomposition of a real symmetric matrix, i.e., A has an O.N. eigenbasis for \mathbb{R}^n , and the diagonal entries of D are the real eigenvalues of A .

62 Positive definite and semi-definite, and indefinite matrices

Definition. Let $A \in \mathbb{R}^{n \times n}$ be a real symmetric matrix, i.e., $A = A^T$.

(1) We call A **positive definite** (P.D.) if

$$x^T \cdot A \cdot x > 0, \quad \forall 0 \neq x \in \mathbb{R}^n.$$

(2) We call A **positive semi-definite** (P.S.D.) if

$$x^T \cdot A \cdot x \geq 0, \quad \forall x \in \mathbb{R}^n.$$

(3) We call A **indefinite** if there exists $x \in \mathbb{R}^{n \times n}$ and $y \in \mathbb{R}^{n \times n}$ such that

$$x^T \cdot A \cdot x > 0 \quad \text{but} \quad y^T \cdot A \cdot y < 0.$$

Theorem 62.1. Let $A \in \mathbb{R}^{n \times n}$ be a real symmetric matrix. The following are equivalent:

- (1) A is positive definite.
- (2) Each eigenvalue of A is positive.
- (3) There is an invertible matrix B such that $A = B^T \cdot B$.

Proof. 1) \Rightarrow 2) Suppose A is P.D. Let $v \in \mathbb{R}^n$ be an eigenvector associated with an eigenvalue λ . Then $Av = \lambda v$ and $v^T Av = v^T(\lambda \cdot v) = \lambda \cdot v^T v$ where $v^T \cdot A \cdot v > 0$ and $v^T v > 0$ because $v \neq 0$.

Since A is positive-definite and $v \neq 0$ then $v^T Av > 0$ and so

$$\lambda = \frac{v^T Av}{v^T v} > 0$$

where $v^T v > 0$. Therefore, each eigenvalue λ of A is positive.

2) \Rightarrow 3): We will use the spectral decomposition. Suppose A is real symmetric and all eigenvalues are positive. Therefore, by the spectral decomposition of A , $A = P^T \cdot D \cdot P$, where P is orthogonal and

$$D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

and $\lambda_i > 0$ for all i .

Define $D^{\frac{1}{2}}$ as

$$D^{\frac{1}{2}} = \begin{bmatrix} \sqrt{\lambda_1} & & & \\ & \sqrt{\lambda_2} & & \\ & & \ddots & \\ & & & \sqrt{\lambda_n} \end{bmatrix}$$

which is a diagonal matrix. Then $D^{\frac{1}{2}} \cdot D^{\frac{1}{2}} = D$ and $D^{\frac{1}{2}}$ is invertible. Therefore,

$$A = P^T \cdot D^{\frac{1}{2}} \cdot D^{\frac{1}{2}} \cdot P = (P^T \cdot D^{\frac{1}{2}}) \cdot (D^{\frac{1}{2}} \cdot P).$$

Let $B = D^{\frac{1}{2}} \cdot P$. Since $D^{\frac{1}{2}}$ and P are invertible, then B is invertible. Also, $B^T = (D^{\frac{1}{2}} \cdot P)^T = P^T (D^{\frac{1}{2}})^T = P^T \cdot D^{\frac{1}{2}}$, therefore $A = B^T \cdot B$.

3) \Rightarrow 1) Suppose $A = B^T \cdot B$, where B is invertible. We must show A is P.D. For any $0 \neq x \in \mathbb{R}^n$,

$$\begin{aligned} x^T A x &= x^T \cdot B^T \cdot B \cdot x \\ &= (Bx)^T \cdot (Bx) = \|Bx\|^2. \end{aligned}$$

Since $x \neq 0$ and B is invertible, then $Bx \neq 0 \Rightarrow \|Bx\|^2 > 0$. Therefore, $x^T \cdot A \cdot x > 0$ for all $0 \neq x \in \mathbb{R}^n$. Therefore, A is positive definite. \square

If A is positive semi-definite then each eigenvalue is non-negative, $A = B^T \cdot B$, where $B \in \mathbb{R}^{n \times n}$, not necessarily invertible.