MATH 487 - Notes
Continuous Dynamical Systems

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Contents

1 Linear Systems of ODEs

A linear system of ODEs can be written as:

$$\dot{x} = Ax$$

where $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$. We will look at the solution to this system as n varies over $0, 1, 2, \ldots$ and the system of equations are coupled are not coupled.

1.1 n = 1

If $n = 1$ we have

$$\begin{aligned} \dot{x} &= ax \\ x(0) &= c \end{aligned}$$

and the solution to this equation is $x(t) = ce^{at}$.

1.2 n = 2 (uncoupled)

If $n = 2$ and the two equations are not coupled then we have

$$\begin{aligned} \dot{x}_1 &= -x_1 \qquad x_1(0) = c_1 \\ \dot{x}_2 &= 2x_2 \qquad x_2(0) = c_2 \end{aligned}.$$

Since the two equations are not coupled we can solve them separately giving the solution

$$\begin{aligned} x_1(t) &= c_1 e^{-t} \\ x_2(t) &= c_2 e^{2t} \end{aligned}.$$

We can rewrite these separate equations as a linear system,

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad x(0) = c, x \in \mathbb{R}^2$$

and the solution to this linear system is

$$x(t) = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix} c \qquad c \in \mathbb{R}^2.$$

[to do: insert graph with phase curves]

To find the equations of the phase curves for this system we can divide both sides of the second equation by the first equation

$$\frac{dx_2}{dx_1} = \frac{2x_2}{-x_1}$$

which is a first order differential equation we can solve by separation of variables. Letting $y = x_2$ and $x = x_1$ and rearranging the equation we have

$$\frac{dy}{dx} = \frac{-2y}{-x} \implies \frac{dy}{y} = \frac{-2}{x} dx$$

and integrating both sides then solving for y we get

$$\begin{aligned} \ln y &= -2 \ln x + c = \ln x^{-2} + c \\ y &= e^c \frac{1}{x^2} = \frac{\hat{c}}{x^2} \end{aligned}$$

where $\hat{c}$ will determine which phase curve we are on in the phase curve diagram.

1.3 n = 3 (uncoupled)

If $n = 3$ and the equations are not coupled then we have

$$\begin{aligned} \dot{x}_1 &= x_1 \\ \dot{x}_2 &= x_2 \\ \dot{x}_3 &= -x_3 \end{aligned}$$

which has the solution

$$\begin{aligned} x_1(t) &= c_1 e^t \\ x_2(t) &= c_2 e^t \\ x_3(t) &= c_3 e^{-t} \end{aligned} \; .$$

These three equations can be rewritten as

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} x \\ x(0) &= c \end{aligned}$$

where $x \in \mathbb{R}^3$ and $c \in \mathbb{R}^3$.

[add phase portrait diagram]

1.4 Classification of 2×2 linear systems based on eigenvalues of A

Let $\dot{x} = Ax$ represents a system of coupled equations. Then A must have some non-zero entries for the off diagonals and therefore A is not diagonal. First we state a useful theorem from linear algebra.

Theorem 1.1. If the eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_n$ of the $n \times n$ matrix A are real and distinct then any set of corresponding eigenvectors form a basis for $\mathbb{R}^n$. In addition, the matrix P whose column vectors are the eigenvectors of A, denoted $P = [v_1, v_2, \ldots, v_n] \in \mathbb{R}^n$ is invertible and

$$P^{-1}AP = \mathrm{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n],$$

where $\mathrm{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n]$ denotes a diagonal matrix whose diagonal entries starting at the upper left corner are $\lambda_1, \lambda_2, \ldots, \lambda_n$.

We now consider different cases for the eigenvalues of A.

1.4.1 Case 1: A has real and distinct eigenvalues

Start with $\dot{x} = Ax$ assuming this represents a system of coupled differential equations and that A has real and distinct eigenvalues. Assume $P = [v_1, v_2, \ldots, v_n]$ so P^{-1} exists (Since $v_1, v_2, \ldots, v_n$ form a basis they are linearly independent, which implies P is invertible). Define

$$y = P^{-1}x. \tag{1}$$

Then left multiplying both sides of the above equation by P we have

$$Py = PP^{-1}x = x$$

so that we have $x = Py$. Now taking the derivatives on both sides of (1) we have

$$\begin{aligned} \dot{y} &= P^{-1}\dot{x} \\ &= P^{-1}Ax \\ &= P^{-1}A(Py)^{\cdot} \\ &= (P^{-1}AP)y \end{aligned} \tag{2}$$

We can have converted the original linear system $\dot{x} = Ax$ to a new system

$$\dot{y} = \Lambda y$$

where $\Lambda = \mathrm{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n]$. This is a system of uncoupled differential equations that has the solution

$$y(t) = \begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{bmatrix} y(0)$$

where $y(0) = P^{-1}x(0)$. Now plugging in this value for y into $x = Py$ we get the solution to the original linear system as

$$x(t) = P \begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{bmatrix} P^{-1}x(0).$$

Example. Consider the linear system $\dot{x} = \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix} x$. Then solving the equation $\det(A - \lambda I) = 0$ we find that $\lambda_1 = 5$ and $\lambda_2 = 4$. Then $v_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 & 2 \end{bmatrix}$.

Now we need to form the matrix $P = \text{diag}[v_1, v_2]$ which is

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

We also need to find P^{-1} which is

$$P^{-1} = \frac{1}{\det P} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}.$$

Now we form the matrix $P^{-1}AP$ as

$$\begin{aligned} P^{-1}AP &= \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 10 & -5 \\ -4 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix}. \end{aligned}$$

Then the solution to $\dot{y} = \Lambda y = \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix}$ is $y(t) = \begin{bmatrix} e^{5t} & 0 \\ 0 & e^{4t} \end{bmatrix}$. Then converting back to x using $x = Py$ we have the solution to our original linear system as

$$x(t) = Py(t) = P \begin{bmatrix} e^{5t} & 0 \\ 0 & e^{4t} \end{bmatrix} P^{-1}x(0).$$

1.4.2 Case 2: A has real repeated eigenvalues

Consider the linear system $\dot{x} = Ax$ where $x \in \mathbb{R}^2$ and $A \in \mathbb{R}^{2\times 2}$. Now consider the case where

$$A = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix}.$$

To find the eigenvalues of A we solve the equation $\det(A - \lambda I) = 0$ for λ, which is $(\lambda_1 - \lambda)^2 = 0$ and so $\lambda = \lambda_1$ and $\lambda = \lambda_1$. To find the eigenvector associated with λ_1 we plug in λ_1 for λ in $A - \lambda I = 0$ and solve the resulting system of linear equations. Doing this we have

$$\begin{bmatrix} \lambda_1 - \lambda_1 & 1 - 0 \\ 0 & \lambda_1 - \lambda_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and then we have the linear system

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This can be rewritten as two equations,

$$\begin{aligned} 0(\alpha_1) + 1(\alpha_2) &= 0 \\ 0(\alpha_1) + 0(\alpha_2) &= 0 \end{aligned}$$

and we see that $\alpha_2 = 0$ and α_1 is a free variable and so we set $\alpha_1 = 1$. Then the eigenvector associated with λ_1 is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Now consider the linear system

$$\begin{aligned} x' &= \lambda_1 x + y \\ y' &= \lambda_1 y \end{aligned}$$

where $y(t) = c_2 e^{\lambda_1 t}$. Now plugging in $y(t)$ into $x' = \lambda_1 x + y$ we have $x' = \lambda_1 x + c_2 e^{\lambda_1 t}$, which is a first order, linear, nonhomogenous equation. The general solution to this equation is $x(t) = x(t)_h + x(t)_p$ where $x(t)_h$ is the solution to the corresponding homogeneous equation $x' = \lambda_1 x$ and $x(t)_p$ is any solution to the nonhomogeneous equation. We have that $x_h(t) = c_1 e^{\lambda_1 t}$ and $x_p(t) = cte^{\lambda_1 t}$. Therefore $\{e^{\lambda_1 t}, te^{\lambda_1 t}\}$ is a fundamental solution set for the differential equation and the general solution is

$$x(t) = c_1 e^{\lambda_1 t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{\lambda_1 t} \begin{bmatrix} t \\ 1 \end{bmatrix}$$

Now what happens to the solution $x(t)$ as $t \to \infty$? We consider two cases (1) $\lambda_1 > 0$ and (2) $\lambda_1 < 0$.

If $\lambda_1 > 0$ then $x(t) \to \mp\infty$.

If $\lambda_1 < 0$ then we need to find $\lim_{t\to\infty} c_2 e^{\lambda_1 t} \begin{bmatrix} t \\ 1 \end{bmatrix} = \begin{bmatrix} \lim_{t\to\infty} c_2 t e^{\lambda_1 t} \\ \lim_{t\to\infty} c_2 e^{\lambda_1 t} \end{bmatrix}$. We know that $\lim_{t\to\infty} c_2 e^{\lambda_1 t} = 0$ since $\lambda < 0$. To find $\lim_{t\to\infty} c_2 t e^{\lambda_1 t}$ we can use L'Hopital's rule,

$$\begin{aligned} \lim_{t\to\infty} c_2 t e^{\lambda_1 t} &= c_2 \lim_{t\to\infty} \frac{\frac{t}{1}}{e^{-\lambda_1 t}} \\ &= c_2 \lim_{t\to\infty} \frac{1}{-\lambda_1 e^{-\lambda_1 t}} \\ &= c_2 \frac{1}{\infty} \\ &= c_2 \cdot 0 \\ &= 0 \end{aligned}.$$

Therefore, if $\lambda_1 < 0$ then $x(t) \to 0$ as $t \to \infty$.

1.4.3 Case 3: A has complex conjugate pair of eigenvalues

Consider the linear system $\dot{x} = Ax$ where $x \in \mathbb{R}^2$ and $A \in \mathbb{R}^{2\times 2}$. Now assume that the eigenvalues $\lambda_\mp$ for A are a complex conjugate pair $\alpha \mp i\beta$ where α and β are real numbers and $i = \sqrt{-1}$. The eigenvectors associated with the complex conjugate pair of eigenvalues are $v\mp = u \mp iw$. Assume the initial conditions are $c\mp = \frac{1}{2}(g \mp ih)$. The solutions $e^{\lambda t}$ will now have the form $e^{(\alpha \mp i\beta)t}$ which we can rewrite using Euler's formula as

$$e^{(\alpha\mp i\beta)t} = e^{\alpha t} e^{\mp i\beta t} = e^{\alpha t}(\cos\beta t \mp i \sin\beta t).$$

2 Classification of phase portrait for 2 x 2 linear systems

In this section we consider linear systems of the form

$$\begin{aligned} \dot{x} &= Ax \\ x(0) &= x_0 \end{aligned}$$

where $A \in \mathbb{R}^{2\times 2}$, $x_0 \in \mathbb{R}^2$, and x is a function $x : \mathbb{R} \to \mathbb{R}^2$ where $x(t) = (x_1(t), x_2(t))$. Considering a specific example this system can be written as

$$\begin{aligned} \dot{x}_1 &= -x_1 \quad x_1(0) = c_1 \\ \dot{x}_2 &= 2x_2 \quad x_2(0) = c_2 \end{aligned}$$

and the solution is

$$\begin{aligned} x_1(t) &= c_1 e^{-t} \\ x_2(t) &= c_2 e^{2t} \end{aligned}.$$

This is a system of decoupled equations that can also be written as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

with initial conditions

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} (0) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

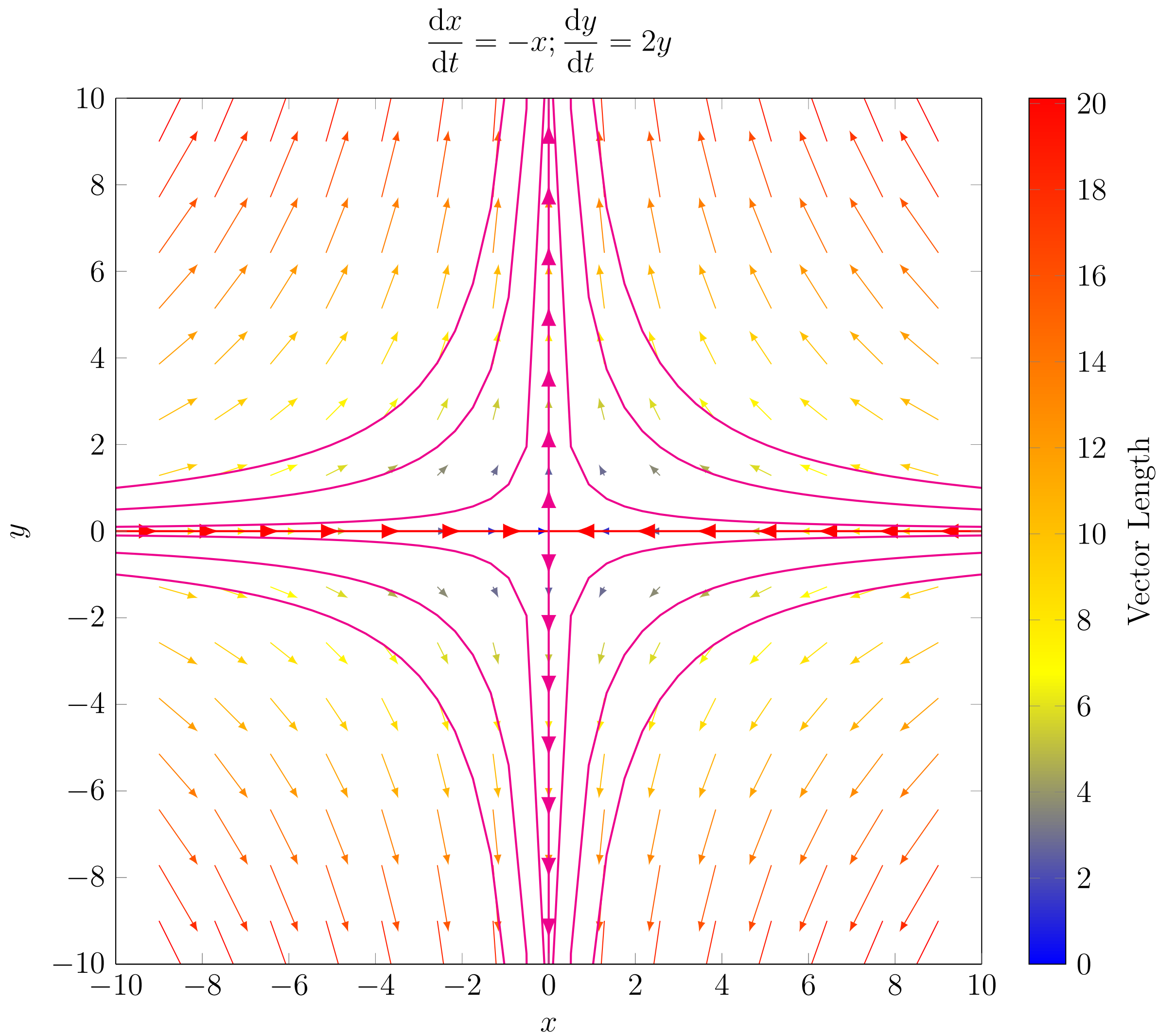
The solution can be written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

The eigenvalues of the matrix $\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$ are $\lambda_1 = -1$ and $\lambda_2 = 2$ with associated eigenvectors $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. So we can also write the solution of the above system as

$$\begin{aligned} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &= c_1 e^{-t} v_1 + c_2 e^{2t} v_2 \\ &= c_1 e^{-t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \\ &= \begin{bmatrix} c_1 e^{-t} \\ c_2 e^{2t} \end{bmatrix} \end{aligned}$$

2.1 Saddle



2.2 Node

2.3 Focus or Spiral

2.4 Degenerate Cases

3 Summary of solutions of 2 × 2 linear systems

3.1 real and distinct

Consider the linear system $\dot{x} = Ax$ where

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$

and the eigenvalues of A are real and distinct.

The solution of this system has the form

$$x(t) = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t}.$$

where λ_1 and λ_2 are eigenvalues of A, v_1 and v_2 are the eigenvectors associated with λ_1 and λ_2, and c_1 and c_2 are constants.

3.2 repeated root

Consider the linear system $\dot{x} = Ax$ where

$$\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$$

and the eigenvalues of A are real and repeated.

The solution to this system has the form

$$x(t) = c_1 v_1 e^{-t} + c_2 v_2 t e^{-t}$$

where $\lambda_1 = -1$ and $\lambda_2 = -1$ are eigenvalues of A, v_1 and v_2 are the eigenvectors associated with $\lambda_1 = -1$ and $\lambda_2 = -1$, and c_1 and c_2 are constants.

3.3 complex conjugate pair

Consider the linear system $\dot{x} = Ax$ where

$$\begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$$

and the eigenvalues of A are $\lambda = \alpha \mp i\beta$ and the eigenvectors associated with λ are $u \mp iv$. The solution to this system has the form

$$x(t) = e^{\alpha t} \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} \cos\beta t & \sin\beta t \\ -\sin\beta t & \cos\beta t \end{bmatrix} \begin{bmatrix} g \\ -h \end{bmatrix}$$

4 Trace-Determinant plane

Consider the generic 2×2 linear system $\dot{x} = Ax$ where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then we have that $\det A = ad - bc$ and $\operatorname{tr} A = a + d$.

To find the eigenvalues of A we solve the equation $\det(A - \lambda I) = 0$. First we calculate $\det(A - \lambda I)$ as

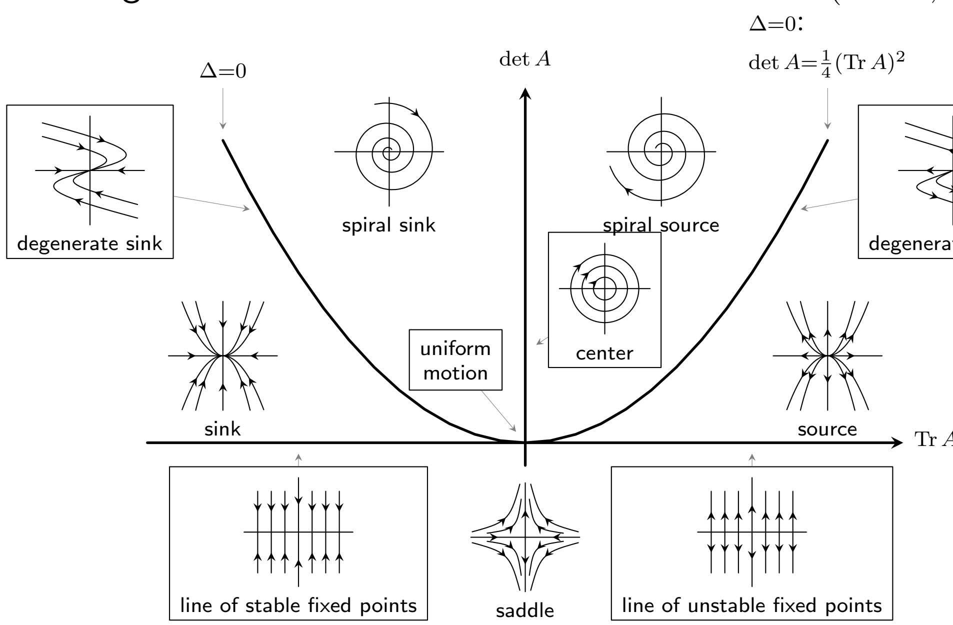
$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix} \\ &= (a - \lambda)(d - \lambda) - bc \\ &= ad - (a + d)\lambda - ad - bc \\ &= \lambda^2 - \operatorname{tr} A\lambda + \det A. \end{aligned}$$

Setting $\lambda^2 - \operatorname{tr} A\lambda + \det A = 0$ and using the quadratic formula we can solve for λ,

$$\lambda = \frac{\operatorname{tr} A \mp \sqrt{(\operatorname{tr} A)^2 - 4\det A}}{2}.$$

The type of eigenvalues for A will depend on the value of $(\operatorname{tr} A)^2$ and $4\det A$. We can use this information to classify the phase diagrams of any 2×2 linear system.

Poincaré Diagram: Classification of Phase Portaits in the $(\det A, \operatorname{Tr} A)$-plane



5 Diagonalization of $n \times n$ matrices

5.1 Jordan Canonical Decomposition

Theorem 5.1. Let A be a real matrix with real eigenvalues $\lambda_j,\ j = 1, \ldots, k$ and complex eigenvalues $\lambda_j = a_j + ib_j,\ j = k+1, \ldots, n$. Then there exists a basis $\{v_1, \ldots, v_k, u_{k+1}v + k + 1, \ldots, u_n v_n\}$ where $v_j,\ j = 1, \ldots, k$ and $w_j,\ j = 1, \ldots, n$ are generalized eigenvalues of A with $u_j = Re(w_j)$ and $v_j = Im(w_j)$, $j = k+1, \ldots, n$ such that $P = [v_1 \ldots v_k \quad v_{k+1}u_{k+1} \ldots v_n u_n]$ is invertible and

$$P^{-1}AP = \begin{bmatrix} B_1 & 0 & \ldots & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & \ldots & \ddots & 0 \\ 0 & 0 & \ldots & B_r \end{bmatrix}$$

where B_j are Jordan blocks of the form

$$\begin{bmatrix} \lambda & 1 & 0 & 0 & 0 & 0 \\ 0 & \ddots & \ddots & 0 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 0 & 0 \\ 0 & 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \lambda & 1 \end{bmatrix}$$

for real λ or

$$\begin{bmatrix} D & I_2 & 0 & 0 & 0 & 0 \\ 0 & \ddots & \ddots & 0 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 0 & 0 \\ 0 & 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & D & I_2 \end{bmatrix}$$

where $D = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ for complex eigenvalues. Jordan form is unique up to the order of the blocks.

Example. Consider a 5×5 matrix with 1 real and distinct eigenvalue, 1 real and repeated eigenvalue, and 1 complex conjugate pair of eigenvalues. Then the Jordan form is

$$\left[\begin{array}{c|cc|cc} \lambda_1 & 0 & 0 & 0 & 0 \\ \hline 0 & \lambda_2 & 1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 & 0 \\ \hline 0 & 0 & 0 & \alpha & \beta \\ 0 & 0 & 0 & -\beta & \alpha \end{array}\right]$$

where B_1 is 1×1, B_2 is 2×2, and B_3 is 2×2 so that

$$P^{-1}AP = \begin{bmatrix} B_1 & 0 & 0 \\ 0 & B_2 & 0 \\ 0 & 0 & B_3 \end{bmatrix}.$$

5.2 Examples of Jordan blocks

5.2.1 2×2 matrices

1. Real and distinct eigenvalues

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

2. Real, repeated eigenvalues with algebraic and geometric multiplicity 2

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{bmatrix}$$

3. Real, repeated eigenvalues with algebraic multiplicity 2 and geometric multiplicity 1

$$\begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix}$$

4. pair of complex conjugate eigenvalues

$$\begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix}$$

5.2.2 3×3 matrices

1. real and distinct eigenvalues

$$\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

2. real and repeated eigenvalues with algebraic and geometric multiplicity of 3

$$\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix}$$

3. real and repeated eigenvalues with algebraic multiplicity 3 and geometric multiplicity 2

$$\begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix}$$

4. real and repeated eigenvalues with algebraic multiplicity 3 and geometric multiplicity 1

$$\begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix}$$

5. two real eigenvalues and one repeated with algebraic multiplicity 2 and geometric multiplicity 1

$$\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{bmatrix}$$

6. two real eigenvalues and one repeated with algebraic multiplicity and geometric multiplicity 2

$$\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}$$

7. one real eigenvalues and one complex conjugate pair of eigenvalues

$$\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & -\beta & \alpha \end{bmatrix}$$

6 Exponentials of operators

If we have the initial value problem $\dot{x} = ax$ with $x(0) = x_0$ and $a \in \mathbb{R}$ then we know the solution is $x(t) = x_0 e^{at}$. Now what if we consider the linear system $\dot{x} = Ax$ with $x(0) = x_0$ but now $x \in \mathbb{R}^2$ and $A \in \mathbb{R}^{2\times 2}$. It would be nice if we could write the solution as $x(t) = e^{At}x_0$, but what is e^{At}?

Definition. Operator Norm. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator. Then the *operator norm* of T is defined to be

$$||T|| = \max_{|x|\leq 1} |T(x)|$$

where $|\cdot|$ denotes the Euclidean norm for $x \in \mathbb{R}^n$.

The operator norm has the following properties:

(a) $||T|| > 0$ and $||T|| = 0$ if and only if $T = 0$.

(b) $||kT|| = |k|||T||$ for $k \in \mathbb{R}$.

(c) $||S + T|| \leq ||S|| + ||T||$

If $T \in \mathcal{L}(\mathbb{R}^n)$ is represented by a matrix A with respect to the standard basis in $\mathbb{R}^n$ then $||A|| \leq \ell\sqrt{n}$ where ℓ is the maximum length of the rows of A.

Definition. Convergence. A sequence of linear operators $T_k \in \mathcal{L}(\mathbb{R}^n)$ converges to a linear operator $T \in \mathcal{L}(\mathbb{R}^n)$ as $k \to \infty$ if $\lim_{k\to\infty} T_k = T$, i.e.,

$$\forall \epsilon > 0 \exists N \in \mathbb{N} \text{ such that } k \geq N \implies ||T_k - T|| < \epsilon.$$

Lemma. *For $S, T \in \mathcal{L}(\mathbb{R}^n)$ and $x \in \mathbb{R}^\ltimes$*

1. $|T(x)| \leq ||T|||x|$

2. $||TS|| \leq ||T||||S||$

3. $||T^k|| \leq (||T||)^k$ *for* $k = 0, 1, 2, \ldots$.

Proof. 1. True for $x = 0$. Assume $x \neq 0$ and define $y = \frac{x}{|x|}$. Then

$$||T|| \geq |T(y)| = \frac{1}{|x|}|T(x)|.$$

2. For $|x| < 1$, 1. implies

$$\begin{aligned} |T(S(x))| &\leq ||T|||S(x)| \\ &\leq ||T||||S|||x|. \end{aligned}$$

3. Follows from 2. (by induction)

□

Definition. Weierstrass M-test. Suppose that f_n is a sequence of real or complex valued functions defined on a set A and that there is a sequence of non-negative numbers M_n such that $|f_n(x)| \leq M_n$ for all $n \geq 1$ and for all $x \in A$ and $\displaystyle\sum_{n=1}^{\infty} M_n$ converges then $\displaystyle\sum_{n=1}^{\infty} f_n(x)$ converges uniformly and absolutely.

Theorem 6.1. Given $T \in \mathcal{L}(\mathbb{R}^n)$ and $t_0 > 0$, the series $\displaystyle\sum_{k=0}^{\infty} \frac{T^k t^k}{k!}$ is absolutely and uniformly convergent for all $|t| \leq t_0$.

Proof. Let $||T|| = a$. Then

$$\left|\left|\frac{T^k t^k}{k!}\right|\right| \leq \frac{||T||^k |t|^k}{k!} \leq \frac{a^k t_0^k}{k!}.$$

So $\displaystyle\sum_{k=0}^{\infty} \frac{a^k t_0^k}{k!} = e^{at_0}$. By Weierstrass M-test $\displaystyle\sum_{k=0}^{\infty} \frac{T^k t^k}{k!}$ is absolutely and uniformly convergent. □

So we define the exponential of the linear operator T to be the absolutely convergent series

$$e^T = \sum_{k=0}^{\infty} \frac{T^k}{k!}$$

only for square matrices. Thus e^T is a linear operator. It follows that

$$||e^T|| \leq e^{||T||}.$$

Lemma. *Let* $A \in \mathbb{R}^n$. *Then* $\dfrac{d}{dt}e^{At} = Ae^{At}$.

Proof. Since A commutes with itself,

$$\begin{aligned}
\frac{d}{dt}e^{At} &= \lim_{h\to 0}\frac{e^{A(t-h)} - e^{At}}{h} \\
&= \lim_{h\to 0} e^{At}\left(\frac{e^{Ah} - I}{h}\right) \\
&= e^{At}\lim_{h\to 0}\frac{1}{h}\lim_{k\to\infty}\left(I + Ah + \frac{(Ah)^2}{2} + \cdots + \frac{(Ah)^k}{k!} - I\right) \\
&= e^{At}\lim_{h\to 0}\frac{1}{h}\lim_{k\to\infty}\left(Ah + \frac{(Ah)^2}{2} + \cdots + \frac{(Ah)^k}{k!}\right) \\
&= e^{At}\lim_{h\to 0}\lim_{k\to\infty}\left(A + \frac{A^2h}{2} + \frac{A^hh^{k-1}}{k!}\right) \\
&= e^{At}\lim_{k\to\infty}\lim_{h\to 0}\left(A + \frac{A^2h}{2} + \frac{A^hh^{k-1}}{k!}\right) \\
&= e^{At}\lim_{k\to} A \\
&= e^{At}A \\
&= Ae^{At}
\end{aligned}$$

□

Remark. In the last line of the above proof we switched A from the right to the left side because A commutes with itself. Therefore

$$\begin{aligned}
Ae^{At} &= A(I + At + \frac{1}{2!}(At)^2 + \cdots \\
&= A + A^2t + \frac{1}{2!}A^2t^2 + \cdots \\
&= (I + At + \frac{1}{2!}(At)^2 + \cdots)A \\
&= e^{At}A
\end{aligned}$$

Also from the third to fourth equality we switched the limits because we have uniform convergence.

Lemma. *If S and T are linear transformations on $\mathbb{R}^n$ which commute, i.e., $ST = TS$, then $e^{S+T} = e^Se^T$.*

Proof. If $ST = TS$ then by the binomial theorem

$$(S+T)^n = n\sum_{j+k=n}\frac{S^jT^k}{j!k!}.$$

Therefore

$$e^{S+T} = \sum_{n=0}^{\infty} \sum_{j+k=n} \frac{S^j T^k}{j!k!} = \left(\sum_{j=0}^{\infty} \frac{S^j}{j!} \right) \left(\sum_{k=0}^{\infty} \frac{T^k}{k!} \right) = e^S e^T.$$

□

Remark. In the second equality above we were able to split the double sum because we have absolute convergence.

Theorem 6.2. Let A be an $n \times n$ matrix, then for a given $x_0 \in \mathbb{R}^n$, the initial value problem $\dot{x} = A$ with $x(0) = x_0$ has the solution $x(t) = e^{At}x_0$.

Proof. If $x(t) = e^{At}x_0$ then $x(t) = Ae^{At}x_0 = Ax(t)$, and $x(0) = Ix_0 = x_0$. Now to show this solution is unique set $y(t) = e^{-At}x(t)$. Then

$$\begin{aligned} y' &= -Ae^{-At}x(t) + e^{-At}x'(t) \\ &= -Ae^{-At}x(t) + e^{-At}Ax(t) = 0. \end{aligned}$$

Thus $y'(t) = 0$ and so y must be a constant, say $y(t) = x_0$. Then $e^{-At}x(t) = x_0$ and so $x(t) = e^{At}x_0$. Therefore $x(t) = e^{At}x_0$ is unique solution to $\dot{x} = Ax$. □

Proposition. *If P and T are linear transformations on $\mathbb{R}^n$ and $S = PTP^{-1}$ then $e^S = Pe^Tp^{-1}$.*

Proof. By definition

$$\begin{aligned} e^S &= \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(PTP^{-1})}{k!} \\ &= \left(I + \frac{PTP^{-1}}{1} + \frac{PTP^{-1})^2}{2!} + \frac{(PTP^{-1})^3}{3!} + \cdots \right) \\ &= P\left(I + T + \frac{T^2}{2} + \frac{T^3}{3} + \cdots \right) P^{-1} \\ &= Pe^TP^{-1}. \end{aligned}$$

□

Remark. In the above proof to go from the second to third equality note that $I = PP^{-1}$ so that

$$\begin{aligned} (PTP^{-1})^2 &= (PTP^{-1})(PTP^{-1}) \\ &= PTP^{-1}PTP^{-1} \\ &= PTTP^{-1} \\ &= PT^2P^{-1} \end{aligned}$$

and

$$\begin{aligned}(PTP^{-1})^3 &= (PTP^{-1})(PTP^{-1})(PTP^{-1})\\ &= PTP^{-1}PTP^{-1}PTP^{-1}\\ &= PTTTP^{-1}\\ &= PT^3P^{-1}\end{aligned}$$

7 Generalized Eigenspaces

If $PAP^{-1} =$ diag$[\lambda_j]$ then $e^{At} = P\text{diag}(e^{\lambda_j t})P^{-1}$ where $\text{diag}(e^{\lambda_j t}) = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t}\end{bmatrix}$ for a 2×2 linear system.

If $A = \begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}$ then

$$\begin{aligned} e^{At} &= I + At + \frac{1}{2!}(At)^2 + \frac{1}{3!}(At)^3 + \cdots\\ &= \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix} + \frac{1}{2!}\begin{bmatrix}\lambda_1^2 t^2 & 0\\ 0 & \lambda_2^2t^2\end{bmatrix} + \cdots\\ &= \begin{bmatrix}1+\lambda_1 t + \frac{1}{2!}(\lambda_1^2t^2)+\cdots & 0\\ 0 & 1+\lambda_2 t+\frac{1}{2!}(\lambda_2^2t^2)+\cdots\end{bmatrix}.\\ &= \begin{bmatrix}e^{\lambda_1 t} & 0\\ 0 & e^{\lambda_2 t}\end{bmatrix}\end{aligned}$$

Let $P = [v_1 \dots v_n]$ where $v_1 \dots v_2$ are the eigenvectors of A. Thus P is non-singular and so P^{-1} exists.

Now for a 2×2 linear system

$$\begin{aligned} A\begin{bmatrix}v_1 & v_2\end{bmatrix} &= \begin{bmatrix}Av_1 & Av_2\end{bmatrix}\\ &= \begin{bmatrix}\lambda_1 v_1 & \lambda_2 v_2\end{bmatrix}\\ &= \begin{bmatrix}v_1 & v_2\end{bmatrix}\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}.\end{aligned}$$

So $AP = A\Lambda$ which implies that $\Lambda = P^{-1}AP$ and then we say that A is diagonalizable or semi-simple.

We say that going from $A \to P^{-1}AP$ is a similarity transform.

Consider $\dot{x} = Ax$. Consider $Py = x$ which implies $y = P^{-1}x$. Then

$$\frac{dy}{dt} = P^{-1}\frac{dy}{dt} = P^{-1}Ax = P^{-1}APy.$$

So $\frac{dy}{dt} = P^{-1}APy$ if A is diagonal Λ.

Then

$$\frac{dy}{dt} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} y$$

and so

$$\frac{dy_1}{dt} = \lambda_1 y_1$$
$$\frac{dy_2}{dt} = \lambda_2 y_2$$

and our solution is $y(t) = e^{\Lambda t}c$ and then we transform back to x to find that $x(t) = Py = Pe^{\Lambda t}c$.

7.1 Non-diagonalizable matrices

Not all matrices are diagonalizable, such as ones with repeated eigenvalues. For example, if the characteristic equation of a matrix is $(\lambda - 1)^2(\lambda - 2)^5 = 0$ then $\lambda_1 = 1$ has an algebraic multiplicity of 2 and $\lambda = 2$ has an algebraic multiplicity of 5. To find λ we solve the equation $(\det(A - \lambda I) = 0$ and so we find the values of λ that make $(A - \lambda I)$ singular and hence $(A - \lambda I)$ has a nontrivial nullspace.

The eigenvector is then exactly the nullspace of $(A - \lambda I)$. When the algebraic and geometric multiplicity of an eigenevalue are we say there is a deficiency and we need a generalized eigenvector.

Definition. Invariant space. A space E is invariant under an operator T if for every $v \in E$ it follows that $T(v) \in E$.

Definition. Generalized eigenspace. Consider $T : E \to E$, with eigenvalues and eigenvector pair where $v \in \ker(T - \lambda I)$.

Suppose λ_k is an eigenvalue of a linear operator T with algebraic multiplicity n_k. The *generalized eigenspace* of λ_k is

$$E_k := \ker\left[(T - \lambda_k I)^{n_k}\right].$$

The generalized eigenspace is an invariant subspace.

Remark. For a 2×2 linear system that has a saddle phase portrait, the x-axis is an invariant subspace. If you start on the x-axis and apply the operator T you stay on the x-axis.

Theorem 7.1. Each of the generalized eigenspaces E_j of a linear operator T is invariant under T, that is, if E_j is a generalized eigenspace, then $T : E_j \to E_j$.

Proof. Suppose $v \in E_j$, so $(T - \lambda_j I)^{n_j})v = 0$. We want to show that $Tv \in E_j$. Compute

$$
\begin{aligned}
(T-\lambda_j I)^{n_j} T v &= (T-\lambda_j I)^{n_j} T v - \lambda_j (T-\lambda_j I)^{n_j} v \\
&= (T-\lambda_j I)^{n_j} [T v - \lambda_j v] \\
&= (T-\lambda_j I)^{n_j} (T-\lambda_j I) v \\
&= (T-\lambda_j I)(T-\lambda_j I)^{n_j} v \\
&= 0.
\end{aligned}
$$

Therefore, whenever $v \in E_j$ then $Tv \in E_j$ and so E_j is invariant under T.

□

Remark. In the above proof for the first equality we can subtract $\lambda_j(T-\lambda_j I)^{n_j} v$ because $v \in E_j$ which means that $(T-\lambda_j I)^{n_j} v = 0$. In the second equality we pull out $(T-\lambda_j I)^{n_j}$ to the left. Then in the last equality we can switch the order of $(T-\lambda_j I)^{n_j}$ and $(T-\lambda_j I)$ because it commutes with itself.

Theorem 7.2. Let T be a linear operator on a complex vector space E with distinct eigenvalues $\lambda_1 \dots \lambda_r$ and let E_j be the generalized eigenspaces of T with eigenvalue λ_j. Then the $\dim(E_j)$ is the algebraic multiplicity of λ_j and the generalized eigenvectors span E,

$$E = E_1 \oplus E_2 \oplus \cdots \oplus E_r.$$

Example. Consider the 3×3 system where

$$A = \begin{bmatrix} 6 & 2 & 1 \\ -7 & -3 & -1 \\ -11 & -7 & 0 \end{bmatrix}.$$

The characteristic polynomial of A is $(\lambda-2)^2(\lambda+1)$. Setting this polynomial equal to zero and solving for λ we find that $\lambda_2 = 2$ with algebraic multiplicity of 2 and $\lambda_1 = -1$ with algebraic multiplicity 1. To find the eigenvector associated with $\lambda_1 = 2$ we solve $(A-\lambda_2 I)v_2 = 0$ for v_2 and find that $v_2 = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$. Likewise, we solve $(A-\lambda_1 I)v_1 = 0$ for v_1 and find that $v_1 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$. Now to find v_3 we must find a generalized eigenvector since there is a deficiency for λ_1 (i.e., the algebraic and geometric multiplicities are different). To find the generalized eigenvector we need to solve $(A-\lambda_2 I)^2 v_3 = 0$. Solving the previous equation for v_3 we find that $v_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

Remark. To find the generalized eigenvector in the previous example we could have also solved $(A-\lambda_2 I)v_3 = v_2$ for v_3. This is because we know $(A-\lambda_2 I)v_1 = 0$ and $(A-\lambda_2 I)^2 v_3 = 0$. Therefore

$$(A-\lambda_1 I)^2 v_3 = (A-\lambda I)v_1 = 0.$$

8 Semi simple Nilpotent Decomposition

Definition. Nilpotent. Let N be an $n \times n$ matrix. Then N is *nilpotent* if there exists a $k \in \mathbb{N}$ such that $N^k = 0$.

Definition. Let A be on $n \times n$ matrix with generalized eigenvalues $v_1 \ldots v_n$ and $P = [v_1 \ldots v_n]$, where P is non-singular since the v's are linearly independent. Let $\Lambda = \operatorname{diag}(\lambda_1 \ldots \lambda_n)$ and define $S = P\Lambda P^{-1}$ with $Sv_i = \lambda_i v_i$. Then S is *semisimple* if there is a nonsingular matrix P such that $P^{-1}SP = \Lambda$. Then $A = S + N$, where N is nilpotent.

Lemma. *Let $N = A - S$ where $S = P\Lambda P^{-1}$. Then N commutes with S and is nilpotent with order at most the maximum of the algebraic multiplicity of the eigenvalues of A.*

Definition. Commuting matrices. We say that the matrices S and S commute if $SN = NS$ or if their commutator $[S, N] = SN - NS$ is equal to zero.

Proof. Consider $[S, N] = [S, A - S] = [S, A] - [S, S] = [S, A]$. For any $v \in E_j$ we have $Sv = \lambda_j v$ and

$$\begin{aligned}[S, A]v &= SAv - ASv \\ &= SAv - A\lambda_j v \\ &= (S - \lambda_j I)Av.\end{aligned}$$

Since the eigenspace E_j is invariant then $Av \in E_j$ and $[S, A]v = (S - \lambda_j I)Av = 0$. Since S has the same eigenvalues and eigenvectors as A and Av is in the null space of $A - \lambda_j I$, so $(S - \lambda_j I)Av = 0$. Note that since Av is in the null space of $A - \lambda_j I$, then $(A - \lambda_j I)Av = 0$.

Recall that $E = E_1 \oplus E_2 \oplus \cdots \oplus E_r$ so any vector $\displaystyle\sum_{k=1}^{n} \alpha_k v_k$, where $v_k \in E_k$ so $[S, A]w = 0$.

Since this is true for any arbitrary vectors w, then $[S, A] = 0$. Since $[S, N] - [S, A] = 0$ then $[S, N] = 0$. So S commutes with N.

To see that N is nilpotent, suppose the maximum algebraic multiplicity of the eigenvalues is m. Then for any $v \in E_j$, since $[S, A] = 0$,

$$\begin{aligned}N^m v = (A - S)^m v &= (A - S)^{m-1}(A - S)v \\ &= (A - S)^{m-1}(Av - \lambda_j v) \\ &= (A - \lambda_j I)(A - S)^{m-1} v \\ &. \\ &. \\ &. \\ &= (A - \lambda_j I)^m v = 0.\end{aligned}$$

Since this holds for all $v \in E$, then $N^m = 0$, so N is nilpotent of order m. □

Theorem 8.1. A matrix A on a complex vector space E has a unique decomposition $A = S + N$ where S is semisimple (or diagonalizable) and N is nilpotent with $[S, N] = 0$.

8.1 Now going back to ODEs...

1. Start with $\dot{x} = Ax$ where $A \in \mathbb{R}^{n \times n}$.
2. Find the (generalized) eigenvectors and eigenvalues of A.
3. Construct $P = [v_1 \dots v_n]$ and $\Lambda = \text{diag}\{\lambda_i\}$.
4. Find $S = P\Lambda P^{-1}$.
5. Find $N = A - S$.
6. Then the general solution to $\dot{x} = Ax$ is

$$\begin{aligned} x(t) &= e^{At}c \\ &= e^{(S+N)t}c \\ &= e^{St}e^{Nt}c \\ &= e^{P\Lambda P^{-1}t}e^{Nt}c \\ &= P\text{diag}\{e^{\lambda_i t}\}P^{-1}(I + Nt + \frac{1}{2!}(Nt)^2 + \cdots \frac{1}{n!}N^m t^m)c \end{aligned}$$

Example. Let $\dot{x} = Ax$ for $A = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$. Then the characteristic equation is

$$\det(A - \lambda I) = \begin{bmatrix} 2-\lambda & 1 \\ -1 & 4-\lambda \end{bmatrix} = (2-\lambda)(4-\lambda) - (-1) = \lambda^2 - 6\lambda + 9.$$

Setting the characteristic equation equal to zero and solving for λ we find that $\lambda = 3$. Now we solve the system $(A - \lambda I) = 0$ to find the eigenvetors for λ. This is

$$\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which reduces to one equation

$$-\alpha_1 + \alpha_2 = 0.$$

This implies that $\alpha_2 = \alpha_1$ and so we let $\alpha_1 = 1 = \alpha_2$ and therefore $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Now we must find a generalized eigenvector because $\lambda = 3$ is a deficient eignevalue. To find a generalized eigenvector we solve $(A - \lambda I)v_2 = v_1$ for v_2. This equation can be written as

$$\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

which reduces to the equation $-\alpha_1 + \alpha_2 = 1$. Therefore, $\alpha_2 = \alpha_1 + 1$. If we set $\alpha_1 = 0$ then $\alpha_2 = 1$ and so $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Thus

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Now we find the matrix $S = P\Lambda P^{-1}$ where $\Lambda = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$. To find P^{-1} we have

$$P = \frac{1}{\det P} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1}{1} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}.$$

Therefore we have that the matrix S is

$$S = P\Lambda P^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}.$$

Now we find the matrix N as

$$N = A - S = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}.$$

We know that N is nilpotent of order 2 so that means that $N^2 = 0$. Now we can construct the general solution of $\dot{x} = Ax$ as

$$\begin{aligned} x(t) = e^{At}x_0 &= Pe^{\Lambda t}P^{-1}e^{Nt}x_0 \\ &= e^{3t}[I + Nt]x_0 \\ &= e^{3t} \begin{bmatrix} 1-t & t \\ -t & 1+t \end{bmatrix} x_0. \end{aligned}$$

Example. Let $\dot{x} = Ax$ for $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ with $x(0) = x_0$. The eigenvalues for A are $\lambda_1 = 1$ and $\lambda_2 = 2 = \lambda_3$. Then $v_1 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Since $\lambda = 2$ is a deficient eigenvalue we need to find a generalized eigenvector. To do this we consider the equation $(A - 2I)^2 v_3 = 0$ and solve for v_3. Thus we have

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix} v_3 = 0$$

which implies that $v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Now we create the matrix P where the column vectors are the eigenvectors v_1, v_2, v_3. So

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ -2 & 1 & 0 \end{bmatrix}.$$

Now we find the matrix S as

$$S = P\Lambda P^{-1} = P \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}.$$

The matrix N is

$$N = A - S = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{bmatrix}.$$

Now we construct our solution of $\dot{x} = Ax$ as

$$\begin{aligned} x(t) &= P \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{2t} \end{bmatrix} P^{-1}(I + Nt)x_0 \\ &= \begin{bmatrix} e^t & 0 & 0 \\ e^t - e^{2t} & e^{2t} & 0 \\ -2e^t + (2-t)e^{2t} & te^{2t} & e^{2t} \end{bmatrix}. \end{aligned}$$

Theorem 8.2. If $2n \times 2n$ real matrix A has $2n$ distinct complex eigenvalues $\lambda_j = a_j + ib_j$, $\bar{\lambda} = a_j - ib_j$ with complex eigenvectors $w_j = u_j + iv_j$ and $\bar{w} = u_j - iv_j$, then $P = [v_1 u_1 \quad v_2 u_2 \dots v_n u_n]$ is a $2n \times 2n$ invertible matrix and $P^{-1}AP = \operatorname{diag} \begin{bmatrix} a_j & -b_j \\ b_j & a_j \end{bmatrix}$.

Example. Consider the linear system

$$\dot{x} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix} x$$

where $x(0) = x_0$. Since A is a diagonal matrix with 2×2 blocks along the diagonal we can find the eigenvalues relatively easily. The first block on the diagonal of A is $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and so $\lambda_{1_\mp} = 1 \mp i$. Since the second block on the diagonal is $\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$ then $\lambda_{1_\mp} = 2 \mp i$. Then the eigenvector $w_{1_\mp}$ associated with $\lambda_{1_\pm}$ is

$$w_{1_\mp} = \begin{bmatrix} \pm i \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \mp i \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

and the eigenvector associated with $\lambda_{2_\mp}$ is

$$w_{2_\mp} = \begin{bmatrix} 0 \\ 0 \\ 1 \mp i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \mp i \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Now we can construct the matrix $P = [v_1 u_1 v_2 u_2]$ as

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and so

$$\Lambda = P^{-1}AP = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Now we can construct our general solution as

$$x(t) = P \begin{bmatrix} e^t \cos t & -e^t \sin t & 0 & 0 \\ e^t \sin t & e^t \cos t & 0 & 0 \\ 0 & 0 & e^{2t} \cos t & -e^{2t} \sin t \\ 0 & 0 & e^{2t} \sin t & e^{2t} \cos t \end{bmatrix} P^{-1} x_0.$$

Note that

$$e^{\begin{bmatrix} a_j & -b_j \\ b_j & a_j \end{bmatrix} t} = \begin{bmatrix} e^{a_j t} \cos b_j t & -e^{a_j t} \sin b_j t \\ e^{a_j t} \sin b_j t & e^{a_j t} \cos b_j t \end{bmatrix} = e^{a_j t} \begin{bmatrix} \cos b_j t & -sin b_j t \\ \sin b_j t & \cos b_j t \end{bmatrix}.$$

Now we can rewrite our solution as

$$x(t) = \begin{bmatrix} e^t \cos t & -e^t \sin t & 0 & 0 \\ e^t \sin t & e^t \cos t & 0 & 0 \\ 0 & 0 & e^{2t}(\cos t + \sin t) & -2e^{2t} \sin t \\ 0 & 0 & e^{2t} \sin t & e^{2t}(\cos t - \sin t) \end{bmatrix} x_0.$$

Remark. For an example of a repeated complex eigenvalue with algebraic multiplicity consider $(\lambda^2 + 2)^2 = 0$ and so $\lambda^2 + 1 - 0$. Therefore $\lambda^2 = -1$ and so $\lambda = \pm i$.

Example. Consider the linear system $\dot{x} = Ax$ where

$$A = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Then $\lambda = \pm i$ with algebraic multiplicity of 2. The eigenvector $w_{1_\pm}$ is

$$w_{1_\pm} = \begin{bmatrix} 0 \\ 0 \\ i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \pm i \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

We must now find a generalized eigenvector because λ is a deficient eigenvalue. To do this we consider $(A - \lambda I)^2 w_{2\pm} = 0$ and solve this equation for $w_{2\pm}$. Doing this we find that

$$w_{2\pm} = \begin{bmatrix} i \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \pm i \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Now the matrix P is

$$P = [v_1 u_1 v_2 u_2] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

and then

$$S = P\Lambda P^{-1} = P \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} P^{-1} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

Now we construct the matrix N as

$$N = A - S = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

We know that N is nilpotent with order 2 so that $N^2 = 0$. Now we can write our general solution as

$$x(t) = P \begin{bmatrix} \cos t & -\sin t & 0 & 0 \\ \sin t & \cos t & 0 & 0 \\ 0 & 0 & \cos t & -sint \\ 0 & 0 & \sin t & \cos t \end{bmatrix} P^{-1}(I + Nt)x_0.$$

9 Fundamental Solution

Theorem 9.1. Let A be an $n \times n$ matrix. Then the initial value problem $\dot{x} = Ax$, $x(0) = x_0$ has the unique solution $x(t) = e^{At}x_0$.