Maximum Sustainable Yield

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Notes on deriving the formula for maximum sustainable yield.

Population Model

Suppose a population grows according to the logistic growth model,

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)$$

where r is the intrinsic rate of increase of the population, N is the number of individuals alive in the population, and K is the maximum population size, also called the carrying capacity. Now we include the effects of harvesting individuals, which removes them from the population,

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - H$$

where H is the amount of individuals removed. This can be represented as a product of the instantaneous fishing mortality rate F and N,

$$H = FN$$

so that our model now can be written as

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - FN.$$

To determine the equilibrium points for equation # we set $\frac{dN}{dt} = 0$ and solve for N,

$$\begin{aligned} rN\left(1 - \frac{N}{K}\right) - FN &= 0 \\ rN - rN\frac{N}{K} - FN &= 0 \\ N(r - r\frac{N}{K} - F) &= 0 \end{aligned}$$

and we see there are two equilibria when

$$N = 0$$

or

$$r - r\frac{N}{K} - F = 0.$$

Solving the second equlibrium for N we find the equilibrium population abundance N_e,

$$\begin{aligned}
r - r\frac{N}{K} - F &= 0 \\
r - F &= r\frac{N}{K} \\
\frac{N}{K} &= \frac{r-F}{r} \\
N_e &= K\left(\frac{r-F}{r}\right) = K\left(\frac{r}{r} - \frac{F}{r}\right) = K\left(1 - \frac{F}{r}\right)
\end{aligned}$$

If we define yield Y, or the number of fish harvested, as $Y = FN$ then the equilibrium yield is

$$Y = FN_e.$$

Substituting equation # into equation #, taking the derivative with respect to F, and then solving for F will give us the equilibrium fishing mortality rate F_e. First we substitute equation # into equation # and take the derivative with respect to F,

$$\begin{aligned}
\frac{dY}{dF} = \frac{d}{dF}Y &= \frac{d}{dF}F\left(K\left(1 - \frac{F}{r}\right)\right) \\
&= \frac{d}{dF}F\left(K - \frac{FK}{r}\right) \\
&= \frac{d}{dF}FK - \frac{F^2K}{r} \\
&= \frac{d}{dF}FK - \frac{d}{dF}\frac{F^2K}{r} \\
&= K - \frac{2FK}{r} \\
&= K\left(1 - \frac{2F}{r}\right)
\end{aligned}$$

Then we set the first derivative equal to zero and solve for F to get the equilibrium fishing mortality rate,

$$\frac{dY}{dT} = K - \frac{2FK}{r} = 0$$

and then

$$\begin{aligned}
\frac{2FK}{r} &= K \\
2FK &= Kr \\
F_e &= \frac{Kr}{2K} \\
F_e &= \frac{r}{2}
\end{aligned}$$

Then we substitute equation # and # into equation # to obtain the equilibrium yield,