MATH 487 - Notes
Continuous Dynamical Systems

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Contents

1 Linear Systems of ODEs

A linear system of ODEs can be written as:

$$\dot{x} = Ax$$

where $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$. We will look at the solution to this system as n varies over $0, 1, 2, \ldots$ and the system of equations are coupled are not coupled.

1.1 n = 1

If $n = 1$ we have

$$\begin{aligned} \dot{x} &= ax \\ x(0) &= c \end{aligned}$$

and the solution to this equation is $x(t) = ce^{at}$.

1.2 n = 2 (uncoupled)

If $n = 2$ and the two equations are not coupled then we have

$$\begin{array}{ll} \dot{x}_1 = -x_1 & x_1(0) = c_1 \\ \dot{x}_2 = 2x_2 & x_2(0) = c_2 \end{array}.$$

Since the two equations are not coupled we can solve them separately giving the solution

$$\begin{aligned} x_1(t) &= c_1 e^{-t} \\ x_2(t) &= c_2 e^{2t} \end{aligned}.$$

We can rewrite these separate equations as a linear system,

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad x(0) = c, x \in \mathbb{R}^2$$

and the solution to this linear system is

$$x(t) = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix} c \qquad c \in \mathbb{R}^2.$$

[to do: insert graph with phase curves]

To find the equations of the phase curves for this system we can divide both sides of the second equation by the first equation

$$\frac{dx_2}{dx_1} = \frac{2x_2}{-x_1}$$

which is a first order differential equation we can solve by separation of variables. Letting $y = x_2$ and $x = x_1$ and rearranging the equation we have

$$\frac{dy}{dx} = \frac{-2y}{-x} \implies \frac{dy}{y} = \frac{-2}{x} dx$$

and integrating both sides then solving for y we get

$$\begin{aligned} \ln y &= -2 \ln x + c = \ln x^{-2} + c \\ y &= e^c \frac{1}{x^2} = \frac{\hat{c}}{x^2} \end{aligned}$$

where $\hat{c}$ will determine which phase curve we are on in the phase curve diagram.

1.3 n = 3 (uncoupled)

If $n = 3$ and the equations are not coupled then we have

$$\begin{aligned} \dot{x}_1 &= x_1 \\ \dot{x}_2 &= x_2 \\ \dot{x}_3 &= -x_3 \end{aligned}$$

which has the solution

$$\begin{aligned} x_1(t) &= c_1 e^{t} \\ x_2(t) &= c_2 e^{t} \\ x_3(t) &= c_3 e^{-t} \end{aligned}.$$

These three equations can be rewritten as

$$\dot{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} x$$
$$x(0) = c$$

where $x \in \mathbb{R}^3$ and $c \in \mathbb{R}^3$.
[add phase portrait diagram]

1.4 Classification of 2 × 2 linear systems based on eigenvalues of A

Let $\dot{x} = Ax$ represents a system of coupled equations. Then A must have some non-zero entries for the off diagonals and therefore A is not diagonal. First we state a useful theorem from linear algebra.

Theorem 1.1. If the eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_n$ of the $n \times n$ matrix A are real and distinct then any set of corresponding eigenvectors form a basis for $\mathbb{R}^n$. In addition, the matrix P whose column vectors are the eigenvectors of A, denoted $P = [v_1, v_2, \ldots, v_n] \in \mathbb{R}^n$ is invertible and

$$P^{-1}AP = \text{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n],$$

where $\text{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n]$ denotes a diagonal matrix whose diagonal entries starting at the upper left corner are $\lambda_1, \lambda_2, \ldots, \lambda_n$.

We now consider different cases for the eigenvalues of A.

1.4.1 Case 1: A has real and distinct eigenvalues

Start with $\dot{x} = Ax$ assuming this represents a system of coupled differential equations and that A has real and distinct eigenvalues. Assume $P = [v_1, v_2, \ldots, v_n]$ so P^{-1} exists (Since $v_1, v_2, \ldots, v_n$ form a basis they are linearly independent, which implies P is invertible). Define

$$y = P^{-1}x. \tag{1}$$

Then left multiplying both sides of the above equation by P we have

$$Py = PP^{-1}x = x$$

so that we have $x = Py$. Now taking the derivatives on both sides of (1) we have

$$\begin{aligned} \dot{y} &= P^{-1}\dot{x} \\ &= P^{-1}Ax \\ &= P^{-1}A(Py) \\ &= (P^{-1}AP)y \end{aligned}^{\cdot} \tag{2}$$

We can have converted the original linear system $\dot{x} = Ax$ to a new system

$$\dot{y} = \Lambda y$$

where $\Lambda = \text{diag}[\lambda_1, \lambda_2, \dots, \lambda_n]$. This is a system of uncoupled differential equations that has the solution

$$y(t) = \begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{bmatrix} y(0)$$

where $y(0) = P^{-1}x(0)$. Now plugging in this value for y into $x = Py$ we get the solution to the original linear system as

$$x(t) = P \begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{bmatrix} P^{-1}x(0).$$

Example. Consider the linear system $\dot{x} = \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix} x$. Then solving the equation $\det(A - \lambda I) = 0$ we find that $\lambda_1 = 5$ and $\lambda_2 = 4$. Then $v_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 & 2 \end{bmatrix}$.

Now we need to form the matrix $P = \text{diag}[v_1, v_2]$ which is

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

We also need to find P^{-1} which is

$$P^{-1} = \frac{1}{\det P} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}.$$

Now we form the matrix $P^{-1}AP$ as

$$\begin{aligned} P^{-1}AP &= \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 10 & -5 \\ -4 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix}. \end{aligned}$$

Then the solution to $\dot{y} = \Lambda y = \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix}$ is $y(t) = \begin{bmatrix} e^{5t} & 0 \\ 0 & e^{4t} \end{bmatrix}$. Then converting back to x using $x = Py$ we have the solution to our original linear system as

$$x(t) = Py(t) = P \begin{bmatrix} e^{5t} & 0 \\ 0 & e^{4t} \end{bmatrix} P^{-1}x(0).$$

1.4.2 Case 2: A has real repeated eigenvalues

Consider the linear system $\dot{x} = Ax$ where $x \in \mathbb{R}^2$ and $A \in \mathbb{R}^{2\times 2}$. Now consider the case where

$$A = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix}.$$

To find the eigenvalues of A we solve the equation $\det(A - \lambda I) = 0$ for λ, which is $(\lambda_1 - \lambda)^2 = 0$ and so $\lambda = \lambda_1$ and $\lambda = \lambda_1$. To find the eigenvector associated with λ_1 we plug in λ_1 for λ in $A - \lambda I = 0$ and solve the resulting system of linear equations. Doing this we have

$$\begin{bmatrix} \lambda_1 - \lambda_1 & 1 - 0 \\ 0 & \lambda_1 - \lambda_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and then we have the linear system

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This can be rewritten as two equations,

$$\begin{aligned} 0(\alpha_1) + 1(\alpha_2) &= 0 \\ 0(\alpha_1) + 0(\alpha_2) &= 0 \end{aligned}$$

and we see that $\alpha_2 = 0$ and α_1 is a free variable and so we set $\alpha_1 = 1$. Then the eigenvector associated with λ_1 is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Now consider the linear system

$$\begin{aligned} x' &= \lambda_1 x + y \\ y' &= \lambda_1 y \end{aligned}$$

where $y(t) = c_2 e^{\lambda_1 t}$. Now plugging in $y(t)$ into $x' = \lambda_1 x + y$ we have $x' = \lambda_1 x + c_2 e^{\lambda_1 t}$, which is a first order, linear, nonhomogenous equation. The general solution to this equation is $x(t) = x(t)_h + x(t)_p$ where $x(t)_h$ is the solution to the corresponding homogeneous equation $x' = \lambda_1 x$ and $x(t)_p$ is any solution to the nonhomogeneous equation. We have that $x_h(t) = c_1 e^{\lambda_1 t}$ and $x_p(t) = cte^{\lambda_1 t}$. Therefore $\{e^{\lambda_1 t}, te^{\lambda_1 t}\}$ is a fundamental solution set for the differential equation and the general solution is

$$x(t) = c_1 e^{\lambda_1 t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{\lambda_1 t} \begin{bmatrix} t \\ 1 \end{bmatrix}$$

Now what happens to the solution $x(t)$ as $t \to \infty$? We consider two cases (1) $\lambda_1 > 0$ and (2) $\lambda_1 < 0$.

If $\lambda_1 > 0$ then $x(t) \to \mp\infty$.

If $\lambda_1 < 0$ then we need to find $\lim_{t\to\infty} c_2 e^{\lambda_1 t}\begin{bmatrix} t \\ 1 \end{bmatrix} = \begin{bmatrix} \lim_{t\to\infty} c_2 t e^{\lambda_1 t} \\ \lim_{t\to\infty} c_2 e^{\lambda_1 t} \end{bmatrix}$. We know that $\lim_{t\to\infty} c_2 e^{\lambda_1 t} = 0$ since $\lambda < 0$. To find $\lim_{t\to\infty} c_2 t e^{\lambda_1 t}$ we can use L'Hopital's rule,

$$\begin{aligned} \lim_{t\to\infty} c_2 t e^{\lambda_1 t} &= c_2 \lim_{t\to\infty} \frac{\frac{t}{1}}{e^{-\lambda_1 t}} \\ &= c_2 \lim_{t\to\infty} \frac{1}{-\lambda_1 e^{-\lambda_1 t}} \\ &= c_2 \frac{1}{\infty} \\ &= c_2 \cdot 0 \\ &= 0 \end{aligned}.$$

Therefore, if $\lambda_1 < 0$ then $x(t) \to 0$ as $t \to \infty$.

1.4.3 Case 3: A has complex conjugate pair of eigenvalues

Consider the linear system $\dot{x} = Ax$ where $x \in \mathbb{R}^2$ and $A \in \mathbb{R}^{2\times 2}$. Now assume that the eigenvalues $\lambda_{\mp}$ for A are a complex conjugate pair $\alpha \mp i\beta$ where α and β are real numbers and $i = \sqrt{-1}$. The eigenvectors associated with the complex conjugate pair of eigenvalues are $v\mp = u \mp iw$. Assume the initial conditions are $c\mp = \frac{1}{2}(g \mp ih)$. The solutions $e^{\lambda t}$ will now have the form $e^{(\alpha \mp i\beta)t}$ which we can rewrite using Euler's formula as

$$e^{(\alpha \mp i\beta)t} = e^{\alpha t} e^{\mp i\beta t} = e^{\alpha t}(\cos\beta t \mp i \sin\beta t).$$

2 Classification of phase portrait for 2 x 2 linear systems

In this section we consider linear systems of the form

$$\begin{aligned} \dot{x} &= Ax \\ x(0) &= x_0 \end{aligned}$$

where $A \in \mathbb{R}^{2\times 2}$, $x_0 \in \mathbb{R}^2$, and x is a function $x : \mathbb{R} \to \mathbb{R}^2$ where $x(t) = (x_1(t), x_2(t))$. Considering a specific example this system can be written as

$$\begin{aligned} \dot{x}_1 &= -x_1 \quad x_1(0) = c_1 \\ \dot{x}_2 &= 2x_2 \quad x_2(0) = c_2 \end{aligned}$$

and the solution is

$$\begin{aligned} x_1(t) &= c_1 e^{-t} \\ x_2(t) &= c_2 e^{2t} \end{aligned}.$$

This is a system of decoupled equations that can also be written as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

with initial conditions

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}(0) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

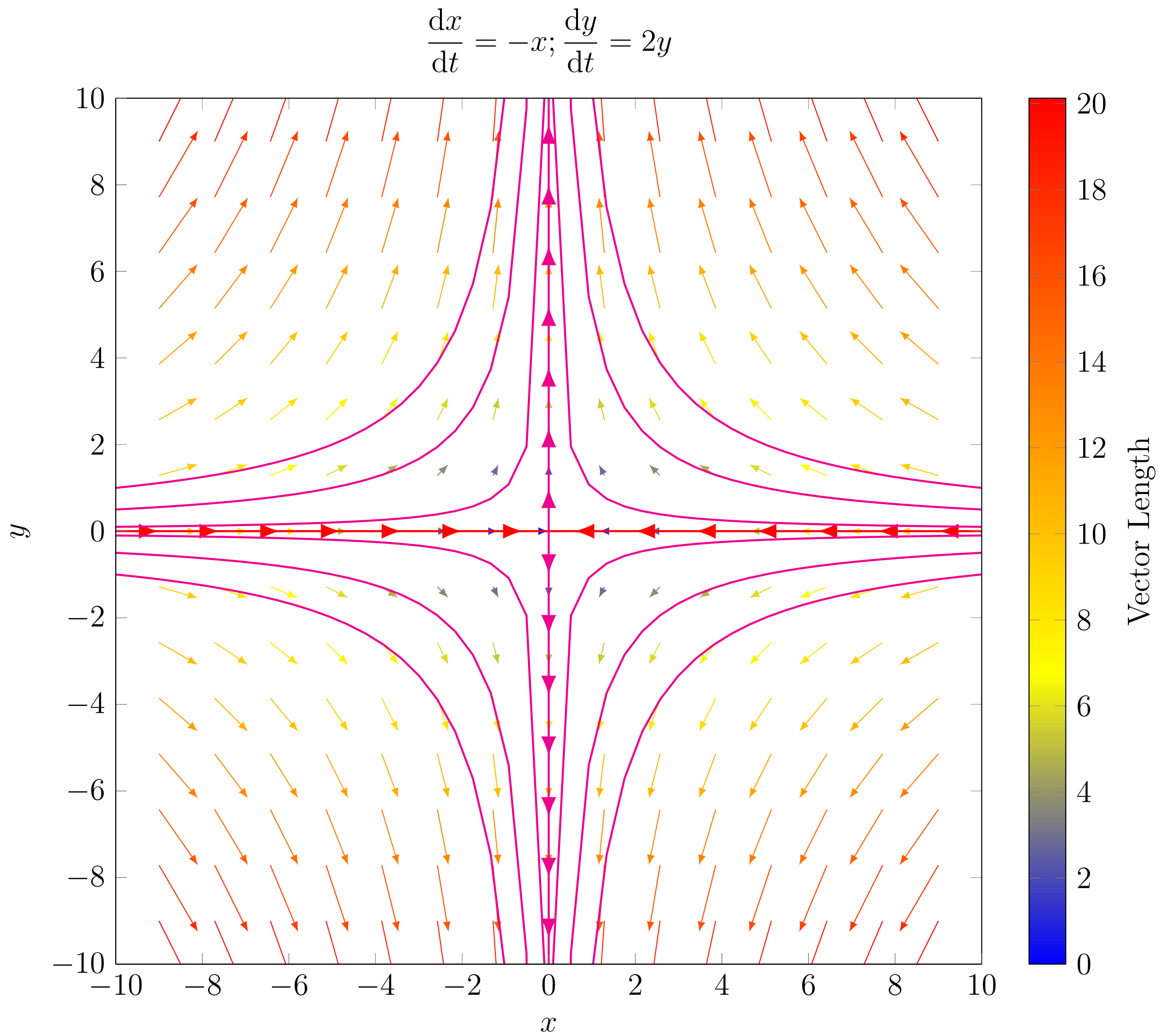
The solution can be written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

The eigenvalues of the matrix $\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$ are $\lambda_1 = -1$ and $\lambda_2 = 2$ with associated eigenvectors $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. So we can also write the solution of the above system as

$$\begin{aligned} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &= c_1 e^{-t} v_1 + c_2 e^{2t} v_2 \\ &= c_1 e^{-t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \\ &= \begin{bmatrix} c_1 e^{-t} \\ c_2 e^{2t} \end{bmatrix} \end{aligned}$$

2.1 Saddle



2.2 Node

2.3 Focus or Spiral

2.4 Degenerate Cases

3 Summary of solutions of 2 × 2 linear systems

3.1 real and distinct

Consider the linear system $\dot{x} = Ax$ where

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$

and the eigenvalues of A are real and distinct.

The solution of this system has the form

$$x(t) = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t}.$$

where λ_1 and λ_2 are eigenvalues of A, v_1 and v_2 are the eigenvectors associated with λ_1 and λ_2, and c_1 and c_2 are constants.

3.2 repeated root

Consider the linear system $\dot{x} = Ax$ where

$$\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$$

and the eigenvalues of A are real and repeated.

The solution to this system has the form

$$x(t) = c_1 v_1 e^{-t} + c_2 v_2 t e^{-t}$$

where $\lambda_1 = -1$ and $\lambda_2 = -1$ are eigenvalues of A, v_1 and v_2 are the eigenvectors associated with $\lambda_1 = -1$ and $\lambda_2 = -1$, and c_1 and c_2 are constants.

3.3 complex conjugate pair

Consider the linear system $\dot{x} = Ax$ where

$$\begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$$

and the eigenvalues of A are $\lambda = \alpha \mp i\beta$ and the eigenvectors associated with λ are $u \mp iv$. The solution to this system has the form

$$x(t) = e^{\alpha t} \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} \cos\beta t & \sin\beta t \\ -\sin\beta t & \cos\beta t \end{bmatrix} \begin{bmatrix} g \\ -h \end{bmatrix}$$

4 Trace-Determinant plane

Consider the generic 2×2 linear system $\dot{x} = Ax$ where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then we have that $\det A = ad - bc$ and $\operatorname{tr} A = a + d$.

To find the eigenvalues of A we solve the equation $\det(A - \lambda I) = 0$. First we calculate $\det(A - \lambda I)$ as

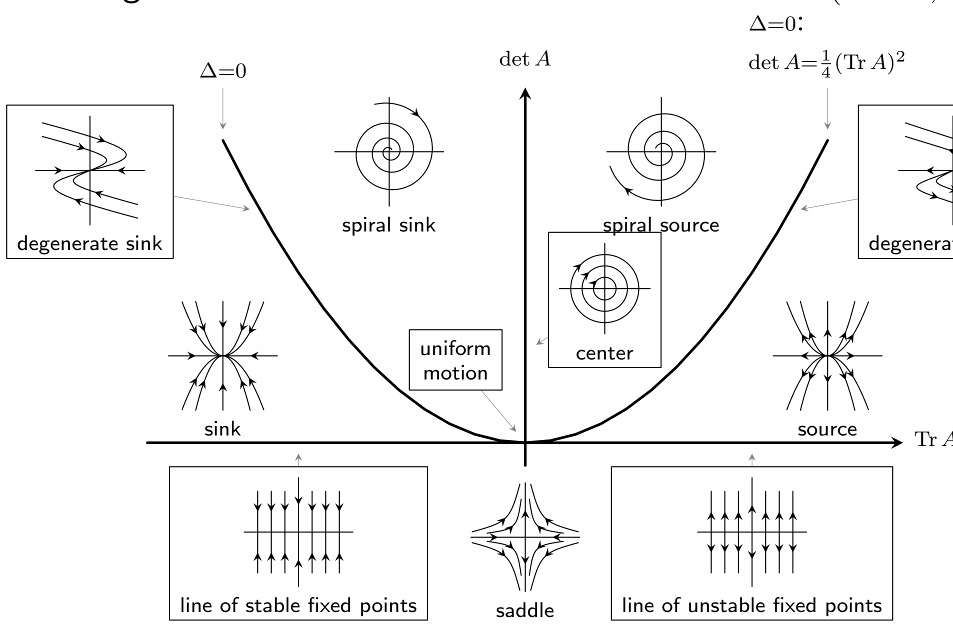
$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix} \\ &= (a - \lambda)(d - \lambda) - bc \\ &= ad - (a + d)\lambda - ad - bc \\ &= \lambda^2 - \operatorname{tr} A\lambda + \det A. \end{aligned}$$

Setting $\lambda^2 - \operatorname{tr} A\lambda + \det A = 0$ and using the quadratic formula we can solve for λ,

$$\lambda = \frac{\operatorname{tr} A \mp \sqrt{(\operatorname{tr} A)^2 - 4\det A}}{2}.$$

The type of eigenvalues for A will depend on the value of $(\operatorname{tr} A)^2$ and $4\det A$. We can use this information to classify the phase diagrams of any 2×2 linear system.

Poincaré Diagram: Classification of Phase Portaits in the $(\det A, \operatorname{Tr} A)$-plane



5 Diagonalization of $n \times n$ matrices

5.1 Jordan Canonical Decomposition

Theorem 5.1. Let A be a real matrix with real eigenvalues $\lambda_j,\ j = 1,\ldots,k$ and complex eigenvalues $\lambda_j = a_j + ib_j,\ j = k+1,\ldots,n$. Then there exists a basis $\{v_1,\ldots,v_k, u_{k+1}v + k+1,\ldots,u_n v_n\}$ where $v_j,\ j = 1,\ldots,k$ and $w_j,\ j = 1,\ldots,n$ are generalized eigenvalues of A with $u_j = Re(w_j)$ and $v_j = Im(w_j)$, $j = k+1,\ldots,n$ such that $P = [v_1 \ldots v_k \quad v_{k+1}u_{k+1}\ldots v_n u_n]$ is invertible and

$$P^{-1}AP = \begin{bmatrix} B_1 & 0 & \ldots & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & \ldots & \ddots & 0 \\ 0 & 0 & \ldots & B_r \end{bmatrix}$$

where B_j are Jordan blocks of the form

$$\begin{bmatrix} \lambda & 1 & 0 & 0 & 0 & 0 \\ 0 & \ddots & \ddots & 0 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 0 & 0 \\ 0 & 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \lambda & 1 \end{bmatrix}$$

for real λ or

$$\begin{bmatrix} D & I_2 & 0 & 0 & 0 & 0 \\ 0 & \ddots & \ddots & 0 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 0 & 0 \\ 0 & 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & D & I_2 \end{bmatrix}$$

where $D = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ for complex eigenvalues. Jordan form is unique up to the order of the blocks.

Example. Consider a 5×5 matrix with 1 real and distinct eigenvalue, 1 real and repeated eigenvalue, and 1 complex conjugate pair of eigenvalues. Then the Jordan form is

$$\left[\begin{array}{c|cc|cc} \lambda_1 & 0 & 0 & 0 & 0 \\ \hline 0 & \lambda_2 & 1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 & 0 \\ \hline 0 & 0 & 0 & \alpha & \beta \\ 0 & 0 & 0 & -\beta & \alpha \end{array}\right]$$

where B_1 is 1×1, B_2 is 2×2, and B_3 is 2×2 so that

$$P^{-1}AP = \begin{bmatrix} B_1 & 0 & 0 \\ 0 & B_2 & 0 \\ 0 & 0 & B_3 \end{bmatrix}.$$

5.2 Examples of Jordan blocks

5.2.1 2×2 matrices

1. Real and distinct eigenvalues

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

2. Real, repeated eigenvalues with algebraic and geometric multiplicity 2

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{bmatrix}$$

3. Real, repeated eigenvalues with algebraic multiplicity 2 and geometric multiplicity 1

$$\begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix}$$

4. pair of complex conjugate eigenvalues

$$\begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix}$$

5.2.2 3×3 matrices

1. real and distinct eigenvalues

$$\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

2. real and repeated eigenvalues with algebraic and geometric multiplicity of 3

$$\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix}$$

3. real and repeated eigenvalues with algebraic multiplicity 3 and geometric multiplicity 2

$$\begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix}$$

4. real and repeated eigenvalues with algebraic multiplicity 3 and geometric multiplicity 1

$$\begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{bmatrix}$$

5. two real eigenvalues and one repeated with algebraic multiplicity 2 and geometric multiplicity 1

$$\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{bmatrix}$$

6. two real eigenvalues and one repeated with algebraic multiplicity and geometric multiplicity 2

$$\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}$$

7. one real eigenvalues and one complex conjugate pair of eigenvalues

$$\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & -\beta & \alpha \end{bmatrix}$$

6 Exponentials of operators

If we have the initial value problem $\dot{x} = ax$ with $x(0) = x_0$ and $a \in \mathbb{R}$ then we know the solution is $x(t) = x_0 e^{at}$. Now what if we consider the linear system $\dot{x} = Ax$ with $x(0) = x_0$ but now $x \in \mathbb{R}^2$ and $A \in \mathbb{R}^{2\times 2}$. It would be nice if we could write the solution as $x(t) = e^{At} x_0$, but what is e^{At}?

Definition. Operator Norm. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator. Then the *operator norm* of T is defined to be

$$||T|| = \max_{|x|\leq 1} |T(x)|$$

where $|\cdot|$ denotes the Euclidean norm for $x \in \mathbb{R}^n$.

The operator norm has the following properties:

(a) $||T|| > 0$ and $||T|| = 0$ if and only if $T = 0$.

(b) $||kT|| = |k|||T||$ for $k \in \mathbb{R}$.

(c) $||S + T|| \leq ||S|| + ||T||$

If $T \in \mathcal{L}(\mathbb{R}^n)$ is represented by a matrix A with respect to the standard basis in $\mathbb{R}^n$ then $||A|| \leq \ell\sqrt{n}$ where ℓ is the maximum length of the rows of A.

Definition. Convergence. A sequence of linear operators $T_k \in \mathcal{L}(\mathbb{R}^n)$ converges to a linear operator $T \in \mathcal{L}(\mathbb{R}^n)$ as $k \to \infty$ if $\lim_{k\to\infty} T_k = T$, i.e.,

$$\forall\epsilon > 0 \exists N \in \mathbb{N} \text{ such that } k \geq N \implies ||T_k - T|| < \epsilon.$$

Lemma. *For $S, T \in \mathcal{L}(\mathbb{R}^n)$ and $x \in \mathbb{R}^\ltimes$*

1. $|T(x)| \leq ||T|||x|$

2. $||TS|| \leq ||T||||S||$

3. $||T^k|| \leq (||T||)^k$ *for* $k = 0, 1, 2, \ldots$.

Proof. 1. True for $x = 0$. Assume $x \neq 0$ and define $y = \frac{x}{|x|}$. Then

$$||T|| \geq |T(y)| = \frac{1}{|x|}|T(x)|.$$

2. For $|x| < 1$, 1. implies

$$\begin{aligned} |T(S(x))| &\leq ||T|||S(x)| \\ &\leq ||T||||S|||x|. \end{aligned}$$

3. Follows from 2. (by induction)

□

Definition. Weierstrass M-test. Suppose that f_n is a sequence of real or complex valued functions defined on a set A and that there is a sequence of non-negative numbers M_n such that $|f_n(x)| \leq M_n$ for all $n \geq 1$ and for all $x \in A$ and $\displaystyle\sum_{n=1}^{\infty} M_n$ converges then $\displaystyle\sum_{n=1}^{\infty} f_n(x)$ converges uniformly and absolutely.

Theorem 6.1. Given $T \in \mathcal{L}(\mathbb{R}^n)$ and $t_0 > 0$, the series $\displaystyle\sum_{k=0}^{\infty} \frac{T^k t^k}{k!}$ is absolutely and uniformly convergent for all $|t| \leq t_0$.

Proof. Let $||T|| = a$. Then

$$\left|\left|\frac{T^k t^k}{k!}\right|\right| \leq \frac{||T||^k |t|^k}{k!} \leq \frac{a^k t_0^k}{k!}.$$

So $\displaystyle\sum_{k=0}^{\infty} \frac{a^k t_0^k}{k!} = e^{at_0}$. By Weierstrass M-test $\displaystyle\sum_{k=0}^{\infty} \frac{T^k t^k}{k!}$ is absolutely and uniformly convergent. □

So we define the exponential of the linear operator T to be the absolutely convergent series

$$e^T = \sum_{k=0}^{\infty} \frac{T^k}{k!}$$

only for square matrices. Thus e^T is a linear operator. It follows that

$$||e^T|| \leq e^{||T||}.$$

Lemma. *Let* $A \in \mathbb{R}^n$. *Then* $\dfrac{d}{dt}e^{At} = Ae^{At}$.

Proof. Since A commutes with itself,

$$\begin{aligned}
\frac{d}{dt}e^{At} &= \lim_{h\to 0}\frac{e^{A(t-h)}-e^{At}}{h} \\
&= \lim_{h\to 0} e^{At}\left(\frac{e^{Ah}-I}{h}\right) \\
&= e^{At}\lim_{h\to 0}\frac{1}{h}\lim_{k\to\infty}\left(I+Ah+\frac{(Ah)^2}{2}+\cdots+\frac{(Ah)^k}{k!}-I\right) \\
&= e^{At}\lim_{h\to 0}\frac{1}{h}\lim_{k\to\infty}\left(Ah+\frac{(Ah)^2}{2}+\cdots+\frac{(Ah)^k}{k!}\right) \\
&= e^{At}\lim_{h\to 0}\lim_{k\to\infty}\left(A+\frac{A^2h}{2}+\frac{A^hh^{k-1}}{k!}\right) \\
&= e^{At}\lim_{k\to\infty}\lim_{h\to 0}\left(A+\frac{A^2h}{2}+\frac{A^hh^{k-1}}{k!}\right) \\
&= e^{At}\lim_{k\to} A \\
&= e^{At}A \\
&= Ae^{At}
\end{aligned}$$

□

Remark. In the last line of the above proof we switched A from the right to the left side because A commutes with itself. Therefore

$$\begin{aligned}
Ae^{At} &= A(I+At+\frac{1}{2!}(At)^2+\cdots \\
&= A+A^2t+\frac{1}{2!}A^2t^2+\cdots \\
&= (I+At+\frac{1}{2!}(At)^2+\cdots)A \\
&= e^{At}A
\end{aligned}$$

Also from the third to fourth equality we switched the limits because we have uniform convergence.

Lemma. *If S and T are linear transformations on $\mathbb{R}^n$ which commute, i.e., $ST = TS$, then $e^{S+T} = e^Se^T$.*

Proof. If $ST = TS$ then by the binomial theorem

$$(S+T)^n = n\sum_{j+k=n}\frac{S^jT^k}{j!k!}.$$

Therefore

$$e^{S+T} = \sum_{n=0}^{\infty} \sum_{j+k=n} \frac{S^j T^k}{j!k!} = \left(\sum_{j=0}^{\infty} \frac{S^j}{j!} \right) \left(\sum_{k=0}^{\infty} \frac{T^k}{k!} \right) = e^S e^T.$$

□

Remark. In the second equality above we were able to split the double sum because we have absolute convergence.

Theorem 6.2. Let A be an $n \times n$ matrix, then for a given $x_0 \in \mathbb{R}^n$, the initial value problem $\dot{x} = A$ with $x(0) = x_0$ has the solution $x(t) = e^{At}x_0$.

Proof. If $x(t) = e^{At}x_0$ then $x(t) = Ae^{At}x_0 = Ax(t)$, and $x(0) = Ix_0 = x_0$. Now to show this solution is unique set $y(t) = e^{-At}x(t)$. Then

$$\begin{aligned} y' &= -Ae^{-At}x(t) + e^{-At}x'(t) \\ &= -Ae^{-At}x(t) + e^{-At}Ax(t) = 0. \end{aligned}$$

Thus $y'(t) = 0$ and so y must be a constant, say $y(t) = x_0$. Then $e^{-At}x(t) = x_0$ and so $x(t) = e^{At}x_0$. Therefore $x(t) = e^{At}x_0$ is unique solution to $\dot{x} = Ax$. □

Proposition. *If P and T are linear transformations on $\mathbb{R}^n$ and $S = PTP^{-1}$ then $e^S = Pe^T p^{-1}$.*

Proof. By definition

$$\begin{aligned} e^S &= \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(PTP^{-1})}{k!} \\ &= \left(I + \frac{PTP^{-1}}{1} + \frac{PTP^{-1})^2}{2!} + \frac{(PTP^{-1})^3}{3!} + \cdots \right) \\ &= P\left(I + T + \frac{T^2}{2} + \frac{T^3}{3} + \cdots \right) P^{-1} \\ &= Pe^T P^{-1}. \end{aligned}$$

□

Remark. In the above proof to go from the second to third equality note that $I = PP^{-1}$ so that

$$\begin{aligned} (PTP^{-1})^2 &= (PTP^{-1})(PTP^{-1}) \\ &= PTP^{-1}PTP^{-1} \\ &= PTTP^{-1} \\ &= PT^2P^{-1} \end{aligned}$$

and

$$\begin{aligned}
(PTP^{-1})^3 &= (PTP^{-1})(PTP^{-1})(PTP^{-1}) \\
&= PTP^{-1}PTP^{-1}PTP^{-1} \\
&= PTTTP^{-1} \\
&= PT^3P^{-1}
\end{aligned}$$

7 Generalized Eigenspaces

If $PAP^{-1} =$ diag$[\lambda_j]$ then $e^{At} = P\text{diag}(e^{\lambda_j t})P^{-1}$ where diag$(e^{\lambda_j t}) = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$ for a 2×2 linear system.

If $A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ then

$$\begin{aligned}
e^{At} &= I + At + \frac{1}{2!}(At)^2 + \frac{1}{3!}(At)^3 + \cdots \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} \lambda_1^2 t^2 & 0 \\ 0 & \lambda_2^2 t^2 \end{bmatrix} + \cdots \\
&= \begin{bmatrix} 1 + \lambda_1 t + \frac{1}{2!}(\lambda_1^2 t^2) + \cdots & 0 \\ 0 & 1 + \lambda_2 t + \frac{1}{2!}(\lambda_2^2 t^2) + \cdots \end{bmatrix}. \\
&= \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}
\end{aligned}$$

Let $P = [v_1 \dots v_n]$ where $v_1 \dots v_2$ are the eigenvectors of A. Thus P is non-singular and so P^{-1} exists.

Now for a 2×2 linear system

$$\begin{aligned}
A \begin{bmatrix} v_1 & v_2 \end{bmatrix} &= \begin{bmatrix} Av_1 & Av_2 \end{bmatrix} \\
&= \begin{bmatrix} \lambda_1 v_1 & \lambda_2 v_2 \end{bmatrix} \\
&= \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.
\end{aligned}$$

So $AP = A\Lambda$ which implies that $\Lambda = P^{-1}AP$ and then we say that A is diagonalizable or semi-simple.

We say that going from $A \to P^{-1}AP$ is a similarity transform.

Consider $\dot{x} = Ax$. Consider $Py = x$ which implies $y = P^{-1}x$. Then

$$\frac{dy}{dt} = P^{-1}\frac{dy}{dt} = P^{-1}Ax = P^{-1}APy.$$

So $\dfrac{dy}{dt} = P^{-1}APy$ if A is diagonal Λ.

Then

$$\frac{dy}{dt} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} y$$

and so

$$\begin{aligned} \frac{dy_1}{dt} &= \lambda_1 y_1 \\ \frac{dy_2}{dt} &= \lambda_2 y_2 \end{aligned}$$

and our solution is $y(t) = e^{\Lambda t}c$ and then we transform back to x to find that $x(t) = Py = Pe^{\Lambda t}c$.

7.1 Non-diagonalizable matrices

Not all matrices are diagonalizable, such as ones with repeated eigenvalues. For example, if the characteristic equation of a matrix is $(\lambda-1)^2(\lambda-2)^5 = 0$ then $\lambda_1 = 1$ has an algebraic multiplicity of 2 and $\lambda = 2$ has an algebraic multiplicity of 5. To find λ we solve the equation $(\det(A-\lambda I) = 0$ and so we find the values of λ that make $(A-\lambda I)$ singular and hence $(A-\lambda I)$ has a nontrivial nullspace.

The eigenvector is then exactly the nullspace of $(A-\lambda I)$. When the algebraic and geometric multiplicity of an eigenevalue are we say there is a deficiency and we need a generalized eigenvector.

Definition. Invariant space. A space E is invariant under an operator T if for every $v \in E$ it follows that $T(v) \in E$.

Definition. Generalized eigenspace. Consider $T : E \to E$, with eigenvalues and eigenvector pair where $v \in \ker(T-\lambda I)$.

Suppose λ_k is an eigenvalue of a linear operator T with algebraic multiplicity n_k. The *generalized eigenspace* of λ_k is

$$E_k := \ker\left[(T-\lambda_k I)^{n_k}\right].$$

The generalized eigenspace is an invariant subspace.

Remark. For a 2×2 linear system that has a saddle phase portrait, the x-axis is an invariant subspace. If you start on the x-axis and apply the operator T you stay on the x-axis.

Theorem 7.1. Each of the generalized eigenspaces E_j of a linear operator T is invariant under T, that is, if E_j is a generalized eigenspace, then $T : E_j \to E_j$.

Proof. Suppose $v \in E_j$, so $(T-\lambda_j I)^{n_j})v = 0$. We want to show that $Tv \in E_j$. Compute

$$\begin{aligned}(T-\lambda_j I)^{n_j}Tv &= (T-\lambda_j I)^{n_j}Tv - \lambda_j(T-\lambda_j I)^{n_j}v\\ &= (T-\lambda_j I)^{n_j}[Tv-\lambda_j v]\\ &= (T-\lambda_j I)^{n_j}(T-\lambda_j I)v\\ &= (T-\lambda_j I)(T-\lambda_j I)^{n_j}v\\ &= 0.\end{aligned}$$

Therefore, whenever $v \in E_j$ then $Tv \in E_j$ and so E_j is invariant under T.

□

Remark. In the above proof for the first equality we can subtract $\lambda_j(T-\lambda_j I)^{n_j}v$ because $v \in E_j$ which means that $(T-\lambda_j I)^{n_j}v = 0$. In the second equality we pull out $(T-\lambda_j I)^{n_j}$ to the left. Then in the last equality we can switch the order of $(T-\lambda_j I)^{n_j}$ and $(T-\lambda_j I)$ because it commutes with itself.

Theorem 7.2. Let T be a linear operator on a complex vector space E with distinct eigenvalues $\lambda_1 \dots \lambda_r$ and let E_j be the generalized eigenspaces of T with eigenvalue λ_j. Then the $\dim(E_j)$ is the algebraic multiplicity of λ_j and the generalized eigenvectors span E,

$$E = E_1 \oplus E_2 \oplus \cdots \oplus E_r.$$

Example. Consider the 3×3 system where

$$A = \begin{bmatrix} 6 & 2 & 1\\ -7 & -3 & -1\\ -11 & -7 & 0\end{bmatrix}.$$

The characteristic polynomial of A is $(\lambda-2)^2(\lambda+1)$. Setting this polynomial equal to zero and solving for λ we find that $\lambda_2 = 2$ with algebraic multiplicity of 2 and $\lambda_1 = -1$ with algebraic multiplicity 1. To find the eigenvector associated with $\lambda_1 = 2$ we solve $(A-\lambda_2 I)v_2 = 0$ for v_2 and find that $v_2 = \begin{bmatrix}-1\\1\\3\end{bmatrix}$. Likewise, we solve $(A-\lambda_1 I)v_1 = 0$ for v_1 and find that $v_1 = \begin{bmatrix}-1\\1\\2\end{bmatrix}$. Now to find v_3 we must find a generalized eigenvector since there is a deficiency for λ_1 (i.e., the algebraic and geometric multiplicities are different). To find the generalized eigenvector we need to solve $(A-\lambda_2 I)^2 v_3 = 0$. Solving the previous equation for v_3 we find that $v_3 = \begin{bmatrix}1\\-1\\0\end{bmatrix}$.

Remark. To find the generalized eigenvector in the previous example we could have also solved $(A-\lambda_2 I)v_3 = v_2$ for v_3. This is because we know $(A-\lambda_2 I)v_1 = 0$ and $(A-\lambda_2 I)^2 v_3 = 0$. Therefore

$$(A-\lambda_1 I)^2 v_3 = (A-\lambda I)v_1 = 0.$$

8 Semi simple Nilpotent Decomposition

Definition. Nilpotent. Let N be an $n \times n$ matrix. Then N is *nilpotent* if there exists a $k \in \mathbb{N}$ such that $N^k = 0$.

Definition. Let A be on $n \times n$ matrix with generalized eigenvalues $v_1 \dots v_n$ and $P = [v_1 \dots v_n]$, where P is non-singular since the v's are linearly independent. Let $\Lambda = \text{diag}(\lambda_1 \dots \lambda_n)$ and define $S = P\Lambda P^{-1}$ with $Sv_i = \lambda_i v_i$. Then S is *semisimple* if there is a nonsingular matrix P such that $P^{-1}SP = \Lambda$. Then $A = S + N$, where N is nilpotent.

Lemma. *Let $N = A - S$ where $S = P\Lambda P^{-1}$. Then N commutes with S and is nilpotent with order at most the maximum of the algebraic multiplicity of the eigenvalues of A.*

Definition. Commuting matrices. We say that the matrices S and S commute if $SN = NS$ or if their commutator $[S, N] = SN - NS$ is equal to zero.

Proof. Consider $[S, N] = [S, A - S] = [S, A] - [S, S] = [S, A]$. For any $v \in E_j$ we have $Sv = \lambda_j v$ and

$$\begin{aligned} [S, A]v &= SAv - ASv \\ &= SAv - A\lambda_j v \\ &= (S - \lambda_j I)Av. \end{aligned}$$

Since the eigenspace E_j is invariant then $Av \in E_j$ and $[S, A]v = (S - \lambda_j I)Av = 0$. Since S has the same eigenvalues and eigenvectors as A and Av is in the null space of $A - \lambda_j I$, so $(S - \lambda_j I)Av = 0$. Note that since Av is in the null space of $A - \lambda_j I$, then $(A - \lambda_j I)Av = 0$.

Recall that $E = E_1 \oplus E_2 \oplus \cdots \oplus E_r$ so any vector $\displaystyle\sum_{k=1}^{n} \alpha_k v_k$, where $v_k \in E_k$ so $[S, A]w = 0$.

Since this is true for any arbitrary vectors w, then $[S, A] = 0$. Since $[S, N] - [S, A] = 0$ then $[S, N] = 0$. So S commutes with N.

To see that N is nilpotent, suppose the maximum algebraic multiplicity of the eigenvalues is m. Then for any $v \in E_j$, since $[S, A] = 0$,

$$\begin{aligned} N^m v = (A - S)^m v &= (A - S)^{m-1}(A - S)v \\ &= (A - S)^{m-1}(Av - \lambda_j v) \\ &= (A - \lambda_j I)(A - S)^{m-1}v \\ &\;\;. \\ &\;\;. \\ &\;\;. \\ &= (A - \lambda_j I)^m v = 0. \end{aligned}$$

Since this holds for all $v \in E$, then $N^m = 0$, so N is nilpotent of order m. $\square$

Theorem 8.1. A matrix A on a complex vector space E has a unique decomposition $A = S + N$ where S is semisimple (or diagonalizable) and N is nilpotent with $[S, N] = 0$.

8.1 Now going back to ODEs...

1. Start with $\dot{x} = Ax$ where $A \in \mathbb{R}^{n \times n}$.
2. Find the (generalized) eigenvectors and eigenvalues of A.
3. Construct $P = [v_1 \dots v_n]$ and $\Lambda = \mathrm{diag}\{\lambda_i\}$.
4. Find $S = P\Lambda P^{-1}$.
5. Find $N = A - S$.
6. Then the general solution to $\dot{x} = Ax$ is

$$\begin{aligned} x(t) &= e^{At}c \\ &= e^{(S+N)t}c \\ &= e^{St}e^{Nt}c \\ &= e^{P\Lambda P^{-1}t}e^{Nt}c \\ &= P\mathrm{diag}\{e^{\lambda_i t}\}P^{-1}(I + Nt + \frac{1}{2!}(Nt)^2 + \cdots \frac{1}{n!}N^m t^m)c \end{aligned}$$

Example. Let $\dot{x} = Ax$ for $A = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$. Then the characteristic equation is

$$\det(A - \lambda I) = \begin{bmatrix} 2-\lambda & 1 \\ -1 & 4-\lambda \end{bmatrix} = (2-\lambda)(4-\lambda) - (-1) = \lambda^2 - 6\lambda + 9.$$

Setting the characteristic equation equal to zero and solving for λ we find that $\lambda = 3$. Now we solve the system $(A - \lambda I) = 0$ to find the eigenvetors for λ. This is

$$\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which reduces to one equation

$$-\alpha_1 + \alpha_2 = 0.$$

This implies that $\alpha_2 = \alpha_1$ and so we let $\alpha_1 = 1 = \alpha_2$ and therefore $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Now we must find a generalized eigenvector because $\lambda = 3$ is a deficient eignevalue. To find a generalized eigenvector we solve $(A - \lambda I)v_2 = v_1$ for v_2. This equation can be written as

$$\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

which reduces to the equation $-\alpha_1 + \alpha_2 = 1$. Therefore, $\alpha_2 = \alpha_1 + 1$. If we set $\alpha_1 = 0$ then $\alpha_2 = 1$ and so $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Thus

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Now we find the matrix $S = P\Lambda P^{-1}$ where $\Lambda = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$. To find P^{-1} we have

$$P = \frac{1}{\det P} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1}{1} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}.$$

Therefore we have that the matrix S is

$$S = P\Lambda P^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}.$$

Now we find the matrix N as

$$N = A - S = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}.$$

We know that N is nilpotent of order 2 so that means that $N^2 = 0$. Now we can construct the general solution of $\dot{x} = Ax$ as

$$\begin{aligned} x(t) = e^{At}x_0 &= Pe^{\Lambda t}P^{-1}e^{Nt}x_0 \\ &= e^{3t}[I + Nt]x_0 \\ &= e^{3t} \begin{bmatrix} 1-t & t \\ -t & 1+t \end{bmatrix} x_0. \end{aligned}$$

Example. Let $\dot{x} = Ax$ for $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ with $x(0) = x_0$. The eigenvalues for A are $\lambda_1 = 1$ and $\lambda_2 = 2 = \lambda_3$. Then $v_1 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Since $\lambda = 2$ is a deficient eigenvalue we need to find a generalized eigenvector. To do this we consider the equation $(A - 2I)^2 v_3 = 0$ and solve for v_3. Thus we have

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix} v_3 = 0$$

which implies that $v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Now we create the matrix P where the column vectors are the eigenvectors v_1, v_2, v_3. So

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ -2 & 1 & 0 \end{bmatrix}.$$

Now we find the matrix S as

$$S = P\Lambda P^{-1} = P \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}.$$

The matrix N is

$$N = A - S = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{bmatrix}.$$

Now we construct our solution of $\dot{x} = Ax$ as

$$\begin{aligned} x(t) &= P \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{2t} \end{bmatrix} P^{-1}(I + Nt)x_0 \\ &= \begin{bmatrix} e^t & 0 & 0 \\ e^t - e^{2t} & e^{2t} & 0 \\ -2e^t + (2-t)e^{2t} & te^{2t} & e^{2t} \end{bmatrix}. \end{aligned}$$

Theorem 8.2. If $2n \times 2n$ real matrix A has $2n$ distinct complex eigenvalues $\lambda_j = a_j + ib_j$, $\bar{\lambda} = a_j - ib_j$ with complex eigenvectors $w_j = u_j + iv_j$ and $\bar{w} = u_j - iv_j$, then $P = [v_1 u_1 \quad v_2 u_2 \ldots v_n u_n]$ is a $2n \times 2n$ invertible matrix and $P^{-1}AP = \operatorname{diag} \begin{bmatrix} a_j & -b_j \\ b_j & a_j \end{bmatrix}$.

Example. Consider the linear system

$$\dot{x} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix} x$$

where $x(0) = x_0$. Since A is a diagonal matrix with 2×2 blocks along the diagonal we can find the eigenvalues relatively easily. The first block on the diagonal of A is $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and so $\lambda_{1_\mp} = 1 \mp i$. Since the second block on the diagonal is $\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$ then $\lambda_{1_\mp} = 2 \mp i$. Then the eigenvector $w_{1_\mp}$ associated with $\lambda_{1_\pm}$ is

$$w_{1_\mp} = \begin{bmatrix} \pm i \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \mp i \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

and the eigenvector associated with $\lambda_{2_\mp}$ is

$$w_{2_\mp} = \begin{bmatrix} 0 \\ 0 \\ 1 \mp i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \mp i \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Now we can construct the matrix $P = [v_1 u_1 v_2 u_2]$ as

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and so

$$\Lambda = P^{-1}AP = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Now we can construct our general solution as

$$x(t) = P \begin{bmatrix} e^t \cos t & -e^t \sin t & 0 & 0 \\ e^t \sin t & e^t \cos t & 0 & 0 \\ 0 & 0 & e^{2t} \cos t & -e^{2t} \sin t \\ 0 & 0 & e^{2t} \sin t & e^{2t} \cos t \end{bmatrix} P^{-1} x_0.$$

Note that

$$e^{\begin{bmatrix} a_j & -b_j \\ b_j & a_j \end{bmatrix} t} = \begin{bmatrix} e^{a_j t} \cos b_j t & -e^{a_j t} \sin b_j t \\ e^{a_j t} \sin b_j t & e^{a_j t} \cos b_j t \end{bmatrix} = e^{a_j t} \begin{bmatrix} \cos b_j t & -sin b_j t \\ \sin b_j t & \cos b_j t \end{bmatrix}.$$

Now we can rewrite our solution as

$$x(t) = \begin{bmatrix} e^t \cos t & -e^t \sin t & 0 & 0 \\ e^t \sin t & e^t \cos t & 0 & 0 \\ 0 & 0 & e^{2t}(\cos t + \sin t) & -2e^{2t} \sin t \\ 0 & 0 & e^{2t} \sin t & e^{2t}(\cos t - \sin t) \end{bmatrix} x_0.$$

Remark. For an example of a repeated complex eigenvalue with algebraic multiplicity consider $(\lambda^2 + 2)^2 = 0$ and so $\lambda^2 + 1 - 0$. Therefore $\lambda^2 = -1$ and so $\lambda = \pm i$.

Example. Consider the linear system $\dot{x} = Ax$ where

$$A = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Then $\lambda = \pm i$ with algebraic multiplicity of 2. The eigenvector $w_{1_\pm}$ is

$$w_{1_\pm} = \begin{bmatrix} 0 \\ 0 \\ i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \pm i \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

We must now find a generalized eigenvector because λ is a deficient eigenvalue. To do this we consider $(A-\lambda I)^2 w_{2\pm} = 0$ and solve this equation for $w_{2\pm}$. Doing this we find that

$$w_{2\pm} = \begin{bmatrix} i \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \pm i \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Now the matrix P is

$$P = [v_1 u_1 v_2 u_2] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

and then

$$S = P\Lambda P^{-1} = P \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} P^{-1} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

Now we construct the matrix N as

$$N = A - S = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

We know that N is nilpotent with order 2 so that $N^2 = 0$. Now we can write our general solution as

$$x(t) = P \begin{bmatrix} \cos t & -\sin t & 0 & 0 \\ \sin t & \cos t & 0 & 0 \\ 0 & 0 & \cos t & -sint \\ 0 & 0 & \sin t & \cos t \end{bmatrix} P^{-1}(I+Nt)x_0.$$

9 Fundamental Solution

Theorem 9.1. Let A be an $n \times n$ matrix. Then the initial value problem $\dot{x} = Ax$, $x(0) = x_0$ has the unique solution $x(t) = e^{At}x_0$.

Proof. First compute

$$
\begin{aligned}
\frac{d}{dt}e^{At} &= \lim_{h\to 0} \frac{e^{(t+h)A} - e^{tA}}{h} \\
&= \lim_{h\to 0} \left(\frac{e^{ht} - I}{h} \right) e^{tA} \\
&= \lim_{h\to 0} \left(\frac{1}{h} \sum_{n=1}^{\infty} \frac{(hA)^n}{n!} \right) e^{tA} \\
&= \lim_{h\to 0} \left(\frac{hA}{h} + \sum_{n=2}^{\infty} h^{n-1} \frac{A^n}{n!} \right) e^{At} \\
&= \lim_{h\to 0} \left(A + h \sum_{j=0}^{\infty} h^j \frac{A^{j+2}}{(j+2)!} \right) e^{At} \\
&= Ae^{At}.
\end{aligned}
$$

So $\frac{d}{dt}e^{At}x_0 = Ae^{At}x_0 = Ax$ and so $x(t) = e^{At}x_0$ is a solution. To show that $x(t)$ is unique, suppose $y(t)$ is also a solution. Then

$$
\begin{aligned}
\frac{d}{dt}\left(e^{-At}y(t)\right) &= -Ae^{-At}y(t) + e^{-tA}Ay(t) \\
&= -Ae^{-tA} + e^{-tA}Ay(t) \\
&= 0.
\end{aligned}
$$

Therefore $e^{-At}y(t) = y_0 =$ a constant then $y(t) = e^{At}y_0$. Since $x_0 = y_0$ the solutions are the same, which is a contradiction. $\square$

We consider a set of initial conditions $x_{j_0} = v_j$ for $j = 1 \ldots n$ with v_j an arbitrary initial vector. Define $\Psi = [v_1 \ldots v_n]$ (an $n \times n$ matrix). Then the solution matrix is $\Psi(t) = [x_1(t) \ldots x_n(t)]$ with $x_j(t) = e^{At}v_j$.

Theorem 9.2. The matrix initial value problem $\frac{d}{dt}\Psi = A\Psi$ with $\Psi(0) = \Psi_0$ has the unique solution $\Psi(t) = e^{tA}\Psi(0)$. When $\Psi_0 = I$, $\Psi(t) = e^{At}$ is the fundamental solution matrix.

10 Linear Stability

Consider $\dot{x} = Ax$. If $\lambda > 0$ then $x(t) = ce^{\lambda t}$ grows unbounded. If $\lambda < 0$ then $x(t) = c^{\lambda t}$ goes to zero. In some sense, $\lambda > 0$ is unstable and $\lambda < 0$ is stable.

10.1 Spectral stability

Definition. Spectral stability. A linear system is **spectrally stable** if none of the eigenvalues have a positive real part.

If $\lambda_j = \alpha_j + \beta_j i$ with $v_j = u_j + iw_j$ then

$$E^u = \text{span}\{u_j, w_j : \text{Re}(\lambda_j) > 0\}$$

is the unstable subspace,

$$E^s = \text{span}\{u_j, w_j : \text{Re}(\lambda_j) < 0\}$$

is the stable subspace,

$$E^c = \text{span}\{u_j, w_j : \text{Re}(\lambda_j) = 0\}$$

is the center subspace, and

$$E = E^u \oplus E^s \oplus E^c.$$

Since the generalized eigenspaces are invariant so are the stable, unstable, and center subspace.

Example. Consider the linear system $\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} x$. The eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 2$ with associated eigenvectors $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. So in this example we have that $E^u = \text{span}\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\}$, $E^s = \text{span}\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\}$, and $E^c = \varnothing$.

Remark. In linear systems the terms manifold is the same as subspace but in non-linear systems there two terms are not equivalent.

Construct a restriction of $A\big|_{E^s}$. If $P = [v_1 \dots v_k]$ (an $n \times k$ matrix) is a basis for E^u (so that P contains all eigenvectors with eigenvalues that have positive real part) then

$$x = \sum_{i=1}^{k} c_i v_i = PC \in E^u.$$

Since E^u is an invariant subspace, each column in AP is in E^k. We define $\mathcal{U} = \{u_{ij}\} = A\big|_{E^u}$ is the $k \times k$ matrix that solves $AP = P\mathcal{U}$.

The dynamics of $x \in E^u$ are determined by the dynamics of C. Since

$$x = \sum_{i=1}^{k} c_i v_i = PC$$

then

$$\begin{aligned} \dot{x} = P\dot{C} &= APC \\ &= P\mathcal{U}C. \end{aligned}$$

Remark. s

Example. Consider $\dot{x} = Ax$ with

$$A = \begin{bmatrix} -1 & 1 & -2 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now the eigenvalue/eigenvectors pairs for A are $\lambda_3 = 1$ and $v_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$, and $\lambda_1 = -1 = \lambda_2$ with $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Definition. hyperbolic system. A **hyperbolic system** is one where all the eigenvalues have non-zero real part.

Give an example of a spectrally stable hyperbolic matrix.
Consider the matrix

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

where all eigenvalues are negative, real, and nonzero. Thus this matrix is spectrally stable and hyperbolic.

Theorem 10.1. If A is an $n \times n$ matrix and $x_0 \in E^S$, the stable subspace of A, then there are constants $K \geq 1$ and $\alpha > 0$ such that $|e^{tA}x_0| \leq Ke^{-\alpha t}|x_0|$, $\forall t \geq 0$. As a result $e^{tA}x_0 \to 0$ as $t \to \infty$.

Proof. The general solution is

$$e^{tA}x_0 = Pe^{t\Lambda}P^{-1}\left(\sum_{j=0}^{n-1} \frac{(tN)^j}{j!}\right) x_0.$$

Since $x_0 \in E^S$ and E^S is invariant with dimension n_S, we consider $e^{At}\big|_{E^S}$. Each element is a linear combination of stable eigenvectors with $t^k e^{a_j t} e^{ib_j t}$, $\lambda_j = a_j \pm ib_j$ with $a_j < 0$ and $k < n_S$.

This restriction has nilpotency at most n_k, the algebraic multiplicity λ_k, which implies that the order of the polynomial in t is n_{k-1}. So

$$\left(e^{tA}\big|_{E^S}\right)_{l_m} = \sum_{j=1}^{n_S} \sum_{k=0}^{n_j-1} t^k e^{a_j t}(c_j k l_m \cos(b_j t) + d_j k l_m \sin(b_j t))$$

for some coefficients $c_j k l_m$ and $d_j k l_m$. Choose $\alpha > 0$ such that $a_j < -\alpha < 0$. Then there exists a K such that

$$t^{n_s} e^{\alpha + a_j)t} \sqrt{c{j_k}^2 l_m + d{j_k}^2 l_m} < \frac{k}{n_S}$$

for all $j \in [1, n_S]$ and $lm \in [1, n]$, for all $t \geq 0$. So each term is bounded by $\frac{k}{n_S} e^{-\alpha t}$ which implies

$$|e^{tA} x_0| \leq K e^{-\alpha t} |x_0|$$

for all $t \geq 0$. This implies that $e^{tA} x_0 \to 0$ as $t \to \infty$. $\square$

A stronger concept of spectral stability is called **linear stability**.

10.2 Linear Stability

Definition. Linear stability A linear system is **linearly stable** if all its solutions are bounded as $t \to \infty$.

Any $x_0 \in E^S$ has bounded solutions for any $t > 0$.
Any $x_0 \in E^U$ has unbounded solutions for $t \to \infty$.
For $x_0 \in E^C$, all centers have bounded solutions, but solutions with $Re(\lambda_i) = 0$ and with algebraic multiplicity > algebraic multiplicity there can be unbounded solutions because of polynomial growth.

Remark. Does spectral stability imply linear stability? **No**, because of $x_0 \in E^C$ with $Re(\lambda_i) = 0$ with algebraic multiplicity > algebraic multiplicity, which results in polynomial growth at $t \to \infty$.

Remark. Does linear stability imply spectral stability? **Yes**.

10.3 Asymptotic Linear Stability

Definition. Asymptotic Linear Stability A linear system is asymptotically linearly stable if all solutions approach 0 as $t \to \infty$.

Remark. If a system is asymptotically linearly stable then $E = E^S$.

The following are true:

1. Asymptotic linear stability implies spectral stability
2. Asymptotic linear stability implies linear stability
3. Spectral stability does not imply asymptotic linear stability

Theorem 10.2. The $\lim_{t \to \infty} e^{At} x_0 = 0$ for all x_0 if and only if all eigenvalues of A have negative real part.

Proof. If all eigenvalues have negative real parts then $e^{tA}x_0 \to 0$ as $t \to \infty$.

Conversely, if there is an eigenvalue with positive real part then there is an initial condition in the eigenspace corresponding to the eigenvalue so that the solution grows without bound. Finally if there is an eigenvalue with 0 real part, then solutions in this subspace have terms of the form $t^j e^{iIm(\lambda_J)t}$ and do not go to zero. $\square$

Example. Consider the system $\dot{x} = Ax$ where

$$A = \begin{bmatrix} -2 & -1 & -2 \\ -2 & -2 & -2 \\ 2 & 1 & 2 \end{bmatrix}.$$

This matrix has characteristic equation $\lambda^3+2\lambda^2 = 0$ and so $\lambda^2(\lambda+2) = 0$. Therefore, $\lambda = -2$ with algebraic multiplicity of 1 and $\lambda = 0$ with algebraic multiplicity of 2.

Is this system:

1. spectrally stable? **Yes**
2. hyperbolic? **No**
3. linearly stable? **No**
4. asymptotically stable? **No**.

For this system, $E^S =$ span $\{V_3\}$ where V_3 is the eigenvector associated with $\lambda = -2$, $E^C =$ span $\{V_1, V_2\} = E\ E^S$, where V_1 and V_2 are eigenvectors associated with $\lambda = 0$, and there is no unstable subspace. The solution for this system is $e^{tA} = Pe^{tA}P^{-1}(I + tN) = e^{0t}(a + bt) + ce^{-2t}$.

11 Nonautonomous Linear Systems and Floquet Theory

Consider $\dot{x} = Ax + f(t)$ where $f(t)$ is for example, an impulse function.

Floquet (1880) studied systems of this type with periodic functions $A(t+T) = A(t)$.

Recall the fundamental solution matrix $\Phi(t, t_0)$ solves $\frac{d}{dt}\Phi = A\Phi$. Note that $\Phi(t, t_0) = \Phi$ when $x(t_0) = x_0$ is $x(t) = \Phi(t, t_0)x_0$.

Recall $\varphi(t, v) = \varphi(t, s)\varphi(s, r)$ for all $s, t, r \in \mathbb{R}$. Therefore, it doesn't matter if you stop in the middle, you end up in the same place.

Now recall that if A is a constant then the solution to the linear system is e^{At} and so maybe when A is nonconstant the solution is $e^{\int_{t_0}^{t} A(s)ds}$? **No**, this solution does not work for nonconstant A. This doesn't work because $A(s)$ doesn't necessarily commute with $A(t)$.

Example. Consider the example where

$$A(t) = \begin{bmatrix} -1 + \alpha \cos^2 t & 1 - \alpha \cos t \sin t \\ -1 - \alpha \cos t \sin tt & -1 + \alpha \sin^2 t \end{bmatrix}$$

and so $A(2\pi + t) = A(t)$. The eigenvalues are $\lambda = 1/2(\alpha - 2 \pm \sqrt{\alpha^2 - 4})$, which implies that when $\alpha <$, the real part of λ is less than zero and so the system is stable.

However, the solution of $\dot{x} = A(t)x$ is

$$x_1(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} e^{(\alpha-1)t}, \quad x_2(t) = \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} e^{-t}.$$

Compare solution and stability conclusions.

But for $\alpha > 1$, $x(t)$ is unbounded and the origin is unstable.

For the system $\dot{x} = Ax$ and for $A(t) = A(t+T)$, we defined the **monodromy matrix** $M = \varphi(T, 0)$.

Then the solution after period T for the initial value problem $x(0) = x_0$ is $x(T) = M_{x_0}$.

So what is $X(2T)$? $X(2T) = M^2 x_0$

$X(T) = Mx_0$

Now let $x_0 = Mx_0$ and then what is the time T map of the initial condition M_{x_0}? $M(Mx_0) = M^2 x_0 = X(2T)$.

So we need to compute M^n. The eigenvalues of M^n are called **floquet multipliers**. If x is an eigenvector of M with eigenvalue μ then

$$x(nT) = \mu^n x_0 = e^{n \ln \mu} x_0$$

where $\ln \mu$ is the **floquet exponent**.

Now go back to the previous example.

If $x_1(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $x_2(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ then evaluating at $t = 2\pi$, given

$$M = \Phi(2\pi, 0) = \begin{bmatrix} e^{2\pi(\alpha-1)} & 0 \\ 0 & e^{-2\pi} \end{bmatrix}.$$

The floquet multipliers are $\mu_1 = e^{2\pi(\alpha-1)}$ and $\mu_2 = e^{-2\pi}$. So when $\alpha > 1$, then $2\pi(\alpha - 1) > 0$ and $-2\pi < 0$.

Theorem 11.1. (Abel) The determinant of the fundamental matrix is

$$\det(\varphi(t, t_0)) = \exp\left(\int_{t_0}^{t} \operatorname{tr}(A(s)) ds\right).$$

Remark. This is a generalization of the Wronskian we saw in MATH 225

12 Logarithm of a Matrix

Floquet exponents are defined to be $\ln \mu_j$. So we can also define $\ln M$.

Recall $\ln(1-x) = -\sum_{j=1}^{\infty} \frac{x_j}{j}$, which converges when $|x| < 1$. Also note that $\ln M = \ln(I - (I - M))$, which converges when $||I - M|| \leq 1||$.

Lemma. *Any nonsingular matrix A has a not unique and possibly complex logarithm*

$$\ln A = P \ln \Lambda P^{-1} - \sum_{j=1}^{n} \frac{(-S^{-1}N)^J}{j}$$

where $A = S + N$, $S = P\Lambda P^{-1}$ is a diagonlizable, N is nilpotent, Λ is a diagonal matrix, P is the matrix of generalized eigenvalues of A.

Proof. First consider a semisimple nonsingular matrix S, so there exists a P such that $P^{-1}SP = \Lambda$, where Λ is diagonal with nonzero but possibly complex entries. Define $\ln \Lambda = \operatorname{diag}(\ln \Lambda_{i,i})$, so we have $e^{\ln \Lambda}$ and $S = Pe^{\ln \Lambda}P^{-1} = \exp(P \ln \Lambda P^{-1})$. So $\ln S = P \ln \Lambda P^{-1}$ and so $\ln S$ exists for S semisimple and nonsingular.

Now let N be nilpotent. Define

$$B = -\sum_{j=1}^{\infty} \frac{(-N)^j}{j} = -\sum_{j=1}^{n-1} \frac{(-N)^j}{j}.$$

Note that this infinite sum converges because it is equal to a finite sum and so is well defined. The equality holds because N is nilpotent.

Claim: $e^B = I + N$.

$$\begin{aligned} e^B &= \sum_{k=0}^{\infty} \frac{1}{k!} \left(-\sum_{j=1}^{\infty} \frac{(-N)^j}{j} \right)^k \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \left(N - \frac{N^2}{2} + \frac{N^3}{3} + \frac{N^4}{4} + \cdots \right)^k \\ &= I + N \end{aligned}$$

Note that $[N^j, N^k] = 0$.

So $B = \ln(I + N)$.

In the general case $A = S + N = S(I + S^{-1}N)$. Note that $N = S^{-1}N$.

Since N is nilpotent $[S, N] = 0$ then $S^{-1}N$ is nilpotent because if $N^k = 0$ then $(S^{-1}N)^k = S^{-k}N^k = 0$. (because $N = 0$)

$\ln A = \ln S + \ln(I + S^{-1}N)$

Note that $[S, I + S^{-1}N] = 0$ so by definition $[\ln S, \ln(I + S^{-1}N)] = 0$.

Then

$$e^B = e^{\ln S + \ln(I+S^{-1}N)} = e^{\ln S} e^{\ln(I+S^{-1}N)} = S(I + S^{-1}N) = A.$$

□

Theorem 12.1. (Floquet 1883) Let M be the monodromy matrix for a T-periodic linear system $\dot{x} = A(t)x$ and $TB = \ln M$. Then there exists a T-periodic matrix P such that the fundamental solution is $\Phi(t, 0) = P(t)e^{tB}$.

Proof. Let $\Psi(t) = \varphi(t+T, 0)$. Since A is periodic, then $\frac{d\Psi}{dt} = A(t+T)\Psi = A(t)\Psi$ with $\Psi(0) = M$. Since φ is the fundamental solution matrix, every solution $\Psi(t)$ is of the form $\varphi(t,0)x(0)$, so $\Psi(t) = \varphi(t,0)M$ and $\varphi(t+T, 0) = \varphi(t,0)M = \Psi(t,0)e^{TB}$.

Since e^{TB} is nonsingular, define $P(t) = \varphi(t,0)e^{-tB}$ so that

$$\begin{aligned} P(t+T) &= \varphi(t+T, 0)e^{-(t+T)B} \\ &= \varphi(t,0)e^{TB}e^{-(t+T)B} \\ &= \varphi(t,0)e^{-tB} \\ &= P(t). \end{aligned}$$

Therefore P is T-periodic. □

Theorem 12.2. Let φ bet he fundamental solution matrix for the T-periodic linear system $\dot{x} = A(t)x$ with $x(t_0) = x_0$. Then there exists a real 2T-periodic matrix A and a real matrix R such that $\varphi(t,0) = Q(t)e^{tR}$.

Proof. For any nonsingular matrix M, there exists a real matrix R such that $M^2 = e^{TR}$. Define $Q(t) = \varphi(t,0)e^{-tR}$. Then

$$\begin{aligned} Q(t+2T) &= \varphi(t+2T, 0)e^{-2TR}e^{-tR} \\ &= \varphi(t,0)M^2M^{-2}e^{-tR} \\ &= Q(t). \end{aligned}$$

Therefore Q is 2T-periodic □

13 Existence and Uniqueness for Non-Linear Systems

Definition. A **metric space** is a vector space V with a distance function $\rho(f,g) : V \times V \to \mathbb{R}$ such that

1. $\rho(f,g) \geq 0$, $\rho(f,g) = 0$ only when $f = g$,
2. $\rho(f,g) = \rho(g,f)$,
3. $\rho(f,h) \leq \rho(f,g) + \rho(g,h)$.

For a normed space we have that $\rho(f,g) = ||f-g||$.

Definition. A **Banach Space** is a normed linear space.

Definition. A **complete space** is a space X where every Cauchy sequence in X converges to an element in X.

Theorem 13.1. Contraction Mapping Theorem. Let $T : X \to X$ be a map in a complete metric space X. If T is a contraction, i.e., for all $f, g \in X$ there exists a constant $0 < c < 1$ such that $\rho(T(f), T(g)) \leq c\rho(f,g)$, then T has a unique fixed point $f^* = T(f^*) \in X$.