Maximum Sustainable Yield

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Notes on deriving the formula for maximum sustainable yield.

1 Population Model

Suppose a population grows according to the logistic growth model,

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) \tag{1}$$

where r is the intrinsic rate of increase of the population, N is the number of individuals alive in the population, and K is the maximum population size, also called the carrying capacity. Now we include the effects of harvesting individuals, which removes them from the population,

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - H \tag{2}$$

where H is the amount of individuals removed. This can be represented as a product of the instantaneous fishing mortality rate F and N,

$$H = FN \tag{3}$$

so that our model now can be written as

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - FN. \tag{4}$$

To determine the equilibrium points for equation (4) we set $\frac{dN}{dt} = 0$ and solve for N,

$$\begin{aligned} rN\left(1 - \frac{N}{K}\right) - FN &= 0 \\ rN - rN\frac{N}{K} - FN &= 0 \\ N(r - r\frac{N}{K} - F) &= 0 \end{aligned}$$

and we see there are two equilibria when

$$N = 0 \tag{5}$$

or

$$r - r\frac{N}{K} - F = 0. \tag{6}$$

Since we are not interested in the population when $N = 0$ we will focus on the second equilibrium point. Solving the second equilibrium for N we find the equilibrium population abundance N_e,

$$N_e = K\left(1 - \frac{F}{r}\right). \tag{7}$$

If we define yield, Y, as the number of fish harvested, then $Y = FN$ and the equilibrium yield is

$$Y = FN_e. \tag{8}$$

Substituting equation (7) into equation (8), taking the first derivative with respect to F, setting the first derivative equal to zero, and then solving for F will give us the equilibrium fishing mortality rate F_e. First we substitute equation (7) into equation (8) and take the derivative with respect to F,

$$\frac{dY}{dF} = \frac{d}{dF}F\left(K\left(1 - \frac{F}{r}\right)\right) = K\left(1 - \frac{2F}{r}\right). \tag{9}$$

Then we set the first derivative equal to zero

$$\frac{dY}{dT} = K - \frac{2FK}{r} = 0 \tag{10}$$

and then solve for F to get the equilibrium fishing mortality rate,

$$F_e = \frac{r}{2} \tag{11}$$

Then we substitute equation (7) and (11) into equation (8) to obtain the equilibrium yield,

$$Y_e = F_e N_e = \frac{r}{2}\left(K\left(1 - \frac{\frac{r}{2}}{r}\right)\right) = \frac{Kr}{2} - \frac{K(\frac{r}{2})^2}{r} = \frac{Kr}{4} \tag{12}$$

2 Appendix

The details of some calculations presented in the main text are here in the appendix.

Solving equation (6) for N to get equilibrium abundance.

$$\begin{aligned}
r - r\frac{N}{K} - F &= 0 \\
r - F &= r\frac{N}{K} \\
\frac{N}{K} &= \frac{r-F}{r} \\
N_e &= K\left(\frac{r-F}{r}\right) = K\left(\frac{r}{r} - \frac{F}{r}\right) = K\left(1 - \frac{F}{r}\right)
\end{aligned}$$

Substituting equation (7) into equation (8), taking the first derivative with respect to F,

$$
\begin{aligned}
\frac{dY}{dF} = \frac{d}{dF}Y = \frac{d}{dF}F\left(K\left(1 - \frac{F}{r}\right)\right) \\
&= \frac{d}{dF}F\left(K - \frac{FK}{r}\right) \\
&= \frac{d}{dF}FK - \frac{F^2K}{r} \\
&= \frac{d}{dF}FK - \frac{d}{dF}\frac{F^2K}{r} \\
&= K - \frac{2FK}{r} \\
&= K\left(1 - \frac{2F}{r}\right)
\end{aligned}
$$

Solve equation (10) for F to get equilibrium fishing mortality rate,

$$
\begin{aligned}
\frac{2FK}{r} &= K \\
2FK &= Kr \\
F_e &= \frac{Kr}{2K} \\
F_e &= \frac{r}{2}
\end{aligned}
$$